Dynamic Inventory Management with Expediting

by

Chiwon Kim

B.S., Mechanical and Aerospace Engineering,
Seoul National University (2002)
M.S., Mechanical Engineering,
Massachusetts Institute of Technology (2004)

Submitted to the Department of Mechanical Engineering
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mechanical Engineering

at the
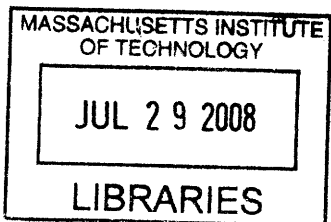
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2008

© Massachusetts Institute of Technology 2008. All rights reserved.

Author . . . .
Department of Mechanical Engineering
March 31, 2008

Certified by . . . .
David Simchi-Levi
Professor, Department of Civil and Environmental Engineering and the Engineering
Systems Division
Thesis Supervisor

Certified by . . . .
Sanjay Sarma
Professor, Department of Mechanical Engineering
Thesis Supervisor

Accepted by . . . .
Lallit Anand
Chairman, Department Committee on Graduate Students

Dynamic Inventory Management with Expediting

by

Chiwon Kim

Submitted to the Department of Mechanical Engineering
on March 31, 2008, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Mechanical Engineering

Abstract

In modern global supply chains, goods travel stochastically from suppliers to their final destinations through several intermediate installations such as ports and distribution facilities. In such an environment, the supply chain must be agile to respond quickly to demand spikes. One way to achieve this objective is by expediting outstanding orders from the intermediate installations through premium delivery. In this research, we study the optimal expediting and regular ordering policies of a serial supply chain with a radio frequency identification deployment at each installation. Radio frequency identification technology allows capturing the state of the system, i.e., the time and location of goods, at any point in time, and thus enables to expedite outstanding orders directly to the destination, which faces stochastic demand.

We identify systems, called *sequential*, that yield simple and tractable optimal policies. For sequential systems, outstanding orders including expediting do not cross in time. For such systems, we find that the optimal policies of expediting and regular ordering are the base stock type policies. The directional sensitivity of the base stock levels with respect to expediting costs is also obtained. We provide an important managerial insight on the radio frequency identification technology: we need to actively use the additional information from the radio frequency identification technology through new business processes such as expediting to unveil more benefits from the supply chain. On the other hand, orders may cross in time for systems that are not sequential, thus in such a case optimal policies are hard to obtain. We propose a heuristic for such systems and discuss its performance and limitation. Lastly, as an extension to the model, we study the optimal policies of expediting and regular ordering when there is an expiry date on outstanding orders. The optimal expediting policy identifies a number of base stock levels depending on the age of the orders, but the structure of the optimal policy remains simple for sequential systems.

Thesis Supervisor: David Simchi-Levi
Title: Professor, Department of Civil and Environmental Engineering and the Engineering Systems Division

Thesis Supervisor: Sanjay Sarma
Title: Professor, Department of Mechanical Engineering

Acknowledgments

First and foremost, I would like to thank my advisor Professor David Simchi-Levi for his valuable guidance throughout my PhD years. He is not only an excellent scholar with deep insights, but also a great advisor on both research and life. I indeed appreciate his wisdom at every key point of my research. I was very fortunate to become his student even though he is famous for being very selective in choosing his students and my home department is rather far from the field of supply chain management. It has truly been an honor and a pleasure to work with him.

Next, I would like to thank my co-advisor Professor Diego Klabjan at Northwestern University. It is difficult to fully acknowledge here the extent of his support and express my gratitude. He is simply an amazing researcher, teacher, and mentor in every aspect of life. I can discuss anything with him, and he has been always there to help me and to encourage me. He was at MIT during 2005-2006 for his sabbatical year, and that is how I started working with him. Even after he left for Northwestern University, he remained as my co-advisor and guided me through every step of my research. Without him, I could not have completed my research.

I would like to thank my thesis committee members, Professors Sanjay Sarma and Jeremie Gallien. Professor Sarma has been a good supporter of my research and always encourages me to keep going forward. Professor Gallien provided insightful comments and suggestions on this research. I appreciate all of their input and time. Also, I am very grateful to my Master's thesis advisor, Dr. Stanley Gershwin. He has always been a great advisor and mentor. I am greatly indebted to him for his help during my early days at MIT.

I would like to thank all of my friends at MIT for making my life really enjoyable: Sunghwan Jung, Youngjae Jang, Hyuksang Kwon, Jaemyung Ahn, Sangwon Byun, and Sang-Il Lee, among others. I cannot name them all here since they are so many. I am equally grateful to every one of them, and my life would never have been the same without them. Finally, I am indebted to my family for their support. My parents and my sister are always supportive for my studies far away from home. My wife, Dooran Kim, gives me endless support, encouragement and love during ups and downs. This thesis could not have been finished without her support.

Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation

Recent globalization has brought increased complexity in supply chains. With facilities in supply chains spread throughout the globe, lead times are growing and becoming more volatile. Fierce competition among global supply chains is observed. According to Lee (2004), an important challenge for competitive advantage is to build *agile, adaptable,* and *aligned* supply chains. Agility, adaptability, and alignment of supply chains mean the following:

- Agility: ability to respond quickly to short-term changes in demand or supply

- Adaptability: ability to adjust supply chain design to accommodate medium or long-term market changes

- Alignment: ability to establish incentives for supply chain partners to improve performance of the entire chain

Among these, our focus is mainly on the issue of improving the agility of today's supply chains. Agility can be improved by promoting the flow of information between suppliers and customers and developing collaborative relationships with suppliers. For instance, if suppliers provide more information on shipments along with more delivery options for different rates, then the agility of supply chains can be improved. In this research, we try to improve the agility of a supply chain through expediting outstanding orders based on extra information about goods in transit from the supplier. Rather than just waiting for regular

orders to be delivered, a firm may expedite partial or complete orders in transit through premium delivery, such as by air, with extra cost, to improve agility.

To see potential benefits, let us consider an inventory system that faces stochastic lead time and demand. Usually, the optimal operation of an inventory system can be achieved through balancing holding and backlogging costs under certain expectations on lead time and demand. If the realized demand is higher than expected, a backlogging cost is incurred. On the other hand, if the demand is lower than expected, a holding cost occurs. The same case happens with stochastic lead time: if the lead time is shorter than expected, a backlogging cost is incurred, and otherwise, a holding cost occurs. Figure 1-1 summarizes 9 possible scenarios due to these uncertainties. Improved agility through expediting can

Demand / Lead Time	Less than expected	Expected	More than expected
Longer than expected	OK	Backlogging cost	High backlogging cost
Expected	Holding cost	OK	Backlogging cost
Shorter than expected	High holding cost	Holding cost	OK

Figure 1-1: Possible scenarios of uncertainties

directly reduce the backlogging cost due either to high demand, short lead time, or both. If demand spikes, we may expedite outstanding orders to meet the excessive demand to reduce the undesirable backlogging cost. Also, we may shorten the undesirably prolonged lead time of certain orders through expediting. Not only the backlogging cost, but also the holding cost, can be reduced by improved agility through expediting, since the supply chain with expediting does not require as much safety stock as the one without expediting options. Reduced safety stock generally lowers the holding cost. Therefore, the improved agility through expediting certainly reduces unexpected costs, both holding and backlogging costs of the supply chain.

However, the practice of expediting incurs expediting costs. Therefore, in order to minimize the total supply chain costs, which include the expediting costs, one has to know how to use the expediting options wisely. In this research, we study how to optimally exploit expediting to increase agility, which is our central focus. More specifically, we address the

9

questions of what the optimal expediting policy is, whether the policy is practical, and what the corresponding optimal regular ordering policy is. Additionally, we discuss the questions of what the effects of expediting costs on the optimal policy are, what information systems we need to support expediting, and how we can extend the model to accommodate more real-world situations. We answer all these questions in the following chapters.

Simple but Nontrivial Illustration[1]

Suppose that a company in South Korea makes a high-value product such as LCD panels. As the leading supplier, it supplies its panels to multiple TV and computer monitor manufacturers spread throughout the US. It operates a distribution center in Long Beach, CA, for operational efficiency. Because of the weight and volume of LCD panels, it usually uses ocean shipping rather than air to transport LCD panels to the distribution center, and then it uses ground transportation to each of the manufacturers. The lead time is stochastic, between 2 and 6 weeks. While there are several manufacturers, our focus is on the manufacturer labeled M-1. To increase agility, M-1 utilizes expediting. M-1 may expedite LCD panels from the supplier in South Korea by air to M-1, or from the distribution center in Long Beach by air to M-1. See Figure 1-2.
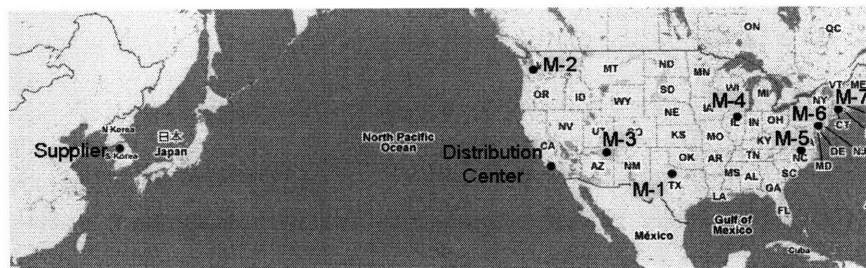
Figure 1-2: An illustration with a two-installation supplier and manufacturing facilities

In this thesis, we discuss more general models than the one just illustrated. However, it is important to remark that, even though it looks simple, this illustration contains all the complex features of much more general models, which may have multiple installations spread all over the world.

---

[1] This illustration is not a real business case.

1.2 Stochastic Inventory Theory: Review

In this section, we give a brief review of stochastic inventory theory. For a broader review, we refer to Simchi-Levi et al. (2004) and Porteus (2002).

Basic inventory model

As the simplest multi-period case, consider a periodic-review, single-item inventory problem. The planning horizon is T time periods. This inventory faces stochastic demand, and the demand distribution is known and independent for each time period. There is a single supplier of the inventory with the procurement cost of c per unit, and the order lead time is instantaneous compared with the time period. Excessive demand is backlogged at cost b per unit per time period and fulfilled in the following time periods. On the other hand, excessive inventory incurs the holding cost of h per unit per time period.

Let us denote the demand by D and the inventory on hand by v. The inventory manager places an order of amount u at the beginning of a time period, which is the decision variable for that time period. At time period $T + 1$, the inventory on hand can be returned to the supplier at the per unit cost of c. For simplicity, we do not discount future costs, and do not consider any fixed ordering costs. For convenience, let us define $L(x) = E[b \cdot (x - D)^- + h \cdot (x - D)^+]$, where $(x)^+ = \max\{x, 0\}$ and $(x)^- = \min\{x, 0\}$.

Let us denote by $J_t(v)$ the cost-to-go at time period t with on-hand inventory v. The dynamic programming optimality equation reads

$$J_t(v) = \min_{u \geq 0}\{cu + L(u + v) + E[J_{t+1}(u + v - D)]\},$$

where $J_{T+1}(v) = -cv$. It is common to introduce $y = u + v$. Then we have

$$J_t(v) = \min_{y \geq v}\{cy + L(y) + E[J_{t+1}(y - D)]\} - cv,$$

where $J_{T+1}(v) = -cv$. Let us first examine $J_T(v)$. We have

$$J_T(v) = \min_{y \geq v}\{cy + L(y) + E[-cy + cD]\} - cv = \min_{y \geq v}\{L(y)\} + c(E[D] - v).$$

Note that $L(\cdot)$ is a convex function since convexity is preserved under expectation. We

11

can find a quantity y_T^* that minimizes $L(y)$, or $y_T^* = \arg\min\{L(y)\}$. The optimal ordering policy at time period T is to order $y_T^* - v$ if $v \leq y_T^*$, and otherwise order nothing. This policy is called *the base stock policy*, and y_T^* is the base stock level of time period T with respect to v. Cost-to-go $J_t(v)$ has a special structure. It is the sum of $L(y_T^*) - cv$ and a monotonically nondecreasing convex function $g(v)$, where $g(v) = 0$ for $v \leq y_T^*$ and $g(v) = L(v) - L(y_T^*)$ for $v > y_T^*$.

Now, assume $J_{t+1}(v)$ is convex for a fixed t such that $t + 1 \leq T$. By the same reasoning, $cy + L(y) + E[J_{t+1}(y - D)]$ is convex and has at least one finite minimizer. Let us denote the minimizer by y_t^*, or $y_t^* = \arg\min_y\{cy + L(y) + E[J_{t+1}(y - D)]\}$. Then the optimal policy is again the base stock policy with the base stock level y_t^*. Finally, $J_t(v)$ is again a convex function under the base stock policy. By the inductive argument, we conclude that the base stock policy with the base stock level y_t^* for time period t is optimal for this inventory problem.

Finite lead time inventory model

In certain cases, lead time cannot be considered instantaneous compared to the review period. In such cases, we consider an inventory with a finite lead time of multiple time periods. Lead time can be either deterministic or stochastic, and here we review the deterministic lead time case. Since the lead time is longer than a review period, we have to keep track of multiple order amounts that are placed within the lead time. Let us denote by v_0 the on-hand inventory and by v_i the outstanding order amount that has i time periods remaining until delivery. Let L be the lead time. Then the state variables are $(v_0, v_1, \cdots, v_{L-1})$. The optimality equation reads

$$J_t(v_0, v_1, \cdots, v_{L-1}) = \min_{u \geq 0}\{cu + L(v_0) + E[J_{t+1}(v_0 + v_1 - D, v_2, \cdots, v_{L-1}, u)]\},$$

where $J_{T+1}(v_0, v_1, \cdots, v_{L-1}) = -c(v_0 + v_1 + \cdots + v_{L-1})$. We again use the transformation of the optimality equation using $y = u + v_0 + v_1 + \cdots + v_{L-1}$. Let $x^{L-1} = v_0 + v_1 + \cdots + v_{L-1}$ be the *inventory position*. The transformed optimality equation only depends on the inventory position rather than on $(v_0, v_1, \cdots, v_{L-1})$ in determining the optimal ordering quantity. The optimal ordering policy is the base stock policy with respect to the inventory position.

Multi-echelon inventory model

The multi-echelon inventory problem is introduced in Clark and Scarf (1960). It has a series of installations, where an installation supplies the next one, and exogenous demand is realized at one end of the chain. Let us denote by I_j the jth installation, $0 \le j \le K$, where I_0 faces exogenous demand, I_K has infinite amount of inventory, and I_j places orders to I_{j+1}. The lead time between two consecutive installations can be either zero or finite time periods. Let the lead time be L time periods.

The notion of an echelon is important. An echelon is a certain subsystem of the entire supply chain. More specifically, by echelon i we mean the subsystem from I_0 to I_i. Therefore, echelon 0 is just I_0, echelon L is the whole system, and thus there are a total of $L+1$ echelons. Echelon stock is the sum of all stock in the corresponding echelon *plus* outstanding orders that are supposed to be delivered within L time periods to the echelon. Let us denote the echelon-i stock by x^i. Inventory position, which is defined above, is simply echelon-$(L-1)$ stock.

The optimal policy of a multi-echelon inventory system is the base stock policy adapted to the multi-echelon setting. Consider echelon i. Echelon i receives stock from I_{i+1} up to the availability in I_{i+1}. The optimal policy for ordering from I_{i+1} for echelon i is the base stock policy with respect to echelon-i stock x^i, but the ordering amount is limited by the current inventory level at I_{i+1}. From echelon 0 to echelon $L-1$, orders are made based on the base stock policies, with different base stock levels for each echelon at each time period.

Finite shelf life inventory model

Nahmias (1975) and Fries (1975) studied a periodic review, zero lead time inventory problem with deterministic shelf life. Here we briefly introduce the approach of Nahmias (1975). Let us denote by x_i the amount of product on hand that will perish exactly i periods into the future. The state of the system can be represented as $\mathbf{x} = (x_{m-1}, x_{m-2}, \cdots, x_1)$. For convenience, let us define $x(i) = (x_i, x_{i-1}, \cdots, x_1)$, i.e., $x(m-1) = \mathbf{x}$, and $w_i = \sum_{j=1}^{i} x_j$. The inventory position is $x = w_{m-1}$. Demand density f is known and independent for each time period. Decision variable y is the fresh order placed at the beginning of the current period, which arrives instantaneously. The next time period state $(s_{m-1}[y, \mathbf{x}, D], \cdots, s_1[y, \mathbf{x}, D])$, where D is the demand in the current period, is given as

- $s_i[y, \mathbf{x}, D] = (x_{i+1} - (D - w_i)^+)^+$ for $1 \le i \le m - 2$,

- $s_{m-1}[y, \mathbf{x}, D] = y - (t - w_{m-1})^+$ (backlogging).

Demand $A_{j,n}[x(j)]$ over j periods is the total demand over periods $n, n+1, n+2, \cdots, n + j - 1$ that cannot be met by allocations of supply, which would have been outdated by the beginning of period $n + j$. Formally we have

- $A_{1,n}[x(1)] = (D_n - x_1)^+$,

- $A_{2,n}[x(2)] = (D_{n+1} + (D_n - x_1)^+ - x_2)^+ = (D_{n+1} + A_{1,n}[x(1)] - x_2)^+$,

- $\vdots$

- $A_{m-1,n}[x(m-1)] = (D_{n+m-2} + A_{m-2,n}[x(m-2)] - x_{m-1})^+$.

Quantity $y - A_{m-1,n}[x(m-1)]$ is the total amount of the fresh order on hand at the start of period $m + n - 1$. The amount of the fresh order that perishes is $R^*_{m,n} = (y - D_{m+n-1} - A_{m-1,n}[x(m-1)])^+$. To get the distribution of $R^*_{m,n}$, Nahmias (1975) first defines $G_{j,n}[x(j)] = Prob(D_{j+n-1} + A_{j-1,n}(x(j-1)) \le x_j)$, which is the probability that there will be outdating at the end of period $n + j - 1$. Then, Nahmias (1975) shows that

$$G_{j,n}[x(j)] = \int_0^{x_j} G_{j-1,n}[v + x_{j-1}, x(j-2)] f(x_j - v) dv,$$

where $G_1(t) = F(x_1)$. From the definition of $G_{m,n}$, it follows that $Prob[R^*_{m,n} \le t] = 1 - G_{m,n}(y - t, \mathbf{x})$, for $t \ge 0$ and $Prob[R^*_{m,n} \le t] = 0$ for $t < 0$. Since demand is a nonnegative random variable, $E[R^*_{m,n}] = \int_0^y G_{m,n}(t, \mathbf{x}) dt$. The single period cost $L_n(\mathbf{x}, y)$ is given by

$$L_n(\mathbf{x}, y) = cy + h \int_0^{x+y} (x + y - t) f(t) dt + r \int_{x+y}^\infty (t - x - y) f(t) dt + \theta \int_0^y G_{m,n}(t, \mathbf{x}) dt.$$

It can be shown that $L_n(\mathbf{x}, y)$ is convex in y for a fixed $\mathbf{x}$. Let $C_n(\mathbf{x})$ be the minimum expected discounted cost when there are n remaining periods. Similarly, let $L_n(\mathbf{x}, y)$ be the cost when there are n remaining periods. We also define

$$B_n(\mathbf{x}, y) = L_n(\mathbf{x}, y) + \alpha \int_o^\infty C_{n-1}[s(y, \mathbf{x}, t)] f(t) dt.$$

14

There is a functional relation of $C_n(\mathbf{x}) = inf_{y \geq 0} B_n(\mathbf{x}, y)$. Let us define $\bar{x} = F^{-1}[\frac{r-(1-\alpha)c}{r+h}]$. Under some mild assumptions, the following holds.

- $B_n(\mathbf{x}, y)$ is convex in y for all $\mathbf{x}$.

- If $x \leq \bar{x}$, then there exists a unique solution of the following equation:

$$\frac{\partial B_n(\mathbf{x}, y)}{\partial y}\Big|_{y=y_n(\mathbf{x})} = 0.$$

- The optimal policy is to order $y_n(\mathbf{x})$ if $x \leq \bar{x}$.

