Geometric Quantization and Dynamical Constructions on the Space of Kähler Metrics

by

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Abstract

This Thesis is concerned with the study of the geometry and structure of the space of Kähler metrics representing a fixed cohomology class on a compact Kähler manifold.

The first part of the Thesis is concerned with a problem of geometric quantization: Can the geometry of the infinite-dimensional space of Kähler metrics be approximated in terms of the geometry of the finite-dimensional spaces of Fubini-Study Bergman metrics sitting inside it? We restrict to toric varieties and prove the following result: Given a compact Riemannian manifold with boundary and a smooth map from its boundary into the space of toric Kähler metrics there exists a harmonic map from the manifold with these boundary values and, up to the first two derivatives, it is the limit of harmonic maps from the Riemannian manifold into the spaces of Bergman metrics. This generalizes previous work of Song-Zelditch on geodesics in the space of toric Kähler metrics.

In the second part of the Thesis we propose the study of certain discretizations of geometric evolution equations as an approach to the study of the existence problem of some elliptic partial differential equations of a geometric nature as well as a means to obtain interesting dynamical systems on certain infinite-dimensional spaces. We illustrate the fruitfulness of this approach in the context of the Ricci flow as well as another flow on the space of Kähler metrics. We introduce and study dynamical systems related to the Ricci operator on the space of Kähler metrics that arise as discretizations of these flows. As an application, we address several questions in Kähler geometry related to canonical metrics, energy functionals, the Moser-Trudinger-Onofri inequality, Nadel-type multiplier ideal sheaves, and the structure of the space of Kähler metrics.

Thesis Supervisor: Gang Tian
Title: Professor of Mathematics, Princeton University

To my parents

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Y.A.R.
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CHAPTER 1

A bird's eye view

This Thesis describes research concerning the geometry and structure of the space of Kähler metrics on a Kähler manifold focused on two aspects: (i) approximation of harmonic maps into this space, (ii) behavior of natural dynamical systems on this space related to canonical metrics. About a quarter of the Thesis' original results are contained in part (i), and about three-quarters are contained in part (ii).

Kähler manifolds form a rich family of complex manifolds whose study brings together techniques mainly from partial differential equations (PDEs), Riemannian geometry, complex analysis, and algebraic geometry. The simplest examples of such manifolds are Riemann surfaces. To a considerable extent, Kähler geometry seeks to generalize to higher dimensions classical facts about Riemann surfaces, for example the theorem that every Riemann surface admits a metric of constant scalar curvature and the study of the geometry of the moduli space of Riemann surfaces of a fixed genus. For surfaces, these objects are known to encode deep geometrical and topological information on the underlying manifold and have attracted the attention of many mathematicians and physicists since the time of Gauss and Riemann. In higher dimensions however our understanding of the most basic questions, such as the existence of canonical metrics and the structure of certain moduli spaces associated to Kähler manifolds, is still very far from complete.

Two major themes in the field emerged from fundamental work of Calabi and Mabuchi in the 1980's: on the one hand, the study of the (moduli) space of all Kähler metrics representing a fixed cohomology class on a given Kähler manifold; on the other hand, the search for canonical metrics on such manifolds. These canonical metrics are distinguished points in the moduli space. These two themes are to a large extent intertwined, as we will attempt to emphasize in this Thesis. One particular unifying aspect of their study is the profound analysis of a variety of complex Monge-Ampère equations. The complex Monge-Ampère equation is one of the principle examples of a fully nonlinear second order elliptic PDE. On a domain in $\mathbb{C}^n$ it takes the form

$(\sqrt{-1}\partial\bar{\partial}u)^n = F(u)dV$, where dV is the Euclidean volume form, F is a nonnegative function and u is the unknown function.

This Thesis is concerned with the basic geometry and structure of the space of Kähler metrics associated to a Kähler manifold and its relation to finite-dimensional spaces of distinguished metrics sitting inside it.

There are two kinds of distinguished finite-dimensional spaces. The first are the canonical metrics on the manifold. These arise most frequently from a variational problem, e.g., Calabi's extremal metrics that are critical points of the average of the scalar curvature squared. Here we sweep under the rug the fact that it is a non-trivial problem to prove that these sets are finite-dimensional and have a particular structure. The second are canonical metrics on another manifold, pulled back to the original manifold via an embedding. The principal construction here is that of Kodaira's projective embeddings that exhibits any Kähler manifold possessing an integral Kähler class as a projective subvariety in an infinite number of projective spaces in a canonical manner. Here again a technical issue is that the integrality condition amounts to restricting to the class of projective Kähler manifolds. The canonical metrics on projective space are the Fubini-Study metrics that are both Kähler and Einstein and their pull-backs on the original manifold are called Bergman metrics.

A fascinating aspect of the theory is that all of the spaces involved have symmetric space structures as Riemannian manifolds, in the sense of Cartan. For the Bergman space of Fubini-Study metrics coming from a fixed projective embedding into $\mathbb{P}^d$ it is manifest that the space is isomorphic to the symmetric space $GL(d,\mathbb{C})/U(d)$. For the space of constant scalar curvature metrics a corollary of recent fundamental work of Chen and Tian shows that this space is either empty or isomorphic to the connected symmetric space dual to the isometry group of one of the constant scalar curvature metrics. Finally, the space of Kähler metrics itself is also a symmetric space when this notion is properly generalized to infinite dimensions, as shown by Semmes and Donaldson.

A word about some additional motivation behind the study of the space of Kähler metrics $\mathcal{H}_\omega$, the main one being its intrinsic beauty and complexity. Fundamental works in the last two decades have shown that the geometry and structure of $\mathcal{H}_\omega$ encapsulate deep geometrical information on the Kähler manifold M itself. In particular, a better understanding of $\mathcal{H}_\omega$ would provide considerable insight into some of the major problems in Kähler geometry and related fields, most notably the study of holomorphic foliation structures induced by Monge-Ampère equations, the existence of canonical metrics and the construction of multiplier ideal sheaves in their absence, the effective approximation of canonical metrics by algebraic and essentially computable objects, the relation between canonical metrics and notions of stability in Geometric Invariant Theory, and the study of the Calabi and Ricci flows.

1.1 A problem of geometric quantization

The first part of the Thesis, that comprises Chapter 3, contains about a quarter of the Thesis' original results. In this part we study a problem of geometric quantization: Is the geometry of the infinite-dimensional space of Kähler metrics approximated in terms of the geometry of the finite-dimensional spaces of Bergman metrics? This line of research goes back to the fundamental result of Tian that exhibited the space of Kähler metrics as the topological closure of the spaces of Bergman metrics. It is further motivated by the fact that both the finite- and infinite-dimensional spaces involved are symmetric, strongly suggesting—in light of Tian's theorem—that it might be true that the geometry of the Bergman spaces approximates the geometry of its topological limit. Moreover, this problem may be fitted into the framework of geometric quantization, a differential geometric theory of quantum mechanics, and the intuition there suggests that such a semi-classical limit/approximation should make sense.

We now describe the setting in order to be able to summarize our results. Let M be a projective Kähler manifold, and let ω be a Kähler form on it. Hodge theory implies that any Kähler form that lies in the same cohomology class as ω may be written in the form $\omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi$ with φ a smooth function on M. Up to an identification between functions φ and the corresponding forms ω_φ, the space of Kähler metrics representing the cohomology class $[\omega]$ is thus an infinite-dimensional space given by

$$\mathcal{H}_\omega = \{\varphi \ : \ \omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi > 0\} \subset C^\infty(M). \tag{1}$$

Following Mabuchi, Semmes and Donaldson this space becomes an infinite-dimensional Riemannian symmetric space when equipped with the L^2 metric defined by

$$g_{L^2}(\psi,\eta)_\varphi := \int_M \psi\eta\,\omega_\varphi^n, \qquad \psi,\eta \in T_\varphi\mathcal{H}_\omega = C^\infty(M).$$

It has been conjectured that the space of Kähler metrics in a fixed cohomology class on a projective Kähler manifold should be well-approximated by the much simpler finite-dimensional spaces of Bergman metrics obtained via pull-back from a succession of embeddings of the manifold into projective spaces of increasing dimension. To describe this more precisely, if $L \to M$ is an ample line bundle with $c_1(L) = [\omega]$, then for each sufficiently large $k \in \mathbb{N}$ the Kodaira Embedding Theorem says that a basis of the vector space $H^0(M, L^k)$ of holomorphic sections of $L^k \to M$ furnishes a holomorphic embedding of M into projective space $\mathbb{P}^{d_k-1}$, where $d_k = \dim_{\mathbb{C}} H^0(M, L^k)$. By pulling back Fubini-Study metrics on $\mathbb{P}^{d_k-1}$ one obtains a finite-dimensional space of so-called Bergman metrics isomorphic to the nonpositively curved symmetric space

$$\mathcal{H}_k = GL(d_k, \mathbb{C})/U(d_k), \tag{2}$$

sitting inside the infinite-dimensional space $\mathcal{H}_\omega$. The conjecture then is that the geometry of the space $\mathcal{H}_\omega$ should be well-approximated by the geometry of the spaces $\mathcal{H}_k$ in the "semi-classical" limit where the Planck constant $1/k$ tends to zero. The fundamental theorem motivating this conjecture, due to Tian, is that the finite-dimensional spaces $\mathcal{H}_k$ do approximate $\mathcal{H}_\omega$ topologically at least, on the level of individual points. Additional impetus is given by the fact that $\mathcal{H}_\omega$ can be regarded as the dual symmetric space to the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms.

A question in this direction was posed by Arezzo-Tian and Donaldson who asked whether geodesics in $\mathcal{H}_\omega$ are approximated by geodesics in $\mathcal{H}_k$, the latter being induced by one-parameter subgroups of $GL(d_k, \mathbb{C})$. Phong-Sturm studied this problem and proved a weak convergence (in an almost-everywhere sense) of a sequence of geodesics in $\mathcal{H}_k$ to a prescribed geodesic of $\mathcal{H}_\omega$. Song-Zelditch proved that the same sequence converges in $C^2([0,1] \times M)$ when the manifold is toric and one restricts to the torus-invariant metrics, and Berndtsson used a different argument to prove that geodesics in $\mathcal{H}_\omega$ can be C^0-approximated by geodesics in spaces of Bergman metrics induced by embeddings by sections of $L^k \otimes K_M$, where K_M is the canonical bundle of M.

More generally, Tian has posed the question whether one may approximate Wess-Zumino-Witten (WZW) maps in a similar manner. These are maps from a Riemann surface into $\mathcal{H}_\omega$ that are a perturbed version of harmonic maps.

We now describe the results of this part of the Thesis. We restrict to the setting of toric varieties and consider the space of torus-invariant potentials

$$\mathcal{H}_\omega(T) := \{\varphi \in \mathcal{H}_\omega : \omega_\varphi > 0, \quad g^\star\varphi = \varphi, \quad \forall g \in T\},$$

with respect to a torus-invariant form Kähler ω. We observe that in this setting the WZW equation reduces to the Laplace equation. Motivated by this observation we pose the problem of approximating harmonic maps of arbitrary dimension into $\mathcal{H}_\omega$. This problem seems difficult in the general projective setting. However our next observation is that for toric varieties the harmonic map equation may be solved rather simply using the Legendre transform that linearizes it just like in the two-dimensional setting of WZW maps. The use of the Legendre transform to study toric manifolds is rooted in the work of Guillemin. Moreover we show that the Legendre transform has the remarkable property of transforming the whole Eells-Sampson harmonic map flow into the usual heat flow (see Theorem 3.28).

Theorem 1.1. *Under the Legendre tranform the Eells-Sampson harmonic map flow on the space of Kähler potentials is mapped to the heat flow on the space of symplectic potentials.*

Next we use some of the important tools developed by Song-Zelditch on the asymptotics of toric monomials to generalize their computations from the one-dimensional case of geodesics to the higher-dimensional one of harmonic maps. By using the Schauder estimates, the computations carry over without major difficulties. This allows us to give the following affirmative answer regarding approximation of harmonic maps on the space of Kähler metrics on a toric variety (see Theorem 3.25).

Theorem 1.2. *Let (M, L, ω) be a polarized toric Kähler manifold, and let (N, f) be a compact oriented smooth Riemannian manifold with smooth boundary ∂N. Let $\psi : \partial N \to \mathcal{H}_\omega(T)$ denote a fixed smooth map. There exists a harmonic map $\varphi : N \to \mathcal{H}_\omega(T)$ with $\varphi|_{\partial N} = \psi$ and harmonic maps $\varphi_k : N \to \mathcal{H}_k(T)$ whose boundary values are obtained from Tian's theorem and one has $\lim_{k\to\infty} \varphi_k = \varphi$ in the $C^2(N \times M)$ topology.*

As a direct corollary we obtain an explicit approximation scheme, in the setting of toric varieties, to the much studied homogeneous complex Monge-Ampère equation

$$\begin{aligned} (\pi_2^\star\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^{n+1} = 0, &\quad \text{on } \Sigma \times M, \\ \varphi = \psi, &\quad \text{on } \partial\Sigma \times M, \end{aligned}$$

(let $\pi_2 : \Sigma \times M \to M$ denote the projection) that describes WZW maps from a Riemann surface Σ with boundary $\partial\Sigma$ isomorphic to S^1 into the space $\mathcal{H}_\omega(T)$ (see Corollary 3.26).

1.2 Discretizations of geometric flows

The second part of the Thesis, that comprises Chapter 4, contains about three-quarters of the Thesis' original results. In this part we propose the systematic study of certain discretizations of geometric flows as an approach to the study of some elliptic PDEs of a geometric nature as well as a means to obtain interesting dynamical systems on certain infinite-dimensional spaces. The applications we have in mind are to the study of how the structure of the space of Kähler metrics is related to the finite-dimensional canonical spaces of Kähler-Einstein metrics or constant scalar curvature metrics inside it.

We illustrate the fruitfulness of this approach in the context of the Ricci flow as well as another flow on the space of Kähler metrics. Quite surprisingly, these discretizations give rise to remarkable new operators on the space of Kähler metrics whose dynamics are still far from understood. While we only make a small step towards the understanding of the limiting behavior of these dynamical systems in this Thesis, we believe that they are highly intersting and merit further study. Nevertheless we are able to use these constructions to address several questions in Kähler and conformal geometry.

Let us summarize the results of this part of the Thesis. In the past, two basic methods were used to construct canonical Kähler metrics. First, the continuity method was used by Aubin and Yau in the 1970's to study Kähler-Einstein metrics of nonpositive Ricci curvature. Second, the Ricci flow was introduced by Hamilton in 1982 and was used by Cao in 1985 to obtain a new proof of the Aubin-Yau results. Each method deforms an arbitrary initial metric in a different way. The first is more "linear" in nature, while the second corresponds to a "heat" flow on $\mathcal{H}_\omega$. In Chapter 4 we introduce a new method—that we call the Ricci iteration—for the construction of Kähler-Einstein metrics by discretizing the Ricci flow and other related flows on the space of Kähler metrics. While the idea is simple, this construction turns out to be instrumental in solving several open problems in Kähler geometry. Given a fixed $\tau > 0$, assume that the first Chern class has a sign $\mu \in \{1, -1, 0\}$. Then the time τ Ricci iteration is defined as the sequence of Kähler metrics $\{\omega_j\}_{j\geq 1}$ satisfying the equations

$$\omega_k - \omega_{k-1} = -\tau \operatorname{Ric}\omega_k + \tau\mu\omega_k, \quad \forall k \in \mathbb{N}, \qquad \omega_0 = \omega. \tag{3}$$

These equations are equivalent to a sequence of non-degenerate complex Monge-Ampère equations.

First, we show that for some values of τ and μ the iteration will converge to a Kähler-Einstein metric when a unique such metric exists (Theorem 4.5).

Theorem 1.3. *Let (M, J) be a compact Kähler manifold admitting a unique Kähler-Einstein metric. Let Ω be a Kähler class such that $\mu\Omega = c_1$ with $\mu \in \mathbb{R}$. Then for any $\omega \in \mathcal{H}_\Omega$ and for any $\tau > 1/\mu$ (when $\mu > 0$) or $\tau > 0$ (when $\mu \leq 0$), the time τ Ricci iteration exists for all $k \in \mathbb{N}$ and converges to a Kähler-Einstein metric.*

The iteration prompts us to introduce a new operator on the space of Kähler metrics of a Fano manifold. We call this operator the inverse Ricci operator (Definition 4.24), and denote it by

$$\operatorname{Ric}^{-1} : \mathcal{H}_{c_1} \to \mathcal{H}_{c_1}.$$

Many of the applications we find are directly related to this operator.

Next, we introduce a family of new energy functionals I_k, $k = 0, \ldots, n$, on the space of Kähler metrics

$$I_k(\omega, \omega_\varphi) = \frac{1}{V}\int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{l=0}^{k-1} \frac{k-l}{k+1}\omega^{n-1-l} \wedge \omega_\varphi^l,$$

that generalize Aubin's functional J (indeed, $I_n = J$) and establish several of their fundamental properties. In passing we also obtain results on a related family of energy functionals J_k defined by Chen-Tian (see §§4.4.1). We apply this to prove a formula relating the Chen-Tian energy functionals E_k to the K-energy E_0 in a succinct manner (see Proposition 4.12).

Theorem 1.4. *The following relation holds on* $\mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$

$$E_k(\omega, \omega_\varphi) = E_0(\omega, \omega_\varphi) + I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) - I_k(\omega, \mathrm{Ric}\,\omega).$$

In addition we demonstrate a basic formula relating the Berger-Moser-Ding functional F_1 to the Ricci energy E_n (see Proposition 4.40):

Theorem 1.5. *The following relation holds on* $\mathcal{H}_{c_1} \times \mathcal{D}_{c_1}$

$$(\mathrm{Ric}^{-1})^\star E_n = F_1,$$

This gives for the first time a geometric interpreation for the functional F_1. This functional has a long history going back to the late 1960's when it was first studied in relation to Nirenberg's problem for prescribing scalar curvature on S^2 and is important in conformal geometry.

We introduce the following canonical subsets of the space $\mathcal{D}_{c_1}$ of smooth closed differential forms representing the first Chern class on a Fano manifold, related to the functionals I_k and E_k:

$$\begin{aligned} \mathcal{A}_k &= \{\omega_\varphi \in \mathcal{H}_{c_1} \,:\, E_k(\omega_{\mathrm{KE}}, \omega_\varphi) \geq 0\}, \\ \mathcal{B}_k &= \{\omega_\varphi \in \mathcal{H}_{c_1} \,:\, I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) \geq 0\}. \end{aligned}$$

The sets $\mathcal{A}_k$ are those for which the functional E_k is nonnegative on a Kähler-Einstein manifold. As a corollary of Theorem 1.4 we obtain a bound on these sets: $\mathcal{B}_k \subseteq \mathcal{A}_k$ and $\mathcal{H}^+_{c_1} \underset{\neq}{\subset} \mathcal{B}_k$ (here $\mathcal{H}^+_{c_1} \subset \mathcal{H}_{c_1}$ denotes the subset of Kähler metrics of positive Ricci curvature), and one may obtain bounds on $\mathcal{B}_k$ in terms of a lower negative bound on the Ricci curvature (see Theorem 4.42). This improves previous results of Bando-Mabuchi and Song-Weinkove.

Next we prove a monotonicity result for the Chen-Tian energy functionals along the Ricci iteration (see Proposition 4.17). The sets $\mathcal{B}_k$ appear here naturally.

Theorem 1.6. *i) The functional* E_0 *is monotonically decreasing along the time* τ *iteration* ($\tau > 0$) *whenever the initial point is not Kähler-Einstein.*
(ii) When $1/\mu = \tau = 1$ *the same is true for* F_1, E_1, *and, when the initial metric lies in* $\mathcal{B}_k \underset{\neq}{\supset} \mathcal{H}^+_{c_1}$, *also for* E_k, $k \geq 2$.

Now we are able to use the theorems above to solve two problems of X.-X. Chen regarding characterizations of analytic stability and semistability in terms of properness and boundedness of the Chen-Tian energy functionals E_j. These can be summarized as follows (Theorem 4.45).

Theorem 1.7. *Let $k \in \{0, \ldots, n\}$. Assume that $\mathrm{Aut}(M, \mathrm{J})$ is finite. The existence of a Kähler-Einstein metric on a Fano manifold is equivalent to the properness of the functional E_k or F_1 on $\mathcal{H}^+_{c_1}$. For F_1, E_0, E_1 one may replace $\mathcal{H}^+_{c_1}$ by $\mathcal{H}_{c_1}$.*

Theorem 1.8. *On a Fano manifold the energy functionals $F_1, E_0, \ldots, E_n$ are bounded from below on $\mathcal{H}^+_{c_1}$ if and only if one of them is. For F_1, E_0, E_1 one may replace $\mathcal{H}^+_{c_1}$ by $\mathcal{H}_{c_1}$.*

This generalizes previous fundamental work of Tian for the K-energy E_0 that was later extended to E_1 by Song-Weinkove. The proof of these last two theorems involves demonstrating an a priori estimate for solutions of a continuity family of Monge-Ampère equations as well as using Theorems 1.4–1.6.

Next, we prove that the Chen-Tian functionals E_k may be expressed in terms of Bott-Chern forms (Proposition 4.22), extending a result of Tian for the K-energy E_0. It then follows from Theorem 1.7 and the work of Tian that when a Kähler-Einstein metric exists then the manifold is GIT stable in the sense of Tian with respect to a family of virtual bundles, one for each k (see §§4.4.5).

The classical Moser-Trudinger-Onofri inequality on S^2

$$\frac{1}{V}\int_{S^2} e^{-\varphi+\frac{1}{V}\int_{S^2}\varphi\omega}\omega \le e^{\frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi}, \tag{4}$$

has been the topic of intensive studies in conformal geometry and describes a critical case of the Sobolev Embedding Theorem. All previous proofs of this inequality used substantially the special structure of S^2 in the form of symmetrization/rearrangement arguments that reduce it to the case of an interval. First, using Theorem 1.5 we are able to give a completely (complex-)geometric proof of this inequality. This proof moreover extends to higher dimensions and enables us to characterize a canonical enlargement MTO_n of the space of Kähler potentials on which the inequality

$$\frac{1}{V}\int_{M} e^{-\varphi+\frac{1}{V}\int_{M}\varphi\omega^n}\omega^n \le e^{J(\omega,\omega_\varphi)}, \tag{5}$$

continues to hold in higher dimensions, thus extending previous work of Ding-Tian. At the same time we also prove a result on the set on which the functionals E_k are bounded from below. For brevity we only state now the result regarding MTO_n (see Theorem 4.42 for more details):

Theorem 1.9. *Let (M, J, ω) be a Kähler-Einstein Fano manifold. The generalized Moser-Trudinger-Onofri inequality (5) holds precisely on the set $MTO_n = \mathrm{Ric}(\mathcal{A}_n)$. Furthermore, $\mathcal{H}_{c_1} \subsetneq MTO_n \subseteq \mathcal{D}_{c_1}$, and there exist $c_n > 0$ depending only on n such that*

$$MTO_n \supseteq \{\varphi \in C^\infty(M) \,:\, \omega_\varphi \ge -c_n \mathrm{Ric}^{-1}\omega_\varphi\},$$

and, e.g., $c_1 = \infty, c_2 \geq 2, c_3 \geq 1$.

Finally, we are able to prove an inequality strengthening the classical Moser-Trudinger-Onofri inequality on S^2 (and also in higher dimensions) using the inverse Ricci operator in a crucial way. We state the result for simplicity on S^2 (see Theorem 4.54 and Corollary 4.55):

Theorem 1.10. *Denote by* $(S^2, \mathrm{J}, \omega = \omega_{\mathrm{FS},2/V})$ *a round sphere of volume* V. *For any function* φ *on* S^2 *in* $W^{1,2}(S^2)$ *one has*

$$\frac{1}{V}\int_{S^2} e^{-\varphi+\frac{1}{V}\int_{S^2}\varphi\omega}\omega \leq e^{\frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi-\sum_{j=1}^{\infty}J(\mathrm{Ric}^{(-j)}\omega_\varphi,\mathrm{Ric}^{(-j+1)}\omega_\varphi)}. \tag{6}$$

Each of the terms in the sum is nonnegative, and equals zero if and only if ω_φ *is obtained from* ω *by a Möbius transformation. This also characterizes when equality holds in (6).*

Next we suppose a Kähler-Einstein metric does not exist on a Fano manifold. We are then able to use the Ricci iteration to describe the limiting behavior of the Ricci iteration in terms of Nadel-type multiplier ideal sheaves (see Theorem 4.58):

Theorem 1.11. *Let* (M, J) *be a Fano manifold not admitting a Kähler-Einstein metric. Let* $\gamma \in (1, \infty)$ *and let* $\omega \in \mathcal{H}_{c_1}$. *Then there exists a subsequence* $\{\psi_{j_k}\}_{k\geq 1}$ *of the time one Ricci iteration such that* $\lim_{k\to\infty}\psi_{j_k} = \psi_\infty \in PSH(M, \mathrm{J}, \omega)$ *and* $\mathcal{I}(\gamma\psi_\infty)$ *is a proper Nadel-type multiplier ideal sheaf.*

The inverse Ricci operator and the monotonicity result Theorem 1.6 combine to answer the following question of Nadel posed in the mid 1990's: Given $\omega \in \mathcal{H}_{c_1}$ define a sequence of metrics $\omega, \mathrm{Ric}\,\omega, \mathrm{Ric}(\mathrm{Ric}\,\omega), \ldots$, as long as positivity is preserved; what are the periodic orbits of this dynamical system? We prove the following result (see Theorem 4.60):

Theorem 1.12. *Let* (M, J, ω) *be a Fano manifold and assume that* $\mathrm{Ric}^l\,\omega = \omega$ *for some* $l \in \mathbb{Z}$. *Then* ω *is Kähler-Einstein.*

Moreover we generalize this question to the context of Kähler-Ricci solitons by defining a twisted inverse Ricci operator (Definition 4.35) and prove an analogous result in that context via a monotonicity result for a twisted energy functional (see Corollary 4.62).

Next, motivated by a new nonparabolic flow that can be defined on $\mathcal{H}_{c_1}$ we define via a discretization a generalized inverse Ricci operator on any Kähler manifold and class (see Definition 4.32). Iteration of this operator defines the dynamical system

$$\mathrm{Ric}\,\omega_k = H_{\omega_{k-1}}\mathrm{Ric}\,\omega_{k-1}, \ \ \forall k \in \mathbb{N}, \qquad \omega_0 = \omega. \tag{7}$$

Constant scalar curvature metrics are fixed points of this iteration. It would be interesting and highly challenging to understand this very natural iteration, canonically associated to each Kähler class. We also point out a counterpart of this for extremal metrics (Remark 4.33).

We end by defining the Ricci index on the space $\mathcal{H}_{c_1}$. It is a new notion of positivity of the Ricci curvature and corresponds to the existence of a canonical nested sequence of sets for which $\mathcal{H}_{c_1}$ and $\mathcal{H}_{c_1}^{+}$ are the first two (see §§4.10.7). We conclude by posing several questions related to it.

1.3 Organization

The Thesis is organized as follows. In Chapter 2 we discuss fundamental topics in Kähler geometry ranging from the very elementary notions to more advanced topics in order to set the stage for the main part of the Thesis in the later chapters. Some familiarity with the notions discussed will be helpful although it is not essential, and we have made considerable efforts to be as self-contained as possible. In some discussions we have emphasized intuition and explicitness to an extreme, almost at the price of some amount of rigor and perhaps at the risk of imprecisions. We have taken this liberty only in discussions concerning background material and not in the rigorous development of our own results. Our intention was to guide along our readers as far as possible, trying to convey as much of our intuition as possible. Here we are following our own preference and inclination, and in some sense it is true that this Thesis is primarily written with the target audience the author himself younger by several years.

In Chapter 3 we describe our results on quantizing harmonic maps into the space of Kähler metrics on a toric variety. While the results on this topic comprise about a quarter of those in the Thesis the length of this chapter is comparable to that of the next one. This is accounted for by the considerable amount of elementary background material that we provide.

Chapter 4 contains our results on discretizing geometric evolution equations in the setting of the space of Kähler metrics and a description of an array of results that we have obtained from this approach.

Finally, a note about the Bibliography. At various places I felt it necessary to point out historical references, remarks, different observations by various authors as well as sources that I have found helpful for developing my own intuition, and so in a certain sense I have included certain references for my own bookkeeping, for future reference. This accounts in part for the length of the bibliographical list.

> *Art does not solve any problems, but makes us aware of their existence.*
>
> — Magdalena Abakanowicz

CHAPTER 2

The space of Kähler metrics

2.1 Introduction to Kähler geometry

2.1.1 Introduction to differential geometry. This opening subsection is intended as a rather vague introduction for the layman, unlike the rest of this monograph that will not be easily accessible to him/her.

Roughly speaking, geometry studies the shape of things. The objects of study are ensembles of points, typically parametrized by a certain number of real numbers ("coordinates"). In order to carry out this task one is interested in introducing quantities that may measure and describe the shape, or in other words, that will distinguish between two objects with different shapes.

This may sound absurd: Once an object is described by its coordinates it is completely determined and distinguishable. There should be no need to introduce any futher invariants. However, more thought reveals that the same object may be parametrized in many different ways, and so one does need invariants that uniquely characterize and capture the shape of the object, regardless of the way it is represented.

One is led to think of such objects in a more abstract manner, not necessarily as subsets given by Cartesian parameters. A basic notion is then that of a "metric space": an ensemble of points together with a function that assigns to each pair of points a nonnegative number, the "distance between them." One then obtains the notion of an r-neighborhood of a point in the space: the set of all points with distance less than or equal to r from that point. Thus, one obtains a topology on the space. In particular, topology is one aspect described by the geometry of a space.

We will always assume that our space has neighborhoods that are all equivalent in the sense that they all look the same—up to stretching or shrinking in a continuous manner. Moreover, we will even assume that this may be done in a very smooth manner. This is called a "differentiable manifold" or sometimes a "smooth manifold." The second word of the name comes to explain in some sense the complexity of the object.

Once we have a working notion of a manifold endowed with a distance function we would like to associate to it invariants that may describe its "shape." A central notion is the curvature associated to a geometry. Much of differential geometry studies consequences of properties of the curvature of a manifold on its geometry and this is one of the invariants that help tell manifolds apart.

One may study special classes of manifolds that have special structures. For example, the complex plane $\mathbb{C}$ is a manifold that has an additional structure beyond the real plane $\mathbb{R}^2$, although as metric spaces they are the same. Indeed one may define holomorphic functions using this structure: these are functions whose directional derivative is independent of the direction. "Complex manifolds" are those smooth manifolds whose neighborhoods all looks like neighborhoods of products of several copies of the complex plane (i.e., are locally parametrized by a collection of complex numbers instead of real ones). They are natural generalizations of the complex plane. The whole arsenal of tools and insights from function theory may then be applied to the study of such manifolds. This is one motivation for studying this special class of manifolds.

Another incentive is the fact one eventually comes to terms with, that spaces or manifolds and the structures they are equipped with, are completely determined by the knowledge of the functions on them that respect those structures. Generally then, the more structures we impose on a manifold, the smaller the ensemble of functions that will respect those structures, hence the simpler it will be to specify and work with such manifolds. Indeed, the collection of holomorphic functions is vastly smaller than that of the smooth functions, itself immensely smaller than that of continuous functions. Therefore, for example, the word "complex" in "complex manifold" comes from the term "complex numbers" and not from the supposed complexity of this manifolds. In fact the opposite is true, the complex structure reduces the complexity of the manifold.

Furthermore, special classes of manifolds lie at the intersection of several branches of mathematics. The study of the manifolds we will concentrate on in this Thesis, called Kähler manifolds, in fact uses tools from Riemannian geometry, complex analysis, partial differential equations, algebraic geometry, and more. This then means that one may pose very interesting problems motivated from several fields and surprising connections oftentimes arise.

A particularly fascinating aspect of differential geometry is the study of moduli spaces of geometric structures. Such moduli spaces group together all possible structures of a specific kind on an equal footing. By studying these spaces—oftentimes infinite-dimensional spaces that themselves have manifold structures—one may rephrase problems regarding the geometry of the manifold itself and gain considerable understanding.

As one expects from their definition, the structure of these infinite-dimensional spaces encodes considerable geometric information regarding M. One may argue that to a large extent differential geometry seeks to devise methods to somehow extract this information from these fantastic spaces.

Finally, as a side remark, it is worth mentioning the point of view that geometric structures oftentimes arise while trying to understand mathematical problems from various fields (both in mathematics and in other subjects) and ultimately one may hope that geometry may provide a unifying language.

A word about differential geometry and notation. Notation helps communicate mathematics to others. Differential geometry has sometimes been referred to as the field of mathematics where the objects are invariant under a change of notation (paraphrasing the commonly used fact expressed by saying that tensors are invariant under a change of coordinate system). Sometimes it requires serious thought from a reader to actually understand completely the geometrical meaning of a very simple equation. As an example consider the statement that a certain differential form α is invariant under a diffeomorphism f, summarized in the equation $f^\star\alpha = \alpha$. Sometimes it is not clear whether the objects considered are being evaluated at a point or not, and at other times it is not clear where the objects live, i.e., implicit identifications are being made so that objects that induce other objects are considered in formally wrong spaces. These and many other instances make differential geometry a difficult subject to read. As with any language, time, patience and curiosity are important ingredients in learning the language.

Before going on to the more technical part of this monograph we refer the interested reader to the rather general audience reflections on geometry by Atiyah [**4**] and Chern [**69,70**] as well as to the Encyclopædia Universalis entries by Russo and Libermann [**233,168**].

2.1.2 Differentiable and Riemannian structures.

The notion of a differentiable (or smooth) manifold is a starting point for modern differential geometry and allows the formal extension of Calculus to objects that are more general than Euclidean space $\mathbb{R}^n$. Historically, the notion of a manifold took a long time to crystalize. This is all the more apparent from Cartan's 1928 statement "La notion générale de variété est assez difficile à definir avec précision" [**54**]. It should be emphasized that this statement, as well as the section "Notion de variété" that it opens, was kept without change,[1] except for an addition of references on "general manifolds", in the

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[1] I would like to thank Sigurdur Helgason for pointing this out to me.

second edition of 1946 [**55**, p. 56]. Until Whitney's work, two concurrent notions were used to define manifolds, and the relationship between them was not clear: "A differentiable manifold is generally defined in one of two ways; as a point set with neighborhoods homeomorphic with Euclidean space E_n, coördinates in overlapping neighborhoods being related by a differentiable transformation, or as a subset of E_n, defined near each point by expressing some of the coördinates in terms of the others by differentiable functions."[2] Whitney unified these by showing that "the first definition is no more general than the second; any differentiable manifold may be imbedded in Euclidean space." [**290**].

We now give a rather precise definition of this notion, more as a summary of the above discussion than as a systematic routine, indeed in the sequel we will mostly rely of the reader's familiarity with the elementary notions in differential geometry that will be encountered and that can be readily found in one of the following books: Berger [**18**], Besse [**21**], Helgason [**130**], Hörmander [**133**], Jost [**138**], Lee [**159,160**], Petersen [**212**], Poor [**218**], Postnikov [**219,220,221**], Warner [**285**]—whenever a notion will not be properly defined we refer the reader to one of these sources for a detailled definition.

Definition 2.1. *(0) A topological manifold of dimension n is the couple (M,T) where: (i) M is a pointed-set, (ii) T is a collection of subsets of M that is*

> *0) a topology on M, namely closed under taking arbitrary unions and finite intersections and such that $M, \emptyset \in T$. Equivalently, this is the information of which functions on M are continuous.*

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[2] The historical development of the notion of a manifold is certainly a matter that can be discussed in great length, which is however not our intention or within our capacity at the moment. Let us just point out a few historical references that the reader might find interesting to compare regarding this point (see also Scholz [**234**]); 1) From around the time of Whitney's work: Cartan [**54,55**] gives a rather vague definition (in similar veins see also König's article [**155**, p. 215]), Burgatti, Boggio and Burali-Forti [**38**] give an inductive (and hence somewhat indirect) definition of a manifold in terms of intersection with planes of complimentary dimension in Euclidean space, Veblen and Whitehead [**283**] gives a definition that is rather cumbersome and clumsy, although an early attempt to formalize this notion (see in this regard also Milnor's commentary [**195**, p. xxi], as well as Scholz' [**234**, p. 55]), and Mayer and Thomas [**190**], shortly after Whitney's work, give a definition that is succinct and accurate; 2) From the turn of the twentieth century: Typical of the time when tensor analysis was considerably more developed than other aspects of differential geometry, Bianchi [**22**] in his second edition of "Lezioni di geometria differenziale" defers the definition of a (arbitrary dimension) manifold until chapter 9 (this tendency still remained for some years, and even until this day in some of the Physics literature), defining it as a parametrized set in Euclidean space.

1) separated, namely, for each pair of distinct points in M there exists a pair of elements of T, each containing one of the points.

2) admits a countable base (also called second countable), namely each element of T may be expressed as a union of elements of a countable subset of T.

3) locally Euclidean, namely for each point in M there exists an element of T containing the point and homeomorphic to Euclidean space $\mathbb{R}^n$ (i.e., there exists a continuous one-to-one map with continuous inverse (such a map is called a homeomorphism) between them), with n independent of the point. Define $\dim_{\mathbb{R}} M$, *the dimension of M, to be n.*

The collection T is called the topology of (M, T) and its elements are called open sets of M. By strict abuse of notation, we sometimes abbreviate the notation (M, T) by the notation M.

(1) A coordinate patch (also called coordinate chart) $(U, \mathbf{x})$ of M is defined by an open contractible set $U \subseteq M$ and a homeomorphism $\mathbf{x}(p) = (x_1(p), \ldots, x_n(p))$ defined on this set whose image is contained in $\mathbb{R}^n$, where $n = \dim_{\mathbb{R}} M$.

(2) An open covering of M is a collection $\{U_\alpha\}_{\alpha \in A}$ (the index set A may be finite or infinite) of open contractible sets satisfying $M = \cup_{\alpha \in A} U_\alpha$.

(3) A smooth atlas for M is a collection $\{(U_\alpha, \mathbf{x}_\alpha)\}_{\alpha \in A}$ of coordinate patches for which $\{U_\alpha\}_{\alpha \in A}$ is an open covering, such that whenever $U_\alpha \cap U_\beta \neq \emptyset$ the map $\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}$ restricted to $\mathbf{x}_\alpha(U_\alpha \cap U_\beta)$ is a smooth one-to-one map with a smooth inverse (also called diffeomorhpism) onto $\mathbf{x}_\beta(U_\alpha \cap U_\beta)$.

(4) A differentiable structure on M is the specification of a smooth atlas for M. Equivalently, it is the specification of which continuous functions on M are smooth.

(5) A differentiable manifold of dimension n is the couple $((M, T), A)$ where: (i) (M, T) is a topological space of dimension n, and (ii) A is a smooth atlas for M. By strict abuse of notation we sometimes abbreviate the notation $((M, T), A)$ by the notation M.

In this form, the above definition was formulated by Ehresmann [**89**] (see also [**169**, p. 38]), who moreover first formulated the notion of the tangent bundle (as a special case of a fiber bundle) in the familiar modern language.[3,4] A tangent bundle is the most natural object to define if one intends to extend Calculus to the setting of differentiable manifolds. We give now however a slightly less common, yet elegant, definition, and then consider the standard one, as in Ehresmann [**89**].

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[3] Fiber bundles were independently defined and studied during the second world war years by Whitney [**291**] and Steenrod [**256**], in which time the connections between the two continents broke [**88**, p. 31].

[4] For an alternative definition we refer to Kriegel and Michor [**158,194**].

Definition 2.2. (See [**71**, p. 76], cf. [**130**, p. 95].) *1) Let M be a differentiable manifold. A tangent vector X to M is a derivation of $C^\infty(M)$, that is a mapping $X : C^\infty(M) \to C^\infty(M)$ satisfying for each $f, g \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$: $X(\alpha f + \beta g) = \alpha X f + \beta X g$, and $X(fg) = gXf + fXg$. Denote by* $\operatorname{diff}(M)$ *the collection of all tangent vectors to M.*
2) Let $p \in M$. Let the tangent space T_pM to M at p be the collection of mappings $X|_p : f \mapsto (Xf)(p)$ with $X \in \operatorname{diff}(M)$.

This description exhibits $\operatorname{diff}(M)$ as a Lie algebra, that is an algebra with an operation (called a bracket) $[\cdot, \cdot] : X, Y \mapsto [X, Y] \equiv \mathrm{Lie}_X Y$ (in our case $[X, Y] := X \circ Y - Y \circ X$), satisfying the Jacobi identity $\mathcal{L}_X[Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$.

Any tangent vector may be represented in a coordinate patch $U, (x_1, \ldots, x_n)$ as a $C^\infty(M)$-linear comination of the locally-defined vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. At a point a tangent vector is represented as an $\mathbb{R}$-linear combination of these vector fields. Accordingly, each coordinate patch induces an isomorphism between a tangent space T_pM and $\mathbb{R}^n$.

An alternative definition of tangent vector involves the notion of a vector bundle. Here one considers the local representations of the tangent spaces with respect to an atlas, with $GL(n, \mathbb{R})$-valued functions over coordinate patch intersections relating the different representations (called the transition functions). The resulting total space is a manifold in its own right (of dimension $2n$), called the tangent bundle to M and denoted by TM [**89**].

We define a Riemannian manifold (M, g) to be a differentiable manifold M equipped with an assignment

$$g : (p, x, y) \in M \times TM \times TM \mapsto g(x, y)(p) \in C^\infty(M),$$

that is positive in the sense that $g(x, x) > 0$ for each $x \neq 0$ and satisfying the symmetry assumption $g(x, y) = g(y, x)$ (also called a Riemannian structure/metric). The angle between the the vectors x and y is then defined by $g(x, y)/\sqrt{g(x, x)g(y, y)}$ and the length of a vector x is defined by $\sqrt{g(x, x)}$.

The notion of a Riemannian manifold goes hand in hand with that of a differentiable manifold since by using a partition of unity we may equip any differentiable manifold with a smooth Riemannian metric; a different way to obtain this would be to note that a Riemannian metric is a smooth section of a $GL(n, \mathbb{R})/O(n)$-fiber bundle and such a section always exists since the fibers are contractible, diffeomorphic to $\mathbb{R}^{n(n+1)/2}$, and so the fiber bundle is diffeomorphic to a vector bundle and such bundles always have sections (e.g., the 0-section) [**89**]. Moreover, this also shows that the space of Riemannian metrics on a given differentiable manifold is infinite-dimensional.

The notion of parallel transport on a Riemannian manifold was introduced by Levi-Civita and Schouten [**161**,**235**] (cf. [**258**]) and enables comparison of vectors on different tangent spaces. This notion is closely related with the notion of the Levi-Civita connection. First, an affine connection is an assignment $D : TM \times TM \to TM$, denoted $D : (X, Y) \mapsto D_X Y$ that is tensorial in the first argument

$$D_{fX} Y = f D_X Y, \qquad f \in C^\infty(M), \quad X, Y \in \mathrm{diff}(M),$$

and satisfies the Leibnitz rule in the second argument

$$D_X(fY) = f D_X Y + df(X)Y.$$

A connection D is called a Levi-Civita connection and denoted by ∇ if it is torsion free, namely satisfies

$$\nabla_x y - \nabla_y x - [x, y] = 0, \qquad x, y \in \mathrm{diff}(M), \tag{8}$$

as well as metric compatibile, namely satisfies

$$xg(y, z) = g(\nabla_x y, z) + g(x, \nabla_x z) \qquad x, y, z \in \mathrm{diff}(M). \tag{9}$$

Such a connection is uniquely determined in terms of g, according to the following computation [**218**, p. 122] which we record for later use,

$$\begin{aligned} g(\nabla_x y, z) &= xg(y, z) - g(y, \nabla_x z) = xg(y, z) - g(y, \nabla_z x + [x, z]) \\ &= xg(y, z) - g(y, [x, z]) - zg(y, x) + g(\nabla_z y, x) \\ &= xg(y, z) - g(y, [x, z]) - zg(y, x) + g(\nabla_y z + [z, y], x) \\ &= xg(y, z) - g(y, [x, z]) - zg(y, x) + g([z, y], x) + yg(z, x) - g(z, \nabla_y x) \\ &= xg(y, z) - g(y, [x, z]) - zg(y, x) + g([z, y], x) + yg(z, x) - g(z, \nabla_x y - [x, y]), \end{aligned} \tag{10}$$

hence,

$$2g(\nabla_x y, z) = xg(y, z) - g(y, [x, z]) - zg(y, x) + g([z, y], x) + yg(z, x) + g(z, [x, y]). \tag{11}$$

Parallel transport of a vector $x \in T_p M$ along a curve $\gamma : [0, 1] \to M$ may now be defined as the operators $P_\gamma(t) : T_p M \to T_{\gamma(t)} M$ with $P_\gamma(t)x := x(t)$ and $x(t)$ is the solution of the first order differential equation

$$\nabla_{d\gamma(t)\frac{d}{dt}} x(t) = 0, \qquad t \in [0, 1].$$

Next, we define the (Riemann-Christoffel) curvature tensor

$$R(x, y)z := \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z, \qquad x, y, z \in \Gamma(M, TM), \tag{12}$$

as well as its $(0,4)$-tensor version

$$R(x, y, z, w) := g(R(x, y)z, w). \tag{13}$$

Since we just saw that ∇ is entirely determined by g, so is the curvature tensor R. Finally, we refer the reader to Riemann's original lecture [**255**, Vol. 2, p. 135].

2.1.3 Almost-complex, complex and symplectic structures. In search of further structures on manifolds one is naturally led to structures involving complex numbers. By a complex manifold $((M, T), A)$ of complex dimension $\dim_{\mathbb{C}} M = n$ we mean simply a topological manifold of real dimension $2n$ endowed with a holomorphic atlas A, that is a differentiable atlas for which different coordinates patches are related to each other over intersections by biholomorphisms, instead of just diffeomorphisms. I shall remark that while the notion of a complex manifold is very natural, its meaning still eludes me.

Endowing the manifold with further structures has the advantage of reducing its complexity. Consider the frame bundle FM over M whose fiber over $p \in M$ parametrizes all possible bases for T_pM. In other words this is a bundle whose fiber is isomorphic to $GL(2n, \mathbb{R})$. The existence of local holomorphic coordinates tells us that there exists a subbundle of FM whose fibers are isomorphic to $GL(n, \mathbb{C})$, that can be viewed as a subgroup of $GL(2n, \mathbb{R})$ (see next paragraph).

A crucial observation however is that the possible reduction of the structure group in such a manner is a more general notion than that of a complex manifold. Following Ehresmann, define an almost-complex manifold to be one for which such a reduction to $GL(n, \mathbb{C})$ is possible. This may be shown to be equivalent to the existence of a tensor field $\mathrm{J} : TM \to TM$ satisfying $\mathrm{J}^2 = -I$ with I the identity endomorphism tensor [**90,91**]. Such manifols are automatically even-dimensional (the only eigenvalues of J are $\pm\sqrt{-1}$, each with the same multiplicity) and oriented since the determinant of the Jacobian matrix of transformation between two basis is always positive, as $GL(n, \mathbb{C})$ embeds in $GL(2n, \mathbb{R})$ via $A + \sqrt{-1}B \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ (alternatively, $GL(2n, \mathbb{R})$ has several connected components and this is a continuous map from a connected space whose image contains the identity matrix, hence is contained in the same component as the identity). (Alternatively, orientability may be seen to be equivalent to the existence of a nowhere zero $2n$-form, and using any Riemannian metric one may create a 2-form from J that is nondegenerate, i.e., its top exterior product is a nowhere zero $2n$-form [**90**, p. 11].)

In the same manner a Riemannian structure allows us to reduce the structure group (from $GL(2n, \mathbb{R})$) to $O(2n)$. Indeed one may find a local orthonormal frame for the tangent bundle on each patch (note that this basis will consist of coordinate vector fields $\frac{\partial}{\partial x^j}$ only if the manifold is locally flat). The change of basis matrices between two local frames will be orthogonal. Hence we obtain a principal $O(2n)$-subbundle of the frame bundle FM.

Ehresmann defined the notion of an almost-complex manifold and pointed out that probably not all such structures come from complex structures [**90**, p. 3].[5] This remained open until 1951 when several authors demonstrated that there exists an almost-complex structure on S^6 that is not complex [**45,86,93**]. The key observation, due to Eckmann and Frölicher, is that a necesarry condition for an almost-complex structure to be integrable is that its Nijenhuis tensor N_{J} vanish, defined as a vector-valued 2-form[6] (cf. [**100**, p. 551; **152**]; the name comes from [**204**, (3.1)])

$$N_{\mathrm{J}}(x,y) := [\mathrm{J}x, \mathrm{J}y] - [x,y] - \mathrm{J}[\mathrm{J}x, y] - \mathrm{J}[x, \mathrm{J}y]. \tag{14}$$

This may be seen geometrically as the requirement, satisfied for any complex structure, that the splitting $T^{1,0}M \oplus T^{0,1}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ respect the bracket operation, or in other words, the distribution $T^{1,0}M$ (as well as $T^{0,1}M$) should be integrable, and since $\Gamma(M, T^{1,0}M) = \{X - \sqrt{-1}\mathrm{J}X \ : \ X \in \Gamma(M, TM)\} = \ker(\mathrm{I} + \sqrt{-1}\mathrm{J})$ this corresponds to

$$(\mathrm{I} + \sqrt{-1}\mathrm{J})[X - \sqrt{-1}\mathrm{J}X, Y - \sqrt{-1}\mathrm{J}Y] = 0, \tag{15}$$

for every real vector fields X, Y. Taking the real part of (15) gives $N_{\mathrm{J}} = 0$. (The reason this is satisfied for any genuine complex structure is that then we also have an isomorphism $\Gamma(U, T^{1,0}M) \cong \{a^k \frac{\partial}{\partial z^k}, : \ a^k \in C^\infty(M, \mathbb{C})\}$ over a local holomorphic patch and so the bracket is visibly of type $(1,0)$.)

In coordinates, $\mathrm{J} = \mathrm{J}^j_k \frac{\partial}{\partial x^j} \otimes dx^k$, and

$$N_{\mathrm{J}} = N^k_{ij} \frac{\partial}{\partial x^k} \otimes dx^i \wedge dx^j, \tag{16}$$

with $N^k_{ij} \frac{\partial}{\partial x^k} = N_{\mathrm{J}}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and (denoting coordinate derivatives by $\mathrm{J}^k_{j,p} := \partial \mathrm{J}^k_j / \partial x^p$)

$$N^k_{ij} = \mathrm{J}^p_i \mathrm{J}^k_{j,p} - \mathrm{J}^p_j \mathrm{J}^k_{i,p} + \mathrm{J}^k_p \mathrm{J}^p_{i,j} - \mathrm{J}^k_p \mathrm{J}^p_{j,i}. \tag{17}$$

Remark 2.3. The condition of Eckmann and Frölicher proceeded an equivalent condition formulated by Ehresmann formulated in terms of integrability of the distribution $T^{0,1}M$ given by Pfaffian equations. Namely, if the 1-forms $\{\omega_j\}_{j=1}^n$ define this distribution on a chart U in the sense that X is a section of $T^{0,1}M|_U$ precisely when $\omega_j(X) = 0$, then a necessary condition for the structure to be complex is that

$$d\omega_j = \sum_{j=1}^{n} \omega_j \wedge \omega_{jk}, \tag{18}$$

---

[5] Hopf also defined at about the same time the notion of an endomorphism of the tangent bundle none of whose eigenvalues are real (which he called a J-structure) [**132**, **143**;**167**, p. 199].

[6] Alternatively N_{J} is $(1,2)$-type tensor that is skew-symmetric in two of its arguments, see (16).

with ω_{jk} in the $\mathbb{C}$-linear span of $\{\omega_j, \overline{\omega_j}\}_{j=1}^n$ (the necessity follows, similarly to before, by taking a local holomorphic trivialization of the cotangent bundle). When the structure is real-analytic so that the forms ω_j may be taken to be real-analytic, the integrability condition (18) above is equivalent to the distribution of vectors being integrable in the sense of being closed under bracket as first observed by de Rham, which allowed Ehresmann to show then that in the real-analytic situation Equation (18) is equivalent to the almost-complex structure being complex [**91**, p. 418; **165**, p. 743].

Example 2.4. *A complex structure on the two-sphere.* Consider the round sphere, embedded in $\mathbb{R}^3$ as the locus of points at distance 1 from the origin. The tangent space to a vector at a point $\vec{x} \in S^2 \subset \mathbb{R}^3$ may be identified with the 2-plane passing through that point and orthogonal to $\vec{x}$. Define $\mathrm{J} : T_{\vec{x}}S^2 \to T_{\vec{x}}S^2$ by

$$\mathrm{J}|_{\vec{x}} : \vec{y} \mapsto \vec{x} \times \vec{y}.$$

This indeed maps the tangent space to itself since $\vec{x} \cdot (\vec{x} \times \vec{y}) = 0$, and $\mathrm{J}^2|_{\vec{x}}\vec{y} = \vec{x} \times (\vec{x} \times \vec{y}) = (\vec{y} \cdot \vec{x})\vec{x} - (\vec{x} \cdot \vec{x})\vec{y} = -\vec{y}$. This structure is the one coming from endowing S^2 with the complex structure containing the atlas $\{(\mathbb{C}, z), (\mathbb{C}, w)\}$ with $w(z) = 1/z$ on $\mathbb{C} \setminus \{0\}$: Consider for example the north pole $N = (0, 0, 1)$ with $z(N) = 0$ in the first chart; its holomorphic tangent space is identified with the copy of $\mathbb{C}$, namely the plane passing through N given by $\{(x, y, 1) : x + \sqrt{-1}y \in \mathbb{C}\}$ that is endowed with a complex structure corresponding to rotation by ninety degrees, just as J. By homogeneity this then follows for all points.

* * *

We now describe a classical and non-trivial example.

Example 2.5. *The six-sphere admits an almost-complex structure that is not complex.* Following Kirchhoff [**143**] one may equip S^6 with an almost-complex structure, as follows. Consider $S^6 \subset \mathbb{R}^7$ again as the round unit sphere and represent each point $x \in S^6$ as a unit imaginary octonion, with $\mathbb{R}^7 \cong \operatorname{Im} \mathbb{O}$. Note that an octonion is represented by $t = u_1 + u_2 i + u_3 j + u_4 k + (v_1 + v_2 i + v_3 j + v_4 k)l =: u + vl$, with $i^2 = j^2 = k^2 = l^2 = -1$, and $ij = -ij = k, jk = -kj = i$, and l anticommuting with each of i, j, k. Let $\bar{t} = u_1 - u_2 i - u_3 j - u_4 k - (v_1 + v_2 i + v_3 j + v_4 k)l$. If $w = x_1 + x_2 i + x_3 j + x_4 k + (y_1 + y_2 i + y_3 j + y_4 k)l =: x + yl$, then $tw = ux - y\bar{v} + (\bar{u}y + xv)l$. For imaginary octonions ($u_1 = x_1 = 0$) the real part of the product equals simply $-\langle (u_2, u_3, u_4, v_1, v_2, v_3, v_4), (x_2, x_3, x_4, y_1, y_2, y_3, y_4) \rangle$, simply minus of the dot product when considered as vectors in $\mathbb{R}^7$. Whenever these two vectors are orthogonal in $\mathbb{R}^7$ we therefore have that $tw = \operatorname{Im}(tw)$, and so the product is still an imaginary octonion, representable in $\mathbb{R}^7$. Define

$$\mathrm{J}|_t w = tw.$$

We first need to show that $t \perp tw$ as vectors in $\mathbb{R}^7$. This is equivalent to showing that as an octonion $t(tw)$ is imaginary. Second, we need to show $t(tw) = -w$, which will also imply the previous conclusion. Compute using the associativity of the quaternions and that $\operatorname{Re} u = 0$ implies both $\bar{u} = -u$ and $u^2 = (\bar{u})^2 = -||u||^2$, and finally the fact that $||t||^2 = 1$,

$$\begin{aligned} t(tw) &= (u + vl)(ux - y\bar{v} + (\bar{u}y + xv)l) \\ &= u(ux - y\bar{v}) - (\bar{u}y + xv)\bar{v} + (\bar{u}(\bar{u}y + xv) + (ux - y\bar{v})v)l \\ &= -(||u||^2 + ||v||^2)(x + yl) = -w. \end{aligned}$$

To show it is not complex, we compute its Nijenhuis tensor, following Eckmann and Frölicher [**86**] (also Frölicher [**99**, p. 61]) (for another approach as well as a study of the automorphisms of this structure we refer to Ehresmann and Libermann [**93,166,167**]). Working with the local Euclidean coordinates coming from the embedding we see that J is locally a linear function in the coordinates u_α, v_α. Consider the north pole N represented by i or by $(1,0,0,0,0,0,0)$. Then locally trivializing the tangent bundle in the w coordinates, for each t near N we may represent J by

$$\begin{aligned} \mathrm{J}|_t : (x_2, x_3, x_4, y_1, y_2, y_3, y_4) \mapsto (&u_3x_4 - u_4x_3 + y_1v_2 - y_2v_1 + y_3v_4 - y_4v_3, \\ &u_4x_2 - u_2x_4 + y_1v_3 - y_3v_1 + y_4v_2 - y_2v_4, \\ &u_2x_3 - u_3x_2 + y_1v_4 - y_4v_1 + y_2v_3 - y_3v_2, \\ &u_2y_2 + u_3y_3 + u_4y_4 - x_2v_2 - x_3v_3 - x_4v_4, \\ &- u_2y_1 - u_3y_4 + u_4y_3 + x_2v_1 + x_3v_4 - x_4v_3, \\ &u_2y_4 - u_3y_1 - u_4y_2 - x_2v_4 + x_3v_1 + x_4v_2, \\ &- u_2y_3 + u_3y_2 - u_4y_1 + x_2v_3 - x_3v_2 + x_4v_1). \end{aligned}$$

This may be represented by the matrix

$$\begin{pmatrix} 0 & -u_4 & u_3 & v_2 & -v_1 & v_4 & -v_3 \\ u_4 & 0 & -u_2 & v_3 & -v_4 & -v_1 & v_2 \\ -u_3 & u_2 & 0 & v_4 & v_3 & -v_2 & -v_1 \\ -v_2 & -v_3 & -v_4 & 0 & u_2 & u_3 & u_4 \\ v_1 & v_4 & -v_3 & -u_2 & 0 & u_4 & -u_3 \\ -v_4 & v_1 & v_2 & -u_3 & -u_4 & 0 & u_2 \\ v_3 & -v_2 & v_1 & -u_4 & u_3 & -u_2 & 0 \end{pmatrix}$$

Consider local coordinates on S^6 near N given by the parametrization

$$(\sqrt{1 - u_3^2 - u_4^2 - v_1^2 - v_2^2 - v_3^2 - v_4^2}, u_3, u_4, v_1, v_2, v_3, v_4).$$

At the point N,

$$\mathrm{J}|_i : (0, x_3, x_4, y_1, y_2, y_3, y_4) \mapsto (0, -x_4, x_3, y_2, -y_1, y_4, -y_3),$$

that is the only nonzero coefficients of J are (the upper index is the row number, with the first row corresponding to u_2, the second to u_3 and the seventh to v_4)

$$\mathrm{J}^{u_4}_{u_3} = -\mathrm{J}^{u_3}_{u_4} = 1, \mathrm{J}^{v_2}_{v_1} = -\mathrm{J}^{v_1}_{v_2} = -1, \mathrm{J}^{v_4}_{v_3} = -\mathrm{J}^{v_3}_{v_4} = -1.$$

Note also that $\mathrm{J}^{v_1}_{u_3} = -v_3$, $\mathrm{J}^{v_1}_{v_4} = u_4$, $\mathrm{J}^{v_2}_{v_4} = -u_3$, $\mathrm{J}^{v_2}_{u_3} = v_4$. Therefore by (17),

$$N^{v_1}_{v_4\,u_3} = \mathrm{J}^{v_3}_{v_4}\mathrm{J}^{v_1}_{u_3,v_3} - \mathrm{J}^{u_4}_{u_3}\mathrm{J}^{v_1}_{v_4,u_4} + \mathrm{J}^{v_1}_{v_2}\mathrm{J}^{v_2}_{v_4,u_3} - \mathrm{J}^{v_1}_{v_2}\mathrm{J}^{v_2}_{u_3,v_4} = 1\cdot(-1) - 1\cdot 1 + 1\cdot(-1) - 1\cdot 1 = -4.$$

The problem, stated already in [**143**], whether S^6 admits a complex structure, remains open to this day.

* * *

A sufficient condition for the integrability of an almost-complex structure in the real-analytic situation was derived by Ehresmann [**90,91**], as well as by Eckmann and Frölicher via a different formulation, demonstrating that $N_{\mathrm{J}} = 0$ is also a sufficient condition when J is real-analytic [**86**]. Newlander and Nirenberg relaxed the regularity of J to a finite number of derivatives [**201,202**] (for a retrospective discussion [**137**]). This theorem was revisited and refined by several authors, e.g., [**74,142,146,149,183,207,286**].

Since integrability of the complex structure is characterized by differential equations the set of complex structures is therefore considerably more restricted than the set of almost-complex structures. For example, the automorphism group of a complex structure is always a finite-dimensional complex Lie group [**28**; **21**, p. 87], while for an almost-complex structure this group may be (real) odd-dimensional or even infinite-dimensional [**165,166**].

A natural question is, when will we be able to find a $U(n) = O(2n) \cap GL(n, \mathbb{C})$-subbundle of the frame bundle FM? This will be the case precisely when the Riemannian and almost-complex structures are compatible, that is when a local basis $e_1, \mathrm{J}e_1, \ldots, e_n, \mathrm{J}e_n$ that reduces the structure group to $GL(n, \mathbb{C})$ is also an orthonormal basis, that is also reduces the structure group to $O(2n, \mathbb{R})$. And this will be the case when

$$g(e_i, \mathrm{J}e_j) = 0, \quad g(\mathrm{J}e_i, \mathrm{J}e_j) = \delta_{ij}, \quad \forall\, i, j \in \{1, \ldots, n\}.$$

In other words precisely when

$$g(x, y) = g(\mathrm{J}x, \mathrm{J}y), \quad \forall\, x, y \in \Gamma(M, TM). \tag{19}$$

The structure is then called almost-Hermitian. Note that given an almost-complex structure it is always possible to find a Riemannian metric compatible with it.

Example 2.6. The almost-complex structure of Example 2.5 is compatible with the round metric of constant sectional curvature on S^6. Indeed the tangent space $T_NS^6 \cong \mathbb{R}^6$ is parametrized by $(u_3, u_4, v_1, v_2, v_3, v_4)$, the metric at N is given by $g|_N = du_3^2 + du_4^2 + dv_1^2 + dv_2^2 + dv_3^2 + dv_4^2$, i.e., is represented by the identity matrix, and the almost-complex structure $\mathrm{J}|_N$ is represented by the orthogonal block-diagonal matrix $\mathrm{diag}(-A, A, A)$ with $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The same is true for Example 2.4, as there $g|_N = dx^2 + dy^2$ and $\mathrm{J}|_N$ is represented by $-A$.

* * *

A skew-symmetric bilinear form α that is nondegenerate (i.e., $\alpha^n \neq 0$) on a (necessarily even-dimensional) vector space V is called a symplectic structure on V. A nondegenerate 2-form ω on M (i.e., ω^n is a nowhere zero $2n$-form) induces a symplectic structure on each tangent space T_pM. The structure (M, ω) is called an almost-symplectic structure on the manifold M [**218**, p. 245] and is equivalent to the reduction of the frame bundle FM to a $Sp(2n, \mathbb{R})$-principal subbundle. When such a structure exists it is possible to further reduce to a $U(n)$-principal subbundle by first finding a compatible almost-complex structure J [**191**, p. 70] and therefore a compatible Riemannian structure $g(\,\cdot\,,\,\cdot\,) := \omega(\,\cdot\,, \mathrm{J}\,\cdot\,)$ (note $U(n) = Sp(2n, \mathbb{R}) \cap O(2n)$). In other words, an almost-symplectic manifold admits an almost-Hermitian structure (see (19)). For the other direction, if g and J are compatible then $\omega(\,\cdot\,,\,\cdot\,) := g(\mathrm{J}\,\cdot\,,\,\cdot\,)$ is a nondegenerate 2-form (this observation goes back to Ehresmann [**91**, p. 414]).

An almost-symplectic structure (M, ω) is called symplectic when $d\omega = 0$. This notion, together with the notions of an almost-Hermitian and almost-Kähler manifolds, were defined by Ehresmann [**91**, p. 415] (cf. [**92**, p. 482]), as both a special class of almost-complex manifolds and as a generalization of the notion of a Kähler manifold. We now turn to discuss this latter notion in some detail, as it will be at the heart of this Thesis.

2.1.4 The Schouten-van Dantzig-Kähler condition. A special class of Riemannian metrics on complex manifolds was introduced by Schouten [**235**], Schouten and van Dantzig [**237**,**238**], and Kähler [**141**] (see also Schouten [**236**, pp. 397–406], Yano [**293**], Chern [**68**], Nijenhuis [**205**, p. 6]). On the one hand, Kähler discovered, motivated partly by Einstein's equation $\mathrm{Ric}g = cg$ on Riemannian manifolds, that for certain metrics g this equation reduces from a set of $n(n+1)/2$ equations (Einstein's equation is an equality between symmetric 2-tensors) to a single equation for a real-valued function. The condition satisfied by such "remarkable" metrics g is that their Kähler form $\omega_g := \omega_{g,\mathrm{J}} = g(\mathrm{J}\,\cdot\,,\,\cdot\,)$ is a closed 2-form,

$$d\omega_g = 0. \tag{20}$$

A remarkable characterization of this condition is that such metrics admit—at least locally—potentials, i.e., $\omega_g|_U = \sqrt{-1}\partial\bar{\partial}u_U$ for some real valued function u_U over a patch $U \subseteq M$. Schouten and van Dantzig, on the other hand, discovered that the same condition is equivalent to parallel transport preserving type of vectors, or in other words respecting the complex structure. They also observed the existence of local potentials thus placing this type of geometry on the intersection of potential theory and differential geometry.

Schouten called this new type of geometry "unitary," and this can be understood from the fact that not only does one obtain a reduction of the structure group to $U(n)$ (this merely characterizes almost-Hermitian manifolds, see (19)) but also the holonomy is reduced from $O(2n)$ to $U(n)$. In retrospect this name seems quite fitting, on a somewhat similar footing to "symplectic geometry," a name first suggested by Ehresmann. However, only a few authors in the 1940's and 1950's were familiar with Schouten-van Dantzig's work. The first few accounts of Hermitian geometry, mainly by Bochner, Eckmann and Guckenheimer referred only to Kähler's works, and the name "Kähler geometry" became rooted. By the time of Yano's survey in 1955 (op. cit.) the name was already well-established. To a large extent Schouten's work seems to have been underread due to his reputation for heavy notation and his strive for utmost generality (to give an example of the latter we recall the fact that Schouten considered geometries where the Christoffel symbols for 1-forms and vectors could be different!). A similar incident was Schouten's discovery of the notion of parallelism, independently of Levi-Civita. The latter was simply far more elegantly written and readable and so became known as Levi-Civita's parallelism [**258**].

The equivalence of (20) to the existence of local potentials may be seen in an elementary way. First $d\omega_g = 0$ implies that $\bar\partial\omega_g = 0 = \partial\omega_g$. By the first equation, Dolbeault cohomology theory (namely $H^{p,q}(\mathbb{C}^n) = 0$ for $p + q > 0$) implies that locally $\omega_g = \bar\partial\alpha$ for some $(1,0)$-form α (as ω_g is of type $(1,1)$; this is also called the complex Poincaré lemma [**136**, p. 46; **203**]) and by the second equation $\partial(\bar\partial\alpha) = -\bar\partial(\partial\alpha) = 0$. Hence locally $\partial\alpha = \bar\partial\beta$ but this is only possible if $\partial\alpha = 0$, $\bar\partial\beta = 0$ by type considerations. But then locally $\alpha = \partial f_U$ for some function f_U from which $\omega_g = \bar\partial\partial f_U$. As ω_g is real we conclude that so is $\sqrt{-1}f_U$ and $\omega_g|_U = \sqrt{-1}\partial\bar\partial(\sqrt{-1}f_U)$, as required. Since ω_g is positive, that is satisfies $\omega_g(v, \bar v)$ for any $(1,0)$ vector v, the functions $\sqrt{-1}f_U$ are plurisubharmonic, namely satisfy $\sqrt{-1}\partial\bar\partial(\sqrt{-1}f_U) > 0$, and locally the matrix

$$\left(\frac{\partial^2(\sqrt{-1}f_U)}{\partial z^i \partial \overline{z^j}}\right)_{i,j=1}^{n}$$

is positive definite. Conversely, the existence of local potentials for ω_g implies (20).

The existence of local potentials for closed $(1,1)$-forms, however, holds on any complex manifold, as can be seen from the proof. A property that is special for Kähler manifolds is the following global version of the previous property:

Lemma 2.7. (See [**114**, p. 149].) *Let α and β be two closed $(1,1)$-forms representing the same cohomology class. Then $\alpha = \beta + \sqrt{-1}\partial\bar\partial f$ for some function $f \in C^\infty(M, \mathbb{C})$.*

This prompted Calabi [**39**,**40**] to define the space of Kähler forms in a fixed cohomology class:

Definition 2.8. *The space of Kähler forms representing* $\Omega \in H^2(M,\mathbb{R})$ *is given by*

$$\mathcal{H}_\Omega := \{\omega \,:\, [\omega] = \Omega, \omega > 0\} \subset \mathcal{D}_\Omega \cong C^\infty(M)/\mathbb{R}.$$

The space of Kähler potentials is given by

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(M) \,:\, \omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi > 0\} \subset C^\infty(M).$$

One has an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{H}_\omega \overset{\omega+\sqrt{-1}\partial\bar\partial(\,\cdot\,)}{\longrightarrow} \mathcal{H}_\Omega \longrightarrow 0, \tag{21}$$

and so in practice it seems that one should hardly distinguishes between the two spaces. However, a good picture to have in mind is that $\mathcal{H}_\Omega$ can be identified (in many different ways) as a hypersurface in $\mathcal{H}_\omega$ given by specifying one "Pfaff form" on $\mathcal{H}_\omega$ (or one "integral", i.e., preserved quantity). One particular place where this can be important is when seeking to prove C^0 estimates, where one has to be careful in going from an equation for curvature on the level of forms to an equation for potentials. This occurs frequently in Chapter 4.

While this is only a definition its impact on the study of Kähler geometry was important since it meant that variational problems could be formulated for Kähler manifolds with a fixed Kähler class in a manner that has no parallel for general complex manifolds. It provides a "moduli space" for such manifolds although the number of "moduli" is infinite.

A natural question is then: are there finite-dimensional subsets which are distinguished? There are two possible senses in which this can be perceived. On the one hand canonical metrics form distinguished finite-dimensional subsets. On the other hand, canonical metrics on spaces in which one can embed the manifold also form distinguished finite-dimensional subsets, by restriction. Each of these finite-dimensional subsets has its own importance in Kähler geometry. Chapter 3 involves the study of the latter, while Chapter 4 is to a large extent centered about the former.

We now list several equivalent characterizations of Kähler manifolds, mostly to illustrate the richness of the structure as lying on the crossroads of complex analysis, Riemannian geometry and sympletic geometry.

Before that a note on terminology: Since Kähler forms and their corresponding Hermitian metrics mutually determine each other one may speak of the space of Kähler forms or the space of Kähler metrics interchangeably, the two spaces being isomorphic in a canonical manner. For example, this is the meaning behind statements like "a Kähler metric in $\mathcal{H}_\Omega$."

By abuse of terminology one often also does not distinguish between the space of Kähler forms $\mathcal{H}_\Omega$ and the space of Kähler potentials $\mathcal{H}_\omega$, even though the latter is only the quotient of the former by the additive group $\mathbb{R}$, as we saw in (21).

2.1.4.1 Riemannian metric, Hermitian metric, and the Kähler form. Given a Riemannian structure (M, g) and a compatible integrable almost complex structure J we may extend the metric $\mathbb{C}$-linearly to be defined on $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ simply by setting $g(v + \sqrt{-1}u, z) := g(v, z) + \sqrt{-1}g(u, z)$. We denote the original Riemannian metric defined on the real vector bundle TM by g_{Riem}.

On a patch introduce local complex linear coordinates

$$z_1 := x_1 + \sqrt{-1}x_{n+1}, \ldots, z_n := x_n + \sqrt{-1}x_{2n},$$

and complex conjugate-linear coordinates

$$\overline{z_1} := x_1 - \sqrt{-1}x_{n+1}, \ldots, \overline{z_n} := x_n - \sqrt{-1}x_{2n}.$$

This means we also have real local coordinates $x_1, \ldots, x_{2n}$ for which $\mathrm{J}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^{i+n}}$. Let $g_{ij} := g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. We observe that J-compatibility implies $g_{i\,i+n} = 0$, an orthogonality relation. Bases for the holomorphic and antiholomorphic tangent bundles are given by $\{\frac{\partial}{\partial z^i} = \frac{1}{2}\frac{\partial}{\partial x^i} - \frac{\sqrt{-1}}{2}\frac{\partial}{\partial x^{i+n}}\}_{j=1}^n$, and $\{\frac{\partial}{\partial \overline{z^i}} = \frac{1}{2}\frac{\partial}{\partial x^i} + \frac{\sqrt{-1}}{2}\frac{\partial}{\partial x^{i+n}}\}_{j=1}^n$, respectively. Define the complex numbers $g_{i\bar{j}} := g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z^j}})$ (with respect to the $\mathbb{C}$-linearly extended Riemannian metric g) satisfying

$$g_{i\bar{j}} = \frac{1}{4}g_{ij} + \frac{1}{4}g_{i+n\,j+n} + \frac{\sqrt{-1}}{4}g_{i\,j+n} - \frac{\sqrt{-1}}{4}g_{j\,i+n} = \frac{1}{2}g_{ij} + \frac{\sqrt{-1}}{2}g_{i\,j+n} \tag{22}$$

(using J-compatibility of the metric).

We may extend the metric to the cotangent bundle (and then to the complexified one) by defining $g^{ij} := g(dx^i, dx^j)$ such that $[g_{ij}][g^{ij}] = I$; equivalently we could define $g(\alpha, \beta) := g(\alpha^\sharp, \beta^\sharp)$ with $\gamma^\sharp$ defined by $g(\gamma^\sharp, \cdot) = \gamma(\cdot)$ (in coordinates, e.g., $(dx^i)^\sharp = \sum_{j=1}^{2n} g^{ij}\frac{\partial}{\partial x^j} =: g^{ij}\frac{\partial}{\partial x^j}$, where the latter defines the Einstein summation convention of suppressing the summation notation $\sum$ whenever an index occurs twice). This corresponds to 'raising' the originally $(0, 2)$-tensor $g = g_{ij}dx^i \otimes dx^j$ to a $(2, 0)$-tensor $g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$.

Now one sees that

$$\begin{aligned}
g &= g_{ij}dx^i \otimes dx^j + g_{i+n\,j}dx^{i+n} \otimes dx^j + g_{i\,j+n}dx^i \otimes dx^{j+n} + g_{i+n\,j+n}dx^{i+n} \otimes dx^{j+n} \\
&= \frac{1}{2}g_{ij}(dz^i \otimes d\overline{z^j} + d\overline{z^i} \otimes dz^j) + \frac{\sqrt{-1}}{2}g_{i\,j+n}(dz^i \otimes d\overline{z^j} - d\overline{z^i} \otimes dz^j) \\
&= g_{i\bar{j}}dz^i \otimes d\overline{z^j} + g_{\bar{i}j}d\overline{z^i} \otimes dz^j
\end{aligned}$$

(remembering to sum over both i and j). The interpretation of this equality lies at the foundation of Kähler geometry. For if we define the induced Hermitian metric from g by $g_{\text{Herm}} := g_{i\bar{j}} dz^i \otimes \overline{dz^j}$ we see that formally $\operatorname{Re} g_{\text{Herm}} = \frac{1}{2} g$. Naturally we wonder what is the corresponding imaginary part. Now

$$\begin{aligned}
\operatorname{Im} g_{\text{Herm}} &= \frac{1}{2\sqrt{-1}} \big(g_{i\bar{j}} dz^i \otimes \overline{dz^j} - \overline{g_{k\bar{l}} dz^k \otimes \overline{dz^l}} \big) \\
&= \frac{1}{2\sqrt{-1}} \big(g_{i\bar{j}} dz^i \otimes \overline{dz^j} - g_{\bar{k}l} \overline{dz^k} \otimes dz^l \big) \\
&= \frac{1}{2\sqrt{-1}} \big(g_{i\bar{j}} dz^i \otimes \overline{dz^j} - g_{\bar{j}i} \overline{dz^j} \otimes dz^i \big) \\
&= -\frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge \overline{dz^j} = -\frac{1}{2} \omega_g .
\end{aligned} \tag{23}$$

Therefore formally $g_{\text{Herm}} = \frac{1}{2} g - \frac{\sqrt{-1}}{2} \omega$ (this formula goes back at least to [**237**, (17)]).

We would like to close the circle and come back to the original metric g. For that one finds using (22) and (23) that

$$\begin{aligned}
\omega\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{j+n}}\Big) &= \sqrt{-1}(-g_{i\bar{j}}\sqrt{-1} - g_{\bar{j}i}\sqrt{-1}) = g_{ij} = g\Big(\mathrm{J}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{j+n}}\Big), \\
\omega\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) &= -g_{ij+n} = -g\Big(\frac{\partial}{\partial x^i}, \mathrm{J}\frac{\partial}{\partial x^j}\Big), \\
\omega\Big(\frac{\partial}{\partial x^{i+n}}, \frac{\partial}{\partial x^{j+n}}\Big) &= -g_{ij+n} = g\Big(\mathrm{J}\frac{\partial}{\partial x^{i+n}}, \frac{\partial}{\partial x^{j+n}}\Big),
\end{aligned}$$

from which we conclude that $\omega(\,\cdot\,,\,\cdot\,) = g(\mathrm{J}\,\cdot\,,\,\cdot\,)$. Finally we remark that the condition (20) may be written in local coordinates as

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} =: g_{i\bar{j},k} = g_{k\bar{j},i} := \frac{\partial g_{k\bar{j}}}{\partial z^i}. \tag{24}$$

2.1.4.2 Parallel almost-complex structure. The condition (20) may be expressed as a restriction on the almost-complex structure:

$$N_{\mathrm{J}} = 0, \quad \text{and} \quad d\omega_g = 0 \quad \Leftrightarrow \quad \nabla \mathrm{J} = 0. \tag{25}$$

To show this first use (8)-(9), and $(\nabla_x \mathrm{J})y = \nabla_x(\mathrm{J}y) - \mathrm{J}\nabla_x y = \nabla_x(\mathrm{J}y) - \mathrm{J}\nabla_y x - \mathrm{J}[x,y]$, to compute

$$\begin{aligned}
d\omega_g(x,y,z) &= x\omega_g(y,z) + y\omega_g(z,x) + z\omega_g(x,y) \\
&\quad - \omega_g([x,y],z) - \omega_g([y,z],x) - \omega_g([z,x],y) \\
&= g(\nabla_x(\mathrm{J}y),z) + g(\mathrm{J}y,\nabla_x z) + g(\nabla_y(\mathrm{J}z),x) + g(\mathrm{J}z,\nabla_y x) + g(\nabla_z(\mathrm{J}x),y) \\
&\quad + g(\mathrm{J}x,\nabla_z y) - g(\mathrm{J}[x,y],z) - g(\mathrm{J}[y,z],x) - g(\mathrm{J}[z,x],y) \\
&= g((\nabla_x \mathrm{J})y,z) + g((\nabla_y \mathrm{J})z,x) + g((\nabla_z \mathrm{J})x,y).
\end{aligned}$$

Since J is skew-symmetric with respect to g so is $\nabla_a \mathrm{J}$ for any a (to see this, parallel translate b, c along an integral curve of a and evaluate at a point the covariant derivative of $g(\mathrm{J}b, c) + g(b, \mathrm{J}c) = 0$ in the direction a). Therefore $g((\nabla_a \mathrm{J})b, c) = -g(b, (\nabla_a \mathrm{J})c)$. Also $\mathrm{J}^2 = -\mathrm{I}$ implies $\nabla \mathrm{J} \circ \mathrm{J} = -\mathrm{J} \circ \nabla \mathrm{J}$. Therefore (25) follows from

$$d\omega_g(x, y, z) - d\omega_g(\mathrm{J}x, \mathrm{J}y, z) = g(z, \mathrm{J}N_{\mathrm{J}}(x, y)) + 2g((\nabla_z \mathrm{J})x, y),$$

where we have used (14) and $\mathrm{J}N_{\mathrm{J}} = (\nabla_x \mathrm{J})y - (\nabla_{\mathrm{J}x}\mathrm{J})\mathrm{J}y - (\nabla_y \mathrm{J})x + (\nabla_{\mathrm{J}y}\mathrm{J})\mathrm{J}x$.

Note that geometrically the condition $\nabla \mathrm{J} = 0$ means that parallel translation preserves type, that is the restriction of the Levi-Civita connection to either $T^{1,0}M$ or $T^{0,1}M$ is a well-defined connection compatible with g_{Herm}. Indeed, suppose $v(t)$ is the parallel translation of $v(0) \in T^{1,0}_{\gamma(0)}M$ along $\gamma(t)$ and $\mathrm{J}v(0) = \sqrt{-1}v(0)$. Then $\mathrm{J}v(t)$ satisfies $\nabla_{\gamma'(t)}(\mathrm{J}v(t)) = \mathrm{J}\nabla_{\gamma'(t)}v(t) = 0$. Now both $\mathrm{J}v(t)$ and $\sqrt{-1}v(t)$ are parallel along $\gamma(t)$ and both agree at $\gamma(0)$, hence for all t.

2.1.4.3 Parallel symplectic form. In light of (25) one also has the following characterization of of when a symplectic manifold is Kähler :

$$\nabla \mathrm{J} = 0 \quad \Leftrightarrow \quad \nabla \omega_g = 0. \tag{26}$$

Indeed, $\nabla \omega_g = 0$ iff $0 = (\nabla_x \omega_g)(y, z) = \nabla_x(\omega_g(y, z)) - \omega_g(\nabla_x y, z) - \omega_g(y, \nabla_x z)$ for all x, y, z (parallel transport preserves ω_g). Equivalently, $x(g(\mathrm{J}y, z)) = g(\mathrm{J}\nabla_x y, z) + g(\mathrm{J}y, \nabla_x z)$ or by (9) $g(\nabla_x(\mathrm{J}y), z)) = g(\mathrm{J}\nabla_x y, z)$, that is $g((\nabla_x \mathrm{J})(y), z)) = 0$, i.e., $\nabla \mathrm{J} = 0$.

2.1.4.4 Holonomy characterization. For Riemannian metrics $\nabla g = 0$, and so parallel translation along a loop γ defines an operator $P_\gamma : T_pM \to T_pM$ satisfying $g(P_\gamma v, P_\gamma w) = g(v, w)$, hence any orthonormal frame is mapped to another one, i.e., $P_\gamma \in O(T_pM) \cong O(2n)$. Similary the additional condition $\nabla \omega_g = 0$ implies $\omega(P_\gamma v, P_\gamma w) = \omega(v, w)$, i.e., a unitary frame $\{e_1, \ldots, e_n, \mathrm{J}e_1, \ldots, \mathrm{J}e_n\}$ is carried to another such and $P_\gamma \in U(n) = O(2n) \cap GL(n, \mathbb{C})$. In detail: $\nabla \omega_g = 0$ is equivalent to $\frac{d}{dt}\omega_g(P_{\gamma_t}v, P_{\gamma_t}w) = \omega_g(\nabla_{\gamma'(t)}(P_{\gamma_t}v), P_{\gamma_t}w) + \omega_g(P_{\gamma_t}v, \nabla_{\gamma'(t)}(P_{\gamma_t}w)) + (\nabla_{\gamma'(t)}\omega_g)(P_{\gamma_t}v, P_{\gamma_t}w) = 0$ as P_{γ_t} by definition produces parallel sections.

Conversely, if for a Riemannian manifold (M, g) parallel translation always produces unitary matrices (we then say that its holonomy group $\mathrm{Hol}(M, g)$ is a subset of $U(n)$) then M is Kähler! This statement amounts to constructing a Kähler structure, or equivalently an almost-complex structure satisfying (25) [**21**, p. 283].

Indeed, at the point $p \in M$ choose a coordinate patch where the Riemannian metric is Euclidean at p and consider the quadratic Hermitian form $\omega(p) = \delta_{i\bar{j}} dz^i \wedge d\bar{z}^j$ on $T^{1,0}_p M$. Extend ω to all of M by parallel translation. By the holonomy condition this construction is well-defined independently of the path we choose to parallel translate along (think of a coordinate patch where P_γ acts on T_pM by a unitary matrix and hence $\omega(\cdot,\cdot) = \omega(P_\gamma\cdot, P_\gamma\cdot))$. By construction $\nabla\omega = 0$. Further, ω is nondegenerate globally, i.e. $\omega^n \neq 0$ since if $\omega^n(q) = 0$ for $q \in M$ then $\nabla(\omega^n) = 0$ would imply $\omega^n = 0$ globally and we know this is not so at p. Now define a complex structure $\mathrm{J}(p)$ on T_pM by the equation $\omega(v,w)|_p = g(\mathrm{J}v,w)|_p$, $\mathrm{J}^2|_p = -\mathrm{I}$, that is just take a complex structure on $\mathbb{R}^{2n}$ compatible with the Euclidean metric, i.e., $\mathrm{J}(p) = \left(\begin{smallmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{smallmatrix}\right)$. Next extend J uniquely by parallel translation—this is well-defined since at p J is defined in terms of ω and g and both are parallel. This also implies that we have $\omega(v,w) = g(\mathrm{J}v,w)$ everywhere. We also have $\mathrm{J}^2 = -\mathrm{I}$ everywhere (since $\mathrm{J}^2 + \mathrm{I}$ is parallel and zero at p). Hence (M, g, J) is Kähler.

We remark that we have also just proved that a Kähler manifold can be characterized as admitting a positive parallel 2-form. We also note that ω just constructed is closed: indirectly by combining (25) and (26), or directly since any parallel form is [**218**, p. 150], to wit,

$$d\alpha(v_0,\ldots,v_r) = \sum_{j=0}^{r}(-1)^j(\nabla_{v_j}\alpha)(v_0,\ldots,\hat{v}_j,\ldots,v_r).$$

2.1.5 Curvature identities on Kähler manifolds. The reader of this section might find Bochner's early account quite relevant and readable [**26**]. Kähler metrics satisfy special curvature identities [**141,238**]. Consider a line bundle $L \xrightarrow{\pi} M$ represented by the Čech cocycle $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ (where A is the index set for a holomorphic atlas) representing a class in $H^1(M, \mathcal{O}^\star)$ [**114**, p. 132]. Endow the bundle with a Hermitian metric h represented locally by positive functions h_α satisfying $h_\alpha = h_\beta|g_{\alpha\beta}|^2$.

A connection D on the bundle is represented by a 1-form $\alpha(p)$ on M with values in endomorphisms of the complex line $L_p = \pi^{-1}(p)$ above each point $p \in M$. One writes

$$D_X s = \alpha(X)s, \qquad \forall X \in \Gamma(M, T_{\mathbb{C}}M), \quad s \in \Gamma(M, L).$$

To explain what such a 1-form resembles, note $h(p)$ is a map $L_p \times \overline{L_p} \to \mathbb{C}$. However we may identify it as a function h_α locally as we did above, via the isomorphism between such maps and maps $L_p \to L_p \cong \overline{L_p}^\star$. So for example dh is such an endomorphism valued 1-form. In analogy to a Levi-Civita connection one would like a distinguished connection on the fibers of $L \xrightarrow{\pi} M$ respecting the metric h, but in addition respecting also the complex structure on the total space L, that is the given one on M and the linear structure of $\mathbb{C}$ on each fiber. So, we require that

$$\begin{gathered} Xh(s,t) = h(D_X s, t) + h(s, D_X t), \qquad \forall X \in \Gamma(M, T_{\mathbb{C}}M), \quad s,t \in \Gamma(M,L). \\ D_X s = 0, \qquad \forall X \in H^0(M, \mathcal{O}_M(T^{1,0}M)), \quad s \in H^0(M, \mathcal{O}_M(L)). \end{gathered}$$

The first equation can be rewritten as one for endomorphism-valued 1-forms

$$d(h(s,t)) = h(\alpha s, t) + h(s, \alpha t).$$

The second equation simply means that the 1-form part of α is of type $(1,0)$. The first equation then splits into two equivalent ones according to type, one of which reads $\partial(h(s,t)) = h(\alpha s, t)$. It follows that $\alpha = h^{-1}\partial h$ (with some abuse of notation; think here of choosing a base e_L for the fibers (one vector since one dimensional) and representing h as the norm of that vector, i.e., as a function, and putting $s = t = e_L$ in the previous equation). This connection is called the complex connection (as well as the Chern connection). This formula is a kind of complex analogue of the Koszul formula for the Levi-Civita connection (11).

Given a connection, its curvature its defined as the exterior differential of its representing 1-form. The motivation for this comes simply from the abstract formula $d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$: think of α here as a one-form with values in the endomorphisms of L and let each side of the equation act on a section s. The right hand side then corresponds to $(D_X D_Y - D_Y D_X - D_{[X,Y]})s$ and so can be associated with the curvature! We call the curvature of an Hermitian metric the curvature of its complex connection, denoted by $R(h)$. Explicitly, it is seen to be an imaginary (1,1)-form

$$R(h) = d(h^{-1}\partial h) = \partial(h^{-1}\partial h) + \bar\partial(h^{-1}\partial h) = \bar\partial(h^{-1}\partial h) = -\partial\bar\partial \log h, \tag{27}$$

since $\partial(h^{-1}\partial h) = h^{-1}\partial\partial h - h^{-1}\partial h \wedge h^{-1}\partial h = 0 - 0 = 0$.

We now specialize to the case $L = \Lambda^n T^{1,0}M =: K^\star_M$, the anticanonical bundle. The special feature is that then the curvature is given in terms of the Hermitian metric on $T^{1,0}M$. First consider $L^\star$ the canonical bundle whose sections are $(n,0)$ forms. Given two such forms represented by $\Omega_i = f_i dz^1 \wedge \cdots \wedge dz^n$, $i = 1, 2$ we have then $h(\Omega_1, \Omega_2) = g(\Omega_1, \Omega_2) = f_1 f_2 \det g^{-1}$ where g here stands for the extension of the Hermitian metric on $T^{1,0\,\star}M$ to the exterior algebra $\Lambda T^{1,0\,\star}M$. It follows that $\det g$ is a Hermitian metric on L (we represent it locally by $\det(g_{i\bar j})$ by choosing $e_L = dz^1 \wedge \cdots \wedge dz^n$), and that

$$R(h) = -\partial\bar\partial \log \det g.$$

We define the Ricci curvature form of (M,g) to be the real closed (1,1)-form

$$\operatorname{Ric}\omega := \sqrt{-1} R_{i\bar j} dz^i \wedge \overline{dz^j} := \frac{\sqrt{-1}}{2\pi} R(h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log\det g. \tag{28}$$

One may show [**271**, p. 6] that this essentially coincides with the Ricci curvature tensor defined in the Riemannian setting in the sense that

$$\operatorname{Ric} g(\,\cdot\,,\,\cdot\,) = \operatorname{Ric}\omega(\,\cdot\,, \mathrm{J}\,\cdot\,).$$

According to the theory of Chern [**114,140**] the cohomology class of $\mathrm{Ric}\,\omega$ is an integral cohomology class independent of the choice of g and depends only on the complex structure (M, J), and one denotes

$$c_1 := c_1(M, \mathrm{J}) := [\mathrm{Ric}\,\omega] \in H^2(M, \mathbb{Z}). \tag{29}$$

In terms of Čech cohomology c_1 is the image of the Čech class of $\Lambda^n T^{1,0} M$ under the map

$$H^1(M, \mathcal{O}^\star) \to H^2(M, \mathbb{Z}),$$

induced from the short exact sequence (63) (see Section 3.2).

The following definition is due to Kähler [**141**].

Definition 2.9. *A Kähler form ω is said to be Kähler-Einstein if* $\mathrm{Ric}\,\omega = a\omega$ *for some $a \in \mathbb{R}$. One then calls (M, ω) a Kähler-Einstein manifold.*

A corollary of (29) that seems to have been first noticed by Calabi is the following cohomological restriction induced by a choice of a complex structure:

Corollary 2.10. *A necessary condition for a Kähler-Einstein metric to exist in $\mathcal{H}_\Omega$ is that $c_1(M, \mathrm{J}) = a\Omega$, for some $a \in \mathbb{R}$.*

A Kähler-Einstein metric is an example of a canonical metric. As a definition of a canonical metric one could take the rather liberal notion of a metric that is uniquely determined by the specification of certain (other, non-metric) structures. A Kähler-Einstein metric is determined solely by a choice of a complex structure.

A more general notion than that of a Kähler-Einstein metric is the one of a Kähler metric of constant scalar curvature (csc). This is a Kähler metric g_ω satisfying the equation

$$s(\omega) := \mathrm{tr}_\omega \mathrm{Ric}\,\omega := \frac{n c_1 \wedge \omega^{n-1}}{\omega^n} = \text{const.} \tag{30}$$

In local coordinates this gives the trace of the matrix obtained from the Ricci tensor times the inverse of the metric tensor

$$s(\omega) = \frac{g^{i\bar{j}}}{\sqrt{-1}} \sqrt{-1} R_{i\bar{j}} = g^{i\bar{j}} R_{i\bar{j}}.$$

Indeed the top exterior product of a $(1,1)$-form is given by the determinant of the matrix representing it and the $(n-1)$-th exterior product gives then—up to a determinant—the matrix of minors.

The cohomological restriction of Corollary 2.10 no longer exists. Thus a constant scalar curvature metric is canonically determined by a choice of a complex structure and a Kähler cohomology class Ω. These metrics admit a characterization in terms of Hodge theory, as observed by Calabi (see Lemma 4.29). For a Kähler-Einstein metric the constant in (30) equals na. Observe that the value of the scalar curvature, if it is constant, is predetermined by the Kähler class and the complex structure, indeed by (30) the average of the scalar curvature for any metric in $\mathcal{H}_\Omega$ equals

$$s_0 = \frac{nc_1.\Omega^{n-1}([M])}{\Omega^n([M])}, \tag{31}$$

2.1.6 The moduli space of cohomologous Kähler metric.

Questions in Kähler geometry can be oftentimes phrased in terms of variational problems on the space of Kähler forms. The most classical example, also one of the motivations for Calabi's Definition 2.8 is the question of existence of extremal metrics in a fixed cohomology class, that is the existence of minimizers of the functional

$$\omega \mapsto \frac{1}{V}\int_M (s(\omega)-s_0)^2\omega^n,$$

with s_0 defined by (31), and where V is the volume of (M,ω),

$$V = \int_M \omega^n = [\omega]^n([M]).$$

Observe that V is even a simpler invariant associated to the space $\mathcal{H}_\omega$ than s_0 (31). Similarly one has the invariants

$$\int_M (\mathrm{Ric}\,\omega)^k \wedge \omega^{n-k}, \tag{32}$$

that come up naturally, see §§4.4.2.

A striking feature of this space, first discovered by Mabuchi, is the existence of simple and beautiful (infinite-dimensional) Riemannanian structure on it. The space $\mathcal{H}_\omega$ is a neighborhood of 0 in $C^\infty(M)$ and hence the tangent space to any point is simply $C^\infty(M)$. The Mabuchi metric on $\mathcal{H}_\omega$ is given by

$$g_{L^2}(\mu,\eta)_\varphi := \frac{1}{V}\int \mu\eta\,\omega^n, \qquad \mu,\eta \in T_\varphi\mathcal{H}_\omega. \tag{33}$$

This structure was rediscovered by Semmes who also proved a uniqueness result about it [**240**, Lemma 4.3], as well as by Donaldson [**79**]. In fact this structure is an example of an infinite-dimensional symmetric space structure, and is associated to the group of Hamiltonian diffeomorphisms of (M,ω). This point of view is so fecund that it has completely transformed Kähler geometry and has pointed out several beautiful connections to other fields in mathematics. This will be briefly touched upon in Section 2.2.

2.1.7 Vector fields and one-forms on the space of Kähler metrics. As on any Riemannian manifold, one may define tensors on $\mathcal{H}_\omega$. Since $\mathcal{H}_\omega$ is a neighborhood of the zero function in $C^\infty(M)$, i.e., an open subset of $C^\infty(M)$, its tangent space at any point is isomorphic to $C^\infty(M)$. Its cotangent space at any point is therefore isomorphic to the space of all $2n$-forms $\Gamma(M, \Lambda^{2n}T^\star M)$.

Therefore, a vector field on $\mathcal{H}_\omega$ is by definition an assignment of a smooth function to each point in $\mathcal{H}_\omega$, while a 1-form is an assignment of a smooth $2n$-form to each point in $\mathcal{H}_\omega$. The pairing between $T_\varphi\mathcal{H}_\omega \cong C^\infty(M)$ and $T^\star_\varphi\mathcal{H}_\omega \cong \Gamma(M, \Lambda^{2n}T^\star M)$ is given by integration:

$$(\psi, \nu)|_\varphi \mapsto \int_M \psi\nu, \qquad \psi \in T_\varphi\mathcal{H}_\omega, \quad \nu \in T^\star_\varphi\mathcal{H}_\omega.$$

Vector fields and one-forms on infinite-dimensional spaces can be used to reformulate succinctly many familiar notions from the calculus of variations and geometric analysis. For example, a geometric evolution equation for the metric (or a "flow") describes the integral curve on a vector field on $\mathcal{H}_\omega$ (see, for example, Section 4.7), and a functional on $\mathcal{H}_\omega$ should really be considered as a function on $\mathcal{H}_\omega$.

One also has the notion of an exterior differential, and hence of closed forms, as in finite dimensions. First, the variation of a function f gives a 1-form df on $\mathcal{H}_\omega$, given by

$$(df(\varphi), \psi)_\varphi = \frac{d}{dt}\Big|_0 f(\varphi + t\psi), \quad \forall\, \psi \in T_\varphi\mathcal{H}_\Omega.$$

Next, given a 1-form α on $\mathcal{H}_\omega$ we define its exterior differential by setting for any two smooth functions ψ_1, ψ_2 on M,

$$\begin{aligned} d\alpha|_\varphi(\psi_1, \psi_2) &:= \psi_1\, \alpha(\psi_2)|_\varphi - \psi_2\, \alpha(\psi_1)|_\varphi \\ &= \frac{d}{dt}\Big|_0 \alpha(\psi_2)|_{\varphi+t\psi_1} - \frac{d}{dt}\Big|_0 \alpha(\psi_1)|_{\varphi+t\psi_2}. \end{aligned} \tag{34}$$

Here ψ_1, ψ_2 are considered as constant vector fields on $\mathcal{H}_\omega$ and so formally $[\psi_1, \psi_2] = 0$. Thus the definition (34) agrees with the finite-dimensional notion of an exterior derivative

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]). \tag{35}$$

This is important since, for example, if we restrict d to a finite-dimensional subspace of $\mathcal{H}_\omega$ we recover the usual notion of an exterior derivative there.

As first observed by Donaldson for a different infinite-dimensional space and then by Mabuchi for $\mathcal{H}_\Omega$, closed 1-forms admit a characterization in terms of (the trivial)

Čech cohomology of $\mathcal{H}_\Omega$ [**77**,**174**]. We have switched to $\mathcal{H}_\Omega$ instead of $\mathcal{H}_\omega$ although this is not at all essential. Namely any closed 1-form α on $\mathcal{H}_\Omega$ is exact since $\mathcal{H}_\Omega$ is simply connected (contractible, in fact), and may therefore be represented by a Čech 1-cocycle $\{f(x,y)\}_{x,y\in\mathcal{H}_\Omega}$ with f a function on $\mathcal{H}_\Omega\times\mathcal{H}_\Omega$ that satisfies the Čech cocycle condition

$$(\delta f)(\omega_1,\omega_2,\omega_3)=f(\omega_1,\omega_2)+f(\omega_2,\omega_3)+f(\omega_3,\omega_1)=0. \tag{36}$$

Here δ is the usual Čech differential [**114**].

Dually, given a closed 1-form α on $\mathcal{H}_\Omega$, one may define a function f on $\mathcal{H}_\Omega\times\mathcal{H}_\Omega$ that satisfies the Čech cocycle condition (36) by setting

$$f(\omega_1,\omega_2):=\int_{\omega_1}^{\omega_2}\alpha, \tag{37}$$

and this is independent of the path since one may prove that, as in finite-dimensions, for any 1-form β

$$\int_{\omega_1}^{\omega_2}\beta=\int_{C(\omega_1,\omega_2)}d\beta,$$

where $C(\omega_1,\omega_2)$ is the image of a disc in $\mathcal{H}_\Omega$ with boundary the loop given by the concatentation of any two paths connecting ω_1 and ω_2. In particular fixing $\omega_0\in\mathcal{H}_\Omega$ we thus obtain a function

$$\omega\mapsto f(\omega_0,\omega), \tag{38}$$

on $\mathcal{H}_\Omega$. This observation has very important consequences, for example for the construction of energy functionals on $\mathcal{H}_\Omega$ (see Section 4.4). We therefore record the following definition:

Definition 2.11. *A function on $\mathcal{H}_\Omega\times\mathcal{H}_\Omega$ is called an exact functional if*

$$(\delta f)(\omega_1,\omega_2,\omega_3)=f(\omega_1,\omega_2)+f(\omega_2,\omega_3)+f(\omega_3,\omega_1)=0. \tag{39}$$

The reader may find it interesting to compare the discussion in this subsection to [**31**,**109**].

2.2 The space of Kähler metrics and symplectic geometry

This section is a very brief account of the basic features of the Riemannian structure of the space of Kähler metrics. We refer the reader to Gauduchon [**109**] for more details. We hope to discuss several interesting properties of this structure in a separate article and therefore will be rather brief here.

2.2.1 Certain classical Riemannian symmetric spaces. Let G be a compact Lie group and let $\mathcal{G} := \mathrm{Lie}\, G$ denote its Lie algebra. Any Lie group has (many) left-invariant Riemannian metrics given by

$$\langle u, v\rangle_g = \langle dL_{g^{-1}}u, dL_{g^{-1}}v\rangle_0, \quad \forall\, u, v \in T_gG,$$

where we have fixed some arbitrary inner product $\langle\,\cdot\,,\,\cdot\,\rangle_0$ on T_eG, and L denotes the left action of G on itself by multiplication (similarly R will denote the right action). Consider the adjoint action of the group on the Lie algebra

$$\mathrm{Ad}_g w = \frac{d}{dt}\Big|_0 g \exp_e(tw) g^{-1}, \ \forall w \in \mathcal{G},\ g \in G.$$

Haar theory for compact topological groups provides for an Ad_G-invariant measure $d\gamma$ on G [**129**, p. 88; **222**, p. 336–340]. Consequently, by averaging a left-invariant metric with respect to the right action one obtains a bi-invariant Riemannian metric (simultaneously right- and left-invariant) h on G given by

$$h(u, v)|_f = \int_G \langle dR_k u, dR_k v\rangle|_{fk} d\gamma(k), \qquad \forall f \in G.$$

Hence h is Ad_G-invariant and therefore with respect to h the adjoint action of $\mathcal{G}$ on $\mathcal{G}$ is skew-symmetric:

$$h([u, v], w) = -h([u, w], v). \tag{40}$$

This is the unique bi-invariant metric (up to scale).

Recall that the Killing form κ is a negative semi-definite bilinear form on $\mathcal{G}$ defined by

$$\kappa(u, v) = \mathrm{tr}(\mathrm{ad}u \circ \mathrm{ad}v),$$

where $(\mathrm{ad}\, u)(x) := [u, x]$, and $(\mathrm{ad}u \circ \mathrm{ad}v)(x) = [u, [v, x]]$. This form is $\mathrm{ad}_{\mathcal{G}}$-invariant, namely satisfies [**130**, p. 131]

$$\kappa(\mathrm{ad}_a u, \mathrm{ad}_a v) \equiv \kappa([a, u], [a, v]) = \kappa(u, v), \qquad \forall a \in \mathcal{G}. \tag{41}$$

The form κ is strictly negative if and only if G is semisimple (under the assumption that G is compact as we assume throughout) [**130**, p. 132; **222**, p. 342]. In that case $-\kappa$ can serve as a biinvariant metric on G as will be seen below.

Now it is known that the Levi-Civita connection is given by $\nabla_u v = \frac{1}{2}[u, v]$ for left-invariant vector fields u, v (which by a slight abuse of notation we note as $u, v \in \mathcal{G}$ via an identification between T_eG and left invariant vector fields on G), and that the curvature tensor can be written as

$$R(u, v)w = -\tfrac{1}{4}[[u, v], w], \quad \forall u, v, w \in \mathcal{G}. \tag{42}$$

We will prove these facts in the next subsection where we verify that these remain valid also in infinite-dimensions (see also [**58**, p. 65]).

Consequently, using (40), the sectional curvatures are nonnegative

$$K(u,v) = \frac{h(R(u,v)v,u)}{h(u,u)h(v,v)-h(u,v)^2} = \frac{1}{4}\cdot\frac{h([u,v],[u,v])}{h(u,u)h(v,v)-h(u,v)^2} \geq 0, \tag{43}$$

and, using (41), with respect to an orthonormal basis $e_1,\ldots,e_{\dim G}$ for T_eG with respect to h the Ricci curvature may be written as

$$\begin{aligned}\mathrm{Ric}\,h\,(u,v) &= \sum_{i=1}^{\dim G} h(-\tfrac{1}{4}[[u,e_i],e_i],v)\\ &= \tfrac{1}{4}\sum_{i=1}^{\dim G} h([u,e_i],[v,e_i])\\ &= -\tfrac{1}{4}\sum_{i=1}^{\dim G} h([v,[u,e_i]],e_i)\\ &= -\tfrac{1}{4}\mathrm{tr}(\mathrm{ad}u\circ\mathrm{ad}v)\\ &= \tfrac{1}{4}(-\kappa)(u,v). \end{aligned}\tag{44}$$

When G is semisimple the Killing form is negative definite, and so if we choose $h=-\kappa$ then the metric is Einstein with constant Ricci curvature $1/4$ [**222**, p. 342] (cf. [**147**, p. 258]).

Following Cartan, a Riemannian manifold is called a symmetric space if for each point there exists an isometry that fixes the point and reverses the orientation of geodesics through the point [**21**,**37**,**130**,**212**]. It follows that one may find an isometry that exchanges any two points, i.e., these spaces are homogeneous, namely the view from any point is the same. It also follows that the space is complete, since if a geodesic is defined up to time T it may be reflected and thus extended up to time $2T-\epsilon$. A characterization of symmetric spaces is that they are simply connected and have parallel curvature tensor. We will switch our original convention and take this as the definition of a symmetric space, since it is this definition that generalizes to infinite-dimensions. We remark that there is a complete classification of symmetric spaces in terms of Lie theory.

Examples of symmetric spaces are the space forms of constant curvature. Compact Lie groups are another class of examples that have in general only constant Ricci curvature, as we saw above. Indeed, the map $g\mapsto hg^{-1}h$ is an isometry that preserves each fixed $h\in G$ and reverses geodesics about h. It is the composition of maps $L_h\circ(\text{ inversion })\circ L_{h^{-1}}$, where the inversion map sends g to g^{-1}.

A remarkable fact is that the compact Lie groups have natural non-compact "dual" symmetric spaces.[7] These are obtained by "complexifying" the group and quotienting the result by the original group. A complexification of a finite-dimensional Lie group G can be defined as a complex Lie group $G^{\mathbb{C}}$ with the universal property that any Lie group homomorphism from G into a complex Lie group H may be uniquely analytically continued to a homomorphism from $G^{\mathbb{C}}$ into H. It follows that $G^{\mathbb{C}}$ is unique up to isomorphism. One may show that any compact connected finite-dimensional Lie group admits a complexification [**37**, p. 184]. We claim that the quotient $G^{\mathbb{C}}/G$ is then a symmetric space. Indeed, $\mathrm{Lie}\, G^{\mathbb{C}} = \mathcal{G} \oplus \sqrt{-1}\mathcal{G}$ and the involution $x + \sqrt{-1}y \mapsto x - \sqrt{-1}y$ can be integrated to give a Lie group involution of $G^{\mathbb{C}}$: alternatively on the Lie group level the involution is given by complex conjugation (this makes sense since $G^{\mathbb{C}}$ embeds in $GL(N, \mathbb{C})$ for some N [**37**, p. 202]). At the identity the tangent space splits into the $+1$ and -1 eigenspaces, the latter being $\sqrt{-1}\mathcal{G} \subset T_e G^{\mathbb{C}}$. Now G acts as a Lie transformation group on $G^{\mathbb{C}}$ by right multiplication and the orbit space $G^{\mathbb{C}}/G$ (whose points are cosets gG with $g \in G^{\mathbb{C}}$) inherits the map induced by the involution on the group level, originally defined on $G^{\mathbb{C}}$ (this is visible also from the description as a matrix group). This inherited map then reflects geodesics about the point eG and its differential is minus the identity on the tangent space $T_{eG}(G^{\mathbb{C}}/G)$. Similarly one may construct by conjugation such reflections about any point. This sketches the proof of the claim.

The symmetric space $G^{\mathbb{C}}/G$ has a natural (left-invariant) Riemannian structure induced from the Riemannian submersion $G^{\mathbb{C}} \to G^{\mathbb{C}}/G$ and one may show that its curvature satisfies the equation (42) with the sign reversed [**58**,**130**]. Therefore it has nonpositive sectional curvature, and under irreducibility assumptions will be Einstein [**21**, pp. 196, 202]. This will be illustrated in some detail in an infinite-dimensional example in the next two subsections.

Example 2.12. The simplest example is $G = S^1 \cong SO(2) \cong U(1)$ for which $G^{\mathbb{C}} = GL(1, \mathbb{C}) = \mathbb{C}^\star$ and the resulting dual is $G^{\mathbb{C}}/G \cong \mathbb{R}_+ \setminus \{0\}$. Another example is $G = S^3 \cong SU(2) \cong O(3)$. Here $G^{\mathbb{C}} = SL(2, \mathbb{C})$ and the noncompact dual of G is $SL(2, \mathbb{C})/SU(2)$ isomorphic to the space of all positive Hermitian matrices. It may be shown that the space has constant negative Ricci curvature [**21**, pp. 196, 202]. In particular since it has nonpositive sectional curvature it is simply connected and since it is 3-dimensional, constant Ricci curvature implies constant curvature. It follows that it is isometric to the 3-dimensional hyperbolic space form.

2.2.2 The group of Hamiltonian diffeomorphisms.

Let Diff(M) denote the infinite-dimensional group of diffeomorphisms of M.[8] Remember M denotes a

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[7] For a precise notion of duality in this context, see Bump [**37**, pp. 213, 216], Cheeger-Ebin [**58**, p. 77].

[8] When performing analysis on such infinite-dimensional objects it is necessary to work with a

differentiable manifold, that is an equivalence class of differentiable structures on a fixed topological manifold and so elements of Diff(M) can be considered as gauge transformations (of the differential structure). Let diff(M) denote the Lie algebra of Diff(M), consisting of all (smooth) derivations on M, equivalently vector fields on M, that is sections of the tangent bundle $\Gamma(M, TM)$, together with the bracket structure $[X, Y] = X \circ Y - Y \circ X$.

Let $\mathrm{Symp}(M, \omega)$ denote the infinite-dimensional closed subgroup of symplectomorphisms of (M, ω),

$$\{\phi \in \mathrm{Diff}(M) : \phi^\star \omega = \omega\},$$

and let $\mathrm{Symp}(M, \omega)_0$ stand for its identity component, namely $\{\phi \in \mathrm{Diff}(M) : \phi^\star \omega = \omega, \phi \text{ homotopic to the identity map id}\}$. (The meaning of this equation is that by inducing a two form at $\phi(p)$ from the form ω at p we obtain precisely the form $\omega(\phi(p))$, which need not be the case for general ω and ϕ.) Denote by $\mathrm{symp}(M, \omega)$ the Lie algebra of ω-symplectic derivations of $C^\infty(M)$, or symplectic vector fields on M (recall Definition 2.2)

$$\mathrm{symp}(M, \omega) := \{X \in \mathrm{diff}(M) \, : \, \mathcal{L}_X \omega = 0\},$$

considered as a Lie subalgebra of diff(M). One may think of $\mathrm{Symp}(M, \omega)$ as the gauge group of the symplectic structure.

Let

$$\psi : \mathbb{R} \to \mathrm{Symp}(M, \omega), \qquad t \mapsto \psi_t, \tag{45}$$

denote a family of symplectomorphisms passing through the identity $\mathrm{id} \in \mathrm{Symp}(M, \omega)$. One has $\psi_t^\star \omega = \omega$ and consequently by the Cartan formula for the Lie derivative $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$ [**46**, p. 36],

$$0 = \frac{d}{dt} \psi_t^\star \omega = \mathcal{L}_{d\psi_t \frac{d}{dt}} \omega \big|_{\psi_t(\cdot)} = d\iota_{d\psi_t \frac{d}{dt}} \omega \big|_{\psi_t(\cdot)}.$$

Therefore $[\iota_{d\psi_t \frac{d}{dt}} \omega] \in H^1(M, \mathbb{R})$. When $[\iota_{d\psi_t \frac{d}{dt}} \omega] = 0$ one may write $\iota_{d\psi_t \frac{d}{dt}} \omega = dH_t$ for some smooth functions, up to constants, H_t. Thus as sets $\mathrm{symp}(M, \omega) \cong C^\infty(M)/\mathbb{R}$. It will be convenient to identify $C^\infty(M)/\mathbb{R}$ with the functions of average zero with respect to the volume form ω^n. The map

$$H : \mathbb{R} \to C^\infty(M)/\mathbb{R}, \qquad t \mapsto H_t, \tag{46}$$

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precise definition of what is meant by a Lie group structure, see, e.g., Milnor [**196**]. However for our purposes this will not be necessary as we will not use the analytical structure (e.g., local models of the space (infinite-dimensional version of coordinate patches)) of the infinite-dimensional spaces as given but rather draw conclusions on a more superficial level. Thus we only consider these as spaces with a group structure. There will however be no problem in regarding their tangent spaces at the identity as (bona fide) Lie algebras.

is called a time-dependent Hamiltonian associated to the one-parameter subgroup $\{\psi_t\}_{t\in\mathbb{R}}$. Also, each of the maps $\psi_t \in \mathrm{Symp}(M,\omega)$ is called a Hamiltonian diffeomorphism (also known as symplectomorphisms exactly homotopic to the identity) and one may show that the set $\mathrm{Ham}(M,\omega)$ of all such forms a closed group [**217**]. Note that unlike Diff(M) or Symp(M,ω) the group Ham(M,ω) is not defined as the gauge group for a certain structure, a fact which makes its study highly interesting and non-trivial [**217**, p. 8]. Let

$$\mathrm{ham}(M,\omega) := \mathrm{Lie}\,\mathrm{Ham}(M,\omega) = \{X \in \mathrm{diff}(M)\ :\ [\iota_X\omega] = 0 \in H^1(M,\mathbb{R})\} \qquad (47)$$

denote the Lie algebra of Hamiltonian vector fields. Write X_h for the Hamiltonian vector field corresponding to the function h, i.e. satisfying $\iota_{X_h}\omega = dh$. Define the Poisson bracket with respect to ω on $C^\infty(M)/\mathbb{R}$ by

$$\{f,g\}_\omega := \omega(X_f, X_g). \qquad (48)$$

We will frequently use the shorthand notation $\{\,\cdot\,,\,\cdot\,\}$. First observe that this is always in $C^\infty(M)/\mathbb{R}$, i.e., of average zero, since we claim that

$$\omega(X_f, X_g)\omega^n/n! = df \wedge dg \wedge \omega^{n-1}/(n-1)!$$

and the latter is an exact form, hence integrates to zero on a compact manifold. Indeed, $\omega^{n+1} = 0$ and so

$$\begin{aligned}0 = \iota_{X_g}\iota_{X_f}\omega^{n+1} &= (n+1)\iota_{X_g}(\iota_{X_f}\omega \wedge \omega^n)\\ &= (n+1)(\iota_{X_g}\iota_{X_f}\omega)\omega^n - n(n+1)\iota_{X_f}\omega \wedge \iota_{X_g}\omega \wedge \omega^{n-1},\end{aligned}$$

proving the claim. We now show that the Poisson bracket defines a Lie algebra structure on $C^\infty(M)$. First using the identity $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$ [**46**, p. 108] one has

$$\begin{aligned}\iota_{[X_f,X_g]}\omega &= [\mathcal{L}_{X_f}, \iota_{X_g}]\omega = \mathcal{L}_{X_f}dg - 0\\ &= d\mathcal{L}_{X_f}g = dX_f g\\ &= d(dg(X_f)) = d\omega(X_g, X_f) = -d\{f,g\}_\omega. \qquad (49)\end{aligned}$$

Hence,

$$[X_f, X_g] = -X_{\{f,g\}_\omega}. \qquad (50)$$

Thus our assignment is a Lie algebra isomorphism once we can show that (48) satisfies the Jacobi identity. To see that use the identity twice again to compute

$$\begin{aligned}d\{f,\{g,h\}\} &= \iota_{X_{\{f,\{g,h\}\}}}\omega = \iota_{[X_{\{g,h\}},X_f]}\omega\\ &= -\mathcal{L}_{X_f}\iota_{X_{\{g,h\}}}\omega = -\mathcal{L}_{X_f}\iota_{[X_h,X_g]}\omega\\ &= -\iota_{[X_f,[X_h,X_g]]}\omega = \iota_{[X_f,[X_g,X_h]]}\omega.\end{aligned}$$

Since $[\,\cdot\,,\cdot\,]$ satisfies the Jacobi identity so does $\{\,\cdot\,,\cdot\,\}$. Thus the map $h \mapsto X_h$ yields a Lie algebra isomorphism $\mathrm{ham}(M,\omega) \cong (C^\infty(M)/\mathbb{R}, -\{\,\cdot\,,\cdot\,\}_\omega)$, whose inverse is given by $X \mapsto \int_{p_0}^{p} \iota_X\omega - \frac{1}{V}\int(\int_{p_0}^{p}\iota_X\omega)\omega^n$ (well-defined independent of choice of paths since $[\iota_X\omega] = 0$).

Now we identify the one-parameter subgroups of $\mathrm{Ham}(M,\omega)$. That is we specialize (45) to the case where $\{\psi_t\}$ is a family generated by integrating the flow of a fixed vector field X up to time t. We use the notation $\psi_t = \exp_{\mathrm{id}} tX$. The map (46) then takes a simple form $t \mapsto H_0 \circ \psi_t^{-1}$, since putting $X_s = d\psi_s \frac{d}{dt}$ we have for any global vector field Y

$$\begin{aligned} dH_0(Y) &= \omega(X_0, Y) = (\psi_t^\star\omega)(X_0, Y) \\ &= \omega(d\psi_t X_0, d\psi_t Y) = \omega(X_t, d\psi_t Y) \\ &= dH_t(d\psi_t Y) = (\psi_t^\star dH_t)(Y), \end{aligned}$$

hence $\psi_t^\star H_t - H_0$ is constant and in particular $H_t = H_0 \circ \psi_t^{-1}$ since we assume the Hamiltonians are normalized to have average zero. These are naturally the one-parameter subgroups of $\mathrm{Ham}(M,\omega)$. For a vector $f \in T_F\mathrm{Ham}(M,\omega)$ left-translation is then given by

$$dL_g f \equiv dg(f) := f \circ g^{-1} \in T_{F\circ g}\mathrm{Ham}(M,\omega).$$

The image of the exponential map corresponds precisely to those Hamiltonian diffeomorphisms which can be generated by time-independent isotopies. It may be shown that this map is not surjective and that moreover it covers no neighborhood of the identity in any reasonable topology. Note that this exponential map can also be defined in a similar manner for the larger group $\mathrm{Diff}(M)$ and in the case of $M = S^1$ its failure to be surjective is discussed in detail in Hamilton [**123**] (cf. Freifeld [**96**]).

Despite the fact that $\mathrm{Ham}(M,\omega)$ is infinite dimensional, it behaves in a formal sense just like a compact group in finite-dimensions. Cartan theory for the Riemannian structure of compact finite-dimensional Lie groups formally carries over and yields the same identical formulas for the curvature as we now describe.

First, one defines the L^2 inner product

$$\langle f, h\rangle\big|_{\mathrm{id}} = \int_M fh\omega^n, \qquad f, h \in T_{\mathrm{id}}\mathrm{Symp}(M,\omega), \tag{51}$$

and correspondingly a metric on all of $\mathrm{Symp}(M,\omega)$ simply by pull-back via left-translations: $\langle f|_g, h|_g\rangle\big|_g := \langle dL_{g^{-1}}(f|_g), dL_{g^{-1}}(h|_g)\rangle\big|_{\mathrm{id}}$. Recall that the adjoint action of the group on itself is given by

$$\mathrm{Ad}_g f = \frac{d}{dt}\Big|_{t=1} g \circ \exp_{\mathrm{id}} tf \circ g^{-1} = dg(f) \equiv dL_g(f) = f \circ g^{-1},$$

and it preserves the metric. Therefore the inner product (51) is biinvariant (note however that the distance function it induces is identically zero [**217**]! In this sense this group is very compact...).

Now we seek to characterize the Levi-Civita connection ∇ associated to the metric. The proof of the Koszul formula (10) carries over to infinite dimensions. Consider the functions f, g, h as constant vector fields on $\mathrm{Symp}(M, \omega)$. One then has

$$2\langle \nabla_f g, h\rangle = \langle \{h, g\}, f\rangle + \langle \{f, g\}, h\rangle - \langle \{f, h\}, g\rangle. \tag{52}$$

The point here is the implicit identification

$$(\mathrm{ham}(M, \omega), [\,\cdot\,,\cdot\,]) \cong (C^\infty(M)/\mathbb{R}, -\{\,\cdot\,,\cdot\,\})$$

that together with the identification of $T_{\mathrm{id}}\mathrm{Ham}(M, \omega)$ and all other tangent spaces with $\mathrm{ham}(M, \omega)$ allows us to make sense of the brackets in (10) as Poisson brackets. The first and the third terms cancel each other due to the identity[9]

$$\{h, g\}f + \{h, f\}g = \{h, fg\} \tag{53}$$

and the fact that this last term has average zero. Alternatively, the cancellation in (52) can be understood as the infinitesimal version of the invariance of the inner product under the adjoing action of the group. We conclude that

$$\nabla_f g = \tfrac{1}{2}\{f, g\}.$$

Note this implies that the one-parameter subgroups are the geodesics, $\nabla_f f = 0$.[10]

Finally, using the Jacobi identity we may compute the curvature tensor,

$$\begin{aligned} R(f,g)k =& \nabla_f \nabla_g k - \nabla_g \nabla_f k - \nabla_{\{f,g\}} k \\ &= \nabla_f \tfrac{1}{2}\{g, k\} - \nabla_g \tfrac{1}{2}\{f, k\} - \tfrac{1}{2}\{\{f, g\}, k\} \\ &= \tfrac{1}{4}\{f, \{g, k\}\} - \tfrac{1}{4}\{g, \{f, k\}\} - \tfrac{1}{2}\{\{f, g\}, k\} \\ &= -\tfrac{1}{4}\{\{f, g\}, k\}. \end{aligned} \tag{54}$$

Now the group $\mathrm{Ham}(M, \omega)$ may be considered as "compact semisimple" since it has no center: if h is a function such that $[X_h, X_f] = 0$ for all functions f then

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[9] To prove this identity take the exterior differential of both sides: $d\{h, fg\} = \iota_{X_{\{h,fg\}}}\omega = -\iota_{[X_h, X_{fg}]}\omega = -[\mathcal{L}_{X_h}, \iota_{X_{fg}}]\omega = -\mathcal{L}_{X_h} d(fg) + 0 = -\mathcal{L}_{X_h}(gdf + fdg) = -\omega(X_h, X_g)df - \omega(X_f, X_h)dg - gd(\omega(X_f, X_h)) - fd(\omega(X_g, X_h)) = \{h, g\}df + \{h, f\}dg + gd\{h, f\} + fd\{h, g\}$.

[10] Remark that there is a unique affine connection on a Lie group which is torsion-free and for which the geodesics are the one-parameter subgroups. In fact from $\nabla_f f = 0$ one has $\nabla_{f+g} f + g = 0 \Leftrightarrow \nabla_f g + \nabla_g f = 0$ which when combined with $\nabla_f g - \nabla_g f - \{f, g\}_\omega = 0$ yields $\nabla_f g = \frac{1}{2}\{f, g\}$. Such a connection is called a Cartan connection [**222**, p. 71–72; **130**, p. 148].

$\{h, f\} = 0$ and so h is constant along the flow of X_f, and therefore locally constant, and since M is connected, constant. The same argument as in finite-dimensions then implies that $\mathrm{Ham}(M, \omega)$ is an "Einstein" space of constant Ricci curvature $1/4$. This fact does not seem to have been noticed in the literature explicitly to the best of our knowledge.

2.2.3 Riemannian geometry of the space of Kähler metrics. The purpose of this subsection is to illustrate the fact that the space of Kähler forms $\mathcal{H}_{[\omega]}$ is the "non-compact" dual of the group $\mathrm{Ham}(M, \omega)$. This fact is the culmination of the present introductory chapter. Although this fact by itself does not prove "hard" results in Kähler geometry it serves as a guide as to what to expect, by analogy with the Cartan theory in finite dimensions. In addition it may be used to explain various facts in Kähler geometry as instances of the geometry of this beautiful infinite-dimensional space. Studying Riemannian structures of moduli spaces of metrics goes back at least to the work of Ebin on the space of all Riemannian metrics [**84**].

A particular instance where this serves as a guide is the case of projective manifolds. As will be explained in detail in the next chapter, there is a sequence of naturally constructed finite-dimensional spaces of the form $GL(d_k, \mathbb{C})/U(d_k)$ that sit inside $\mathcal{H}_\omega$ and whose dimensions tend to infinity. One is therefore led to ask whether these spaces approximate $\mathcal{H}_\omega$ in some sense and this is precisely the starting point of the theory of Kähler quantization.

Recall from §§2.1.6 the definition (33) of a Riemannian structure g_{L^2} on the space of Kähler metrics $\mathcal{H}_\omega$. That metric is "universal " in a certain sense and to a certain extent the fact that it corresponds to a symmetric space structure should be expected: By definition in the moduli space of Kähler forms equipped with this metric all Kähler structures are on the same footing, and so it—as well as its curvature—should look the same viewed from any one of them. We now sketch a verification of this fact, following Mabuchi [**176**], Semmes [**240**] and Donaldson [**79**].

To find the Levi-Civita connection of the metric g_{L^2} we follow a slightly different argument than that in the literature (cf. [**44,79,176,240**]), that seems simpler. We use the Koszul formula as in finite dimensions to find the expression for the Levi-Civita connection. Indeed the same proof as in finite dimensions shows uniqueness of the Levi-Civita connection, namely that if it exists it must take the form (10). Now out of the six terms in that equation only three terms are nonvanishing and these are precisely the terms that were zero when computing the Levi-Civita connection of $\mathrm{Ham}(M, \omega)$ in (52). We will then have to verify that the object we get is in fact the Levi-Civita.

Let D denote the candidate Levi-Civita connection of $(\mathcal{H}_\omega, g_{L^2})$. Indeed now we regard the functions c, e, f as constant vector fields rather than left-invariant vector fields and so the bracket terms all vanish. Therefore (10) gives

$$2g_{L^2}(D_c e, f)|_\varphi = c\, g_{L^2}(e, f) - f\, g_{L^2}(e, c) + e\, g_{L^2}(f, c).$$

Since

$$c\,g_{L^2}(e,f) = \frac{d}{dt}\Big|_0 \frac{1}{V}\int_M ef(\omega_\varphi + t\sqrt{-1}\partial\bar{\partial}c)^n = \frac{1}{V}\int_M ef\Delta_\varphi c\,\omega_\varphi^n,$$

we have

$$\begin{aligned} 2g_{L^2}(D_c e, f)|_\varphi &= \frac{1}{V}\int_M (ef\Delta_\varphi c - ce\Delta_\varphi f + fc\Delta_\varphi e)\,\omega_\varphi^n \\ &= \frac{1}{V}\int_M (ef\Delta_\varphi c - f\Delta_\varphi(ce) + e\Delta_\varphi(fc))\,\omega_\varphi^n. \\ &= -\frac{1}{V}\int_M g_\varphi(\nabla c, \nabla e) f\,\omega_\varphi^n. \end{aligned}$$

It follows that

$$D_c e = -\tfrac{1}{2} g_\varphi(\nabla c, \nabla e). \tag{55}$$

Now we check that this expression, that is the only possible candidate for a Levi-Civita connection, is in fact one. The fact that the expression (55) is symmetric implies D is torsion-free. It is also compatible with g_{L^2} since $f\,g_{L^2}(c,e) = g_{L^2}(D_f c, e) + g(c, D_f e)$ is just

$$\frac{1}{V}\int_M ec\Delta_{\omega_\varphi} f\omega^n = \frac{1}{V}\int_M ec\tfrac{1}{2}\Delta_{g_\varphi} f\omega^n = -\frac{1}{V}\int_M \tfrac{1}{2}(e\nabla f\cdot\nabla c + c\nabla f\cdot\nabla e)\omega^n,$$

using the notation $\nabla f\cdot\nabla c = g_\varphi(\nabla f, \nabla c)$ with respect to the Riemannian metric and gradient (thus the integration by parts involves the Riemannian Laplacian that is twice the complex one).

The covariant derivative of a not necessary constant vector field e along a path $\gamma : [0,1] \to \mathcal{H}_\omega$ is then

$$D_{\dot\gamma} e = \dot{e}(\gamma(t)) - \tfrac{1}{2} g_\varphi(\nabla c, \nabla e). \tag{56}$$

As a corollary we obtain the following equation for geodesics $\{\varphi_t\}$ of $(\mathcal{H}_\omega, g_{L^2})$,

$$\ddot\varphi - \tfrac{1}{2} g_\varphi(\nabla\dot\varphi, \nabla\dot\varphi) = 0.$$

A direct computation [**176,240,79**] then gives the following expression for the curvature tensor R of $(\mathcal{H}_\omega, g_{L^2})$

$$R(c,e)f|_\varphi = \tfrac{1}{4}\{\{c,e\}_{\omega_\varphi}, f\}_{\omega_\varphi}, \qquad c,e,f \in T_\varphi\mathcal{H}_\omega,$$

and the analogy with (54) is established. Moreover, the space of Kähler metrics is seen to be an Einstein space of constant Ricci curvature $-1/4$ as in finite-dimensions, in duality to the fact observed at the end of §§2.2.2.

Bravo! Tu sais donc, en ce cas-là...
car tu n'as pas appris les mathématiques et la philosophie sans un peu d'histoire...
— Alexandre Dumas, *Le vicomte de Bragelonne* (1850)

CHAPTER 3

Quantization of harmonic maps into the space of Kähler metrics

The results of this chapter comprise about a quarter of the results of this Thesis. Part of the results of this chapter have been obtained in collaboration with Steve Zelditch and will appear in a forthcoming joint publication [**232**]. I would like to thank him for his encouragement to include these results here.