- Denote by $y^{(i)}$ differentiating with respect to $i-$th argument. Then $-1 \leq y_n^{(1)}(\mathbf{x}) \leq y_n^{(2)}(\mathbf{x}) \leq y_n^{(3)}(\mathbf{x}) \leq \cdots \leq y_n^{(m-1)}(\mathbf{x}) \leq 0$. This means that if the initial stock of inventory at any age level is increased by one unit, the optimal order quantity decreases, but by less than a single unit. Furthermore, the optimal order quantity is more sensitive with respect to the newer inventory.

- If demand is backlogged, then $y_n(\mathbf{x}) = y_n(0) + |x_{m-1}|$.

Nahmias (1982) states that the actual computation is impractical if $m \geq 3$.

1.3 Road Map

This thesis consists of four main topics, each of which is introduced independently in the respective chapters. Since the problem of finding an optimal expediting policy is quite demanding, in Chapter 2 we first restrict our attention to the problem with a deterministic lead time. Even though the lead time is deterministic, finding an optimal policy is still challenging, and it requires a careful treatment. The concept of *sequential systems* appears first in this chapter.

In Chapter 3, we extend the model so that the lead time is stochastic. With stochastic lead time, we have to capture the locations of outstanding orders in order to expedite them. Radio Frequency Identification (RFID) is introduced for this purpose in this chapter. Also, the concept of sequential systems is generalized to accommodate stochastic lead time. The optimal policies are simple and elegant. The solution methodology is complex, but manageable for sequential systems.

In Chapter 4, we consider systems that are not sequential, and perform a numerical study on non-sequential systems with a proposed heuristic policy. The heuristic policy is quite robust for the systems that are *close* to being sequential. We discuss its performance and limitations.

Chapter 5 extends the model of Chapter 3 so that orders in transit can have a certain expiry date until delivery, which is the most general model treated in the thesis. The optimal policy for sequential systems identifies a number of parameters, but the structure of the optimal policy remains simple.

We conclude in Chapter 6 with a detailed discussion of contributions made in this thesis and directions for further research.

Chapter 2

Deterministic Lead Time Model

2.1 Introduction

We consider a supply chain that consists of a supplier and a manufacturing facility. Between them, there are multiple intermediate installations such as ports and distribution centers. The manufacturer faces stochastic demand, periodically reviews inventory on hand, and places orders at the supplier. In regular delivery, orders pass through all installations with a deterministic lead time. In addition to regular delivery, expedited delivery is available with extra cost for all or part of the outstanding orders in the pipeline. The manufacturer may expedite orders based on the current inventory status and the demand forecast. When expedited, orders instantly arrive at the manufacturing facility and they are ready to fulfill upcoming demand. In our setting, all decisions are made by the manufacturer, and it is assumed that the manufacturer cannot influence inventory among installations other than into the manufacturing facility. As a consequence, expediting from any installation is allowed only when the destination is the manufacturing facility. This is reasonable when the manufacturing facility is an independent company from the remaining installations and thus cannot instantiate expediting between two other installations. Without expediting, it is well known that the optimal regular ordering follows the base stock policy with respect to the inventory position.

In general, the problem of finding an optimal inventory control policy with respect to regular ordering and expediting is difficult, and it depends critically on the system parameters such as the expediting costs. We introduce the notion of sequential systems, where it is never optimal to expedite from an installation before expediting all outstanding

orders in the downstream installations. The optimal regular and expedited orders preserve their sequence in time until eventual delivery, and thus they never cross in time. We show that in sequential systems the regular ordering policy is the base stock policy with respect to the inventory position and the expediting policy is a variant of the base stock policy that involves multiple base stock levels with respect to echelon stocks. Sequential systems are easy to identify since the expediting cost must be convex with respect to installations.

To summarize, there are three major contributions of this chapter. First, we find that simple optimal policies for regular ordering and expediting can be obtained when both regular and expedited orders do not cross in time. We identify a class of systems based on the expediting costs that has this sequential delivery property. Second, we find that the optimal policies for sequential systems are variants of the base stock policies with respect to inventory position and echelon stocks. Furthermore, the structure of an optimal expediting policy is to expedite everything up to a certain point in the pipeline, and nothing beyond. We provide simple recursion equations to compute the base stock levels. Finally, the modeling and proof techniques are novel. We propose an alternative optimality equation appropriate for sequential systems, and the main results are derived from the alternative optimality equation. Furthermore, standard inductive arguments coupled with separability of the cost-to-go function as often done in the literature cannot be carried out in our context. Indeed, our proof technique is based on studying the difference in the cost-to-go function with different states as well as induction arguments.

In Section 2.2 we formally state the model together with the general optimality equation. We characterize sequential systems and derive an alternative optimality equation appropriate for them in Section 2.3. Section 2.4 presents the optimal policies for sequential systems.

Literature review for deterministic lead time model

Our problem has similarities with multi-supplier inventory problems. One supplier with a much shorter lead time can be used as the expedited mode while the other one with possibly longer lead time as the regular mode. Barankin (1961), Daniel (1963), Neuts (1964), and Veinott (1966) have considered the inventory system with two supply modes of instantaneous and one period lead time. Their model is a special case of our model in this chapter, and thus both models have the same optimal policy structure. Fukuda (1964)

18

extends this model to the case where the lead times are k and $k+1$ periods. Whittemore and Saunders (1977) generalize the two supply mode problem to arbitrary lead times, however the optimal ordering policies are no longer simple functions if the difference in the lead times is more than one period. They also give conditions on optimality of using a single supplier. The stochastic lead time model of zero or one period is considered by Anupindi and Akella (1993). While most of the literature for multiple supply modes addresses the two supply mode case, some researchers, including Fukuda (1964), Zhang (1996), and Feng et al. (2005), consider the three supply mode case. Their optimal policies are generally not base stock type policies.

In the same spirit, models with emergency orders relate to our problem, since expediting has a similar effect. The periodic review inventory model with emergency supply is considered by Chiang and Gutierrez (1996, 1998), Tagaras and Vlachos (2001), and Huggins and Olsen (2003b). Chiang and Gutierrez (1998) allow placing multiple emergency orders within a review period, while the others allow placing a single emergency order per cycle. Huggins and Olsen (2003b) consider a two-stage supply chain system where shortages are not allowed, so the shortage must be fulfilled by some form of expediting such as overtime production. They found that the optimal regular ordering policy is the (s, S) type policy, but the expediting policy is not a base stock type policy. Related research in this area includes Groenevelt and Rudi (2003), where a manufacturing order can be split into fast and slow shipping modes, and Vlachos and Tagaras (2001), where there is a capacity cap on the size of an emergency order. Both multi-supplier and emergency order models in the literature differ significantly from our model since the realized lead time can be any number between 0 and the regular lead time in our model, and it varies dynamically.

The multi-echelon inventory system with expediting has been studied by Lawson and Porteus (2000) who extend the work by Clark and Scarf (1960) by introducing expedited delivery with zero lead time between two consecutive installations. Our model resembles the model in Lawson and Porteus (2000) because a unit can be expedited through several intermediate installations at the same time in both models. Also, their optimal policy is a base stock type policy for each echelon. However, our model is substantially different from Lawson and Porteus (2000) in that we do not allow expediting between two consecutive intermediate installations. As we have already pointed out, in our model expediting can only occur from an installation to the manufacturing facility. This corresponds to situations

in which the manufacturer may request expediting from an installation to the manufacturing facility, but the manufacturer does not have any control to move inventory between any two other installations. The model in Lawson and Porteus (2000) cannot capture the same situation as ours, since in order to prevent prohibited expediting from an installation i to an intermediate installation, the associated expediting per unit cost needs to be set to a high value. However, this high cost also prevents any expediting from upstream of installation i to downstream of installation i. Therefore, their model simply addresses different situations from those captured by our model. Muharremoglu and Tsitsiklis (2003a) generalize Lawson and Porteus (2000) further by allowing super modular expediting cost instead of a linear one. However, their model is different from our model by the same reason.

2.2 Model Statement

We consider a serial supply chain that consists of $L + 1$ installations, numbered from 0 to L, where installation 0 is the manufacturing facility, and installation L is the supplier. A unit of goods can pass through all the installations from the supplier to the manufacturing facility and stays for one period at each installation. Expedited delivery of a fraction or all of outstanding orders is available at each installation, and the lead time is instantaneous. Therefore the actual lead time for a unit is dynamic with the maximum of L time periods and the minimum of 0. The per unit expediting cost from installation i at time period k is $d_{i,k}$. The total planning horizon is T time periods. Figure 2-1 depicts the model.
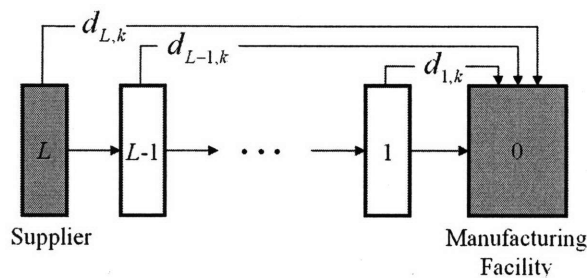

Figure 2-1: The underlying inventory system

Demand D_k for period k is a nonnegative continuous random variable. (It can also be a discrete random variable with a finite support.) At the manufacturing facility, excess demand is backlogged and incurs a backlogging cost, while excess inventory incurs a holding

cost. We require that the holding/backlogging cost function is convex in the amount of inventory. Let $r_k(\cdot)$ be any convex holding/backlogging cost function and for ease of notation let $L_k(x) = E[r_k(x - D_k)]$. Clearly, $L_k(\cdot)$ is convex. An intermediate installation may charge per unit holding or processing cost, but at present we assume that there is no holding or processing cost. We discuss this generalization in Section 2.5.

The sequence of events is as follows. At the beginning of time period k, the manufacturer first places a new regular order at the supplier at cost c_k per unit, and next decides how much to expedite from each installation. The manufacturer may also expedite from the supplier up to the amount of the regular order just placed. After the expedited deliveries of the outstanding orders are received, demand realizes at the manufacturing facility. Holding or backlogging cost is accounted for at the end of time period k. After cost accounting, the outstanding orders at installations 1 through L move to the next downstream installation instantaneously and then the next time period begins.

The problem is to determine an optimal regular ordering quantity and optimal expediting quantities from each of the installations 1 to L at the beginning of each time period. Let us denote by v_i the amount of inventory at installation i at the beginning of a time period before expediting for $i = 0, 1, \cdots, L - 1$. Since the supplier has no inventory at the beginning of a time period, $(v_0, v_1, \cdots, v_{L-1})$ is the current state of the system. Let $J_k(v_0, v_1, \cdots, v_{L-1})$ be the value of the cost-to-go function at the beginning of time period k under optimal regular ordering and expediting. For simplicity, we do not discount any future costs. After time period T, holding and backlogging costs are assumed to be zero, thus the terminal cost J_{T+1} at time $T + 1$ is zero. The optimality equation reads

$$
\begin{aligned}
J_k(v_0, v_1, \cdots, v_{L-1}) = \min_{\substack{u, e_1, \cdots, e_L \\ u \geq e_L \geq 0 \\ v_i \geq e_i \geq 0 \\ i=1, \cdots, L-1}} \{ &\sum_{i=1}^{L} d_{i,k} e_i + L_k(v_0 + \sum_{i=1}^{L} e_i) + c_k u \\
&+ E[J_{k+1}(v_0 + v_1 + \sum_{i=2}^{L} e_i - D, v_2 - e_2, \cdots, v_{L-1} - e_{L-1}, u - e_L)] \},
\end{aligned}
\tag{2.1}
$$

where u is the regular ordering quantity, and e_i is the expediting quantity from installation i. Note that after expediting e_i from installation i, $v_i - e_i$ units remain at installation i and move to installation $i - 1$ in the next time period as shown in Figure 2-2.

For ease of exposition, we consider only stationary demand distributions and cost co-
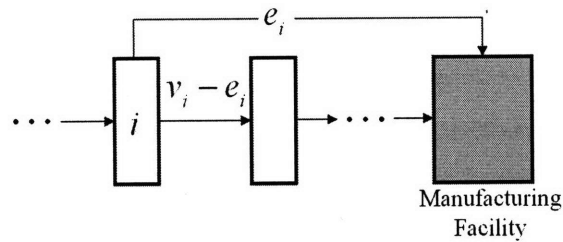
Figure 2-2: Expediting a partial order from installation i

efficients. All presented results hold also in the nonstationary case as discussed in Section 2.5. Therefore we drop time index k from the demand variables and the cost coefficients. We also use $L(\cdot)$ for stationary systems instead of $L_k(\cdot)$.

2.3 Sequential Systems

Optimality equation (2.1) is hard to analyze. To obtain analytical results, we have to confine our interest to a special class of systems. In this section, we explore systems that are analytically manageable and derive structural results for such systems. First, we formally define sequential systems using expediting costs.

Sequential systems A system is *sequential*, if expediting cost coefficients d_i's satisfy $d_i - d_{i-1} \le d_{i+1} - d_i$ for $1 \le i \le L - 1$, where $d_0 = 0$.

For sequential systems, the expediting cost coefficients are increasing convex in installation i. Sequential systems can be found in situations similar to the following explanatory example. Consider a supply chain system with a supplier in Portland, Oregon and a manufacturing facility in Boston, Massachusetts. In between the two locations, there is an installation in St. Louis, Missouri. The review period is one week. The regular delivery lead times between the supplier and the intermediate installation and between the intermediate installation and the manufacturing facility are one week by ground. The expedited shipment by overnight air is available from the supplier with cost d_2 and the intermediate installation with cost d_1. The freight air market between Portland and Boston is much weaker than the high volume market between St. Louis (a logistics hub) and Boston. Therefore, the economies of scale imply that the expediting cost can be much higher in Portland than

22

in St. Louis. As a result we could have $d_2 - d_1 \geq d_1$, or equivalently $d_2 \geq 2d_1$.

The following is a key theorem to derive the optimal policies for regular ordering and expediting.

Theorem 1. *Sequential systems preserve the sequence of orders in time when operated optimally.*

To prove this theorem, we need the following lemma proved in Appendix.

Lemma 1. *For a sequential system, $d_i \geq d_j + d_{i-j}$ for all i and $1 \leq j \leq i-1$.*

Proof of Theorem 1. Expediting has no lead time, thus expediting multiple units can be decomposed to multiple decisions of expediting a unit from a certain installation, until there is no further need of expediting. Consider two nonempty installations i and j, $i > j$, and the two following actions at the current time period.

Action 1: Expediting a unit from installation i

Action 2: Expediting a unit from installation j

We show that there exists a suboptimal strategy that starts with Action 2, costs no more, but replicates the effect of Action 1. Action 1 has an effect of raising the inventory of the manufacturing facility by 1 unit for i time periods compared to no action. Similarly, Action 2 has an effect to raise the inventory for j time periods. Since installation i is nonempty, there is at least a unit, and let us denote it by A. Consider a strategy that starts with Action 2 and expedite unit A after j time periods from the current time period. After j time periods, unit A is in installation $i - j$. Since Action 2 raises the inventory for j time periods and expediting unit A raises the inventory for further $i - j$ time periods, this strategy raises inventory for i time periods, which replicates the effect of Action 1.

Now consider the expediting cost. Action 1 costs d_i while the replicating strategy costs $d_j + d_{i-j}$. For sequential systems, Lemma 1 indicates that Action 1 is more costly or at least of equal cost to the replicating strategy. Therefore the replicating strategy costs no more and is obviously suboptimal. The existence of the suboptimal strategy implies that any strategies that start with Action 1 cannot be optimal. In other words, if expediting is necessary in sequential systems, it is optimal to expedite from the nonempty installation that is closest to the manufacturing facility. Therefore, orders preserve sequence in time under an optimal expediting policy for sequential systems. This completes the proof. □

23

In sequential systems, it is never optimal to expedite from installation i before expediting all the outstanding orders at the downstream installation of installation i. Using this fact, we formulate an alternative optimality equation equivalent to (2.1). Let x^i be the sum of the inventory from installation 0 to installation i: $x^i = \sum_{j=0}^{i} v_j$. Let $\bar{0}^i = (0, 0, \cdots, 0)$ be a vector containing i zeros. For $1 \leq j \leq L$, let $J_k^j(\cdot)$ be the optimal cost-to-go that can be achieved by a restricted control space, in which expediting from installations $j + 1, j + 2, \cdots, L$ in time period k is not allowed. The control space for J_k^j is restricted in time period k, but unrestricted after time period k. Note that $J_k^L(\cdot) = J_k(\cdot)$. We utilize $J_k^j(\cdot)$ with respect to a fictitious state $(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{L-1})$, where installation 0 has inventory x^{i-1}, and installations $1, 2, \cdots, i - 1$ are empty. The optimality equation for $J_k^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{L-1})$ is given by

$$
\begin{aligned}
J_k^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{L-1}) = \min_{x^{i-1} \leq y_i \leq x^i, z \geq x^{L-1}} & \{ d_i(y_i - x^{i-1}) + L(y_i) + c(z - x^{L-1}) \\
& + E[J_{k+1}(y_i - D, \bar{0}^{i-2}, x^i - y_i, v_{i+1}, \cdots, z - x^{L-1})] \},
\end{aligned}
\tag{2.2}
$$

where y_i and z are decision variables: $y_i - x^{i-1}$ is the expediting amount from installation i and $z - x^{L-1}$ is the regular ordering amount.

An alternative formulation of the optimality equation for J_k of sequential systems is given by

$$
\begin{aligned}
J_k(v_0, v_1, v_2, \cdots, v_{L-1}) = \min\{ & J_k^1(x^0, v_1, v_2, \cdots, v_{L-1}), \\
& d_1 v_1 + J_k^2(x^1, 0, v_2, \cdots, v_{L-1}), \\
& d_1 v_1 + d_2 v_2 + J_k^3(x^2, 0, 0, v_3, \cdots, v_{L-1}), \\
& \cdots, \\
& \sum_{i=1}^{L-1} d_i v_i + J_k^L(x^{L-1}, \bar{0}^{L-1}) \}.
\end{aligned}
\tag{2.3}
$$

At time period k, the first term corresponds to expediting partially or fully from installation 1 and no expediting beyond, the second term captures expediting everything from installation 1, expediting partially or fully from installation 2, and no expediting beyond, and so forth. Since the system is sequential, the eventual optimal decisions for regular ordering and expediting are determined by the minimum term in (2.3). For example, if the j-th term achieves the minimum in (2.3), the optimal decision for expediting is to expedite all

outstanding orders in installations $1, 2, \cdots, j-1$ and to expedite $y_j - x^{j-1}$ from installation j and nothing beyond installation j. The optimal regular ordering decision is to place a regular order in the amount $z - x^{L-1}$ that is determined in the j-th term.

2.4 Optimal Policies for Sequential Systems

First we introduce preliminary results needed to derive optimal policies for sequential systems, and then we present main results.

Preliminaries

The following lemma from Lawson and Porteus (2000), which originates in Karush (1959), is used frequently throughout the chapter.

Lemma 2. *Let f be convex and have a finite minimizer on $\mathbb{R}$. Let $y^* = \arg\min f(x)$. Then, $\min\limits_{x_1 \le x \le x_2} f(x) = a + g(x_1) + h(x_2)$, where $a = f(y^*)$, and penalty functions $g(x_1)$ and $h(x_2)$ are*

$$g(x_1) = \begin{cases} 0 & x_1 \le y^* \\ f(x_1) - a & x_1 > y^* \end{cases} \quad and \quad h(x_2) = \begin{cases} f(x_2) - a & x_2 \le y^* \\ 0 & x_2 > y^* \end{cases}.$$

For a nondecreasing convex f, we define $a = 0$, $g(x) = f(x)$, and $h(x) = 0$. On the other hand, for a nonincreasing convex f, we define $a = 0$, $g(x) = 0$, and $h(x) = f(x)$.