Let us outline the contents of this chapter. The original results are contained in Sections 3.5–3.8, while the earlier sections serve to explain to the reader the necessary background. In Section 3.1 we discuss the general notion of quantization in the languange of differential geometry and try to convey some of the intuition behind it. We then describe in some detail the construction of a prequantizing line bundle in the general compact symplectic setting. In Section 3.2 we specialize to the setting of compact Kähler manifolds and discuss the two foundational results of Kähler quantization, the Kodaira Embedding Theorem and the Tian Asymptotic Isometry Theorem. Section 3.3 is devoted to background on the asymptotic expansion of the Szegő kernel that provides one framework to prove these results. Here we also provide some detail on the Boutet de Monvel-Sjöstrand description of the singularity of the Szegő kernel as an oscillatory integral with a complex phase function. We try to emphasize the simple ideas behind this deep analytical result. This result is the heart of the matter and the principal analytical tool for this chapter. Next, we specialize to the setting of toric symplectic manifolds. In the rather lengthy Section 3.4 we give a brief crash course in toric geometry trying to highlight the intuition and key ideas with several worked-out examples. The section ends with a description of the recent results of Song-Zelditch on asymptotics of toric monomials that are of central importance later. In Section 3.5 we state our main result concerning the approximation of harmonic maps from any compact Riemannian manifold with boundary into the space of Kähler metrics on a toric variety using harmonic maps into the finite-dimensional spaces of Bergman metrics, generalizing Song-Zelditch's theorem for approximation of geodesics. In Section 3.6 we turn to the proof of the main result. As a first step we describe how the Legendre transform can be used to linearize and hence solve the harmonic map equation. We also prove a generalization of a well-known formula from convex analysis whose immediate consequence is that the Legendre tranform actually transforms the entire Eells-Sampson harmonic map flow into the usual heat flow in our setting. In Section 3.7 we describe the "quantum" sequence of harmonic maps into the finite-dimensional Bergman spaces approximating the harmonic map obtained in the previous section. Finally, in Section 3.8 we verify the C^2 convergence of the sequence to its limit. Since many of the calculations in this section are simply more complicated versions of the one-dimensional case of Song-Zelditch we try to concentrate on the novel features in the higher-dimensional harmonic map case and the reader would benefit from some familiarity with their work although most of the background is provided and complete details are provided for our calculations. This section is split into several subsections since each derivative has to be treated separately.

3.1 Introduction to geometric quantization

Before we delve into concrete problems in this field related to Kähler geometry let us take a moment to describe some of the intuition behind this beautiful theory as well as a result that lies at its foundations.

Geometric quantization borrows ideas from quantum mechanics in an attempt to relate (global) algebro-geometric objects to differential-geometric ones via an approximation scheme in much the same way that polynomials may be used to approximate smooth functions, and goes back to work of Kostant, Souriau, and others **[121,156,157,254]** (see also **[24,120,249,292]**). For a reference on quantum mechanics see, e.g., **[125]**.

3.1.1 Prequantization of symplectic manifolds. In the Hamiltonian formulation of classical mechanics one prescribes a smooth function on phase space, say $T^\star\mathbb{R}^3$, and considers the dynamics induced by it, namely the trajectory of a particle in $\mathbb{R}^3$ under the constraint that the prescribed function is a constant of motion (energy conservation). Here the fiber directions of the bundle $T^\star\mathbb{R}^3$ describe the velocity vector at any given point in $\mathbb{R}^3$. The simplest example would be

$$f(x,p) = \tfrac{1}{2}K|x|^2 + \frac{|p|^2}{2m},$$

with $K, m > 0$ constants, x the space coordinates and p the momentum coordinates. To simplify matters let us consider $\mathbb{R}$ instead of $\mathbb{R}^3$, and put $K = 1, m = 1$. With respect to the symplectic form $\omega(x,p) = dx \wedge dp$ on $T^\star\mathbb{R} \cong \mathbb{R}^2$ one then find that the Hamiltonian vector field X_f defined by $\iota_{X_f}\omega = df$ is given by

$$X_f = p\frac{\partial}{\partial x} - x\frac{\partial}{\partial p},$$

and the equations of motion of the particle in phase space are

$$\frac{d(x(t),p(t))}{dt} = X_f,$$

or in other words

$$\frac{dx(t)}{dt} = p,$$
$$\frac{dp(t)}{dt} = -x.$$

The solution of these is the familiar expression for a harmonic oscillator satisfying the equations $\ddot{x} + x = 0 = \ddot{p} + p$.

Functions on phase space are on the one hand the Hamiltonian functions generating dynamics as described above, and on the other hand the so called "observables"; these are the quantities associated to a particle that can be classically (that is in the theory of classical mechanics) measured or observed.

We now come back to the phase space $T^\star\mathbb{R}^3 \cong \mathbb{R}^6$. In quantum mechanics observables are replaced by self-adjoint operators acting on a Hilbert space of quantum states, that in our case would be a Hilbert space H (however in general not a Hilbert subspace of $L^2(\mathbb{R}^6,\mathbb{C})$). The classical states were simply the points of phase space (particular momentum at a particular position), equivalently of the given symplectic manifold. A state of a particle at a given time was then the specification of the position and velocity of a particle at a given moment. That is by measuring or observing an observable at that time. The quantum measurements are the expectation values of the self-adjoint operators "acting" on a quantum state. For example, the expected quantum measurement of the observable A (a self-adjoint operator on the Hilbert space H) at a quantum state f is given by the real number

$$\langle Af, f\rangle_H / \langle f, f\rangle_H.$$

One possible choice for H would be the square integrable functions that depend only on the variables x_j, isomorphic to $L^2(\mathbb{R}^3,\mathbb{C})$. For example, to compute the expected value of the first position coordinate of a particle in a state f one evaluates

$$\langle \hat{x}_1 f, f\rangle = \int_{\mathbb{R}^3} x_1 \cdot |f|^2 dx_1 \wedge dx_2 \wedge dx_3.$$

Here the operator $\hat{x}_1$ is multiplication by x_1. The operator to measure momentum is given by $\hat{p}_j := \frac{1}{k\sqrt{-1}}\partial/\partial x_j$, with $k \in \mathbb{N}$. In quantum theory one makes these choices for quantizing the functions $f(x,p) = x_j$ and $f(x,p) = p_j$ motivated by the Dirac commutation axioms: Given two classical observables and their corresponding self-adjoint operators, the bracket of the latter should be equal—up to the factor $\sqrt{-1}\hbar$, with $\hbar := 1/k$—to the self-adjoint operator corresponding to the Poisson bracket of the classical observables. For example, indeed $[\hat{p}_1, \hat{x}_1] = \frac{\sqrt{-1}}{k}\{\widehat{p_1, x_1}\}$, since $\hat{p}_1(\hat{x}_1 f) - \hat{x}_1(\hat{p}_1 f) = \frac{1}{k\sqrt{-1}}f$ and $\{p_1, x_1\} = -1$. In particular if we try to quantize functions that depend on both x and p we run into a problem: should one quantize xp as $\hat{x} \circ \hat{p}$ or as $\hat{p} \circ \hat{x}$ (they are different by the previous computation)?

A consequence of the choice for quantizing the functions x_j and p_j as $\hat{x}_j$ and $\hat{p}_j$ is the Heisenberg Uncertainty Principle [**125**, p. 85] that holds for all functions that depend only on x:

$$\left[\frac{\langle(\hat{x}_j)^2 f, f\rangle}{\langle f, f\rangle} - \frac{\langle \hat{x}_j f, f\rangle^2}{\langle f, f\rangle^2}\right] \cdot \left[\frac{\langle(\hat{p}_j)^2 f, f\rangle}{\langle f, f\rangle} - \frac{\langle \hat{p}_j f, f\rangle^2}{\langle f, f\rangle^2}\right] \geq \frac{\hbar^2}{4}, \quad \forall f = f(x) \in L^2(\mathbb{R}^3, \mathbb{C}). \tag{57}$$

This certainly does not hold for all functions in $L^2(T^\star\mathbb{R}^3, \mathbb{C})$ since they can be arbitrary localized about a point in phase space and therefore have arbitrary small uncertainty regarding their classical state (i.e., the values $x_1, x_2, x_3, p_1, p_2, p_3$ may be determined to arbitrary precision, independently of k). This motivates the reduction of the Hilbert space from all the L^2 integrable functions on phase space to functions that depend on half of the variables. Naturally then one is inclined to take the largest possible Hilbert subspace for which the principles above hold and the problems mentioned above do not arise. In the complex geometric situation later a natural choice will be to restrict to the holomorphic functions or sections.

Geometric quantization seeks to describe methods to perform such classical-to-quantum replacements on abstract phase spaces, namely on symplectic manifolds, and to study the relation between the classical and quantum pictures that results.

Let us try to describe this more precisely. Consider a compact closed symplectic manifold (M, ω) (i.e., a compact phase space). The goal is to find a good quantum analogue for a function on M (in the form of some operator on some Hilbert space).

A natural answer to this problem can be described geometrically. The very first observation is the following. A function on M generates a Hamiltonian flow. In other words any function is a Hamiltonian function. Now let $\psi_t := \exp_{\mathrm{id}} tX_f$ be the flow generated by the Hamiltonian vector field X_f corresponding to f. One may think of this flow as acting on f by pull-back. Indeed, f is preserved along the flow, and this precisely means that f at a point $\psi_t(p)$ is obtained from pulling back f at the point p by ψ_t, hence f is constant along the trajectory. We thus obtain operators that act on the Hilbert space $L^2(M, \omega)$. While this is not even a "prequantization" in the sense to be described below, this idea of obtaining operators from functions via Hamiltonian vector fields is an important ingredient in the construction of a prequantization below.

A natural procedure in geometry is to replace functions by sections of a $\mathbb{C}$-line bundle. If we assume that $[\omega]$ represents an integral class such a line bundle $L \xrightarrow{\pi} M$ is naturally associated to (M, ω). (The argument is sketched in Section 3.2 below in the setting where M also has a complex structure for which the construction actually produces a holomorphic $\mathbb{C}$-line bundle. However, in the general setting of a symplectic manifold where one only aspires to construct a plain $\mathbb{C}$-line bundle the same argument works by replacing $\mathcal{O}_M$ by $\mathcal{E}_M$, the sheaf of germs of smooth functions, and noting that the latter is a fine sheaf and hence automatically $H^2(M, \mathcal{E}_M) = 0$ [**114**, p. 42].) The idea is then to identify in some manner functions on M with operators acting on the space $L^2(M, L)$ of L^2 section of L over M (with respect to some (and hence any) Hermitian metric).

Note that what we described earlier was an identification between smooth functions up to a constant with the Lie algebra $\mathrm{ham}(M, \omega)$ of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$. The latter was a group acting on our old Hilbert space. Now we seek an identification between the space of all smooth functions and a set of operators acting on $L^2(M, L)$. Since we have the short exact sequence of Lie algebras (with all maps being Lie algebra homomorphisms)

$$\{0\} \to (\mathbb{R}, 0) \to (C^\infty(M), -\{\ ,\ \}_\omega) \to (\mathrm{ham}(M, \omega), [\ ,\ ]) \to \{0\},$$

it is natural to seek for a Lie group, call it $\widehat{\mathrm{Ham}}(M, \omega)$, whose Lie algebra will be the middle term, and that will therefore fit in the sequence

$$\{\mathrm{id}\} \to S^1 \to \widehat{\mathrm{Ham}}(M, \omega) \xrightarrow{\gamma} \mathrm{Ham}(M, \omega) \to \{\mathrm{id}\}. \tag{58}$$

Such a group $\widehat{\mathrm{Ham}}(M, \omega)$ is called a central extension of $\mathrm{Ham}(M, \omega)$ since on the level of Lie algebras we extend the Lie algebra structure to the direct sum $\mathrm{ham}(M, \omega) \oplus \mathbb{R}\mathbf{x}$ where $\mathbf{x}$ is a generator of the trivial Lie algebra $(\mathbb{R}, 0)$ in such a way that $\mathbf{x}$ is central, that is the new Lie bracket satisfies $[\mathbf{x}, \cdot\,]_{\mathrm{new}} = 0$. The point is that for elements $X, Y \in \mathrm{ham}(M, \omega)$ one now has $[X, Y]_{\mathrm{new}} = [X, Y]_{\mathrm{old}} + c(X, Y)\mathbf{x}$, with $[\,\cdot\,,\cdot\,]_{\mathrm{old}} = [\,\cdot\,,\cdot\,]$ (the usual bracket on vector fields) in our situation and with $c(X, Y)$ a number. This extra freedom precisely produces a prequantization.

The following theorem, going back at least to Kostant, shows how to construct a central extension for $\mathrm{Ham}(M, \omega)$. This construction is called a prequantization and the line bundle L is called the prequantizing line bundle [**121**, p. 223]. A comment on the reason for this name will proceed the proof. The proof should give some feeling as to what the word "quantization" refers to in our differential geometric setting. While the result itself will not be used below we hope its proof will give some feeling for the notions of a line bundle and sections of a line bundle as these will be central for this chapter.

Theorem 3.1. *Let (M, ω) be a compact closed symplectic manifold and suppose that $[\omega] \in H^2(M, \mathbb{Z})$. Let $L \xrightarrow{\pi} M$ be a $\mathbb{C}$-line bundle with Hermitian metric h and metric compatible connection D whose curvature F_D equals $-2\pi\sqrt{-1}\omega$, hence $c_1(L) = [\omega]$. For each such choice (unique up to a constant) there is a choice of central extension*

$$\widehat{\mathrm{Ham}}(M, \omega) = \{ \textit{ line bundle automorphisms preserving } h \textit{ and } D \,\} =: \mathrm{Aut}(L, h, D),$$

and a corresponding representation on $L^2(M, L)$.

Proof. First observe that $\mathrm{Aut}(L, h, D)$ is indeed a group (infinite-dimensional). The proof will proceed in two steps. First, we will explain why (58) holds. Then we will prove that there is a Lie algebra isomorphism $\mathrm{Lie}\,\mathrm{Aut}(L, h, D) \cong (C^\infty(M), -\{\,\cdot\,,\cdot\,\}_\omega)$.

Let $\hat{F} \in \mathrm{Aut}(L, h, D)$. We also denote by $\hat{F}$ the induced (by restriction) automorphism of the principal $\mathbb{C}^\star$ bundle L_0. Since $\hat{F}$ preserves the metric it also induces an automorphism of the principal $U(1)$-bundle L_1, and this will also be denoted by $\hat{F}$. Let Θ denote the connection form on (the total space) L_0 and let $\iota^\star\Theta$ denote its restriction to L_1, via $\iota : L_1 \hookrightarrow L_0$. The map $\hat{F}$ preserves the connection, $\hat{F}^\star\Theta = \Theta$, hence $\hat{F}^\star d\Theta = d\Theta = -\sqrt{-1}\pi^\star\omega$, that is

$$\hat{F}^\star\pi^\star\omega = \pi^\star\omega.$$

Since $\hat{F}$ is a bundle map it covers a diffeomorphism F of M, more specifically one has the commutative diagram

$$\begin{tikzcd} L \arrow[r, "\hat{F}"] \arrow[d, "\pi"] & L \arrow[d, "\pi"] \\ M \arrow[r, "F"] & M \end{tikzcd}$$

i.e., $F \circ \pi = \pi \circ \hat{F}$, and in particular $F := \pi \circ \hat{F} \circ \pi^{-1}$ is well-defined. Since for each $p \in L_1$ the map $\pi_\star : T_pL_1 \to T_{\pi(p)}M$ is a surjection we see that for all $X, Y \in TM$,

$$\omega(F_\star X, F_\star Y) = \omega(X, Y),$$

that is, F is a symplectomorphism. In particular this tells how to define the map $\widehat{\mathrm{Ham}}(M, \omega) \xrightarrow{\gamma} \mathrm{Ham}(M, \omega)$ in (58): we set $\gamma : \hat{F} \mapsto F$. Maps that cover the identity map act as multiplication on each fiber separately. Since these maps preserve h this must be multiplication by a factor of norm 1 and since they preserve D this factor must be a fixed constant independent of the fiber. We conclude that $\ker\gamma \cong S^1$ and (58) holds, in other words we have constructed a central extension $\mathrm{Aut}(L, h, D)$ of $\mathrm{Ham}(M, \omega)$ once we show that its Lie algebra is isomorphic to $(C^\infty(M), -\{\,\cdot\,,\cdot\,\}_\omega)$.

We now turn to prove the Lie algebra isomorphism. To that end we would like to identify the relation between the infinitesimal action upstairs and the Hamiltonian vector field downstairs.

To that end we recall the decomposition of the tangent space of the total space of the bundle into horizontal and vertical components. Let $\{\hat{F}_t\}$ denote a one-parameter subgroup of $\mathrm{Aut}(L, h, D)$ through the identity map on L, $\hat{F}_0 = \mathrm{id} := \mathrm{id}_L$, and let $X \in T_{\mathrm{id}}\mathrm{Aut}(L, h, D)$ denote the corresponding vector field on L_0. Let $X = \mathrm{hor}X + \mathrm{ver}X$ be the decomposition of X into the horizontal and vertical parts, where $\pi_\star \mathrm{ver}X = 0$ and $\mathrm{hor}X$ is in the kernel of connection 1-form defined on the total space [**138**, p. 14]. Since h is preserved under the action, each of the $\hat{F}_t$ acts linearly on the fibers of L by multiplication by $e^{\sqrt{-1}f_t}$, with f_t a real function on M. With respect to a choice of a local unit frame s and a corresponding complex coordinate z on the fiber one has $z(\hat{F}_t(p)) = e^{\sqrt{-1}f_t}z(p)$ (note that the action on a section by pull-back has therefore the effect of multiplication by $e^{-\sqrt{-1}f_t}$. More precisely the action is $s \mapsto \hat{F}_{-t} \circ s \circ F_t$.). On L_1 the tangent space in the fiber direction (i.e., the vertical direction) is generated by $\frac{\partial}{\partial(\sqrt{-1}\theta)}$ via our conventions (see after (59)). So $\mathrm{ver}X(p)(z) = \sqrt{-1}dz(\mathrm{ver}X)(p) = \sqrt{-1}\frac{d(z \circ \hat{F}_{-t})}{dt}(p)\big|_0 = \sqrt{-1}(-\sqrt{-1})\dot{f}_0 z(p)$. Therefore $\mathrm{ver}X = \dot{f}_0 z\frac{\partial}{\partial z} - \dot{f}_0 \bar{z}\frac{\partial}{\partial \bar{z}} = \dot{f}_0 \frac{\partial}{\partial(\sqrt{-1}\theta)}$. For simplicity put $f := \dot{f}_0$.

We remark that this shows somewhat more explicitly how the action of a diffeomorphism in $\mathrm{Aut}(L, h, D)$ can be understood as a combination of a symplectomorphism on M in addition to multiplication of the fibers of $L \xrightarrow{\pi} M$ above.

An equivalent manner to describe a the above splitting of TL_1 into a vertical and a horizontal space is via the connection 1-form Θ on the total space of the principal S^1 bundle L_1 (that is specifying a notion of parallel transport). To describe this point of view we start with the connection 1-form (for the bundle $L \xrightarrow{\pi} M$) regarded as a 1-form α_s on M in this local trivialization, (the last notation comes to specify that this local representation of the connection form α depends on the local choice of a trivializing unit section s with respect to h). It is a purely imaginary form: For any tangent vector X, $0 = Xh(s, s) = h(\alpha_s(X)s, s) + h(s, \alpha_s(X)s) = 2\mathrm{Re}\,\alpha_s(X) \cdot 1$ (here $h(s, s) = 1$). The induced connection 1-form on L_0 is then written over U as $\Theta = \pi^\star\alpha_s + dz/2z - d\bar{z}/2\bar{z}$ and on L_1 it restricts to

$$\Theta = \pi^\star\alpha_s + \sqrt{-1}d(\sqrt{-1}\theta). \tag{59}$$

Indeed, first, the composition of maps $\sqrt{-1}\mathbb{R} \cong u(1) \to TL_0 \to T_pL_0 \to u(1)$ is the identity, where the first map is induced from the right principal $U(1)$ action on L_0, the second is the restriction and the third is the pairing with Θ. And second, Θ is preserved under the principal $U(1)$ action (it is pulled-back to itself, more precisely), which is simply rotation of each S^1 fiber by a certain angle (a unitary gauge transformations). (There is an equivalent way to define the connection form on L_1 by decreeing that Θ be invariant under the principal action and that for any section t holds

$$D_Y t = (t^\star \Theta)(Y) \cdot t. \tag{60}$$

Then one sees that $t^\star\Theta = \alpha_t$. Then $(ft)^\star\Theta = \alpha_{ft} = f(\alpha_t + df/f)$. If we write $f = re^{\sqrt{-1}\theta}$ we recover (59).)

Now that we understood the connection and the decomposition into horizontal and vertical spaces, we would like to see how does $X \in \operatorname{Lie}\operatorname{Aut}(L,h,D)$ uniquely determine a function in $C^\infty(M)$. To that end, recall that D, equivalently Θ, is preserved under the action, hence

$$0 = \mathcal{L}_X\Theta = \frac{d}{dt}|_0 \hat{F}_t^\star \Theta \circ F_t = d\big(\Theta(X)\big) + \iota_X d\Theta. \tag{61}$$

Then using the fact that $\pi^\star\omega = \sqrt{-1}d\Theta$ (compare with Equation (71) below where different conventions are used) we have

$$\iota_X d\Theta = \iota_X(-\sqrt{-1}\pi^\star\omega) = -\sqrt{-1}\iota_{\pi_\star X}\omega.$$

Recall that $\mathrm{ver}X = f\frac{\partial}{\partial(\sqrt{-1}\theta)}$ for some smooth function f on M. Therefore from (59) and (61) we obtain

$$-\sqrt{-1}\iota_{\pi_\star X}\omega = -d\big(\Theta(X)\big) = -d\big(\Theta(\mathrm{ver}X)\big) = -\sqrt{-1}df$$

(here $f = \dot{f}_0$ is the function determined from the multiplication on the fibers induced by $\mathrm{ver}X$ as explained earlier) from which we see that F is in fact Hamiltonian and that $\mathrm{hor}X = \widetilde{X_f}$, where by definition $\widetilde{X_f}$ stands for the horizontal lift of X_f, that is the unique vector field on L such that $\pi_\star\widetilde{X_f} = X_f$ and $\widetilde{X_f} \in \ker\Theta$. Since $f \in C^\infty(M)$ was arbitrary we see that as vector spaces $C^\infty(M) \cong T_{\mathrm{id}}\operatorname{Aut}(L,h,D) \subseteq \Gamma(L,TL)$ with the isomorphism given by

$$C^\infty(M) \ni f \mapsto \hat{X}_f := \widetilde{X_f} + f\frac{\partial}{\partial(\sqrt{-1}\theta)} \in \operatorname{Lie}\operatorname{Aut}(L,h,D). \tag{62}$$

We remark that this can be also rephrased in terms of an infinitesimal action on sections induced from the pull-back action on sections given by $s \mapsto \hat{X}_f s = (D_{X_f} - \sqrt{-1}f)s$. Indeed, if t is a section, in coordinates given by $t(x) = (x, r(t(x))e^{\sqrt{-1}\theta(t(x))})$ we see that $f\frac{\partial}{\partial\theta}$ acts as the operator of multiplication by $\sqrt{-1}f(x)$ on the fiber above x, sending t to $\sqrt{-1}ft$ (compare to the formula mentioned previously $s \mapsto \hat{F}_{-t}\circ s\circ F_t$).

Finally, we would like to show that (62) is a Lie algebra isomorphism. To show this we compute for any section t, using (50) and $\widetilde{X_f}g = X_f g = dg(X_f) = \omega(X_g, X_f) =$

$-\{f,g\}$,

$$\begin{aligned}
-[\hat{X}_f,\hat{X}_g]t-\hat{X}_{\{f,g\}_\omega}t& \\
&=-\hat{X}_f(\widetilde{X_g}t-\sqrt{-1}gt)+\hat{X}_g(\widetilde{X_f}t-\sqrt{-1}ft)-\widetilde{X_{\{f,g\}}}t+\sqrt{-1}\{f,g\}t\\
&=\widetilde{X_f}(\sqrt{-1}gt)-\widetilde{X_g}(\sqrt{-1}ft)-[\widetilde{X_f},\widetilde{X_g}]t+\sqrt{-1}f\widetilde{X_g}t-\sqrt{-1}g\widetilde{X_f}t\\
&\quad+\widetilde{[X_f,X_g]}t+\sqrt{-1}\{f,g\}_\omega t\\
&=-[\widetilde{X_f},\widetilde{X_g}]t+\widetilde{[X_f,X_g]}t-\sqrt{-1}\{f,g\}_\omega t.
\end{aligned}$$

Now for any horizontal vector field $\widetilde{Y}$ and section s, $\widetilde{Y}s$ is again a section (one thinks of s as a function on L which is constant on fibers). Moreover, by (60) we have $D_Y s=\widetilde{Y}s$. Hence the first two terms equal $-([D_{X_f},D_{X_g}]-D_{[X_f,X_g]})s=\sqrt{-1}\omega(X_f,X_g)s=\sqrt{-1}\{f,g\}_\omega s$ and thus

$$[\hat{X}_f,\hat{X}_g]=-\hat{X}_{\{f,g\}_\omega}=\widehat{[X_f,X_g]},$$

proving that $f\mapsto\hat{X}_f$ is a Lie algebra isomorphism, as claimed. □

We remark that for a fixed Hermitian metric h a connection D as in the statement of the Theorem always exists: Given any connection $\tilde{D}$ its curvature form represents $c_1(L)$ and if it not equal to ω then it equals $\omega+d\alpha$. Then the connection D corresponding to the global 1-form $\Theta_{\tilde{D}}+\sqrt{-1}\pi^\star\alpha$ on L has curvature form ω. It is also possible to say what are all the different choices of connection forms that prequantize (M,ω) (i.e., that have curvature ω), see [**120**, p. 92].

The reason for the name "prequantization" is that this procedure produces the unitary operators needed however one still needs to choose a subspace of the Hilbert space on which they will act. As remarked earlier in our situation below this will simply correspond to restricting from the space of all L^2 sections to all holomorphic L^2 sections.

3.2 Kodaira's embedding theorem and Tian's asymptotic isometry theorem

The study of semi-classical approximation of Kähler metrics by projectively embedded Fubini-Study metrics (Bergman metrics) can be traced back to Tian's thesis work [**265,266**] (for a survey see [**29**]). This beautiful result illustrated the many geometric insights that may be obtained from the classical construction of Kodaira and paved the road for many subsequent investigations, including the one we wish to describe in the present Chapter.

We now state Kodaira's Embedding theorem. Let the Kähler cone $\mathcal{K}_M$ denote the cone inside the vector space $H^2(M,\mathbb{R})$ of cohomology classes that may be represented by Kähler forms.

Theorem 3.2. (See [**148**].) *The Kähler cone $\mathcal{K}_M$ contains a point of $H^2(M,\mathbb{Z})$ if and only if M is projective.*

We recall the lines of the proof. First, we recall the elementary fact that an integral second cohomology class induces a line bundle [**114**, p. 163].[11] To that end recall the 'exponential' short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\star \to 0. \tag{63}$$

This induces a long exact sequence on the level of Čech cohomology groups, in particular,

$$\operatorname{Pic}(M) := H^1(M,\mathcal{O}^\star) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \xrightarrow{\pi^{0,2}} H^2(M,\mathcal{O}) \cong H^{0,2}_{\bar\partial}(M,\mathbb{C}). \tag{64}$$

Here we defined the Picard group $\operatorname{Pic}(M)$ to be the group of all holomorphic $\mathbb{C}$-line bundles over M, up to biholomorphisms. Now let $\Omega \in \mathcal{K}_M$ be a point in $\mathcal{K}_M \cap H^2(M,\mathbb{Z})$ (here we view $H^2(M,\mathbb{Z})$ as a lattice inside $H^2(M,\mathbb{R})$ in which $\mathcal{K}_M$ also sits). Since M is Kähler the Hodge decomposition $H^2(M,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} \cong H^{2,0}_{\bar\partial}(M,\mathbb{C}) \oplus H^{1,1}_{\bar\partial}(M,\mathbb{C}) \oplus H^{0,2}_{\bar\partial}(M,\mathbb{C})$ constructs for us projection maps onto each factor, and $\pi^{0,2}$ in (64) is simply defined as the projection onto the last factor, as $H^2(M,\mathcal{O}) \cong H^{0,2}_{\bar\partial}(M,\mathbb{C})$. Since Ω is a Kähler class we have $\Omega \in H^{1,1}_{\bar\partial}(M,\mathbb{C})$, that is $\pi^{0,2}(\Omega) = 0$, the exactness of (64) implies that Ω is in the image of c_1. In other words, there exists a holomorphic line bundle $L \xrightarrow{\pi} M$ such that $c_1(L) = \Omega$.

Let $H^0(M,L^k) := H^0(M,\mathcal{O}_M(L^k))$ denote the $\mathbb{C}$-vector space of holomorphic sections of $L^k \xrightarrow{\pi} M$. Second, one shows that high enough tensor powers of L furnish projective embeddings of M into $\mathbb{P}^{d_k-1} := \mathbb{P}(H^0(M,L^k)^\star)$, with $d_k = \dim_{\mathbb{C}} H^0(M,L^k)$. Note that by Kodaira's vanishing theorem [**114**, pp. 154–160] for every large enough k and for each $q > 0$ we have $H^q(M,L^k) = 0$. Hence the dimension of the space of holomorphic sections equals the Euler characteristic, and the latter can be calculated from the Riemann-Roch formula [**162**, p. 119]

$$\begin{aligned} d_k + 1 &= \dim H^0(M,L^k) = \sum_{q=0}^{n} (-1)^q \dim H^q(M,L^k) \\ &= \chi(\mathcal{O}_M(L^k)) = \operatorname{ch}(L^k).\operatorname{td}(M)([M]) = e^{kc_1(L)}.(1 + \frac{c_1(M)}{2} + \ldots)([M]) \\ &= k^n \frac{c_1(L)^n}{n!}([M]) + k^{n-1}\frac{c_1(L)^{n-1}}{(n-1)!}.\frac{c_1(M)}{2}([M]) + O(k^{n-2}). \end{aligned} \tag{65}$$

---

[11] This is known as the Lefschetz Theorem on $(1,1)$-classes. The argument above is due to Kodaira and Spencer [**151**, p. 21].

So we see that the number of holomorphic sections grows polynomially (that is, quite fast) and one hopes that for large enough k one will have enough sections to separate points on M via the map

$$p \in M \overset{\iota_k}{\mapsto} [s_0(p) : \ldots : s_{d_k}(p)] \in \mathbb{P}^{d_k}, \tag{66}$$

(this map is well-defined into projective space independently of the local representation of the sections: the sections over a coordinate patch all transform in the same way when switching to a neighboring coordinate patch (by multiplication by a nonzero local holomorphic function)). This is indeed the case. As a corollary of the proof one then obtains a more descriptive version of Theorem 3.2 also due to Kodaira:

Theorem 3.3. *Let $L \xrightarrow{\pi} M$ be a line bundle with $c_1(L) > 0$. Then there exists $K \in \mathbb{N}$ such that for each $k \geq K$ the holomorphic map $M \overset{\iota_k}{\hookrightarrow} \mathbb{P}^{d_k}$ is an embedding.*

An analytical proof of this theorem goes back at least to Tanaka [**262**] (for other analytical proofs see [**30,173,244**]).

A remarkable fact proven by Tian in his thesis is that this tower of projective embeddings has a differential geometric interpretation in terms of geometric quantization. More specifically, given ω, there exists a canonical sequence of projective embeddings that are asymptotically isometric! We state the theorem somewhat vaguely now since a considerable part of the next sections will be devoted to explaining this result (see Theorem 3.11). Let $\mathcal{H}_k$ denote the pull-backs of all Fubini-Study metrics on $\mathbb{P}^{d_k}$. It is a finite-dimensional space of metrics, consisting of so-called height k Bergman metrics.

Theorem 3.4. (See [**266,56,299**].) *The Bergman metrics are dense in the space of Kähler metrics, namely*

$$\overline{\bigcup_{k \geq K} \mathcal{H}_k} = \mathcal{H}$$

in the C^l-topology for any l.

3.3 The asymptotic expansion of the Szegő kernel

A natural question is: to what extent is the Kodaira embedding map ι_k (66) an isometry (equivalently, a symplectomorphism)? To analyze this question requires a careful examination of the construction of the previous section, that we now outline.

3.3.1 The dual disc bundle and its boundary. The following observation goes back at least to Grauert [**113,** p. 341].

Lemma 3.5. *The dual disc bundle of a positive Hermitian line bundle* (L, h) *is a pseudoconvex domain inside the dual bundle* $L^\star$.

Proof. By definition, we need to exhibit a plurisubharmonic exhaustion function, that is a function ρ such that the equation $\rho < 0$ defines the domain, $\sqrt{-1}\partial\bar\partial\rho > 0$ on the domain (i.e., the Levi form is positive), and $d\rho \neq 0$ on its boundary.

Let $\rho(z, W) := h^{-1}(z)(W, W) - 1$ denote a function defined on all points (z, W) in $L^\star$ (here $z \in M,\ W \in \pi^{-1}(z)$). The dual disc bundle to (L, h) is then by definition

$$D^\star := \{\rho < 0\}, \tag{67}$$

and the dual circle bundle is then

$$X := \{\rho = 0\} = \partial D^\star. \tag{68}$$

Note that $d\rho|_X \neq 0$ since $h^{-1}(z)(W, W) = h^{-1}(z)|W|^2$ (by abuse of notation we use the same notation to denote the abstract Hermitian metric h^{-1} and its local representation) and so the derivative of this expression with respect to W is not zero at X. To see that ρ is plurisubharmonic as a function of $n+1$ variables note that in general if f is psh so is e^f, and since $\sqrt{-1}\partial\bar\partial\log h^{-1} = -\sqrt{-1}\partial\bar\partial\log h > 0$ in the z variables, we have $\sqrt{-1}\partial\bar\partial h^{-1} > 0$. The full Levi form is then positive since it has n positive eigenvalues as well as a positive determinant equal to

$$h^{-1}\cdot\det\left[\frac{\partial^2 h^{-1}}{\partial z^i \partial \bar{z}^{\bar{j}}}\right] + \sqrt{-1}\partial\bar\partial h^{-1}((\partial h^{-1})^\sharp, (\bar\partial h^{-1})^\sharp). \qquad \square$$

A neat feature of considering the dual bundle $L^\star$ is that sections of L may be identified with functions that are $\mathbb{C}$-linear in the fiber direction (i.e., functions respecting the fiber structure), via the map:

$$f \in \Gamma(M, L) \quad \mapsto \quad \tilde f \in C^\infty(L^\star), \quad \text{defined by } \tilde f((z, W)) := (f(z), W),$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the pairing between L and $L^\star$. Restricting such a function $\tilde f$ to X, one obtains a function on X that is S^1-equivariant (with respect to the principal S^1 action of the circle bundle X). Moreover, if f is chosen to be holomorphic, an element of $H^0(M, L)$, then $\tilde f$ will be holomorphic on $L^\star$.

In addition, holomorphic functions on $L^\star$ are sent to other holomorphic functions under the $\mathbb{C}^\star$-action: under this action a function $(z, W) \mapsto b(z, W)$ is sent (pulled-back) to the function $(z, W) \mapsto b(z, \lambda W)$ with λ a fixed constant in $\mathbb{C}^\star$. (The only reason one considers a $\mathbb{C}^\star$-action instead of a multiplicative $\mathbb{C}$-action is that the latter is not a group action (0 has no inverse).) In sum, the S^1-action commutes with the operator $\bar\partial$. For convenience we will restrict to the compact subset (with boundary) $\overline{D^\star} = D^\star \cup X$ of $L^\star$. The S^1-action preserves this set (i.e., maps it to itself), and so maps the space $H^2(\overline{D^\star}) := \ker \bar\partial \cap L^2(\overline{D^\star})$ of square-integrable holomorphic functions on $\overline{D^\star}$ to itself (note that the action actually preserves the L^2 norm). In other words, we have a representation of the group S^1 on the (infinite-dimensional) vector space $H^2(\overline{D^\star})$ (note that this discussion carries over to $L^2(\overline{D^\star})$) and so we expect this representation to split into a direct sum of irreducible representations. To be precise, denote the S^1-action by the assignment $\lambda \in S^1 \mapsto M_\lambda \in \operatorname{Aut}(L)$, with $M_\lambda : (z, W) \mapsto (z, \lambda W)$. Irreducible representations of S^1 are indexed by $\mathbb{Z}$ and the k-th piece in our case is

$$H^2_k(\overline{D^\star}) := \{\, b \in H^2(\overline{D^\star}) \ : \ b \circ M_\lambda = \lambda^k b\}, \tag{69}$$

(fix a $\lambda \in S^1$, then M_λ is a bounded self-adjoint operator acting on a Hilbert space, however the decomposition is independent of λ) with [**223**, Chapter VII]

$$H^2(\overline{D^\star}) = \bigoplus_{k\geq 0} H^2_k(\overline{D^\star}). \tag{70}$$

The zeroth piece of this decomposition are the constants. We have identified previously the first piece with sections of L. What about the higher pieces? Naturally, these may be identified with holomorphic sections of $L^k := L^{\otimes k}$, where $L^{\otimes k} := L \otimes \cdots \otimes L$ (k times), via the map

$$f \in \Gamma(M, L^k) \quad \mapsto \quad \tilde f \in C^\infty(L^\star), \quad \text{defined by } \ \tilde f((z,W)) := (f(z), W^{\otimes k}),$$

now the pairing $(\,\cdot\,,\,\cdot\,)$ being between L^k and $(L^\star)^k$.

Let

$$\alpha := -\sqrt{-1}\partial\rho|_X.$$

It is a contact form for X, namely $(d\alpha)^n \wedge \alpha$ is a volume form on X as we now show. Compute, writing $W = re^{\sqrt{-1}\theta}$ for the fiber coordinate, and recalling that $h^{-1}|W|^2 = 1$ (but not $dr = 0$) on X,

$$\begin{aligned}
\alpha &= -\sqrt{-1}|W|^2\frac{\partial h^{-1}}{\partial z^k}dz^k - \sqrt{-1}h^{-1}\bar W dW = -\sqrt{-1}|W|^2\frac{\partial h^{-1}}{\partial z^k}dz^k - \sqrt{-1}\frac{dW}{W}\\
&= -\sqrt{-1}h\frac{\partial h^{-1}}{\partial z^k}dz^k + d\theta - \sqrt{-1}\frac{dr}{r} = \sqrt{-1}\partial\log h + d\theta - \frac{\sqrt{-1}}{2}d\log h\\
&= -\frac{1}{2}d^c\log h + d\theta.
\end{aligned}$$

Therefore

$$d\alpha = -\sqrt{-1}\partial\bar\partial\log h = \pi^\star\omega \tag{71}$$

(here h is considered as a function on X), and $(d\alpha)^n \wedge \alpha = \pi^\star\omega^n \wedge d\theta$, a volume form.

At this point we throw-in more structure and equip our vector spaces with L^2 metrics (a Hilbert space structure), given by integration of the pointwise Hermitian inner products:

$$\langle f, g\rangle := \frac{1}{V}\int_M h^k(f,g)(k\omega)^n, \quad \forall\, f, g \in \Gamma(M, \mathcal{O}_M(L^k)), \tag{72}$$

$$\langle a, b\rangle := \frac{1}{2\pi V}\int_X a\bar{b}(d\alpha)^n \wedge \alpha, \quad \forall\, a, b \in C^\infty(L^\star). \tag{73}$$

Up to a factor of k^n the map $f \mapsto \tilde{f}$ is an isometry: $\tilde{f}$ is an S^1-invariant function and so

$$\frac{1}{2\pi V}\int_X \tilde{f}\cdot\bar{\tilde{g}}\,(d\alpha)^n \wedge \alpha = \frac{1}{V}\int_{\pi^{-1}(M)\cap X\cap\{\theta=\mathrm{const}\}} \tilde{f}\circ\pi\cdot\bar{\tilde{g}}\circ\pi\cdot\pi^\star\omega^n = \frac{1}{V}\int_M f\cdot\bar{g}\,\omega^n.$$

One typical construction is the orthogonal projection (with respect to the inner product (72)) of $L^2(D^\star)$ onto $H^2(D^\star)$, the kernel of $\bar\partial$. Equivalently, one may work with boundary values of such functions, defined on X, that make up the kernel of the operator $\bar\partial_b$ which is defined on functions on X by $\bar\partial_b f := \pi^{0,1}\circ df$ where $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}\frac{\partial}{\partial\theta}$ and $\pi^{0,1}$ is defined as the projection onto the second factor. (This projector is described for the model case of the boundary of an upper half-space (the set $\{\operatorname{Im} z_{n+1} = \sum_{j=1}^n |z_j|^2\} \subset \mathbb{C}^{n+1}$) by Stein [**257,** Chapter 12, Section 2].) We denote this projection by

$$\tilde\Pi : L^2(X) \to \ker\bar\partial_b \cap L^2(X) =: H^2(X). \tag{74}$$

It is called the Szegő projector.

From (70) it follows that

$$\tilde\Pi = \sum_{k\geq 0}\tilde\Pi_k, \tag{75}$$

with

$$\tilde\Pi_k : L^2(X) \to H^2_k(X). \tag{76}$$

This projection acts on a function f in L^2 by integration against a "kernel" function defined on $X \times X \setminus \{(x,x) : x \in X\}$ that we still denote by $\tilde\Pi$, as follows:

$$f \mapsto (\tilde\Pi f)(\cdot) := \frac{1}{2\pi V}\int_X f(y)\tilde\Pi(\,\cdot\,, y)((d\alpha)^n \wedge \alpha)(y). \tag{77}$$

One may regard $\tilde\Pi$ as a distribution on all of $X \times X$, its singular set being the diagonal $\{(x,x) : x \in X\} \subset X \times X$. If one wants to make effective use of (77) an analysis of the precise description of the behavior of $\tilde{\Pi}$ near the diagonal must be carried out. We will return to this matter later.

A natural formal expression for the kernel function is in terms of an orthonormal basis $\{a_j\}_{j=0}^{\infty}$ for the closed subspace onto which we are projecting is:

$$\tilde{\Pi}(x,y) = \sum_{j\geq 0} a_j(x)\overline{a_j(y)}. \tag{78}$$

We may write for the kernel functions, similarly to (75),

$$\tilde{\Pi}(x,y) = \sum_{k\geq 0} \tilde{\Pi}_k(x,y), \tag{79}$$

and we may express each piece by (remembering (78))

$$\tilde{\Pi}_k(x,y) = \int_0^{2\pi} e^{-k\sqrt{-1}\theta}\tilde{\Pi}(M_{e^{\sqrt{-1}\theta}}x,y)\frac{d\theta}{2\pi} = \int_0^{2\pi} e^{k\sqrt{-1}\theta}\tilde{\Pi}(x,M_{e^{\sqrt{-1}\theta}}y)\frac{d\theta}{2\pi}. \tag{80}$$

Naturally then, $\tilde{\Pi}_k(x,y)$ are the "Fourier components" of $\tilde{\Pi}(x,y)$.

In an analogous manner one may define the projection of L^2 sections of L^k onto the holomorphic ones. Here we use an orthonormal basis of the latter subspace. If $\{s_j\}_{j=0}^{d_k}$ is such a basis of $H^0(M,\mathcal{O}_M(L^k))$ then $\{s_j^\star\}_{j=0}^{d_k}$ is a basis for the space of holomorphic sections of $(L^\star)^k$ (here $s^\star$ is defined by agreeing that $(s^\star,f) = h(s,f)$ for any section f), and

$$\Pi_k(x,y) = \sum_{j=0}^{d_k} s_j(x)\otimes s_j^\star(y) \tag{81}$$

is such a projector for $L^2(M,L)$, acting as follows:

$$f \mapsto \sum_{j=0}^{d_k}\Big(\frac{1}{V}\int_M h^k(f,s_j)(k\omega)^n\Big)s_j = \frac{1}{V}\int_M \Pi_k(x,y)f(x)(k\omega(x))^n.$$

Now we would like to relate $\tilde{\Pi}_k$ to Π_k. By our discussion above the functions $\{k^{n/2}\tilde{s}_j\}_{j=0}^{d_k}$ form an orthonormal basis for $H_k^2(X)$, and hence the desired relation between the projections is

$$\tilde{\Pi}_k = k^n\sum_{j=0}^{d_k}\tilde{s}_j(x)\overline{\tilde{s}_j(x)} = k^n\pi^\star\Pi_k \equiv k^n\Pi_k\circ\pi. \tag{82}$$

It is important to note that while the kernel $\tilde{\Pi}$ is singular on the diagonal, the individual pieces $\tilde{\Pi}_k$ (and hence also Π_k) are smooth functions on $X\times X$, indeed they are expressed as a finite sum of smooth functions on the compact manifold $X \times X$ (this is true in some generality when projecting onto a finite-dimensional subspace). Moreover it is precisely their value on the diagonal that will be important below. The singularity in $\tilde{\Pi}$ comes from the fact that the sum in Equation (78) is infinite (indeed restricting to the diagonal and integrating (78) we get a contribution of 1 for each j).

3.3.2 Bergman metrics and the Szegő kernel on the diagonal. The key point of the analysis described above is the relation of kernel functions to the projective embeddings furnished by the holomorphic sections. Indeed the projective embedding is given by a basis of sections. Consider the map

$$M \ni p \mapsto [s_0(p), \ldots, s_{d_k}(p)] \in \mathbb{C}^{d_k+1},$$

given by an orthonormal basis of sections. It is not well-defined. Nevertheless for large enough k the space of sections $H^0(M, L^k)$ is base-point free (base points are the common zeros in M of all the sections) and therefore the map

$$M \ni p \overset{\iota_k}{\mapsto} [s_0(p) : \ldots : s_{d_k}(p)] \in \mathbb{P}^{d_k},$$

is well-defined. The Fubini-Study form on $\mathbb{P}^{d_k}$ (for historical references see Fubini [**101**] and Study [**259**]) is defined in terms of the homogeneous "coordinates" $[Z_0 : \ldots : Z_{d_k}]$ [**127**, p. 4] by

$$\omega_{\text{FS}} := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{d_k}|Z_j|^2, \tag{83}$$

and its restriction to $\iota_k(M)$ is therefore

$$\begin{aligned}\iota_k^\star\omega_{\text{FS}} &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{d_k}|s_j|^2 = -k\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{d_k}|s_j|_{h^k}^2\\ &= k\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Pi_k(x,x),\end{aligned}$$

(note this is well-defined), since indeed the kernel function (81) simplifies on the diagonal $x = y$ and is equal, using the pairing between L and $L^\star$, to $\sum_{j=0}^{d_k} h^k(s_j, s_j) \equiv \sum_{j=0}^{d_k} |s_j|_{h^k}$, i.e., it is identified with a function on M. In sum, the Szegő kernel on the diagonal measures the difference $\frac{1}{k}\iota_k^\star\omega_{\text{FS}} - \omega$ between the Kähler form and its projective approximations.

3.3.3 The singularity of the Szegő kernel of a CR domain. The boundary of a strictly pseudoconvex domain (e.g., X) is called a CR manifold (Cauchy-Riemann or complex-real). We now study the kernel $\tilde{\Pi}$ near the diagonal, in order to study $\tilde{\Pi}_k$ (or Π_k) on the diagonal, asymptotically in k. As we are only interested in the behavior near the diagonal, where $\tilde{\Pi}$ is unbounded, it suffices to evaluate $\tilde{\Pi}$ up to a bounded function on $X \times X$ (that is a negligible error for our purposes).

The simplest example is that of the unit circle in the complex plane for which $\tilde{\Pi}(x, y) = 1/(1 - x\bar{y})$. Indeed an orthonormal basis is given by the trigonometric polynomials $e^{m\sqrt{-1}\theta} =: x^m$ with $m \in \mathbb{N} \cup \{0\}$. Therefore, by (78), $\tilde{\Pi}(x, y) = 1 + x\bar{y} + x^2\bar{y}^2 + \ldots = 1/(1 - x\bar{y})$. The projector simply picks out the nonnegative Fourier coefficients of a function.

Let us now describe the general case of a boundary of a pseudoconvex domain, due to Boutet de Monvel and Sjöstrand. First some preliminaries.

3.3.3.1 Almost-analytic extensions. First, one constructs an "almost-analytic extension" (or "almost-holomorphic") of the exhaustion function ρ to $X \times X$. Here we view the diagonal in $X \times X$ as a copy of X and $X \times X$ as its "complexification".

Let ω_0 be a reference real-analytic Kähler form on M. Note that such a metric exists in our case since M is projective and the pull-back of a Fubini-Study form (itself real-analytic on projective space as may be seen from the explicit formula defining it) via any holomorphic embeddings is a real-analytic Kähler form in the given Kähler class (in short, each of the Bergman forms is real-analytic). Locally then $\omega_0 = -\sqrt{-1}\partial\bar{\partial}\log h_0$, with h_0 real-analytic on each coordinate neighborhood. Therefore $h_0(z + a) = \sum \frac{\partial h_0}{\partial z^{\vec{\alpha}} \partial \overline{z^{\vec{\beta}}}}(z) a^{\vec{\alpha}} \overline{a^{\vec{\beta}}}$, whenever $z + a$ is still in the neighborhood (we assume here that we have taken the coordinate neighborhood small enough so that it is contained in the domain of convergence about any point in it). Now we consider the function $h_{0,\mathbb{C}}$ on $U \times U$ defined by the expression

$$h_{0,\mathrm{c}}(z + a, z + b) = \sum \frac{\partial h_0}{\partial z^{\vec{\alpha}} \partial \overline{z^{\vec{\beta}}}}(z) a^{\vec{\alpha}} \overline{b^{\vec{\beta}}}. \tag{84}$$

This makes sense and converges within the same radius of convergence as before, and so is a well-defined function on $U \times U$. Moreover it agrees with h_0 on the diagonal and is holomorphic in the first direction and antiholomorphic in the second. (Implicit in this discussion is the fact that the diagonal in $M \times M$ is a totally-real submanifold.)

Consider now an arbitrary (smooth) Kähler form ω, and write $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ with $\varphi \in C^\infty(M)$ and $h = h_0 e^{-\varphi}$ a local smooth function on U. Since φ is no longer real-analytic, we cannot "complexify" it as before, however there is a non-unique alternative construction. For simplicity let us consider first the simplest case of a totally-real submanifold, namely $\mathbb{R} \subset \mathbb{C}$.

Lemma 3.6. (See [**187**].) *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support. There exists a smooth function $F : \mathbb{C} \to \mathbb{C}$ satisfying:*

(i) $F|_{\mathbb{R}} = f$,
(ii) $\left.\dfrac{\partial^{p+q}F}{\partial z^p \partial \bar{z}^q}\right|_{\mathbb{R}} = 0$ *for each* $(p,q) \in \mathbb{N} \cup \{0\} \times \mathbb{N}$.

Proof. Let $\eta : \mathbb{R} \to [0,1]$ be a smooth function that is zero outside $[-1,1]$ and constant 1 in a neighborhood of 0. Consider the function

$$F(z) = \int_{\mathbb{R}} \eta(\xi \operatorname{Im} z) e^{\sqrt{-1}z\xi} \hat{f}(\xi) d\xi. \tag{85}$$

It satisfies (i) since $\eta(0) = 1$. To check it is smooth write $\operatorname{Im} z = (z - \bar{z})/2\sqrt{-1}$ and note that differentiating F p times in z and q times in $\bar{z}$ has the effect of replacing $\eta(\xi \operatorname{Im} z)$ in the integral above by a polynomial $a(\xi, z)$ of degree $p+q$ in ξ with smooth coefficients depending on z. Moreover this vanishes identically whenever $\xi \operatorname{Im} z \geq 1$. But $|e^{\sqrt{-1}z\xi}| < e$ when $\xi \operatorname{Im} z < 1$, and so

$$\left|\frac{\partial^{p+q}F}{\partial z^p \partial \bar{z}^q}(z)\right| \leq e \int_{\mathbb{R}} \left|a(\xi, z) e^{\sqrt{-1}\operatorname{Re} z \cdot \xi} \hat{f}(\xi)\right| d\xi \leq C ||f||_{C^{p+q}(\mathbb{R})}.$$

Finally, (ii) holds, that is $\bar{\partial} F$ vanishes to infinite order on $\mathbb{R}$ since differentiating once with respect to $\bar{z}$ under the integral sign we obtain

$$\frac{\partial F}{\partial \bar{z}} = \int_{\mathbb{R}} \frac{\sqrt{-1}\xi}{2} \eta'(\xi \operatorname{Im} z) e^{\sqrt{-1}z\xi} \hat{f}(\xi) d\xi,$$

vanishing on $\mathbb{R}$ since $\eta'(0) = 0$. □

Corollary 3.7. *Let* $\varphi \in C^\infty(M)$. *Let* $D_M := \{(p,p) : p \in M\} \subset M \times M$, *and let* $\iota_M : M \to D_M$ *denote the map* $p \mapsto (p,p)$. *There exists a function* $\varphi_{\mathbb{C}} \in C^\infty(M \times M, \mathbb{C})$ *satisfying:*
(i) $\varphi_{\mathbb{C}}|_{D_M} = \varphi \circ \iota_M^{-1}$,
(ii) $\left.\dfrac{\partial^{p+q+r+s}\varphi_{\mathbb{C}}}{\partial z^p \partial \bar{z}^q \partial w^r \partial \bar{w}^s}\right|_{D_M} = 0$ *for each* $(p,q,r,s) \in \mathbb{N} \cup \{0\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \cup \{0\}$.

Proof. We regard D_M as a totally-real submanifold of $(M \times M, \mathrm{J} \oplus -\mathrm{J})$. Let $U \subset M$ be such that $(\iota_M(U), x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a a coordinate patch on $\iota_M(U) \subset D_M$. By a generalization of Lemma 3.6 with $\mathbb{R} \subset \mathbb{C}$ replaced by $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$ there exists a function Φ_U defined on $U \times U$ with holomorphic coordinates $((z_1, \ldots, z_n), (\bar{w}_1, \ldots, \bar{w}_n))$ such that $\operatorname{Re} z_j = x_j, \operatorname{Re} w_j = y_j$, satisfying $\Phi|_{\iota_M(U)} = \varphi \circ \iota_M^{-1}|_{\iota_M(U)}$ and vanishing to infinite order with respect to $z_1, \ldots, z_n$ and $\bar{w}_1, \ldots, \bar{w}_n$ on $\iota_M(U) \subset U \times U$. We may construct such a Φ_U for each U in M, however they need not agree on intersections.

To remedy this we choose a finite covering of M by coordinate patches and a subordinate partition of unity [**285**, p. 8] and work with the resulting pieces of φ that sum up to φ and that are each supported in a single coordinate patch. This is valid since the construction of Lemma 3.6 is linear with respect to the input real-valued function (see (85)). In sum, we have constructed a function that satisfies both (i) and (ii) but that is only defined on a open neighborhood of D_M in $M \times M$. To conclude we can just extend this function to all of $M \times M$ in a smooth manner. □

From Corollary 3.7 we conclude that there exists an extension of h to $M \times M$—and hence of ρ to $L^\star \times L^\star$—that behaves similarly to an analytic extension. Note that in the real-analytic case the analytic extension is symmetric in the sense that

$$\varphi_{\mathrm{c}}(z,w) = \overline{\varphi_{\mathrm{c}}(w,z)} \tag{86}$$

(this follows from (84) noting that φ is real). In the rather arbitrary manner we have constructed our almost-analytic extensions in the non real-analytic case this will no longer hold in general. However if we consider $(\varphi_{\mathrm{c}}(z,w) + \overline{\varphi_{\mathrm{c}}(w,z)})/2$ we will obtain a function that satisfies (86) as well as (i) and (ii) of Corollary 3.7. From now on we will call a function satisfying these three conditions an almost-analytic extension of φ, and denote it by φ_{c}.

3.3.3.2 Pseudoconvexity and the complex phase function. Let $h_{\mathrm{c}}, \rho_{\mathrm{c}}$ be almost-analytic extensions. The following Lemma follows from (and can be even used to characterize) pseudoconvexity.

Lemma 3.8. (See [**35**, p. 127].) *Let $d_h : L^\star \times L^\star \to \mathbb{R}_+$ be the distance function on $L^\star$ induced from the Hermitian metric $g_\omega(z) + h^{-1}(z)\sqrt{-1}dW \otimes d\overline{W}$. There exists a constant $C > 0$ depending only on (L, h) such that for any $x, y \in X$ one has*

$$\operatorname{Re}\rho_{\mathbb{C}}(x,y) \leq -Cd_h(x,y)^2 + O(d_h(x,y)^3). \tag{87}$$

Proof. First we describe the proof for $M = \mathbb{C}^{n-1}, L^\star = \mathbb{C}^n$ and $X = \{\rho = 0\}$ the boundary of a pseudoconvex domain in $\mathbb{C}^n$. In this case one may add and subtract points freely and so the following identity holds

$$\rho_{\mathrm{c}}(x,y) + \rho_{\mathrm{c}}(y,x) - \rho_{\mathrm{c}}(x,x) - \rho_{\mathrm{c}}(y,y) = -\sum_{j,l} \frac{\partial^2 \rho}{\partial z^j \partial \overline{z^l}}\Big|_x (x-y)^j \overline{(x-y)^l} + O(|x-y|^3). \tag{88}$$

To verfiy this identity Taylor expand about the point (x,x) (up to third order error term) and observe that the lowest order contribution that remains after cancellations comes from the fourth term, and is equal to the first term in the right hand side.

Therefore, using (86),(88), and since $\rho(x) = \rho_{\mathbb{C}}(x,x) = 0$ and the same for y, we have:

$$\begin{aligned} 2\mathrm{Re}\,\rho_{\mathbb{C}}(x,y) &= \rho_{\mathbb{C}}(x,y) + \overline{\rho_{\mathbb{C}}(x,y)} = \rho_{\mathbb{C}}(x,y) + \rho_{\mathbb{C}}(y,x) \\ &= \rho_{\mathbb{C}}(x,x) + \rho_{\mathbb{C}}(y,y) - \sum_{j,l} \frac{\partial^2 \rho}{\partial z^j \partial \overline{z^l}}\Big|_x (x-y)^j \overline{(x-y)^l} + O(|x-y|^3) \\ &\leq -C|x-y|^2 + O(|x-y|^3), \end{aligned}$$

where the last inequality follows from the strict pseudoconvexity, and C can be taken as the smallest eigenvalue of the Levi form restricted to the compact domain X.

Now the general manifold situation is no more complicated: Whenever x and y are both contained in the same coordinate neighborhood for $L \longrightarrow M$ (on which ω has a local Kähler potential and the line bundle is trivial) the same argument as before works. And this may always be arranged on a connected manifold: take a neighborhood of a simple (non-self-intersecting) path connecting x and y. It is contractible and may be endowed with a coordinate patch structure. (Note however that this Lemma is only of use when x, y are close, i.e., (x, y) is near the diagonal.) □

3.3.3.3 An oscillatory integral expression for the singularity. Boutet de Monvel and Sjöstrand [**35**] expressed the singularity of the Szegő kernel in terms of an oscillatory integral with a geometrically defined phase (to be defined in the next subsection) (cf. Fefferman [**94**] and Beals, Fefferman and Grossman [**16**]):

Theorem 3.9. *There exist smooth functions $\{s_k\}_{k\geq 0}$ on $X \times X$ such that*

$$\tilde{\Pi}(x,y) - \int_{\mathbb{R}_+} e^{t\rho_{\mathbb{C}}(x,y)} \sum_{k\geq 0} t^{n-k} s_k(x,y) dt \tag{89}$$

is a smooth function on $X \times X$. In addition, $s_0(x,x) = \mathrm{const}$.

In analogy with the notion of a parametrix for an operator [**133**, p. 170], one may refer to this oscillatory integral as a parametrix for the Szegő kernel, since it differs from it by a smooth function.

3.3.3.4 Oscillatory integrals and the method of stationary phase. Going back to (80), Equation (89) now allows us to evaluate the functions $\tilde{\Pi}_k(x,y)$ near the diagonal, using the method of complex stationary phase, that we now describe closely following Hörmander [**133**, §7.6–7.8]. An expression of the form

$$\int_{\mathbb{R}} u(x) e^{\sqrt{-1}kf(x)} dx$$

is called an oscillatory integral. When f is real-valued and has no critical points, and u has compact support, this expression tends to zero as $k \in \mathbb{R}_+$ tends to infinity:

$$\begin{aligned}\int_{\mathbb{R}} u(x)e^{\sqrt{-1}kf(x)}dx &= \int_{\mathbb{R}} \frac{u(x)}{\sqrt{-1}kf'(x)}\frac{d}{dx}e^{\sqrt{-1}kf(x)}dx \\ &= -\int_{\mathbb{R}} \frac{d}{dx}\Big(\frac{u(x)}{\sqrt{-1}kf'(x)}\Big)e^{\sqrt{-1}kf(x)}dx,\end{aligned}$$

and this may be repeated to show that our original integral is $O(k^{-l})$ for any $l \in \mathbb{N}$.

The procedure of integration by parts may be formalized via the operator

$$L_{k,f}(\,\cdot\,) := \frac{1}{\sqrt{-1}k}\frac{\nabla f \cdot \nabla(\,\cdot\,)}{|\nabla f|^2},$$

that "reproduces" the function $e^{\sqrt{-1}kf}$ in the sense that $L_{k,f}e^{\sqrt{-1}kf} = e^{\sqrt{-1}kf}$. Let $L^t_{k,f}$ denote the transpose operator to $L_{k,f}$ with respect to the $L^2(\mathbb{R})$ inner product. One then writes

$$\begin{aligned}\int_{\mathbb{R}} u(x)e^{\sqrt{-1}kf(x)}dx &= \int_{\mathbb{R}} u(x)L_{k,f}e^{\sqrt{-1}kf(x)}dx \\ &= \int_{\mathbb{R}} e^{\sqrt{-1}kf(x)}L^t_{k,f}u(x)dx,\end{aligned}$$

and this may be repeated any number of times.

When f has critical points the same argument shows that the integral, asymptotically in k, localizes to the these points where the phase function f is stationary. The contributions from these points may be evaluated by the "method of stationary phase" that we now describe. Near such a point x_0, $f - f(x_0)$ is approximated, up to a third order error, by its Hessian. For more generality let us now pass to the multi-variable situation, $x \in \mathbb{R}^n$. Regardless of the error term, one has the formula [**133**, p. 219]

$$\int_{\mathbb{R}^n} u(x)e^{\sqrt{-1}k\langle \nabla^2 f|_{x_0}\cdot(x-x_0),x-x_0\rangle}dx = \frac{e^{-\sqrt{-1}\langle(\nabla^2 f|_{x_0})^{-1}\nabla,\nabla\rangle/4k}u(x_0)}{\sqrt{\det(\frac{k\nabla^2 f|_{x_0}}{\pi\sqrt{-1}})}}.$$

Grouping the error term together with u under the integral sign one may then obtain a precise asymptotic expansion (expanding the exponential on the right hand side) [**133**, p. 220]. We note that the discussion above carries over to complex-valued phase functions f satisfying $\operatorname{Im} f \geq 0$ where the contributions come from where the phase is both stationary and real (when it is not real one has some exponential decay). We may also somewhat relax the condition on the support of u.

Theorem 3.10. *Let $k > 0$ and $f, u \in C^{\infty}(\mathbb{R}^n)$. Assume that x_0 is the only point where ∇f vanishes, that $\operatorname{Im} f(x_0) = 0$, and that for all x outside a compact set $K \subset \mathbb{R}^n$ containing x_0 and all $l > l_0$, after l-times repeated integration by parts one obtains $k^l (L^t_{k,f})^l u \in L^1(\mathbb{R}^n \setminus K)$. Then for each $R \in \mathbb{N}$,*

$$\Big| \int_{\mathbb{R}^n} u(x) e^{\sqrt{-1}k(f(x)-f(x_0))} dx - \sum_{j=0}^{R-1} k^{-j} \frac{(L_j u)(x_0))}{\sqrt{\det(\frac{k\nabla^2 f|_{x_0}}{2\pi\sqrt{-1}})}} \Big| \leq Ck^{-R} ||u||_{C^{2R}(K)}, \tag{90}$$

where L_j is a differential operator of order $2j$ defined by

$$L_j u = (\sqrt{-1})^{-j} \sum_{\substack{b=j+a \\ 2b \geq 3a}} \langle (\tfrac{1}{2}\nabla^2 f|_{x_0})^{-1} \nabla, \nabla \rangle^b (r_{x_0} u)/(a!b!), \tag{91}$$

where r_{x_0} is the third order Taylor remainder

$$r_{x_0}(x) := f(x) - f(x_0) - \langle (\tfrac{1}{2}\nabla^2 f|_{x_0})^{-1}(x - x_0), x - x_0 \rangle. \tag{92}$$

One sometimes expresses the content of (90) by saying that the oscillatory integral has an asymptotic expansion in k.

3.3.3.5 An asymptotic expansion. We are finally in a position to obtain an asymptotic expansion for Π_k. In the first part of this paragraph we will obtain such an expansion on the diagonal, and in the second part off of the diagonal. From (80) and (89) we have

$$\tilde{\Pi}_k(x, x) = \sum_{k \geq 0} \int_{\mathbb{R}_+ \times S^1} t^{n-k} s_k(M_\theta x, x) e^{t\rho_{\mathbb{C}}(M_\theta x, x) - k\sqrt{-1}\theta} dt \wedge \frac{d\theta}{2\pi}. \tag{93}$$

Since $\rho((z, W)) = h^{-1}(z)|W|^2 - 1$ with $x = (z, W) \in X$, the almost-analytic extension of ρ is obtained by the product of the almost-analytic extension of h^{-1} and that of $|\cdot|^2$, namely

$$\rho_{\mathbb{C}}(x, y) = \rho_{\mathbb{C}}((z_1, W_1), (z_2, W_2)) = h_{\mathbb{C}}^{-1}(z_1, z_2) W_1 \overline{W_2} - 1. \tag{94}$$

And, since $\rho|_X = 0$,

$$\rho_{\mathbb{C}}((z, We^{\sqrt{-1}\theta}), (z, W)) = h^{-1}(z, z)|W|^2 e^{\sqrt{-1}\theta} - 1 = e^{\sqrt{-1}\theta} - 1.$$

The phase function in (93) may therefore be expressed by

$$f(t, \theta) = \sqrt{-1}t(1 - e^{\sqrt{-1}\theta}) - k\theta,$$

and since $\nabla f = (\sqrt{-1}(1-e^{\sqrt{-1}\theta}), te^{\sqrt{-1}\theta}-k)$, the only stationary phase point is $(k,0)$. Now $\nabla^2 f = \left(\begin{smallmatrix} 0 & e^{\sqrt{-1}\theta} \\ e^{\sqrt{-1}\theta} & \sqrt{-1}te^{\sqrt{-1}\theta} \end{smallmatrix}\right)$ and $(\nabla^2 f|_{(k,0)})^{-1} = \left(\begin{smallmatrix} -\sqrt{-1}k & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\langle(\frac{1}{2}\nabla^2 f|_{x_0})^{-1}\nabla, \nabla\rangle = \frac{1}{2\pi\sqrt{-1}}\frac{\partial^2}{\partial t^2} + \frac{\pi}{k}\frac{\partial^2}{\partial t\partial\theta}$. Evaluating (91) at $(k,0)$ we then obtain by Theorem 3.10 an asymptotic expansion of leading order k^n, whose first term is in fact a constant times k^n, according to Theorem 3.9. This constant equals 1, by integrating over M and comparing with (65), and the factor of k^n cancels when passing from $\tilde{\Pi}_k$ to Π_k in light of (82). In sum we have the following statement, due to Catlin and Zelditch, that refined Tian's original result (for an earlier refinement we refer to Ruan [**225**]).

Theorem 3.11. (See [**56,266,299**].) *Let $(L,h) \xrightarrow{\pi} (M,\omega)$ be an ample Hermitian line bundle over a projective manifold and let $\{s_j\}_{j=0}^{d_k}$ be an orthonormal basis for $H^0(M,L^k)$ with respect to the inner product (72). Then there exist smooth functions $\{a_j\}_{j\geq 1}$ on M such that*

$$\sum_{j=0}^{d_k} |s_j|^2_{h^k} = \Pi_k(z,z) = 1 + \sum_{j=1}^{\infty} \frac{a_j(z)}{k^j}. \tag{95}$$

Moreover this expansion may be differentiated termwise any number of times and the functions a_j depend in a smooth uniform manner on ω.

Historically, Tian's work was motivated by two main applications. On the one hand, obtaining a local version of the Riemann-Roch theorem, in some sense inspired by Index Theory. On the other hand, being able to translate problems in Kähler geometry (such as extremization of energy functionals or computation of Tian's complex singularity exponent (the 'α invariant') towards the construction of canonical metrics) to problems on finite-dimensional spaces of Bergman metrics, obtaining a solution for each k and then taking the limit. This motivation is explained, e.g., in Tian's thesis [**265,266**] and has turned out to be very fruitful, for example in Tian's solution of the Calabi conjecture for complex surfaces [**267**], and Donaldson's and Mabuchi's work on construction and uniqueness of constant scalar curvature metrics on projective manifolds [**81,181,182**]. It is worth mentioning that the question of which metrics can be approximated by Bergman metrics was first raised by Yau [**297**, p. 139].

Lu has computed the first few terms of the expansion, in particular a_1 equals the scalar curvature up to a constant (cf. [**56**]), and each of the a_j is given by functions of the curvature tensor and involves $2(j-1)$ derivatives of the curvature tensor [**172**]. In a precise sense (95) can be considered as a local version of the Riemann-Roch count, as each term in the expansion integrates to the corresponding term of (65), divided by k^n.

For many geometrical problems it is necessary to have asymptotic expressions for the Szegő kernel on an asymptotically small neighborhood of the diagonal. More specifically we will be interested later in the regions where special sections concentrate in L^2 (or "peak"). From a physical standpoint, as k tends to infinity the "uncertainty" associated to such a special section (a quantum state) tends to zero (see (57)). From (81) we have that for $s \in H^0(M, L^k)$

$$s(z) = \frac{1}{V} \int_M s(w) \Pi_k(z, w)(k\omega(w))^n.$$

If we can understand the off-diagonal decay of Π_k we will then have a rather precise estimate on the concentration of the section.

Such off-diagonal asymptotics are again a corollary of Theorem 3.9 via a both rescaling and a stationary phase argument. This time however the critical points of the phase may lie outside the diagonal in $X \times X$ (or in some sense "disappear to the complex domain," cf. Melin-Sjöstrand [**192**, p. 123]). To see this first we compute the almost-analytic extension of the defining function of $D^\star$ off the diagonal. We specialize (94) to the case where both W_1 and W_2 have the same angle in S^1. There is no loss in generality since Π_k is equivariant under the principal S^1 action and so we could always make the two angles θ_1, θ_2 equal at the expense of a factor $e^{\sqrt{-1}(\theta_1 - \theta_2)}$. Now $|W_1|^2 = h(z_1), |W_2|^2 = h(z_2)$ and since by our assumption $W_1 \bar{W}_2$ is real we also have $W_1 \bar{W}_2 = \sqrt{|W_1|^2 |W_2|^2} = \sqrt{h(z_1) h(z_2)}$. Hence under our assumption

$$\rho_{\mathbb{C}}(x, y) = \rho_{\mathbb{C}}((z_1, W_1), (z_2, W_2)) = \frac{\sqrt{h(z)h(w)}}{h_{\mathbb{C}}(z_1, z_2)} - 1 = e^{\varphi_{\mathbb{C}}(z_1, z_2) - \varphi(z_1)/2 - \varphi(z_2)/2} - 1. \tag{96}$$

Thus,

$$\rho_{\mathbb{C}}(M_\theta x, y) = e^{\varphi_{\mathbb{C}}(z_1, z_2) - \varphi(z_1)/2 - \varphi(z_2)/2} e^{\sqrt{-1}\theta} - 1.$$

Next, we have by (89) as in (93)

$$\tilde{\Pi}_k(x, y) = \sum_{k \geq 0} \int_{\mathbb{R}_+ \times S^1} t^{n-k} s_k(M_\theta x, y) e^{t\rho_{\mathbb{C}}(M_\theta x, y) - k\sqrt{-1}\theta} dt \wedge \frac{d\theta}{2\pi}. \tag{97}$$

In other words the phase is now

$$f(t, \theta) = \sqrt{-1} t (1 - e^{\varphi_{\mathbb{C}}(z_1, z_2) - \varphi(z_1)/2 - \varphi(z_2)/2} e^{\sqrt{-1}\theta}) - k\theta,$$

and its critical points satisfy

$$0 = \frac{\partial f}{\partial \theta} = t e^{\varphi_{\mathbb{C}}(z_1, z_2) - \varphi(z_1)/2 - \varphi(z_2)/2} e^{\sqrt{-1}\theta} - k,$$

$$0 = \frac{\partial f}{\partial t} = \sqrt{-1}(1 - e^{\varphi_{\mathbb{C}}(z_1, z_2) - \varphi(z_1)/2 - \varphi(z_2)/2} e^{\sqrt{-1}\theta}).$$

Dividing the equations we obtain $t = k$ and then $\sqrt{-1}\theta = \varphi(z_1)/2 + \varphi(z_2)/2 - \varphi_{\mathbb{C}}(z_1, z_2)$. The point is that since the right hand side is no longer zero or real there is no real θ that satisfies this equation. Therefore the usual stationary phase method [**133**] does not apply verbatim. The modification needed has been carried out by Shiffman-Zelditch [**244**] (cf. [**252**, §4], [**56**]). It is more convenient to state the result on the lifted Szegő kernel since the Szegő kernel with respect to the sections is not identified with a function off the diagonal in such a simpler manner as on the diagonal. One then has:

Theorem 3.12. (See [**244,252**].) *Let $L \xrightarrow{\pi} M$ be an ample line bundle over a projective manifold and let $\{s_j\}_{j=0}^{d_k}$ be an orthonormal basis for $H^0(M, L^k)$ with respect to the inner product (72). Then there exist smooth complex-valued functions $\{a_j\}_{j\geq 1}$ on $M \times M$ such that whenever $|z_1 - z_2|$ is small one has a complete asymptotic expansion*

$$\begin{aligned}\hat{\Pi}_k((z_1,0),(z_2,0)) &= \sum_{j=0}^{d_k} \hat{s}_j((z_1,0))\overline{\hat{s}_j((z_2,0))} \\ &= e^{k(\varphi_{\mathbb{C}}(z_1,z_2)-\varphi(z_1)/2-\varphi(z_2)/2)} \sum_{j=0}^{\infty} \frac{a_j(z_1,z_2)}{k^j} + O(k^{-\infty}). \quad (98)\end{aligned}$$

Here by $O(k^{-\infty})$ we mean asymptotically vanishing as $O(k^{-M})$ for every $M > 0$ (i.e., decays faster than any polynomial). By a complete asymptotic expansion in k one means an expansion that may be differentiated termwise with the same asymptotic remainder. For an alternative derivation of this asymptotic expansion we refer to [**173**].

3.4 Asymptotics of toric Bergman metrics

3.4.1 Introduction to symplectic toric manifolds. Symplectic toric manifolds (or simply toric manifolds) are compact Kähler manifolds that are the phase space of a completely integrable system. The purpose of this section is to describe this concept. For more examples and details we refer to Cannas da Silva [**46,47**].

Let us assume that the automorphism group $\mathrm{Aut}(M, \mathrm{J})$ contains a complex torus $(\mathbb{C}^\star)^n$ and that the orbit of generic points is in fact isomorphic to $(\mathbb{C}^\star)^n$. Further we assume that this action is the complexification of the action of a real torus $(S^1)^n$ by isometries of the Riemannian metric, equivalently fixing the Kähler form, and we moreover assume that this is a Hamiltonian action, namely there exist functions μ^X for each $X \in \mathrm{Lie}((S^1)^n)$ such that $\iota_X\omega = d\mu^X$ and $\{\mu^X, \mu^Y\} = \mu^{[X,Y]} = \mu^0 = 0$.

The map $\mu : \mathrm{Lie}((S^1)^n) \to C_0^\infty(M) \cong \mathrm{Lie}(\mathrm{Ham}(M,\omega))$ is called a comoment map and is a Lie algebra homomorphism. Let $\frac{\partial}{\partial\sqrt{-1}\theta^1},\ldots,\frac{\partial}{\partial\sqrt{-1}\theta^n}$ denote a basis for the Lie algebra $\mathbb{R}^n$ of the real torus. The image of the map

$$M \ni p \mapsto (\mu^{\frac{\partial}{\partial\sqrt{-1}\theta^1}}(p),\ldots,\mu^{\frac{\partial}{\partial\sqrt{-1}\theta^n}}(p)) \in \mathbb{R}^n,$$

can be shown to be a compact convex polytope P, called the moment polytope, that depends only the cohomology class of ω (we will return to this fact in the next paragraph).

We make the further restriction that this polytope be a Delzant polytope, that is: (i) at each vertex meet exactly n edges, (ii) each edge is the set of points $\{p + tu_{p,j} : t \geq 0\}$ with p a vertex, $u_{p,j} \in \mathbb{Z}^n$ and $\mathrm{span}\{u_{p,1},\ldots,u_{p,n}\} = \mathbb{Z}^n$. Equivalently, there exist outward pointing normal vectors $\{v_j\}_{j=1}^d \subset \mathbb{Z}^n$ that are primitive (i.e., their components have no common factor) to the d facets in ∂P and P may be written as

$$P = \{x \in \mathbb{R}^n : l_j(x) := \langle x, v_j\rangle - \lambda_j \leq 0, \quad j = 1,\ldots,d\}, \tag{99}$$

with $\lambda_j = \langle p, v_j\rangle \in \mathbb{Z}$ with p any vertex on the j-th facet (this does not depend on such p as $u_{p,k} \perp v_j$, with $j \in \{1,\ldots,d\}$, $k \in \{1,\ldots,n\}$). This implies that $[\omega] \in H^2(M,\mathbb{Z})$ **[119]** (part of the Delzant conditions follow though solely from the smoothness). Coming back to the fact that P is realized as the image of the moment map with respect to some symplectic representative of $[\omega]$ it is now more intuitive why P does not depend on the choice of representative: By the Delzant conditions the vertices of P are rational points and when varying ω continuously therefore these will not change. Since P is the convex hull of its vertices (by a theorem of Atiyah and Guillemin-Sternberg [**5**,**122**,**118**]) it follows that P does not depend on he choice of representative in $\mathcal{H}_\omega$.

3.4.1.1 Coordinate choices. The coordinates

$$z = e^{\rho/2+\sqrt{-1}\theta} = (e^{\rho_1/2+\sqrt{-1}\theta_1},\ldots,e^{\rho_n/2+\sqrt{-1}\theta_n}) \in (\mathbb{C}^n)^\star$$

that parametrize the complex torus $(\mathbb{C}^\star)^n$ then provide for coordinates on an open orbit $M_{\mathrm{open}} \cong (\mathbb{C}^\star)^n$, that is an open dense set in M whose complement is a complex submanifold of codimension 1 (that is itself also toric). The complex structure, namely the distribution $T^{1,0}M$ then corresponds to these coordinates and the operator ∂ is then defined by $\partial f = \frac{\partial f}{\partial z^j} dz^j$. More frequently, however, one works with the logarithmic coordinates $\log z = \rho/2 + \sqrt{-1}\theta$ (identifying $(\mathbb{C}^\star)^n$ with $\mathbb{C}^n$), and simply with the coordinates ρ when dealing with torus-invariant objects. Let $D := M \backslash M_{\mathrm{open}}$ denote the divisor at infinity, and let N_D denote some fixed neighborhood of D in M. For example, we define the C^1 norm of a torus-invariant function f by

$$||f||_{C^1(M_{\mathrm{open}}\backslash N_D)} = \sum_{j=1}^n \sup_M \Big|\frac{\partial f}{\partial \rho_j}\Big|. \tag{100}$$

Higher order norms are defined similarly using the vector fields

$$\frac{\partial}{\partial \rho_1}, \ldots, \frac{\partial}{\partial \rho_n}. \tag{101}$$

Now these coordinates are only valid on the open orbit and one needs a good choice of coordinates at the divisor at infinity (i.e., the extra codimension one piece that comes from taking the closure of the open orbit in some projective space; it will be the points obtained in the limit $z_j \to 0$). Such a choice is given by the coordinates $r_j = e^{\rho_j/2} = |z_j|$ that are valid near a point for which $\rho_j = -\infty$. In fact, some contemplation reveals that a point for which p of its coordinates z_j vanish precisely lies in the preimage of a codimension p face of the polytope. These coordinates are useful and necessary for performing calculations near the boundary, since the vector fields (101) that are nowhere vanishing on the open orbit (and thus form a basis (in fact give a trivialization) for the tangent space $TM|_{M_{\text{open}}}$) vanish identically on the divisor at infinity (here it is easier to think about $1/\rho_j$ and the fact that this is constant on the divisor at infinity) while

$$\frac{\partial}{\partial r_j} = \frac{\rho_j}{2}\frac{\partial}{\partial \rho_j}, \tag{102}$$

no longer vanishes at $\rho_j = -\infty$. Thus for a point for which some of its z_j coordinates vanish we define C^k norms using the vectors $\partial/\partial r_j$ in those directions. We denote the resulting norms by $||\cdot||_{C^k(M)}$, for example,

$$||f||_{C^1(M)} := ||f||_{C^1(M_{\text{open}})} + \sum_{j=1}^{n} \sup_{M\setminus M_{\text{open}}} \left|\frac{\partial f}{\partial r_j}\right|. \tag{103}$$

It is worth pointing out that there is a subtle point here in using the vector fields $\frac{\partial}{\partial r_j}$ that are not technically valid at $r_j = 0$. However one may check that for functions which depend only on r_j^2 as in our situation below this is valid (cf. [**252**]).

Coordinates for the polytope are simpler and are simply the Euclidean ones, denoted by $x = (x_1, \ldots, x_n)$, whether inside or on the boundary of the polytope. We will see below that these coordinates can be described in terms of the moment map for the real torus $(S^1)^n$ (Equation (112)).

3.4.1.2 The Legendre transform. Convex functions on $\mathbb{R}^n$ transform to convex functions (on a different domain) under the Legendre transform [**224**]. Now the Kähler form is $\partial\bar\partial$-exact over the open orbit M_{open} and so $\omega|_{M_{\text{open}}} = -\frac{\sqrt{-1}}{2}\partial\bar\partial\varphi = \frac{\sqrt{-1}}{2}\partial\bar\partial \log e^{-\varphi}$ (we emphasize that φ is only defined on $M_{\text{open}} \subset M$). Since ω is fixed by the torus action so is φ, implying that φ depends only on the ρ coordinates.

Thus φ is a convex function on $\mathbb{R}^n$. When restricted to the space of such Kähler potentials (inside the space of all convex functions on $\mathbb{R}^n$) the Legendre transform may be shown to have special properties that we now describe **[1,118]**.

The gradient of a strictly convex function f on $\mathbb{R}^n$ is a one-to-one mapping of $\mathbb{R}^n$ onto its image $\operatorname{Im}\nabla f$. Therefore $(\nabla f)^{-1}$ exists and is a one-to-one onto mapping from $\operatorname{Im}\nabla f$ to $\mathbb{R}^n$. The remarkable fact is that there exists a function $g : \operatorname{Im}\nabla f \to \mathbb{R}$ such that $\nabla g = (\nabla f)^{-1}$. Such a function can be constructed by the formula

$$g(y) = (\mathcal{L}f)(y) := \langle y, (\nabla f)^{-1}(y)\rangle - f((\nabla f)^{-1}(y)). \tag{104}$$

Indeed,

$$\nabla g(y) = (\nabla f)^{-1}(y) + \sum_{j=1}^{n} \langle y, \nabla_{\frac{\partial}{\partial y^j}} (\nabla f)^{-1}(y))\rangle - \nabla(f((\nabla f)^{-1}(y))). \tag{105}$$

In the last term put $x = (\nabla f)^{-1}(y)$ and compute ∇f at $x = (\nabla f)^{-1}(y)$:

$$\begin{aligned}
&(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_1}, \ldots, \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_n} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_n}) \\
&= (\langle(\nabla f)|_{x=(\nabla f)^{-1}(y)}, \frac{\partial x}{\partial y_1}\rangle, \ldots, \langle(\nabla f)|_{x=(\nabla f)^{-1}(y)}, \frac{\partial x}{\partial y_n}\rangle), \\
&= (\langle y, \frac{\partial x}{\partial y_1}\rangle, \ldots, \langle y, \frac{\partial x}{\partial y_n}\rangle)
\end{aligned}$$

and plugging this back into (105) we see that

$$\nabla(g(y)) = (\nabla f)^{-1}(y). \tag{106}$$

The function $g = \mathcal{L}f$ is called the Legendre dual, or Legendre transform, of f.

Example 3.13. Let $f(x) = e^{tx}$ with $t > 0$ a fixed constant. Then $\nabla f = \partial f/\partial x = te^{tx}$ and $\operatorname{Im}\nabla f = \mathbb{R}_+$. The Legendre dual is then $g(y)$ defined on $\mathbb{R}_+$ by letting $y = \nabla f(x) = te^{tx}$ in the formula

$$g(y) = x \cdot y(x) - f(x) = x \cdot te^{tx} - e^{tx} = y \cdot \frac{1}{t}\ln\frac{y}{t} - \frac{y}{t}. \tag{107}$$

~ ~ ~

Consider a family of convex functions $f(t,\,\cdot\,)$ parametrized by $t \in K$ defined on a common domain $D \in \mathbb{R}^n$ with the property that the image of their gradients coincide (i.e., their Legendre transforms $g(t,\,\cdot\,)$ have the same domain). An important aspect of the Legendre tranform is that a simple formula relates the variation of each family of convex functions:

$$\frac{df}{dt}\bigg|_{(\nabla f)^{-1}(y)} = -\frac{dg}{dt}\bigg|_{y}. \tag{108}$$

Intuitively, if one raises the parabola $f(x) = tx^2$ then naturally the dual parabolas $g(y) = \frac{1}{4t}y^2$ are lowered! And, putting $y = \partial f/\partial x = 2tx$, precisely lowered by the same amount raised (infinitesimally)

$$\frac{d(tx^2)}{dt} = x^2 = \frac{y^2}{4t^2}, \qquad \frac{d(y^2/4t)}{dt} = -\frac{y^2}{4t^2}.$$

To prove (108) we put $x = x(y) = (\nabla f)^{-1}(y)$ and take a variation of (104) in t to get

$$\frac{\partial g}{\partial t}\bigg|_y = \frac{dg}{dt}\bigg|_y = \sum_{j=1}^{n} y_j \frac{\partial x_j}{\partial t} - \frac{\partial f}{\partial t}\bigg|_x - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial t} = -\frac{\partial f}{\partial t}\bigg|_x,$$

since $\nabla f(x) = y$.

We further discuss this principle (central to this chapter) in the following example.

Example 3.14. In Example 3.13 the family of functions indeed has a common image of the gradient and so the discussion of the previous paragraph applies. We note that

$$\frac{dg(y)}{dt} = \frac{d}{dt}\Big(\frac{y}{t}\ln\frac{y}{t} - \frac{y}{t}\Big) = -\frac{y}{t^2}\ln\frac{y}{t}, \tag{109}$$

and on the other hand,

$$\frac{df(x)}{dt} = xe^{tx}, \tag{110}$$

in accordance with (108). We remark that a common confusion could have arisen by differentiating the third expression in (107) with respect to t. That would be to compute

$$\frac{\partial}{\partial t}(x \cdot te^{tx} - e^{tx}) = xe^{tx} + tx^2e^{tx} - xe^{tx},$$

which does not agree with (109). However,

$$\begin{aligned}\frac{d}{dt}(x \cdot te^{tx} - e^{tx}) &= \frac{\partial x}{\partial t}\cdot te^{tx} + xe^{tx} + xte^{tx}\Big(x + t\frac{\partial x}{\partial t}\Big) - xe^{tx} - t\frac{\partial x}{\partial t}e^{tx}\\ &= xte^{tx}\Big(x + t\frac{\partial x}{\partial t}\Big).\end{aligned}$$

Since $x = \frac{1}{t}\ln\frac{y}{t}$ one has though

$$\frac{\partial x}{\partial t} = -\frac{1}{t^2}\ln\frac{y}{t} + \frac{1/t}{y/t}\Big(-\frac{y}{t^2}\Big) = -\frac{1}{t^2}\ln\frac{y}{t} - \frac{1}{t^2} = -\frac{x}{t} - \frac{1}{t^2}.$$

Thus $x + t\frac{\partial x}{\partial t} = -\frac{1}{t}$ and we conclude that indeed

$$\frac{dg(y)}{dt} = \frac{dg(x(y,t))}{dt} = -\frac{df(x)}{dt},$$

explaining the confusion, indeed in (108) one must compute full derivatives with respect to t.[12]

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We will explore below more beautiful properties of the Legendre transform; for the moment though let us come back to our previous situation. We have that $u_\varphi := \mathcal{L}(\varphi)$ with

$$\begin{aligned} u_\varphi(x) &= \langle x, (\nabla\varphi)^{-1}(x)\rangle - \varphi((\nabla\varphi)^{-1}(x)), \\ u_\varphi(\nabla\varphi(\rho)) &= \langle \rho, (\nabla\varphi)(\rho)\rangle - \varphi(\rho). \end{aligned}$$

In particular it means that u_φ is a convex function on $\operatorname{Im}\nabla\varphi = P$: in coordinates one has

$$\omega = \frac{\sqrt{-1}}{2}\partial\bar\partial\varphi = \frac{1}{2}\sqrt{-1}\sum_{j,l=1}^{n} \frac{\partial^2\varphi}{\partial\rho_j\partial\rho_l}\frac{dz_j}{z_j} \wedge \frac{\overline{dz_l}}{\bar z_l},$$

(remember $\sqrt{-1}\partial\bar\partial$ is an operator defined independently of the coordinates and may be computed with respect to either the coordinates $\log z$ (as we have) or z) and since

$$\frac{\partial}{\partial\sqrt{-1}\theta_m} = \frac{\partial z}{\partial\sqrt{-1}\theta_m}\frac{\partial}{\partial z} + \frac{\partial \bar z}{\partial\sqrt{-1}\theta_m}\frac{\partial}{\partial \bar z} = z_m\frac{\partial}{\partial z_m} - \bar z_m\frac{\partial}{\partial \bar z_m} \tag{111}$$

(no summation in this equation), we have

$$\iota_{\frac{\partial}{\partial\sqrt{-1}\theta_m}}\omega = \frac{1}{2}\sum_{l=1}^{n}\Big(\frac{\partial^2\varphi}{\partial\rho^m\partial\rho^l}\overline{dz^l} - -\frac{\partial^2\varphi}{\partial\rho^m\partial\rho^l}dz^l\Big) = d\frac{\partial\varphi}{\partial\rho_m},$$

and this in turn means that the comoment map is given by

$$M \ni p \mapsto \Big(\frac{\partial\varphi}{\partial\rho_1}(p), \ldots, \frac{\partial\varphi}{\partial\rho_n}(p)\Big) = \nabla_\rho\varphi|_p \in P. \tag{112}$$

The second derivative of the Legendre tranform of a function may also be neatly related to that of the function itself. Indeed, ∇f is an isomorphism between $\mathbb{R}^n$ (with coordinates x) and $\operatorname{Im}\nabla f$ (with coordinates y), and pulls-back the Euclidean measure $dy^1\wedge\cdots\wedge dy^n$ to the measure $(\nabla f)^\star(dy^1\wedge\cdots dy^n) = \det\nabla^2 f\, dx^1\wedge\cdots\wedge dx^n$. Pulling-back this measure under $(\nabla f)^{-1}$ ought to give us back the same measure we

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[12] I am indebted to B. Klartag for a lively discussion on this point.

started with, $dy^1 \wedge \cdots \wedge dy^n$, and since $(\nabla f)^{-1} = \nabla g$ we get $(\nabla g)^\star\big((\nabla f)^\star(dy^1 \wedge \cdots dy^n)\big) = (\nabla g)^\star\big(\det \nabla^2 f\, dx^1 \wedge \cdots \wedge dx^n\big) = \det \nabla^2 g \cdot \det \nabla^2 f\, dy^1 \wedge \cdots \wedge dy^n$. It follows that $\det \nabla^2 f = (\det \nabla^2 g)^{-1}$. This argument only gives an equality on the level of determinants, however it may be used to give an equality also on the level of matrices: $dy^j = (\nabla g)^\star \circ (\nabla f)^\star dy^j = (\nabla^2 g)|_y (\nabla^2 f)|_{(\nabla f)^{-1}(y)} dy^j$ for each j and thus

$$\nabla^2 g|_y = (\nabla^2 f)^{-1}|_{(\nabla f)^{-1}(y)} \tag{113}$$

(each j gives an equality of one row).

Finally, we compute the second variation formula for the Legendre tranform for a family of convex functions $f(t, \cdot\,)$ with identical gradient image. Differentiating (108) and using (106) one has

$$\begin{aligned} \frac{\partial^2 g}{\partial t^2}\bigg|_y &= -\frac{\partial^2 f}{\partial t^2}\bigg|_x - \sum_{j=1}^n \frac{\partial^2 f}{\partial t \partial x_j} \frac{\partial((\nabla f)^{-1}(y))_j}{\partial t} \\ &= -\frac{\partial^2 f}{\partial t^2}\bigg|_x - \sum_{j=1}^n \frac{\partial^2 f}{\partial t \partial x_j} \frac{\partial(\partial g/\partial y_j)}{\partial t} \\ &= -\frac{\partial^2 f}{\partial t^2}\bigg|_x - \langle \nabla(\partial f/\partial t)|_x, \nabla(\partial g/\partial t)|_y \rangle, \end{aligned}$$

or more succinctly

$$-\ddot{f} = \ddot{g} + \langle \nabla \dot{f}, \nabla \dot{g} \rangle. \tag{114}$$

3.4.1.3 Toric monomials. Now let $L \xrightarrow{\pi} M$ denote a very ample line bundle whose first Chern class equals $[\omega]$. In other words, any basis of the vector space $H^0(M, L^k) \cong \mathbb{C}^{d_k}$ induces a Kodaira embedding into $\mathbb{P}^{d_k - 1}$. We would like to identify a canonical basis that is toric. Such a basis turns out to exist and is completely determined by P as we now describe.

Recall that $\mathbb{R}^n$ is identified with the dual of the Lie algebra of the real torus $(S^1)^n$, and hence the lattice points $\mathbb{Z}^n \subset \mathbb{R}^n$ are in one-to-one correspondence with the characters of the complex torus $(\mathbb{C}^\star)^n$, all such characters being given by the Lie group homomorphisms

$$(\mathbb{C}^\star)^n \ni z \mapsto \chi_\alpha(z) := z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathbb{C}^\star. \tag{115}$$

Let $H \xrightarrow{\pi} \mathbb{P}^N$ denote the hyperplane bundle. First we recall the following general result:

Lemma 3.15. (See [**110**, p. 169].) *Given a toric line bundle $L \xrightarrow{\pi} M$ embedded in a nondegenerate manner in $H \xrightarrow{\pi} \mathbb{P}^N$ there exists a subset $\{\alpha_j\}_{j=0}^N \subset \mathbb{Z}^n$ containing $N+1$ elements such that M is realized as the Zariski closure of the monomial embedding*

$$(\mathbb{C}^\star)^n \ni z \mapsto [x_0 z^{\alpha_0}, \ldots, x_N z^{\alpha_N}] \in \mathbb{P}^N, \tag{116}$$

with $[x_0 : \ldots : x_N]$ *any point in the open orbit of* M *that of necessity satisfies* $x_j \neq 0$.

Proof. The nondegeneracy assumption means that the intersection of M with any hyperplane has dimension less than n (that is M is not contained in any smaller linear subspace of $\mathbb{P}^N$). By assumption the complex torus acts on the whole line bundle L simply by (constant) linear transformations. When embedded in $H \xrightarrow{\pi} \mathbb{P}^N$ such an action is given by constant matrices in $GL(\mathbb{N}+1, \mathbb{C})$ and so extends to all of $\mathbb{C}^{N+1}$. A torus action is by definition given by a commuting family of matrices, that is matrices that all share the same eigenvectors. Representing the vector space $\mathbb{C}^{N+1}$ with respect to those eigenvectors the matrices representing the action are then diagonal and thus are given by (116) for some characters—or "toric monomials"—of $(\mathbb{C}^\star)^n$. Since by assumption the embedding is nondegenerate we must have that $x_j \neq 0$ for all j in (116). □

To get a better idea of the relation between the subset of $\mathbb{Z}^n$ given by Lemma 3.15 and the polytope the best thing to do is to consider some simple examples.

Example 3.16. *Toric monomials for* $\mathbb{P}^n$. The Riemann sphere $\mathbb{P}^1$ may be embedded in $\mathbb{P}^N$ for each $N \in \mathbb{N}$ by the Veronese map $[Z_0 : Z_1] \mapsto [Z_0^N : Z_0^{N-1} Z_1 : \ldots : Z_1^N]$. On the open orbit given by the affine coordinate where $Z_0 \neq 0$, put $z_1 := Z_1/Z_0$. The map is then written as $[1 : z_1] \mapsto [1 : z_1 : \ldots : z_1^N]$. Hence the corresponding set of lattice points is $\{0, 1, \ldots, N\}$. Notice that the fact that the moment polytope equals $[0, N]$ can be computed by calculating the image of the gradient of the open-orbit Kähler potential $\varphi = \log(1 + |z_1|^2 + \ldots + |z_1|^{2N})$ obtained by pulling-back the Fubini-Study potential from projective space. Indeed, since $e^{\rho_1} = |z_1|^2$ one has

$$\frac{\partial \varphi}{\partial \rho_1} = \frac{0 \cdot 1 + 1 \cdot |z_1|^2 + \ldots + N \cdot |z_1|^{2N}}{1 + |z_1|^2 + \ldots + |z_1|^{2N}} \in [0, N].$$

For $\mathbb{P}^2$ the Veronese embeddings are

$$[Z_0 : Z_1 : Z_2] \mapsto [Z_0^N : Z_0^{N-1} Z_1 : Z_0^{N-1} Z_2 : Z_0^{N-2} Z_1^2 : \ldots : Z_2^N] \in \mathbb{P}^{\binom{N+2}{2}-1}.$$

Again on the patch where $Z_0 \neq 0$ the map may be written as $[1 : z_1 : z_2] \mapsto [1 : z_1 : z_2 : z_1^2 : z_2^2 : z_1 z_2 : \ldots : z_2^N]$. Hence the collection of lattice points is the set $\mathbb{Z}^2 \cap \{(x, y) : x, y, x+y \in [0, N]\}$.

Similarly for $\mathbb{P}^n$ one considers the embeddings $\mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{N+n}{n}-1} \cong \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H^N))$ [**114**, p. 166] and obtains the lattice points given by $\mathbb{Z}^n \cap A$ with $A = \{(x_1, \ldots, x_n) : 0 \leq x_j \leq N, \sum_{j=1}^n x_j \leq N\}$ the n-simplex with edges of length N.

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An interesting feature of this example is the fact that as N tends to infinity one exhausts all the possible monomials in the positive orthant of $\mathbb{Z}^n$. This fact is true in general for any toric manifold as we will see below.

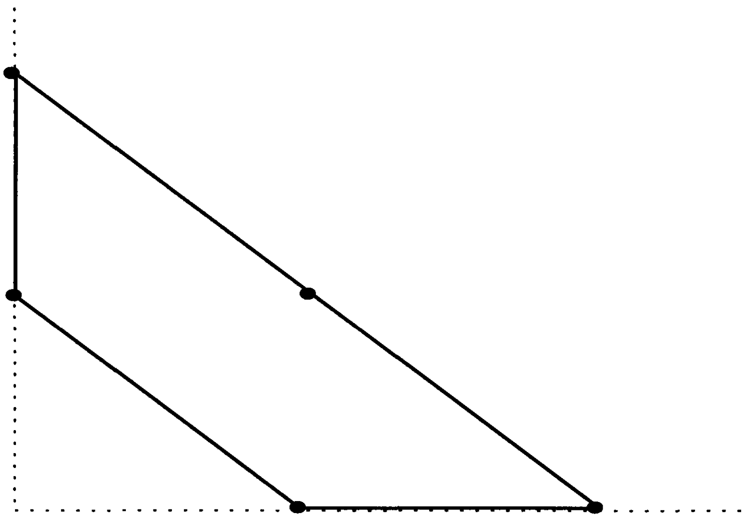

Figure 1. The moment polytope of $\mathbb{P}^2\#\overline{\mathbb{P}^2}$.

Another interesting property of this example that the toric monomials are indexed by a convex set restricted to $\mathbb{Z}^n$, in fact precisely by the moment polytope P! This also is a universal feature. Instead of proving this fact we illustrate its proof on a two-dimensional example. This example precisely explains how one can understand this fact as we hope will be visible: We start with a toric variety and a monomial embedding and then show how from this data we can construct the moment polytope, precisely as the convex hull of the lattice points of the given toric monomials. One then understands that conversely, given a moment polytope, the monomials that will give such a moment polytope must correspond to the lattice points contained in the moment polytope. Indeed given a lattice point outside P, say whose first coordinate is bigger then the maximum allowed for points in P, if the corresponding monomial would be part of the lattice points inducing the monomial embedding then considering the torus action corresponding to the angle variable θ_1 we would get that the moment map must extend in the x_1 direction up to that point outside P, which is impossible. We turn to the explicit example.

Example 3.17. We consider the blow-up of $\mathbb{P}^2$ at a point whose moment polytope P is known to be given by the intersection of the half-spaces $0 \leq x, 0 \leq y, 1 \leq x+y \leq 2$

(see Figure 1). We will derive this directly. To describe this manifold $M =: \mathbb{P}^2\#\overline{\mathbb{P}^2}$ we consider $\mathbb{P}^2$ with its usual homogeneous coordinates $[Z_0 : Z_1 : Z_2]$ and perform a blow-up at the point $p = [0 : 0 : 1]$, that is we replace smoothly this point by all infinitesimal directions at it [**114**, p. 182]. In other words, in the blow-up two short curves passing at p no longer intersect if they represented different tangent vectors in T_pM. One then has a projection map from $\pi : M \longrightarrow \mathbb{P}^2$ that is one-to-one everywhere except for $\pi^{-1}(p) \cong \mathbb{P}^1 = \mathbb{P}T_p\mathbb{P}^2$. Therefore M may be expressed as a submanifold of $\mathbb{P}^2 \times \mathbb{P}^1$ as follows:

$$M = \{([Z_0 : Z_1 : Z_2], [Y_0, Y_1]) \, : \, Z_0Y_1 = Z_1Y_0\}. \tag{117}$$

Now the Segre map embeds $\mathbb{P}^2 \times \mathbb{P}^1$ in $\mathbb{P}^5$ by

$$([Z_0 : Z_1 : Z_2], [Y_0 : Y_1]) \mapsto [Z_0Y_0 : Z_0Y_1 : Z_1Y_0 : Z_1Y_1 : Z_2Y_0 : Z_2Y_1],$$

and correspondingly M is the intersection of this map with the hyperplane $W_1 - W_2 = 0$ in $\mathbb{P}^5$ (with homogeneous coordinates $[W_0 : \ldots : W_5]$). Switching to affine coordinates about p, $z_0 = Z_0/Z_2, z_1 = Z_1/Z_2$, the relation in (117) turns to $z_0y_1 = z_1$ and the embedding of M in $\mathbb{P}^5$ is given by, assuming $z_0, z_1 \neq 0$,

$$\begin{aligned}([z_0 : z_1 : 1], [1 : y_1]) \mapsto & [z_0 : z_0y_1 : z_1 : z_1y_1 : 1 : y_1] \\ &= [z_0 : z_1 : z_1 : z_1y_1 : 1 : y_1] \\ &= [z_0^2 : z_0z_1 : z_0z_1 : z_0z_1y_1 : z_0 : z_0y_1] \\ &= [z_0^2 : z_0z_1 : z_0z_1 : z_1^2 : z_0 : z_1].\end{aligned}$$

This is a degenerate embedding and induces the following non-degenerate monomial embedding

$$(z_0, z_1) \mapsto [z_0^2 : z_0z_1 : z_1^2 : z_0 : z_1] \in \mathbb{P}^4 \tag{118}$$

(this is the orbit of the point $x = [1 : 1 : 1 : 1 : 1]$ in the notation of Lemma 3.15). Note that the blow-up had the effect of cutting out a corner from the polytope: now the point $(0, 0) \in \mathbb{Z}^2$ is no longer in the polytope and we only have 5 monomials. The $(\mathbb{C}^\star)^2$-action is then the one induced from this map, given by

$$\begin{aligned}r_\theta : & [z_0^2 : z_0z_1 : z_1^2 : z_0 : z_1] \\ & \mapsto [e^{2\sqrt{-1}\theta_0}z_0^2 : e^{\sqrt{-1}\theta_0+\sqrt{-1}\theta_1}z_0z_1 : e^{2\sqrt{-1}\theta_1}z_1^2 : e^{\sqrt{-1}\theta_0}z_0 : e^{\sqrt{-1}\theta_1}z_1].\end{aligned}$$

The restriction of the Hermitian metric on $H \xrightarrow{\pi} \mathbb{P}^{|P\cap\mathbb{Z}^n|-1}$ to the image of the previous map is $(|z_0|^4 + |z_0|^2|z_1|^2 + |z_1|^4 + |z_0|^2 + |z_1|^2)^{-1}$ inducing the open-orbit Kähler potential $\varphi = \log(|z_0|^4 + |z_0|^2|z_1|^2 + |z_1|^4 + |z_0|^2 + |z_1|^2)$. The torus action is generated by the vector fields $\frac{\partial}{\partial\sqrt{-1}\theta_j} = z_j\frac{\partial}{\partial z^j} - \bar{z}_j\frac{\partial}{\partial \bar{z}^j}$ (see (111)). We are now able to compute the two Hamiltonians, and consequently the image of the moment map. Note that $\iota_{\frac{\partial}{\partial z^j}}\sqrt{-1}\partial\bar{\partial}\varphi = \sqrt{-1}\bar{\partial}\frac{\partial\varphi}{\partial z_j}$, either directly, or since $\mathcal{L}_{X_j}\sqrt{-1}\partial\bar{\partial}\varphi = \sqrt{-1}\bar{\partial}(X_j\varphi)$ for any holomorphic vector field. Similarly $\mathcal{L}_{\bar{X}_j}\sqrt{-1}\partial\bar{\partial}\varphi = -\sqrt{-1}\partial(\bar{X}_j\varphi)$ Thus, putting $X_j = z_j\frac{\partial}{\partial z^j}$,

$$\iota_{\sqrt{-1}z_j\frac{\partial}{\partial z^j}-\sqrt{-1}\bar{z}_j\frac{\partial}{\partial \bar{z}^j}}\sqrt{-1}\partial\bar{\partial}\varphi = \sqrt{-1}\bar{\partial}(X_j\varphi) - -\sqrt{-1}\partial(\bar{X}_j\varphi).$$

Note that from the special form of X and the fact that φ depends only on $|z_j|^2$ we have $X_j\varphi = \bar{X}_j\varphi$. Thus

$$\iota_{\frac{\partial}{\partial\sqrt{-1}\theta_j}}\sqrt{-1}\partial\bar{\partial}\varphi = d(X_j\varphi).$$

We compute for $j = 0, 1$ to get the two Hamiltonians

$$X_0\varphi = \frac{2|z_0|^4 + |z_0|^2|z_1|^2 + |z_0|^2}{|z_0|^4 + |z_0|^2|z_1|^2 + |z_1|^4 + |z_0|^2 + |z_1|^2} \in [0, 2],$$
$$X_1\varphi = \frac{|z_0|^2|z_1|^2 + 2|z_1|^4 + |z_1|^2}{|z_0|^4 + |z_0|^2|z_1|^2 + |z_1|^4 + |z_0|^2 + |z_1|^2} \in [0, 2].$$

And crucially:

$$1 \leq X_0\varphi + X_1\varphi \leq 2.$$

This precisely means that the image of these two Hamiltonians will be the polytope P described initially. If we would not have been missing the monomial 1 corresponding to the lattice point $(0, 0)$ then the denominators would contain also 1 and then we would have $X_0\varphi + X_1\varphi \in [0, 2]$ as for $\mathbb{P}^2$. Now $[\omega] = c_1(L) = \iota^\star[H|_{\mathbb{P}^{|P\cap\mathbb{Z}^n|-1}}]$. Since every section in $H^0(M, L)$ is induced from a section of H on the projective space we have therefore identified a distinguished basis of $H^0(M, L) \cong H^0(\mathbb{P}^4, H) \cong \mathbb{C}^5$.

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Some remarks regarding the previous example might be helpful as it is a source for some intuition.

Remark 3.18. First notice that the divisor at infinity corresponds precisely to the case when one of z_0 or z_1 vanishes. When both vanish we get the exceptional divisor $\pi^{-1}(p) \cong \mathbb{P}^1$ while when each vanishes separately we get two additional copies of a lower dimensional toric manifold in fact two copies of $\mathbb{P}^1$ (all of these pieces come from taking the Zariski closure of the complex torus). A fourth copy of $\mathbb{P}^1$ comes from switching coordinates to the r_j coordinates centered at p. These four copies of $\mathbb{P}^1$ precisely correspond to the four faces of the polytope P. Notice that the blow-up operation introduced a corner, or in other words, a new face, corresponding to the exceptional divisor, an additional piece in the divisor at infinity of the open orbit.

Remark 3.19. It is worthwhile to point out that equivalent polytopes may be obtained by blowing up different points on $\mathbb{P}^2$. Let us choose now $[1 : 0 : 0]$. This is equivalent to choosing to work on different affine coordinate charts. Switching in (118) to homogeneous coordinates we get

$$[Z_0 : Z_1 : Z_2] \mapsto [Z_0^2 : Z_0Z_1 : Z_1^2 : Z_0Z_2 : Z_1Z_2],$$

and working now on the patch $Z_0 \neq 0$ instead of $Z_2 \neq 0$ as before we get the map

$$(z_1, z_2) \mapsto [1 : z_1 : z_1^2 : z_2 : z_1z_2].$$

The same calculations will now give the moment polytope $0 \leq x, 0 \leq y \leq 1, x+y \leq 2$. This normalization is a bit more convenient since when a 1 is present in the denominator then a corner will be cut precisely in the direction where a highest monomial power is missing (here $|z_2|^2$ is missing). Another observation of interest is that the corner cut will come from X_2 now, while previously it came from $X_0 + X_1$. This is no coincidence, in fact on $\mathbb{P}^2$ itself one has $-X_2 = X_0+X_1$ since $X_0+X_1+X_2$ generates the rescaling action on $\mathbb{C}^N$ that is quotiented out when forming $\mathbb{P}^{N-1}$.

To summarize, we have the following result:

Proposition 3.20. *Let (M, L) be a toric very ample line bundle. Let P be the image of the moment map with respect to any Kähler form representing $c_1(L)$. Then*

$$H^0(M, L) = \operatorname{span}_{\mathbb{C}}\{\chi_\alpha\}_{\alpha\in P\cap\mathbb{Z}^n}. \tag{119}$$

This formula is remarkable since it implies that a rather complicated combinatorial count of lattice points may be performed by invoking the Riemann-Roch formula we encountered earlier (cf. [**118**] and references therein). In addition, as will be important below, it provides for a canonical basis for the space of holomorphic sections. While each of these toric monomials is equivariant with respect to the torus action the sum of such is not any longer, so not all sections are equivariant as might seem from a naïve look at (119) (each monomial is multiplied by a different factor under an action of a fixed element of the torus; in other words each monomial is an eigenvector for each fixed action but with different eigenvalues for different monomials in general).

3.4.1.4 Toric Bergman metrics. Objects defined on M are called toric if they are invariant under the real torus action. Let $T = (S^1)^n$ denote the real torus. The space of global toric Kähler potentials is denoted by

$$\mathcal{H}_\omega(T) := \mathcal{H}(T) := \{\varphi \in \mathcal{H}_\omega \, : \, \omega_\varphi > 0, \quad g^\star\varphi = \varphi, \quad \forall g \in T\}. \tag{120}$$

By abuse of notation we will frequently identify $\mathcal{H}(T)$ with the local torus-invariant Kähler potentials defined on the open orbit, see §3.6.

As we saw in (72), a Hermitian metric h on L defines a Hermitian metric h^k on L^k and induces an inner product on $H^0(M, L^k)$. This inner product is represented by a matrix and is denoted by $\mathrm{Hilb}_k(h)$. Also, given an inner product structure on $H^0(M, L^k)$ (i.e., a positive Hermitian $(d_k+1) \times (d_k+1)$ matrix) we pick an orthonormal basis for $\{s_j\}$ and construct a corresponding Bergman metric on M by pulling-back a Fubini-Study metric induced by the Kodaira embedding corresponding to this basis. Let h_0 be the Hermitian metric such that

$$-\sqrt{-1}\partial\bar{\partial}\log h_0 = \omega.$$

The map sending an inner product A to a point in $\mathcal{H}_k$ will be denoted by

$$\mathrm{FS}_k(A) := \frac{1}{k}\log\sum_j |s_j|^2_{h_0^k} \in \mathcal{H}_k.$$

In this notation Theorem 3.11 may be rephrased as $\lim_{k\to\infty} \mathrm{FS}_k \circ \mathrm{Hilb}_k(h) = h$.

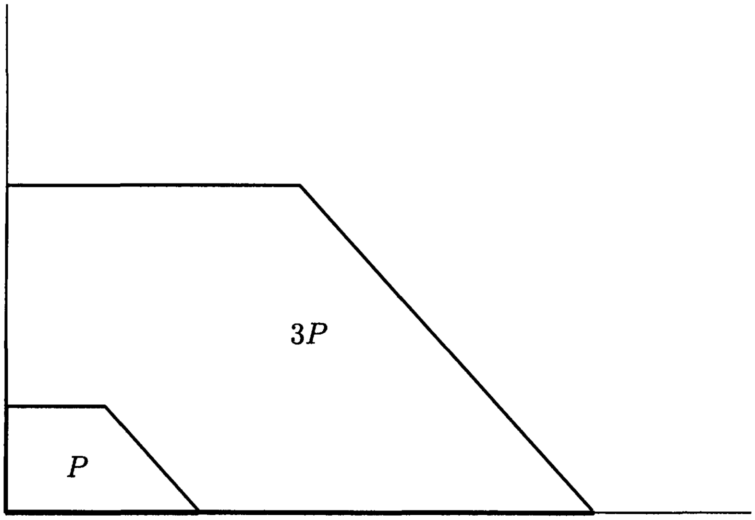

Figure 2. A polytope and a three-times rescaled polytope.

According to Proposition 3.20 on a toric variety the space $H^0(M, L^k)$ has a distinguished basis given by the collection of toric monomials $\{\chi_\alpha(z) := z^a\}_{\alpha\in kP\cap\mathbb{Z}^n}$, where kP denotes the k-times rescaled polytope (see Figure 2), and $kP\cap\mathbb{Z}^n$ are the lattice points contained in it. As a check there are $O(k^n)$ such lattice points and on the other hand $\dim H^0(M, L^k) = O(k^n)$ according to (65). The space of toric Bergman potentials, defined as those potentials induced by Kodaira embeddings consisting of invariant sections, is denoted by $\mathcal{H}_k(T)$ and may therefore be written as

$$\mathcal{H}_k(T) = \{\varphi \in \mathcal{H}(T) \, : \, \varphi(z) = \frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^n} c_\alpha |\chi_\alpha(z)|^2, \quad c_\alpha > 0\}. \tag{121}$$

Note that $\mathcal{H}_k(T) \cong (\mathbb{R}_+ \setminus \{0\})^{|kP \cap \mathbb{Z}^n|}$.

The main feature of the analysis of the Bergman kernel on toric manifolds is that the toric monomials $\chi_\alpha(z) = z^\alpha$ form a basis for $H^0(M, L^k)$ that is orthogonal with respect to any toric inner product. More precisely we have the following result that is analogous to the fact that $\{z^l\}_{l \in \mathbb{Z}_+}$ form an orthogonal basis for the holomorphic functions in the complex plane (considered as sections of the trivial bundle over $\mathbb{C}$) with respect to any Hermitian metric that depends only on $|z|$.

Lemma 3.21. *The toric monomials $\{\chi_\alpha(z) := z^\alpha\}_{\alpha \in kP \cap \mathbb{Z}^n}$ are an orthogonal basis of $H^0(M, L^k)$ with respect to any element of $\mathcal{H}_k(T)$.*

Proof. The toric monomials form a basis of $H^0(M, L^k)$ by Proposition 3.20. It remains to check they are orthogonal with respect to any toric Hermitian metric on this finite-dimensional vector space. A toric Hermitian metric h^k (i.e., an element of $\mathcal{H}_k(T)$) induces a Hilbert space structure on $H^0(M, L^k)$ by setting (see (72))

$$\langle s, t \rangle := \frac{1}{V} \int_M s\,\bar{t} \cdot h^k \, (k\omega_h)^n.$$

One may integrate on the full measure open orbit diffeomorphic to $(S^1)^n \times P$ instead to obtain

$$\begin{aligned} \langle \chi_\alpha, \chi_\beta \rangle &= \frac{1}{V} \int_{(\mathbb{C}^\star)^n} \chi_\alpha \bar{\chi}_\beta h^k (k\omega_h)^n \\ &= \frac{1}{V} \int_{(S^1)^n \times P} z^\alpha \overline{z^\beta} h^k dx \wedge d\theta_1 \wedge \cdots \wedge d\theta_n. \end{aligned}$$

Writing $z^\alpha = e^{\langle \rho, \alpha \rangle /2 + \sqrt{-1} \langle \theta, \alpha \rangle}$ we then observe that the previous integral vanishes unless $\alpha = \beta$ since if, say, $\alpha_j \neq \beta_j$, one has

$$\int_0^{2\pi} e^{\sqrt{-1}(\alpha_j - \beta_j)\theta_j} d\theta_j = 0. \qquad \square$$

We will further explore the simplification resulting from this result in the next subsection.

3.4.2 Norming constants, normalized monomials, and peak values. Having recalled the most basic torus geometric notions we next recall some additional fundamental facts relevant to our setting. We refer the reader to Cannas da Silva [**47**], Shiffman-Tate-Zelditch [**243**], and Song-Zelditch [**253**] for more details.

While a general Bergman metric may be represented by a positive Hermitian matrix, i.e., an inner product on $H^0(M, L^k) \cong \mathbb{C}^{d_k+1}$, a toric Bergman metric is, according to Lemma 3.21, represented by a positive diagonal matrix, namely one only needs to specify the norms each toric monomial has with respect to the metric. This is a significant simplification and it will be crucial below.

Equivalently, a toric inner product, or a point in $\mathcal{H}_k(T)$, is completely determined by the L^2 norms (divided by k^n), or "norming constants" of the toric monomials

$$\mathcal{Q}_{h^k}(\alpha) := ||\chi_\alpha||^2_{h^k} = \frac{1}{V}\int_{(\mathbb{C}^*)^n} |z^\alpha|^2 e^{-k\varphi}\omega_h^n,$$

and $\{\chi_\alpha/\sqrt{k^n\mathcal{Q}_{h^k}(\alpha)}\}_{\alpha\in P\cap\mathbb{Z}^n}$ is an orthonormal basis for $(H^0(M, L^k), \mathrm{Hilb}_k(h))$.

Define the normalized norms of the monomials

$$\mathcal{P}_{h^k}(\alpha, z) := \frac{|\chi_\alpha(z)|^2_{h^k}}{||\chi_\alpha||^2_{h^k}}. \tag{122}$$

Thus

$$\frac{1}{V}\int_M \mathcal{P}_{h^k}(\alpha, z)(k\omega)^n = 1. \tag{123}$$

According to the Riemann-Roch formula (65) and (119) the number of monomials increases polynomially. Also when scaling down the polytope kP to P the lattice points $kP \cap \mathbb{Z}^n$ become asymptotically dense in P. In analogy from Quantum Mechanics, one expects the sections to localize in the limit and so to tend to Dirac delta functions about a dense set of rational points in P. The guiding example, and the simplest one, is $\mathbb{C}$ with the Hermitian metric $e^{-k|z|^2}$, which we now explore to gain more intuition.

Example 3.22. *The linear (Bargmann-Fock) model.* The k-th Bargmann-Fock model on $\mathbb{C}$ is the vector space of holomorphic functions on $\mathbb{C}$ that are L^2 with respect to the Hermitian metric $h^k_{\mathrm{BF}} := e^{-k|z|^2}$ and a basis is given by all monomials z^α with $\alpha \in \mathbb{Z}_+$. More intrinsically we are considering the trivial line bundle over $\mathbb{C}$ equipped with the metric h_{BF} and taking tensor powers of it. In other words we are quantizing the non-compact Kähler manifold $(\mathbb{C}, \sqrt{-1}dz\wedge d\bar{z})$ and $\sqrt{-1}dz\wedge d\bar{z} = \sqrt{-1}\partial\bar{\partial}\varphi$ with $\varphi = -\frac{1}{k}\log h^k_{\mathrm{BF}}$.

We now check whether the sections do in fact asymptotically localize. By localization of a section s we will mean the localization of the globally defined function $|s|^2_{h^k}/||s||^2_{h^k}$ associated to it. In our case we therefore seek the maximum of the function

$$\mathrm{const}\cdot\mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha, z) = |z^\alpha|^2 e^{-k|z|^2}. \tag{124}$$

We write $r = |z|$ and compute

$$0 = \frac{d}{dr}\left(r^{2\alpha}e^{-kr^2}\right) = 2\alpha r^{2\alpha-1}e^{-kr^2} - 2kr^{2\alpha+1}e^{-kr^2},$$

to get $r^2 = \alpha/k$, or in other words, the function $\mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha, z)$ achieves its maximum (this must be a maximum since the function is an exponentially decaying Gaussian perturbed by a polynomial term) at the points $(\nabla\varphi)^{-1}(\alpha/k)$. This set is generically an n-dimensional real torus $(S^1)^n$ when $\alpha/k \in \overset{\circ}{P}$ but in general is $(S^1)^m$ with $m \leq n$ if α/k lies in a codimension m face of ∂P.
Next, let us compute the peak value of the functions $\mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha, z)$. This will follow from plugging-in in (124) once we compute the constant that appears there (namely, the norming constant $\mathcal{Q}_{h^k_{\mathrm{BF}}}(\alpha)$). We have, using $\sqrt{-1}dz \wedge d\overline{z} = 2rdr \wedge d\theta$, that

$$\begin{aligned}\mathcal{Q}_{h^k_{\mathrm{BF}}}(\alpha) = ||\chi_\alpha||^2_{h^k_{\mathrm{BF}}} &= \int_{\mathbb{C}^\star} |z^\alpha|^2 e^{-k|z|^2}\sqrt{-1}dz \wedge d\overline{z}/2\pi \\ &= \int_{\mathbb{R}_+} r^{2\alpha}e^{-kr^2}2rdr \\ &= \frac{d^\alpha}{dk^\alpha}\int_{\mathbb{R}_+}(-1)^\alpha e^{-kr^2}2rdr \\ &= \frac{d^\alpha}{dk^\alpha}(-1)^\alpha\Big[-\frac{1}{k}e^{-kr^2}\Big]\Big|_0^\infty \\ &= \frac{d^\alpha}{dk^\alpha}(-1)^\alpha\frac{1}{k} = \frac{\alpha!}{k^{\alpha+1}}(-1)^{2\alpha} = \frac{\alpha!}{k^{\alpha+1}}.\end{aligned} \tag{125}$$

Therefore putting $|z|^2 = \alpha/k$ in (124) we obtain

$$\mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha) = \mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha, (\nabla\varphi)^{-1}(\alpha/k)) = ke^{-\alpha}\alpha^\alpha/\alpha!. \tag{126}$$

Note that these computations immediately give the corresponding higher dimensional ones since the Bargmann-Fock model on $\mathbb{C}^n$ with Kähler form $\sqrt{-1}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ and Hermitian metric $e^{-k|z|^2}$ (with $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$) is a product of one-dimensional ones. For example, (126) generalizes to

$$\mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha) = k^n\alpha^\alpha e^{-|\alpha|}/\alpha!, \tag{127}$$

using the vector notation, i.e., $\alpha! = \alpha_1!\cdots\alpha_n!$ and $\alpha^\alpha = \alpha_1^{\alpha_1}\cdots\alpha_n^{\alpha_n}$. Let us remark that in general there is another way to compute the coefficients $\mathcal{P}_{h^k}(\alpha)$ that will be explained in Example 3.31.

~ ~ ~

As we saw above the sections in the linear model tend to peak. Since locally every compact Kähler manifold looks like $(\mathbb{C}^n, \sqrt{-1}\sum_{j=1}^n dz_j \wedge d\bar{z}_j)$ the same (local in nature) phenomenon will occur in general (this behavior was explored by Tian [**266**]). In our toric situation there is even a rather natural candidate for the rough/approximate peak value of a toric monomial given by

$$\mathcal{P}_{h^k}(\alpha) := \frac{|\chi_\alpha(\mu_{h^k}^{-1}(\frac{\alpha}{k}))|_{h^k}^2}{||\chi_\alpha||_{h^k}^2}. \tag{128}$$

This localization phenomenon is very useful. For example, consider a sum of the form $\sum_{\alpha\in P\cap\mathbb{Z}^n} B_k(\alpha)\mathcal{P}_{h^k}(\alpha, z)$ where the coefficients $B_k(\alpha)$ grow only polynomially in k. This type of sum occurs very often in the analysis of families of Szegő kernels, indeed the Szegő kernel of a single metric is given by

$$\sum_{\alpha\in P\cap\mathbb{Z}^n} \mathcal{P}_{h^k}(\alpha, z),$$

and studying families of metrics introduces coefficients in the sum in certain situations. Then one may for all practical purposes throw away all of the normalized monomials except those $\mathcal{P}_{h^k}(\alpha, z)$ for which $(\nabla\varphi)^{-1}(\alpha/k)$ is relatively close to $\log|z|^2$! This is suggested intuitively and can be shown directly in the Bargmann-Fock model. The general toric situation can be reduced to the linear model using the asymptotic expansion of the Szegő kernel off the diagonal and the simple formula

$$\mathcal{P}_{h^k}(\alpha, z) = \int_{(S^1)^n} \Pi_k(e^{\sqrt{-1}\theta}z, z)e^{-\sqrt{-1}\langle\alpha,\theta\rangle}d\theta, \tag{129}$$

that express the normalized monomials as 'Fourier components' of the Szegő kernel of the circle bundle associated to $L^\star$ [**252**, §3,5]. It is now that one uses the off-diagonal asymptotics of Theorem 3.12 to conclude from (129) the asymptotics of the normalized section. Namely one demonstrates by repeated integration by parts in the oscillatory integral (129) (this is essentially the main tool available to study oscillatory integrals, due to their definition) that $\mathcal{P}_{h^k}(\alpha, z)$ is of order k^{-M} for any $M > 0$ provided that $|\alpha/k - (\nabla\varphi)(z)| \geq Mk^{\delta-\frac{1}{2}}$ (we emphasize that in these asymptotic expressions one should think of a sequence of lattice points α depending on k). In sum one has the following result of Song-Zelditch.

Lemma 3.23. (See [**252**, Lemma 1.2].) *Let K be a compact set and let $B_k(y,\alpha) : K \times kP\cap\mathbb{Z}^n \to \mathbb{C}$ be a family of lattice point functions satisfying $|B_k(y,\alpha)| \leq C_0k^M$ for some $C_0, M \geq 0$. Fix $\delta \in (0, 1/2)$. Then for every $C > 0$ one has*

$$\sum_{kP\cap\mathbb{Z}^n} B_k(y,\alpha)\mathcal{P}_{h_y^k}(\alpha, z) = \sum_{\alpha:|\frac{\alpha}{k}-\mu_y(z)|\leq k^{\delta-\frac{1}{2}}} B_k(y,\alpha)\mathcal{P}_{h_y^k}(\alpha, z) + O(k^{-C}).$$

3.4.2.1 Asymptotics of the peak values. We saw that rather precise information regarding the region of concentration of the toric monomials may be obtained (asymptotically). It is also very useful to be able to estimate (asymptotically) the peak value itself. These are quite naturally two complementary issues when dealing with convergence of lattice point sums. Such an estimate has been developed by Song-Zelditch, and it is the second main tool that we will make crucial use of. We emphasize again that one should think of a sequence of α as parametrizing a sequence of lattice points $\{\alpha_k\}$ depending on k when computing asymptotics of $\mathcal{P}_{h^k}(\alpha)$.

One may argue that it is the possibility to have such a precise expression for the peak values as the feature that essentially distinguishes the toric situation from the general Kähler manifold situtation.

Asymptotically these peak values are determined by the geometry of the polytope in the following manner: if $\lim_k \alpha_k/k$ is contained in a codimension m face of P then the contribution from the m "lost" directions will be exactly as in an m-dimensional linear model, while the contribution from the remaining $n-m$ "interior" directions will be precisely given by the symplectic potential reduced to those directions.

In our situation this will be of great use since will we try to approximate a harmonic map whose explicit expression we will be able to determine. This should become more transparent to the reader when we go into the details of the proof of our main theorem later.

We now state the result of Song-Zelditch more precisely. Let $\delta_k := \frac{1}{\sqrt{k}\log k}$. Denote by $\mathcal{F}_{\delta_k}(x) = \{r : l_r(x) < \delta_k\}$ the index set for those facets to which x is $\frac{1}{\sqrt{k}\log k}$-close, and let $\delta_k^\sharp(x)$ denote the cardinality of this set. Set

$$\mathcal{G}_\varphi(x) := \left(\delta_\varphi(x) \cdot \Pi_{j \notin \mathcal{F}_{\delta_k(x)}} l_j(x)\right)^{-1},$$

where $\delta_\varphi(x)$ is defined in (140) below and put

$$\mathcal{P}_{\mathrm{BF},\delta_k}(\alpha) := \Pi_{j \in \mathcal{F}_{\delta_k(x)}} \mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha_j).$$

These two terms are the far and near contributions to the asymptotics of the peak values $\mathcal{P}_{h^k}(\alpha)$:

Lemma 3.24. (See [**252**, Propositions 6.1,6.5].) *Let $\delta_k = \frac{1}{\sqrt{k}\log k}$. Let $\{h_t\}_{t\in K}$ be a family of metrics with K compact. Then there exist $C > 0$ independent of t such that for any $\delta \in (0, \frac{1}{2})$*

$$\mathcal{P}_{h_t^k}(\alpha) = Ck^{\frac{1}{2}(m-\delta_k^\sharp(\frac{\alpha}{k}))}\sqrt{\mathcal{G}_\varphi(\tfrac{\alpha}{k})}\mathcal{P}_{\mathrm{BF},\delta_k}(\tfrac{\alpha}{k})(1 + R_k(\tfrac{\alpha}{k}, h_t)), \tag{130}$$

where $R_k = O(k^{\delta-\frac{1}{2}})$. This expansion is uniform in t and may be differentiated twice to give for $j = 1, 2$ and for some amplitudes S_j of order zero the expansion

$$\left(\frac{\partial}{\partial t}\right)^j \mathcal{P}_{h_t^k}(\alpha) = C_m k^{\frac{1}{2}(m-\delta_k^\sharp(\frac{\alpha}{k}))} \sqrt{\mathcal{G}_\varphi(\tfrac{\alpha}{k})} \mathcal{P}_{\mathrm{BF},\delta_k}(\tfrac{\alpha}{k})(S_j(t,\alpha,k) + R_k(\tfrac{\alpha}{k}, h_t)). \quad (131)$$

3.5 Statement of the quantization result

Having described most of the necessary preliminary ingredients we may now state the main result of this chapter.

In light of Tian's asymptotic isometry theorem a natural question is: to what extent can the geometry of the space of Kähler metrics $\mathcal{H}$ be approximated by that of the spaces $\mathcal{H}_k$ of algebraic metrics? Indeed, Tian's theorem states that the topology of $\mathcal{H}$ is approximated by that of the Bergman spaces, and the work of Catlin and Zelditch gives a complete geometric asymptotic expansion for this approximation. Now we have seen in §§2.2.3 that $\mathcal{H}$ has a natural Riemannian structure as an infinite-dimensional symmetric space, and so do the Bergman spaces $\mathcal{H}_k \cong GL(d_k + 1, \mathbb{C})/U(d_k + 1)$. It is therefore reasonable to suspect that the geometries of the infinite and finite-dimensional spaces could be closely related.