In Lemma 2, g is nondecreasing convex, while h is nonincreasing convex. The following functions are required later in the derivation of the optimal policies. For $1 \le i \le L$ and $k \le T$, let us recursively define

$$f_{i,k}(x) = d_i x + L(x) + E[S^1_{i-1,k+1}(x - D)], \tag{2.4}$$

$$S^0_{i,k} = a_{i,k} + S^0_{i-1,k+1},$$

$$S^1_{i,k}(x) = g_{i,k}(x) - d_i x, \tag{2.5}$$

$$S^2_{i,k}(x) = h_{i,k}(x) - L(x) + E[S^2_{i-1,k+1}(x - D)],$$

where $S^0_{0,k} = S^1_{0,k}(\cdot) = S^2_{0,k}(\cdot) = 0$ for all k, and $S^0_{i,T+1} = S^1_{i,T+1}(\cdot) = S^2_{i,T+1}(\cdot) = 0$ for all i. Here, $a_{i,k}$, $g_{i,k}$, and $h_{i,k}$ are defined according to Lemma 2 with respect to $f_{i,k}$. Functions

25

$f_{i,k}$ and $S_{i,k}^j$ are well defined, and starting from the last time period T, they can be obtained recursively. In particular, from (2.4) we can compute $f_{i,T}$, then from (2.5) we obtain $S_{i,T}^1$ for all i. Next we compute $f_{i,T-1}$ from (2.4), and in turn, $S_{i,T-1}^1$ from (2.5) for all i. We repeat this procedure to define all $f_{i,k}$ and $S_{i,k}^1$. For $S_{i,k}^0$ and $S_{i,k}^2$ we use a similar procedure.

We use the following lemma in deriving the optimal policies. The proof is provided in Appendix.

Lemma 3. *a. For sequential systems, $f_{i,k}(\cdot)$ is convex for all k and i.*

b. For all k and i we have $S_{i,k}^0 + S_{i,k}^1(x) + S_{i,k}^2(x) = 0$.

c. Let f_1 be convex and $b \in \mathbb{R}$. We have $\min\limits_{b \leq x \leq y} \{f_1(x) + f_2(y)\} = a_1 + g_1(b) + \min\limits_{b \leq y}\{h_1(y) + f_2(y)\}$, where a_1, h_1, and g_1 are defined as in Lemma 2 with respect to f_1.

Let us denote by $y_{i,k}^*$ a minimizer of $f_{i,k}(x)$: $y_{i,k}^* = \arg\min f_{i,k}(x)$. The following theorem is an important property of $f_{i,k}(\cdot)$ for sequential systems. The proof can be found in Appendix.

Theorem 2. *a. For sequential systems, $y_{i,k}^*$'s are nonincreasing in i for a fixed k. That is, $y_{i,k}^* \geq y_{i+1,k}^*$ for all i and k.*

b. For sequential systems, function $g_{i,k}(x) + S_{i-1,k}^2(x)$ is convex for all i and k.

Optimal Policies

The optimal policies for sequential systems are given by the following theorem.

Theorem 3. *For sequential systems, the following properties hold.*

a. The optimal expediting policy for expediting orders from installation i is the base stock policy with respect to echelon stock x^i. The base stock level is given by $y_{i,k}^* = \arg\min f_{i,k}(x)$ for time period k.

b. The optimal regular ordering policy is the base stock policy with respect to inventory position x^{L-1}. The base stock level is given by $z_k^* = \arg\min\{h_{L,k}(z) + cz + E[S_{L-1,k+1}^2(z - D) + H_{k+1}(z - D)]\}$ for time period k, where $H_k(x)$ follows $H_k(x) = \min\limits_{z \geq x}\{h_{L,k}(z) + cz + E[S_{L-1,k+1}^2(z - D) + H_{k+1}(z - D)]\} - S_{L,k}^2(x) - cx$, and $H_{T+1}(\cdot) = 0$.

c. For all i and k, $J_k(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{L-1}) - J_k(x^i, \bar{0}^i, v_{i+1}, \cdots, v_{L-1}) = S_{i,k}^0 + S_{i,k}^1(x^{i-1}) + S_{i,k}^2(x^i)$.

Part (a) of Theorem 2 indicates that the expediting base stock levels are nonincreasing in i. On the other hand, echelon stock x^i is nondecreasing in i, thus there exists only one i^* such that $y_{i^*,k}^* - x^{i^*-1} > 0$ and $y_{i^*+1,k}^* - x^{i^*} \leq 0$. From part (a) of Theorem 3 we conclude that the optimal expediting policy is to expedite everything from installations $1, \cdots, i^* - 1$, and partially from installation i^*, and nothing beyond. This sequential expediting structure agrees with Theorem 1.

Additionally, the following lemma is used in the proof of Theorem 3. This lemma is proved concurrently with Theorem 3 in an induction step as shown below.

Lemma 4. *For sequential systems,*

a. $H_k(x) = J_k(x, \bar{0}^{L-1})$, *and*

b. $S_{L-1,k}^2(x) + H_k(x)$ *is convex.*

Proof of Theorem 3 and Lemma 4. We prove Theorem 3 and Lemma 4 by induction. In the base case of the induction, when $k = T + 1$, the optimal expediting policy and the optimal regular ordering policy are null. We can safely set the base stock levels for expediting and regular ordering at $-\infty$. Also, part (c) of Theorem 3 and all the properties in Lemma 4 trivially hold when $k = T + 1$ because they are all zero.

Now we continue with the induction step. Let us assume that on and after time $k + 1 \leq T + 1$, the theorem and the three properties hold. Note that we only need to show the results at time period k.

First, we prove part (a) of Lemma 4. Consider $J_k^L(x^{L-1}, \bar{0}^{L-1})$ in (2.3) which is the same as $J_k(x^{L-1}, \bar{0}^{L-1})$. Vector $(x^{L-1}, \bar{0}^{L-1})$ is a state in which we may expedite $y_L - x^{L-1}$ only from the supplier up to the amount of the regular order $z - x^{L-1}$ because there is no outstanding order in any installation. The recursive relationship from (2.2) with $i = L$ is

$$J_k(x^{L-1}, \bar{0}^{L-1})$$
$$= \min_{x^{L-1} \leq y_L \leq z} \{d_L(y_L - x^{L-1}) + L(y_L) + c(z - x^{L-1}) + E[J_{k+1}(y_L - D, \bar{0}^{L-2}, z - y_L)]\}$$
$$= \min_{x^{L-1} \leq y_L \leq z} \{d_L(y_L - x^{L-1}) + L(y_L) + c(z - x^{L-1})$$
$$+ S_{L-1,k+1}^0 + E[S_{L-1,k+1}^1(y_L - D) + S_{L-1,k+1}^2(z - D) + J_{k+1}(z - D, \bar{0}^{L-1})]\},$$

where we use part (c) of Theorem 3 for time period $k+1$. Using the definition of $f_{L,k}$ and part (c) of Lemma 3, we have

$$
\begin{aligned}
J_k(x^{L-1}, \bar{0}^{L-1}) &= \min_{x^{L-1} \leq y_L \leq z} \{f_{L,k}(y_L) + cz + E[S^2_{L-1,k+1}(z-D) + J_{k+1}(z-D, \bar{0}^{L-1})]\} \\
&\quad + S^0_{L-1,k+1} - d_L x^{L-1} - cx^{L-1} \\
&= \min_{x^{L-1} \leq z} \{h_{L,k}(z) + cz + E[S^2_{L-1,k+1}(z-D) + J_{k+1}(z-D, \bar{0}^{L-1})]\} \quad (2.6) \\
&\quad + S^0_{L-1,k+1} - d_L x^{L-1} - cx^{L-1} + g_{L,k}(x^{L-1}) + a_{L,k}.
\end{aligned}
$$

Rearranging the terms and using part (b) of Lemma 3 lead to $J_k(x^{L-1}, \bar{0}^{L-1}) = \min_{x^{L-1} \leq z} \{h_{L,k}(z) + cz + E[S^2_{L-1,k+1}(z-D) + H_{k+1}(z-D)]\} - S^2_{L,k}(x^{L-1}) - cx^{L-1}$, which is the definition of $H_k(x^{L-1})$. Therefore, part (a) of Lemma 4 is proved.

Next, we prove part (b) of Lemma 4. From (2.6) and part (a) of Lemma 4, we have

$$
\begin{aligned}
S^2_{L-1,k}(x^{L-1}) + H_k(x^{L-1}) &= \min_{x^{L-1} \leq z} \{h_{L,k}(z) + cz + E[S^2_{L-1,k+1}(z-D) + H_{k+1}(z-D)]\} \\
&\quad + S^0_{L-1,k+1} - d_L x^{L-1} - cx^{L-1} + g_{L,k}(x^{L-1}) + a_{L,k} + S^2_{L-1,k}(x^{L-1}).
\end{aligned}
$$

Because $g_{L,k}(x^{L-1}) + S^2_{L-1,k}(x^{L-1})$ is convex by part (b) of Theorem 2, and $S^2_{L-1,k+1}(z - D) + H_{k+1}(z - D)$ is convex by the induction hypothesis, we conclude that $S^2_{L-1,k}(x^{L-1}) + H_k(x^{L-1})$ is convex. This shows that part (b) of Lemma 4 holds.

Now we prove part (a) of Theorem 3. Let us consider (2.2). By applying part (c) of Theorem 3 with time period $k+1$ to $J_{k+1}(y_i - D, \bar{0}^{i-2}, x^i - y_i, v_{i+1}, \cdots, z - x^{L-1})$ in (2.2), we obtain $J_{k+1}(y_i - D, \bar{0}^{i-2}, x^i - y_i, v_{i+1}, \cdots, z - x^{L-1}) = S^0_{i-1,k+1} + S^1_{i-1,k+1}(y_i - D) +$

$S_{i-1,k+1}^2(x^i - D) + J_{k+1}(x^i - D, \bar{0}^{i-1}, v_{i+1}, \cdots, z - x^{L-1})$. Applying this repeatedly, we have

$$J_{k+1}(x^i - D, \bar{0}^{i-1}, v_{i+1}, \cdots, z - x^{L-1})$$

$$= S_{i-1,k+1}^0 + S_{i-1,k+1}^1(y_i - D) + S_{i-1,k+1}^2(x^i - D) + \sum_{j=i}^{L-2}\{S_{j,k+1}^0 + S_{j,k+1}^1(x^j - D)$$

$$+ S_{j,k+1}^2(x^{j+1} - D)\} + J_{k+1}(x^{L-1} - D, \bar{0}^{L-2}, z - x^{L-1})$$

$$= S_{i-1,k+1}^0 + S_{i-1,k+1}^1(y_i - D) + S_{i-1,k+1}^2(x^i - D) + \sum_{j=i}^{L-2}\{S_{j,k+1}^0 + S_{j,k+1}^1(x^j - D)$$

$$+ S_{j,k+1}^2(x^{j+1} - D)\} + S_{L-1,k+1}^0 + S_{L-1,k+1}^1(x^{L-1} - D)$$

$$+ S_{L-1,k+1}^2(z - D) + J_{k+1}(z - D, \bar{0}^{L-1}).$$

Substituting this into (2.2) yields

$$J_k^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{L-1}) = \min_{x^{i-1} \leq y_i \leq x^i}\{d_i y_i + L(y_i) + E[S_{i-1,k+1}^1(y_i - D)]\}$$

$$+ \min_{z \geq x^{L-1}}\{cz + E[S_{L-1,k+1}^2(z - D) + J_{k+1}(z - D, \bar{0}^{L-1})]\} - d_i x^{i-1} - cx^{L-1}$$

$$+ S_{i-1,k+1}^0 + E[S_{i-1,k+1}^2(x^i - D)] + E\sum_{j=i}^{L-2}\{S_{j,k+1}^0 + S_{j,k+1}^1(x^j - D) \tag{2.7}$$

$$+ S_{j,k+1}^2(x^{j+1} - D)\} + S_{L-1,k+1}^0 + E[S_{L-1,k+1}^1(x^{L-1} - D)].$$

From (2.7), the optimal expediting amount from installation i at time k is determined from

$$\min_{x^{i-1} \leq y_i \leq x^i}\{d_i y_i + L(y_i) + E[S_{i-1,k+1}^1(y_i - D)]\} = \min_{x^{i-1} \leq y_i \leq x^i} f_{i,k}(y_i).$$

By part (a) of Lemma 3, $f_{i,k}(y_i)$ is a convex function. Therefore, the optimal expediting policy from installation i at time k is the base stock policy with the base stock level $y_{i,k}^* = \arg\min f_{i,k}(y_i)$. Note that we can only expedite up to what we have in installation i. This completes the proof of part (a) of Theorem 3.

Next, we proceed to prove part (b) of Theorem 3. We consider the optimal regular ordering policy. If the last term in (2.3) attains the minimum, then it is determined by (2.6), or equivalently

$$\min_{z \geq x^{L-1}}\{h_{L,k}(z) + cz + E[S_{L-1,k+1}^2(z - D) + J_{k+1}(z - D, \bar{0}^{L-1})]\}, \tag{2.8}$$

or otherwise from (2.7) it is determined by

$$\min_{z \geq x^{L-1}} \{cz + E[S_{L-1,k+1}^2(z-D) + J_{k+1}(z-D, \bar{0}^{L-1})]\}. \tag{2.9}$$

Note that $h_{L,k}(z)$ is nonincreasing convex and $h_{L,k}(z) = 0$ for $z \geq y_{L,k}^*$. Therefore, if $z_k^* \geq y_{L,k}^*$, then (2.8) and (2.9) lead to the same minimizer z_k^*. If $z_k^* < y_{L,k}^*$, from part (a) of Theorem 2, we have $z_k^* < y_{i,k}^*$ for all i, which results in expediting everything in the supply chain including the fresh regular order in the current time period. In this case, (2.8) determines the regular ordering quantity because we are now expediting from the supplier. As a result, (2.8) determines the optimal regular ordering in any case.

Since $S_{L-1,k+1}^2(z) + H_{k+1}(z)$ is convex by part (b) of Lemma 4, and $J_{k+1}(z, \bar{0}^{L-1}) = H_{k+1}(z)$ by part (a) of Lemma 4, $h_{L,k}(z) + cz + E[S_{L-1,k+1}^2(z-D) + J_{k+1}(z-D, \bar{0}^{L-1})]$ is convex. Therefore, (2.8) indicates that the optimal regular ordering policy is the base stock policy with the base stock level z_k^* with respect to the inventory position x^{L-1}. Furthermore, H_k is well defined since $h_{L,k}(z) + cz + E[S_{L-1,k+1}^2(z-D) + H_{k+1}(z-D)]$ is convex for all k. The proof of part (b) of Theorem 3 is thus completed.

It remains to show part (c) of Theorem 3 in time period k. Since we know optimal policies in time period k in an induction step, we use the optimal policies in proving this part. We compare $J_k(x^i, \bar{0}^i, v_{i+1}, \cdots, v_{L-1})$ and $J_k(x^{i+1}, \bar{0}^{i+1}, v_{i+2}, \cdots, v_{L-1})$. If $y_{i+1,k}^* \leq x^i$, then no expediting is necessary from installation $i+1$ and beyond, therefore

$$J_k(x^i, \bar{0}^i, v_{i+1}, v_{i+2}, \cdots, v_{L-1})$$

$$= L(x^i) + \min_{z \geq x^{L-1}} \{c(z - x^{L-1}) + E[J_{k+1}(x^i - D, \bar{0}^{i-1}, v_{i+1}, \cdots, v_{L-1}, z - x^{L-1})]\}$$

$$= L(x^i) + \min_{z \geq x^{L-1}} \{c(z - x^{L-1}) + E[S_{i,k+1}^0 + S_{i,k+1}^1(x^i - D) + S_{i,k+1}^2(x^{i+1} - D)]$$

$$+ E[J_{k+1}(x^{i+1} - D, \bar{0}^i, v_{i+2}, \cdots, v_{L-1}, z - x^{L-1})]\},$$

where we used part (c) of Theorem 3. Since $y_{i+1,k}^* \leq x^i \leq x^{i+1}$, no expediting is necessary, thus we have

$$J_k(x^{i+1}, \bar{0}^{i+1}, v_{i+2}, \cdots, v_{L-1})$$

$$= L(x^{i+1}) + \min_{z \geq x^{L-1}} \{c(z - x^{L-1}) + E[J_{k+1}(x^{i+1} - D, \bar{0}^i, v_{i+2}, \cdots, v_{L-1}, z - x^{L-1})]\}.$$

30

Therefore, $J_k(x^i, \bar{0}^i, v_{i+1}, v_{i+2}, \cdots, v_{L-1}) - J_k(x^{i+1}, \bar{0}^{i+1}, v_{i+2}, \cdots, v_{L-1}) = L(x^i) - L(x^{i+1}) + E[S_{i,k+1}^0 + S_{i,k+1}^1(x^i - D) + S_{i,k+1}^2(x^{i+1} - D)]$.

Next, if $x^i < y_{i+1,k}^* \leq x^{i+1}$, then expediting from installation $i+1$ is necessary, but not from upstream installations. We have

$$J_k(x^i, \bar{0}^i, v_{i+1}, v_{i+2}, \cdots, v_{L-1}) = d_{i+1}(y_{i+1,k}^* - x^i) + L(y_{i+1,k}^*) + \min_{z \geq x^{L-1}}\{c(z - x^{L-1})$$

$$+ E[J_{k+1}(y_{i+1,k}^* - D, \bar{0}^{i-1}, x^{i+1} - y_{i+1,k}^*, v_{i+2}, \cdots, z - x^{L-1})]\}$$

$$= d_{i+1}(y_{i+1,k}^* - x^i) + L(y_{i+1,k}^*) + \min_{z \geq x^{L-1}}\{c(z - x^{L-1}) + E[S_{i,k+1}^0 + S_{i,k+1}^1(y_{i+1,k}^* - D)$$

$$+ S_{i,k+1}^2(x^{i+1} - D)] + E[J_{k+1}(x^{i+1} - D, \bar{0}^i, v_{i+2}, \cdots, v_{L-1}, z - x^{L-1})]\}, \text{ and}$$

$$J_k(x^{i+1}, \bar{0}^{i+1}, v_{i+2}, \cdots, v_{L-1}) = L(x^{i+1}) + \min_{z \geq x^{L-1}}\{c(z - x^{L-1})$$

$$+ E[J_{k+1}(x^{i+1} - D, \bar{0}^i, v_{i+2}, \cdots, v_{L-1}, z - x^{L-1})]\}.$$

Therefore, $J_k(x^i, \bar{0}^i, v_{i+1}, v_{i+2}, \cdots, v_{L-1}) - J_k(x^{i+1}, \bar{0}^{i+1}, v_{i+2}, \cdots, v_{L-1}) = d_{i+1}y_{i+1,k}^* + L(y_{i+1,k}^*) - d_{i+1}x^i - L(x^{i+1}) + E[S_{i,k+1}^0 + S_{i,k+1}^1(y_{i+1,k}^* - D) + S_{i,k+1}^2(x^{i+1} - D)]$.

Finally, if $y_{i+1,k}^* > x^{i+1}$, then we expedite everything in installation $i+1$. Thus the only cost difference is $d_{i+1}v_{i+1} = d_{i+1}x^{i+1} - d_{i+1}x^i$, and we obtain $J_k(x^i, \bar{0}^i, v_{i+1}, v_{i+2}, \cdots, v_{L-1}) - J_k(x^{i+1}, \bar{0}^{i+1}, v_{i+2}, \cdots, v_{L-1}) = d_{i+1}x^{i+1} - d_{i+1}x^i$.