To put our result in context let us recall some previous work in the area. Donaldson and Arezzo-Tian raised the question whether geodesics of $\mathcal{H}$ could be well-approximated by the one-parameter subgroup geodesics of $\mathcal{H}_k$ [**3**,**81**]. Recall from §§2.2.3 that geodesics of $\mathcal{H}$ are given by solutions of

$$\ddot{\varphi} - \tfrac{1}{2}|\nabla\dot{\varphi}|^2 = 0, \quad \varphi(0,\,\cdot\,) = \varphi_0,\ \varphi(1,\,\cdot\,) = \varphi_1, \quad \varphi_0, \varphi_1 \in \mathcal{H}, \qquad (132)$$

where φ is considered as a map from $[0,1]$ to $\mathcal{H}$, or equivalently as a function on $[0,1] \times M$. Phong-Sturm studied this question and proved an almost everywhere weak convergence for a sequence of geodesics in $\mathcal{H}_k$ to a prescribed geodesic of $\mathcal{H}$ [**215**]. Song-Zelditch proved that the same sequence converges in $C^2([0,1] \times M)$ when the manifold is toric and one restricts to the torus-invariant metrics [**252**], and Berndtsson used a different argument to prove that geodesics in $\mathcal{H}$ can be C^0-approximated by geodesics in spaces of Bergman metrics induced by embeddings by sections of $L^k \otimes K_M$, where K_M is the canonical bundle of M [**20**]. In addition, Phong-Sturm and Song-Zelditch proved approximation results for geodesic rays constructed from test configurations [**216**,**253**],

Now the geometry of a Riemannian manifold is reflected to a large extent by its geodesics and more generally by the specification of the harmonic maps into it, involving the analysis of certain nonlinear elliptic PDEs. In this chapter we describe how solutions to these PDEs on $\mathcal{H}$ can be approximated in an algebro-geometric manner by a sequence of solutions to PDEs on $\mathcal{H}_k$, in the setting of a toric variety.

Recall that a harmonic map between two Riemannian manifolds (N, f) and $(\tilde{N}, \tilde{f})$ is a critical point of the energy functional

$$E(a) = \int_N |da|^2_{f \otimes a^\star \tilde{f}} \, dV_{N,f},$$

on the space of smooth maps a from N to $\tilde{N}$ [**87**]. The problem we study in this chapter is whether higher dimensional harmonic maps of general compact Riemannian manifolds N with boundary ∂N into $\mathcal{H}$ admit similar kinds of 'algebro-geometric' approximations. For maps to toric metrics on toric Kähler manifolds, we obtain an affirmative solution at the same level of precision as in the case of geodesics studied by Song-Zelditch [**252**].

Let (N, f) be a compact oriented Riemannian manifold with smooth boundary and let $G(y, q)$ denote the positive Dirichlet Green kernel for the Laplacian Δ_f (see §§3.8.1 for more precise details), and let $dV_{\partial N, f}$ denote the induced measure on ∂N from the restriction of the Riemannian volume form $dV_{N,f}$ from N to ∂N.

The main result of this chapter is the following quantization result:

Theorem 3.25. *Let (M, L, ω) be a polarized toric Kähler manifold, and let (N, f) be a compact oriented smooth Riemannian manifold with smooth boundary ∂N. Let $\psi : \partial N \to \mathcal{H}(T)$ denote a fixed smooth map. There exists a harmonic map $\varphi : N \to \mathcal{H}(T)$ with $\varphi|_{\partial N} = \psi$ and harmonic maps $\varphi_k : N \to \mathcal{H}_k(T)$ with $\varphi_k|_{\partial N} = FS_k \circ Hilb_k(\psi)$, given on the open orbit by*

$$\varphi_k(y, z) = \frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^n} |\chi_\alpha(z)|^2_{h_0^k} \exp\Big(\int_{\partial N} \partial_{\nu_q} G(y, q) \log ||\chi_\alpha||^2_{h^k_{\psi(q)}} dV_{\partial N, f}(q) \Big), \tag{133}$$

and one has

$$\lim_{k \to \infty} \varphi_k = \varphi,$$

in the $C^2(N \times M)$ topology.

A motivating special case is the unit disc $N = D := \{z \in \mathbb{C} : |z| \leq 1\}$. It has been the subject of intensive studies (e.g., [**59,66,81**]). Then the map φ corresponds to certain foliations by holomorphic discs arising from a solution of a certain homogeneous complex Monge-Ampère (HCMA) equation. To describe this, let $\pi_2 : D \times M \to M$ denote the projection onto the second factor and consider the HCMA equation,

$$(\pi_2^\star \omega + \sqrt{-1}\partial\bar{\partial}\varphi)^{n+1} = 0, \quad \text{on} \quad D \times M, \tag{134}$$

$$(\pi^\star \omega + \sqrt{-1}\partial\bar{\partial}\varphi)|_{\{t\} \times M} > 0, \quad \forall t \in D, \tag{135}$$

$$\varphi = \psi, \quad \text{on} \quad \partial D \times M. \tag{136}$$

One may show that this HCMA is the Euler-Lagrange equation of an infinite-dimensional version of a Wess-Zumino-Witten model, given by the energy functional

$$E_{\sigma}^{WZW}(b) = \frac{1}{2}\int_D |\nabla b|^2 + \int_Z \theta,$$

on the space of maps $b \in C^{\infty}(D, G^{\mathbb{C}}/G)$, where $\sigma : \partial D \to G^{\mathbb{C}}/G$ is a fixed map σ [**79**]. The Lie bracket of G determines a 3-form θ and Z is any cochain with boundary $b(D) - b_0(D)$ for some fixed reference map b_0 with the same boundary conditions ψ. The Euler-Lagrange equations for this functional are the WZW equations

$$d^{\star} d\, b + [b_{\star}\tfrac{\partial}{\partial q}, b_{\star}\tfrac{\partial}{\partial s}] = 0,$$

in Euclidean coordinates $q + \sqrt{-1}s \in D$, and where $d^{\star}$ maps sections of $T^{\star}D \otimes b^{\star}TG^{\mathbb{C}}/G$ to sections of $b^{\star}TG^{\mathbb{C}}/G$. Finally, when G and $G^{\mathbb{C}}/G$ are replaced by $\mathrm{Symp}(M, \omega)$ and $\mathcal{H}$, the Christoffel symbols are given by $\Gamma(\zeta, \eta)|_{\varphi} = -\frac{1}{2}g_{\varphi}(\nabla\zeta, \nabla\eta)$ (refer to (55)), and the WZW equation is

$$\varphi_{qq} + \varphi_{ss} - \tfrac{1}{2}|\nabla\varphi_q|^2 - \tfrac{1}{2}|\nabla\varphi_s|^2 + \{\varphi_q, \varphi_s\}_{\omega_{\varphi}} = 0.$$

It is a perturbation of the usual harmonic map equation by a Poisson bracket term. Coming back to the toric situation and restricting to the space of torus-invariant Kähler potentials $\mathcal{H}(T) \subseteq \mathcal{H}$ the functions φ_q and φ_s are commuting Hamiltonians and hence the WZW equation reduces to the harmonic map equation. The finite-dimensional WZW equation on $GL(d_k + 1, \mathbb{C})/U(d_k + 1)$ may be written similarly

$$T^{-1}T_{qq} + T^{-1}T_{ss} - (T^{-1}T_q)^2 - (T^{-1}T_s)^2 + \sqrt{-1}[T^{-1}T_q, T^{-1}T_s] = 0.$$

The torus-invariance then corresponds to restriction to diagonal matrices and again the last term vanishes and the equation reduces to the harmonic map equation. Geometrically, the curvature of $\mathcal{H}$ comes from the Poisson bracket and when we restrict to the flat subspace $\mathcal{H}(T)$ the noncommutativity disappears.

In the case of the unit disc, the normal derivative of the Green kernel is the Poisson kernel, whose restriction to $D \times \partial D$ takes the form

$$P(re^{\sqrt{-1}\theta}, e^{\sqrt{-1}\gamma}) = P_r(\theta - \gamma) = \frac{1}{2\pi}\frac{1 - r^2}{1 - 2r\cos(\theta - \gamma) + r^2}$$

(our convention is that the Green function be nonnegative, as explained in §§3.8.1). Then we have the following more explicit statement of Theorem 3.25:

Corollary 3.26. *Let (M, L, ω) be a polarized toric Kähler manifold. Let φ be a solution of the HCMA equation (134)-(136) with $\psi : S^1 \to \mathcal{H}(T)$ a smooth map, and let $\varphi_k : N \to \mathcal{H}_k(T)$ be given on the open orbit by*

$$\varphi_k(re^{\sqrt{-1}\gamma}, z) = \frac{1}{k}\log \sum_{\alpha \in kP \cap \mathbb{Z}^n} |\chi_{\alpha}(z)|^2_{h_0^k} \exp\Big(\int_{\partial D} P_r(\theta - \gamma)\log ||\chi_{\alpha}||^2_{h^k_{\psi(\theta)}}\, d\theta\Big).$$

Then $\lim_{k\to\infty}\varphi_k = \varphi$ *in the* $C^2(D\times M)$ *topology.*

We mention that in essence Theorem 3.25 is an interpolation result for Kähler manifolds. We hope to return to this point of view in the future. The proof of Theorem 3.25 builds upon the machinery developed by Song-Zelditch for the study of geodesics in $\mathcal{H}(T)$. In the geodesic case, i.e., $N=[0,1]$, one has the expression

$$\varphi_k(t,z)=\frac{1}{k}\log\sum_{\alpha\in kP\cap\mathbb{Z}^n}|\chi_\alpha(z)|^2_{h_0^k}\,e^{-(1-t)\log||\chi_\alpha||^2_{\mathrm{Hilb}_k(h_0)}-t\log||\chi_\alpha||^2_{\mathrm{Hilb}_k(h_1)}}.$$

for the approximating Bergman Kähler potentials. We see that the straight line segment in the case $N=[0,1]$ is replaced by the harmonic extension of the boundary L^2 norming constants in the general case. Aside from justifying the general formula, we need to modify the estimates of **[252]** to apply to harmonic functions on N rather than linear functions on $[0,1]$. Using the localization lemma of **[252]**, the uniform convergence in C^2 reduces to a verification of orders of amplitudes where the analysis is carried out separately in the interior of the polytope and near its boundary.

Let us make note of one more relation between the geodesic segment problem and the harmonic mapping problem. In both cases a key aspect of the toric situation is that the Legendre transform linearizes the harmonic map equation. This was known previously for geodesics **[79,116,240]** (see also **[252]** for a simple proof), but is observed for the first time here for general harmonic maps. We refer the reader to the next Section where we also observe a generalization (146) of a well known formula from convex analysis and show that the Eells-Sampson harmonic map flow is Legendre transformed to the usual heat flow. It follows that one can explicitly solve the WZW and harmonic map equations in terms of the associated symplectic potentials. We make crucial use of this in proving the convergence of the Bergman harmonic maps, and it is the reason why, unlike in the general case, we do not need to work with weak solutions nor worry about the regularity of the limit harmonic map. Our results give the first proof of convergence for higher dimensional harmonic maps into spaces of Kähler metrics. The author hopes to discuss in a future article convergence results for WZW maps into general projective manifolds.

3.6 Legendre linearizing harmonic maps

We now turn to the proof of Theorem 3.25, that will occupy the rest of the present chapter. As a first step, the rest of this section is devoted to explaining how to solve the harmonic map equation and obtain the map $\varphi : N\to\mathcal{H}(T)$ alluded to in Theorem 3.25. The principle tool in this section is the Legendre transform. The Legendre transform first appeared in the context of toric manifolds in the work of Guillemin **[118]** and his approach lies at the heart of this section.

We let $h = e^{-\varphi}$ with φ a local Kähler potential on the open orbit (that does not extend globally). Over the open orbit of M, a toric Kähler potential may be identified with a convex function on $\mathbb{R}^n$, $\varphi = \varphi(\rho)$ in the logarithmic coordinates. By abuse of notation we will frequently identify $\mathcal{H}(T)$ (see (120)) with the local torus-invariant Kähler potentials defined on the open orbit. The gradient of $\varphi(\rho) = \varphi(e^\rho)$ is a one-to-one map, identified with the moment map μ, whose image is P. The Legendre transform takes Kähler potentials φ on the open orbit to symplectic potentials $u_\varphi := \mathcal{L}\varphi$, that are defined as convex functions on P with logarithmic singularities on ∂P, and relates the moment map, local symplectic potential $u_\varphi(x) = u_\varphi(\mu(\rho))$ and local Kähler potential as follows:

$$\begin{aligned} u_\varphi(x) &:= \langle x, 2\log\mu^{-1}(x)\rangle - \varphi(\mu^{-1}(x)) \\ &= \langle x, (\nabla\varphi)^{-1}(x)\rangle - \varphi\big((\nabla\varphi)^{-1}(x)\big). \end{aligned} \tag{137}$$

Any symplectic potential u can be written as $u_0 + f$, with respect to the canonical potential

$$u_0(x) = \sum_{k=1}^{d} l_k(x)\log l_k(x) \tag{138}$$

introduced by Guillemin, with f smooth up to the boundary [**118**]. Just as for Kähler potentials we may define the space of global symplectic potentials:

$$\mathcal{L}\mathcal{H}(T) = \{\, f \in C^\infty(P) \,:\, u_0 + f = \mathcal{L}\varphi \ \text{ with } \ \varphi \in \mathcal{H}(T)\}. \tag{139}$$

We will sometimes, by abuse of notation, identify elements of this space with their local symplectic potentials in the same manner as with $\mathcal{H}(T)$ itself.

Letting $G_\varphi = \nabla_x^2 u_\varphi$ we have

$$\det G_\varphi^{-1} = \delta_\varphi(x) \cdot \Pi_{r=1}^{d} l_r(x), \tag{140}$$

for some positive smooth function δ_φ defined on the whole polytope P [**1**].

Several authors observed previously that the Legendre transform linearizes the geodesic equation, so that a geodesic φ_t with endpoints φ_0, φ_1 is given by $\phi_t = \mathcal{L}^{-1}(\mathcal{L}\varphi_0 + t(\mathcal{L}\varphi_1 - \mathcal{L}\varphi_0))$ [**79,116,240**]. We now observe that under the Legendre transform, a harmonic map into $\mathcal{H}(T)$ is mapped to a family of symplectic potentials that are harmonic functions in the N variables.

Proposition 3.27. *Let $\psi : \partial N \to \mathcal{H}(T)$ be a smooth map. There exists a unique harmonic map φ from N to $\mathcal{H}(T)$ that agrees with ψ on ∂N. Moreover, $\varphi = \mathcal{L}^{-1}u$ where $u \in C^\infty(N \times P \setminus \partial P)$ satisfies $\Delta_N u = 0$ and $u|_{\partial N} = \mathcal{L}\psi$.*

Proof. The proof of the one-dimensional case [**252**, Proposition 2.1], carries over without difficulty. Indeed, harmonic maps into $\mathcal{H}(T)$ are stationary points of the functional,

$$E(\varphi) = \int_N |d\varphi|^2 dV_{N,f} = \int_{N\times M} f^{ab} \frac{\partial\varphi}{\partial y^a}\frac{\partial\varphi}{\partial y^b}\omega_\varphi^n \wedge dV_{N,f}. \tag{141}$$

Torus invariance allows us to integrate instead over the polytope: namely using (106) and (113),

$$\begin{aligned}(\nabla\varphi)_\star(\omega_\varphi^n) &= (\nabla\varphi)_\star\big((\det\nabla^2\varphi)d\rho_1\wedge d\theta_1\wedge\cdots\wedge d\rho_n\wedge d\theta_n\big)\\ &= (\nabla u)_\star^{-1}\big((\det\nabla^2\varphi)d\rho_1\wedge d\theta_1\wedge\cdots\wedge d\rho_n\wedge d\theta_n\big)\\ &= (\nabla u)^\star\big((\det\nabla^2\varphi)d\rho_1\wedge d\theta_1\wedge\cdots\wedge d\rho_n\wedge d\theta_n\big)\\ &= (\det\nabla^2\varphi)(\det\nabla^2 u)dx^1\wedge\cdots\wedge dx^n = dx.\end{aligned}$$

Considering a variation of (137) yields according to (108)

$$\left.\frac{\partial u}{\partial y^a}\right|_x = -\left.\frac{\partial\varphi}{\partial y^a}\right|_{\rho(x)}.$$

Therefore the functional $E(\varphi)$ equals

$$E(u) = \int_{N\times P} f^{ab}\frac{\partial u}{\partial y^a}\frac{\partial u}{\partial y^b}dx\wedge dV_{N,f} = \int_N |du|^2 dV_{N,f}. \tag{142}$$

Now the target space $\mathcal{LH}(T)$ is now flat with vanishing Christoffel symbols as can be seen by considering the metric

$$\langle\psi_1,\psi_2\rangle_{L^2(P,dx)}|_u = \int_P \psi_1\psi_2 dx, \qquad \psi_1,\psi_2\in T_u\mathcal{LH}(T),$$

that is independent of the point $u\in\mathcal{LH}(T)$. The Euler-Lagrange equation for $E(u)$ is therefore $\Delta_N u = 0$. Finally, note that since $u|_{\partial N} = \mathcal{L}\psi|_{\partial N}$ is convex on the boundary it is also convex in the interior of N; observe that from (152) it follows that the Hessian of u (in the M variables) for every $y\in N$ is given by

$$\nabla^2 u(y) = -\int_{\partial N}\nabla^2 u(q)\partial_{\nu(q)}G(y,q)dV_{\partial N,f}(q),$$

and since $-\partial_{\nu(q)}G(y,q)\geq 0$ (see paragraph after (152)) it follows that $\nabla^2 u(y)$ is therefore a positive-definite matrix. Therefore $\varphi := \mathcal{L}^{-1}u$ is in $\mathcal{H}_\omega(T)$ and solves the harmonic map equation with boundary values ψ, as required. □

In essence then, the Legendre tranform is an isometry from $(\mathcal{H}(T), g_{L^2})$ to the space $(\mathcal{LH}(T), \langle\,\cdot\,,\cdot\,\rangle_{L^2(P,dx)})$. While both spaces are flat with respect to their metrics the connection on the former is not trivial while the connection on the latter is identically zero. Hence all harmonic map equations on the latter involve only the geometry of the domain, i.e., reduce to the Laplace equation with respect to the Laplacian of N.

3.6.1 Transforming the Eells-Sampson flow to the heat flow. In this subsection we describe a rather neat duality between two geometric evolution equations on two different infinite-dimensional spaces. This result is not needed for the proof of the main result of this chapter, however it completes our previous results and it seems to the author that its elegance merits a discussion.

Let Γ^c_{ab} denote the Christoffel symbols of (N, f) with respect to local coordinates $y^1, \ldots, y^n$. Recall from (55) that the Christoffel symbols of $(\mathcal{H}, g_{L^2})$ are given by $\Gamma(\zeta, \eta)|_\varphi = -\frac{1}{2} g_\varphi(\nabla\zeta, \nabla\eta)$, $\forall \zeta, \eta \in T_\varphi\mathcal{H}$. The Eells-Sampson harmonic map heat flow [**87**] on the space of smooth maps from (N, f) to $(\mathcal{H}(T), g_{L^2})$ is given by

$$\partial_t\varphi = f^{ab}\partial_{y^a}\partial_{y^b}\varphi - f^{ab}\Gamma^c_{ab}\partial_{y^c}\varphi - \tfrac{1}{2} f^{ab} g(\nabla\partial_{y^a}\varphi, \nabla\partial_{y^b}\varphi), \tag{143}$$

while the heat flow on the space of symplectic potentials $\mathcal{LH}(T)$ is given by

$$\partial_t u = \Delta_N u. \tag{144}$$

Note that Equations (143) and (144) hold without change for the global Kähler and symplectic potentials, respectively (write all potentials with respect to a fixed Kähler form that does not depend on the parameters (variables of N)).

We record the following result that describes a basic property of the Legendre transform.

Theorem 3.28. *Under the Legendre tranform the Eells-Sampson harmonic map flow (143) on the space of Kähler potentials $\mathcal{H}(T)$ is mapped to the heat flow (144) on the space of symplectic potentials $\mathcal{LH}(T)$.*

Proof. As above, taking a variation of (137) yields

$$\frac{\partial u}{\partial t}|_x = -\frac{\partial \varphi}{\partial t}|_{\rho(x)}. \tag{145}$$

Intuitively, the equality of the energy functionals (141) and (142) then suggests that their Euler-Lagrange equations should coincide, however up to a sign, coming from the fact that an infinitesimal variation $\delta\psi(\rho)$ in one corresponds to an infinitesimal variation $-\delta\psi(x)$ in the second. More precisely, a direct computation gives:

$$-f^{ab}\partial_{y^a}\partial_{y^b}u + f^{ab}\Gamma^c_{ab}\partial_{y^c}u = f^{ab}\partial_{y^a}\partial_{y^b}\varphi - f^{ab}\Gamma^c_{ab}\partial_{y^c}\varphi - \tfrac{1}{2} f^{ab} g(\nabla\partial_{y^a}\varphi, \nabla\partial_{y^b}\varphi). \tag{146}$$

This can be seen as follows. First, the terms containing only first derivatives on each side are equal to each other by the first variation formula (145) for the Legendre transform. Next, choose coordinates where $f^{ab} = \delta^{ab}$. Then (114) gives

$$-\partial_{y^a}\partial_{y^a} u = \partial_{y^a}\partial_{y^b}\varphi + \langle \nabla \partial_{y^a} u, \nabla \partial_{y^a}\varphi \rangle.$$

But now

$$\partial_{x_j}(\partial_{y^a} u(x)) = -\partial_{x_j}(\partial_{y^a}\varphi(\rho)) = -\partial_{\rho_k}\partial_{y^a}\varphi \cdot \partial_{x_j}\rho_k = -\partial_{\rho_k}\partial_{y^a}\varphi \cdot \partial_{x_j}(\nabla u)_k.$$

Therefore using (113) and the fact that g_φ is represented in coordinates on the open orbit by $\nabla^2\varphi$ we see that (146) holds. Thus, the Legendre transform sends solutions of (143) to solutions of (144). □

This fact seems rather remarkable since in general one does not expect the Euler-Lagrange equations of two equal functionals defined on two different spaces to transform to each other. In essence what it says is that the Legendre tranform eliminates the Christoffel symbols not only in a variational sense but pointwise.

Observe that Equation (146) generalizes the well-known formula (114) from convex analysis

$$-\ddot{u} = \ddot{\varphi} + \langle \nabla \dot{u}, \nabla \dot{\varphi} \rangle$$

for the second variation a family of convex functions on $\mathbb{R}^n$ parametrized by $(\mathbb{R}, dx)$ that have the same gradient image. The factor $\frac{1}{2}$ in our formula comes from the conventions we used to relate the Riemannian and Kähler metrics.

3.7 Bergman approximation of harmonic maps

3.7.1 The quantum sequence of harmonic maps. Given a harmonic map $\varphi : N \longrightarrow \mathcal{H}(T)$ we now define the purported approximating sequence of harmonic maps $\varphi_k : N \rightarrow \mathcal{H}_k(T)$. First, given a family of toric Kähler metrics ψ parametrized by ∂N we project the family pointwise by $\mathrm{FS}_k \circ \mathrm{Hilb}_k$ onto $\mathcal{H}_k(T)$ to obtain a family of toric Bergman metrics parameterized by ∂N. Each of these metrics is determined by its L^2 norming constants, hence by the diagonal matrices

$$\mathrm{diag}(\mathcal{Q}_{h^k_{\psi(q)}}(\alpha))_{\alpha \in kP \cap \mathbb{Z}^n}, \quad q \in \partial N.$$

For each α, we solve the boundary problem

$$\begin{aligned} \Delta\lambda_\alpha(y) &= 0, \quad y \in N, \\ \lambda_\alpha &= \log \mathcal{Q}_{h^k_{\psi(q)}}(\alpha), \quad q \in \partial N. \end{aligned}$$

We then map back to $\mathcal{H}_k$ via FS_k to obtain the family

$$\varphi_k(y,z) = \frac{1}{k}\log \sum_{\alpha\in kP\cap\mathbb{Z}^n} e^{-\lambda_\alpha(y)}|\chi_\alpha(z)|^2_{h_0^k} \in \mathcal{H}_k(T)$$

of harmonic maps alluded to in Theorem 3.25. This may be written somewhat more explicitly in terms of the Green kernel:

$$\varphi_k(y,z) = \frac{1}{k}\log \sum_{\alpha\in kP\cap\mathbb{Z}^n} |\chi_\alpha(z)|^2_{h_0^k} \exp\Big(\int_{\partial N}\partial_{\nu(q)}G(y,q)\log||\chi_\alpha||^2_{h_{\psi(q)}}dV_{\partial N,f}(q)\Big).$$

3.7.2 An expression for the metric ratios. Our first aim is to prove the C^0 convergence by showing

$$\varphi_k(y,z) - \varphi(y,z) =$$

$$\frac{1}{k}\log \sum_{\alpha\in kP\cap\mathbb{Z}^n} |\chi_\alpha(z)|^2_{h^k_{\varphi(y)}} \exp\Big(\int_{\partial N}\partial_{\nu(q)}G(y,q)\log||\chi_\alpha||^2_{h_{\psi(q)}}dV_{\partial N,f}(q)\Big) = O(\tfrac{\log k}{k}). \tag{147}$$

We begin by rewriting the sum in a convenient way.

Put

$$\mathcal{R}_k(y,\alpha) := \exp\Big(-\int_{\partial N}\partial_{\nu(q)}G(y,q)\log\frac{\mathcal{Q}_{h^k_{\varphi(y)}}(\alpha)}{\mathcal{Q}_{h^k_{\psi(q)}}(\alpha)}dV_{\partial N,f}(q)\Big). \tag{148}$$

Then proving (147) is equivalent to proving

$$\frac{1}{k}\log \sum_{\alpha\in kP\cap\mathbb{Z}^n} \mathcal{R}_k(y,\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z) = O(\log k/k). \tag{149}$$

Put $u_y := u_{\varphi(y)}$, for $y\in N$. In light of the results in the geodesic case **[252]** we expect the asymptote of $\mathcal{R}_k$ to be the following:

Definition 3.29. *Let the metric volume ratio be the function on $N\times P$ defined by*

$$\mathcal{R}_\infty(y,x) := \exp\Big(-\tfrac{1}{2}\int_{\partial N}\partial_{\nu(q)}G(y,q)\log\frac{\det\nabla^2 u_y(x)}{\det\nabla^2 u_q(x)}dV_{\partial N,f}(q)\Big).$$

Note that by (140) we have

$$\mathcal{R}_\infty(y,x) = \exp\Big(-\tfrac{1}{2}\int_{\partial N}\partial_{\nu(q)}G(y,q)\log\frac{\delta_{\psi(q)}(x)}{\delta_{\varphi(y)}(x)}dV_{\partial N,f}(q)\Big),$$

and therefore $\mathcal{R}_\infty \in C^\infty(N \times P)$ (up to the boundary).

In light of Lemma 3.24 it is useful to express the ratio $\mathcal{R}_k$ in terms of the functions $\mathcal{P}_{h^k}(\alpha)$ (128) in the following form:

Lemma 3.30. *One has*

$$\mathcal{R}_k(y,\alpha) = \exp\Big(- \int_{\partial N} \partial_{\nu(q)} G(y,q) \log \frac{\mathcal{P}_{h^k_{\psi(q)}}(\alpha)}{\mathcal{P}_{h^k_{\varphi(y)}}(\alpha)} dV_{\partial N, f}(q)\Big).$$

Proof. By definition,

$$\mathcal{R}_k(y,\alpha) = \exp\Big(\int_{\partial N} \partial_{\nu(q)} G(y,q) \log \frac{\mathcal{Q}_{h^k_{\psi(q)}}(\alpha)}{\mathcal{Q}_{h^k_{\varphi(y)}}(\alpha)} dV_{\partial N, f}(q)\Big).$$

Specializing (137) to the lattice point α one has

$$u_\varphi(\alpha) = \langle \alpha, 2\log \mu^{-1}(\alpha)\rangle - \varphi(\mu^{-1}(\alpha)),$$

implying that

$$\log \mathcal{Q}_{h^k}(\alpha)\mathcal{P}_{h^k}(\alpha) = ku(\alpha/k). \tag{150}$$

Since u is harmonic in y it follows that

$$\log \mathcal{Q}_{h^k_{\varphi(y)}}(\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha) = - \int_{\partial N} \partial_{\nu(q)} G(y,q) \log \mathcal{Q}_{h^k_{\psi(q)}}(\alpha)\mathcal{P}_{h^k_{\psi(q)}}(\alpha) dV_{\partial N, f}(q),$$

which together with the definition concludes the proof. □

Example 3.31. We illustrate Lemma 3.30 in the linear (Bargmann-Fock) model, giving a different manner to compute the peak values than the one given in Example 3.22. Note that in that example it was simple to compute the values directly due to the explicitness of the model. However, the method we illustrate now is sometimes convenient in the more general toric situation.
Since $\varphi_{\mathrm{BF},k} = -\frac{1}{k}\log e^{-k|z|^2}$, we have by (107) $u_{\mathrm{BF}}(x) = \mathcal{L}(|z|^2) = \mathcal{L}(e^\rho) = x\log x - x$. By Lemma 3.30 we have the relation

$$\log \mathcal{P}_{h^k}(\alpha)\mathcal{Q}_{h^k}(\alpha) = ku(\alpha/k).$$

Therefore using (125) we have

$$\mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha) = \frac{k^{\alpha+1}}{\alpha!} e^{\alpha \log \alpha/k - \alpha} = \frac{k\alpha^\alpha}{\alpha!} e^{-\alpha}. \tag{151}$$

3.8 Convergence of the Bergman approximations

3.8.1 Convergence of the metric ratios. Let $n_0 = \dim_{\mathbb{R}} N$ and denote by $y_1, \ldots, y_{n_0}$ local coordinates over some coordinate patch $U \subset N$. We assume that N is oriented as a manifold with boundary, i.e., that the orientation on ∂N is the one induced from N. Recall that there always exists a Green function $G(y, q) \in C^\infty(N \times N \setminus \operatorname{diag}(N))$ for the Laplacian Δ_N on such a manifold and that the usual elliptic regularity theory applies [**9**,**112**]. In other words, if $v \in C^\infty(\partial N)$, the equations

$$\Delta_N u = 0, \text{ on } N,$$

$$u = v, \text{ on } \partial N,$$

have a unique smooth solution

$$u(\,\cdot\,) = -\int_{\partial N} v(q)\partial_{\nu(q)}G(\,\cdot\,, q)dV_{\partial N,f}(q) \tag{152}$$

(our convention will be that $G(y, q)$ is positive in the interior and vanishes when q is on the boundary), with $\nu(q) = \nu^i(q)\frac{\partial}{\partial y^i}|_q$ an outward unit normal to ∂N in N (coming from a Riemannian splitting $TN|_{\partial N} = T\partial N \oplus N_{\partial N}$, where $N_{\partial N}$ is the normal bundle to ∂N in N), and where we let $\partial_{\nu(q)}G(y, q) := \nu(q)G(y, q) \leq 0$ be the normal derivative with respect to the second argument, for all $q \in \partial N$. And secondly, there exists $C = C(N, f)$ such that

$$||u||_{C^{2,\frac{1}{2}}(N)} \leq C(||u||_{C^0(N)} + ||v||_{C^{2,\frac{1}{2}}(\partial N)}). \tag{153}$$

Moreover, the maximum principle implies that $||u||_{C^0(N)} \leq ||v||_{C^0(\partial N)}$, and so the estimates are only in terms of $||v||_{C^{2,\frac{1}{2}}(\partial N)}$.

First we need the following asymptotic regularity for the coefficients $\mathcal{R}_k(t, \alpha)$.

Lemma 3.32. *There exists a positive constant $C > 0$ such that for all k, y, α one has*

$$1/C < \mathcal{R}_k(y, \alpha) < C. \tag{154}$$

Moreover, $\log \mathcal{R}_k(y, \alpha)$ is uniformly bounded in $C^2(N)$, and for each $\delta \in (0, 1/2)$ we have

$$||\mathcal{R}_k(y, \alpha) - \mathcal{R}_\infty(y, \alpha/k)||_{C^2(N)} = O(k^{\delta - \frac{1}{2}}).$$

Proof. The Bargmann-Fock terms in (130) depend only on the geometry of P and not on $y \in N$ and so are cancelled in the ratio $\mathcal{R}_k(y, \alpha)$. Therefore, by Lemma 3.30,

$$\begin{aligned}\log \mathcal{R}_k(y,\alpha) &= \int_{\partial N} -\partial_{\nu(q)} G(y,q) \log \mathcal{P}_{h^k_{\psi(q)}} dV_{\partial N,f}(q) - \log \mathcal{P}_{h^k_{\varphi(y)}} \\ &= \frac{1}{2}\int_{\partial N} -\partial_{\nu(q)} G(y,q) \log\Big(\mathcal{G}_{\psi(q)}(\alpha/k)\Big) dV_{\partial N,f}(q) - \frac{1}{2}\log\Big(\mathcal{G}_{\varphi(y)}(\alpha/k)\Big) \\ &+ \int_{\partial N} -\partial_{\nu(q)} G(y,q) \log\Big(1 + R_k(\alpha/k, h_{\psi(q)})\Big) dV_{\partial N,f}(q) \\ &- \log\Big(1 + R_k(\alpha/k, h_{\varphi(y)})\Big).\end{aligned}$$

The first two terms simplify to

$$\frac{1}{2}\int_{\partial N} -\partial_{\nu(q)} G(y,q) \log \frac{\delta_{\varphi(y)}}{\delta_{\psi(q)}} dV_{\partial N,f}(q),$$

and this is uniformly in $C^2(N)$ according to the Schauder estimates (153). The fact that this is in $C^0(N)$, together with Lemma 3.24 imply the uniform estimate (154).

We now turn to prove the higher derivative estimates. A first derivative of the fourth term yields, according to (131),

$$\frac{S_1(y,\alpha,k) + R_k(\frac{\alpha}{k}, h_{\varphi(y)})}{1 + R_k(\frac{\alpha}{k}, h_{\varphi(y)})},$$

and this is uniformly bounded according to Lemma 3.24. In a similar fashion it follows that second derivatives are uniformly bounded as well. Finally, the Schauder estimates (153) may be invoked again for the third term and these will be uniform since the same argument as for the fourth term implies that $|| \log(1+R_k(\frac{\alpha}{k}, h_{\psi(q)}))||_{C^2(\partial N)}$ is uniformly bounded. □

3.8.1.1 The uniform convergence of the harmonic maps. Note that the estimate (154) immediately implies the C^0 convergence of φ_k to φ with the remainder as in (147) since we have an asymptotic expansion for the Szegő kernel that to first order equals

$$\Pi_{h^k_{\varphi(y)}}(z,z) = \sum_{\alpha \in kP\cap\mathbb{Z}^n} \mathcal{P}_{h^k_{\varphi(y)}}(\alpha, z) = 1 + O(k^{-1}). \tag{155}$$

Therefore in (149) we simply estimate each coefficient $\mathcal{R}_k(y,\alpha)$ by a uniform constant C to obtain (as y varies in a compact set N)

$$|\varphi(y,z) - \varphi_k(y,z)| \leq C\log k/k + \frac{1}{k}\log \Pi_{h^k_y}(z,z) \leq C\log k/k + C/k^2 = O(\log k/k).$$

This concludes the proof of the uniform convergence.

Remark 3.33. Note that here we get a better convergence rate than what we will for the higher order derivatives, ($O(\log k/k)$ instead of $O(k^{\delta-\frac{1}{2}})$ for $\delta > 0$). This seems to be an artifact of the choice of the size of the localizing balls in Lemma 3.23 rather than an essential difficulty. However if one localizes to smaller balls the remainder terms are more complicated to handle and so this does not seem to be worth the trouble, especially since this extra precision is of no additional help in getting C^3 convergence, the method seems to still fail for third and higher derivatives regardless of a possible refinement of the localization result. The problem is that one should explore more cancellations instead of the rather crude localization. This seems to be a very interesting problem to explore in the future, especially due to the statistical mechanical interpretation of the C^k-convergence problem as an absence of phase transitions of order less than or equal to k. We hope to return to this issue in the future.

3.8.2 Convergence of harmonic maps in the C^2-topology.

We now turn to the main technical part of the proof of Theorem 3.25 and show convergence of the first and second derivatives with precise asymptotics. In other words, our aim is now to show that the $C^2(N \times M)$ norm of the left hand side of (149) is still $O(k^{\delta-\frac{1}{2}})$. In order to prove these estimates it is crucial to make use of some cancellations. These can be understood as follows. When one replaces all the coefficients $\mathcal{R}_k(y,\alpha)$ by a constant, one reduces to the case of a zero-dimensional map, or equivalently to the known asymptotic expansion of the Szegő kernel that may be differentiated any number of times with a small error. Now there are two cases. When a coefficient $\mathcal{R}_k(y,\alpha)$ or a derivative thereof only multiplies a normalized monomial $\mathcal{P}_{h^k_{\varphi(y)}}(\alpha, z)$ it is enough to use the uniform estimates given by Lemma 3.32 and one does not need to keep track of error terms. However, as is usually the case, if the coefficient $\mathcal{R}_k(y,\alpha)$ or a derivative thereof multiplies another term that itself depends on k, one needs to keep track of the remainder of order $O(k^{\delta-\frac{1}{2}})$ given by Lemma 3.32. When such an error is introduced we simultaneously apply Lemma 3.23 to localize to those lattice points satisfying $|\alpha| \leq k^{\frac{1}{2}+\delta}$. Remembering the overall factor of $\frac{1}{k}$ one then estimates the remainders thus introduced.

3.8.2.1 Convergence of first and second space derivatives. Let us now consider derivatives solely in the M-directions. A derivative of (149) in the ρ_j directions amounts to multiplying each coefficient in the sum (149) by a factor of

$$k((\nabla\varphi_y)(z) - \tfrac{\alpha_j}{k})_j = k(\mu_y(z) - \tfrac{\alpha}{k})_j.$$

(recall that the moment map μ_y is the gradient of the open orbit Kähler potential

$\varphi(e^{\rho})$). Namely in the interior of P one has,

$$\frac{\partial}{\partial\rho_j}(\varphi_k-\varphi)(y,z)=\frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^n}k(\mu_y(z)-\frac{\alpha}{k})_j\mathcal{R}_k(y,\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha)}{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\mathcal{R}_k(y,\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)}.$$

The factor of k is cancelled by the overall factor of $\frac{1}{k}$ and the coefficients $\mathcal{R}_k(y,\alpha)$ are uniformly bounded due to (154). Thus by Lemma 3.23 one may restrict to those α such that $|\mu_y(z)-\frac{\alpha}{k}|\le k^{\delta-\frac{1}{2}}$ (introducing an error $O(k^{-M})$ for some large $M>0$). It follows then that

$$\frac{\partial}{\partial\rho_j}(\varphi_k-\varphi)(y,z)=O(k^{\delta-\frac{1}{2}}).$$

Near the boundary of P one performs the same computation but with respect to the slice-orbit coordinates (the same remark applies to all the computations in this Section). Note that the argument reduced to the one in [**252**, §7.2] once we had (154).

Next, we consider second derivatives in the M-directions. Symmetrizing sums (see [**252**, §8]) one obtains in the interior of P,

$$\begin{aligned}\frac{\partial^2}{\partial\rho_i\partial\rho_j}(\varphi_k-\varphi)(y,z)&=-\frac{\partial^2\varphi(y,z)}{\partial\rho_i\partial\rho_j}\\&+\frac{1}{k}\frac{\frac{1}{2}\sum_{\alpha,\beta\in kP\cap\mathbb{Z}^n}(\alpha_i-\beta_i)(\alpha_j-\beta_j)\mathcal{R}_k(y,\alpha)\mathcal{R}_k(y,\beta)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)\mathcal{P}_{h^k_{\varphi(y)}}(\beta,z)}{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\mathcal{R}_k(y,\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)}.\end{aligned} \tag{156}$$

Once again (154) allows to reduce the computations to those in the case $N=[0,1]$: After localizing (keeping only those α that are close to $\mu_y(z)$) we have the estimate $\frac{1}{k}(\alpha_i-\beta_i)(\alpha_j-\beta_j)=O(k^{2\delta})$. We then replace the coefficients $\mathcal{R}_k(y,\alpha)$ by the uniform constant (independent of α) $\mathcal{R}_\infty(y,\mu_y(z))$ at the price of an error $O(k^{3\delta-\frac{1}{2}})$. But now what is left is then precisely cancelled by $-\frac{\partial^2\varphi(y,z)}{\partial\rho_i\partial\rho_j}$ (up to an error of $O(k^{-2})$) due to the complete asymptotics of the Szegő kernel of a single metric. To prove this last claim, we consider the situation of a family of Szegő kernels parametrized by a compact manifold N, corresponding to the family of Hermitian metrics h_y, $y\in N$. In the toric situation this may be written explicitly as

$$\Pi_k(z,z):=\Pi_{h^k_y}(z,z)=\sum_{\alpha\in kP\cap\mathbb{Z}^n}\frac{|\chi_\alpha(z)|^2_{h^k_y}}{\mathcal{Q}_{h^k_y}(\alpha)}=\sum_{\alpha\in kP\cap\mathbb{Z}^n}\frac{e^{\langle\alpha,\rho\rangle-k\varphi_y}}{\mathcal{Q}_{h^k_y}(\alpha)}.$$

Then, $\varphi(y,z)+\frac{1}{k}\log\Pi_{h^k_y}(z,z)=O(k^{-1})$ (note the minus sign convention $\omega=-\frac{\sqrt{-1}}{8}\partial\bar\partial\varphi$ of §§§3.4.1.2) has a complete asymptotic expansion, and a first space derivative gives

$$\frac{\partial\varphi(y,z)}{\partial\rho_j} + O(k^{-2}) = \frac{\partial\varphi(y,z)}{\partial\rho_j} + \frac{1}{k}\frac{\partial\log\Pi_k(z,z)}{\partial\rho_j} = \frac{(\Pi_k(z,z))^{-1}}{k}\sum_{\alpha\in kP\cap\mathbb{Z}^n}\alpha_j\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)}. \tag{157}$$

Similarly a second space derivative takes the form

$$\begin{aligned}\frac{\partial^2\varphi(y,z)}{\partial\rho_i\partial\rho_j} + O(k^{-2}) &= \frac{(\Pi_k(z,z))^{-2}}{k}\Big[\sum_{\alpha\in kP\cap\mathbb{Z}^n}\alpha_i\alpha_j\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)}\sum_{\beta\in kP\cap\mathbb{Z}^n}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\beta)} \\ &\quad - \sum_{\alpha\in kP\cap\mathbb{Z}^n}\alpha_i\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)}\sum_{\beta\in kP\cap\mathbb{Z}^n}\beta_j\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\beta)}\Big] \\ &= \frac{(\Pi_k(z,z))^{-2}}{k}\frac{1}{2}\sum_{\alpha,\beta\in kP\cap\mathbb{Z}^n}(\alpha_i-\beta_i)(\alpha_j-\beta_j)\frac{e^{\langle\alpha+\beta,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)\mathcal{Q}_{h_y^k}(\beta)} \end{aligned} \tag{158}$$

(by symmetrizing sums). In conclusion we have proved the claim, and hence the convergence of the second space derivatives.

3.8.2.2 Convergence of first parameter derivatives. First we consider one derivative in the N-directions. One has

$$\begin{aligned}&\frac{\partial}{\partial y^a}(\varphi_k-\varphi)(y,z) \\ &= -\frac{\partial\varphi}{\partial y^a} + \frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}\partial_{y^a}\log\frac{\mathcal{R}_k(y,\alpha)}{\mathcal{Q}_{h^k(y)}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}}.\end{aligned} \tag{159}$$

Since the asymptotic expansion (155) can be differentiated and is uniform over compact families **[299]** we have

$$O(k^{-2}) = \frac{1}{k}\frac{\partial}{\partial y^a}\log\Pi_{h^k_{\varphi(y)}}(z,z) = -\frac{\partial\varphi}{\partial y^a} + \frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}\partial_{y^a}\log\frac{1}{\mathcal{Q}_{h^k(y)}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}}. \tag{160}$$

First note that the term

$$\frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}\partial_{y^a}\log\mathcal{R}_k(y,\alpha)}{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}}$$

is of order $O(k^{-1})$ by Lemma 3.32. Thus, we are left with the task of comparing the last term of (160) with

$$\frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}\partial_{y^a}\log\frac{1}{\mathcal{Q}_{h^k(y)}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^n}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}}$$

We now localize the sums about the image of the moment map using Lemma 3.23 introducing negligible errors (of arbitarily high order $O(k^{-M})$). Then we use Lemma 3.32 to replace each occurrence of $\mathcal{R}_k(y,\alpha)$ by $\mathcal{R}_\infty(y,\mu_y(z))$ plus an error of order $O(k^{\delta-1/2})$. In the terms involving $\mathcal{R}_\infty(y,\mu_y(z))$ the factors of $\mathcal{R}_\infty(y,\mu_y(z))$ actually cancel and so cancel with the last term of (160) after localizing the latter.

It remains to show that the error term overall contributes $O(k^{\delta-1/2})$ to the sum (159). To that end, because of the factor of $1/k$ it is enough to show that there exists a uniform constant $C>0$ independent of k such that

$$|\partial_{y^a}\log\mathcal{Q}_{h^k(y)}(\alpha)|\leq Ck. \tag{161}$$

Recall that the duality (150) of Lemma 3.30 implies

$$\partial_{y^a}\log\mathcal{Q}_{h^k_{\varphi(y)}}(\alpha)=k\partial_{y^a}u_y(\alpha/k)-\partial_{y^a}\log\mathcal{P}_{h^k_{\varphi(y)}}(\alpha). \tag{162}$$

The second term of the right hand side is uniformly bounded by applying (131). To evaluate the first term recall that

$$u_y=-\int_{\partial N}u_q\partial_{\nu(q)}G(y,q)dV_{\partial N,f}(q),\quad y\in N. \tag{163}$$

In terms of canonical symplectic potential u_0 of (138) one may write $u_y=u_0+f_y$ for some globally smooth function $f_y\in\mathcal{LH}(T)$ on P and thus we have

$$\partial_{y^a}u_y=\partial_{y^a}f_y=-\partial_{y^a}\int_{\partial N}f_q\partial_{\nu(q)}G(y,q)dV_{\partial N,f}(q),\quad y\in N. \tag{164}$$

This is uniformly bounded according to the Schauder estimates. Combining the above the estimate (161) follows.

In sum we have shown that

$$\frac{\partial}{\partial y^a}(\varphi_k-\varphi)(y,z)=O(k^{\delta-1/2}),$$

which concludes the case of a single N-derivative.

3.8.2.3 Convergence of mixed derivatives. We now consider the case of mixed second derivatives. We will always assume $\alpha, \beta \in kP \cap \mathbb{Z}^n$ and so omit that from the summation notation in what follows. To simplify the notation further we will fix a point $(y, z) \in N \times M$ and use the following abbreviations:

$$\partial_a := \frac{\partial}{\partial y^a},\ \partial_{ab} := \frac{\partial^2}{\partial y^a \partial y^b},$$
$$\mathcal{R}_\alpha := \mathcal{R}_k(y, \alpha),\ \mathcal{Q}_\alpha := \mathcal{Q}_{h^k_{\varphi(y)}}(\alpha),\ \mathcal{P}_\alpha := \mathcal{P}_{h^k_{\varphi(y)}}(\alpha),\ \widetilde{\mathcal{P}}_\alpha := \frac{e^{\langle \alpha, \rho \rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}.$$

Symmetrizing sums again, it follows that

$$\frac{\partial^2}{\partial y^a \partial \rho^j}(\varphi_k - \varphi)(y, z) = -\frac{\partial^2 \varphi(y, z)}{\partial y^a \partial \rho^j}$$
$$+\frac{1}{k}\frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha_j - \beta_j)\frac{\partial}{\partial y^a}\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{Q}_\alpha}\frac{\mathcal{Q}_\beta}{\mathcal{R}_\beta}\right)\mathcal{R}_\alpha \mathcal{R}_\beta \widetilde{\mathcal{P}}_\alpha \widetilde{\mathcal{P}}_\beta}{\left(\sum_\alpha \mathcal{R}_\alpha \widetilde{\mathcal{P}}_\alpha\right)^2}. \tag{165}$$

Localizing sums exchanges the term $(\alpha_j - \beta_j)$ for $O(k^{\frac{1}{2}+\delta})$ up to an error of $O(k^{-M})$ for some large $M > 0$. By applying Lemma 3.32 the term

$$\frac{1}{k}\frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha_j - \beta_j)\frac{\partial}{\partial y^a}\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right)\mathcal{R}_\alpha \mathcal{R}_\beta \widetilde{\mathcal{P}}_\alpha \widetilde{\mathcal{P}}_\beta}{\left(\sum_\alpha \mathcal{R}_\alpha \widetilde{\mathcal{P}}_\alpha\right)^2}$$

is $O(k^{\delta-1/2})$. Now we replace the coefficients $\mathcal{R}_k$ in (165) for the lattice points α that remain (near $\mu_y(z)$) by the uniform constant $\mathcal{R}_\infty(y, \mu_y(z))$ plus an error of order $O(k^{\delta-1/2})$. Any term that does not multiply $\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}$ is of order $O(k^{\delta-1/2})$, or smaller, by using Lemma 3.32. Now there are two kinds of terms left to estimate. The first involve $\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}$ with all $\mathcal{R}_k$ replaced by $\mathcal{R}_\infty(y, \mu_y(z))$. Then the coefficients $\mathcal{R}_\infty(y, \mu_y(z))$ cancel out and we are left with the Szegő kernel approximation of $\frac{\partial^2 \varphi(y,z)}{\partial y^a \partial \rho^j}$ (up to $O(k^{-2})$) and this cancels with $-\frac{\partial^2 \varphi(y,z)}{\partial y^a \partial \rho^j}$ appearing in (165). The second type of contributions comes from error terms of order $O(k^{\delta-1/2})$ multiplying $\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}$. By using (162) we may express $\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}$ in terms of the global symplectic potentials and using in addition the fact that α and β are localized to a neighborhood of size comparable to $k^{1/2+\delta}$ about $k\mu_y(z)$ we obtain

$$\left|\partial_a \log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\right| \leq C + k|\partial_a f_y(\beta/k) - \partial_a f_y(\alpha/k)| \leq C + kC_1|(\alpha - \beta)/k| = O(k^{\frac{1}{2}+\delta}), \tag{166}$$

where C_1 is the Lipschitz constant of the smooth function $\partial_a f_y$ (as a function of N). By the maximum principle C_1 is uniformly bounded in terms of the boundary data (i.e., $\mathrm{FS}_k \circ \mathrm{Hilb}_k(\psi)$) since $\partial_a f_y$ is in fact harmonic in the N-variables. This implies that the second type of contributions are of order $O(k^{2\delta-1})$, due to the overall factor of $1/k$.

Note that the symmetrization of the sums was crucial here. In sum,

$$\frac{\partial^2}{\partial y^a \partial \rho^j}(\varphi_k - \varphi)(y, z) = O(k^{\delta-\frac{1}{2}}).$$

3.8.2.4 Convergence of second parameter derivatives. Finally, we consider the case of two derivatives in the N-directions. This case is somewhat more involved than the previous ones and unlike in the case $N = [0, 1]$ we also need to consider mixed N-derivatives.

We have

$$\begin{aligned}\partial_{ab}(\varphi_k - \varphi)(y, z) &= -\partial_{ab}\varphi \\ &+ \frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\mathcal{R}_\beta\widetilde{\mathcal{P}}_\beta \cdot \left[\partial_{ab}\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{Q}_\alpha}\right) + \frac{1}{2}\partial_a\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{Q}_\alpha}\frac{\mathcal{Q}_\beta}{\mathcal{R}_\beta}\right)\partial_b\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{Q}_\alpha}\frac{\mathcal{Q}_\beta}{\mathcal{R}_\beta}\right)\right]}{\left(\sum_\alpha\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}}. \end{aligned} \tag{167}$$

We rewrite this as

$$\partial_{ab}(\varphi_k - \varphi)(y, z) = -\partial_{ab}\varphi + A + B + C,$$

where

$$A = \frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\mathcal{R}_\beta\widetilde{\mathcal{P}}_\beta \cdot \left[\partial_{ab}\log\left(\frac{1}{\mathcal{Q}_\alpha}\right) + \frac{1}{2}\partial_a\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\partial_b\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\right]}{\left(\sum_\alpha\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}}, \tag{168}$$

$$B = \frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\mathcal{R}_\beta\widetilde{\mathcal{P}}_\beta \cdot \left[\partial_{ab}\log\left(\mathcal{R}_\alpha\right) + \frac{1}{2}\partial_a\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right)\partial_b\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right)\right]}{\left(\sum_\alpha\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}}, \tag{169}$$

$$C = \frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\mathcal{R}_\beta\widetilde{\mathcal{P}}_\beta \cdot \left[\partial_a\log\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\partial_b\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right]}{\left(\sum_\alpha\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}}. \tag{170}$$

By differentiating (155) we obtain similarly (analogously to the computations leading to (158)),

$$O(k^{-2}) = -\partial_{ab}\varphi + \frac{1}{k}\frac{\sum_{\alpha,\beta}\widetilde{\mathcal{P}}_\alpha\widetilde{\mathcal{P}}_\beta \cdot \left[\partial_{ab}\log\left(\frac{1}{\mathcal{Q}_\alpha}\right) + \frac{1}{2}\partial_a\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\partial_b\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\right]}{\left(\sum_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}}. \tag{171}$$

We now localize the sums to a ball of radius $k^{\frac{1}{2}+\delta}$ about $k\mu_y(z)$ introducing a negligible remainder/error and replace the two occurrences of $\mathcal{R}_\gamma$ outside of the square brackets in (167) as well as in the denominator by the lattice-point-independent constant $\mathcal{R}_\infty(y, \mu_y(z))$ (in what follows we refer to this operation as "replacement") introducing an error of order $O(k^{\delta-\frac{1}{2}})$ for each replacement. First, observe that the replacement in the denominator is negligible. What remains to be checked is that the overall error introduced by replacements elsewhere is of order $O(k^{\delta-\frac{1}{2}})$.

In the term A these replacements introduce a term that is cancelled by substituting the expression for $-\partial_{ab}\varphi$ given by (171) into (167). In addition we introduce error terms. Let $\epsilon_\gamma := \mathcal{R}_\gamma - \mathcal{R}_\infty(\mu_y(z)) = O(k^{\delta-\frac{1}{2}})$ for each lattice point γ in a localized sum. The highest order remainders are terms of the form $\frac{1}{k}\epsilon_\gamma \cdot (B_1 + B_2 + B_3)$ where

$$\begin{aligned}
B_1 &:= B_{1,1} + B_{1,2} := \partial_{ab} \log\left(\frac{1}{\mathcal{Q}_\alpha}\right) + \tfrac{1}{2}\partial_a \log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right) \partial_b \log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right), \\
B_2 &:= \partial_{ab} \log \mathcal{R}_\alpha + \tfrac{1}{2}\partial_a \log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right) \partial_b \log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right), \\
B_3 &:= \partial_a \log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right) \partial_b \log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right).
\end{aligned}$$

The errors introduced in the replacements in the term A are of the form $\frac{1}{k}\epsilon_\gamma B_1$. The errors introduced in the replacements of the terms B and C are $\frac{1}{k}\epsilon_\gamma B_2$ and $\frac{1}{k}\epsilon_\gamma B_3$, respectively.

First, $\frac{1}{k}\epsilon_\gamma B_2 = O(k^{\delta-\frac{3}{2}})$ by Lemma 3.32. To bound B_1 and B_3 we will use the estimate (166) for the derivatives of the norming constants $\mathcal{Q}_\alpha$. First, it gives directly that $\frac{1}{k}\epsilon_\gamma B_3 = O(k^{2\delta-1})$. Second, by squaring (166) we also obtain $\frac{1}{k}\epsilon_\gamma B_{1,2} = O(k^{3\delta-\frac{1}{2}})$. Finally, the maximum principle also gives, similarly to the argument proving (161), that

$$\left|\partial_{ab} \log \mathcal{Q}_\alpha\right| \le C + k|\partial_{ab} f_y(\alpha/k)| \le C_2 k,$$

and thus $\frac{1}{k}\epsilon_\gamma B_{1,1} = O(k^{\delta-\frac{1}{2}})$. Altogether we have shown that all the remainders introduced by the replacements are of order $O(k^{\delta-\frac{1}{2}})$.

Hence we have shown that

$$\partial_{ab}(\varphi_k - \varphi)(y, z) = O(k^{\delta-\frac{1}{2}}),$$

and this concludes the proof of Theorem 3.25.

Ma tal mutamento
È credibile appena
— Giuseppe Verdi, *Un ballo in maschera* (1859)

E tu m'appronta un abito
Da pescator
— Giuseppe Verdi, *Un ballo in maschera* (1859)

CHAPTER 4

Canonical metrics and dynamical constructions

The results of this chapter comprise about three-quarters of the results of this Thesis. Our main purpose in this chapter is to propose the systematic use of certain discretizations of geometric evolution equations as an approach to the study of the existence problem of certain elliptic partial differential equations of a geometric nature as well as a means to obtain interesting dynamics on certain infinite-dimensional spaces. We illustrate the fruitfulness of this approach in the context of the Ricci flow, as well as other flows, in Kähler geometry. We describe how in this context this approach gives a new method for the construction of canonical Kähler metrics. We also introduce a number of canonical dynamical systems on the space of Kähler metrics that we believe merit further study. The majority of the results and constructions described here were described in detail in the author's articles [**227**,**228**,**230**]. This chapter is an expanded union of these articles. Some more in-depth discussions are given for some topics, however with the exception of the results of §§4.4.5 on energy functionals and geometric stability in terms of GIT the rest of the results are contained in the aforementioned articles.

We now outline the contents of this chapter. Except Section 4.2 each section contains new contributions. In Section 4.1 we explain the main idea of this chapter of studying geometric PDEs via discretization of natural geometric flows. In Section 4.2 we give a very brief tour of the study of canonical metrics in Kähler geometry. Then in Section 4.3 we introduce the Ricci iteration, obtained from discretizing the Ricci flow, and discuss its most basic properties as well as state a result on its convergence. In the rather lengthy Section 4.4 we first give an introduction to the study of energy functionals on the space of Kähler metrics. Then we introduce a family of energy functionals complementary to one introduced previously by Chen-Tian and study some of its properties (§§4.4.1). Next we state and prove a formula that succinctly relates the Chen-Tian functionals to the K-energy and the Ricci energy (§§4.4.2), and the Berger-Moser-Ding functional to the Ricci energy (§§4.4.3). These are the main results of this section. We make use of these results to prove monotonicity of these functionals along the iteration (§§4.4.4). We end this Section with a detailled discussion of Bott-Chern forms and prove that the Chen-Tian functionals can be expressed in terms of such forms (§§4.4.5). We discuss a relation between this result and geometric stability in terms of GIT. In Sections 4.5–4.6 we prove a result on the convergence of the Ricci iteration (Theorem 4.5) dividing the proof into cases according to the sign of the first Chern class. In Section 4.7 we introduce the Ricci iteration for a general Kähler class, while in Section 4.8 we introduce a geometric flow whose discretization provides a generalized inverse Ricci operator. Section 4.9 discusses twisted versions of the earlier constructions, where we also observe a monotonicity result for a twisted energy functional along this flow.

In the second part of this chapter, Section 4.10, we discuss applications of the first part to several well-studied objects in Kähler and conformal geometry. In §§4.10.1 we provide a new proof and entirely geometric proof of the the classical Moser-Trudinger-Onofri inequality on S^2. This involves a new geometric interpretation of the Berger-Moser-Ding energy functional introduced originally for the study of the Nirenberg problem in the late 1960's. We also find a canonical enlargement of the space of Kähler metrics that we call the Moser-Trudinger-Onofri neighborhood on which this inequality continues to hold in higher dimensions and we prove an estimate for its size in terms of a lower bound on Ricci curvature. In addition we show that the Ricci energy is unbounded from below for any Fano manifold of dimension greater than one. In §§4.10.2 we prove a new characterization of Fano Kähler-Einstein manifolds in terms of the Chen-Tian energy functionals. We also obtain new criterion for the existence of almost-Kähler-Einstein metrics. The results here answer two questions posed by X.-X. Chen. In §§4.10.3 we obtain a new Moser-Trudinger-Onofri inequality on Fano Kähler-Einstein manifolds using the Ricci iteration. This result is new also for S^2. In §§4.10.4 we prove a result on the limiting behavior of the Ricci iteration in the absence of a Kähler-Einstein metric in the language of multiplier ideal sheaves. In §§4.10.5 we point out a relation to Donaldson's work on balanced metrics. In §§4.10.6 we return to a question that motivated this chapter posed by Nadel on the existence of periodic orbits of the Ricci operator. We generalize this question to the Bakry-Émery operator and show that no nontrivial periodic orbits exist. Finally, in §§4.10.7 we introduce the notion of the Ricci index and pose several questions regarding its properties and the structure of the space of Kähler metrics.

4.1 Discretization of geometric evolution equations

Given an elliptic partial differential equation, several classical methods are available to approach the problem of existence of solutions. In essence, standard elliptic theory reduces the existence problem to the demonstration of certain a priori estimates for solutions. The main difficulty lies therefore in devising methods to obtain these estimates.

One common method, that goes back at least to Bernstein and Poincaré, is the continuity method.[13] In this approach one continuously deforms the given elliptic operator to another (oftentimes in a linear fashion), for which the existence problem is known to have solutions. Ellipticity provides for existence of solutions for small perturbations of this easier problem. In order to prove existence for the whole deformation path one then seeks to establish a priori estimates, uniform along the deformation, for solutions of the family of elliptic problems.

Another approach, drawing some of its motivation from Physics, is the heat flow method, going back to Fourier. Here the idea is study a deformation of the elliptic problem according to a parabolic heat equation whose equilibrium state is precisely a solution to the original elliptic equation. Much of the standard elliptic theory has a parabolic counterpart. First, one makes use of the latter in order to establish short-time existence. Long-time existence and convergence then hinge upon establishing a priori estimates, as before.

A third approach, going back, among others, to Euler and Cauchy, is the discretization method, that can be considered as a blend of the two above. Here the idea is to replace an evolution equation (or "flow") by a countable set of elliptic equations that arise by repeatedly solving a difference equation corresponding to discretizing the flow equation in the time variable. This approach provides common and elementary numerical algorithms, the Euler method and its variants, and is widely used in the "real world," for example to numerically integrate differential equations.

In the present work we wish to explore this third approach in the context of certain geometric evolution equations. To the best of our knowledge, it seems that it has not been used before in a systematic manner in this context. In this Thesis we will concentrate on the Ricci flow and other related flows in the context of Kähler geometry. We hope that this circle of ideas might find applications also in other geometric situations involving other flows. For example, to mention one, we hope to explore in the future these ideas in the context of the Yang-Mills flow and Hermitian-Einstein metrics.

We would like to emphasize that when a particular elliptic equation has a solution one morally expects all three methods to converge towards such a solution. Therefore one should not take as a surprise the fact that the discretization method converges in some of the cases we consider. The crux is thus not the convergence itself but rather the new point of view and insights that this method provides; both to the study of the original elliptic problem as well as to the understanding of the evolution equation, the continuity method and the relation between the two. In addition, one may obtain in this way non-trivial canonical discrete dynamical systems on infinite-dimensional spaces that may be of some interest in their own right.

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[13] For some historical background regarding PDEs we refer to Brezis and Browder [**36**].

Let us consider as a simple illustration the Laplace equation on a bounded smooth domain Ω in $\mathbb{R}^n$. The elliptic problem is then to find a function u satisfying

$$\begin{aligned} \Delta u &= 0, \quad \text{on } \Omega, \\ u &= \psi, \quad \text{on } \partial\Omega. \end{aligned} \tag{172}$$

Consider then the difference equations

$$\begin{aligned} u_k - u_{k-1} &= \Delta u_k, \quad \text{on } \Omega, \\ u_k &= \psi, \quad \text{on } \partial\Omega, \\ u_0 &= u, \end{aligned}$$

where u is any smooth function that agrees with the smooth function ψ on the boundary. Thus, one may write

$$u_k = (-\Delta + 1)^{-1} \circ \cdots \circ (-\Delta + 1)^{-1} u.$$

One may then readily show that the sequence $\{u_k\}_{k \geq 0}$ exists for each $k \in \mathbb{N}$ and converges exponentially fast in k to the unique solution of (172).

To give further intuition as to why this method works we consider the following finite-dimensional problem: Given a positive semi-definite matrix A and a vector v, find the projection of v onto the zero eigenspace of A. One possible solution is to consider the sequence of vectors defined iteratively by

$$\begin{aligned} v_k &= (A + I)^{-1} v_{k-1}, \\ v_0 &= v. \end{aligned}$$

Then $\lim_{k \to \infty} v_k$ exists and is the required projection. This algorithm is nothing but the discretization of the flow

$$\begin{aligned} \frac{dv(t)}{dt} &= -Av(t), \\ v(0) &= v. \end{aligned}$$

Discretizations corresponding to different time steps will produce equivalent dynamical systems

$$\begin{aligned} v_k &= (\tau A + I)^{-1} v_{k-1}, \\ v_0 &= v, \end{aligned}$$

whose convergence is faster the larger the time-step $\tau \in (0, \infty)$, with τ and the first non-zero eigenvalue of A controlling the exponential factor of the speed of convergence.

4.2 Constructing canonical metrics in Kähler geometry

In this chapter we wish to apply the method described in the previous section towards the study of canonical Kähler metrics and the space of Kähler metrics. In this section we very briefly describe the problem and some background. We refer to **[9,21,106,247,271]** for more background.