The three cases above can be summarized as

$$J_k(x^i, \bar{0}^i, v_{i+1}, v_{i+2}, \cdots, v_{L-1}) - J_k(x^{i+1}, \bar{0}^{i+1}, v_{i+2}, \cdots, v_{L-1})$$

$$= a_{i+1,k} + g_{i+1,k}(x^i) + h_{i+1,k}(x^{i+1}) - d_{i+1}x^i - L(x^{i+1}) + S_{i,k+1}^0 + E[S_{i,k+1}^2(x^{i+1} - D)]$$

$$= S_{i+1,k}^0 + S_{i+1,k}^1(x^i) + S_{i+1,k}^2(x^{i+1}).$$

Therefore, part (c) of Lemma 4 at time period k is proved. This completes the induction step of the entire proof. $\qquad \square$

A Numerical Example of the Policy

Consider a supply chain system with 6 installations including the supplier and the manufacturing facility. Figure 2-3 illustrates the mechanism behind the base stock policy for regular ordering and the base stock policies for expediting. Note that the echelon stock x^i is nondecreasing in i, and $y_{i,k}^*$ is nonincreasing in i by part (a) of Theorem 2. Therefore,

there can be at most one intersection between these two curves. At the beginning of time
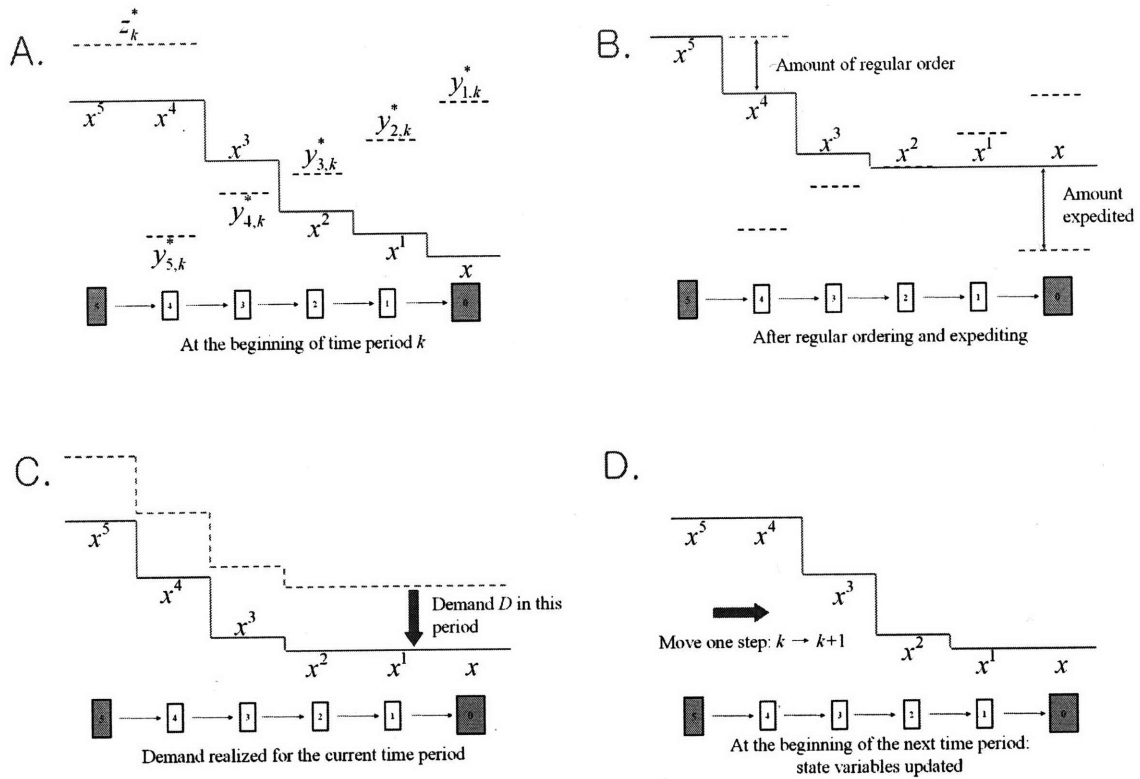

Figure 2-3: An illustration of the optimal policies

period k, we compare echelon stocks with base stock levels (Figure 2-3, A). The optimal regular ordering quantity is $z_k^* - x^4$, and optimal expediting is to expedite everything from installations 1 and 2, and $y_{3,k}^* - x^2$ from installation 3 (Figure 2-3, B). After the expedited quantities arrive, demand D realizes, and pushes the echelon stock levels down by D at the end of the period (Figure 2-3, C). At the beginning of the next period, the echelon stock levels move forward by one step, and the next period begins (Figure 2-3, D). Table 2.1 illustrates the policy by means of a numerical example. The realized demand at time k is 40, and $x^5 = x^4 + max(z^* - x^4, 0)$.

A Numerical Example of Base Stock Levels with Nonstationary Demand Distribution

Consider a three-installation sequential supply chain with stationary cost parameters facing a nonstationary stochastic demand. The three-installation system has all of the typical

32

Table 2.1: A numerical example

	z^*	y_5^*	y_4^*	y_3^*	y_2^*	y_1^*
base stock levels at time period k	100	40	52	65	72	85
	x^5	x^4	x^3	x^2	x^1	x^0
time period k, before decisions		85	72	61	55	48
time period k, after decisions	100	85	72	65	65	65
time period k, after realized demand	60	45	32	25	25	25
time period $k+1$, before decisions		60	45	32	25	25

features of general length systems. Figure 2-4 shows the corresponding base stock levels from Theorem 3.
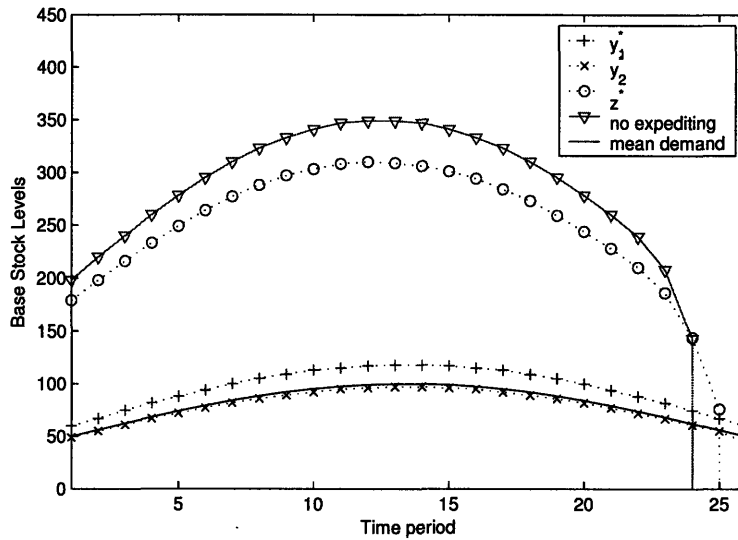

Figure 2-4: The Base Stock Levels with Nonstationary Demand

In Figure 2-4, the solid line is the mean of the nonstationary demand distribution at each time period, and the line with triangles corresponds to the base stock levels without the expediting option. Also, the line with circles corresponds to the regular ordering base stock levels with the expediting options, the line with pluses corresponds to the base stock levels for expediting from stage 1, and the line with crosses corresponds to the expediting base stock levels for expediting from stage 2. The planning horizon is 26 periods.

We observe several interesting points. As it approaches the last time period, specifically in time periods 25 and 26, the regular ordering base stock levels without the expediting options become large negative numbers, which makes sense since new orders would never

33

arrive at the destination (the lead time L is 2). However, with the expediting options, the regular ordering base stock level at time period 25 is not a large negative number (indeed, it is positive in this example). This also makes sense since we may expedite orders placed in time period 25.

Other than the time periods close to the end of the planning horizon, the regular ordering base stock levels with expediting options are smaller than those without the expediting options. This is due to the increased agility of the supply chain resulting from the expediting options, hence decreased need for safety stock in the pipeline. Furthermore, the expediting options effectively reduce lead times. Therefore, as the mean of the demand increases, the increment of the regular ordering base stock levels with expediting options is not as pronounced as that of the base stock levels without the expediting options. It implies that the decreased realized lead times with the expediting options reduce the variability in the regular ordering amount. As for expediting base stock levels, they follow well the mean demand curve.

2.5 Additional Results

We first generalize our results to the case of nonzero per unit holding or processing cost at intermediate installations. If a linear holding or processing cost is incurred at each intermediate installation, we apply the following transformation. Let the linear holding or processing cost be $h_i \geq 0$ at installation i, and let the actual procurement cost be c'. Let also the actual expediting cost be d_i' for expediting a unit from installation i.

Step 1 Let us define $c = c' + h_1 + h_2 + \cdots + h_{L-1}$. Then c can be used as the hypothetical per unit procurement cost in our model. It means that we pay all the holding costs in advance when we place an order.

Step 2 We can use the hypothetical expediting cost $d_i = d_i' - h_i - h_{i-1} - \cdots - h_1$. If $d_i < 0$, then it is always better to expedite from installation i to the manufacturing facility than to pay more expensive holding or processing costs at installations $i, \cdots, 1$. In this case, we never use installations $i, ..., 1$, which leads to shorter lead time.

This transformation is possible because a unit stays exactly one period at each installation if it is not expedited. This is the main difference from the multi-echelon model of Clark

34

and Scarf (1960). In other words, this transformation works only in our setting.

We now derive additional insight in the stationary case, i.e., the demand distribution and all the cost coefficients are stationary. We provide the proof of the following lemma in Appendix.

Lemma 5. *If the demand distribution and cost coefficients are stationary, then for $1 \leq i \leq L$ and $k \leq T - i + 1$, we have $y_{i,k}^* = y_{i,1}^*$.*

Lemma 5 states that the expediting levels are independent of k for $k \leq T - L$. Note that in practice T is much larger than L. Therefore, for a stationary system the base stock levels become constant in time for most of the time periods except a few periods at the end of the planning horizon. This leads to a simple set of optimal parameters.

Another interesting observation can be made in a stationary system. Let z^* and y_i^* be the base stock levels before time $T - L$. If $z^* < y_i^*$, then we never use installations 1 to i at least until time $T - L$, because all units are always expedited on and before arriving at installation i. As a special case, if $z^* < y_L^*$, then we always expedite the entire regular order directly from the supplier, and never use any of the intermediate installations at least until time $T - L$.

Our last remark is about nonstationary systems. If the system parameters are nonstationary, the appropriate definition of sequential systems is the following.

Nonstationary Sequential Systems A nonstationary system is *sequential*, if $d_{i,k} - d_{i-1,k+1} \leq d_{i+1,k} - d_{i,k+1}$, for $1 \leq i \leq L - 1$ and $1 \leq k \leq T$, where $d_{0,k} = 0$ and $d_{i,T+1} = 0$.

In a nonstationary setting, all theorems hold with only minor modifications in the proofs due to the added time indices.

Chapter 3

Stochastic Lead Time Model

3.1 Introduction

Radio Frequency Identification (RFID) is a wireless sensing technology that consists of tags called also transponders, which are tiny computer chips with limited memory, and readers or interrogators. It is frequently referred to as the next generation bar-code. When a tagged item comes in the read range of a reader, the reader reads the data on the tag (e.g., location, time, a unique identifier, etc.) and passes these information to an information system. In supply chains RFID substantially increases inventory visibility and has potentials to improve overall efficiency of supply chains. Among many benefits, labor savings, improved forecasts, and reduced stock-outs are often cited as direct benefits of RFID. The value of RFID may also come indirectly in combination with new business practices that are impossible without RFID. One such practice is expediting outstanding orders in a supply chain in presence of stochastic lead time. In order to substantiate this indirect value opportunity with expediting in an RFID-enabled supply chain, we perform an analytical analysis.

We consider a periodic review, single item inventory problem with a single supplier and a manufacturer where the manufacturer periodically places regular orders at the supplier. The stochastic demand is fulfilled by the manufacturer and excessive demand is backlogged. The supplier's chain consists of multiple installations, and orders progress from one installation to another until delivered to the manufacturer. The movements of outstanding orders among installations are stochastic, hence the overall lead time is stochastic. More specifically, multiple movement patterns of outstanding orders are captured in the model, and one of the patterns is chosen stochastically at each time period. We assume that there exists an

exogenous random variable with a known distribution that chooses the movement pattern that occurs at the current time period.

On top of this, we consider an option to expedite orders from installations to the manufacturer for extra per unit cost according to the current demand situation. Since the lead time is stochastic, under traditional techniques and processes the exact locations of outstanding orders are expensive for the manufacturer to obtain; thus it can be costly to expedite outstanding orders. Under an RFID deployment, with tags attached on units of goods (e.g., pallets or cases) and readers installed at each installation, the real-time location information of outstanding orders is now easily available to the manufacturer. While this is possible with other techniques such as GPS, RFID does not pose a significant capital investment. Tags are currently around 10 cents and reader costs range in few thousand U.S. dollars. Therefore, expediting orders from installations under RFID is now a feasible business proposition.

In order to asses the value of RFID, it is important to develop models capable of exploiting data resulting from RFID, and to find out optimal policies of expediting and regular ordering in such models. In the absence of optimal policies, it is hard to guarantee additional value of RFID to the supply chain. As a result, we focus on deriving the optimal expediting and regular ordering policies under RFID. Since the setting of our model is quite general and the modeling scope is large, finding the optimal policies in general is difficult. They generally depend on state variables, hence they are nonintuitive and complex. However, analytical results can be obtained for a certain subset of serial systems. We characterize conditions for a system to allow simple optimal policies, and call such systems *sequential* since orders do not cross in time under the optimal control. The sequential delivery property plays a key role in analyzing the optimal policies. We note that the concept of sequential systems in the current chapter is more general than the one in Chapter 2. The key difference of the model from Chapter 2 is the stochastic lead time of regular orders, and the concept of sequential systems here accommodates this difference. We also provide sufficient and necessary conditions to facilitate the identification of sequential systems. Within the sequential systems, the optimal regular ordering and expediting policies are derived. The optimal regular ordering policy is the base stock policy with respect to the inventory position, and the optimal expediting policy is a variant of the base stock policy with respect to the echelon stock up to a certain installation. In addition, we find that as the expediting

37

cost of a certain installation increases, the underlying expediting base stock level associated with the installation is nonincreasing, which is intuitive. Interestingly enough, we also derive that as the expediting cost for an installation increases, the expediting base stock levels for installations beyond the installation in question are nondecreasing.

The contributions of this chapter are several. First, to the best of our knowledge, the presented work is the first one to derive an optimal expediting policy of a stochastic lead time model, which is a significant advancement over deterministic ones. Second, the proof technique is novel and nontraditional even though we rely on induction. After characterizing the sequential systems, we formulate the optimality equation suited for these systems using the sequential delivery property, and this leads to simple optimal policies. Optimality of these policies is proved in an induction loop by studying the difference in the cost-to-go for different states. Third, we find interesting directional dependencies of expediting base stock levels on expediting costs. Finally, an important managerial insight – that the value of RFID can be elevated, if utilized actively with innovative processes such as expediting – can be inferred from this work. Firms should look for creative business processes in order to extract more value from RFID.

In Section 3.2, we formally state the underlying model. We delineate the class of systems in which orders do not cross in time in Section 3.3, and discuss the scope of such sequential systems in the same section. We derive the corresponding optimal policies for the sequential systems in Section 3.4. In Section 3.5, we discuss additional results on the optimal policies.

Literature review for stochastic lead time model

The most related models in the literature are divided in two groups: the stochastic lead time models and the multi supply mode models. Among the early work on the stochastic lead time models, Kaplan (1970a), Nahmias (1979), and Ehrhardt (1984) consider stochastic lead time that is determined by a realization of a random variable. In particular, if the age of an order exceeds the realized value of the random variable, then the order arrives at the destination. Song and Zipkin (1996) and Muharremoglu and Tsitsiklis (2003b) are more recent publication on stochastic lead time models. In their models, the supply system is Markov modulated to describe the supply condition. They also define an exogenous random variable, which determines the lead time of an order, but their modeling of the stochastic lead time is more comprehensive than the earlier works since the random variables determine

38

the progress status of outstanding orders. Our model resembles the stochastic lead time description of Song and Zipkin (1996) and Muharremoglu and Tsitsiklis (2003b), however, they do not consider expediting.

The multi supply mode models such as emergency ordering or expediting models with deterministic movement transitions include Barankin (1961), Neuts (1964), Daniel (1963), Fukuda (1964), and Veinott (1966) as the early works. They consider inventory systems with two supply modes of instantaneous and one period lead time. Models with emergency orders among others include Chiang and Gutierrez (1998) and Huggins and Olsen (2003a), but their modeling of emergency orders is different from ours (emergency and expediting have different scopes). More related recent works are Lawson and Porteus (2000) and Muharremoglu and Tsitsiklis (2003a). Lawson and Porteus (2000) extend the multi-echelon model by Clark and Scarf (1960) by allowing expediting between consecutive installations, and their optimal policy is a base stock type policy. Muharremoglu and Tsitsiklis (2003a) generalize Lawson and Porteus (2000) by allowing super modular expediting cost instead of a linear one.

Both Lawson and Porteus (2000) and Muharremoglu and Tsitsiklis (2003a) allow expediting between arbitrary two installations. However, our model does not allow this since in our case orders can be expedited only to the manufacturer. This corresponds with the situation where the manufacturer and the supplier are independent companies, and thus it is prohibitive for the manufacturer to manipulate inventories inside the supplier's chain. The manufacturer may only expedite orders to its own facility based on the inventory information from RFID at each installation. It is important to note that it is nontrivial to prevent expediting between intermediate installations using the models of Lawson and Porteus (2000) and Muharremoglu and Tsitsiklis (2003a). Therefore, our model simply addresses a different situation from their models. Furthermore, the stochastic lead time modeling considered here is a fundamental leap from the deterministic cases in their models.

Gaukler et al. (2005) consider emergency ordering under RFID in a supply chain with multiple stages, where the lead time is stochastic. RFID is used in a similar context as ours, i.e., to gain real-time location information. However, their model is simpler than ours since they allow at most one outstanding order at any point in time, which significantly limits the modeling power. Furthermore, rather than dealing with optimal policies, they confine their study to base stock policies. Therefore, the optimality is not guaranteed, and the nature

of their work is distinct from ours. For further literature review on RFID related inventory models, we refer the reader to Lee and Özer (2007).

3.2 Model Statement

We consider a single supplier with a single-item manufacturing facility facing random demand with known distribution, and $\bar{K} - 1$ serial intermediate installations between them. The supplier is denoted as installation $\bar{K}$ [1] and the manufacturing facility is installation 0. The intermediate installations are numbered from 1 (next to the manufacturing facility) to $\bar{K} - 1$ (next to the supplier). The manufacturer periodically reviews the inventory on hand and places a regular order at the supplier by paying per unit procurement cost c. Unsatisfied demand is backlogged and excessive inventory at the manufacturing facility is penalized. The planning horizon consists of T time periods. For simplicity, we assume that the system is stationary.

A movement pattern w describes the destination installation of outstanding orders for each installation in the next time period. We define multiple movement patterns. For example, consider a supply chain with $\bar{K} = 5$, which has three illustrative movement patterns: slow, normal, and fast. In the normal pattern, orders at installation i move to installation $i - 1$ for $i = 1, \cdots, 5$. In the slow pattern, orders at installations 1, 3, and 5 fail to progress, thus orders at these installations stay at the current location one more time period while orders at the remaining installations move to the next downstream installation. In the fast mode, orders in installations 2 and 3 move to installations 0 and 1 respectively while orders in the other installations move to the next downstream installation. Let us denote by $\mathbf{W}$ the set of all movement patterns, i.e., $\mathbf{W} = \{w_1, w_2, w_3, \cdots\}$. There is an exogenous random variable W with known distribution that selects a movement pattern in $\mathbf{W}$. At each time period, W realizes, and according to the realized movement pattern w, the outstanding orders at installation $i, 1 \le i \le \bar{K}$, move to installation $j = M(i, w)$, $0 \le j \le i$, where $M(\cdot)$ is a function that takes the origin installation i and the realized movement pattern w as arguments. Note that orders are not allowed to go backward to the upstream installations in this definition. We define $M(0, w) = 0$, and before W is realized we denote the corre-

---

[1]In Chapter 2, we denote by L the index of the last installation, which is again the lead time of regular orders. In this chapter, we use $\bar{K}$ instead to emphasize the stochastic lead time of regular orders, and reserve l for a random lead time function.

40

sponding random variable by $M(i, W)$. The lead time of a regular order is stochastic and determined by multiple realized movement patterns until delivery. The departure process for outstanding orders in installation i to the downstream is geometrically distributed with parameter $Prob(M(i, W) < i)$, which is the departure probability.

Let v_i be the amount of inventory at installation i for $0 \leq i \leq \bar{K}$ and $(v_0, v_1, v_2, \cdots, v_{\bar{K}})$ the state vector. Without an RFID deployment at all installations, it is extremely hard to observe the state of the system. RFID is definitely a technology that enables better visibility at a much lower cost. Based on the current state of the system, the manufacturer expedites outstanding orders if need be by paying per unit delivery cost d_i for expediting orders from installation i.

The sequence of events in a time period is as follows. At the beginning of the time period, the state information is given. Then the manufacturer places a regular order with the supplier (installation $\bar{K}$). Next, the manufacturer makes decisions on expediting for each installation, and the expedited orders arrive at the manufacturing facility instantaneously. After that, demand D realizes for the current time period. Inventory holding or backlogging cost is accounted for at the manufacturing facility after demand realization. Finally, W realizes and regular delivery occurs just before the end of the time period. Then the next time period begins.