The search for a canonical metric representative of a fixed Kähler class has been at the heart of Kähler geometry since its birth. Indeed, in his visionary article Kähler defined the eponymous manifold motivated by the fact that in this setting Einstein's equation simplifies considerably and reduces to a second order partial differential equation for a single function **[141]**, namely, the local potential u which represents the Kähler form ω on the open domain U via $\omega|_U = \sqrt{-1}\partial\bar\partial u$ must satisfy

$$\det\left[\frac{\partial^2 u}{\partial z^i \partial \bar z^j}\right] = e^{-\mu u}, \quad \text{on } U,$$

where μ is the Einstein constant. Two decades later, following Chern's fundamental work on characteristic classes, Calabi introduced the concept of the space of Kähler metrics in a fixed cohomology class and formulated the problem on a compact closed manifold as an equation for a global smooth function (Kähler potential) φ,

$$\omega_\varphi^n = \omega^n e^{f_\omega - \mu\varphi},$$

where f_ω satisfies $\sqrt{-1}\partial\bar\partial f_\omega = \operatorname{Ric}\omega - \mu\omega$. This showed that a necessary condition for the existence of solutions is that the first Chern class be definite or zero. Calabi proposed that this equation should always admit a unique solution in each Kähler class when $\mu = 0$. In addition he suggested the study of a more general notion, that of an extremal metric **[39,40]**. Since then much progress has been made towards understanding when such metrics exist. Regarding Kähler-Einstein metrics, the most general result in this direction is given by the work of Aubin in the negative Ricci curvature case **[7]** and by Yau in the case of nonpositive Ricci curvature which provided a solution to Calabi's conjecture **[41,296]**. The latter result, the Calabi-Yau Theorem, is one of the cornerstones of Kähler geometry and can be stated as follows.

Theorem 4.1. (See **[296]**.) *Let (M, J) be a compact closed Kähler manifold. Let F be a smooth function on M. The Monge-Ampère equation*

$$\omega_\varphi^n = e^F \omega^n$$

has a unique smooth solution (up to a constant) φ provided that $\int_M e^F\omega^n = \int_M \omega^n$.

The key steps in proving this result are to show an a priori estimate for $||\varphi||_{L^\infty(M)}$ depending only on M and $||F||_{L^\infty(M)}$, an a priori estimate for $||\Delta_\omega\varphi||_{L^\infty(M)}$ in terms of the L^∞ estimate, and finally an a priori estimate for $||\varphi||_{C^{2,\alpha}(M,\omega)}$ in terms of the previous two estimates (some additional references for the theory of elliptic complex Monge-Ampère equations in Kähler geometry are [**25,247,271**]). As a corollary one has the following solution of the Calabi conjecture:

Corollary 4.2. (See [**296**].) *Let* (M, J, ω) *be a compact closed Kähler manifold and let* α *be a smooth closed form satisfying* $[\alpha] = c_1$. *There exists a unique smooth function* φ *(up to a constant) such that*

$$\mathrm{Ric}\,\omega_\varphi = \alpha.$$

We mention as a side remark that one of the ideas of the present Chapter is that the Calabi-Yau Theorem may be exploited to many more uses than it has been up to now. The Calabi-Yau Theorem has been widely used to solve Monge-Ampère equations related to Kähler-Einstein metrics. We will suggest (see Section 4.8) that in fact the Calabi-Yau Theorem could possibly be able to construct also constant scalar curvature metrics and extremal metrics via iterative dynamical constructions that use it an infinite number of times (instead of just once). At the moment though the technology to bring this line of research to complete fruition is still not available.

Following the solution of the Calabi conjecture much work has gone into understanding the positive case, notably by Tian who provided a complete solution for complex surfaces, in addition to establishing an analytic characterization of Kähler-Einstein manifolds and a theory of stability [**267,270**]. For general extremal metrics however a general existence theory is not presently available although a conjectural picture, the so-called Yau-Tian-Donaldson conjecture, suggests that it should be intimately related with notions of stability in algebraic geometry [**263**].

The principal tool in the study of Kähler-Einstein metrics has been the continuity method, as suggested initially by Calabi [**41**], and later studied by Aubin and Yau. In the remaining case ($\mu > 0$) Bando and Mabuchi showed that the continuity method will converge to a Kähler-Einstein metric when one exists [**15**]. Another important tool has been the Ricci flow introduced by Hamilton [**124**] in the more general setting of Riemannian manifolds. Cao has shown that the continuity method proofs for the cases $\mu \leq 0$ may be phrased in terms of the convergence of the Ricci flow [**48**]. Later, much work has gone into understanding the Ricci flow on Fano manifolds and recently Perelman and Tian and Zhu proved that the analogous convergence result holds in this case [**280**].

The idea that there might be another way of approaching canonical metrics, in the form of a discrete infinite-dimensional iterative dynamical system, was suggested by Nadel [**200**]. More recently, Donaldson has proposed a program for the construction of constant scalar curvature Kähler metrics on projective manifolds using finite-dimensional iteration schemes and balanced metrics which are in essence computable [**83**].

The main motivation for our work came from trying to approach Nadel's basic problem: Find an (infinite-dimensional) iterative dynamical system on the space of Kähler metrics that converges to a Kähler-Einstein metric. In his note Nadel suggested one such dynamical system on Fano manifolds which, as we explain below, is not suited to the problem (see §§4.10.6). Nevertheless his idea is related to the right answer, and it is our purpose here to describe our approach to Nadel's problem and some of its consequences.

4.3 The Ricci iteration

In this section we introduce the Ricci iteration and describe some of its elementary properties.

Hamilton's Ricci flow on a Kähler manifold of definite or zero first Chern class is defined as the the set $\{\omega(t)\}_{t\in\mathbb{R}_+}$ satisfying the evolution equations

$$\begin{aligned} \frac{\partial\omega(t)}{\partial t} &= -\mathrm{Ric}\,\omega(t)+\mu\omega(t), \quad t\in\mathbb{R}_+, \\ \omega(0) &= \omega\in\mathcal{H}_\Omega, \end{aligned} \tag{173}$$

where Ω is a Kähler class satisfying $\mu\Omega=c_1$ for some $\mu\in\mathbb{R}$ (see, e.g., [**72**]). We will sometimes refer to the equations (173) themselves as the Ricci flow.

We introduce the following dynamical system that is our main object of study in this chapter. It is a discrete version of this flow.

Definition 4.3. *Let Ω be a Kähler class satisfying $\mu\Omega=c_1$ for some $\mu\in\mathbb{R}$. Given a Kähler form $\omega\in\mathcal{H}_\Omega$ and a number $\tau>0$ define the time τ Ricci iteration to be the sequence of forms $\{\omega_{k\tau}\}_{k\geq0}\subseteq\mathcal{H}_\Omega$, satisfying the equations*

$$\begin{aligned} \omega_{k\tau} &= \omega_{(k-1)\tau}+\tau\mu\omega_{k\tau}-\tau\mathrm{Ric}\omega_{k\tau}, \quad k\in\mathbb{N}, \\ \omega_0 &= \omega, \end{aligned} \tag{174}$$

for each $k\in\mathbb{N}$ for which a solution exists in $\mathcal{H}_\Omega$.

We pose the following elementary conjecture concerning the limiting behavior of the Ricci iteration in the presence of fixed points.

Conjecture 4.4. *Let (M,J) be a compact Kähler manifold admitting a Kähler-Einstein metric. Let Ω be a Kähler class such that $\mu\Omega=c_1$ with $\mu\in\mathbb{R}$. Then for any $\omega\in\mathcal{H}_\Omega$ and for any $\tau>0$, the time τ Ricci iteration exists for all $k\in\mathbb{N}$ and converges in the sense of Cheeger-Gromov to a Kähler-Einstein metric.*

Regarding this conjecture we prove in this chapter the following result.[14]

Theorem 4.5. *Let* (M, J) *be a compact Kähler manifold admitting a unique Kähler-Einstein metric. Let* Ω *be a Kähler class such that* $\mu\Omega = c_1$ *with* $\mu \in \mathbb{R}$. *Then for any* $\omega \in \mathcal{H}_\Omega$ *and for any* $\tau > 1/\mu$ *(when* $\mu > 0$*) or* $\tau > 0$ *(when* $\mu \leq 0$*), the time* τ *Ricci iteration exists for all* $k \in \mathbb{N}$ *and converges to a Kähler-Einstein metric in the* $C^l(M)$*-topology for any* $l \in \mathbb{N}$.

Now, let the Ricci potential $f : \varphi \in \mathcal{H}_\omega \to f_{\omega_\varphi} \in C^\infty(M)$ be the vector field on $\mathcal{H}_\omega$ satisfying

$$\sqrt{-1}\partial\bar{\partial} f_{\omega_\varphi} = \mathrm{Ric}\omega_\varphi - \omega_\varphi, \quad \frac{1}{V}\int_M e^{f_{\omega_\varphi}}\omega_\varphi^n = 1. \tag{175}$$

For each k write

$$\omega_{k\tau} = \omega_{\psi_{k\tau}}, \qquad \text{with } \psi_{k\tau} = \sum_{l=1}^{k} \varphi_{l\tau}.$$

$$\omega_{k\tau} = \omega_{(k-1)\tau} + \sqrt{-1}\partial\bar{\partial}\varphi_{k\tau}.$$

The iteration (174) on $\mathcal{H}_\Omega$ can be written as the following system of complex Monge-Ampère equations on $\mathcal{H}_\omega$,

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega + \frac{1}{\tau}\varphi_{k\tau} - \mu\psi_{k\tau}} = \omega^n_{\psi_{(k-1)\tau}} e^{(\frac{1}{\tau}-\mu)\varphi_{k\tau} - \frac{1}{\tau}\varphi_{(k-1)\tau}}, \quad k \in \mathbb{N} \tag{176}$$

(implicit in this equation is also a normalization for $\varphi_{k\tau}$ that eliminates the ambiguity in passing from an equation on $\mathcal{H}_\Omega$ to one on $\mathcal{H}_\omega$). Note that the term $\frac{1}{\tau}\varphi_{k\tau}$ replaces the term $\dot{\varphi}$ for the Ricci flow.

We now mention some basic features of the iteration.

The most elementary one is that at each step one gains regularity (two derivatives). This is a discrete version of the infinite smoothing property of heat equations.

Another distinctive feature of the iteration is that it can be used to turn the solution of each type of Monge-Ampère equation into the next simplest one. Indeed, to find a Kähler-Einstein metric of negative scalar curvature $-n$ one needs to solve the equation

$$\omega^n_\varphi = \omega^n e^{f_\omega + \varphi}. \tag{177}$$

The corresponding time one Ricci iteration requires solving at each step the equation

$$\omega^n_\varphi = \omega^n e^{f_\omega + 2\varphi}. \tag{178}$$

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[14] We refer the reader to the sequel **[231]** where we intend to discuss some of the remaining cases. These cases that we also find of significant geometric interest require more involved and detailled analysis.

Similarly, the Calabi-Yau equation

$$\omega_{\varphi}^{n} = \omega^{n} e^{f_\omega}, \tag{179}$$

is traded for a sequence of equations of the previous type (177), and finally, the most difficult equation,

$$\omega_{\varphi}^{n} = \omega^{n} e^{f_\omega - \varphi}, \tag{180}$$

for a Kähler-Einstein metric of positive scalar curvature n, is turned into a sequence of Calabi-Yau equations (179) via the time one iteration, or to a sequence of equations of the type (177) for smaller time steps. For large time steps it remains an equation of the same type.

We now discuss the link the iteration creates between classical continuity method paths and the heat equation. In several places in the literature on canonical metrics it is mentioned that the continuity method and the heat flow method are morally equivalent (see, e.g., [**138**,**247**]). The following discussion comes to make this statement somewhat more explicit.

Indeed, note that one may also consider the time step of the iteration as a dynamical parameter and study the continuity method path it defines. More precisely, the time τ Ricci iteration is given by a sequence $\{\omega_{k\tau}\}_{k\geq 1}$ of Kähler forms in $\mathcal{H}_\Omega$ satisfying

$$\omega_{k\tau} = \omega_{(k-1)\tau} + \tau\mu\omega_{k\tau} - \tau\mathrm{Ric}\,\omega_{k\tau}, \qquad \omega_0 = \omega, \tag{181}$$

for each $k \in \mathbb{N}$ for which a solution exists. Now set $k = 1$ and consider the path $\{\omega_\tau := \omega_{\varphi_\tau}\}_{\tau\geq 0}$ in $\mathcal{H}_\Omega$ (for each τ for which it exists). In $\mathcal{H}_\omega$ we obtain the path

$$\omega_{\varphi_\tau}^{n} = \omega^{n} e^{f_\omega + (\frac{1}{\tau} - \mu)\varphi_\tau}, \quad \tau \in (0, \infty). \tag{182}$$

Let us compare this path to others that appeared previously in the literature.

In the case $\mu = 1$, when restricted to the segment $\tau \geq 1$ this is just a reparametrization of Aubin's path [**8**] given by

$$\omega_{\varphi_s}^{n} = \omega^{n} e^{f_\omega - s\varphi_s}, \quad s \in [0, 1], \tag{183}$$

via $s = 1 - \frac{1}{\tau}$. Here the solution for (183) at $s = 0$ is given by the Calabi-Yau Theorem. Namely, one typically first solves the family of equations introduced by Calabi [**41**, (11); **296**]

$$\omega_{\varphi_s}^{n} = \omega^{n} e^{(s+1)f_\omega + c_s}, \quad s \in [-1, 0], \tag{184}$$

and then continues to work with the path (183).

For the path (182) we still need to invoke the Calabi-Yau Theorem to show closedness at $\tau \to 1^-$. However, this path can be viewed as a continuity version of the Ricci flow and has various monotonicity properties that (184) does not. When studying the Ricci iteration for the case $\Omega = c_1$ this will be useful (note also that this path may be used in place of (184) to prove the Calabi-Yau Theorem).

Remark 4.6. We note that Calabi's path (184) can in fact be interpreted as a continuity path arising from a flow, however not the Ricci flow, see (263) below.

In light of this relation to the continuity method, the Ricci iteration is seen to interpolate between the continuity method ($\tau = \infty$) and the Ricci flow ($\tau = 0$). Cao and Perelman proved that when a Kähler-Einstein metric exists, the flow will converge to it in the sense of Cheeger-Gromov. Aubin, Bando-Mabuchi and Yau proved the analogous result for the continuity method. These results are the main motivation for Conjecture 4.4.

Next, in the case $\mu = -1$, the continuity path (182) that arises from the Ricci iteration equation is the same as the continuity path considered by Tian and Yau in their study of Kähler-Einstein metrics of negative Ricci curvature on some non-compact manifolds [**275**, p. 586]. The innovative idea of Tian and Yau was to consider a continuity parameter "starting from infinity" observing that along this path one has a uniform lower bound for the Ricci curvature.

Now an open problem concerning the flow equation (173) with $\mu > 0$ is whether one has a uniform lower bound for the Ricci curvature depending only on the initial data in the absence of a Kähler-Einstein metric. Along the time τ iteration one does have such a bound, depending on τ, namely, $\mathrm{Ric}\,\omega_{k\tau} > \frac{\tau\mu-1}{\tau}\omega_{k\tau}$ (this is, by the way, another important feature of the iteration that will be used crucially in our analysis). One possible approach to this problem might be to show that the Ricci flow stays asymptotically close to the time τ Ricci iteration for some range of time steps τ.

Remark 4.7. We remark that when $\mu = 1$ another path has been considered previously by Demailly and Kollár [**73**, (6.2.3)], given by $\omega^n_{\varphi_t} = \omega^n e^{tf_\omega - t\varphi_t}$, $t \in [0, 1]$. As written, this path also does not require to start from a solution to a Calabi-Yau equation. Yet in order to get openess for it one assumes $\mathrm{Ric}\,\omega > 0$ and this involves solving a Calabi-Yau equation (indeed there is no way to produce a Kähler-Einstein metric without entering $\mathcal{H}^+_{c_1}$, and the Calabi-Yau Theorem amounts to $\mathcal{H}^+_{c_1} \neq \emptyset$). This path can also be explained in terms of a discretization; see (264).

Remark 4.8. Another relation between a continuity path, defined by Tian and Zhu, and a discretized flow, this time a modified Ricci flow, will be discussed in Section 4.9.

4.4 Some energy functionals on the space of Kähler metrics

We describe in detail some energy functionals on the space of Kähler metrics. These will be one of the key players in this chapter.

We call a real-valued function A defined on a subset $\mathrm{Dom}(A)$ of $\mathcal{D}_\Omega \times \mathcal{D}_\Omega$ an energy functional if it is zero on the diagonal restricted to $\mathrm{Dom}(A)$. By an exact energy functional, we will mean an energy functional that satisfies the cocycle condition $A(\omega_1, \omega_2) + A(\omega_2, \omega_3) = A(\omega_1, \omega_3)$ with each of the pairs appearing in the formula belonging to $\mathrm{Dom}(A)$ (recall Definition 2.11) [**77**,**174**,**271**]. We will occasionally refer to both of these simply as functionals and exact functionals, respectively. Note that if an exact functional is defined on $U \times W$ with $U \subseteq W$ then there exists a unique exact functional defined on $W \times W$ extending it.

Let $V := \int_M \omega^n = [\omega]^n([M])$. The energy functionals I, J, introduced by Aubin [**8**], are defined for each pair $(\omega, \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi) \in \mathcal{D}_\Omega \times \mathcal{D}_\Omega$ by

$$I(\omega, \omega_\varphi) = V^{-1} \int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{l=0}^{n-1} \omega^{n-1-l} \wedge \omega_\varphi^l = V^{-1} \int_M \varphi(\omega^n - \omega_\varphi^n), \quad (185)$$

$$J(\omega, \omega_\varphi) = \frac{V^{-1}}{n+1} \int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{l=0}^{n-1} (n-l)\omega^{n-l-1} \wedge \omega_\varphi^l. \quad (186)$$

One may also define them via a variational formula. Connect each pair $(\omega, \omega_{\varphi_1} := \omega + \sqrt{-1}\partial\bar{\partial}\varphi_1)$ with a piecewise smooth path $\{\omega_{\varphi_t}\}_{t\in[0,1]}$ (we regard this path as a function on $M \times [0,1]$ and occasionally suppress the subscript t). Then we have for any such path

$$(I - J)(\omega, \omega_{\varphi_1}) = -\frac{1}{V} \int_{M\times[0,1]} \varphi_t n\sqrt{-1}\partial\bar{\partial}\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1} \wedge dt, \quad (187)$$

$$J(\omega, \omega_{\varphi_1}) = \frac{1}{V} \int_{M\times[0,1]} \dot{\varphi}_t(\omega^n - \omega_{\varphi_t}^n) \wedge dt. \quad (188)$$

On $\mathcal{H}_\omega \times \mathcal{H}_\omega$ the functionals I, J and $I - J$ are all nonnegative (and hence non-exact) and equivalent, namely,

$$\frac{1}{n^2}(I - J) \le \frac{1}{n(n+1)} I \le \frac{1}{n} J \le I - J \le \frac{n}{n+1} I \le nJ. \quad (189)$$

Note that pulling-back both arguments of these functionals by an automorphism of (M, J) does not change their value. It is important to understand the behavior of these functionals also outside the subspace $\mathcal{H}_\omega$, as will become clear in §§4.10.1 (see Theorem 4.43).

Lemma 4.9. *Let $\omega \in \mathcal{H}_\omega$. Then $I(\omega, \cdot)$ is unbounded from above on $\mathcal{H}_\omega$ and, when $n > 1$, unbounded on $\mathcal{D}_\Omega$.*

Proof. Fix a holomorphic coordinate patch

$$\psi : U \to \mathbb{C}^n,\ \psi(q) = \mathbf{z}(q) := (z^1(q), \dots, z^n(q)),\ \forall\, q \in U \subseteq M.$$

Let $a > 0$ be such that $\psi^{-1}(\{v \in \mathbb{C}^n \ : \ |v| < 3a\}) \subseteq U$. For the first statement, define $\tilde{\varphi}_b$ by letting $\tilde{\varphi}_b = b|\mathbf{z}|^2$ on $\psi^{-1}(\{v \in \mathbb{C}^n \ : \ a < |v| < 2a\})$ and constant elsewhere on U in such a way that it is continuous. Approximate $\tilde{\varphi}_b$ by smooth functions $\varphi_{b,m}$ that agree with it outside the set $\psi^{-1}\{\{v \in \mathbb{C}^n \ : \ |v| \in (a - \frac{1}{m}, a + \frac{1}{m}) \cup (2a - \frac{1}{m}, 2a + \frac{1}{m})\}\})$ and that satisfy $|\tilde{\varphi}_b - \varphi_{b,m}| < \frac{1}{m}$ on U. Given $a_2 > 0$ there exists b and a corresponding m such that $\varphi_{b,m} \in \mathcal{H}_\omega$ and $I(\omega, \omega_{\varphi_{b,m}}) > a_2$.

For the second statement, construct similarly functions, as above, now setting $\tilde{\varphi}_b = -b(|z_1|^2 + |z_2|^2)$ on $\psi^{-1}(\{v \in \mathbb{C}^n \ : \ a < |v| < 2a\})$. Again one may approximate using functions $\varphi_{b,m}$. Expanding $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_{b,m})^l$ using the binomial formula it then follows that up to a term that is uniformly bounded for m sufficiently large, $I(\omega, \omega_{\varphi_{b,m}})$ equals $V^{-1}\int_M \sqrt{-1}\partial\varphi_{b,m} \wedge \bar{\partial}\varphi_{b,m} \wedge \omega^{n-2} \wedge (a_2\omega + a_3\sqrt{-1}\partial\bar{\partial}\varphi_{b,m})$ for some $a_2, a_3 > 0$. We then see that given any $a_4 > 0$ there exists b and a corresponding m such that $I(\omega, \omega_{\varphi_{b,m}}) < -a_4$. □

We say that an exact functional A is bounded from below on $U \subseteq \mathcal{H}_\omega$ if for every ω such that $(\omega, \omega_\varphi) \in \mathrm{Dom}(A)$ and $\omega_\varphi \in U$ holds $A(\omega, \omega_\varphi) \geq C_\omega$ with C_ω independent of ω_φ. We say it is proper (in the sense of Tian) on a set $U \subseteq \mathcal{H}_\Omega(G)$ if for each $\omega \in \mathcal{H}_\Omega(G)$ there exists a smooth function $\tau_\omega : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{s \to \infty} \tau_\omega(s) = \infty$ such that

$$A(\omega, \omega_\varphi) \geq \tau_\omega((I - J)(\omega, \omega_\varphi)) \tag{190}$$

for every $\omega_\varphi \in U$. This is well-defined, in other words depends only on $[\omega]$ since the failure of $I - J$ to satisfy the cocycle condition is under control with respect to the two base metrics, ω, ω_1 say, to wit,

$$(I - J)(\omega, \omega_2) - (I - J)(\omega_1, \omega_2) = (I - J)(\omega, \omega_1) - \frac{1}{V}\int_M \varphi_1(\omega_2^n - \omega_1^n),$$

with the last term controlled by the oscillation of φ_1. Properness of a functional implies it has a lower bound.

4.4.1 A family of generalized Aubin energy functionals.

We introduce the following collection of energy functionals for each $k \in \{0, \dots, n\}$

$$\begin{aligned} I_k(\omega, \omega_\varphi) &= \frac{1}{V}\int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{l=0}^{k-1} \frac{k-l}{k+1}\omega^{n-1-l} \wedge \omega_\varphi^l \\ &= \frac{V^{-1}}{k+1}\int_M \varphi(k\omega^n - \sum_{l=1}^{k} \omega^{n-l} \wedge \omega_\varphi^l). \end{aligned} \tag{191}$$

Note that $I_n = J$, $I_{n-1} = ((n+1)J - I)/n$. We call these the "staircase" functionals. See Figure 3 for an explanation for the choice of name.

Chen and Tian **[63]** defined another such family

$$J_k(\omega, \omega_{\varphi_1}) = V^{-1} \int_{M \times [0,1]} \dot{\varphi}_t(\omega^k_{\varphi_t} \wedge \omega^{n-k} - \omega^n_{\varphi_t}) \wedge dt, \quad k = 0, \dots, n. \tag{192}$$

(Note that $J_{n-k-1}/(k+1)$ in their article corresponds to J_k in our notation.) The following computation relates these two families of functionals.

Lemma 4.10. *The following relation holds on $\mathcal{H}_\omega \times \mathcal{H}_\omega$:*

$$I_k(\omega, \omega_\varphi) = J(\omega, \omega_\varphi) - J_k(\omega, \omega_\varphi).$$

Proof. Given a path $\{\omega_{\varphi_t}\}_{t \in [0,1]}$ we compute the variational equation for I_k.

$$\begin{aligned}
(k+1)\frac{d}{dt} I_k(\omega, \omega_{\varphi_t}) = & -\frac{1}{V} \int_M \sum_{l=0}^{k-1} \Big(2\dot{\varphi}\sqrt{-1}\partial\bar{\partial}\varphi \wedge (k-l)\omega^{n-1-l} \wedge \omega^l_\varphi \\
& + \varphi\sqrt{-1}\partial\bar{\partial}\varphi \wedge \sqrt{-1}\partial\bar{\partial}\dot{\varphi} \wedge l(k-l)\omega^{n-1-l} \wedge \omega^{l-1}_\varphi \Big) \\
= & -\frac{1}{V} \int_M \dot{\varphi}\sqrt{-1}\partial\bar{\partial}\varphi \wedge \sum_{l=0}^{k-1} \Big(2(k-l)\omega^{n-1-l} \wedge \omega^l_\varphi \\
& + (\omega_\varphi - \omega) \wedge l(k-l)\omega^{n-1-l} \wedge \omega^{l-1}_\varphi \Big) \\
= & -\frac{1}{V} \int_M \dot{\varphi}\sqrt{-1}\partial\bar{\partial}\varphi \wedge \Big(\sum_{l=0}^{k-1} 2(k-l)\omega^{n-1-l} \wedge \omega^l_\varphi \\
& + \sum_{l=1}^{k-1} l(k-l)\omega^{n-1-l} \wedge \omega^l_\varphi \\
& - \sum_{l=0}^{k-2} (k-l-1)(l+1)\omega^{n-1-l} \wedge \omega^l_\varphi \Big) \\
= & -(k+1)\frac{1}{V} \int_M \dot{\varphi}\sqrt{-1}\partial\bar{\partial}\varphi \wedge \sum_{l=0}^{k-1} \omega^{n-1-l} \wedge \omega^l_\varphi,
\end{aligned}$$

and putting $\sqrt{-1}\partial\bar{\partial}\varphi = \omega_\varphi - \omega$ we have

$$\frac{d}{dt} I_k(\omega, \omega_{\varphi_t}) = V^{-1} \int_M \dot{\varphi}_t(\omega^n - \omega^{n-k} \wedge \omega^k_{\varphi_t}). \tag{193}$$

Combining with (192) and (188) we conclude. $\square$

Note that from the definitions it follows that

$$0 \leq I_k(\omega, \omega_\varphi) \leq J(\omega, \omega_\varphi), \qquad \text{on } \mathcal{H}_\omega \times \mathcal{H}_\omega. \tag{194}$$

As a corollary of Lemma 4.10 we have therefore $0 \leq J_k(\omega, \omega_\varphi) \leq J(\omega, \omega_\varphi)$ on $\mathcal{H}_\omega \times \mathcal{H}_\omega$. We point out that this upper bound improves [**63**, Corollary 4.5] while the lower bound appears to be new. Also from (191)

$$\frac{k+2}{k+1} I_{k+1} \geq \frac{k+1}{k} I_k, \qquad \text{on } \mathcal{H}_\omega \times \mathcal{H}_\omega. \tag{195}$$

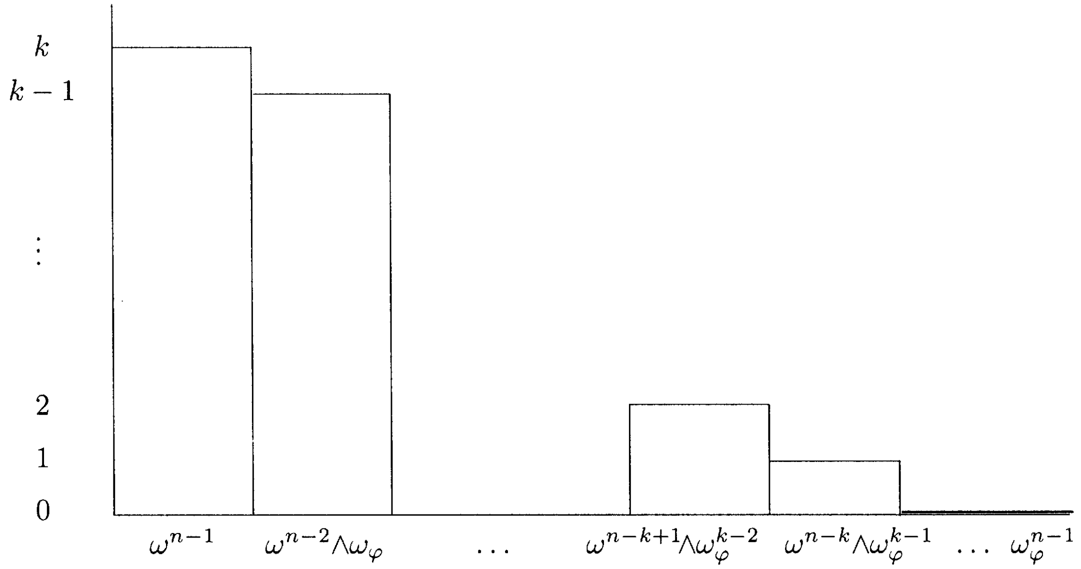

Figure 3. The "staircase" functionals $(k+1)I_k$. The vertical axis corresponds to the coefficient of the term $\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^{n-1-l} \wedge \omega_\varphi^l$ in the sum (191) forming $(k+1)I_k$.

Note that in particular $I_{k+1} \geq I_k$ and so by Lemma 4.10 $J_k \geq J_{k+1}$. We note in passing that this lemma also yields the following formula

$$\begin{aligned} J_k(\omega, \omega_\varphi) = \frac{V^{-1}}{n+1} \int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \Big(\frac{n-k}{k+1} \sum_{l=0}^{k-1} (l+1) \omega^{n-1-l} \wedge \omega_\varphi^l \\ + \sum_{l=k}^{n-1} (n-l) \omega^{n-1-l} \wedge \omega_\varphi^l \Big). \end{aligned} \tag{196}$$

The functionals J_k are therefore more complicated than the I_k. Rather than a staircase they are better described by a "mountain" since when we draw a figure for them as we did for I_k above we obtain a mountain with peak value at the coefficient of $\omega^{n-1-k} \wedge \omega_\varphi^k$.

4.4.2 An "interpolation" formula for the Chen-Tian energy functionals. The Chen-Tian energy functionals $E_k,\ k = 0, \dots, n$, are defined by

$$E_k(\omega, \omega_{\varphi_1}) = V^{-1} \int_{M \times [0,1]} \Delta_{\varphi_t} \dot{\varphi}_t \mathrm{Ric}(\omega_{\varphi_t})^k \wedge \omega_{\varphi_t}^{n-k} \wedge dt \tag{197}$$

$$- \frac{n-k}{k+1} V^{-1} \int_{M \times [0,1]} \dot{\varphi}_t (\mathrm{Ric}(\omega_{\varphi_t})^{k+1} - \mu_k \omega_{\varphi_t}^{k+1}) \wedge \omega_{\varphi_t}^{n-1-k} \wedge dt, \tag{198}$$

where

$$\mu_k := \frac{c_1^{k+1} \cup [\omega]^{n-k-1}([M])}{[\omega]^n([M])}. \tag{199}$$

(Recall the discussion surrounding (32).) This gives rise to well-defined exact energy functionals [**63**] (note that $E_k/(k+1)$ in Chen-Tian's article corresponds to E_k in our notation). The K-energy, E_0, was introduced by Mabuchi [**174**], while E_n, which we refer to as the 'Ricci energy', was introduced by Bando and Mabuchi[3] [**14**].

We note that (198) can rewritten as (although we will not make use of it)

$$E_k(\omega, \omega_\varphi) = \frac{1}{k+1} \frac{1}{V} \int_{M \times [0,1]} \left[\mu \omega^{k+1} - (\mathrm{Ric} \omega_\varphi - \Delta \dot{\varphi})^{k+1}\right] \wedge (\omega_\varphi + \dot{\varphi})^{n-k} \wedge dt. \tag{200}$$

While these expressions are useful for computations, it is useful to understand the functionals E_k as the potentials of certain geometrically defined "universal" closed 1-forms on the space of Kähler potentials. To start with the simplest two, the K-energy E_0 satisfies

$$dE_0(\omega_0, \omega) = -\mathrm{tr}_\omega(\mathrm{Ric}\,\omega - s_0/n \cdot \omega)\, \omega^n = -(s(\omega) - s_0)\, \omega^n = -n(\mathrm{Ric}\,\omega - \omega) \wedge \omega^{n-1}$$

---

[3] Kähler-Einstein forms are the only critical points of these two functionals when $\Omega = \mu c_1, \mu \in \{\pm 1\}$: For E_0 see [**271**, p. 19] while for E_n the critical forms satisfy $(\mu \mathrm{Ric}\omega)^n = \omega^n$ and writing $\mu \mathrm{Ric}\omega = \omega + \sqrt{-1}\partial\bar{\partial} f$ we see that $\mu \mathrm{Ric}\omega > 0$ at the minimum of f. Since the smallest eigenvalue of a Hölder continuous matrix-valued function is also Hölder continuous [**2**, p. 438] we conclude that $\mu \mathrm{Ric}\omega > 0$ implying that f is constant by the uniqueness argument of Calabi (for a different proof see [**186**, §8]). However when $c_1 = 0$ there are nontrivial solutions of $(\mathrm{Ric}\omega)^n = 0$ if the manifold is a product. For $\mu = 1$ critical points of E_k with nonnegative Ricci curvature are necessarily Kähler-Einstein [**281**].

where the exterior differential is with respect to the ω variable (recall the discussion of §§2.1.7) and ω_1 is any fixed point in $\mathcal{H}_\omega$. The Ricci energy E_n satisfies

$$dE_n(\omega_0,\omega) = \Delta_\varphi(\det_\omega \mathrm{Ric}\,\omega)\,\omega^n := \Delta_\omega\Big(\frac{(\mathrm{Ric}\,\omega)^n}{\omega^n}\Big)\omega^n.$$

Similarly, defining the normalized elementary symmetric polynomials of the eigenvalues of $\mathrm{Ric}\,\omega$ (with respect to ω) by

$$\sigma_k(\omega) = \binom{n}{k}^{-1}\frac{(\mathrm{Ric}\,\omega)^k\wedge\omega^{n-k}}{\omega^n},$$

and recalling (199), one has for each $k\in\{0,\ldots,n\}$

$$dE_k(\omega_0,\omega) = \Big[\Delta_\omega\sigma_k - \frac{n-k}{k+1}\Big(\sigma_{k+1}-\mu_{k+1}\Big)\Big]\omega^n. \tag{201}$$

In other words while $(\sigma_1(\omega)-\mu_1)\omega^n$ is closed the 1-forms $(\sigma_k(\omega)-\mu_k)\omega^n$ are not closed for $k\neq 1$ and its exterior derivative equals $k\Delta_\omega\sigma_{k-1}/(n-k+1)\omega^n$. Combining $(\sigma_k(\omega)-\mu_k)\omega^n$ with this correction yields an overall closed form. We will devote a separate subsection to a more conceptual framework to prove such results using the theory of Bott-Chern forms (§§4.4.5). This formulation will have an application to GIT and geometric stability that we will describe in §§4.4.5.

For each element of $\mathcal{H}_\omega$ these functionals (being exact) induce a (real) Lie group homomorphism $\mathrm{Aut}(M,\mathrm{J})_0\to\mathbb{R}$ given by $h\mapsto E_k(\omega,h^\star\omega)$. The corresponding Lie algebra homomorphism $\mathrm{aut}(M,\mathrm{J})\to\mathbb{R}$ is given by $X\mapsto \frac{d}{dt}\big|_0 E_k(\omega,(\exp tX)^\star\omega)$. This naturally extends to a complex Lie algebra homomorphism

$$X\mapsto\mathcal{F}_k(X;\omega) := \frac{d}{dt}\Big|_0 E_k(\omega,(\exp tX)^\star\omega) - \sqrt{-1}\frac{d}{dt}\Big|_0 E_k(\omega,(\exp tJX)^\star\omega). \tag{202}$$

Changing ω within a fixed cohomology class does not change the homomorphism [**63,186**]. This is an extention of the Bando-Calabi-Futaki Theorem, the case $k=0$ [**21,42,105**] (the construction was further generalized by Futaki [**107**]). One calls these homomorphisms Futaki characters (or invariants). When (M,J,ω) is Fano Kähler-Einstein it follows from (197) that $\mathcal{F}_k$ is trivial and hence $E_k(\omega,\omega_\varphi)=0$ if ω_φ is Kähler-Einstein, since the set of Kähler-Einstein metrics is equal to an $\mathrm{Aut}(M,\mathrm{J})_0$-orbit of ω [**15**].

Bando and Mabuchi derived the following elegant formula.

Proposition 4.11. (See [**15**, (1.8.1)].) *For every* $(\omega,\omega_\varphi)\in\mathcal{H}_{c_1}\times\mathcal{H}_{c_1}$,

$$E_n(\omega,\omega_\varphi) = E_0(\omega,\omega_\varphi) + J(\omega_\varphi,\mathrm{Ric}\,\omega_\varphi) - J(\omega,\mathrm{Ric}\,\omega).$$

We now show that Proposition 4.11 can be generalized as follows in terms of the functionals I_k introduced above.

Proposition 4.12. *Let $k \in \{0,\dots,n\}$. For every $(\omega,\omega_\varphi) \in \mathcal{H}_{c_1}\times\mathcal{H}_{c_1}$,*

$$E_k(\omega,\omega_\varphi) = E_0(\omega,\omega_\varphi) + I_k(\omega_\varphi,\mathrm{Ric}\,\omega_\varphi) - I_k(\omega,\mathrm{Ric}\,\omega), \tag{203}$$

$$= E_n(\omega,\omega_\varphi) - J_k(\omega_\varphi,\mathrm{Ric}\,\omega_\varphi) + J_k(\omega,\mathrm{Ric}\,\omega), \tag{204}$$

$$= \big((1-\tfrac{l}{k+1})E_0 + \tfrac{l}{k+1}E_n\big)(\omega,\omega_\varphi) + (I_k - \tfrac{l}{k+1}J)(\omega_\varphi,\mathrm{Ric}\,\omega_\varphi) \tag{205}$$

$$-\,(I_k - \tfrac{l}{k+1}J)(\omega,\mathrm{Ric}\,\omega), \quad \forall\, l \in \{0,\dots,k+1\}.$$

Proof. First, in order to establish formula (203) we show that the variations of both sides of the equation agree.

$$\begin{aligned} -(k+1)V\frac{d}{dt}I_k(\omega_\varphi,\mathrm{Ric}\,\omega_\varphi) &= \frac{d}{dt}\int_M f_{\omega_\varphi}\sqrt{-1}\partial\bar\partial f_{\omega_\varphi}\wedge\sum_{l=0}^{k-1}(k-l)\omega_\varphi^{n-1-l}\wedge(\mathrm{Ric}\,\omega_\varphi)^l \\ &= \frac{d}{dt}\int_M f_{\omega_\varphi}(\mathrm{Ric}\,\omega_\varphi - \omega_\varphi)\wedge\sum_{l=0}^{k-1}(k-l)\omega_\varphi^{n-1-l}\wedge(\mathrm{Ric}\,\omega_\varphi)^l \\ &= \frac{d}{dt}\int_M f_{\omega_\varphi}\Big(-k\omega_\varphi^n + \sum_{l=1}^{k}\omega_\varphi^{n-l}\wedge(\mathrm{Ric}\,\omega_\varphi)^l\Big) \end{aligned}$$

$$= \int_M \dot f_{\omega_\varphi}\Big(-k\omega_\varphi^n + \sum_{l=1}^{k}\omega_\varphi^{n-l}\wedge(\mathrm{Ric}\,\omega_\varphi)^l\Big) \tag{206}$$

$$+\int_M f_{\omega_\varphi}\sqrt{-1}\partial\bar\partial\dot\varphi\wedge\Big(\sum_{l=1}^{k}(n-l)\omega_\varphi^{n-l-1}\wedge(\mathrm{Ric}\,\omega_\varphi)^l - kn\omega_\varphi^{n-1}\Big) \tag{207}$$

$$-\int_M f_{\omega_\varphi}\sqrt{-1}\partial\bar\partial\Delta_\varphi\dot\varphi\sum_{l=1}^{k} l\omega_\varphi^{n-l}\wedge(\mathrm{Ric}\,\omega_\varphi)^{l-1}. \tag{208}$$

First, we write (206) as

$$\int_M \dot f_{\omega_\varphi}\Big(-k\omega_\varphi^n + \sum_{l=1}^{k}\omega_\varphi^{n-l}\wedge(\mathrm{Ric}\,\omega_\varphi)^l\Big) =: \iota_1 + \mu_1.$$

We will evaluate (207) and (208) by substituting once again $\sqrt{-1}\partial\bar{\partial} f_{\omega_\varphi} = \operatorname{Ric}\omega_\varphi - \omega_\varphi$. For (207) we get

$$
\begin{aligned}
&\int_M \dot{\varphi}(\operatorname{Ric}\omega_\varphi - \omega_\varphi) \wedge \Big(-kn\omega_\varphi^{n-1} + \sum_{l=1}^{k} (n-l)\omega_\varphi^{n-l-1} \wedge (\operatorname{Ric}\omega_\varphi)^l \Big) \\
&\quad = \int_M \dot{\varphi}\Big(kn\omega_\varphi^{n} - kn\omega_\varphi^{n-1} \wedge \operatorname{Ric}\omega_\varphi \\
&\qquad\qquad - (n-1)\omega_\varphi^{n-1} \wedge \operatorname{Ric}\omega_\varphi + \sum_{l=2}^{k} \omega_\varphi^{n-l} \wedge (\operatorname{Ric}\omega_\varphi)^l \\
&\qquad\qquad + (n-k)\omega_\varphi^{n-k-1} \wedge (\operatorname{Ric}\omega_\varphi)^{k+1} \Big) \\
&\quad = \int_M \dot{\varphi}\Big([-(n-k) + (k+1)n - k]\omega_\varphi^{n} - (k+1)n\omega_\varphi^{n-1} \wedge \operatorname{Ric}\omega_\varphi \\
&\qquad\qquad + \sum_{l=1}^{k} \omega_\varphi^{n-l} \wedge (\operatorname{Ric}\omega_\varphi)^l + (n-k)\omega_\varphi^{n-k-1} \wedge (\operatorname{Ric}\omega_\varphi)^{k+1} \Big) \\
&\quad =: (\kappa_1 + \lambda_1 + \iota_2) + \lambda_2 + \mu_2 + \kappa_2.
\end{aligned}
$$

For (208) we get

$$
\begin{aligned}
&\int_M \Delta_\varphi\dot{\varphi}(\omega_\varphi - \operatorname{Ric}\omega_\varphi) \wedge \sum_{l=1}^{k} l\omega_\varphi^{n-l} \wedge (\operatorname{Ric}\omega_\varphi)^{l-1} \\
&\quad = \int_M \Delta_\varphi\dot{\varphi}\Big(\omega_\varphi^{n} + \sum_{l=1}^{k-1} \omega_\varphi^{n-l} \wedge (\operatorname{Ric}\omega_\varphi)^l - k\omega_\varphi^{n-k} \wedge (\operatorname{Ric}\omega_\varphi)^k \Big) \\
&\quad = \int_M \Delta_\varphi\dot{\varphi}\Big(\sum_{l=1}^{k} \omega_\varphi^{n-l} \wedge (\operatorname{Ric}\omega_\varphi)^l - (k+1)\omega_\varphi^{n-k} \wedge (\operatorname{Ric}\omega_\varphi)^k \Big) \\
&\quad =: \mu_3 + \kappa_3.
\end{aligned}
$$

Noting that $\dot{f}_{\omega_\varphi} = -\Delta_\varphi\dot{\varphi} - \dot{\varphi} + c$ with c a constant yields $\iota_1 + \iota_2 = -kcV$ and $\mu_1 + \mu_2 + \mu_3 = kcV$. Note that $\kappa_1 + \kappa_2 + \kappa_3 = -(k+1)V\frac{d}{dt}E_k(\omega, \omega_\varphi)$ and $\lambda_1 + \lambda_2 = (k+1)V\frac{d}{dt}E_0(\omega, \omega_\varphi)$. This completes the proof of (203).

Formulas (204) and (205) now follow: first use (203) with $k = n$ to express E_0 in terms of E_n and J, and then substitute this expression back into (203) and apply Lemma 4.10. □

Let us note that one way one could arrive at the formula would be to use the expression for $(k+1)I_k - kI_{k-1}$ (see (191)) and Lemma 4.14 together with the observation

$$\frac{d}{dt}((k+1)E_k - kE_{k-1})(\omega, \omega_{\varphi_t}) = -\frac{1}{V}\int_M \dot{\varphi}_t \omega_{\varphi_t}^n - \frac{d}{dt}\Big(\frac{1}{V}\int_M f_{\omega_{\varphi_t}} (\mathrm{Ric}\,\omega_{\varphi_t})^k \wedge \omega_{\varphi_t}^{n-k}\Big). \tag{209}$$

The functionals E_k are thus seen to be described as 'Kähler-Ricci' energies, "interpolating" between the Kähler energy E_0 and the Ricci energy E_n.

One particularly visible consequence of Proposition 4.12 is the fact that the homomorphisms $\mathcal{F}_k$ all coincide, a result first proved by Maschler [**186**, (17)] using an equivariant formulation and later by Liu [**170**, §3] by a direct computation (see also [**163**]). For other explicit expressions for the functionals E_k see [**63**,**163**,**210**,**250**].

We note that there exist counterparts of the formulas above for some other classes, according to the following statement. Since the proof is identical we omit it.

Proposition 4.13. *Let $\mu \neq 0$. Let $k \in \{0, \dots, n\}$. For every $(\omega, \omega_\varphi) \in \mathcal{H}_{\mu c_1} \times \mathcal{H}_{\mu c_1}$,*

$$\mu^{k+1} E_k(\omega, \omega_\varphi) = \mu E_0(\omega, \omega_\varphi) + I_k(\omega_\varphi, \mu \mathrm{Ric}\,\omega_\varphi) - I_k(\omega, \mu \mathrm{Ric}\,\omega). \tag{210}$$

An interesting (and somewhat surprising at first) consequence is that for manifolds with $c_1 < 0$ the Kähler-Einstein metric is not necessarily a minimum for the functionals E_k but either a minimum or a maximum depending on the parity of k.

4.4.3 A formula for the Berger-Moser-Ding functional. We will initially assume that (M, J) is Fano and let $\omega_\varphi \in \mathcal{H}_{c_1}$. Let $f_{\omega_\varphi} \in C^\infty(M)$ denote the unique function satisfying

$$\sqrt{-1}\partial\bar{\partial} f_{\omega_\varphi} = \mathrm{Ric}\,\omega_\varphi - \omega_\varphi,$$

and

$$V^{-1}\int_M e^{f_{\omega_\varphi}} \omega_\varphi^n = 1.$$

Following Ding [**75**], define an exact functional on $\mathcal{H}_{c_1} \times \mathcal{D}_{c_1}$ by

$$F_1(\omega, \omega_\varphi) = J(\omega, \omega_\varphi) - \frac{1}{V}\int_M \varphi \omega^n - \log \frac{1}{V}\int_M e^{f_\omega - \varphi} \omega^n. \tag{211}$$

For Riemann surfaces this functional goes back to Berger and Moser [**19**,**198**] (see also §§4.10.1). The critical points of this functional are the Kähler-Einstein metrics. We state the following relation between the functionals E_0 and F_1.

Lemma 4.14. (See [**76**].) *Let $(\omega, \omega_\varphi) \in \mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$. Then*

$$F_1(\omega, \omega_\varphi) = E_0(\omega, \omega_\varphi) + \frac{1}{V}\int_M f_{\omega_\varphi} \omega_\varphi^n - \frac{1}{V}\int_M f_\omega \omega^n.$$

Note that by Jensen's inequality (or the Arithmetic Mean-Geometric Mean inequality)

$$\frac{1}{V}\int_M f_{\omega_\varphi}\omega_\varphi^n \leq \frac{1}{V}\int_M e^{f_{\omega_\varphi}}\omega_\varphi^n - 1 = 0. \tag{212}$$

Note also that one may define a Lie algebra homomorphism corresponding to F_1 similarly to the construction for E_k in (202). Lemma 4.14 implies that this homomorphism will coincide with $\mathcal{F}_0$.

An equivalent form of the following was stated by Bando and Mabuchi [**15**, (1.5)]. Note that however at their time Ding's functional was yet to be defined and so their somewhat indirect formulation in terms of a commutative diagram is different than ours. We emphasize that our formulation given below turns out to have rather striking applications, as will be highlighted in Section 4.10.

Lemma 4.15. *For every $(\omega, \omega_\varphi) \in \mathcal{H}_{c_1}^+ \times \mathcal{H}_{c_1}$ one has*

$$E_n(\omega, \omega_\varphi) = F_1(\mathrm{Ric}\omega, \mathrm{Ric}\omega_\varphi).$$

Note that by exactness this formula completely determines E_n on $\mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$, as remarked earlier.

Proof. Let $\{\varphi_t\}$ denote a smooth family of functions such that $\omega_{\varphi_0} = \omega$, $\omega_{\varphi_1} = \omega_\varphi$. Write $\mathrm{Ric}\omega_{\varphi_t} = \mathrm{Ric}\omega + \sqrt{-1}\partial\bar\partial \log \frac{\omega^n}{\omega_{\varphi_t}^n}$. Then $f_{\mathrm{Ric}\,\omega} = \log \frac{\omega^n}{(\mathrm{Ric}\,\omega)^n}$. Thus for each $t \in [0,1]$,

$$F_1(\mathrm{Ric}\,\omega, \mathrm{Ric}\,\omega_{\varphi_t}) = J(\mathrm{Ric}\,\omega, \mathrm{Ric}\,\omega_{\varphi_t}) - \frac{1}{V}\int_M \log \frac{\omega^n}{\omega_{\varphi_t}^n}(\mathrm{Ric}\,\omega)^n.$$

Hence,

$$\frac{d}{dt}F_1(\mathrm{Ric}\,\omega, \mathrm{Ric}\,\omega_{\varphi_t}) = -V^{-1}\int_M (-\Delta_t\dot\varphi_t)(\mathrm{Ric}\,\omega_{\varphi_t})^n = \frac{d}{dt}E_n(\omega, \omega_{\varphi_t}),$$

from which we conclude by integration. □

We will later interpret this computation in light of the inverse Ricci operator in a manner that we will be quite useful (see Proposition 4.40).

Remark 4.16. Consider the slightly more general case where c_1 may be either positive, negative or zero. For each $\mu \in \mathbb{R}$ define the Ding functional

$$F_\mu(\omega, \omega_\varphi) = \begin{cases} -\dfrac{1}{n+1}\dfrac{1}{V}\displaystyle\int_M \varphi\sum_{l=0}^{n}\omega^{n-l}\wedge\omega_\varphi^l - \frac{1}{\mu}\log\frac{1}{V}\int_M e^{f_\omega - \mu\varphi}\omega^n, & \text{for } \mu \neq 0, \\ -\dfrac{1}{n+1}\dfrac{1}{V}\displaystyle\int_M \varphi\sum_{l=0}^{n}\omega^{n-l}\wedge\omega_\varphi^l + \frac{1}{V}\int_M \varphi e^{f_\omega}\omega^n, & \text{for } \mu = 0. \end{cases} \tag{213}$$

When $\mu = 1$ we recover (211) and we set $F := F_1$. The critical points of these functionals are Kähler-Einstein metrics [**75**]. However, there is an important difference between the two cases in (213). While for the first the functional is exact, for $\mu = 0$ this is not true. This is because the second term of F_0 is not exact on the space $\mathcal{H}_\omega$, while the first is. This rather peculiar phenomenon is reflected also by a property of a generalized Ding functional (see the end of §§4.10.3).

4.4.4 Monotonicity of energy functionals along the Ricci iteration. In this subsection we will obtain a monotonicity result along the Ricci iteration for a family of energy functionals. Besides its application later in the chapter this result has independent interest, and it seems interesting to compare it with corresponding studies for the Ricci flow (see Remark 4.19).

We now introduce certain subsets of the space of Kähler forms on a Fano manifold; these will be defined more systematically in §§4.10.1 below, however in the present subsection we will make a first use of them. We set

$$\mathcal{A}_k(\omega) = \{\omega_\varphi \in \mathcal{H}_{c_1} \, : \, E_k(\omega, \omega_\varphi) \geq 0\}, \tag{214}$$

$$\mathcal{B}_k = \{\omega_\varphi \in \mathcal{H}_{c_1} \, : \, I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) \geq 0\}. \tag{215}$$

When a Kähler-Einstein form ω exists we denote $\mathcal{A}_k := \mathcal{A}_k(\omega)$. This is well-defined and does not depend on the choice of the Kähler-Einstein form. We recall that both the sets $\mathcal{A}_k$ and $\mathcal{B}_k$ strictly contain the set $\mathcal{H}^+_{c_1}$, of Kähler forms of positive Ricci curvature (see Theorem 4.42 below for a more precise statement).

The following monotonicity result will be useful later. For E_0, F_μ and E_1 with $1/\mu = \tau = 1$ this was proven before by several authors [**12,76,250**].

Proposition 4.17. *(i) The functional E_0 is monotonically decreasing along the time τ iteration ($\tau > 0$) whenever the initial point is not Kähler-Einstein.*
(ii) When $1/\mu = \tau = 1$ the same is true for F_1, E_1, and, when the initial metric lies in $\mathcal{B}_k \supset \mathcal{H}^+_{c_1}$, also for E_k, $k \geq 2$.

Proof. By exactness, it suffices to show monotonicity at each step of the iteration.
(i) For concreteness we derive the result only for $\tau = 1$ but explicitly compute the the energy decrease along the iteration (in general see Lemma 4.36, stated for $\mu = 1$, that works for all μ). Consider the equation $\omega^n_{\varphi_1} = \omega^n e^{f_\omega - (\mu - 1)\varphi_1}$. One has [**60**; **269,** p. 254; **270,** (5.14)]

$$E_0(\omega, \omega_{\varphi_1}) = \frac{1}{V} \int_M \log \frac{\omega^n_{\varphi_1}}{\omega^n} \omega^n_{\varphi_1} - \mu(I - J)(\omega, \omega_{\varphi_1}) + \frac{1}{V} \int_M f_\omega(\omega^n - \omega^n_{\varphi_1}). \tag{216}$$

First, let $\mu = 1$. One has,

$$E_0(\omega, \omega_{\varphi_1}) = -(I - J)(\omega, \omega_{\varphi_1}) + \frac{1}{V} \int_M f_\omega \omega^n. \tag{217}$$

This is nonpositive by the definition of f_ω and Jensen's inequality.

When $\mu = -1$ one has

$$\begin{aligned} E_0(\omega, \omega_{\varphi_1}) &= \frac{1}{V}\int_M 2\varphi_1 \omega_{\varphi_1}^n + (I-J)(\omega, \omega_{\varphi_1}) + \frac{1}{V}\int_M f_\omega \omega^n \\ &= \frac{1}{V}\int_M 2\varphi_1 \omega_{\varphi_1}^n + \frac{1}{V}\int_M \varphi_1(\omega^n - \omega_{\varphi_1}^n) - J(\omega, \omega_{\varphi_1}) + \frac{1}{V}\int_M f_\omega \omega^n \\ &= -(I+J)(\omega, \omega_{\varphi_1}) + \frac{1}{V}\int_M (f_\omega + 2\varphi_1)\omega^n. \end{aligned}$$

Each term is nonpositive, once again by using the normalization inherent in (178).

When $\mu = 0$ one has

$$\begin{aligned} E_0(\omega, \omega_{\varphi_1}) &= \frac{1}{V}\int_M \varphi_1 \omega_{\varphi_1}^n + \frac{1}{V}\int_M f_\omega \omega^n \\ &= -I(\omega, \omega_{\varphi_1}) + \frac{1}{V}\int_M (f_\omega + \varphi_1)\omega^n, \end{aligned}$$

and we may argue as before.

(ii) Using (185)-(188) and (213) one has

$$\begin{aligned} F_1(\omega, \omega_{\varphi_1}) &= -(I-J)(\omega, \omega_{\varphi_1}) - \frac{1}{V}\int_M \varphi_1 \omega_{\varphi_1}^n - \log \frac{1}{V}\int_M e^{f_\omega - \varphi_1}\omega^n \qquad (218) \\ &= -(I-J)(\omega, \omega_{\varphi_1}) - \frac{1}{V}\int_M \varphi_1 \omega_{\varphi_1}^n - \log \frac{1}{V}\int_M e^{-\varphi_1}\omega_{\varphi_1}^n. \qquad (219) \end{aligned}$$

By Jensen's inequality the last two terms combined are nonpositive, and so we conclude by the positivity of $I - J$. (Alternatively, the iteration stays on the submanifold of $\mathcal{H}_\omega$ defined by the equation $\frac{1}{V}\int_M e^{f_\omega - \varphi}\omega^n = 1$ and hence the third term is identically zero, while the second one is negative, again by a special case of Jensen's inequality: $1 - \frac{1}{V}\int_M \varphi_1 \omega_1^n \leq \frac{1}{V}\int_M e^{-\varphi_1}\omega_1^n = \frac{1}{V}\int_M \omega_2^n = 1$.)

Next, when $1/\mu = \tau = 1$ Proposition 4.12 gives

$$E_k(\omega, \omega_{\varphi_1}) = E_0(\omega, \omega_{\varphi_1}) + I_k(\omega_{\varphi_1}, \mathrm{Ric}\,\omega_{\varphi_1}) - I_k(\omega, \mathrm{Ric}\,\omega). \qquad (220)$$

Next, recall the inequality $I_k \leq J$ (Equation (194)). Since in our case $\mathrm{Ric}\,\omega_{\varphi_1} = \omega$ we deduce from (220) and (217) that

$$E_k(\omega, \omega_{\varphi_1}) \leq -(I-J)(\omega, \omega_{\varphi_1}) + J(\omega_{\varphi_1}, \omega) - I_k(\omega, \mathrm{Ric}\,\omega).$$

Since $J(\omega, \omega_{\varphi_1}) + J(\omega_{\varphi_1}, \omega) = I(\omega, \omega_{\varphi_1})$, one has $E_k(\omega, \omega_{\varphi_1}) \leq 0$ if $I_k(\omega, \mathrm{Ric}\,\omega) \geq 0$, or in other words, if $\omega \in \mathcal{B}_k$. Finally, note that the subspace $\mathcal{B}_k$ is preserved under the iteration since, in fact, after the first step the iteration will stay in $\mathcal{H}_{c_1}^+$. □

Remark 4.18. An alternative derivation of the second part of (ii) could be to choose a particular path connecting ω and ω_1 and note that each of the contributions has a preferred sign. For example, choosing Calabi's continuity path (184) produces three terms of which two are evidently nonnegative. However then one still needs to manipulate the third term which comes up, $-\frac{V^{-1}}{k+1}\int_M \sum_{i=1}^{k} \binom{i+1}{k+1} f_\omega(\sqrt{-1}\partial\bar{\partial} f_\omega)^i \wedge \omega^{n-i}$, and argue that it equals precisely $I_k(\omega, \mathrm{Ric}\omega)$ and then use the results of **[227]** as above. However to derive (i) for all time steps one needs to use instead the path (182) along which E_0 is monotonic, as alluded to after (184) above.

Remark 4.19. Here it is interesting to compare with the Ricci flow. One knows that F_1, E_0 are monotonically decreasing along the flow and that as long as $\mathrm{Ric}\omega > -\omega$ the same is true for E_1 [**63**, §§3.3, Proposition 4.9]. However an analogous result is not known along the Ricci flow for E_k, $k \geq 2$. In the case that a Kähler-Einstein metric exists one knows that the flow will converge. One also knows that when restricted to the space $\mathcal{B}_k$ the functional E_k attains a minimum precisely on the space of Kähler-Einstein metrics, however that outside this space it is not true that these functionals are bounded from below on $\mathcal{H}_{c_1}$ [**227**, §5] (see also §§4.10.2 below). Thus all that is apparent at the present moment is that once the flow stays in $\mathcal{B}_k$, the functional E_k will eventually decrease, however even then we do not know whether this will happen monotonically.

4.4.5 A Bott-Chern expression for the Chen-Tian functionals.

In this subsection we will describe the theory Bott-Chern forms and energy functionals as first explored by Donaldson **[77]**. The inspiration for the constructions can be traced to work of Bott and Chern **[34]**. This will be applied to showing that the Chen-Tian functionals have an expression in terms of Bott-Chern forms. It follows that they are exact functionals as claimed in §§4.4.2.[15] In addition, as an application we will relate the Chen-Tian functionals to geometric stability and GIT in the spirit of the work of Tian.

The main idea of Bott-Chern forms is that given a moduli space of Hermitian metrics on a bundle one may construct canonically defined "universal" functions on it associated to curvature. These functions arise via a "potential" for the curvature form of a "universal" bundle over the whole moduli space.

Let $E \to M$ be a holomorphic vector bundle of rank r.[16] A vector bundle represents a Čech cohomology class in $H^1(M, \mathcal{O}_M(GL(r, \mathbb{C})))$. Here by $\mathcal{O}_M(GL(r, \mathbb{C}))$

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[15] Strictly speaking, the results of this subsection only apply to integral Kähler classes since Bott-Chern forms are constructed using line bundles polarizing Kähler classes. In most of the applications explored in this Thesis $\Omega = c_1$ is an integral class. An alternative proof of exactness for all classes was outlined in subsection 4.4.2.

[16] For the general discussion of Bott-Chern forms we will only make use of the fact that M is

we mean the sheaf of germs of holomorphic functions to $GL(r,\mathbb{C})$. When $r=1$ it is denoted by $\mathcal{O}^\star_M$. Let us identify E with its Čech class representative, i.e., by a collection of transition functions $g=\{g_{\alpha\beta}\}$ that are holomorphic maps from the intersection of any two coordinate neighborhoods $U_\alpha, U_\beta \subset M$ to $GL(r,\mathbb{C})$, $g_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(r,\mathbb{C})$, satisfying the Čech cocycle conditions [**114**, p. 66]

$$(\delta g)_{\alpha\beta\gamma} := g_{\alpha\beta}\cdot g_{\beta\gamma}\cdot g_{\gamma\alpha} = \mathrm{I}.$$

Note that here the groups comprising the sheaf have a multiplicative structure (and not an additive one) and hence $\delta g = \mathrm{I}$ expresses closedness, i.e., $[g]$ represents a Čech cohomology class.

Denote by $\mathcal{H}_E$ the space of all Hermitian metrics on E. Let $\mathrm{Herm}(r)$ denote the space of positive Hermitian $r\times r$ matrices. Any Hermitian metric $H \in \mathcal{H}_E$ can be represented by smooth maps $H_\alpha : U_\alpha \to \mathrm{Herm}(r)$ such that with respect to local bases of sections one has $(H_\alpha)_{i\bar{j}} = (g_{\alpha\beta})_{ik}(H_\beta)_{k\bar{l}}\overline{(g_{\alpha\beta})_{jl}}$, or simply $H_\alpha = g^\star_{\alpha\beta}H_\beta g_{\alpha\beta}$. This in summarized in the notation $H=\{H_\alpha\}$.

To every $H \in \mathcal{H}_E$ there is associated a unique complex connection D_H. Its construction in the case of a line bundle was explained in §§2.1.5. For a general holomorphic vector bundle the connection D_H is a 1-form on M with values in the bundle $\mathrm{End}(E)$ of endomorphisms of E. One may write globally

$$D_H = H^{-1}\circ\partial\circ H.$$

The derivation of this formula follows the same argument as in the line bundle case. The exact meaning of how to understand this and other similar expressions involving compositions of endomorphisms and differential operators will be explained in detail below in several computations. With respect to a local holomorphic frame $e_1,\ldots,e_r$ over $U_\alpha \subset M$ the endomorphism H may be represented by a matrix and one has the special expression $(D_H|_{U_\alpha})^i_j = \partial h_{j\bar{l}}\cdot h^{\bar{l}i}$. This expression is not valid with respect to an arbitrary frame.

The global expression for the curvature is then

$$F_H := D_H\circ D_H = \bar\partial\circ H^{-1}\circ\partial\circ H. \tag{221}$$

Implicit in this notation as well as in the sequel is the convention of working with endomorphisms with values in the exterior algebra of differential forms on M. Locally with respect to a holomorphic frame one has the special expression $F_H|_{U_\alpha} = \bar\partial(\partial h_{j\bar{l}}\cdot h^{\bar{l}i})$. This expression is not valid with respect to an arbitrary frame. However it

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complex (rather than Kähler). In our applications we will always work with line bundles, i.e., $r=1$. If one were interested only in that case the discussion below could be slightly simplified. However we have chosen to maintain this level of generality.

demonstrates that F_H is $(1,1)$-form on M (with values in $\mathrm{End}(E)$), since the type is independent of the choice of frame.

Let ϕ denote an elementary symmetric polynomial on $gl(r,\mathbb{C}) \times \cdots \times gl(r,\mathbb{C})$ (p times) that is invariant under the adjoint action of $GL(r,\mathbb{C})$ (conjugation). The idea behind Chern classes is that one may plug into such polynomials matrices that have differential forms as their entries, for example F_H. Since the polynomials are $GL(r,\mathbb{C})$-invariant one obtains a differential form that is invariant under a change of local trivializations for the bundle, hence is intrinsically defined. Moreover it turns out that such forms are closed hence define intrinsic cohomology classes that depend only on the complex structure of $E \to M$.

Now we come back to our original task of constructing functions on $\mathcal{H}_E$. One may show that $\phi(F_H) := \phi(F_H,\dots,F_H)$ is a closed $2p$-form. It certainly depends on the metric H, however its cohomology class in $H^{2p}(M,\mathbb{Z})$ does not. This means that the difference

$$\phi(F_{H_0}) - \phi(F_{H_1})$$

is exact. Moreover, and here we arrive at the main point of Bott-Chern theory, one may find a $(p-1,p-1)$-form $\mathbf{BC}(\phi; H_0, H_1)$, well-defined up to ∂- and $\bar\partial$-exact forms, such that

$$\bar\partial\partial\mathbf{BC}(\phi; H_0, H_1) = \phi(F_{H_1}) - \phi(F_{H_0}).$$

The form $\mathbf{BC}(\phi; H_0, H_1)$ may then be integrated against a $(n-p+1, n-p+1)$-form on M to give a number. Fixing H_0 and letting H_1 vary we therefore obtain a function on $\mathcal{H}_E$ as desired. And, if $p-1=n$, we even do not need to make a choice of such a form in order to integrate (this will be the case in our applications). We now show how to construct the Bott-Chern form $\mathbf{BC}(\phi; H_0, H_1)$.

Proposition 4.20. (See [**77**, Proposition 6].) *Let ϕ be a $GL(r,\mathbb{C})$-invariant elementary symmetric polynomial. Given $H_0, H_1 \in \mathcal{H}_E$ and any path $\{H_t\}_{t\in[0,1]}$ in $\mathcal{H}_E$ connecting them, the $(p-1,p-1)$-form*

$$\mathbf{BC}(\phi; H_0, H_1) := p(\sqrt{-1})^{p-1}\int_{[0,1]} \phi(H_t^{-1}\dot H_t, F_{H_t},\dots,F_{H_t})dt \mod \operatorname{Im}\partial + \operatorname{Im}\bar\partial, \tag{222}$$

is well-defined, namely does not depend on the choice of path. In addition,

$$\mathbf{BC}(\phi; H_0, H_1) + \mathbf{BC}(\phi; H_1, H_2) + \mathbf{BC}(\phi; H_2, H_0) = 0, \tag{223}$$

and

$$\bar\partial\partial\mathbf{BC}(\phi; H_0, H_1) = \phi(F_{H_1}) - \phi(F_{H_0}). \tag{224}$$

Notice that the first argument of ϕ is an endomorphism while the rest of its arguments are endomorphism-valued 2-forms.

Proof. Note that $\mathbf{BC}(\phi; H_0, H_1)$ is given by integration over a path connecting H_0 and H_1 of a globally defined 1-form on $\mathcal{H}_E$ with values in $(p-1, p-1)$-forms on M. We call this form θ. To show independence of path we show that this form is closed modulo ∂- and $\bar{\partial}$-exact terms. Let $H \in \mathcal{H}_E$. Let $h, k \in T_H\mathcal{H}_E$ and extend them to constant vector fields near H. Now we compute, similarly to as done in (34)-(35),

$$d\theta(h,k) = k\,\theta|_H(h) - h\,\theta|_H(k). \tag{225}$$

First we obtain an expression for the infinitesimal change of the curvature under a variation of a Hermitian metric. Write $H + tk = H \circ (\mathrm{I} + tH^{-1}k) =: H \circ f$. We write $\circ$ to emphasize that composition of endomorphisms is taking place (in coordinates multiplication of matrices). According to (221) we have

$$\begin{aligned} F_{H\circ f} &= \bar{\partial}\big((H\circ f)^{-1} \circ \partial(H \circ f)\big) \\ &= \bar{\partial}\big(f^{-1} \circ (H^{-1} \circ \partial H) \circ f + f^{-1} \circ \partial f\big) \\ &= \bar{\partial} \circ f^{-1} \circ D_H^{1,0} \circ f. \end{aligned} \tag{226}$$

Here it should be emphasized that D_H decomposes according to type (of its 1-form part) into $D_H^{1,0}$ and $D_H^{0,1}$ and that while originally the connection D_H was defined on E it may be extended naturally to $\mathrm{End}(E)$ and it is this extension that we use in the equation above (f is a section of $\mathrm{End}(E)$ and not of E). The same applies to the operator $\bar{\partial}$ that we also extend to act on $\mathrm{End}(E)$. To understand the last equation better we let the endomorphism it defines act on a holomorphic section s of E, and compute:

$$\begin{aligned} F_{H\circ f}s &= \bar{\partial} \circ f^{-1} \circ D_H^{1,0}(fs) \\ &= \bar{\partial} \circ f^{-1} \circ \big((D_H^{1,0}f)s + fD_H^{1,0}s\big) \\ &= \Big(\bar{\partial}(f^{-1}(D_H^{1,0}f)) + \bar{\partial} \circ D_H^{1,0}\Big)s. \end{aligned}$$

Therefore we write

$$F_{H\circ f} = F_H + \bar{\partial}\big(f^{-1}(D_H^{1,0}f)\big),$$

and the second term should be understood to be distinct from (226). This subtle notational issue can be a cause for great confusion when consulting the literature on vector bundles and Yang-Mills theory. Putting now $f = \mathrm{I} + tH^{-1}k$ we obtain

$$F_{H+tk} = F_H + t\bar{\partial}\big(D_H^{1,0}(H^{-1}k)\big) + O(t^2).$$

Hence the first term in (225) is given by

$$\begin{aligned} \frac{1}{p(\sqrt{-1})^{p-1}} k\theta|_H(h) &= \frac{d}{dt}\Big|_0 \phi((H+tk)^{-1}h, F_{H+tk}, \ldots, F_{H+tk}) \\ &= \phi(-H^{-1}kH^{-1}h, F_H, \ldots, F_H) \\ &\quad + \textstyle\sum \phi(H^{-1}h, F_H, \ldots, \bar{\partial}D_H^{1,0}(H^{-1}k), \ldots, F_H). \end{aligned}$$

Therefore one has, putting $\sigma = H^{-1}h,\ \tau = H^{-1}k$,

$$\frac{1}{p(\sqrt{-1})^{p-1}} d\theta(h,k) = \phi([\sigma,\tau], F_H, \ldots, F_H) + \sum \phi(\sigma, F_H, \ldots, \bar{\partial} D_H^{1,0}\tau, \ldots, F_H) \\ - \sum \phi(\tau, F_H, \ldots, \bar{\partial} D_H^{1,0}\sigma, \ldots, F_H). \tag{227}$$

σ and τ are sections of the endomorphism bundle of E. Note that as operators on this bundle one has

$$\bar{\partial} \circ D_H^{1,0} + D_H^{1,0} \circ \bar{\partial} = (\bar{\partial} + D_H^{1,0}) \circ (\bar{\partial} + D_H^{1,0}) = D_H^2 = F_H,$$

since we already saw that the curvature is of type $(1,1)$. Hence for example

$$\bar{\partial} \circ D_H^{1,0}\sigma = -D_H^{1,0} \circ \bar{\partial}\sigma + [F_H, \sigma]. \tag{228}$$

Note $[F_H, \sigma] \equiv F_H\sigma$, and the bracket notation simply emphasizes that we have extended F_H to act on the endomorphism bundle and so the endomorphism part of F_H will actually act by bracket on the endomorphism σ (the 2-form part will simply be multiplied along). By the Bianchi identity $D_H F_H = 0$ and so $D_H^{1,0} F_H = 0,\ \bar{\partial} F_H = 0$. Now,

$$\begin{aligned} \phi(\sigma, \bar{\partial} D_H^{1,0}\tau, F_H, \ldots, F_H) &= \bar{\partial}\phi(\sigma, D_H^{1,0}\tau, F_H, \ldots, F_H) \\ &\quad - \phi(\bar{\partial}\sigma, D_H^{1,0}\tau, F_H, \ldots, F_H) \\ &\quad - \sum \phi(\sigma, D_H^{1,0}\tau, F_H, \ldots, \bar{\partial} F_H, \ldots, F_H) \\ &= \bar{\partial}\phi(\sigma, D_H^{1,0}\tau, F_H, \ldots, F_H) \\ &\quad - \phi(\bar{\partial}\sigma, D_H^{1,0}\tau, F_H, \ldots, F_H). \end{aligned} \tag{229}$$

Using (228) the corresponding term in the second sum of (227) is

$$\begin{aligned} -\phi(\tau, \bar{\partial} D_H^{1,0}\sigma, F_H, \ldots, F_H) &= -\phi(\tau, -D_H^{1,0}\bar{\partial}\sigma + [F_H, \sigma], F_H, \ldots, F_H) \\ &= -\phi(\tau, [F_H, \sigma], F_H, \ldots, F_H) + \partial\phi(\tau, \bar{\partial}\sigma, F_H, \ldots, F_H) \\ &\quad - \phi(D_H^{1,0}\tau, \bar{\partial}\sigma, F_H, \ldots, F_H) \\ &\quad - \sum \phi(\tau, \bar{\partial}\sigma, F_H, \ldots, D_H^{1,0} F_H, \ldots, F_H). \\ &= \phi(\tau, [\sigma, F_H], F_H, \ldots, F_H) + \partial\phi(\tau, \bar{\partial}\sigma, F_H, \ldots, F_H) \\ &\quad - \phi(D_H^{1,0}\tau, \bar{\partial}\sigma, F_H, \ldots, F_H). \end{aligned} \tag{230}$$

Note that it is not necessarily true that

$$-\phi(D_H^{1,0}\tau, \bar{\partial}\sigma, F_H, \ldots, F_H) \tag{231}$$

cancels with

$$-\phi(\bar\partial\sigma, D_H^{1,0}\tau, F_H, \dots, F_H).$$

But, eventually taking the equations (229) and (230) for all pairs appearing in the sums (227) then, e.g., the term (231) will cancel with the term

$$-\phi(\bar\partial\sigma, F_H, \dots, F_H, D_H^{1,0}\tau).$$

Indeed we are only allowed to permute the arguments of ϕ cyclically (e.g., for three matrices A, B, C one has $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$, but in general $\operatorname{tr}(ABC)$ is different from $\operatorname{tr}(BAC)$). Note also that while ϕ does not change when permuting matrices cyclically, when we permute cyclically matrix valued 1-forms a sign appears, as usual. This explains the cancellation above.

Hence, modulo ∂- and $\bar\partial$-exact terms, we are left with

$$\sum \phi(\tau, F_H, \dots, [\sigma, F_H], F_H, \dots, F_H)$$

which cancels with the first term in (227); this can be seen by using the invariance of ϕ under the action of $GL(r, \mathbb{C})$ by conjugation

$$\frac{d}{dt}\bigg|_0 \phi(e^{-tB}A_1e^{tB}, \dots, e^{-tB}A_pe^{tB}) = \sum \phi(A_1, \dots, [A_j, B], \dots, A_p). \qquad \square$$

Example 4.21. Let $E = L$ denote an ample line bundle polarizing a Kähler class $\Omega = c_1(L)$. We identify $\mathcal{H}_E$ with $\mathcal{H}_\omega$ where $\omega = -\sqrt{-1}\partial\bar\partial \log h = \sqrt{-1}\bar\partial(h^{-1}\partial h) = \sqrt{-1}F_h$ is a Kähler form with $h \in \mathcal{H}_E$ (and hence $[\omega] = \Omega$). Now $r = 1$ and so no traces are needed (the matrices are all one-dimensional). Put

$$\phi(A_1, \dots, A_{n+1}) := A_1 \cdots A_{n+1}. \tag{232}$$

Take a path of Hermitian metrics $h_t = e^{-\varphi_t}h$. Then $F_{h_t} = F_h + \partial\bar\partial\varphi_t = \omega_{\varphi_t}/\sqrt{-1}$. Then the Bott-Chern form is

$$\mathbf{BC}(\phi; h_0, h_1) = (n+1)\int_0^1 (\sqrt{-1})^n \phi(-\dot\varphi_t, F_{h_t}, \dots, F_{h_t})dt = -(n+1)\int_0^1 \dot\varphi_t\, \omega_{\varphi_t}^n \wedge dt. \tag{233}$$

This expresses a $2n$-form on M. Integrating this gives the following function on $\mathcal{H}_\omega$,

$$F^0(\varphi) := -(n+1)\int_{M\times[0,1]} \dot\varphi_t\, \omega_{\varphi_t}^n \wedge dt.$$

By Proposition 4.20 this is independent of the choice of path since any two choices differ by ∂- and $\bar\partial$-exact terms and hence by d-exact terms since our expressions are real. The function F^0 goes back to the work of Bando-Mabuchi [**15**, (1.4.1)] and Futaki [**104**] (in the latter this function was only defined on the finite-dimensional orbits of the automorphism group). For a relation between this functional and GIT we refer to [**211**] and references therein.

* * *

Now we turn to the setting of a Kähler manifold with an integral Kähler class $[\omega]$. Let L be a line bundle polarizing the Kähler class, namely $c_1(L) = [\omega]$. Let K_M^{-1} denote the anticanonical bundle polarizing the class c_1. Given a Hermitian metric h on L of positive curvature, one obtains a metric $\det F_h = \det(\omega/\sqrt{-1})$ on K_M^{-1} (we mean that locally $\det F_h = \det(g_{i\bar{j}})$ if $\omega = \sqrt{-1}g_{i\bar{j}}dz^i\wedge\overline{dz^j}$) where $\omega = -\sqrt{-1}\partial\bar{\partial}\log h$. We write $h \in \mathcal{H}_\omega =: \mathcal{H}_L^+$. Note that the Bott-Chern forms defined below are defined on $\mathcal{H}_L^+$ rather than on all of $\mathcal{H}_L$ (or to be more precise on a set isomorphic to $\mathcal{H}_L^+$, see [**272**, p. 214]).

The main result of this subsection is the following expression for the Chen-Tian functionals in terms of Bott-Chern forms. While this generalizes a theorem of Tian for the K-energy, the case $k = 0$, the computations are almost identical to that case [**272**, §2]. In essence, the Chen-Tian functionals are realized as a linear combination of Bott-Chern forms, one for each of the line bundles $E_j = K_M^{-1} \otimes L^{n-2j}$ for $j = 0, \ldots, n$. One of these terms (the simplest contribution) is a multiple of the functional of Example 4.21.

Proposition 4.22. *Let $k \in \{0, \ldots, n\}$ and let ϕ be defined by (232). Let (M, J, ω) be a projective Kähler manifold, and let L be a line bundle with $c_1(L) = [\omega]$. Let μ_k be given by (199). One has*

$$\begin{aligned} dE_k = {} & \frac{2^{-n}}{(n+1)!}\binom{n}{k}^{-1}\frac{1}{V}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-2j)^k\,\mathbf{BC}(\phi; h_0^{n-2j}\det F_{h_0}, h_1^{n-2j}\det F_{h_1}) \\ & - \frac{1}{V}\frac{\mu_{k+1}}{n+1}\frac{n-k}{k+1}\mathbf{BC}(\phi; h_0, h_1). \end{aligned} \tag{234}$$

Proof. One has $F_{h_t} = F_h e^{\log \omega_{\varphi_t}^n/\omega_\varphi^n}$ and

$$\begin{aligned} \sqrt{-1}F_{\det F_{h_t}} &= \mathrm{Ric}\,\omega_{\varphi_t} = \mathrm{Ric}\,\omega_\varphi - \sqrt{-1}\partial\bar{\partial}\log\omega_{\varphi_t}^n/\omega_\varphi^n \\ &= -\sqrt{-1}\partial\bar{\partial}\log\omega_{\varphi_t}^n/\omega_\varphi^n + \sqrt{-1}F_{\det F_h}. \end{aligned}$$

Let $H_t(j) := h_t^{n-2j}\det F_{h_t}$. Observe that $H_t(j)$ is a Hermitian metric on $E_j := K_M^{-1} \otimes L^{n-2j}$. We have

$$H_t(j)^{-1}\dot{H}_t(j) = \Delta_{\varphi_t}\dot{\varphi}_t - (n-2j)\dot{\varphi}_t,$$

and

$$F_{H_t(j)} = F_{\det F_{h_t}} + (n-2j)F_{h_t}.$$

It follows that,

$$\begin{aligned}
&\mathbf{BC}(\phi; \det(R(h_0))h_0^{n-2j}, \det(R(h_1))h_1^{n-2j}) \\
&= \int_{[0,1]} (n+1)(\sqrt{-1})^n \phi(H_t(j)^{-1}\dot{H}_t(j), F_{H_t(j)}, \ldots, F_{H_t(j)})dt \\
&= \int_{[0,1]} (n+1)(\Delta_{\varphi_t}\dot{\varphi}_t - (n-2j)\dot{\varphi}_t)(\mathrm{Ric}\,\omega_{\varphi_t} + (n-2j)\omega_{\varphi_t})^n dt. \quad (235)
\end{aligned}$$

Since we only keep expressions that are $2n$-forms (others integrate to zero on the $2n$-dimensional manifold M) the integrand splits as

$$(n+1)\Delta_{\varphi_t}\dot{\varphi}_t \sum_{l=0}^{n} \binom{n}{l} (n-2j)^l (\mathrm{Ric}\,\omega_{\varphi_t})^{n-l} \wedge \omega_{\varphi_t}^l,$$

and

$$-(n+1)\dot{\varphi}_t \sum_{l=0}^{n} \binom{n}{l} (n-2j)^{l+1} (\mathrm{Ric}\,\omega_{\varphi_t})^{n-l} \wedge \omega_{\varphi_t}^l.$$

Now

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-2j)^l = \begin{cases} 0 & \text{for } l < n \text{ or } l = n+1, \\ 2^n n! & \text{for } l = n. \end{cases}$$

Hence if we take the j-th Bott-Chern form (235) with weight $(-1)^j\binom{n}{j}(n-2j)^k$ and sum over $j = 0, \ldots, n$ we get the following coefficient for $\Delta_{\varphi_t}\dot{\varphi}_t$

$$\begin{aligned}
A_1 :=&(n+1)\sum_{l=0}^{n} \binom{n}{l} (\mathrm{Ric}\,\omega_{\varphi_t})^{n-l} \wedge \omega_{\varphi_t}^l \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-2j)^{k+l} \\
=& 2^n (n+1)! \binom{n}{k} (\mathrm{Ric}\,\omega_{\varphi_t})^k \wedge \omega_{\varphi_t}^{n-k},
\end{aligned}$$

and the following coefficient for $\dot{\varphi}_t$

$$\begin{aligned}
A_2 :=& -(n+1)\sum_{l=0}^{n} \binom{n}{l} (\mathrm{Ric}\,\omega_{\varphi_t})^{n-l} \wedge \omega_{\varphi_t}^l \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-2j)^{k+1+l} \\
=& -2^n (n+1)! \binom{n}{k+1} (\mathrm{Ric}\,\omega_{\varphi_t})^{k+1} \wedge \omega_{\varphi_t}^{n-k-1}.
\end{aligned}$$

Dividing by $2^n(n+1)!\binom{n}{k}$ we therefore obtain, according to (201),

$$\begin{aligned}&\frac{2^{-n}}{(n+1)!}\binom{n}{k}^{-1}\frac{1}{V}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-2j)^k\,\mathbf{BC}(\phi; h_0^{n-2j}\det F_{h_0}, h_1^{n-2j}\det F_{h_1})\\ &=\frac{2^{-n}}{(n+1)!}\binom{n}{k}^{-1}(A_1\Delta_{\varphi_t}\dot{\varphi}_t + A_2\dot{\varphi}_t) = dE_k - \mu_{k+1}\frac{n-k}{k+1}\omega^n.\end{aligned}$$

The Proposition now follows by applying (233). ☐

Tian [**269**, p. 255–257; **270**, (8.5); **272**, (2.5)] gave an interpretation of the "complex Hessian" of the K-energy E_0 in terms of a certain "universal" Hermitian metric **h** (see the references above for the notation and definitions):

$$-\sqrt{-1}\partial\bar{\partial}E_0 = \frac{1}{V}\int_M(\sqrt{-1})^{n+1}\Big(F_{\det F_{\mathbf{h}}} - \frac{n\mu_0}{n+1}F_{\mathbf{h}}\Big)\wedge(F_{\mathbf{h}})^n. \tag{236}$$

We note in passing the following generalization of this formula to the Chen-Tian functionals as a simple corollary of Proposition 4.22 and Tian's original derivation of (236):

$$-\sqrt{-1}\partial\bar{\partial}E_k = \frac{1}{V}\int_M(\sqrt{-1})^{n+1}\Big((F_{\det F_{\mathbf{h}}})^{k+1} - \frac{(n-k)\mu_k}{(k+1)(n+1)}(F_{\mathbf{h}})^{k+1}\Big)\wedge(F_{\mathbf{h}})^{n-k}. \tag{237}$$

Another application of Proposition 4.22 for the Chen-Tian functionals is to generalize Tian's theorem on the relation between geometric stability in terms of Geometric Invariant Theory (GIT) and properness of E_0 to the functionals E_k. Since the proof itself involves no new ideas beyond Proposition 4.22 and the ideas in Tian's proof for $k=0$ we will not go into details here, and simply mention that (using the notation in Tian's article [**272**]) if one uses the virtual bundle

$$(n+1)(k+1)(\mathcal{K}^{-1}-\mathcal{K})^{k+1}\otimes(\mathcal{L}-\mathcal{L}^{-1})^{n-k} - (n-k)\mu_k(\mathcal{L}-\mathcal{L}^{-1})^{n+1},$$

one may obtain a version of Tian's theorem [**272**, Theorem 3.1] for E_k.

Note that one example where this is applicable is the case of an anticanonically polarized Fano manifold admitting a Kähler-Einstein manifold, according to Theorem 4.45 (ii) proven below. In particular, the existence of a Kähler-Einstein metric thus implies that (M, J) is stable in the sense of Tian's theorem. Moreover when such a metric exists, the functional E_n is proper on the space of Kähler metrics with positive Ricci curvature representing any fixed Kähler class, and so one may apply this result also to these cases, relating the existence of "central" metrics in the sense of Maschler [**186**] to GIT. Central metrics are critical points of E_n and satisfy the equation $(\mathrm{Ric}\,\omega)^n = c\,\omega^n$ (see also the footnote in the first paragraph of §§4.4.2).

Whenever a Kähler-Einstein metric ω_{KE} exists they exist by the Calabi-Yau Theorem and are simply the solutions of $\omega^n = \omega^n_{\text{KE}}/c$, i.e., $\text{Ric}\,\omega = \omega_{\text{KE}}$, in each Kähler class with c a cohomologically determined constant (and hence are unique up to automorphisms according to Bando-Mabuchi [**15**]). This is explained by the fact that the functional E_n is identical on all the spaces of Kähler metrics on a Fano manifold since the functional only depends on the volume form of each metric and each of the spaces of Kähler metrics is isomorphic to the space of all volume forms. Moreover the isomorphism between the spaces of Kähler metrics is given by the Calabi-Yau Theorem and is in fact an isometry with respect to the Mabuchi metrics g_{L^2} (33) of each space.

4.5 The Ricci iteration for negative and zero first Chern class

In this section we prove the existence and convergence of the Ricci iteration in the case that either $c_1 < 0$ and $\Omega = -c_1$, or that $c_1 = 0$ and Ω is an arbitrary Kähler class.

We start with a result that is a simple consequence of the theory of elliptic complex Monge-Ampère equations. This result is the existence part of Theorem 4.5 in the cases under consideration.

Lemma 4.23. *Let (M, J) be a compact Kähler manifold whose first Chern class is negative or zero. When $c_1 = 0$ denote by Ω a Kähler class; otherwise let $\Omega = \mu c_1$ denote a Kähler class with $\mu < 0$. Then for any $\omega \in \mathcal{H}_\Omega$, the time τ Ricci iteration exists for all $k \in \mathbb{N}$ and all $\tau \in (0, \infty)$.*

Proof. It is enough to show existence for one step of the iteration in order to show the iteration exists for each $k \in \mathbb{N}$ (by repeating the argument at each step).

Fix $\tau \in (0, \infty)$. The existence of ω_1 amounts to solving the equation

$$\omega_1 = \omega_0 + \tau\mu\omega_1 - \tau\text{Ric}\,\omega_1.$$

Let $\omega_{\varphi_1} = \omega_1$ with $\varphi_1 \in \mathcal{H}_\omega$. This can be written as a complex Monge-Ampère equation:

$$\omega^n_{\varphi_1} = \omega^n e^{f_\omega + (\frac{1}{\tau} - \mu)\varphi_1}, \qquad \int_M \omega^n e^{f_\omega + (\frac{1}{\tau} - \mu)\varphi_1} = V. \tag{238}$$

Under the assumption $\mu < 1/\tau$, and hence in particular if $\mu \leq 0$, the maximum principle gives an a priori L^∞ estimate on φ_1. Then the work of Aubin and Yau [**7**,**296**] immediately applies to give higher-order estimates. We conclude that a unique solution $\omega_{\varphi_1} \in \mathcal{H}_\Omega$ exists. □

We now turn to the proof of the convergence statement of Theorem 4.5 in the cases under consideration.

Proof of Theorem 4.5 ($\mu \le 0$). Assume first that $c_1 < 0$ and let $\Omega = -c_1$. We have the following system of Monge-Ampère equations:

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega + \psi_{k\tau} + \frac{1}{\tau}\varphi_{k\tau}}, \quad k \in \mathbb{N}. \tag{239}$$

We first prove an a priori uniform bound, independent of k in an inductive manner. The first equation reads $\omega^n_{\varphi_\tau} = \omega^n e^{f_\omega + (1+\frac{1}{\tau})\varphi_\tau}$. At the maximum of φ_τ we have $\omega_{\varphi_\tau} \le \omega$ and thus $(1+\frac{1}{\tau}) \sup \varphi_\tau \le -\inf f_\omega$. A similar argument at the minimum of φ_τ gives $-(1+\frac{1}{\tau}) \inf \varphi_\tau \le \sup f_\omega$. The second equation reads $\omega^n_{\varphi_\tau + \varphi_{2\tau}} = \omega^n_{\varphi_\tau} e^{-\frac{1}{\tau}\varphi_\tau + (1+\frac{1}{\tau})\varphi_{2\tau}}$. The maximum/minimum principle now gives $(1+\frac{1}{\tau}) \sup \varphi_{2\tau} \le \frac{1}{\tau} \sup \varphi_\tau$ and $-(1+\frac{1}{\tau}) \inf \varphi_{2\tau} \le -\frac{1}{\tau} \inf \varphi_\tau$ or $\sup \varphi_{2\tau} \le -\frac{\tau}{(1+\tau)^2} \inf f_\omega$ and $-\inf \varphi_{2\tau} \le \frac{\tau}{(1+\tau)^2} \sup f_\omega$. We then have $\sup \psi_{k\tau} \le -\inf f_\omega$, $-\inf \psi_{k\tau} \le \sup f_\omega$. This uniform bound implies the existence of an a priori $C^{2,\alpha}$ bound on $\psi_{k\tau}$, independently of k. Such a claim would follow directly from Aubin and Yau's arguments if the term $\varphi_{k\tau}$ did not appear in the right hand side of (239).[17] To justify the claim in our context where such a term does appear we will argue differently in order to obtain a uniform estimate for $\Delta_\omega \varphi_{k\tau}$. Such an estimate then implies a uniform $C^{2,\alpha}$ estimate using standard elliptic regularity techniques for the nondegenerate Monge-Ampère equation [**25**, §5]. To obtain the Laplacian estimate we follow Bando-Kobayashi's derivation of a Laplacian estimate in a different context, that adapts to our setting [**13**, p. 179]. Let $f : (Q, \alpha_1) \to (R, \alpha_2)$ be a map between two complete Kähler manifolds. The Chern-Lu inequality in the context of Yau's Schwarz Lemma gives [**171,295**]

$$\Delta_{\alpha_1} \log |\partial f|^2 \ge \frac{\operatorname{Ric} \alpha_1(\partial f, \bar\partial f)}{|\partial f|^2} - \frac{\operatorname{Bisect}_{\alpha_2}(\partial f, \bar\partial f, \partial f, \bar\partial f)}{|\partial f|^2}.$$

Let now f be the identity map from $(M, \omega_{\psi_{k\tau}})$ to (M, ω). In our setting we have

$$\operatorname{Ric} \omega_{\psi_{k\tau}} = -(1+1/\tau)\omega_{\psi_{k\tau}} + 1/\tau\, \omega_{\psi_{(k-1)\tau}} > -(1+1/\tau)\omega_{\psi_{k\tau}},$$

that is the Ricci curvature is uniformly bounded from below along the iteration. Since $|\partial f|^2 = \operatorname{tr}_{\omega_{\psi_{k\tau}}} \omega = n - \Delta_{\omega_{\psi_{k\tau}}} \psi_{k\tau}$ it follows that

$$\Delta_{\omega_{\psi_{k\tau}}} \log(n - \Delta_{\omega_{\psi_{k\tau}}} \psi_{k\tau}) \ge -C(1 + n - \Delta_{\omega_{\psi_{k\tau}}} \psi_{k\tau}),$$

for some uniform constant $C > 0$, and hence

$$\Delta_{\omega_{\psi_{k\tau}}} \Big(\log(n - \Delta_{\omega_{\psi_{k\tau}}} \psi_{k\tau}) - (C+1)\psi_{k\tau} \Big) \ge -n - C(1+n) + (n - \Delta_{\omega_{\psi_{k\tau}}} \psi_{k\tau}). \tag{240}$$

We may now apply Yau's maximum principle [**295**] to obtain a uniform upper bound for $\operatorname{tr}_{\omega_{\psi_{k\tau}}} \omega$, using the fact that we already have a uniform L^∞ estimate for $\psi_{k\tau}$.

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[17] I am grateful to V. Tosatti for pointing this out to me.

Now observe the uniform L^∞ on $\psi_{k\tau}$ together with (239) implies that the volume forms $\omega^n_{\psi_{k\tau}}$ converge in L^∞ to a fixed uniformly positive and bounded volume form. Namely, $\omega^n_{\psi_{k\tau}}/\omega^n = \det_\omega \omega_{\psi_{k\tau}}$ is uniformly bounded. This then implies that also $\operatorname{tr}_\omega \omega_{\psi_{k\tau}}$ is uniformly bounded, namely we have a uniform bound for $n + \Delta_\omega \psi_{k\tau}$ as required.

As a result, by elliptic regularity theory, a subsequence converges to a smooth solution which we denote by ψ_∞. In fact the convergence is exponentially fast and there is no need to take a subsequence: $||\psi_{k\tau} - \psi_{(k-1)\tau}||_{C^{2,\alpha}} \leq C\tau(1+\tau)^{-k}$.

Now, by Proposition 4.17, we notice that unless ω_0 is itself Kähler-Einstein, the functional E_0 is strictly decreasing along the iteration. In particular, since ω_∞ is a fixed point of the iteration it must be Kähler-Einstein.

We now consider the case $\mu = 0$, for which we have the following system of equations,

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega + \frac{1}{\tau}\varphi_{k\tau}}, \quad k \in \mathbb{N}. \tag{241}$$

We may rewrite this as $\omega^n_{\psi_{k\tau}} = \omega^n_{\psi_{(k-1)\tau}} e^{-\frac{1}{\tau}\varphi_{(k-1)\tau} + \frac{1}{\tau}\varphi_{k\tau}}$ from which we have $\sup \varphi_{k\tau} \leq \sup \varphi_{(k-1)\tau} \leq \ldots \leq -\tau \inf f_\omega$. Therefore we have

$$||e^{f_\omega + \frac{1}{\tau}\varphi_{k\tau}}||_{L^\infty(M)} \leq e^{\operatorname{osc} f_\omega}, \quad \forall\, k \in \mathbb{N}.$$

Now, by Yau's work it follows that there exists an a priori L^∞ bound on $\psi_{k\tau}$, independently of k. We may now invoke the same arguments as before since once again the Ricci curvature is uniformly bounded from below (this time by $-1/\tau$), and again the uniform L^∞ estimate together with (241) implies that the volume ratios $\omega^n_{\psi_{k\tau}}/\omega^n$ are uniformly bounded. It follows that a uniform $C^{2,\alpha}$ estimate holds.

Combined with the monotonicity result it follows, as before, that a subsequence converges to a Kähler potential of a Kähler-Einstein metric. Moreover, since the Kähler-Einstein metric is unique (in each fixed Kähler class) **[41]** any converging subsequence will necessarily converge to the same limit point. This then implies that our original sequence converges to this limit. □

4.6 The Ricci iteration for positive first Chern class

We turn to the study of the iteration on Fano manifolds that, as noted in the Introduction, is our main motivation for introducing the Ricci iteration. Most of the applications described in Section 4.10 are for this class of manifolds.

We first introduce an operator that arises very naturally although it seems to have not been defined previously in the literature. It exists and is well-defined by the Calabi-Yau Theorem **[296]**.

Definition 4.24. *Define the inverse Ricci operator* $\mathrm{Ric}^{-1}:\mathcal{D}_{c_1}\to\mathcal{H}_{c_1}$ *by letting* $\mathrm{Ric}^{-1}\omega:=\omega_\varphi$ *with* ω_φ *the unique Kähler form in* $\mathcal{H}_{c_1}$ *satisfying* $\mathrm{Ric}\,\omega_\varphi=\omega$. *Similary denote higher order iterates of this operator by* Ric^{-l} *for each* $l\in\mathbb{N}$. *Let* $\mathrm{Ric}^0:=\mathrm{Id}$ *denote the identity operator.*

There exists a generalization of this operator to any Kähler manifold (Definition 4.32). For another direction in which this operator may be generalized see Definition 4.35.

We then see that the dynamical system corresponding to the time one Ricci iteration on a Fano manifold with $\mu=1$ is nothing but the evolution of iterates of the inverse Ricci operator,

$$\omega_l=\mathrm{Ric}^{-l}\omega_0.$$

The following result concerns the "allowed" time steps in the iteration for any Fano manifold and is well-known. Note that unlike in the previous, unobstructed, cases, the allowed range for the time step is restricted unless an analytic "semi-stability" condition holds.

Define

$$\tau_M(G)=\sup\{\ t\ :\ \text{(182) has a solution for each } \tau\in(0,t) \text{ and } \omega\in\mathcal{H}_{c_1}(G)\}. \tag{242}$$

By definition this is a holomorphic invariant. Recall also the definition of the following holomorphic invariants studied by Tian [**264**,**268**]

$$\alpha_M(G)=\sup\{\ a\ :\ \sup_{\varphi\in\mathcal{H}_\omega(G)}\frac{1}{V}\int_M e^{-a(\varphi-\sup\varphi)}\omega^n<\infty\}, \tag{243}$$

$$\tilde\beta_M(G)=\sup\{\ b\ :\ \mathrm{Ric}\,\omega\geq b\omega,\ \omega\in\mathcal{H}_\Omega(G)\}, \tag{244}$$

where in (243) ω is any element of $\mathcal{H}_{c_1}(G)$.

Lemma 4.25. *Let* (M,J) *be a Fano manifold and let* G *be a compact subgroup of* $\mathrm{Aut}(M,\mathrm{J})$.
(i) For any $\omega\in\mathcal{H}_{c_1}(G)$, *the time* τ *Ricci iteration exists for all* $k\in\mathbb{N}$ *and all* $\tau\in[0,\tau_M(G))$. *One has* $\frac{1}{1-\tilde\beta_M(G)}\geq\tau_M(G)\geq\left|\frac{1}{\max\{1-(n+1)\alpha_M(G)/n,0\}}\right|>1$.
(ii) Assume that E_0 *is bounded from below on* $\mathcal{H}^+_{c_1}(G)$. *Then* $\tau_M(G)=\infty$.

Proof. (i) By the Calabi-Yau Theorem $\tau_M(G)\geq1$. According to Tian [T1] the path (183) exists for each $s\in[0,(n+1)\alpha_M(G)/n)\cap[0,1]$ whenever $\omega\in\mathcal{H}_{c_1}(G)$. Note that $\tau=1/(1-s)$ and that $\alpha_M(G)>0$.
(ii) This is equivalent to a result of Bando and Mabuchi [**15**, Theorem 5.7]. □

Combined with Theorem 4.44 we therefore obtain the existence part of Theorem 4.5 for $\mu>0$.

Corollary 4.26. *Let (M, J) be a Kähler-Einstein Fano manifold. Then for any $\omega \in \mathcal{H}_{c_1}$, the time τ Ricci iteration exists for all $k \in \mathbb{N}$ and all $\tau \in (0, \infty)$.*

We now turn to the proof of the remaining part of Theorem 4.5. We assume for simplicity, as in the statement of the theorem, that $\mathrm{aut}(M, \mathrm{J}) = \{0\}$ (for the additional details necessary for the general case we refer to **[231]** (see also §4.9)). We set $\mu = 1$. The computations for other values of $\mu > 0$ will then follows by rescaling. We consider the following system of equations on $\mathcal{H}_\omega$ corresponding to the Ricci iteration on $\mathcal{H}_{c_1}$:

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega - \psi_{k\tau} + \frac{1}{\tau}\varphi_{k\tau}}, \quad k \in \mathbb{N}. \tag{245}$$

Let $G_{k\tau}$ be a Green function for $-\Delta_{k\tau} = -\Delta_{\bar{\partial}, \omega_{\psi_{k\tau}}}$ satisfying $\int_M G_{k\tau}(\cdot, y)\omega^n_{\psi_{k\tau}}(y) = 0$. Set $A_{k\tau} = -\inf_{M\times M} G_{k\tau}$. Since $-n < \Delta_0 \psi_{k\tau}$ and $n > \Delta_{k\tau}\psi_{k\tau}$ the Green formula gives

$$\psi_{k\tau}(x) - \frac{1}{V}\int_M \psi_{k\tau}\omega^n_0 = -\frac{1}{V}\int_M G_0(x,y)\Delta_0\psi_{k\tau}(y)\omega^n_0(y) \leq nA_0, \tag{246}$$

$$\psi_{k\tau}(x) - \frac{1}{V}\int_M \psi_{k\tau}\omega^n_{\psi_{k\tau}} = -\frac{1}{V}\int_M G_{k\tau}(x,y)\Delta_{k\tau}\psi_{k\tau}(y)\omega^n_{\psi_{k\tau}}(y) \geq -nA_{k\tau}. \tag{247}$$

Hence

$$\sup \psi_{k\tau} - \inf \psi_{k\tau} =: \operatorname{osc} \psi_{k\tau} \leq n(A_0 + A_{k\tau}) + I(\omega_0, \omega_{\psi_{k\tau}}). \tag{248}$$

Since by Theorem 4.44 (ii) E_0 is proper on $\mathcal{H}_{c_1}$ in the sense of Tian, if $E_0(\omega, \cdot)$ is uniformly bounded from above on a subset of $\mathcal{H}_{c_1}$ so is $I(\omega, \cdot)$. By the monotonicity of E_0 along the iteration we conclude that $I(\omega, \omega_{\psi_{k\tau}})$ is uniformly bounded independently of k.

To get a uniform bound on the oscillation it remains to bound $A_{k\tau}$. This can be done using a special case of Bando and Mabuchi's Green's function estimate that we now state.

Theorem 4.27. (See **[15**, Theorem 3.2].) *Let (N, h) be a connected compact closed Riemannian manifold of nonnegative Ricci curvature. Let G_h denote the Green function of $d^{\star h} \circ d + d \circ d^{\star h}$ satisfying $\int_N G(x,y)dV_h(y) = 0$ for each $x \in N$ and let $A_h = -\inf_{M\times M} G_h$. Then*

$$A_h \leq c_n \frac{\mathrm{diam}(N,h)^2}{\mathrm{Vol}(N,h)},$$

with c_n depending only on n.

Now, along the iteration it holds $\mathrm{Ric}\,\omega_k > (\tau - 1)\omega_k > 0$. By Myers' Theorem **[212**, p. 245] then

$$\mathrm{diam}(M, \omega_k)^2 \leq \pi^2(2n-1)/(\tau - 1). \tag{249}$$

It follows that

$$\operatorname{osc}\psi_{k\tau} \leq C, \tag{250}$$

with C a positive constant indpendent of k.

We now rewrite (245) as

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega - (1-\frac{1}{\tau})\psi_{k\tau} - \frac{1}{\tau}\psi_{(k-1)\tau}}. \tag{251}$$

Since $\frac{1}{V}\int_M e^{f_\omega}\omega^n = 1$ (see §§4.4.3) it follows that the function $(1-\frac{1}{\tau})\psi_{k\tau} + \frac{1}{\tau}\psi_{(k-1)\tau}$ changes signs. Since $\tau > 1$ then in particular we have

$$(1-1/\tau)\sup\psi_{k\tau} + 1/\tau\sup\psi_{(k-1)\tau} \geq 0, \tag{252}$$

$$(1-1/\tau)\inf\psi_{k\tau} + 1/\tau\inf\psi_{(k-1)\tau} \leq 0. \tag{253}$$

The first inequality implies that at least one of the numbers $\sup\psi_{k\tau}, \sup\psi_{(k-1)\tau}$ is nonnegative. Assume without loss of generality that $\sup\psi_{k\tau} \geq 0$. Then one of the following cases must arise: (i) $\sup\psi_{k\tau} \geq 0 \geq \inf\psi_{k\tau}$ and $\sup\psi_{(k-1)\tau} \geq \inf\psi_{(k-1)\tau} \geq 0$, or (ii) $\sup\psi_{k\tau} \geq 0 \geq \inf\psi_{k\tau}$ and $0 \geq \sup\psi_{(k-1)\tau} \geq \inf\psi_{(k-1)\tau}$, or (iii) $\sup\psi_{k\tau} \geq \inf\psi_{k\tau} \geq 0$ and $\sup\psi_{(k-1)\tau} \geq 0 \geq \inf\psi_{(k-1)\tau}$, or (iv) $\sup\psi_{k\tau} \geq \inf\psi_{k\tau} \geq 0$ and $0 \geq \sup\psi_{(k-1)\tau} \geq \inf\psi_{(k-1)\tau}$, or (v) $\sup\psi_{k\tau} \geq 0 \geq \inf\psi_{k\tau}$ and $\sup\psi_{(k-1)\tau} \geq 0 \geq \inf\psi_{(k-1)\tau}$. In case (v) both of the functions change signs and so the oscillation estimate implies an L^∞ bound on each of them. Let us now consider the other cases.

In case (i) we have $||\psi_{k\tau}||_{L^\infty} \leq C$ with C as in (250). By (253) we have that $\frac{1}{\tau}\inf\psi_{(k-1)\tau} \leq -(1-\frac{1}{\tau})\inf\psi_{k\tau} < (1-\frac{1}{\tau})C$, and combined with (250) we have $||\psi_{(k-1)\tau}||_{L^\infty} \leq \tau C$. In case (ii) we again have $||\psi_{k\tau}||_{L^\infty} \leq C$. By (252) then $\frac{1}{\tau}\sup\psi_{(k-1)\tau} \geq -(1-\frac{1}{\tau})\sup\psi_{k\tau} > -(1-\frac{1}{\tau})C$, and combined with (250) we have $||\psi_{(k-1)\tau}||_{L^\infty} \leq \tau C$. In case (iii) we have $||\psi_{(k-1)\tau}||_{L^\infty} \leq C$. By (253) then $(1-\frac{1}{\tau})\inf\psi_{k\tau} \leq -\frac{1}{\tau}\inf\psi_{(k-1)\tau} < \frac{1}{\tau}C$, and using (250) we have $||\psi_{k\tau}||_{L^\infty} \leq \frac{\tau}{\tau-1}C$.

Now consider case (iv). If $\sup\psi_{(k+1)\tau} \geq 0 \geq \inf\psi_{(k+1)\tau}$ the same argument as in case (i) then implies that $||\psi_{k\tau}||_{L^\infty} \leq \tau C$. So we may assume that $0 \geq \sup\psi_{(k+1)\tau} \geq \inf\psi_{(k+1)\tau}$. Furthermore, we may assume that for some k_0 and all $l \geq k_0$ the functions $\psi_{l\tau}$ do not change signs and that for the sequence $\{\psi_{l\tau}\}_{l\geq k_0}$ each nonnegative function is followed by a nonpositive function. Note $k_0 \geq 2$ since the function $\psi_{1\tau}$ itself changes signs as can be seen from the equation $\omega^n_{\psi_{1\tau}} = \omega^n e^{f_\omega - (1-\frac{1}{\tau})\psi_{1\tau}}$ and the fact that $\frac{1}{V}\int_M e^{f_\omega}\omega^n = 1$. We now argue inductively. Assume without loss of generality that $k_0 = 2$. One has $||\psi_{1\tau}||_{L^\infty} \leq C$. First, assume $\psi_{2\tau}$ is nonnegative. Then by case (iii) $\inf\psi_{2\tau} \leq \frac{1}{\tau-1}C$ and $\sup\psi_{2\tau} \leq (\frac{1}{\tau-1}+1)C$. Next, $\psi_{3\tau}$ is nonpositive and $(1-\frac{1}{\tau})\sup\psi_{3\tau} \geq -\frac{1}{\tau}\sup\psi_{2\tau}$. Hence $\sup\psi_{3\tau} \geq -\frac{1}{\tau-1}(\frac{1}{\tau-1}+1)C$ and $-||\psi_{3\tau}||_{L^\infty} = \inf\psi_{3\tau} \geq -\big(\frac{1}{\tau-1}(\frac{1}{\tau-1}+1)+1\big)C$. By induction it follows that $||\psi_{k\tau}||_{L^\infty} \leq C\sum_{j=0}^{k-1}\frac{1}{(\tau-1)^j} < \frac{\tau-1}{\tau-2}C$, when $\tau > 2$. Second, if we assume instead that

$\psi_{2\tau}$ is nonpositive then $(1-\frac{1}{\tau})\sup\psi_{2\tau} \geq -\frac{1}{\tau}\sup\psi_{1\tau} \geq -\frac{1}{\tau}C$ so $\sup\psi_{2\tau} \geq -\frac{1}{\tau-1}C$ and $\inf\psi_{2\tau} \geq -(\frac{1}{\tau-1}+1)C$. Now $\psi_{3\tau}$ is nonnegative and $(1-\frac{1}{\tau})\inf\psi_{3\tau} \leq -\frac{1}{\tau}\inf\psi_{2\tau}$, and so $\sup\psi_{3\tau} \leq (\frac{1}{\tau-1}(\frac{1}{\tau-1}+1)+1)C$. As before it follows that when $\tau > 2$ we have $||\psi_{k\tau}||_{L^\infty} < \frac{\tau-1}{\tau-2}C$. In sum, when $\tau > 2$ we have a uniform estimate on $||\psi_{k\tau}||_{L^\infty}$, independently of k.

Now when τ is not necessary larger than 2 we will need to normalize $\psi_{k\tau}$, namely put $\tilde{\psi}_{k\tau} := \psi_{k\tau} - \frac{1}{V}\int_M \psi_{k\tau}\omega^n$. The estimate (250) implies that $||\tilde{\psi}_{k\tau}||_{L^\infty} \leq C$. We now want to show uniform higher derivative estimates for $\tilde{\psi}_{k\tau}$. The key now is to show that the volume forms $\omega^n_{\tilde{\psi}_{k\tau}} = \omega^n_{\psi_{k\tau}}$ are uniformly positive and bounded. By (251) that is equivalent to showing that $(1-\frac{1}{\tau})\psi_{k\tau} + \frac{1}{\tau}\psi_{(k-1)\tau}$ is uniformly bounded. We may assume that we are in case (iv) and that each nonnegative function in $\{\psi_{l\tau}\}_{l\geq 2}$ is followed by a nonpositive one. We may also assume $\psi_{k\tau}$ is positive and $\psi_{(k-1)\tau}$ is negative, that $B_l := ||\psi_{l\tau}||_{L^\infty}$ satisfies $\lim_{\mathbb{N}\ni l\to\infty} B_l = \infty$, and that B_{k-1} is much larger than C, say $B_{k-1} > 2519((\tau+1)C+1)$. We have $-\inf\psi_{(k-1)\tau} = B_{k-1}$. Then $-\sup\psi_{(k-1)\tau} \geq B_{k-1} - C$. Now from (252) we have $(1-\frac{1}{\tau})\sup\psi_{k\tau} > -\frac{1}{\tau}\sup\psi_{(k-1)\tau}$, namely $\sup\psi_{k\tau} \geq \frac{1}{\tau-1}(B_{k-1}-C)$, and so $\inf\psi_{k\tau} \geq \frac{1}{\tau-1}(B_{k-1}-C) - C$. From (253) we have that $(1-\frac{1}{\tau})\inf\psi_{k\tau} < -\frac{1}{\tau}\inf\psi_{(k-1)\tau}$, thus $\inf\psi_{k\tau} < \frac{1}{\tau-1}B_{k-1}$, and so $\sup\psi_{k\tau} < \frac{1}{\tau-1}B_{k-1} + C$. Combining these inequalities we have shown that $\frac{1}{\tau-1}(B_{k-1}-C) - C \leq \psi_{k\tau} \leq \frac{1}{\tau-1}B_{k-1} + C$. Since $-B_{k-1} \leq \psi_{(k-1)\tau} \leq -B_{k-1} + C$ it follows that

$$-C \leq (1-1/\tau)\psi_{k\tau} + 1/\tau\,\psi_{(k-1)\tau} \leq C,$$

as required. Thus we have shown that $1/C' < \omega^n_{\tilde{\psi}_{k\tau}}/\omega^n < C'$ for some constant $C' > 0$, independently of k.

Now the Laplacian estimate goes through for $\tilde{\psi}_{k\tau}$ just as in (240) since that inequality is not sensitive to changing $\psi_{k\tau}$ by a constant, and since, as before, we have a uniform lower bound for the Ricci curvature along the iteration. This gives a uniform estimate on $\operatorname{tr}_{\omega_{\tilde{\psi}_{k\tau}}}\omega = n - \Delta_{\omega_{\tilde{\psi}_{k\tau}}}\tilde{\psi}_{k\tau}$. Using the volume ratio estimate proven in the previous paragraph this implies a uniform estimate on $\operatorname{tr}_\omega \omega_{\tilde{\psi}_{k\tau}} = n + \Delta_\omega \tilde{\psi}_{k\tau}$ and subsequently by elliptic regularity also $C^{2,\alpha}$ and higher estimates, as explained in Section 4.5. We may thus extract from $\{\tilde{\psi}_{k\tau}\}$ a converging subsequence in $C^{2,\alpha}$ that converges to a smooth limit. Thanks to the monotonicity of E_0 it must converge to a Kähler potential for a Kähler-Einstein metric. Since such a metric is unique [**15**, Remark 9.3] the same argument as before gives the convergence of the full orbit $\{\omega_{\psi_{k\tau}}\}_{k\geq 0}$ of the Ricci iteration. □

4.7 The Kähler-Ricci flow and the Ricci iteration for a general Kähler class

A natural question is whether on an arbitrary Kähler manifold one may define an iteration scheme generalizing the Ricci iteration. To answer this question of course one first needs to generalize the Ricci flow itself. In this section we recall one such possibility. We end with a conjecture regarding the convergence of this iteration.

A flow on the space of Kähler forms $\mathcal{H}_\Omega$ can be considered as an integral curve of a vector field on this space. A vector field χ on $\mathcal{H}_\Omega$ is an assignment $\omega \mapsto \chi_\omega \in C^\infty(M)/\mathbb{R}$. The Ricci flow describes the dynamics of minus the Ricci potential vector field $-f$. Recall that the vector field f is the assignment $\omega \mapsto f_\omega$ with f_ω defined by $\operatorname{Ric}\omega - \mu\omega = \sqrt{-1}\partial\bar\partial f_\omega$, $\mu \in \mathbb{R}$, where $\mu\Omega = c_1$.

The Ricci iteration in turn can be thought of as a piecewise linear trajectory in $\mathcal{H}_\Omega$ induced from the Ricci potential vector field $-f$ and approximating its integral curves.

Motivated by this one is naturally led to extend the definition of the Ricci flow (173) to an arbitrary Kähler manifold, simply by defining the flow lines to be integral curves of minus the Ricci potential vector field $-f$ on $\mathcal{H}_\Omega$, with Ω an arbitrary Kähler class. Recall that the Ricci potential is defined in general by $\operatorname{Ric}\omega - H_\omega \operatorname{Ric}\omega = \sqrt{-1}\partial\bar\partial f_\omega$. The resulting flow equation can be written as

$$\begin{aligned} \frac{\partial \omega(t)}{\partial t} &= -\operatorname{Ric}\omega(t) + H_t \operatorname{Ric}\omega(t), \quad t \in \mathbb{R}_+, \\ \omega(0) &= \omega, \end{aligned} \tag{254}$$

for each t for which a solution exists in $\mathcal{H}_\Omega$ (throughout subscripts are meant to indicate that the relevant object corresponds to the metric indexed by that subscript). This flow, introduced by Guan, is part of the folklore in the field although it has not been much studied.[18]

Corresponding to this flow we introduce the following dynamical system on $\mathcal{H}_\Omega$ which generalizes Definition 4.3.

Definition 4.28. *Given a Kähler form $\omega \in \mathcal{H}_\Omega$ let the time τ Ricci iteration be the sequence of forms $\{\omega_{k\tau}\}_{k\geq 0}$, satisfying the equations*

$$\begin{aligned} \omega_{k\tau} &= \omega_{(k-1)\tau} + \tau H_{k\tau}\operatorname{Ric}\omega_{k\tau} - \tau \operatorname{Ric}\omega_{k\tau}, \quad k \in \mathbb{N}, \\ \omega_0 &= \omega, \end{aligned} \tag{255}$$

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[18] It seems that Guan first considered this flow in unpublished work in the 90's (see references to **[115]**). After completing the present work and posting **[230]** I also became aware, thanks to G. Székelyhidi, of a recent preprint **[117]** posted by Guan on his webpage in which this flow is studied. We hope that the elementary discussion in this section is still of some interest even though it was written before learning of **[115,117]**. For a different but related flow see **[246]**.

for each $k \in \mathbb{N}$ for which a solution exists in $\mathcal{H}_\Omega$.

As in Section 4.3, setting $k = 1$ and varying τ defines a continuity path that is of independent interest.

An observation that goes back to Calabi characterizes the equilibrium state of the flow and the iteration.

Lemma 4.29. (See [**39**, Theorem 1].) *The Ricci form of a Kähler metric is a harmonic representative of c_1 with respect to the metric if and only if its scalar curvature is constant.*

Proof. One has

$$n\mathrm{Ric}\,\omega \wedge \omega^{n-1} = \mathrm{tr}_\omega \mathrm{Ric}\,\omega\ \omega^n = s(\omega)\omega^n.$$

Let L denote the Lefschetz operator that takes a form and wedges it with ω. Since ω is a harmonic representative of its class, the identity $[L, \Delta] = 0$ [**114**, p. 115] implies that $s(\omega)$ is harmonic, i.e., constant, if and only if $\mathrm{Ric}\,\omega$ is. □

An infinitesimal automorphism $X \in \mathrm{aut}(M, \mathrm{J})$ naturally induces a vector field ψ^X on $\mathcal{H}_\Omega$ given by

$$\psi^X : \omega \mapsto \psi^X_\omega \in C^\infty(M)/\mathbb{R}, \quad \text{where } \mathcal{L}_X\omega = \sqrt{-1}\partial\bar\partial\psi^X_\omega. \tag{256}$$

Recall the following generalization of the notion of a constant scalar curvature Kähler metric, due to Guan. Alternatively it may be seen as a generalization of the notion of a Kähler-Ricci soliton to an arbitrary class.

Definition 4.30. (See [**115**].) *Let $X \in \mathrm{aut}(M, \mathrm{J})$. A Kähler metric ω will be called a Kähler-Ricci soliton if it satisfies*

$$\mathrm{Ric}\,\omega - H_\omega \mathrm{Ric}\,\omega = \mathcal{L}_X\omega. \tag{257}$$

Equivalently, if the vector field $\psi^X - f$ on $\mathcal{H}_{[\omega]}$ has a zero at ω.

We remark that the notion of a Ricci soliton for the case of definite first Chern class goes back at least to the work of Friedan [**97**,**98**].

Motivated by the results for Kähler-Einstein manifolds we believe the following conjecture should hold.

Conjecture 4.31. *Let (M, J) be a compact closed Kähler manifold, and assume that there exists a constant scalar curvature Kähler metric representing the class Ω. Then for any $\omega \in \mathcal{H}_\Omega$, the Kähler-Ricci flow (254) and the Ricci iteration (255) exist and converge in an appropriate sense to a constant scalar curvature metric.*

Similarly, we believe an analogous result should hold for Kähler-Ricci solitons (257) using the twisted constructions of Section 4.9.

4.8 Another flow and the inverse Ricci operator for a general Kähler class

Our purpose in this section is to explain why the inverse Ricci operator—that appeared as a very singular iterative construction for anticanonically polarized Fano manifolds—is in fact a special case of a more general construction on any Kähler manifold. This gives another application of our approach explained in the Introduction since it involves a discretization of another geometric flow equation.

To that end, given a Kähler form ω let us consider the flow equations

$$\begin{aligned} \frac{\partial \mathrm{Ric}\,\omega(t)}{\partial t} &= -\mathrm{Ric}\,\omega(t) + H_t \mathrm{Ric}\,\omega(t), \quad t \in \mathbb{R}_+, \\ \omega(0) &= \omega, \end{aligned} \tag{258}$$

for each t for which a solution exists.

The following brief and informal discussion comes to motivate this definition. Consider the case when the first Chern class is definite ($\mu \in \mathbb{R} \setminus \{0\}$ with $\Omega = \mu c_1$), or zero (Ω is arbitrary), and take $\omega \in \mathcal{H}_\Omega$. The evolution equation then becomes

$$\frac{\partial \mathrm{Ric}\,\omega(t)}{\partial t} = -\mathrm{Ric}\,\omega(t) + \mu\omega(t). \tag{259}$$

Assume momentarily that the flow preserves the Kähler class and that it exists on some time interval $[0, T]$. Then on the level of potentials it can be written as

$$-\Delta_t \dot{\varphi}_t = \log \frac{\omega_{\varphi_t}^n}{\omega^n} + \mu\varphi_t - f_\omega + a_t, \quad \varphi_0 = \text{const}, \tag{260}$$

or as a Monge-Ampère equation

$$\omega_{\varphi_t}^n = \omega^n e^{f_\omega - \mu\varphi_t - \Delta_t\dot{\varphi}_t - a_t}, \tag{261}$$

with a_t a certain normalizing constant. Set $u := \Delta_t \dot{\varphi}_t$. A time derivative of (260) gives

$$\frac{du}{dt} = -u + \mu G_t u + b_t,$$

with b_t another normalizing constant. One may show that

$$||u||_{L^\infty(M\times[0,T])} < Ce^{-t},$$

when $\mu \leq 0$ and that $||u||_{L^\infty(M\times[0,T])} < Ce^{(\mu/\lambda_1(t)-1)t}$, when $\mu > 0$, where $\lambda_1(t)$ is the first nonzero eigenvalue of $-\Delta_t$. The constant C depends a priori on t. Going back to (261) one may show an a priori estimate $||\varphi_t||_{L^\infty(M\times[0,T])} < C_1$, with C_1 depending only on ω, whenever $\mu \leq 0$. This then should imply a priori estimates on higher order derivatives under some assumptions. Finally, take a converging subsequence. Along this subsequence λ_1 is uniformly bounded away from zero. Going back to the exponential decay of u we apply uniform Schauder estimates to conclude that $\dot{\varphi}_t$ is uniformly decaying. It then follows that the limit is a Kähler-Einstein metric. By uniqueness of the metric one then argues, as earlier on, that the flow itself converges exponentially fast to a Kähler-Einstein metric. On the other hand, the case $\mu > 0$ would require more work, quite likely in the spirit of the corresponding result for the Ricci flow [**48**,**280**] (cf. also [**8**,**15**]).

Motivated by this discussion, we introduce the following dynamical system on $\mathcal{H}_\Omega$ obtained as the time one Euler method for this flow:

$$\begin{aligned} \operatorname{Ric}\omega_{k+1} &= H_k \operatorname{Ric}\omega_k, \quad k \in \mathbb{N}, \\ \omega_0 &= \omega. \end{aligned} \tag{262}$$

It can be thought of as describing the dynamics of a generalized inverse Ricci operator. This motivates the following definition, generalizing Definition 4.24 to an arbitrary Kähler manifold.

Definition 4.32. *Define the inverse Ricci operator* $\operatorname{Ric}_\Omega^{-1} : \mathcal{H}_\Omega \to \mathcal{H}_\Omega$ *by letting* $\operatorname{Ric}_\Omega^{-1}\omega := \omega_\varphi$ *with* ω_φ *the unique Kähler form in* $\mathcal{H}_\Omega$ *satisfying* $\operatorname{Ric}\omega_\varphi = H_\omega \operatorname{Ric}\omega$. *Similary we denote higher order iterates of this operator by* $\operatorname{Ric}_\Omega^{-l}$ *for each* $l \in \mathbb{N}$.

Calabi-Yau manifolds are singled-out as those manifolds for which this operator is a constant map. In general the dynamics of this operator seems intriguing.

We end this section with two remarks regarding continuity method paths induced from the flow (259), directly continuing the discussion in Section 4.3. First, it is interesting to note that discretizing this flow for time steps $\tau \in [0,1]$ gives rise to the well-known continuity path of the Calabi-Yau Theorem (here $\mu = 0$) introduced by Calabi [**41**, (11)],[19]

$$\operatorname{Ric}\omega_{\varphi_\tau} - \operatorname{Ric}\omega = -\tau \operatorname{Ric}\omega \quad \Longrightarrow \quad e^{\tau f_\omega + d_\tau}\omega^n = \omega_{\varphi_\tau}^n, \tag{263}$$

with $d_\tau = -\log \frac{1}{V}\int_M e^{\tau f_\omega}\omega^n$, $\tau \in [0,1]$.

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[19] To obtain this path in the equivalent setting of the search for a Kähler metric with prescribed Ricci form, one considers the flow obtained by replacing the harmonic projection term in (258) by a prescribed form representing c_1.

The K-energy decreases along this path, however not monotonically in general. This is in contrast to the continuity path arising from the Ricci iteration and fits-in well with what we would expect: the former arises from the Euler method (as opposed to the backwards Euler method) and so one does not expect monotonicity, nor convergence for large enough time steps.

Also, we remark that the backwards Euler method of the same evolution equation (259) yields the continuity path

$$\omega_{\varphi_\tau}^n = \omega^n e^{\frac{\tau}{1+\tau}(f_\omega - \mu\varphi)}, \quad \tau \geq 0, \tag{264}$$

that coincides in the case $\mu = 1$, after reparametrization, with the continuity path used by Demailly and Kollár alluded to earlier (Remark 4.7).

Remark 4.33. One may also define an analogous iteration whose fixed points are extremal metrics. Once again it arises from discretizing a flow

$$\begin{aligned} \frac{\partial \mathrm{Ric}\, \omega(t)}{\partial t} &= -\mathrm{Ric}\, \omega(t) + \Pi_t \mathrm{Ric}\, \omega(t), \quad t \in \mathbb{R}_+, \\ \omega(0) &= \omega. \end{aligned} \tag{265}$$

Here Π is a projection operator such that $\Pi_\omega \mathrm{Ric}\, \omega = \mathrm{Ric}\, \omega$ if and only if ω is extremal **[246]**. One then obtain an iteration given by

$$\begin{aligned} \mathrm{Ric}\, \omega_{k+1} &= \Pi_k \mathrm{Ric}\, \omega_k, \quad k \in \mathbb{N}, \\ \omega_0 &= \omega. \end{aligned} \tag{266}$$

Note that this construction does not generalize (262) and Definition 4.32, but is a counterpart of it. The constructions (262) and (266) coincide when there are no holomorphic vector fields, but not in general. In some sense, when $\mathrm{Aut}(M, \mathrm{J})$ has positive dimension, one should "know" a priori whether a constant scalar curvature metric or an extremal metric exists in order to pick the right dynamical system to study.

4.9 The twisted Ricci iteration and a twisted inverse Ricci operator

When searching for canonical metrics, the presence of continuous symmetries has traditionally required additional analysis. Although the arguments are very similar to the previous sections, there are certain differences. In this section we merely introduce some of the dynamical constructions relevant to this case which will be further used and studied in the sequel **[231]** in the setting of convergence towards

Kähler-Einstein metrics with continuous symmetries and Kähler-Ricci solitons (or multiplier Hermitian structures). We also state a monotonicity result that will be used in §§4.10.6.

In the presence of holomorphic vector fields one oftentimes modifies the flow equation by a time-dependent family of automorphisms [**63**,**280**]. More generally, one may study the dynamics of a perturbation of the vector field $-f$ by an arbitrary vector field χ. Adapting the point of view of either Section 4.7 or Section 4.8 yields two ways to obtain discrete dynamics. The following definition corresponds to the former.

Definition 4.34. *Given a vector field* $\chi : \omega \mapsto \chi_\omega \in C^\infty(M)/\mathbb{R}$ *on* $\mathcal{H}_\Omega$ *define the* χ*-twisted time* τ *Ricci iteration to be the sequence of forms* $\{\omega_{k\tau}\}_{k\geq 0}$ *satisfying the equations*

$$\begin{aligned} \omega_{k\tau} &= \omega_{(k-1)\tau} + \tau H_{k\tau}\mathrm{Ric}\,\omega_{k\tau} - \tau\mathrm{Ric}\,\omega_{k\tau} + \tau\sqrt{-1}\partial\bar{\partial}\chi_{\omega_{k\tau}}, \quad k \in \mathbb{N}, \\ \omega_0 &= \omega, \end{aligned} \tag{267}$$

for each $k \in \mathbb{N}$ *for which a solution exists in* $\mathcal{H}_\Omega$.

The construction in Section 4.7 corresponds to the zero vector field. The case $\chi = \psi^X$, with X an infinitesimal automorphism, will be useful when studying convergence towards solitons.

When $\Omega = c_1, \tau = 1$ this iteration takes on a special form, giving a certain generalized inverse Ricci operator (cf. Definition 4.24).

Definition 4.35. *Given a vector field* $\chi : \omega \mapsto \chi_\omega \in C^\infty(M)/\mathbb{R}$ *on* $\mathcal{H}_{c_1}$ *define the* χ*-twisted Ricci operator* $\mathrm{Ric}_\chi : \mathcal{H}_{c_1} \rightarrow \mathcal{D}_{c_1}$ *by letting* $\mathrm{Ric}_\chi\omega := \mathrm{Ric}\,\omega - \sqrt{-1}\partial\bar{\partial}\chi_\omega$. *Define the* χ*-twisted inverse Ricci operator* $\mathrm{Ric}_\chi^{-1} : \mathcal{H}_{c_1} \rightarrow \mathcal{H}_{c_1}$ *by letting* $\mathrm{Ric}_\chi^{-1}\omega := \omega_\varphi$ *whenever there exists a unique Kähler form* ω_φ *in* $\mathcal{H}_{c_1}$ *satisfying* $\mathrm{Ric}_\chi\omega_\varphi = \omega$. *Denote higher-order iterates of these operators by* Ric_χ^l *for* $l \in \mathbb{Z}$, *setting* $\mathrm{Ric}_\chi^0 := \mathrm{Id}$.