We need the following assumption stating that regular orders should not cross in time. Except for certain situations in which a time period is short and the variability of the lead time is high, this assumption is probably not a severe restriction. This assumption is standard in the stochastic lead time literature that includes Kaplan (1970b), Nahmias (1979), and Muharremoglu and Tsitsiklis (2003a), among many.

Assumption 1 (Orders not crossing in time). $M(i, w) \geq M(i - 1, w)$ *for all i and* $w \in \mathbf{W}$.

Let us define a related movement function $N(j, w) = \max\{i : M(i, w) \leq j, 0 \leq i \leq \bar{K}\}$ for all j and $w \in \mathbf{W}$, and let $N(j, W)$ be the corresponding random variable before W is realized. Under Assumption 1, a one-to-one mapping between M and N exists as the following example illustrates.

Example Consider an 8 installation system including the supplier and the manufacturer ($\bar{K} = 7$). At time t, assume that realized w of W drives the following movement.

41

i	0	1	2	3	4	5	6	7
$M(i,w)$	0	0	0	1	1	1	4	5

An equivalent information of the above movement can be expressed by $N(j,w)$ as follows (See Figure 3-1).

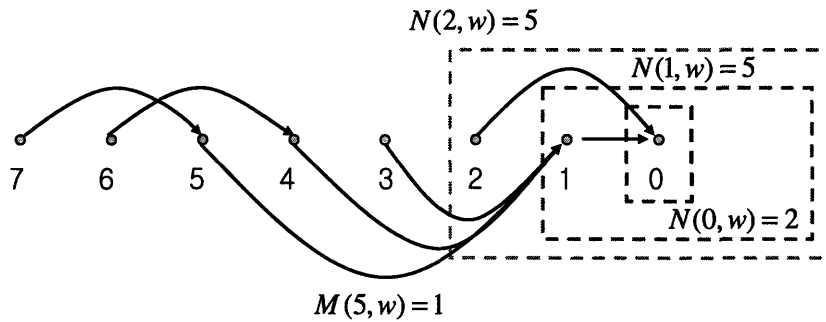| j	0	1	2	3	4	5	6	7
$N(j,w)$	2	5	5	5	6	7	7	7

Figure 3-1: A regular movement driven by a realized w of W

Given installation j, we find $N(j,w)$ by observing the farthest installation whose movement leads to installation j or any downstream installation of j. □

Let us denote by $M^n(i,W)$ the n-period random movement function that represents the location (an installation) after n regular movements of the outstanding orders at installation i. Formally, $M^1(i,W) = M(i,W)$ and $M^n(i,W) = M(M^{n-1}(i,W),W)$. We denote the stochastic lead time of an order at installation i by $l(i,W) = \min\{n : M^n(i,W) = 0, n \geq 1\}$. In particular, $l(\bar{K},W)$ is the regular delivery lead time. Note that the minimum regular delivery lead time is 1 in our model. For convenience, we define $L(x) = E[r(x-D)]$, where $r(\cdot)$ is a convex holding/backlogging cost function, and let $Q^i(W)$ denote $N(M(\bar{K},W) - i,W)$. Let the echelon stock x^i be the sum of the inventory from installation 0 to installation i: $x^i = \sum_{j=0}^{i} v_j$, and let $\bar{0}^i = (0,0,\cdots,0)$ be a vector containing i zeros.

If there is no expediting, the state after a regular movement is a random vector $(x^{N(0,W)} - D, x^{N(1,W)} - x^{N(0,W)}, \cdots, x^{Q^1(W)} - x^{Q^2(W)}, x^{Q^0(W)} - x^{Q^1(W)} + u, \bar{0}^{\bar{K}-M(\bar{K},W)})$, where u is the regular ordering amount. Let e_i denote the expediting amount from installation i.

Including expediting, the next state NS is

$$NS = (x^{N(0,W)} + \sum_{i=N(0,W)+1}^{\bar{K}} e_i - D, x^{N(1,W)} - x^{N(0,W)} - \sum_{i=N(0,W)+1}^{N(1,W)} e_i, \cdots,$$

$$x^{N(i,W)} - x^{N(i-1,W)} - \sum_{i=N(i-1,W)+1}^{N(i,W)} e_i, \cdots,$$

$$x^{Q^1(W)} - x^{Q^2(W)} - \sum_{i=Q^2(W)+1}^{Q^1(W)} e_i, x^{Q^0(W)} - x^{Q^1(W)} + u - \sum_{i=Q^1(W)+1}^{Q^0(W)} e_i, 0^{\bar{K}-M(\bar{K},W)}).$$

Figure 3-2 illustrates the inventory at installation i after a regular movement $x^{N(i,W)} - x^{N(i-1,W)} - \sum_{i=N(i-1,W)+1}^{N(i,W)} e_i$. The complete optimality equation of the dynamic program reads

$$J_t(v_0, v_1, \cdots, v_{\bar{K}}) = \min_{\substack{u, e_1, \cdots, e_{\bar{K}} \\ 0 \leq e_{\bar{K}} \leq u + v_{\bar{K}} \\ 0 \leq e_i \leq v_i \\ i=1,\cdots,\bar{K}-1}} \{\sum_{i=1}^{\bar{K}} d_i e_i + L(x + \sum_{i=1}^{\bar{K}} e_i) + cu + E[J_{t+1}(NS)]\}, \tag{3.1}$$

where J_t is the cost-to-go at the beginning of time period t. Also, $J_{T+1}(v_0, v_1, \cdots, v_{\bar{K}})$ can be any convex function of $x^{\bar{K}}$. Solving this optimality equation directly is difficult because of its complexity. In order to analyze (3.1), we need to introduce further assumptions.
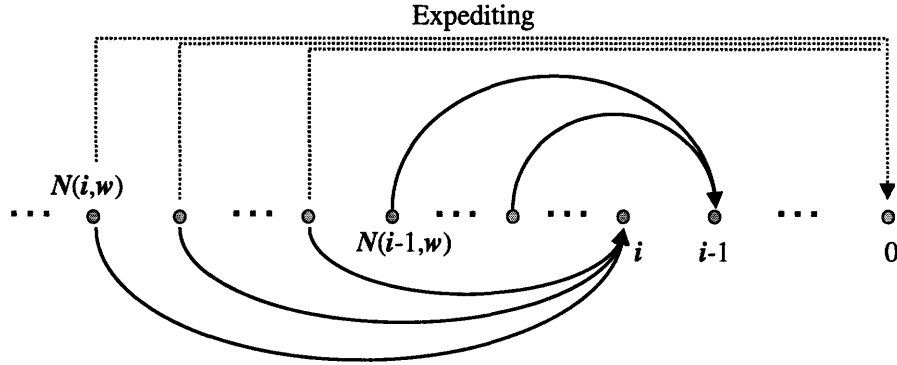

Figure 3-2: The next state transition

In the next section, we characterize a class of systems for which (3.1) has an alternative form that leads to tractable policies.

3.3 Sequential Systems

The following realistic assumption requires that orders almost surely reach installation 0.

Assumption 2 (Eventual delivery of regular orders). $Prob[\cup_{n=1}^{\infty}\{w : M^n(i,w) = 0\}] = 1$ *for every installation i.*

In terms of the finite state Markov Chain theory, Assumption 2 requires that installation 0 is the only *recurrent installation*, and all the other installations are *transient installations*. In order to analyze the system, we need the following assumption.

Assumption 3 (Nondecreasing time value of delayed expediting). $d_i - E[d_{M(i,W)}] \geq d_{i-1} - E[d_{M(i-1,W)}]$ *for all i, where $d_0 = 0$.*

Consider a unit at installation i. If we expedite it at the current time period, it costs d_i. If we defer expediting by a time period, the expected cost of expediting is $E[d_{M(i,W)}]$. Therefore, $d_i - E[d_{M(i,W)}]$ is *the time value of delayed expediting* of a unit at installation i by a time period. Assumption 3 implies that this time value of expediting does not decrease as installation number i increases. Next, we define a class of systems, in which all three assumptions hold.

Sequential systems A system is *sequential* if Assumptions 1, 2, and 3 hold.

If the lead time is deterministic, then the definition above reduces to the definition of the sequential systems in Chapter 2. The following theorem shows a crucial property of sequential systems.

Theorem 4. *Under the optimal control of regular ordering and expediting, sequential systems preserve the sequence of orders in time, i.e., the no cross-over property holds.*

Assumption 1 guarantees that regular orders with no expediting do not cross in time. When expediting is introduced, in general, orders might easily cross even under Assumption 1. Theorem 4 states that this is not the case for sequential systems. To prove this, we require the following lemma whose proof is in Appendix.

Lemma 6. *In sequential systems, $d_i - d_j \geq E[d_{M^n(i,W)} - d_{M^n(j,W)}]$, for any i and j, $i \geq j$, and $n \geq 1$.*

Proof of Theorem 4. Since expediting is instantaneous, expediting multiple units at a time period consists of multiple decisions of expediting a unit from a certain installation, until there is no further need of expediting. Consider two nonempty installations i and j, $i > j$, and let us denote a unit in installation i as u_1 and a unit in installation j as u_2. Now, consider the following two actions.

Action 1: Expedite u_1 in the current time period.

Action 2: Expedite u_2 in the current time period

Consider also the following *replicating strategy*.

1. Set a new index k to be j at the current time period.

2. If the realized value of $M(k, W)$ is 0, expedite u_1 in the subsequent time period and terminate the strategy.

3. Otherwise, update k with the realized value of $M(k, W)$, i.e., $k \leftarrow M(k, W)$, and proceed to the next time period. Go to step 2.

We show that the replicating strategy in combination with Action 2 costs no more, but replicates the effect of Action 1. Let w be a realized movement pattern of W. Action 1 has the effect of raising the inventory at the manufacturing facility by a unit for $l(i, w)$ time periods compared to no action. Similarly, Action 2 raises inventory for $l(j, w)$ time periods. The described strategy is to expedite u_1 at $l(j, w)$ time periods later than the current time period. Thus the strategy raises the inventory for $l(M^{l(j,w)}(i, w), w)$ time periods, since the location of u_1 is installation $M^{l(j,w)}(i, w)$ at the moment of its expediting. From the definition of the lead time, and due to the fact that after $l(j, w)$ time periods there are $l(M^{l(j,w)}(i, w), w)$ time periods for u_1 to arrive, we have $l(i, w) = l(j, w) + l(M^{l(j,w)}(i, w), w)$, for $i \geq j$. Therefore Action 2 with the described strategy replicates the effect of Action 1.

However, for sequential systems the associated cost is different between Action 1 and the combination of Action 2 and the replicating strategy. The cost of Action 1 is d_i, while the expected cost of Action 2 with the replicating strategy is $d_j + E[d_{M^{l(j,w)}(i,W)}]$. Since Lemma 6 holds for any $n \geq 0$, the following

$$d_i - d_j \geq E[d_{M^{l(j,w)}(i,W)} - d_{M^{l(j,w)}(j,W)}] = E[d_{M^{l(j,w)}(i,W)} - d_0] = E[d_{M^{l(j,w)}(i,W)}]$$

45

holds. Therefore, Action 1 costs more than or equal to the combination of Action 2 and the replicating strategy. It implies that any strategies that start with Action 1 cannot be optimal. In other words, if expediting is necessary in sequential systems, it is optimal to expedite from the nonempty installation that is closest to the manufacturing facility. Thus, for sequential systems orders preserve sequence in time under an optimal expediting policy, which completes the proof. $\qquad\square$

For $1 \leq j \leq \bar{K}$, let $J_t^j(\cdot)$ be the optimal cost-to-go that can be achieved by a restricted control space, in which expediting from installations $j+1, j+2, \cdots, \bar{K}$ in time period t is not allowed. Note that the control space for J_t^j is restricted only in time period t, but unrestricted after time period t. Note also that $J_t^{\bar{K}}(\cdot) = J_t(\cdot)$. We utilize $J_t^j(\cdot)$ with respect to a fictitious state $(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{\bar{K}})$, where installation 0 has inventory x^{i-1}, and installations $1, 2, \cdots, i-1$ are empty. The optimality equation for $J_t^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{\bar{K}})$, $1 \leq i \leq \bar{K} - 1$, is given by

$$
J_t^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{\bar{K}}) = \min_{x^{i-1} \leq y_i \leq x^i, z \geq x^{\bar{K}}} \{d_i y_i + L(y_i) - d_i x^{i-1} - cx^{\bar{K}} + cz
$$
$$
+ E[J_{t+1}(y_i - D, \bar{0}^{M(i,W)-1}, x^{N(M(i,W),W)} - y_i, x^{N(M(i,W)+1,W)} - x^{N(M(i,W),W)}, \quad (3.2)
$$
$$
\cdots, x^{N(M(\bar{K},W)-1,W)} - x^{N(M(\bar{K},W)-2,W)}, z - x^{N(M(\bar{K},W)-1,W)}, \bar{0}^{\bar{K}-M(\bar{K},w)})]\},
$$

where y_i and z are decision variables: $y_i - x^{i-1}$ is the expediting amount from installation i and $z - x^{\bar{K}}$ is the regular ordering amount. For $i = \bar{K}$, the constraints in (3.2) become $x^{i-1} \leq y_i \leq z, z \geq x^{\bar{K}}$ in order to allow expediting regular orders that have just been placed. Note that the equation should be read appropriately, if $M(i, w) = 0$ for a realized value w of W.

By Theorem 4, in sequential systems expediting orders from installation i is never optimal before expediting all the outstanding orders at the downstream installation of installation i. With this fact, an alternative optimality equation equivalent to (3.1) is given by

$$J_t(v_0, v_1, v_2, \cdots, v_{\bar{K}}) = \min\{J_t^1(x^0, v_1, v_2, \cdots, v_{\bar{K}}),$$

$$d_1 v_1 + J_t^2(x^1, 0, v_2, \cdots, v_{\bar{K}}),$$

$$d_1 v_1 + d_2 v_2 + J_t^3(x^2, 0, 0, v_3, \cdots, v_{\bar{K}}),$$

$$\cdots, \tag{3.3}$$

$$\sum_{i=1}^{\bar{K}-1} d_i v_i + J_t^{\bar{K}}(x^{\bar{K}-1}, \bar{0}^{\bar{K}-1}, v_{\bar{K}}),$$

$$\sum_{i=1}^{\bar{K}} d_i v_i + J_t(x^{\bar{K}}, \bar{0}^{\bar{K}}), \}.$$

The first term $J_t^1(\cdot)$ corresponds to expediting partially or fully from only installation 1, the second term $d_1 v_1 + J_t^2(\cdot)$ captures expediting everything from installation 1, expediting partially or fully from installation 2, and no expediting beyond, and so forth. The eventual optimal decisions for regular ordering and expediting are determined by the minimum term in (3.3) since the system is sequential. If the j-th term achieves the minimum in (3.3), the optimal decision for expediting is to expedite all outstanding orders in installations 1, 2, $\cdots, j-1$ and to expedite $y_j - x^{j-1}$ from installation j and nothing beyond installation j, where $y_j - x^{j-1}$ is derived from the j-th term. The optimal regular ordering decision is to place a regular order in the amount $z - x^{\bar{K}}$ that is determined in the j-th term.

Characterization of Sequential Systems

In this subsection, we discuss how to identify sequential systems. We derive first a necessary condition and then a sufficient condition for a system to be sequential. The following lemma, whose proof is given in Appendix, is used later.

Lemma 7. *Under Assumption 2, the following holds:*

(a) $\lim_{n\to\infty} Prob[M^n(i, W) = 0] = 1$ *for all i,*

(b) $\lim_{n\to\infty} Prob[M^n(i, W) = k] = 0$, $k \neq 0$ *for all i.*

The expediting costs should be nondecreasing in order for a system to be sequential as the next proposition states.

Proposition 1. *Sequential systems satisfy $d_i \geq d_{i-1}$, for all i.*

47

Proof. Using $j = i - 1$ in Lemma 6 results in $d_i - d_{i-1} \geq E[d_{M^n(i,W)} - d_{M^n(i-1,W)}]$. On the other hand, $E[d_{M^n(i,W)}] = \sum_k d_k Prob[M^n(i,W) = k] = \sum_{k \neq 0} d_k Prob[M^n(i,W) = k] + d_0 Prob[M^n(i,W) = 0]$. By taking $\lim_{n \to \infty}$ and applying Lemma 7 we get

$$\lim_{n \to \infty} E[d_{M^n(i,W)}] = d_0 = 0.$$

Therefore, $d_i - d_{i-1} \geq 0$ for all i. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Next we identify a sufficient condition.

Proposition 2. *Suppose the followings are true for all i and $w \in \mathbf{W}$:*

- $d_i - d_{i-1} \geq d_{i-1} - d_{i-2}$, *and*

- $E[M(i,W) - M(i-1,W)] \leq 1$.

Then, the system is sequential.

Proof. Because of Assumption 1, $M(i,w) - M(i-1,w)$ is a nonnegative integer. Recall that orders do not go backward, i.e., $M(i,W) \leq i$. The first condition in the proposition implies $d_{M(i,w)} - d_{M(i-1,w)} \leq (M(i,w) - M(i-1,w))(d_i - d_{i-1})$. Therefore, by taking expectations on both sides, we have $E[d_{M(i,W)} - d_{M(i-1,W)}] \leq E[(M(i,W) - M(i-1,W))(d_i - d_{i-1})] \leq d_i - d_{i-1}$, which is Assumption 3. $\qquad\qquad\qquad\qquad$ $\square$

We call the first property in Proposition 2 convexity since it implies that the expediting cost differences are convex. Proposition 2 gives only sufficient conditions. We provide an example of a system that is sequential but nevertheless is not convex. In other words, sequential systems also include systems with non-convex expediting costs.

Example Consider a 5 installation system including the manufacturer and the supplier with four movement patterns: w_1, w_2, w_3, and w_4. More specifically,

- w_1: normal mode with probability p_1 such that $M(i,w_1) = i - 1$ for $i = 1, 2, 3, 4$,

- w_2: with probability p_2 such that $M(i,w_2) = i - 1$ for $i = 1, 3, 4$, and $M(2,w_2) = 0$,

- w_3: with probability p_3 such that $M(i,w_3) = i - 1$ for $i = 1, 2, 4$, and $M(3,w_3) = 1$, and

- w_4: with probability p_4 such that $M(i, w_4) = i - 1$ for $i = 1, 2, 3$, and $M(4, w_4) = 2$,

as shown in Figure 3-3. The associated probability distribution is $p_1 = \frac{1}{10}, p_2 = \frac{1}{10}, p_3 = \frac{3}{10}$,
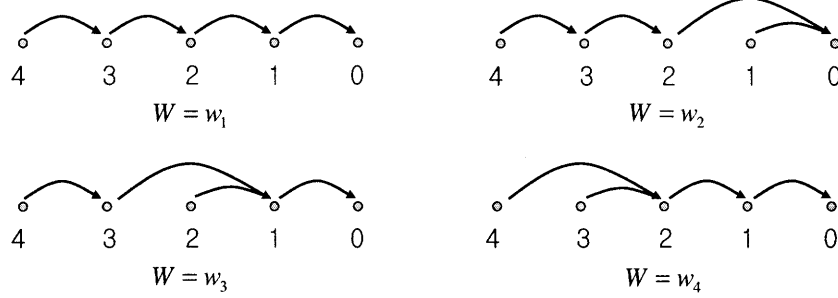

Figure 3-3: The movement patterns

and $p_4 = \frac{1}{2}$. The system is clearly non-convex if the expediting costs are $d_1 = 10, d_2 = 19, d_3 = 27$, and $d_4 = 34$. To check that the system is sequential, let us compute $d_i - E[d_{M(i,W)}]$ for $i = 1, 2, 3$, and 4. We have $d_1 - E[d_{M(1,W)}] = d_1 - 0 = 10$, $d_2 - E[d_{M(2,W)}] = d_2 - p_1 d_1 - p_2 d_0 - p_3 d_1 - p_4 d_1 = 10$, $d_3 - E[d_{M(3,W)}] = d_3 - p_1 d_2 - p_2 d_2 - p_3 d_1 - p_4 d_2 = 10.7$, and $d_4 - E[d_{M(4,W)}] = d_4 - p_1 d_3 - p_2 d_3 - p_3 d_3 - p_4 d_2 = 11$. Since $d_i - E[d_{M(i,W)}] \geq d_{i-1} - E[d_{M(i-1,W)}]$ for all i, the system is sequential. $\square$

3.4 Optimal Policies for Sequential Systems

In this section, we focus on identifying optimal policies for sequential systems.