Recall that the Bakry-Émery Ricci form associated to a pair $(\omega, a) \in \mathcal{H}_\Omega \times C^\infty(M)$ is the form $\mathrm{Ric}\,\omega - \sqrt{-1}\partial\bar{\partial}a$, that can viewed as the Ricci form of the Kähler manifold (M, J) equipped with a Kähler form whose top exterior product equals $e^{2\pi a}\omega^n$ [**11**, (4b)]. The twisted Ricci operator is thus an assignment of a Bakry-Émery Ricci form to each Kähler form determined by a vector field on $\mathcal{H}_\Omega$. The simplest examples include the zero vector field and the Ricci potential vector field that yield the Ricci operator and the identity operator, respectively. Note that the fixed points of the twisted Ricci operator are for certain choices of χ the multiplier Hermitian structures defined by Mabuchi [**180**]. The twisted inverse Ricci operator is not defined for general χ, however it is for some geometrically significant vector fields. Assume that

X belongs to a reductive Lie subalgebra of $\mathrm{aut}(M, \mathrm{J})$ and that the one-parameter subgroup $T_{\mathrm{J}X}$ generated by $\mathrm{J}X$ is a compact torus in $\mathrm{Aut}(M, \mathrm{J})$. When $\chi = \psi^X$ the operator $\mathrm{Ric}^{-1}_{\psi^X}$ restricted to $\mathcal{H}_{c_1}(T_{\mathrm{J}X})$ exists and is well-defined according to a theorem of Zhu [**300**]. More generally, this is still true when χ is a smooth function of ψ^X under some assumptions [**180**].

First, continuing the discussion of Section 4.3 (Remark 4.8) observe that when $\Omega = c_1$ and $\chi = \psi^X$ the continuity method path obtained by setting $k = 1$ and letting τ vary in the segment $[1, \infty)$ coincides with the Tian-Zhu continuity path [**277**, (1.4)]

$$\omega^n_{\varphi_s} = \omega^n e^{f_\omega - \psi^X_{\omega_{\varphi_s}} - s\varphi_s}, \quad s \in [0, 1], \tag{268}$$

via the reparametrization $s = 1 - \frac{1}{\tau}$, discretizing the ψ^X-twisted Kähler-Ricci flow [TZ4, (4.4)]

$$\begin{aligned} \frac{\partial \omega(t)}{\partial t} &= -\mathrm{Ric}\,\omega(t) + \omega(t) + \mathcal{L}_X \omega(t), \quad t \in \mathbb{R}_+, \\ \omega(0) &= \omega \in \mathcal{H}_{c_1}(T_{\mathrm{J}X}). \end{aligned} \tag{269}$$

In fact, more generally Mabuchi's continuity path [**180**, (5.1.4)] in the context of multiplier Hermitian structures is obtained in the same manner from (267) as a result of discretizing the corresponding twisted Kähler-Ricci flow.

We now discuss briefly the special case of Kähler-Ricci solitons. This is mainly done for the sake of concreteness since, due to the work of Mabuchi, the relevant computations go through also for general multiplier Hermitian structures.

In their study of Kähler-Ricci solitons on Fano manifolds Tian and Zhu introduced a twisted version of the functional E_0 [**278**]. To define it we first recall some relevant facts [**105**, §2.4; **180**,**277**]. Given $X \in \mathrm{aut}(M, \mathrm{J})$, let $L^{\psi^X}_\omega$ denote the elliptic operator $L^{\psi^X}_\omega \phi := \Delta_\omega \phi + X\phi$. This operator is self-adjoint with respect to the $L^2(M, e^{\psi^X_\omega} \omega^n)$ inner product denoted by $\langle \cdot\,, \cdot \rangle_{\psi^X}$. The vector field ψ^X on $\mathcal{H}_{c_1}$ induces a vector field on the space of Kähler potentials (that we still denote by the same notation) by decreeing that $\frac{1}{V} \int_M e^{\psi^X_\omega} \omega^n = 1$ for each $\omega \in \mathcal{H}_{c_1}$. One then has $\psi^X_{\omega_\varphi} = \psi^X_\omega + X\varphi$ since $\frac{d}{dt} \frac{1}{V} \int_M e^{\psi^X_\omega + X(t\nu)} \omega^n_{t\nu} = \langle L^{\psi^X}_{\omega_{t\nu}} \nu, 1 \rangle_{\psi^X} = 0$. Define a functional on $\mathcal{H}_{c_1}(T_{\mathrm{J}X}) \times \mathcal{H}_{c_1}(T_{\mathrm{J}X})$ by

$$E^{\psi^X}_0(\omega, \omega_\varphi) = \frac{1}{V} \int_{[0,1]} \langle \dot{\varphi}_t, L^X_{\omega_{\varphi_t}} (\psi^X_{\omega_{\varphi_t}} - f_{\omega_{\varphi_t}}) \rangle_{\psi^X} dt \tag{270}$$

This functional is well-defined independently of a choice of path and exact. Its critical points are Kähler-Ricci solitons.

Lemma 4.36. *Assume that X belongs to a reductive Lie subalgebra of* $\mathrm{aut}(M, \mathrm{J})$ *and that the one-parameter subgroup $T_{\mathrm{J}X}$ generated by $\mathrm{J}X$ is a compact torus in*

Aut(M, J). *The functional* $E_0^{\psi^X}$ *is monotonically decreasing along the* ψ^X*-twisted time* τ *Ricci iteration for each* $\tau > 0$ *for which the iteration exists.*

Proof. Let $\omega \in \mathcal{H}_{c_1}(T_{\mathrm{J}X})$. Whenever the ψ^X-twisted time τ Ricci iteration exists the same is true for smaller time step iterations. Hence the continuity path

$$\omega_{\varphi_t}^n = \omega^n e^{f_\omega - \psi^X_{\omega_{\varphi_t}} + (\frac{1}{t}-1)\varphi_t}, \quad t \in [0, \tau], \tag{271}$$

exists. Differentiating equation (271) gives $(L^{\psi^X}_{\omega_{\varphi_t}} + 1 - \frac{1}{t})\dot{\varphi}_t = -\frac{1}{t^2}\varphi_t$. Hence one has

$$\begin{aligned} E_0^{\psi^X}(\omega_0, \omega_\tau) &= \frac{1}{V}\int_{[0,\tau]} \frac{1}{t}\langle \dot{\varphi}_t, L^{\psi^X}_{\omega_{\varphi_t}}\varphi_t\rangle_{\psi^X} dt \\ &= -\frac{1}{V}\int_{[0,\tau]} t\langle L^{\psi^X}_{\omega_{\varphi_t}}\dot{\varphi}_t, (L^{\psi^X}_{\omega_{\varphi_t}} + 1 - \frac{1}{t})\dot{\varphi}_t\rangle_{\psi^X} dt \leq 0. \end{aligned}$$

When $\tau \leq 1$ the last inequality is a just a consequence of the ellipticity of $L^{\psi^X}_{(\,\cdot\,)}$. When $\tau > 1$ it follows since $L^{\psi^X}_{\omega_{\varphi_t}} + 1 - \frac{1}{t}$ is still elliptic [**277**, Lemma 2.2 (ii)]. □

4.10 Some applications

In this section we describe several applications of the Ricci iteration and the inverse Ricci operator to some classical objects and problems in Kähler and conformal geometry.

4.10.1 The Moser-Trudinger-Onofri inequality on the Riemann sphere and its higher dimensional analogues. Let $\omega_{\mathrm{FS},c}$ denote the Fubini-Study form of constant Ricci curvature c on (S^2, J), the Riemann sphere, given locally by

$$\omega_{\mathrm{FS},c} = \frac{\sqrt{-1}}{c\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

Here $V = \int_{S^2}\omega_{\mathrm{FS},c} = c_1([M])/c = 2/c$. For $c = 1/2\pi$ it is induced from restricting the Euclidean metric on $\mathbb{R}^3$ to the radius 1 sphere $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$. Denote by $W^{1,2}(S^2)$ the space of functions on S^2 that are square-summable and so is their gradient (with respect to some Riemannian metric). The Moser-Trudinger-Onofri inequality states:

Theorem 4.37. (See [**198**,**208**,**282**].) *For $\omega = \omega_{\mathrm{FS},2/V}$ and any function φ on S^2 in $W^{1,2}(S^2)$ one has*

$$\frac{1}{V}\int_{S^2} e^{-\varphi+\frac{1}{V}\int_{S^2}\varphi\omega}\omega \le e^{\frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi}. \tag{272}$$

Equality holds if and only if ω_φ is the pull-back of ω by a Möbius transformation.

Several proofs of this classical result have been given in the literature, and we list here the ones we are aware of, chronologically: Onofri [**208**], Hong [**131**], Osgood-Phillips-Sarnak [**209**], Beckner [**17**], Carlen and Loss [**50**,**51**], Ghigi [**111**] (for more background we refer to Chang [**57**]). All of these proofs use crucially some symmetrization/rearrangement arguments that reduce the problem to a single dimension.

Observe that a function satisfies (272) if and only if

$$F_1(\omega,\omega_\varphi) = \frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi - \frac{1}{V}\int_{S^2}\varphi\omega - \log\frac{1}{V}\int_{S^2}e^{-\varphi}\omega \ge 0.$$

This functional was studied already by Berger and Moser [**19**,**198**] in connection with the Nirenberg Problem of prescribing scalar curvature on S^2. Moser, extending work of Trudinger, was able to show that $F_1(\omega,\omega_\varphi) \ge -C$. Then, Onofri showed that $C = 0$ and characterized the cases of equality, resulting in Theorem 4.37.

In higher dimensions, Aubin first suggested a connection between the classical inequality (272) and Kähler-Einstein manifolds [**8**, (4)]. Following this, Ding [**75**] showed how to generalize the functional F_1 to higher-dimensional Fano manifolds—see Equation (213)—using Aubin's functional J. Using this observation, and modifying the proof of a fundamental result of Bando and Mabuchi concerning the boundedness of the K-energy, Ding and Tian proved an analogue of Theorem 4.37 for all dimensions but only for those functions that belong to the subspace $\mathcal{H}_\omega \subset W^{1,2}(S^2)$. We state both results and their corollary. The corollary is Ding and Tian's restricted generalization[20] of the Moser-Trudinger-Onofri inequality to higher-dimensional Kähler-Einstein manifolds.

Theorem 4.38. (See [**15**, Theorem A; **12**, Theorem 1; **76**, Theorem 1.1.]) *Let (M,J,ω) be a Kähler-Einstein Fano manifold. Then $E_0(\omega,\omega_\varphi), F_1(\omega,\omega_\varphi) \ge 0$ for all $\omega_\varphi \in \mathcal{H}_{c_1}$ with equality if and only if $\omega_\varphi = h^\star\omega$ with $h \in \mathrm{Aut}(M,\mathrm{J})_0$.*

Corollary 4.39. (See [**76**].) *Let (M,J,ω) be a Kähler-Einstein Fano manifold with $\omega \in \mathcal{H}_{c_1}$. Then for each $\varphi \in \mathcal{H}_\omega$ holds*

$$\frac{1}{V}\int_M e^{-\varphi+\frac{1}{V}\int_M\varphi\omega^n}\omega^n \le e^{J(\omega,\omega_\varphi)}. \tag{273}$$

---

[20] This term is meant to emphasize that this generalized a restricted version of the classical inequality.

Equality holds if and only if ω_φ is the pull-back of ω by a holomorphic transformation.

One should note that the subspace of Kähler potentials can be considered as a rather small "ball" sitting inside $C^\infty(S^2) \subset W^{1,2}(S^2)$ since in general a large enough multiple of an element of $\mathcal{H}_\omega$ will no longer belong to $\mathcal{H}_\omega$. Following the work of Ding and Tian it remained an open problem how to extend their techniques and provide a complex-geometric proof of the Moser-Trudinger-Onofri inequality. The key hurdle in proving Theorem 4.37 is to extend the argument to the set $C^\infty(M) \setminus \mathcal{H}_\omega$ which a priori has no clear geometric significance as it represents indefinite forms rather than Kähler forms.

Alternatively, what is missing is a geometric interpretation of the Berger-Moser-Ding functional F_1.[21] The following result is the key ingredient in our proof of (272) and is a restatement of Lemma 4.15.

Proposition 4.40. *Let $\Omega = c_1$. The following relation holds*

$$(\mathrm{Ric}^{-1})^\star E_n = F_1, \quad on \;\; \mathcal{H}_{c_1} \times \mathcal{D}_{c_1}.$$

This provides a geometric interpretation for F_1. Indeed, the functional E_n is the potential for the Laplacian of the determinant of the Ricci tensor, considered as a 1-form on $\mathcal{H}_\Omega$, i.e., $dE_n(\omega, \omega_\varphi) = \Delta_{\omega_\varphi}\Big(\frac{(\mathrm{Ric}\,\omega_\varphi)^n}{\omega_\varphi^n}\Big)\omega_\varphi^n$.

Before proceeding to analyze further implications of this result we now note that our work provides a new and succinct proof of the original Moser-Trudinger-Onofri inequality entirely within the framework of exact energy functionals. This is the first proof that does not use symmetrization/rearrangement arguments.

Proof of Theorem 4.37. By Theorem 4.38 and Proposition 4.12 $E_1(\omega, \cdot) \geq 0$ on $\mathcal{H}_\omega$. Given $\varphi \in C^\infty(S^2)$ there exists $\psi \in \mathcal{H}_\omega$ such that $\mathrm{Ric}\,\omega_\psi = \omega_\varphi$ by solving the Poisson equation on S^2. Thus by Proposition 4.40 $F_1(\omega, \cdot) \geq 0$ on $\mathcal{D}_\Omega$. Using the definition of F, for any smooth function φ we obtain (272). Since $C^\infty(S^2)$ is dense in $W^{1,2}(S^2)$ we conclude. □

We now turn to describe the implications of our approach for higher-dimensional Fano Kähler-Einstein manifolds. The result explains the geometric meaning the set $C^\infty(M) \setminus \mathcal{H}_\omega$ plays in the Moser-Trudinger-Onofri inequality. Namely, a function will satisfy this inequality if and only if it represents the Ricci form of a Kähler metric whose Ricci energy E_n is nonnegative with respect to a Kähler-Einstein metric. It now becomes important to understand the sets (first defined in §§4.4.4)

$$\mathcal{A}_n = \{\omega_\varphi \in \mathcal{H}_{c_1} \, : \, E_n(\omega, \omega_\varphi) \geq 0\}.$$

---

[21] Osgood-Phillips-Sarnak gave a different geometric interpretation strictly in the case $n = 1$ using the determinant of the Laplacian **[209]**.

Naturally, we introduce the following definition.

Definition 4.41. *Let the Moser-Trudinger-Onofri neighborhood of* $\mathcal{H}_\omega$ *be the subset*

$$MTO_n = \{\varphi \in C^\infty(M) : \varphi \text{ satisfies (273) on the Fano manifold } (M, \mathrm{J}), \dim_{\mathbb{C}} M = n\}. \tag{274}$$

More generally we would like to understand the maximal set on which $E_k(\omega, \cdot) \geq 0$ with respect to a Kähler-Einstein form ω. To that end we introduce similarly the sets

$$\mathcal{A}_k := \{\omega_\varphi \in \mathcal{H}_{c_1} \ : \ E_k(\omega, \omega_\varphi) \geq 0\}. \tag{275}$$

In fact, a direct corollary of our formula for the Chen-Tian functionals proven in Proposition 4.12 implies

$$\mathcal{A}_k \supseteq \mathcal{B}_k := \{\omega_\varphi \in \mathcal{H}_{c_1} \ : \ I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) \geq 0\}. \tag{276}$$

For example, for $k = 1$ this gives $\mathcal{A}_1 = \mathcal{H}_{c_1}$, when $k = 2$ we have

$$\mathcal{A}_2 \supseteq \mathcal{B}_2 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1} \ : \ \mathrm{Ric}\,\omega_\varphi + 2\omega_\varphi \geq 0\},$$

for $k = 3$

$$\mathcal{A}_3 \supseteq \mathcal{B}_3 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1} \ : \ \mathrm{Ric}\,\omega_\varphi + \omega_\varphi \geq 0\},$$

and for arbitrary k one may readily obtain an explicit bound (depending on k) on the set $\mathcal{B}_k$, and hence on $\mathcal{A}_k$, in terms of a lower bound on the Ricci curvature, using the definition of our functionals I_k (191).

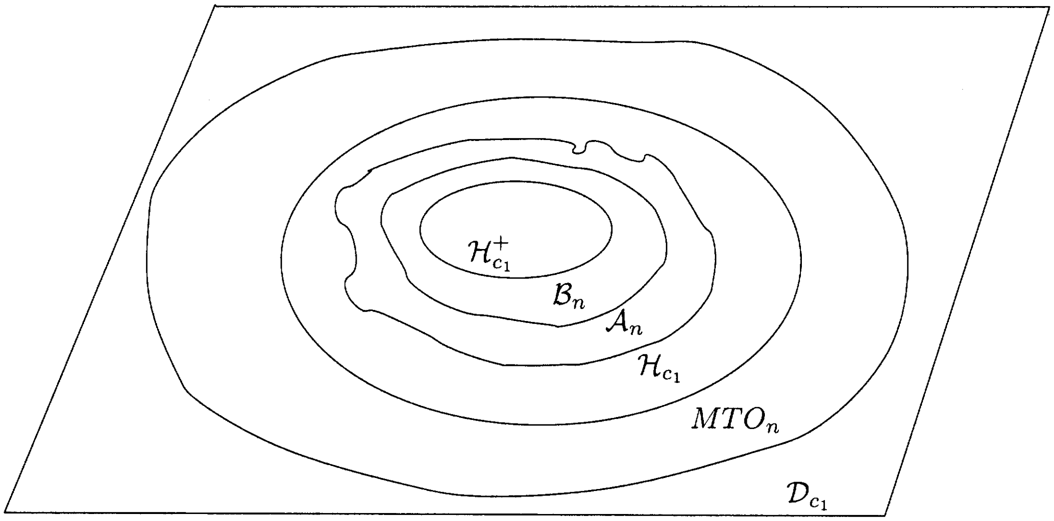

Figure 4. Subspaces inside the space of closed forms representing the first Chern class of a Fano manifold.

We are now in a position to state our generalization of Corollary 4.39 that is optimal in higher dimensions as well as some practical bounds. The result says that the Moser-Trudinger-Onofri inequality holds in higher dimensions on a canonically defined set MTO_n that is strictly larger than the space of Kähler potentials $\mathcal{H}_\omega$ (part (i) in the statement of the theorem below) and is geometrically related to Ricci curvature (part (iii)). Part (ii) summarizes our discussion in the previous paragraph on the proprty of the Chen-Tian functionals E_k.

Theorem 4.42. *Let (M, J, ω) be a Kähler-Einstein Fano manifold.*
(i) The generalized Moser-Trudinger-Onofri inequality (273) holds precisely on the set $MTO_n = \mathrm{Ric}(\mathcal{A}_n)$. Furthermore, $\mathcal{H}_{c_1} \subsetneq MTO_n \subseteq \mathcal{D}_{c_1}$.
(ii) Define the sets $\mathcal{B}_k := \{\omega_\varphi \in \mathcal{H}_{c_1} : I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) \geq 0\}$. Then one has $\mathcal{H}^+_{c_1} \subsetneq \mathcal{B}_k \subseteq \mathcal{A}_k$.
(iii) One has $\mathcal{A}_1 = \mathcal{B}_1 = \mathcal{H}_{c_1}$, $\mathcal{B}_2 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1} : \mathrm{Ric}\,\omega_\varphi + 2\omega_\varphi \geq 0\}$, $\mathcal{B}_3 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1} : \mathrm{Ric}\,\omega_\varphi + \omega_\varphi \geq 0\}$, and for each k one may readily obtain an explicit bound on the set $\mathcal{B}_k$, hence on $\mathcal{A}_k$, in terms of a lower bound on the Ricci curvature using (191). In particular there exist $c_n > 0$ depending only on n such that,

$$MTO_n \supseteq \{\varphi \in C^\infty(M) : \omega_\varphi \geq -c_n \mathrm{Ric}^{-1}\omega_\varphi\},$$

and, e.g., $c_1 = \infty, c_2 \geq 2, c_3 \geq 1$.

In this languange the proof we gave of the Moser-Trudinger-Onofri inequality may rephrased even more succinctly: Observe that $MTO_1 = \mathrm{Ric}(\mathcal{A}_1) = \mathrm{Ric}(\mathcal{H}_{c_1}) = \mathcal{D}_{c_1}$. The last equality requires solving the equation[22] $\mathrm{Ric}\,\omega_\varphi = \omega_\psi$ for φ, equivalently Poisson's equation $\Delta_\omega \varphi = e^{f_\omega - \psi} - 1$, which is always possible, so we are done.

Finally, we state a result that at the time the author proved it seemed surprising. To the author's understanding, a common guiding belief at that time (2006) was that on a Fano Kähler-Einstein manifold all the functionals E_k should be bounded from below. Note that the result holds regardless of whether a Kähler-Einstein metric exists.

Theorem 4.43. *Let (M, J, ω) be a Fano manifold. Then the Ricci energy E_n is unbounded from below on $\mathcal{H}_{c_1}$ if and only if $n > 1$.*

Proof. Recall Jensen's inequality $\frac{1}{V}\int_M e^{-\varphi + \frac{1}{V}\int_M \varphi\omega^n}\omega^n \geq 1$ [**126**]. Now observe that in higher dimensions Lemma 4.9 implies that the functional J is unbounded from below on the whole space of forms. Therefore inequality (273) cannot be extended to

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[22] This is the classical n=1 version of the Calabi-Yau Theorem whose proof goes back at least to Wallach and Warner [**284**].

all of $C^\infty(M)$. Invoking Proposition 4.40 then implies that E_n must be unbounded from below on $\mathcal{H}_{c_1}$, precisely becuase by the Calabi-Yau Theorem $\mathrm{Ric}\,\mathcal{H}_{c_1} = \mathcal{D}_{c_1}$. □

4.10.2 An analytic characterization of Kähler-Einstein manifolds and an analytic criterion for almost-Kähler-Einstein manifolds. In the first part of this subsection we explain how the inverse Ricci operator can be used to solve two additional open problems concerning energy functionals on the space of Kähler forms.

Chen and Tian's generalization of Mabuchi's Kähler energy, E_0, and of Bando and Mabuchi's Ricci energy, E_n, to a family of functionals $\{E_k\}_{k=0}^n$ (see Section 4.4 for definitions) naturally raised the question of whether Tian's analytic characterization of Kähler-Einstein manifolds in terms of E_0 generalizes to these functionals. In addition it raised the question whether Bando and Mabuchi's criterion for almost-Kähler-Einstein manifolds in terms of E_0 generalizes to these functionals. These questions were also independently raised by Chen [**61**, p. 37; **62**, §1.3]. We now recall both of these fundamental results and explain how to generalize them. This provides an answer to these questions. It shows that the answer is both "yes" and "no": these criteria extend to the other functionals $\{E_k\}$, however they fail to extend in an identical manner. The subtlety comes from the appearance of the inverse Ricci operator as we will see below.

Theorem 4.44. *Let (M, J) be a Fano manifold.*
(i) (See [**12,15,76**].) *If either F_1 or E_0 is bounded from below on $\mathcal{H}_{c_1}$ then for each $\epsilon > 0$ there exists a Kähler metric $\omega_\epsilon \in \mathcal{H}_{c_1}$ satisfying $\mathrm{Ric}\,\omega_\epsilon > (1-\epsilon)\omega_\epsilon$.*
(ii) (See [**270,271,278**].) *Assume that $\mathrm{Aut}(M, \mathrm{J})$ is finite.*[23] *Then the properness of F_1 (or E_0) on $\mathcal{H}_{c_1}$ is equivalent to the existence of a Kähler-Einstein metric.*

Our strategy in extending these results to the functionals $\{E_k\}_{k=0}^n$ will be: (a) use our formula (Proposition 4.12) that expresses E_k in terms of the sum of E_0 and another the exact energy functional $(\omega, \omega_\varphi) \mapsto I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) - I_k(\omega, \mathrm{Ric}\,\omega)$ introduced in §§4.4.1 (Proposition 4.12) to show

$$
\begin{aligned}
F_1 \text{ bounded from below on } \mathcal{H}_{c_1} &\Rightarrow E_0 \text{ bounded from below on } \mathcal{H}_{c_1}\\
&\Rightarrow E_1 \text{ bounded from below on } \mathcal{H}_{c_1}\\
&\Rightarrow E_2 \text{ bounded from below on } \mathcal{H}^+_{c_1}\\
&\quad \vdots\\
&\Rightarrow E_n \text{ bounded from below on } \mathcal{H}^+_{c_1}.
\end{aligned}
$$

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[23] In the general case a slightly more involved statement holds (see [**227**] for details).

(b) Next use Proposition 4.40 to conclude:

$$E_n \text{ bounded from below on } \mathcal{H}^+_{c_1} \Rightarrow F_1 \text{ bounded from below on } \mathcal{H}_{c_1}.$$

(c) Finally, some additional arguments were needed in order to prove that the properness of E_n on $\mathcal{H}^+_{c_1}$ implies the existence of a Kähler-Einstein metric.

We can now state more precisely our generalizations of the theorems of Bando-Mabuchi and Tian to the energy functionals $\{E_k\}$. The case $k = 1$ was proved before by Chen-Li-Wang and Song-Weinkove in a different manner [**62**,**250**].

Theorem 4.45. *Let (M, J) be a Fano manifold.*
(i) If either F_1 or E_k (for some $k \in \{0, \ldots, n\}$) is bounded from below on $\mathcal{H}^+_{c_1}$ then for each $\epsilon > 0$ there exists a Kähler metric $\omega_\epsilon \in \mathcal{H}_{c_1}$ satisfying $\mathrm{Ric}\,\omega_\epsilon > (1-\epsilon)\omega_\epsilon$.
(ii) Assume that $\mathrm{Aut}(M, \mathrm{J})$ is finite.[24] *Then the properness of F_1 or of E_k (for some $k \in \{0, \ldots, n\}$) on $\mathcal{H}^+_{c_1}$ is equivalent to the existence of a Kähler-Einstein metric. For F_1, E_0, and E_1 one may replace $\mathcal{H}^+_{c_1}$ by $\mathcal{H}_{c_1}$.*

Two preliminaries are in place before going into the details of the proof. First, it is important to note that the appearance of the inverse Ricci operator in step (b) is a crucial ingredient. The discrepancy between the behavior of F_1, E_0, E_1 and that of the functionals $E_2, \ldots, E_n$ can be explained using the time one Ricci iteration: the first three are unconditionally monotone along the iteration, while for the latter $n-1$ this is true if and only if one assumes that the initial point lies in $\mathcal{B}_k$, and $\mathcal{H}^+_{c_1} \subsetneq \mathcal{B}_k \subseteq \mathcal{A}_k \subsetneq \mathcal{H}_{c_1}$ (Proposition 4.17). Furthermore, along the first step of the iteration the functionals E_k may increase by an arbitrary amount! This is a consequence of Theorem 4.43 for $k = n$:[25] The Ricci energy E_n is bounded from below on $\mathcal{H}_{c_1}$ if and only if $n = 1$. We conclude that the assumption in Theorem 4.45 (ii) is essential and cannot be weakened from $\mathcal{H}^+_{c_1}$ to $\mathcal{H}_{c_1}$. This explains our remark earlier on the subtlety present when $k \geq 2$.

Second, we recall the fundamentals of the continuity method approach that we will be using to prove Theorem 4.45.

4.10.2.1 Continuity method approach. Consider the path $\{\omega_{\varphi_t}\} \subseteq \mathcal{H}_{c_1}$ given implicitly by

$$\begin{aligned} \omega^n_{\varphi_t} &= e^{(t+1)f_\omega + c_t}\omega^n, \qquad && t \in [-1, 0], \\ \omega^n_{\varphi_t} &= e^{f_\omega - t\varphi_t}\omega^n, \qquad && t \in [0, 1], \end{aligned} \tag{277}$$

---

[24] See footnote to Theorem 4.44.

[25] We conjecture that Theorem 4.43 should hold for E_k for each $2 \leq k \leq n$.

with the normalizations $\int_M e^{(t+1)f_\omega+c_t}\omega^n = V$ for $t \in [-1,0]$ and $\int_M e^{f_\omega - t\varphi_t}\omega^n = V$ for $t \in [0,1]$. Note that the first segment always exists by the proof of the Calabi-Yau Theorem [**296**] while the second, when it exists, deforms the metric to a Kähler-Einstein metric [**7**]:

$$\operatorname{Ric}\omega_{\varphi_t} - \omega_{\varphi_t} = -(1-t)\sqrt{-1}\partial\bar{\partial}\varphi_t, \qquad t \in [0,1]. \tag{278}$$

We will make use of the following Proposition:

Proposition 4.46. (See [**15**, Theorem 5.7].) *Assume that* (M, J) *is Fano and let* G *be a compact subgroup of* $\operatorname{Aut}(M, \mathrm{J})$. *Assume that* E_0 *is bounded from below on* $\mathcal{H}^+_{c_1}(G)$ *and let* $\omega \in \mathcal{H}_{c_1}(G)$. *Then (277) has a unique smooth solution for each* $t \in [0,1)$.

Note that by Lemma 4.14 and (212) the same conclusion holds with E_0 replaced by F. In particular, Theorem 4.44 (i) is a direct corollary of Proposition 4.46 combined with this observation (one obtains a version of Theorem 4.45 (i) with the free choice of a subgroup G, although this, as opposed to the refinement of Theorem 4.45 (ii) mentioned in the footnote to Theorem 4.44 (ii), should not be considered as a gain in generality). We also note that one of the important ingredients in the proof of Proposition 4.46 is the fact that $(I-J)(\omega,\cdot)$ is nondecreasing along the continuity path (277) [**15**, Theorem 5.1; **264**, p. 232; **271**, Lemma 6.25].

It is worth noting that Bando has shown that if $\omega \in \mathcal{H}_{c_1}(G)$ satisfies $\operatorname{Ric}\omega > (1-\epsilon)\omega$, $\epsilon > 0$, then "flowing" it along the Ricci flow will produce another metric in $\mathcal{H}_{c_1}(G)$ whose scalar curvature differs from n by at most a fixed constant times ϵ [Ba]. Therefore, the existence of a lower bound for E_0 or for F implies the existence of Kähler metrics in $\mathcal{H}_{c_1}(G)$ whose scalar curvature is as close to a constant as desired (the original result of Bando extends to the G-invariant setting since its proof makes use of a Kähler-Ricci flow which, like the continuity method, preserves $\mathcal{H}_{c_1}(G)$). These can be thought of as "almost Kähler-Einstein" metrics since a Kähler metric of constant scalar curvature in $\mathcal{H}_{c_1}$ is necessary Kähler-Einstein.

It is an open problem to give a precise characterization of Fano manifolds for which the K-energy or equivalently the functional F (note that this equivalence is proved below (Corollary 4.49 and Remark 4.50)) are bounded from below. We pose the following problem (see also the author's article [**229**, §3] where additional motivation is given for this problem).

Problem 4.47. *On a Fano manifold, determine whether the lower boundedness of the functional* F *(equivalently, of Mabuchi's K-energy* E_0*) on* $\mathcal{H}_{c_1}$ *is equivalent to* $\sup\{\, b : \operatorname{Ric}\omega \geq b\omega, \omega \in \mathcal{H}_{c_1}, b \in \mathbb{R}\} = 1$.

Our understanding of these manifolds is incomplete. In fact there are presently no examples of Fano manifolds for which the K-energy is bounded from below however not proper (therefore, by Tian's properness theorem, not Kähler-Einstein). We therefore also pose the following problem.

Problem 4.48. *Find examples of Fano manifolds that do not admit a Kähler-Einstein metric for which the K-energy is bounded from below.*

4.10.2.2 Boundedness and properness properties of energy functionals. Let us turn to the proof of Theorem 4.45 and begin with (ii). Assume that a Kähler-Einstein form ω exists. Then F_1 is proper on $\mathcal{H}_{c_1}^+$ by Theorem 4.44 (ii). By Lemma 4.14 and (212) so is E_0. From Proposition 4.12 we have

$$E_{k+1}(\omega,\omega_\varphi) = E_k(\omega,\omega_\varphi) + (I_{k+1}-I_k)(\omega_\varphi,\operatorname{Ric}\omega_\varphi) - (I_{k+1}-I_k)(\omega,\operatorname{Ric}\omega),$$

with $I_{k+1}\geq I_k$ on $\mathcal{H}_{c_1}\times\mathcal{H}_{c_1}$ as noted after (195). It follows that if E_k is proper on $\mathcal{H}_{c_1}^+$ so is E_{k+1}. We conclude that E_n is proper on $\mathcal{H}_{c_1}^+$.

Assume that E_n is proper on $\mathcal{H}_{c_1}^+$. Then from Lemma 4.15 and the Calabi-Yau Theorem we see that F is bounded from below on $\mathcal{H}_{c_1}$ and from Lemma 4.14 and (212) it follows that so is E_0. Therefore from Proposition 4.46, given $\omega\in\mathcal{H}_{c_1}$, the continuity path (277) extends for all $t<1$.

From the properness and exactness of E_n there exists a function τ_ω as in (190) satisfying $E_n(\omega_{\varphi_0},\omega_{\varphi_t})\geq\tau_\omega(I(\omega,\omega_{\varphi_t}))-E_n(\omega,\omega_{\varphi_0})$. Hence it suffices now to show that $E_n(\omega_{\varphi_0},\omega_{\varphi_t})$ is uniformly bounded from above for all $t>t_0$ with t_0 depending only on (M,J,ω). We will then have that $I(\omega,\omega_{\varphi_t})$ is uniformly bounded independently of $t\in[0,1)$. This will entail a uniform bound on $||\varphi_t||_{L^\infty}$ [**9**, Proposition 7.35; **271**, Lemma 6.19] and hence a uniform bound on $||\varphi_t||_{C^{2,\beta}(M,g_\omega)}$ for some $\beta\in(0,1)$ [**7**,**296**]. By the continuity method arguments therein one then concludes that a unique smooth solution exists at $t=1$ that is a Kähler potential for a Kähler-Einstein form.

In fact we will find such a t_0 depending only on n for each E_k. The computation that follows involves expressions similar to those that figure in the work of Song and Weinkove; using Proposition 4.12 considerably simplifies our calculations compared to the ones there.

Fix $\tau\in[0,1]$. First, from (278) and the definition of E_0 we have

$$\begin{aligned}E_0(\omega_{\varphi_0},\omega_{\varphi_\tau}) &= \int_{[0,\tau]}\frac{d}{dt}E_0(\omega_{\varphi_0},\omega_{\varphi_t})dt\\ &= \frac{1}{V}\int_{M\times[0,\tau]}(1-t)n\dot{\varphi}_t\sqrt{-1}\partial\bar\partial\varphi_t\wedge\omega_{\varphi_t}^{n-1}\wedge dt\end{aligned}$$

$$
\begin{aligned}
&= - \int_{[0,\tau]} (1-t)\frac{d}{dt}(I-J)(\omega, \omega_{\varphi_t})dt \\
&= - (1-\tau)(I-J)(\omega, \omega_{\varphi_\tau}) \\
&\quad + (I-J)(\omega, \omega_{\varphi_0}) - \int_{[0,\tau]} (I-J)(\omega, \omega_{\varphi_t})dt. \qquad (279)
\end{aligned}
$$

From Proposition 4.12, (189) and (194) we therefore conclude that there exists a constant c_ω depending only on (M, J, ω) for which

$$(n+1)E_k(\omega_{\varphi_0}, \omega_{\varphi_\tau}) \leq -(1-\tau)I(\omega, \omega_{\varphi_\tau}) + nI(\omega_{\varphi_\tau}, \mathrm{Ric}\,\omega_{\varphi_\tau}) + c_\omega. \qquad (280)$$

From (278)

$$
\begin{aligned}
I(\omega_{\varphi_\tau}, \mathrm{Ric}\,\omega_{\varphi_\tau}) &= (1-\tau)^2\frac{1}{V}\int_M \sqrt{-1}\partial\varphi_\tau \wedge \bar{\partial}\varphi_\tau \wedge \sum_{l=0}^{n-1}\omega_{\varphi_\tau}^{n-l-1} \wedge (\tau\omega_{\varphi_\tau} + (1-\tau)\omega)^l \\
&= (1-\tau)^2\frac{1}{V}\int_M \sqrt{-1}\partial\varphi_\tau \wedge \bar{\partial}\varphi_\tau \wedge \sum_{l=0}^{n-1}\sum_{j=0}^{l}\binom{l}{j}\tau^{l-j}(1-\tau)^j\omega_{\varphi_\tau}^{n-j-1} \wedge \omega^j \\
&= (1-\tau)^2\frac{1}{V}\int_M \sqrt{-1}\partial\varphi_\tau \wedge \bar{\partial}\varphi_\tau \wedge \sum_{j=0}^{n-1}(1-\tau)^j\sum_{l=j}^{n-1}\binom{l}{j}\tau^{l-j}\omega_{\varphi_\tau}^{n-j-1} \wedge \omega^j.
\end{aligned}
$$

Note that

$$(1-\tau)^j\sum_{l=j}^{n-1}\binom{l}{j}\tau^{l-j} \leq (1-\tau)^j(n-1)\binom{n-1}{j}. \qquad (281)$$

We may choose $t_1 \in [0,1)$ depending only on n in such a way that for all $\tau \in [t_1, 1]$ the expression on the right hand side of (281) is smaller than n for each $j = 0, \ldots, n-1$. We conclude that

$$I(\omega_{\varphi_\tau}, \mathrm{Ric}\,\omega_{\varphi_\tau}) \leq n(1-\tau)^2 I(\omega, \omega_{\varphi_\tau}), \qquad \forall \tau \in [t_1, 1). \qquad (282)$$

Returning to (280) we then see that $E_k(\omega_{\varphi_0}, \omega_{\varphi_\tau}) \leq c_\omega/(n+1)$ whenever $\tau \in [\max\{t_1, 1-\frac{1}{n^2}\}, 1)$. This concludes the proof of Theorem 4.45 (ii). □

As a corollary of the proof we record the following fact.

Corollary 4.49. *Let (M, J) be a Fano manifold. If one of the functionals $F, E_0, \ldots, E_n$ is bounded from below on $\mathcal{H}_{c_1}^+$ so are the rest.*

Combining Corollary 4.49 with Theorem 4.44 (i) concludes the proof of Theorem 4.45 (i). □

We end this section with several remarks.

Remark 4.50. Our methods imply that the refined version of Theorem 4.44 (ii) also extends to each of the functionals E_k.

Remark 4.51. Note that one may state Corollary 4.49 with $\mathcal{H}^+_{c_1}$ replaced by $\mathcal{H}_{c_1}$ for F, E_0 and E_1. Indeed, recall that once F is bounded from below on $\mathcal{H}^+_{c_1}$ so are each of the E_k while a lower bound for E_n on $\mathcal{H}^+_{c_1}$ implies a lower bound for F on $\mathcal{H}_{c_1}$ (by Lemma 4.15) which, in turn, implies the same for E_0 (using Lemma 4.14) and for E_1 (using Proposition 4.12). Some special cases of Corollary 4.49 appeared previously, namely the fact that when F is bounded from below so is E_0 [**76**] and vice versa [**164**], and the fact that when E_0 is bounded from below so is E_1 [**210**] and vice versa [**62**]. We would like to further make note of the fact that these previous results were often proved by appealing to results on the Ricci flow. This suggests that in this context the Ricci iteration rather than the flow is perhaps more suited.

Remark 4.52. Assume that the functionals F and E_k, $k \in \{0, \ldots, n\}$ are bounded from below on $\mathcal{H}^+_{c_1}$ and for each $\omega \in \mathcal{H}_{c_1}$ set $l(\omega) = \inf_{\omega_\varphi \in \mathcal{H}_{c_1}} F(\omega, \omega_\varphi)$ and

$$l_k(\omega) = \begin{cases} \inf_{\omega_\varphi \in \mathcal{H}_{c_1}} E_k(\omega, \omega_\varphi), & \text{for } k = 0, 1, \\ \inf_{\omega_\varphi \in \mathcal{H}^+_{c_1}} E_k(\omega, \omega_\varphi), & \text{for } k = 2, \ldots, n. \end{cases}$$

Then the following relations hold between the various lower bounds:

$$l(\omega) + \frac{1}{V}\int_M f_\omega \omega^n = l_0(\omega) = l_k(\omega) + I_k(\omega, \operatorname{Ric}\omega). \tag{283}$$

This generalizes the relation between l and l_0 [**164**] and between l and l_1 [**62**] that appeared recently; our proof, given below, appears considerably simpler.

Proof. By Lemma 4.14, (212) and Proposition 4.12

$$l(\omega) + \frac{1}{V}\int_M f_\omega \omega^n \le l_0(\omega) \le l_k(\omega) + I_k(\omega, \operatorname{Ric}\omega). \tag{284}$$

(For the second inequality we used (203) and the fact that $I_k(\omega_\varphi, \operatorname{Ric}\omega_\varphi) \ge 0$ for $\omega_\varphi \in \mathcal{H}^+_{c_1}$.) On the other hand, note first that from (279) it follows that $\int_{[0,1]}(I - J)(\omega, \omega_{\varphi_t})dt$ is bounded. As remarked in §§§4.10.2.1 the function $(I - J)(\omega, \omega_{\varphi_t})$ is nondecreasing in t. Hence

$$(1-\tau)(I - J)(\omega, \omega_{\varphi_\tau}) \le \int_{[\tau,1]} (I - J)(\omega, \omega_{\varphi_t})dt,$$

and therefore [**76**, p. 67]

$$\lim_{\tau\to 1^-}(1-\tau)(I-J)(\omega,\omega_{\varphi_\tau})=0. \tag{285}$$

Going back to (279) and using the identity $E_0(\omega,\omega_{\varphi_0})+(I-J)(\omega,\omega_{\varphi_0})=V^{-1}\int_M f_\omega\omega^n$ we have

$$\lim_{\tau\to 1^-}E_0(\omega,\omega_{\varphi_t})=\frac{1}{V}\int_M f_\omega\omega^n-\int_{[0,1]}(I-J)(\omega,\omega_{\varphi_t})dt.$$

By a theorem of Ding and Tian we have [**76**, Theorem 1.2]

$$l(\omega)=\lim_{t\to 1^-}F(\omega,\omega_{\varphi_t})=-\int_{[0,1]}(I-J)(\omega,\omega_{\varphi_t})dt. \tag{286}$$

Combining with (284) we conclude that

$$l_0(\omega)=\lim_{t\to 1^-}E_0(\omega,\omega_{\varphi_t})=l(\omega)+\frac{1}{V}\int_M f_\omega\omega^n.$$

Finally, using (194), (189), (282) and (285) it follows that

$$\lim_{t\to 1^-}I_k(\omega_{\varphi_t},\operatorname{Ric}\omega_{\varphi_t})=0.$$

Therefore, using Proposition 4.12 (203) again we have the inequality $l_0(\omega)\geq l_k(\omega)+I_k(\omega,\operatorname{Ric}\omega)$. □

Remark 4.53. Note that from Proposition 4.12 it follows that if F is proper on $\mathcal{H}_{c_1}$ (equivalently on $\mathcal{H}^+_{c_1}$) with $F(\omega,\omega_\varphi)\geq\tau_\omega((I-J)(\omega,\omega_\varphi))$ (recall (190)) then we have the inequality $E_k(\omega,\omega_\varphi)\geq\tau_\omega((I-J)(\omega,\omega_\varphi))-I_k(\omega,\operatorname{Ric}\omega)$ on $\mathcal{H}^+_{c_1}$ (and for $k=0,1$ on $\mathcal{H}_{c_1}$). On the determination of explicit functions τ_ω we refer to [**214,270,271**].

4.10.3 A new Moser-Trudinger-Onofri inequality on the Riemann sphere and a family of energy functionals. In the first part of this subsection we prove results that improve on the restricted generalized Moser-Trudinger-Onofri inequality (Corollary 4.39) in a different direction than that explored in §§4.10.1 (Theorem 4.42). Namely, we show that the inequality holds even when one adds certain negative terms to the exponent on the right hand side. This is done by expressing the excess in the inequality in geometric terms, namely in terms of the inverse Ricci operator. This is different from Tian's approach to a strenghtened inequality on Kähler-Einstein manifolds [**271**, Theorem 6.21] and in particular involves sharp constants and a precise characterization of the case of equality. In the future we hope to address the relation between these two approaches. In the second part we introduce a family of energy functionals and explain their relation to the improved inequality.

We now state the main result of this subsection.

Theorem 4.54. *Let (M, J, ω) be a Fano Kähler-Einstein manifold. Then for each $\varphi \in MTO_n$ holds*

$$\frac{1}{V}\int_M e^{-(\varphi-\frac{1}{V}\int_M\varphi)}\omega^n \leq e^{J(\omega,\omega_\varphi)-\sum_{j=1}^{\infty}J(\mathrm{Ric}^{-j}\omega_\varphi,\mathrm{Ric}^{-j+1}\omega_\varphi)}. \tag{287}$$

Each of the terms in the sum is nonnegative precisely when $\omega_\varphi \in \mathrm{Ric}(\mathcal{B}_n) \supsetneq \mathcal{H}_{c_1}$.[26] *Equality holds if and only if ω_φ is the pull-back of ω by a holomorphic transformation.*

Recall that Theorem 4.42 strengthened the restricted generalized Moser-Trudinger-Onofri inequality (Corollary 4.39) by optimally enlarging the set of functions on which it holds to a set strictly containing $\mathcal{H}_\omega$. Theorem 4.54 further strengthens Theorem 4.42: it shows that the sets MTO_n (see (274)) are characterized by an inequality stronger than (273). A version of this result holds also under the assumption that the K-energy is bounded from below. For simplicity we only state the result in the Kähler-Einstein setting.

Proof. By definition $F_1(\omega,\omega_\varphi) \geq 0$ for each $\varphi \in MTO_n$. Observe that by Theorem 4.42 (i) it follows that Ric^{-1} preserves MTO_n. Therefore for each $l \in \mathbb{N}$,

$$F_1(\omega, \mathrm{Ric}^{-l}\omega_\varphi) \geq 0, \quad \forall\, \varphi \in MTO_n.$$

By exactness of F we obtain

$$F_1(\omega,\omega_\varphi) + F_1(\omega_\varphi, \mathrm{Ric}^{-1}\omega_\varphi) + \ldots + F_1(\mathrm{Ric}^{-l+1}\omega_\varphi, \mathrm{Ric}^{-l}\omega_\varphi) \geq 0,$$

that is,

$$F_1(\omega,\omega_\varphi) \geq \sum_{j=1}^{l} F_1(\mathrm{Ric}^{-j}\omega_\varphi, \mathrm{Ric}^{-j+1}\omega_\varphi). \tag{288}$$

Now, using (185)-(188) and (219) one has

$$F_1(\omega,\omega_\varphi) = J(\omega,\omega_\varphi) - \frac{1}{V}\int_M \varphi\omega^n - \log\frac{1}{V}\int_M e^{f_\omega-\varphi}\omega^n. \tag{289}$$

---

[26] More precisely, each of the terms with $j \geq 2$ is nonnegative for all $\varphi \in MTO_n$, while the term with $j=1$ is nonnegative precisely when $\omega_\varphi \in \mathrm{Ric}\,(\mathcal{B}_n)$. Therefore, after possibly omitting the first term in the sum, (287) is an improvement over (273) for all $\varphi \in MTO_n$ and not just for the subset $\mathrm{Ric}\,(\mathcal{B}_n) \subseteq MTO_n$ (both strictly contain $\mathcal{H}_{c_1}$).

It follows that for any $\alpha \in \mathcal{H}_{c_1}$ holds

$$F_1(\alpha, \operatorname{Ric}\alpha) = J(\alpha, \operatorname{Ric}\alpha) - \frac{1}{V}\int_M f_\alpha \alpha^n. \tag{290}$$

Combining (288)-(290), and letting l tend to infinity, yields

$$\begin{aligned} \frac{1}{V}\int_M e^{-(\varphi - \frac{1}{V}\int_M \varphi)}\omega^n \leq\; & e^{J(\omega,\omega_\varphi) - \sum_{j=1}^\infty J(\operatorname{Ric}^{-j}\omega_\varphi, \operatorname{Ric}^{-j+1}\omega_\varphi)} \\ & \cdot e^{\sum_{j=1}^\infty \frac{1}{V}\int_M f^{(j)}_{\omega_\varphi}(\operatorname{Ric}^{-j}\omega_\varphi)^n}, \end{aligned} \tag{291}$$

where $f^{(j)}$ is the push-forward of the vector field f under Ric^{-j}. Since by (175) the second term in (290) is nonnegative the desired inequality now follows from (291).

The last statement follows from the fact that $I_n = J$, Proposition 4.40, and the definition of $\mathcal{B}_n$ (215). □

In the case of the Riemann sphere S^2, Theorem 4.42 (iii) implies $\operatorname{Ric}(\mathcal{B}_1) = \operatorname{Ric}(\mathcal{A}_1) = MTO_1 = C^\infty(S^2)$. Therefore we have the following improvement of the classical Moser-Trudinger-Onofri inequality (Theorem 4.37). For notation we refer to §§4.10.1.

Corollary 4.55. *Denote by $(S^2, \mathrm{J}, \omega = \omega_{\mathrm{FS},2/V})$ a round sphere of volume V. For any function φ on S^2 in $W^{1,2}(S^2)$ one has*

$$\frac{1}{V}\int_{S^2} e^{-\varphi + \frac{1}{V}\int_{S^2}\varphi\omega}\omega \leq e^{\frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi - \sum_{j=1}^\infty J(\operatorname{Ric}^{(-j)}\omega_\varphi, \operatorname{Ric}^{(-j+1)}\omega_\varphi)}. \tag{292}$$

Each of the terms in the sum is nonnegative, and equals zero if and only if ω_φ is obtained from ω by a Möbius transformation. This also characterizes when equality holds in (292).

Note that the smoothing property of the iteration (see page 126) implies that the extra terms in the sum are meaningful under the assumption $\varphi \in W^{1,2}(S^2)$.

Motivated by Proposition 4.40 we define the following family of energy functionals. For each $k \in \{0, \ldots, n\}$ and $l \in \mathbb{N} \cup \{0\}$ let $E_{k,l}$ denote the pull-back by Ric^{-l} of the Chen-Tian functional E_k (see (200)). That is

$$E_{k,l}(\omega, \omega_\varphi) = E_k(\operatorname{Ric}^{-l}\omega, \operatorname{Ric}^{-l}\omega_\varphi). \tag{293}$$

For example, $E_{n,1} = F_1$, and

$$E_{k,1}(\omega,\omega_\varphi) = F_1(\omega,\omega_\varphi) - (J - I_k)(\operatorname{Ric}^{-1}\omega_\varphi, \omega_\varphi) + (J - I_k)(\operatorname{Ric}^{-1}\omega, \omega), \quad k = 0, \ldots, n. \tag{294}$$

In light of this, Theorem 4.54 is seen to be a corollary of the following inequality:

$$E_{n,l+1}(\omega,\,\cdot\,)=(\mathrm{Ric}^{-l})^{\star}F_1\,\big|_{\{\omega\}\times MTO_n}\geq 0,\quad \forall\, l\in\mathbb{N}. \tag{295}$$

Finally, we remark that in light of Definition 4.32 and Proposition 4.40 one may also extend the definition of Ding's functional to an arbitrary Kähler manifold and class. This might have some future applications.

4.10.4 Construction of Nadel-type obstruction sheaves. Up until this point we have scarcely concerned ourselves with the behavior of the various dynamical systems constructed in the absence of a smooth fixed point. In this subsection we show that in this situation, and in the Fano setting, the Ricci iteration will produce Nadel-type obstruction sheaves, similarly to the continuity method and the Ricci flow. The basic references for this subsection are Demailly-Kollár **[73]** and Nadel **[199]**.

Let $PSH(M,\mathrm{J},\omega)\subseteq L^1_{\mathrm{loc}}(M)$ denote the set of ω-plurisubharmonic functions. For $\varphi\in PSH(M,\mathrm{J},\omega)$ define the multiplier ideal sheaf associated to φ as the sheaf $\mathcal{I}(\varphi)$ defined for each open set $U\subseteq M$ by local sections

$$\mathcal{I}(\varphi)(U)=\{h\in\mathcal{O}_M(U): |h|^2e^{-\varphi}\in L^1_{\mathrm{loc}}(M)\}. \tag{296}$$

Such sheaves are coherent. Such a sheaf is called proper if it is neither zero nor the structure sheaf $\mathcal{O}_M$.

Nadel showed that in the absence of a Kähler-Einstein metric the continuity method (183) will produce a certain family of multiplier ideal sheaves. Phong, Šešum and Sturm showed that certain multiplier ideal sheaves can be obtained also from the Ricci flow

$$\omega^n_{\varphi_t}=\omega^n e^{f_\omega-\varphi_t+\dot{\varphi}_t},\quad \varphi(0)=\text{const}. \tag{297}$$

Denote by $\lfloor x\rfloor$ the largest integer not larger than x.

Theorem 4.56. (See **[213]**.) *Let (M,J) be a Fano manifold not admitting a Kähler-Einstein metric. Let $\gamma\in(1,\infty)$ and let $\omega\in\mathcal{H}_{c_1}$. Then there exists an initial condition $\varphi(0)$ and a subsequence $\{\varphi_{t_j}\}_{j\geq 0}$ of solutions of (297) such that $\lim_{j\to\infty}\varphi_{t_j}=\varphi_\infty\in PSH(M,\mathrm{J},\omega)$ and $\mathcal{I}(\gamma\varphi_\infty)$ is a proper multiplier ideal sheaf satisfying*

$$H^r(M,\mathcal{I}(\gamma\varphi_\infty)\otimes K_M^{-\lfloor\gamma\rfloor})=0,\quad \forall\, r\geq 1. \tag{298}$$

Their proof relies on some of Perelman's estimates for the Ricci flow as well as the following theorem of Kołodziej.

Theorem 4.57. (See **[153]**.) *Let* $F \in L^p(M,\omega), p > 1$ *be a positive continuous function with* $\frac{1}{V}\int_M F\omega^n = 1$. *There exists a bounded solution* φ *to the equation* $\omega_\varphi^n = F\omega^n$ *on* M *which satisfies* $\operatorname{osc}\varphi \le C$ *with* C *depending only on* $||F||_{L^p(M,\omega)}, p$ *and* (M,ω).

Let $\tau = 1/\mu = 1$. The following simple result is a discrete analogue of Theorem 4.56. Its very simple proof compared to that of the analogous result for the Ricci flow is our main motivation for including it here. Moreover, the sheaves produced in this way are essentially computable (see the next subsection).

Theorem 4.58. *Let* (M, J) *be a Fano manifold not admitting a Kähler-Einstein metric. Let* $\gamma \in (1,\infty)$ *and let* $\omega \in \mathcal{H}_{c_1}$. *Then there exists a subsequence* $\{\psi_{j_k}\}_{k\ge 1}$ *of solutions of (176) such that* $\lim_{k\to\infty}\psi_{j_k} = \psi_\infty \in PSH(M,\mathrm{J},\omega)$ *and* $\mathcal{I}(\gamma\psi_\infty)$ *is a proper multiplier ideal sheaf satisfying (298).*

Proof. Indeed, since the iteration takes the form

$$\omega_{\psi_{l+1}}^n = \omega^n e^{f_\omega - \psi_l}, \quad l \in \mathbb{N},$$

Theorem 4.57 can be directly applied (observe that from (175) an estimate on $\operatorname{osc}\psi_l$ implies one on $||\psi_l||_{L^\infty(M)}$) to construct sheaves with $\gamma > 1$, making use of Proposition 4.17 (for more details see the author's article **[229**, §2 (iv)**]**). □

Remark 4.59. One may also construct multiplier ideal sheaves for the Ricci iteration with other time steps and for exponents in the range $(n/(n+1), 1)$ much the same as the continuity method sheaves constructed by Nadel as well as the analogous ones constructed in **[229]** for the Ricci flow (we hope to discuss this in more detail in the sequel; note that the latter construction strenghthened Theorem 4.56).

We remark that in the context of this section, it is also interesting to study the limiting behavior of the inverse Ricci operator (Section 4.8) under iteration.

4.10.5 Relation to balanced metrics. In this paragraph we describe an immediate corollary of the work of Donaldson. It gives with no further work an algorithm for computing Kähler-Einstein metrics using balanced metrics: Given a polarized Hodge manifold (X, L) and a volume form ν Donaldson **[83]** constructs a sequence of pull-backs of Fubini-Study metrics in $\mathcal{H}_{c_1(L)}$ induced from Kodaira embeddings that converge to a solution of the Calabi-Yau equation $\omega_\varphi^n = \nu$. Since in the Fano case our time one Ricci iteration consists precisely of solving a Calabi-Yau equation at each iteration we see that repeated application of Donaldson's constructions approximates the Ricci iteration and in this sense provides a quantization of the Ricci flow.

Another consequence is the possibility to numerically construct Nadel-type sheaves on Fano manifolds admitting no Kähler-Einstein metrics, by §§4.10.4.

Note that more generally one may approximate in the same manner the orbits of the iteration given by the inverse Ricci operator (Definition 4.32) on an arbitrary Kähler manifold with $\Omega \in H^2(M, \mathbb{Z})$.

Finally, it would be interesting to find more relations between discretizations of other geometric flows and iteration schemes involving Bergman metrics.

4.10.6 A question of Nadel. As explained in Section 4.2 one of the original motivations for our work was a question raised by Nadel [**200**]: Given $\omega \in \mathcal{H}_{c_1}$ define a sequence of metrics $\omega, \operatorname{Ric}\omega, \operatorname{Ric}(\operatorname{Ric}\omega), \ldots$, as long as positivity is preserved; what are the periodic orbits of this dynamical system? The cases $k = 2, 3$ in the following theorem are due to Nadel.

Theorem 4.60. *Let* (M, J, ω) *be a Fano manifold and assume that* $\operatorname{Ric}^l \omega = \omega$ *for some* $l \in \mathbb{Z}$. *Then* ω *is Kähler-Einstein.*

Proof. The theorem follows from Proposition 4.17. Indeed, note that the nonexistence of periodic fixed points of negative order implies that of positive order, and vice versa. Therefore assume that for some $\omega \in \mathcal{H}_{c_1}$ and some $l \in \mathbb{N}$ one has $\operatorname{Ric}^{-l}\omega = \omega$. By the cocycle condition we thus have

$$0 = E_0(\omega, \operatorname{Ric}^{-l}\omega) = \sum_{i=0}^{l-1} E_0(\operatorname{Ric}^{-i}\omega, \operatorname{Ric}^{-i-1}\omega). \tag{299}$$

By Proposition 4.17 one has

$$E_0(\operatorname{Ric}^{-i}\omega, \operatorname{Ric}^{-i-1}\omega) < 0,$$

unless $\operatorname{Ric}^{-i}\omega = \operatorname{Ric}^{-i-1}\omega$. Therefore each of the terms in (299) vanishes and ω is Kähler-Einstein. □

Moreover, from the proof we have the following stronger conclusion:

Corollary 4.61. *Let* (M, J, ω) *be a Fano manifold with trivial Futaki character. Assume that* $\operatorname{Ric}^l \omega = h^\star\omega$ *for some* $l \in \mathbb{Z}$ *and some* $h \in \operatorname{Aut}(M, \mathrm{J})$. *Then* $h = \mathrm{id}$ *and* ω *is Kähler-Einstein.*

Lemma 4.36 implies the following natural generalization of Theorem 4.60 to the setting of solitons.

Corollary 4.62. *Let* (M, J, ω) *be a Fano manifold and assume that* X *belongs to a reductive Lie subalgebra of* $\operatorname{aut}(M, \mathrm{J})$ *and that the one-parameter subgroup* $T_{\mathrm{J}X}$

generated by $\mathrm{J}X$ *is a compact torus in* $\mathrm{Aut}(M, \mathrm{J})$. *Let* $\omega \in \mathcal{H}_{c_1}(T_{\mathrm{J}X})$. *Assume that* $\mathrm{Ric}^{l}_{\psi X}\,\omega = \omega$ *for some* $l \in \mathbb{Z}$. *Then* ω *is a Kähler-Ricci soliton.*

In addition, under the assumption that the Tian-Zhu character [**279**] is trivial one has a statement analogous to Corollary 4.61. Also, as noted in Section 4.9, and using a generalized character introduced by Futaki [**106**], this result extends to the setting of multiplier Hermitian structures.

To conclude this subsection we remark that what now becomes apparent is that Nadel's iteration scheme is precisely the Euler method for the conjugate Ricci flow and is thus dual to our iteration that corresponds to the backwards Euler method for the Ricci flow.

Remark 4.63. R. Hamilton has informed the author that D. DeTurck has also raised the problem of whether the Ricci operator possesses nontrivial fixed points in the general Riemannian setting in the 1980's. He called solutions of $\mathrm{Ric}\,(\mathrm{Ric}\,\omega) = \omega$ Zweistein metrics, solutions of $\mathrm{Ric}\,(\mathrm{Ric}\,(\mathrm{Ric}\,\omega)) = \omega$ Dreistein metrics, and in general higher fixed points k-Stein metrics. It is a very challenging problem to understand whether k-Stein metrics exist in the general Riemannian setting.

Remark 4.64. In light of Theorem 4.60 perhaps it would be interesting to re-examine Nadel's generalized maximum principle which was used to provide a completely different proof for the cases $k = 2, 3$.

4.10.7 The Ricci index and a canonical nested structure on the space of Kähler metrics.

In this subsection we describe a new canonical structure inherent in the space of Kähler forms determined by the complex structure and the Kähler class alone.

Consider first the case of a Fano manifold. As we saw earlier the iteration of the inverse Ricci operator on $\mathcal{H}_{c_1}$ has the advantage of possessing infinite orbits starting at any initial points. The Ricci operator on the other hand lacks this property, according to the Calabi-Yau theorem. This motivates the following definition.

Definition 4.65. *Let* (M, J) *be a Fano manifold. For each* $l \in \mathbb{N} \cup \{0\}$ *denote by* $\mathcal{H}^{(l)}_{c_1}$ *denote the domain of definition of* Ric^{l}.

One has

$$\mathcal{D}_{c_1} = \mathcal{H}^{(0)}_{c_1} \supset \mathcal{H}_{c_1} = \mathcal{H}^{(1)}_{c_1} \supset \mathcal{H}^{(2)}_{c_1} = \mathcal{H}^{+}_{c_1} \supset \cdots \supset \mathcal{H}^{(l)}_{c_1} \supset \cdots. \tag{300}$$

In other words, we may define on $\mathcal{H}_{c_1}$ an integer-valued function

$$\omega \mapsto r(\omega), \tag{301}$$

where $r(\omega)$ is the unique positive integer satisfying $\omega \in \mathcal{H}_{c_1}^{(r(\omega))} \setminus \mathcal{H}_{c_1}^{(r(\omega)+1)}$. When no such number exists we set $r(\omega) = \infty$. We call the function $r : \mathcal{H}_{c_1} \to \mathbb{N}$ the Ricci index. The number $r(\omega)$ is a Riemannian invariant of the manifold (M, J, ω). It may also be defined for general Riemannian manifolds however it seems hard to study in such generality.

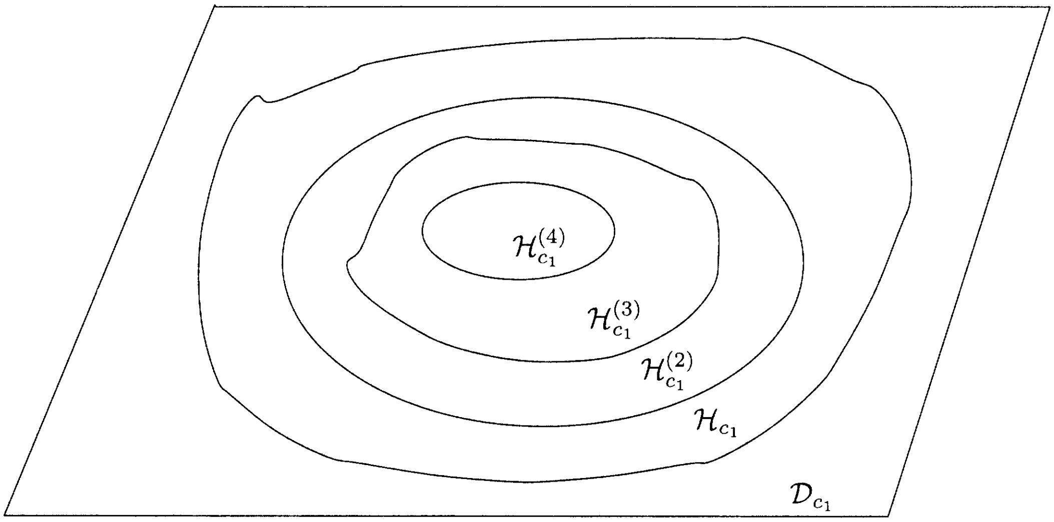

Figure 5. Nested subspaces inside the space of closed forms representing the first Chern class on a Fano manifold.

One may extend such a construction to a general Kähler manifold in at least two ways, using either the Ricci iteration or the inverse Ricci operator. Choosing the latter we obtain the following extension of Definition 4.65.

Definition 4.66. *Let (M, J) be a Kähler manifold and let Ω denote a Kähler class. For each $l \in \mathbb{N}$ denote by $\mathcal{H}_{\Omega}^{(l)}$ the image of $\mathcal{H}_{\Omega}$ under $\mathrm{Ric}_{\Omega}^{-l+1}$.*

Several natural questions arise that we hope to touch upon in the future. What is $\mathcal{H}_{\Omega}^{(\infty)} := \bigcap_{l=1}^{\infty} \mathcal{H}_{\Omega}^{(l)}$? How to asymptotically relate the Ricci index to the time parameter of the Ricci flow? Also, how to relate the Ricci index, on the one hand, to the metric structure g_{L^2} on the space $\mathcal{H}_{\Omega}$ (Equation (33)) and, on the other hand, to sublevel sets of Calabi's energy and Mabuchi's K-energy? Finally, what is the relation between the Ricci index and positivity?

> *D'Artagnan n'était pas un millionnaire; il espérait bien le devenir un jour, mais le temps qu'il se fixait lui-même pour cet heureux changement était assez éloigné.*
>
> — Alexandre Dumas, *Les trois mousquetaires* (1844)

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