3.4.1 Preliminaries

We frequently use Lemma 2, which is reintroduced as the following lemma just for convenience.

Lemma 8 (Lemma 2). *Let f be convex and have a finite minimizer on $\mathbb{R}$. Let $y^* = \arg\min f(x)$. Then, $\min\limits_{x_1 \leq x \leq x_2} f(x) = a + g(x_1) + h(x_2)$, where $a = f(y^*)$, and penalty functions $g(x_1)$ and $h(x_2)$ are*

$$g(x_1) = \begin{cases} 0 & x_1 \leq y^* \\ f(x_1) - a & x_1 > y^* \end{cases} \quad and \quad h(x_2) = \begin{cases} f(x_2) - a & x_2 \leq y^* \\ 0 & x_2 > y^* \end{cases}.$$

For a nondecreasing convex f, we define $a = 0$, $g(x) = f(x)$, and $h(x) = 0$. On the other hand, for a nonincreasing convex f, we define $a = 0$, $g(x) = 0$, and $h(x) = f(x)$.

In Lemma 8, g is nondecreasing convex, while h is nonincreasing convex. The following lemma is an extension of Lemma 8, and identical to Part (c) of Lemma 3.

Lemma 9. *Let f_1 be convex and $b \in \mathbb{R}$. We have $\min_{b \leq x \leq y} \{f_1(x) + f_2(y)\} = a_1 + g_1(b) + \min_{b \leq y}\{h_1(y) + f_2(y)\}$, where a_1, h_1, and g_1 are defined as in Lemma 8 with respect to f_1.*

The following functions are required later in the derivation of the optimal policy. For $1 \leq i \leq \bar{K}$ and $t \leq T$, let us recursively define

$$f_{i,t}(y) = d_i y + L(y) + E[S^1_{M(i,W),t+1}(y - D)], \tag{3.4}$$

$$S^0_{i,t} = a_{i,t} + E[S^0_{M(i,W),t+1}],$$

$$S^1_{i,t}(x) = g_{i,t}(x) - d_i x, \tag{3.5}$$

$$S^2_{i,t}(x) = h_{i,t}(x) - L(x) + E[S^2_{M(i,W),t+1}(x - D)],$$

where $S^0_{0,t} = S^1_{0,t}(\cdot) = S^2_{0,t}(\cdot) = 0$ for all t, and $S^0_{i,T+1} = S^1_{i,T+1}(x) = S^2_{i,T+1}(x) = 0$ for all i. Here, $a_{i,t}$, $g_{i,t}$, and $h_{i,t}$ are defined according to Lemma 8 with respect to $f_{i,t}$. Starting from the last time period T, functions $f_{i,t}$ and $S^j_{i,t}$ can be obtained recursively. In particular, from (3.4) we can compute $f_{i,T}$, then from (3.5) we obtain $S^1_{i,T}$ for all i. Next we compute $f_{i,T-1}$ from (3.4), and in turn, $S^1_{i,T-1}$ from (3.5) for all i. We repeat this procedure to define all $f_{i,t}$ and $S^1_{i,t}$. For $S^0_{i,t}$ and $S^2_{i,t}$ we use a similar procedure. It is easy to check for all i and t that $f_{i,t}(\cdot)$ is convex for sequential systems, and $S^0_{i,t} + S^1_{i,t}(x) + S^2_{i,t}(x) = 0$.

Let us denote by $y^*_{i,t}$ a minimizer of $f_{i,t}(x)$: $y^*_{i,t} \in \arg\min f_{i,t}(x)$. The following theorem is an important property of $f_{i,t}$ for sequential systems. The proof can be found in Appendix.

Theorem 5. *For sequential systems we have $y^*_{i,t} \geq y^*_{i+1,t}$ for all i and t.*

The lemma shown below is used later in the derivation of the optimal policy. The proof is in Appendix.

Lemma 10. *For sequential systems, function $g_{i,t}(x) + S^2_{M(i,w),t}(x)$ is convex for all i and t, and for all $w \in \mathbf{W}$.*

3.4.2 Optimal Policies

The optimal policy for sequential systems is highly structured and given in the following theorem.

Theorem 6. *For sequential systems, the following policy is optimal.*

a. *Optimal expediting is determined by a set of base stock levels. Each base stock level corresponds to $y_{i,t}^*$ for expediting from installation i at time t. The expediting policy compares x^{i-1} and $y_{i,t}^*$ as follows.*

- *If $x^{i-1} < y_{i,t}^*$, then we expedite $\min\{x^i - x^{i-1}, y_{i,t}^* - x^{i-1}\}$ from installation i.*

- *Otherwise, if $x^{i-1} \geq y_{i,t}^*$, we do not expedite anything from installation i.*

b. *The optimal regular ordering policy is the base stock policy with respect to inventory position $x^{\bar{K}}$. The base stock level for regular ordering is $z_t^* = \arg\min\{h_{\bar{K},t}(z) + cz + E[H_{t+1}(z-D) + S_{M(\bar{K},W),t+1}^2(z-D)]\}$ for all i and t, where $H_t(x)$ is convex and follows the recursive equation $H_t(x) = \min_{z \geq x}\{h_{\bar{K},t}(z) + cz + E[H_{t+1}(z-D) + S_{M(\bar{K},W),t+1}^2(z-D)]\} - S_{\bar{K},t}^2(x) - cx$, and $H_{T+1}(x) = J_{T+1}(x, \bar{0}^{\bar{K}})$. We place a regular order in the amount of $\max(0, z_t^* - x^{\bar{K}})$ at time period t.*

To better understand the policy, we consider the following illustrative example consisting of five installations. There are three movement patterns: w_1, w_2, and w_3 with probabilities p_1, p_2, and $1 - p_1 - p_2$, respectively. More specifically,

- w_1: $M(i, w_1) = i - 1$ for $i = 1, 2, 3, 4$,

- w_2: $M(i, w_2) = i - 1$ for $i = 1, 3$, and $M(2, w_2) = 0$ and $M(4, w_2) = 2$, and

- w_3: $M(i, w_3) = i - 1$ for $i = 2, 3$, and $M(1, w_3) = 1$ and $M(4, w_3) = 4$

as shown in Figure 3-4. Suppose that the regular ordering base stock level is $z^* = 210$ and
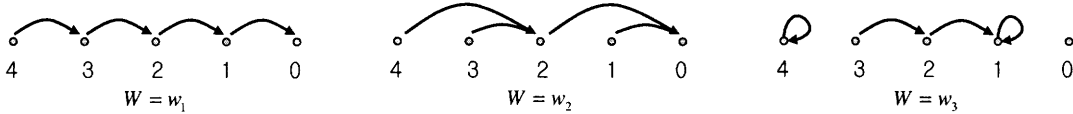

Figure 3-4: The movement patterns

the expediting base stock levels are $y_4^* = 20$, $y_3^* = 50$, $y_2^* = 85$, and $y_1^* = 110$. In the following table, we summarize the mechanics of the optimal policy for a certain time period.

Installation i	4 (Suppl.)	3	2	1	0 (Manuf.)
v_i (x^i) before decisions	60 (185)	45 (125)	50 (80)	40 (30)	-10 (-10)
Regular Ordering max($z^* - x^4, 0$)	25				
Expediting max(min($y_i^* - x^{i-1}, v_i$),0)	0	0	50	40	
Realized demand D					65
v^i (x^i) after decisions and demand	85 (145)	45 (60)	0 (15)	0 (15)	15 (15)
v_i if the movement pattern $W = w_1$	0	85	45	0	15
v_i if the movement pattern $W = w_2$	0	0	130	0	15
v_i if the movement pattern $W = w_3$	85	0	45	0	15

In order to prove Theorem 6, we need the following proposition, which is proved concurrently with Theorem 6 within an induction loop in the subsequent proof. For convenience, we refer the items in the following proposition as (c) and (d).

Proposition 3. *For sequential systems, the following is true*

c. For every $i = 1, \cdots, \bar{K}$, $0 \leq e \leq v_i$, and $t \geq \tilde{t}$, we have

$$J_t(x^{i-1}, \bar{0}^{i-1}, v_i, v_{i+1}, \cdots, v_{\bar{K}}) - J_t(x^{i-1} + e, \bar{0}^{i-1}, v_i - e, v_{i+1}, \cdots, v_{\bar{K}})$$
$$= S_{i,t}^0 + S_{i,t}^1(x^{i-1}) + S_{i,t}^2(x^{i-1} + e).$$

d. Function $S_{M(\bar{K},w),t}^2(x) + H_t(x)$ is convex for all t and $w \in \mathbf{W}$.

Because x^i is nondecreasing in i and $y_{i,t}^*$ is nonincreasing in i, by Theorem 5, there exists at most one $i^* \in \{1, 2, \cdots, \bar{K}\}$ such that $x^{i^*-1} \leq y_{i^*,t}^*$ and $x^{i^*} \geq y_{i^*+1,t}^*$. Theorem 6 states that we expedite everything up to installation $i^* - 1$, min$\{x^{i^*} - x^{i^*-1}, y_{i^*,t}^* - x^{i^*-1}\}$ from installation i^*, and nothing beyond installation i^*. If such an i^* does not exist, then we do not expedite at all, or we expedite everything up to installation $\bar{K}$.

Proof of Theorem 6 and Proposition 3. We use induction. In the base case $t = T + 1$, the optimal expediting and regular ordering policies are null. We can safely set the base stock levels for expediting and regular ordering at negative infinity. Also, part (c) and (d) trivially hold when $t = T + 1$. In the proof, we also show that $H_t(x) = J_t(x, \bar{0}^K)$ for all t.

Now we continue with the induction step. Let us assume that on and after time $t + 1 \leq T + 1$, all parts in the theorem and proposition hold, and $H_{t+1}(x) = J_{t+1}(x, \bar{0}^K)$, and it

is convex. Now we need to show the results for time period t. As the first step, we study $J_t(x, \bar{0}^K)$ in order to show $H_t(x) = J_t(x, \bar{0}^K)$ and that it is convex. In state $(x, \bar{0}^K)$, at the beginning of time period t we place a regular order of $z - x$ units and expedite $y_{\bar{K}} - x$ units out of the just placed regular order. Therefore, we have

$$
\begin{aligned}
J_t(x, \bar{0}^K) &= \min_{x \le y_{\bar{K}} \le z} \{d_{\bar{K}} y_{\bar{K}} + L(y_{\bar{K}}) + cz + E[J_{t+1}(y_{\bar{K}} - D, \bar{0}^{M(\bar{K},W)-1}, z - y_{\bar{K}}, \bar{0}^{\bar{K}-M(\bar{K},w)})]\} \\
&\quad - d_{\bar{K}} x - cx \\
&= \min_{x \le y_{\bar{K}} \le z} \{d_{\bar{K}} y_{\bar{K}} + L(y_{\bar{K}}) + cz + E[J_{t+1}(z - D, \bar{0}^K) + S^0_{M(\bar{K},W),t+1} \\
&\quad + S^1_{M(\bar{K},W),t+1}(y_{\bar{K}} - D) + S^2_{M(\bar{K},W),t+1}(z - D)]\} - d_{\bar{K}} x - cx \\
&= \min_{x \le y_{\bar{K}} \le z} \{f_{\bar{K},t}(y_{\bar{K}}) + cz + E[J_{t+1}(z - D, \bar{0}^K) + S^2_{M(\bar{K},W),t+1}(z - D)]\} \\
&\quad + E[S^0_{M(\bar{K},W),t+1}] - d_{\bar{K}} x - cx.
\end{aligned}
$$

Note that the induction hypothesis on part (c) at $t + 1$ is used in the above derivation. By using Lemma 9, we have

$$
\begin{aligned}
J_t(x, \bar{0}^K) &= \min_{z \ge x} \{h_{\bar{K},t}(z) + cz + E[J_{t+1}(z - D, \bar{0}^K) + S^2_{M(\bar{K},W),t+1}(z - D)]\} \\
&\quad + a_{\bar{K},t} + E[S^0_{M(\bar{K},W),t+1}] + g_{\bar{K},t}(x) - d_{\bar{K}} x - cx.
\end{aligned}
\tag{3.6}
$$

Note that $S^0_{\bar{K},t} + S^1_{\bar{K},t}(x) + S^2_{\bar{K},t}(x) = 0$ and $H_{t+1}(x) = J_{t+1}(x, \bar{0}^{\bar{K}})$ from the induction hypothesis. We have

$$
J_t(x, \bar{0}^K) = \min_{z \ge x} \{h_{\bar{K},t}(z) + cz + E[H_{t+1}(z - D) + S^2_{M(\bar{K},W),t+1}(z - D)]\} - S^2_{\bar{K},t}(x) - cx.
$$

Since this coincides with the definition of $H_t(x)$, we conclude that $H_t(x) = J_t(x, \bar{0}^{\bar{K}})$. Furthermore, from the induction hypothesis on part (d) the right-hand side of (3.6) is convex, hence $H_t(x)$ is convex.

Let us now prove part (d). By adding $S^2_{M(\bar{K},w),t}(x)$ to both sides of (3.6), we get

$$
\begin{aligned}
S^2_{M(\bar{K},w),t}(x) + J_t(x, \bar{0}^K) &= \min_{z \ge x} \{h_{\bar{K},t}(z) + cz + E[J_{t+1}(z - D, \bar{0}^K) + S^2_{M(\bar{K},W),t+1}(z - D)]\} \\
&\quad + a_{\bar{K},t} + E[S^0_{M(\bar{K},W),t+1}] + S^2_{M(\bar{K},w),t}(x) + g_{\bar{K},t}(x) - d_{\bar{K}} x - cx,
\end{aligned}
$$

which is convex because $S^2_{M(\bar{K},w),t}(x) + g_{\bar{K},t}(x)$ is convex by Lemma 10 and $E[J_{t+1}(z -$
$D, \bar{0}^{\bar{K}}) + S^2_{M(\bar{K},W),t+1}(z - D)]$ is convex by the induction hypothesis. This completes the
proof of part (d).

To prove parts (a) and (b), we apply part (c) to

$$J_{t+1}(y_i - D, \bar{0}^{M(i,w)-1}, x^{N(M(i,w),w)} - y_i, x^{N(M(i,w)+1,w)} - x^{N(M(i,w),w)}, \cdots)$$

in (3.2) for $i < \bar{K}$ with a realized value w of W on and after time period $t+1$ repeatedly
to obtain

$$J_{t+1}(y_i - D, \bar{0}^{M(i,w)-1}, x^{N(M(i,w),w)} - y_i, x^{N(M(i,w)+1,w)} - x^{N(M(i,w),w)}, \cdots)$$

$$= S^0_{M(i,w),t+1} + S^1_{M(i,w),t+1}(y_i - D) + S^2_{M(i,w),t+1}(x^{N(M(i,w),w)} - D)$$

$$+ J_{t+1}(x^{N(M(i,w),w)} - D, \bar{0}^{M(i,w)}, x^{N(M(i,w)+1,w)} - x^{N(M(i,w),w)}, \cdots)$$

$$= S^0_{M(i,w),t+1} + S^1_{M(i,w),t+1}(y_i - D) + S^2_{M(i,w),t+1}(x^{N(M(i,w),w)} - D)$$

$$+ \sum_{j=M(i,w)+1}^{M(\bar{K},w)-1} \{S^0_{j,t+1} + S^1_{j,t+1}(x^{N(j-1,w)} - D) + S^2_{j,t+1}(x^{N(j,w)} - D)\}$$

$$+ J_{t+1}(x^{N(M(\bar{K},w)-1,w)} - D, \bar{0}^{M(\bar{K},w)-1}, z - x^{N(M(\bar{K},w)-1,w)}, \bar{0}^{\bar{K}-M(\bar{K},w)})$$

$$= S^0_{M(i,w),t+1} + S^1_{M(i,w),t+1}(y_i - D) + S^2_{M(i,w),t+1}(x^{N(M(i,w),w)} - D)$$

$$+ \sum_{j=M(i,w)+1}^{M(\bar{K},w)-1} \{S^0_{j,t+1} + S^1_{j,t+1}(x^{N(j-1,w)} - D) + S^2_{j,t+1}(x^{N(j,w)} - D)\}$$

$$+ S^0_{M(\bar{K},w),t+1} + S^1_{M(\bar{K},w),t+1}(x^{N(M(\bar{K},w)-1,w)} - D) + S^2_{M(\bar{K},w),t+1}(z - D)$$

$$+ J_{t+1}(z - D, \bar{0}^{\bar{K}}).$$

Let us gather in Q all of the terms in the above equation that only contain constants and
state variables not involving any decision variables. Then we can rewrite

$$J_{t+1}(y_i - D, \bar{0}^{M(i,w)-1}, x^{N(M(i,w),w)} - y_i, x^{N(M(i,w)+1,w)} - x^{N(M(i,w),w)}, \cdots)$$

$$= S^1_{M(i,w),t+1}(y_i - D) + S^2_{M(\bar{K},w),t+1}(z - D) + J_{t+1}(z - D, \bar{0}^{\bar{K}}) + Q.$$

Substituting this into (3.2) and w for W, we obtain

$$J_t^i(x^{i-1}, \bar{0}^{i-1}, v_i, \cdots, v_{\bar{K}})$$

$$= \min_{x^{i-1} \leq y_i \leq x^i, z \geq x^{\bar{K}}} \{d_i y_i + L(y_i) - d_i x^{i-1} - cx^{\bar{K}} + cz$$

$$+ E[S_{M(i,W),t+1}^1(y_i - D) + S_{M(\bar{K},W),t+1}^2(z - D) + J_{t+1}(z - D, \bar{0}^{\bar{K}}) + Q]\} \qquad (3.7)$$

$$= \min_{x^{i-1} \leq y_i \leq x^i, z \geq x^{\bar{K}}} \{f_{i,t}(y_i) + cz + E[S_{M(\bar{K},W),t+1}^2(z - D) + J_{t+1}(z - D, \bar{0}^{\bar{K}})]\}$$

$$+ E[Q] - d_i x^{i-1} - cx^{\bar{K}},$$

for $i < \bar{K}$. When $i = \bar{K}$, we have

$$J_t^{\bar{K}}(x^{\bar{K}-1}, \bar{0}^{\bar{K}-1}, v_{\bar{K}}) = \min_{x^{\bar{K}-1} \leq y_{\bar{K}} \leq z, z \geq x^{\bar{K}}} \{d_{\bar{K}} y_{\bar{K}} + L(y_{\bar{K}}) - d_{\bar{K}} x^{\bar{K}-1} - cx^{\bar{K}} + cz$$

$$+ E[J_{t+1}(y_{\bar{K}} - D, \bar{0}^{M(\bar{K},W)-1}, z - y_{\bar{K}}, \bar{0}^{\bar{K}-M(\bar{K},w)})]\}$$

$$= \min_{x^{\bar{K}-1} \leq y_{\bar{K}} \leq z, z \geq x^{\bar{K}}} \{d_{\bar{K}} y_{\bar{K}} + L(y_{\bar{K}}) - d_{\bar{K}} x^{\bar{K}-1} - cx^{\bar{K}} + cz$$

$$+ E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S_{M(\bar{K},W),t+1}^0 + S_{M(\bar{K},W),t+1}^1(y_{\bar{K}} - D) \qquad (3.8)$$

$$+ S_{M(\bar{K},W),t+1}^2(z - D)]\}$$

$$= \min_{x^{\bar{K}-1} \leq y_{\bar{K}} \leq z, z \geq x^{\bar{K}}} \{f_{\bar{K},t}(y_{\bar{K}}) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}})$$

$$+ S_{M(\bar{K},W),t+1}^2(z - D)]\} + E[S_{M(\bar{K},W),t+1}^0] - d_{\bar{K}} x^{\bar{K}-1} - cx^{\bar{K}}.$$

By applying Lemma 9, we have

$$J_t^{\bar{K}}(x^{\bar{K}-1}, \bar{0}^{\bar{K}-1}, v_{\bar{K}}) = \min_{z \geq x^{\bar{K}}} \{h_{\bar{K},t}(z) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S_{M(\bar{K},W),t+1}^2(z - D)]\}$$

$$+ a_{\bar{K},t} + E[S_{M(\bar{K},W),t+1}^0] + g_{\bar{K},t}(x^{\bar{K}-1}) - d_{\bar{K}} x^{\bar{K}-1} - cx^{\bar{K}}.$$

$$(3.9)$$

We now consider part (a) of the statement. From (3.7) for $i < \bar{K}$ the optimal expediting quantity is determined by

$$\min_{x^{i-1} \leq y_i \leq x^i} \{f_{i,t}(y_i)\},$$

and for $i = \bar{K}$ from (3.8) by

$$\min_{x^{\bar{K}-1} \leq y_{\bar{K}} \leq \max\{x^{\bar{K}}, z_t^*\}} \{f_{\bar{K},t}(y_{\bar{K}})\}.$$

Because $f_{i,t}(y_i)$ is convex for all i, this states that the base stock policy is optimal for expediting for every i. This completes the proof of part (a).

Next, we show part (b) of the theorem using (3.3). Note that the optimal regular ordering is determined by $J_t^i(\cdot)$, $1 \leq i \leq \bar{K}$ or $J_t(x^{\bar{K}}, \bar{0}^{\bar{K}})$ which corresponds to the minimum term in (3.3). Now we show that all of these lead to the same optimal decision. From (3.6) and (3.9), the optimal regular ordering quantity for $J_t^{\bar{K}}(x^{\bar{K}-1}, \bar{0}^{\bar{K}-1}, v_{\bar{K}})$ and $J_t(x^{\bar{K}}, \bar{0}^{\bar{K}})$ is determined by

$$\min_{z \geq x^{\bar{K}}} \{h_{\bar{K},t}(z) + cz + E[J_{t+1}(z - D, \bar{0}^{\bar{K}}) + S^2_{M(\bar{K},W),t+1}(z - D)]\}. \tag{3.10}$$

On the other hand, if the minimum term is attained at $i < \bar{K}$, the optimal regular ordering quantity is determined from (3.7) by

$$\min_{z \geq x^{\bar{K}}} \{cz + E[S^2_{M(\bar{K},W),t+1}(z - D) + J_{t+1}(z - D, \bar{0}^{\bar{K}})]\}. \tag{3.11}$$

Note that $h_{\bar{K},t}(z)$ is nonincreasing convex, and $h_{\bar{K},t}(z) = 0$ for $z \geq y^*_{\bar{K},t} (\in \arg\min\{f_{\bar{K},t}(y)\})$. Therefore, if $z_t^* \geq y^*_{\bar{K},t}$, then (3.10) and (3.11) lead to the same minimizer z_t^*. If $z_t^* < y^*_{\bar{K},t}$, from Theorem 5, we have $z_t^* < y^*_{i,t}$ for all i, which results in expediting everything in the supply chain including the fresh regular order at the current time period by part (a). In this case, (3.10) determines the regular ordering quantity since we are expediting from the supplier. As a result, (3.10) always determines the optimal regular ordering. Because $H_{t+1}(z - D) = J_{t+1}(z - D, \bar{0}^{\bar{K}})$ and $H_{t+1}(z - D) + S^2_{M(\bar{K},w),t+1}(z - D)$ is convex for any realization w of W by part (d), the unconstrained minimizer z_t^* of (3.10) is well defined, and (3.10) states that the optimal regular ordering policy, which is the base stock policy with respect to $x^{\bar{K}}$. Hence part (b) is proved.

Finally, let us prove part (c). At time period t, we know that parts (a), (b), and (d) hold. Also, from the induction hypothesis, we assume (c) holds on and after time period $t + 1$. We show in Appendix that (c) holds at time period t using all these results. Once part (c) is proved at time t with all the induction hypothesis, the induction step of the entire proof is completed. □

3.5 Results on the Expediting Base Stock Levels of Sequential Systems

In this section, we provide an insightful result on the variation of the magnitude of the expediting base stock levels as the expediting cost varies. As expediting cost varies, we expect the expediting base stock levels to also vary. For example, as d_i increases, $y_{i,t}^*$ should be nonincreasing to compensate for the higher cost of expediting. However, this increment in d_i might increase the need for expediting from elsewhere. Indeed, we show that the expediting base stock levels are nondecreasing for installations beyond installation i as d_i increases. On the other hand, the variation in d_i does not effect the base stock levels of the downstream installations.

The results in this section are applicable only when derivatives and integrals in expectations are interchangeable. If the holding and backlogging cost functions have bounded derivatives, all functions under consideration have this interchangeability property, since all functions considered are Lipschitz. We assume in this section that this is the case. By Lemma 3.2 in Glasserman and Tayur (1995), derivatives and integrals in expectations are interchangeable. The main result of this section follows.

Theorem 7. *For a sequential system we have*

$$\frac{\partial y_{i,t}^*}{\partial d_i} \leq 0 \quad and \quad \frac{\partial y_{i,t}^*}{\partial d_j} \geq 0$$

for $j < i$.

The following diagram illustrates this theorem.

$$\text{as } d_i \uparrow \quad \left\{ \begin{array}{ll} \vdots & \\ y_{i-2}^* & \text{no change} \\ y_{i-1}^* & \text{no change} \\ y_i^* & \downarrow \\ y_{i+1}^* & \uparrow \\ y_{i+2}^* & \uparrow \\ \vdots & \end{array} \right.$$

Sequential systems have monotonic base stock levels as in Figure 3-5. As d_i increases, y_i^*
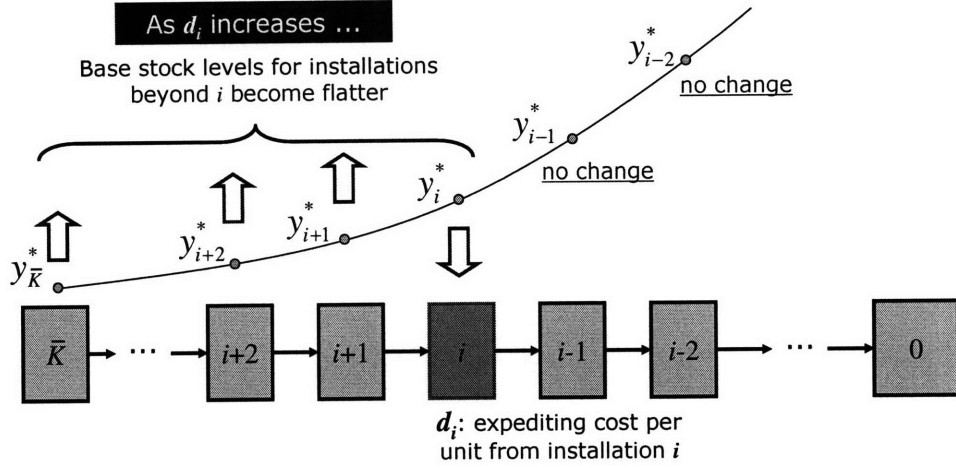
Figure 3-5: Directional sensitivity of base stock levels

decreases because higher d_i directly discourages expediting from installation i. However, the reduced y_i^* results in less safety stock in the manufacturing facility, which again calls for more expediting from beyond installation i, and hence increased y_j^* for $j > i$. The fact that y_t^* for $t \leq i - 1$ do not change follows from their definition since in order to derive them, d_i is not needed. We prove this in several steps using the following two lemmas.

Lemma 11. *In sequential systems, for $i \geq j \geq 1$,*

$$-1 \leq \frac{\partial}{\partial d_j} \frac{\partial S_{i,t}^1(y)}{\partial y} \leq 0.$$

Proof. We use induction. Note that $S_{i,T}^1(y) = g_{i,T}(y) - d_i y$ and $f_{i,T}(y) = d_i y + L(y)$. In the base case we have $-1 \leq \frac{\partial}{\partial d_j} \frac{\partial S_{i,T}^1(y)}{\partial y} \leq 0$ since when $y \leq y_T^*$ and $j = i$, $\frac{\partial}{\partial d_j} \frac{\partial S_{i,T}^1(y)}{\partial y} = -1$, and otherwise $\frac{\partial}{\partial d_j} \frac{\partial S_{i,T}^1(y)}{\partial y} = 0$.

Assume that $-1 \leq \frac{\partial}{\partial d_j} \frac{\partial S_{i,t+1}^1(y)}{\partial y} \leq 0$ for a given i and all j such that $i \geq j \geq 1$. We have

$$\frac{\partial f_{i,t}(y)}{\partial y} = d_i + \frac{\partial L(y)}{\partial y} + E[\frac{\partial}{\partial y} S_{M(i,W),t+1}^1(y - D)].$$

When $y \leq y_{i,t}^*$, we have $S_{i,t}^1(y) = -d_i y$ since $g_{i,t}(y) = 0$. Therefore $\frac{\partial}{\partial d_j} \frac{\partial S_{i,t}^1(y)}{\partial y} = 0$ for $j < i$, and $\frac{\partial}{\partial d_i} \frac{\partial S_{i,t}^1(y)}{\partial y} = -1$ for $j = i$. On the other hand, when $y > y_{i,t}^*$, we have

$S^1_{i,t}(y) = f_{i,t}(y) - d_i y - a_{i,t}$. For $j \leq i$, since $M(i, W) \leq i$ by definition, it follows that

$$-1 \leq \frac{\partial}{\partial d_j} \frac{\partial S^1_{i,t}(y)}{\partial y} = E[\frac{\partial}{\partial d_j} \frac{\partial}{\partial y} S^1_{M(i,W),t+1}(y - D)] \leq 0.$$

Note that we interchanged integrals and derivatives on several occasions. □

Lemma 12. *In sequential systems, for all i we have*

$$\frac{\partial}{\partial d_i} \frac{\partial f_{i,t}(y)}{\partial y} \geq 0,$$

and for $i > j \geq 1$,

$$\frac{\partial}{\partial d_j} \frac{\partial f_{i,t}(y)}{\partial y} \leq 0.$$

Proof. From Lemma 11 for all $j \leq i$ we obtain

$$-1 \leq E[\frac{\partial}{\partial d_j} \frac{\partial}{\partial y} S^1_{M(i,W),t+1}(y - D)] \leq 0.$$

If $j = i$, we have

$$\frac{\partial}{\partial d_i} \frac{\partial f_{i,t}(y)}{\partial y} = 1 + E[\frac{\partial}{\partial d_i} \frac{\partial}{\partial y} S^1_{M(i,W),t+1}(y - D)] \geq 0,$$

and, otherwise if $j < i$, we have

$$\frac{\partial}{\partial d_j} \frac{\partial f_{i,t}(y)}{\partial y} = E[\frac{\partial}{\partial d_j} \frac{\partial}{\partial y} S^1_{M(i,W),t+1}(y - D)] \leq 0.$$

This establishes the proof. □

Proof of Theorem 7. First it is obvious that changes in d_i do not affect $f_{j,t}$ for $j = 1, 2, \ldots i$ for all t since $M(k, W) \leq k$ for all k. From Lemma 12, we have $\frac{\partial}{\partial d_i} \frac{\partial f_{i,t}(y)}{\partial y} \geq 0$, and this implies $\frac{\partial y^*_{i,t}}{\partial d_i} \leq 0$ for all i. Similarly, $\frac{\partial}{\partial d_j} \frac{\partial f_{i,t}(y)}{\partial y} \leq 0$ implies $\frac{\partial y^*_{i,t}}{\partial d_j} \geq 0$, for $j < i$ by Lemma 17, which is in Appendix. The proof is complete. □

Chapter 4

Non-Sequential Systems

4.1 Introduction

So far we have considered sequential systems. For sequential systems, the optimal policy for expediting and regular ordering is well structured. This is basically because of the existence of an alternative optimality equation for sequential systems.

On the other hand, if a system is not sequential, the structure of the optimal policies is complex, and analytical results are hard to obtain. Previous research on stochastic lead time models, which includes Kaplan (1970b), Nahmias (1979), and Ehrhardt (1984), assumes that the orders do not cross in time. If order crossing does occur, the resulting policy is complex. However, we cannot avoid order crossing in time in our model, if the expediting cost structure is not sequential. In other words, we cannot guarantee that it is optimal to expedite the outstanding orders whose delivery time to the manufacturing facility is shorter.

Consider the following counter intuitive example involving a supply chain system with only two time periods. There are a supplier, an intermediate installation, and a manufacturing facility. For simplicity, let us assume that the demand is deterministic in both time periods; 15 in period 1, and 25 in period 2. Let us also assume that the penalty cost component of the holding/backlogging cost is much larger than the procurement cost and the expediting cost. The expediting cost (d_1) from the intermediate installation is 2, and from the manufacturing facility (d_2) is 3. At the beginning of period 1, the initial inventory is 10 units at the manufacturing facility, and 10 at the intermediate installation.

There are two options. The first one is to place a regular order of 20 units at time 1,

expedite 5 units from the intermediate installation at time 1, and expedite 20 units from the intermediate installation at time 2. The second option is to place a regular order of 20 units at time 1, expedite 5 units from the supplier at time 1, and expedite 15 units from the intermediate installation at time 2. The total cost of the first option is $5 \cdot 2 + 20 \cdot 2 + 20c = 50 + 20c$, and of the second option is $5 \cdot 3 + 15 \cdot 2 + 20c = 45 + 20c$, where c is the per unit procurement cost. Therefore, even though $d_2 > d_1$, it is better to use the more expensive expediting option at time 1.

The main contribution of this chapter is that we propose a heuristic policy, the extended heuristic, for nonsequential systems that do not allow simple optimal policies. By means of a computational study, we find that the extended heuristic achieves a local optimum for a much wider class of systems that includes all sequential systems. In this chapter, we follow notations from Chapter 2 though the heuristic policy also applies to the stochastic lead time models.

4.2   The Extended Heuristic for Nonsequential Systems

We discuss nonsequential systems through a three-installation system consisting of a supplier, a manufacturing facility, and an intermediate installation between them. The lead time of regular orders is deterministic of one time period between consecutive installations. Two expediting options are available; one is to expedite from the intermediate installation (installation 1) at d_1 per unit, and the other one is to expedite from the supplier (installation 2) at the per unit cost of d_2. Though this three-installation system is simple, it is nontrivial and shares complex features with general length nonsequential systems, hence its analytical results are hard to obtain. After all, this three-installation system is a good starting point to understand general length systems.

Instead of trying to derive optimal policies, which is a daunting task due to the complexity and state dependency, we confine our interest to the set of all base stock policies. We attempt to find the best base stock levels since base stock policies are particulary useful due to their simple structure. We next propose a heuristic policy that gives base stock levels for nonsequential systems and evaluate its performance numerically using three-installation systems.

The Extended Heuristic The extended heuristic is to apply the base stock policies with the base stock levels as described in Theorem 3 to nonsequential systems. Note that the definitions of $f_{i,k}$ and $S^j_{i,k}$ do not require systems to be sequential, hence the extended heuristic is well defined. Also, if the system is sequential, the extended heuristic finds an optimal control. Note that the extended heuristic is also applicable to general length systems.

In order to evaluate the performance of the extended heuristic, we use the following *derivative method* introduced in Glasserman and Tayur (1995).

The Derivative Method The derivative method is a numerical method to find the sensitivity of the cost-to-go under the base stock policies as the base stock levels vary. The method can be used to find locally optimal base stock levels within the set of all base stock policies using simulation and optimization. Here we briefly explain the procedure of the derivative method customized to our three-installation systems.

Step 1. Set the initial base stock levels: y_1, y_2, and z.

Step 2. Compute the derivatives of the dynamic programming optimality equation (2.1) with respect to the base stock levels. We get recursive equations of the derivatives of the cost-to-go with respect to each of y_1, y_2, and z.

Step 3. Evaluate the cost to go at time period 1 using simulation with the given base stock levels. Evaluate also the derivatives of the cost-to-go at time period 1 using the recursive equations from Step 2. The derivatives give the steepest decent direction of the cost-to-go at time period 1.

Step 4. Search linearly along the steepest decent direction for the best step size, and then set the new base stock levels using the result of the line search.

Step 5. Evaluate the derivatives of the cost-to-go with respect to the base stock levels from Step 4 using simulation. If the norm of the derivatives is smaller than a given threshold α, then terminate. Otherwise, go to Step 3.

After the termination of the derivative method, we get the locally optimal base stock levels and the corresponding cost-to-go at time period 1. Mathematically, the derivative method only works when derivatives and integrals in expectation are interchangeable. All

systems under consideration in this thesis have this interchangeability property. In what follows, we always start the derivative method with the base stock levels from the extended heuristic, hence the derivative method never produces an inferior solution. When the derivative method does not improve the initial solution, we conclude that the extended heuristic achieves a local optimum.

Numerical Data and Results

The following are the detailed data for the numerical study: the procurement cost $c = 100$, the holding cost is 50 per unit, and the backlogging cost is 150 per unit. The demand distribution is triangular with (mean, min, max) = (50, 0, 100). Expediting cost d_2 varies from 10 to 120, while d_1 varies so that d_1/d_2 ranges from 0.4 to 2.4.

Figure 4-1 summarizes the numerical results. The horizontal axis is d_1/d_2, which measures the degree how close a system is to a sequential system. The vertical axis shows the improvement in percentage of the cost-to-go using the base stock levels from the derivative method over the cost-to-go of the extended heuristic. When the gap is zero, it means that the extended heuristic produces a locally optimal solution. Different trend lines in the figure stand for different values of d_2, and within a line, d_1 varies. The 95% confidence interval for any data point in the figure is within 0.05% of its value.

If $d_1/d_2 \leq 0.5$, then the system is sequential, thus the extended heuristic is optimal, and the gap is 0%. Interestingly, we observe that even though the system is nonsequential for $1 \leq d_1/d_2 > 0.5$, the extended heuristic achieves a local optimum among all the base stock policies. As d_1/d_2 increase above 1, we observe a gradual departure from local optimality of the extended heuristic.

In the figure, we see some lines that are always close to zero regardless of the value of d_1/d_2. These lines correspond to the case of d_2 being too small or too large compared to the other costs in the system. As a result, we always expedite everything or do not use expediting at all. In these extreme cases, the extended heuristic performs well regardless of d_1/d_2.

These numerical results show that the extended heuristic performs well for a larger set of systems (systems with $d_1/d_2 \leq 1$) than the set of sequential systems (systems with $d_1/d_2 \leq 0.5$). For systems with $2.4 \geq d_1/d_2 \geq 1$, the gap is always less than 4.5%, which is encouraging and acceptable. On the downside, the gap keeps increasing with increasing
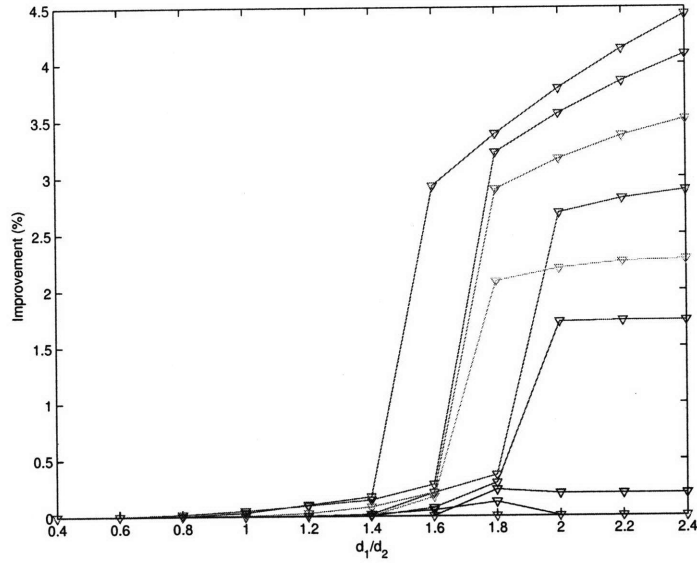
Figure 4-1: Improvements in cost-to-go of the derivative method with respect to the extended heuristic

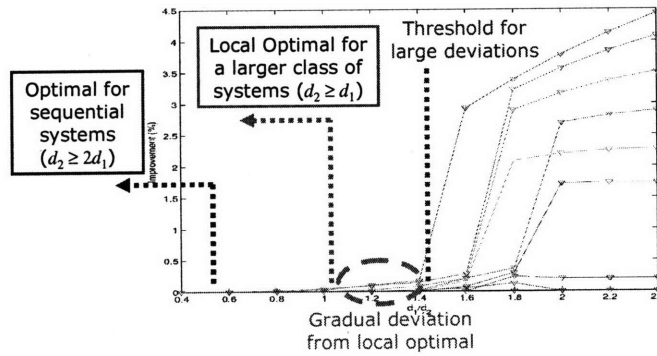d_1/d_2 and it seems that it can be arbitrarily large. See Figure 4-2 for a summary.

Figure 4-2: Local optimality and limitation of the extended heuristic

Clearly this result considers only the case when there are three installations in the system. Nevertheless, we are confident that the observation in Figure 4-2 generalizes to systems with more than three installations. Since the extended heuristic performs well for systems with nondecreasing expediting costs in installation number i, the extended heuristic likely works for most of practical systems. To conclude, though sequential systems constitute a subset of all possible systems, it gives us a valuable guide on what to use as an

64

appropriate heuristic for operating a supply chain with expediting options.

4.3 Intractability of an Optimal Policy and a Lower Bound

In this section, we discuss the intractability of nonsequential systems, and compute a lower bound of the cost-to-go for such systems. Also, we introduce another heuristic, *the rolling heuristic*, to compare with the extended heuristic. Recall that we considers three-installation systems.

At time k, the inventory controller first places a regular order for $z - x^1$ units, and decides to expedite $y_1 - x$ from the intermediate installation, and $y_2 - x^1$ from the supplier, where $x \leq y_1 \leq x^1$ and $x^1 \leq y_2 \leq z$. Then,

$$
\begin{aligned}
J_k(x, v_1) = \min_{x \leq y_1 \leq x^1 \leq y_2 \leq z} & \{d_1(y_1 - x) + d_2(y_2 - x^1) + L(y_1 + y_2 - x^1) \\
& + c(z - x^1) + E[J_{k+1}(y_2 - D, z - y_2)]\}.
\end{aligned}
\tag{4.1}
$$

By rearranging and setting $f_1(y) = d_1 y + L(y)$, we have

$$
\begin{aligned}
J_k(x, v_1) = \min_{x \leq y_1 \leq x^1 \leq y_2 \leq z} & \{f_1(y_1 + y_2 - x^1) + (d_2 - d_1)y_2 + cz \\
& + E[J_{k+1}(y_2 - D, z - y_2)]\} - (d_2 - d_1 + c)x^1 - d_1 x.
\end{aligned}
$$

By applying Lemma 2, we obtain

$$
\begin{aligned}
J_k(x, v_1) = \min_{x^1 \leq y_2 \leq z} & \{g_1(y_2 - v_1) + h_1(y_2) + (d_2 - d_1)y_2 + cz \\
& + E[J_{k+1}(y_2 - D, z - y_2)]\} + a_1 - (d_2 - d_1 + c)x^1 - d_1 x.
\end{aligned}
\tag{4.2}
$$

Let the optimal decisions be $y^*_{2,k}(x^1, v_1)$ and $z^*_k(x^1, v_1)$. From (4.2), because v_1 only appears in $g_1(y_2 - v_1)$ and g_1 is nondecreasing convex, $y^*_{2,k}(x^1, v_1)$ is nondecreasing in v_1, for fixed x^1. Therefore, the optimal expediting amount from the supplier does not follow the base stock policy with respect to the echelon inventory, and the optimal decisions cannot be simple.

An insight is that, while the total inventory x^1 is fixed, we should possibly expedite more in order to cover the increasing backlogging cost as inventory at the intermediate installation (v_1) increases and on hand inventory at the manufacturing facility (x) decreases. Another

interesting observation is that the optimal regular ordering quantity can also depend on v_1. As an example, consider $k = T$. Because $J_{T+1} = 0$, $z_T^*(x^1, v_1) = y_{2,T}^*(x^1, v_1) = \arg\min_{x^1 \leq y_2}\{g_1(y_2 - v_1) + h_1(y_2) + (d_2 - d_1)y_2 + cy_2\}$. In contrast to the previous models with expediting such as Lawson and Porteus (2000) and Muharremoglu and Tsitsiklis (2003a), in which regular ordering from the supplier is a function of the echelon inventory only, in our model regular ordering explicitly depends on v_1 as well as echelon inventory x^1.

Next we obtain a lower bound on J_k given by (4.2). We require the lower bound to be decomposable with respect to echelon stocks, so that its computation is easy. The following lemma can be found in Simchi-Levi et al. (2004).

Lemma 13. *For a convex function $u(x)$, $U_v(x) = u(x) - u(x - v)$ is nondecreasing in x for $v \geq 0$.*

Let us define $R_v(x) = r(x) - r(x - v)$, for $v \geq 0$, where $r(x)$ is the convex holding/backlogging function. To guarantee the lower bound, we need the following assumption.

Assumption 4. *Function $R_v(x)$ is bounded by $r_1 v$ for some constant $r_1 \geq 0$. Formally, $R_v(x) \leq r_1 v$ for every x and v.*

It is easy to show by using Lemma 13 that a linear holding and backlogging cost function $r(x) = r_1(x)^+ + r_2(-x)^+$ satisfies Assumption 4. The proof of the following lemma is given in Appendix.

Lemma 14. *Under Assumption 4, we have $g_1(y - v) \geq g_1(y) - (d_1 + r_1)v$ for $v \geq 0$.*

Let us define J_k^{LB} by

$$
\begin{aligned}
J_k^{LB}(x, v_1) &= \min_{x^1 \leq y_2 \leq z}\{g_1(y_2) - (r_1 + d_1)v_1 + h_1(y_2) + (d_2 - d_1)y_2 + cz \\
&\quad + E[J_{k+1}^{LB}(y_2 - D, z - y_2)]\} + a_1 - (d_2 - d_1 + c)x^1 - d_1 x \\
&= \min_{x^1 \leq y_2 \leq z}\{g_1(y_2) + h_1(y_2) + (d_2 - d_1)y_2 + cz \\
&\quad + E[J_{k+1}^{LB}(y_2 - D, z - y_2)]\} + a_1 - (d_2 + r_1 + c)x^1 + r_1 x.
\end{aligned}
$$

Then, we have the following proposition, whose proof can be established by induction.

Proposition 4. $J_k^{LB}(x, v_1) = p_k^0(x) + p_k^1(x^1)$ *for convex functions p_k^0 and p_k^1.*

Lemma 14 implies $J_k^{LB}(x, v) \leq J_k(x, v)$, therefore J_k^{LB} is a lower bound of J_k, which is decomposable by Proposition 4. We note that a better decomposable lower bound cannot be obtained because Lemma 14 holds with equality for sufficiently large y.

A physical interpretation generating this lower bound is to expedite the outstanding order at the intermediate installation to the manufacturing facility without charging any expediting cost and at the same time not to charge any holding cost for the expedited order.

The Rolling Heuristic

Next we present another reasonable heuristic, which is used for the comparison with the extended heuristic.

The rolling heuristic: This heuristic is based on decoupling of expediting and regular ordering. Note that it is known that the optimal regular ordering policy without expediting is the base stock policy with respect to the inventory position and the computations of the base stock levels are easy because of the decomposability of the cost-to-go function.

The rolling heuristic is as follows. First, place a regular order following the conventional regular ordering base stock policy without considering expediting. Next, we make expediting decisions for all installations at the current time period by assuming that there is no expediting option in the future. This involves solving a nonlinear optimization problem with L decision variables. This is a convex problem regardless of the expediting cost structure, therefore it is tractable. At the next time period, we repeat the same procedure. The detailed mathematical derivation of this heuristic policy can be found in Appendix. The rolling heuristic provides a simple but nontrivial way of enjoying the benefits of expediting. Note that the rolling heuristic also allows order crossing in time for its outstanding orders.

Numerical result

Here, we provide a computational study of the two heuristics and the lower bound for non-stationary systems that consist of a supplier, a manufacturing facility, and an intermediate installation with the general expediting cost structure. The nonstationary systems have 26 time periods with the triangular demand distribution of (min, mean, max) = (0, 100, 200). Let us define the degree of non-sequentialness as $d_r = \frac{d_{1,k} + d_{1,k+1}}{2d_{2,k}}$ for $k < T$ and $d_r = \frac{d_{1,T}}{d_{2,T}}$. If $d_r \leq 0.5$, the expediting cost structure is sequential, and otherwise it is nonsequential.

As d_r increases, the system becomes more nonsequential, which means that the cost for expediting over longer distances becomes relatively cheap.

The cost parameters are randomly generated in the following way. We first generate procurement cost c_k for time period k based on $uf(50, 150)$, a uniform random variable on $[50, 150]$, for all k. Then, we generate the backlogging cost as $r_{2,k} = uf(1, 2) \cdot c_k$ and the holding cost as $r_{1,k} = uf(0, 0.4) \cdot r_{2,k}$ for all k. Finally, we generate $d_{1,k} = uf(0.5, 1.5) \cdot c_k$ for all k, and we assign $d_{2,k} = \frac{d_{1,k} + d_{1,k+1}}{2d_r}$ if $k < T$ and $d_{2,T} = \frac{d_{1,T}}{d_r}$, otherwise. After generating all the cost components, we normalize the expediting costs in order to compare nonstationary systems with different d_r's. Let us define $SC = \frac{1}{T} \sum_{k=1}^{T} c_k$ and $SD = \frac{1}{2T} \sum_{k=1}^{T} (d_{1,k} + d_{2,k}/2)$. Quantity SC is the average procurement cost and SD is the average expediting cost per unit distance. We normalize the expediting costs so that its average is 50% of the average procurement cost. Therefore, we scale the current expediting cost to $d_{1,k} \leftarrow \frac{SC}{2SD} d_{1,k}$ and $d_{2,k} \leftarrow \frac{SC}{2SD} d_{2,k}$.

Performance of the heuristics

We use simulation to measure the performance of the two heuristics as we vary d_r. Let us define $J_1^{EH}(0,0)$ and $J_1^{RH}(0,0)$ to be the average expected cost over the planning horizon with respect to the extended and the rolling heuristic at time period 1, respectively. In our numerical study, we use 30 different randomly generated nonstationary systems for a fixed d_r to measure the average performance. Each nonstationary system is again tested with 30 different simulation runs, each one with a different demand scenario, over 26 time periods. The confidence interval (both sides) of $J_1^{EH}(0,0)$ and $J_1^{RH}(0,0)$ of each nonstationary system is on average 5.02% of its mean with the maximum of 8.79%.

Figure 4-3 summarizes the simulation results. The plot on the left shows the relative magnitudes of $J_1^{EH}(0,0)$, $J_1^{RH}(0,0)$, and $J_1^{LB}(0,0)$ as d_r increases from 0.1 to 1. The plot implies that the extended heuristic is consistently outperforming the rolling heuristic regardless of d_r because $\frac{J_1^{EH}(0,0)}{J_1^{RH}(0,0)} < 100\%$ in all cases.

Note that the extended heuristic is actually optimal for $d_r \leq 0.5$. Nevertheless, in this case its gap with respect to the lower bound is approximately 140%. We conclude that the lower bound is weak and that we can easily attribute approximately 140% of the gap to the lower bound. If we focus now on cases with $d_r > 0.5$, they show a gap of the extended heuristic of slightly more than 140% and as we have just argued, we can easily attribute
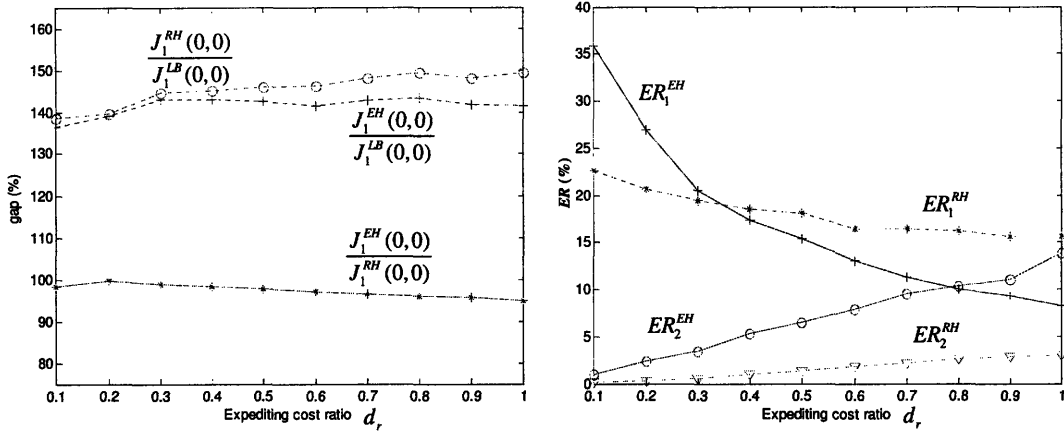
68

Figure 4-3: Simulation results

most of this gap to the weak lower bound. We believe it is safe to conclude that the extended heuristic is only a few percents from the optimal value.

The plot on the right in Figure 4-3 summarizes the total relative quantity of expedited orders from the supplier and the intermediate installation for each heuristic. We denote by ER_i^{EH} and ER_i^{RH} the ratio of total expedited amount from installation i to the total ordered amount in the extended and the rolling heuristic, respectively. In the extended heuristic, as the system becomes more nonsequential (as d_r increases), ER_1^{EH} and ER_2^{EH} are getting closer to each other, and eventually they cross, which is desirable if the system is highly nonsequential. On the other hand, in the rolling heuristic, the cross-over between ER_1^{RH} and ER_2^{RH} does not happen because the expediting decisions are myopic. Based on this numerical study, we conclude that the extended heuristic shows good performance even under the general expediting cost structure.

Base stock levels

Base stock levels of the extended and the rolling heuristic for a nonstationary system with $d_r = 0.8$ are shown in Figure 4-4. Because $d_r > 0.5$, this system is nonsequential, and therefore the base stock levels for expediting from the supplier are higher than for expediting from the intermediate installation for most of the time periods in this figure. This implies that more orders are expedited from the supplier than from the intermediate installation.
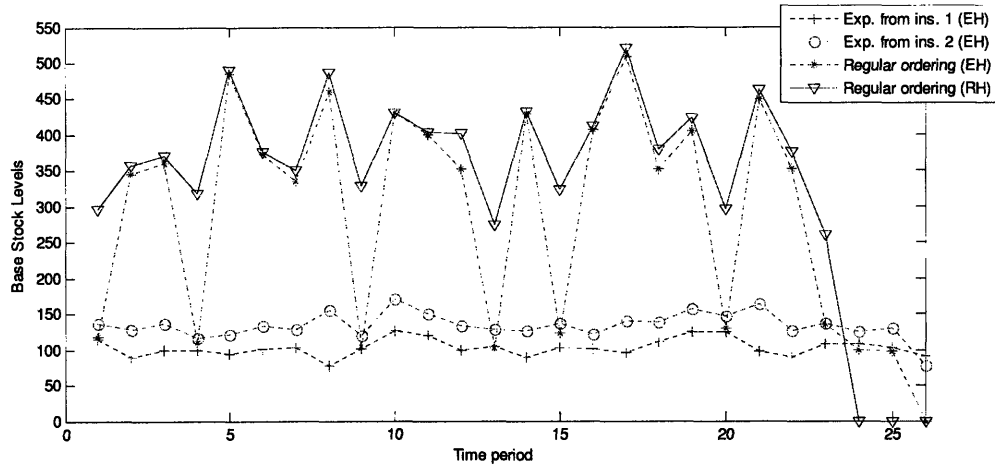
Therefore, orders may cross in time.

Figure 4-4: The base stock levels

Another important observation is that the base stock levels for regular ordering in EH are lower than those in RH because of the existence of the expediting options in EH, while the expediting decisions are separated from the regular ordering decisions in RH. To put it differently, less safety stock is required in EH due to the increased agility in the supply chain by the expediting options.

Chapter 5

Raw Materials with an Expiry Date

5.1 Introduction

In this chapter, we consider the same model as in Chapter 3 with one difference: orders (or raw materials) have a deterministic expiry date. The lead time is stochastic and raw materials are good only until the expiry date. If an order is not delivered within the expiry date, the manufacturer must expedite it. Since expediting is instantaneous, it implies that orders always arrive at the manufacturer before the expiry date. The manufacturer could also scrap the order, however this is not considered here. When an order is expedited just before it expires, we call this process *mandatory expediting*.

Another important modeling assumption in this chapter is that we do not impose any expiry date on delivered orders at the manufacturing facility. Once an order is delivered, we assume that the order is processed and transformed into a nonperishable product. For example, canned products of meats, fish, and/or produce can be applications of the model. After canning, the expiry date becomes much longer than that of raw materials, usually ranging up to several years. Similarly, any processed product with preservatives falls within the application category.

Even though the model is similar, the added control constraint of mandatory expediting brings additional complexity to the solution structure, and a careful treatment is required. For instance, because of the expiry dates, we have to keep track of the ages of all outstanding

orders, which means many more dimensions in the state space. To cope with the increased state space dimensionality, we have to restructure the state space. To optimally capture control with the expiry dates, the control scheme is much more sophisticated.

Our contributions in this chapter are three-fold. First, this is the first work considering expiry dates in a stochastic lead time setting, not to mention expediting. Second, the optimal policy for sequential systems is obtained, and it shows a simple structure. The optimal policy for expediting is again a variant of the base stock policy, and it identifies a number of base stock levels. Finally, we show that if the expiry dates approach to infinity, the optimal control policy converges to the optimal control policy with no expiry presented in Chapter 3.

In Section 5.2, we describe the model. Sequential systems are defined in Section 5.3. Optimal policies are derived and illustrated in Section 5.4.

Literature review for raw materials with expiry

There are two major directions of the literature that consider both the expiry dates and the exact optimal policies. One thread considers deterministic expiry dates and the other considers random expiry dates. Most of the random expiry models consider continuous review, and we refer to Nahmias (1982) for a complete review. For models with deterministic expiry dates and periodic review, there are only a few notable works. For convenience, let us denote the expiry dates on shelf (*shelf life*) by m, and the lead time by l.

The model with arbitrary deterministic $m \geq 1$ and $l = 0$, where unmet demand is backlogged, is studied by Nahmias (1975) and Fries (1975) independently. These works are analytical, and they found that the optimal policy is complex, and depends on the initial amount of stock at time period 1. Nahmias (1975) is reviewed in Chapter 1.

On the other hand, the model with $m = 2$ and arbitrary deterministic $l \geq 0$, where unmet demand is lost, is studied by Williams and Patuwo (1999). Their work is computational, and also shows that the optimal policy is complex.

All other documents consider special cases either of Nahmias (1975), Fries (1975) , or Williams and Patuwo (1999). For approximation models, we also refer to Nahmias (1982).

5.2 Model Statement

Consider the stochastic lead time model with Assumptions 1 and 2 from Chapter 3, where we add an additional constraint on the control space: if an order is not delivered within R time periods, then the manufacturer must expedite it. This process is called mandatory expediting.

Mandatory expediting happens at the end of a time period after the stochastic regular movements occurs. Therefore, the mandatory expediting cost depends on the realization of the regular movements at that time period. For instance, if an order about to expire is delivered to the manufacturing facility by the regular movements, we do not have to mandatorily expedite it. The sequence of other events in a time period is the same as in Chapter 3, and it is shown in Figure 5-1.
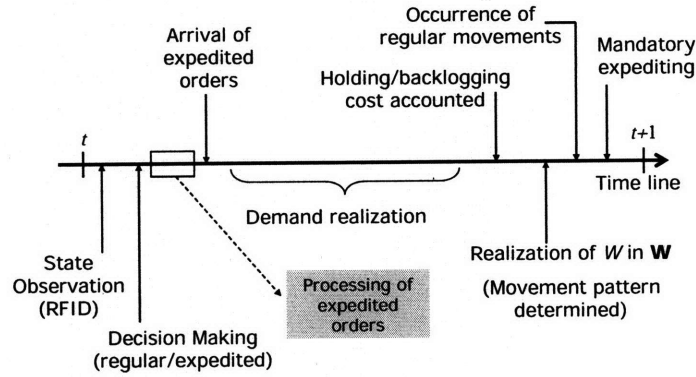
Figure 5-1: Sequence of events

Let us call the order that has j remaining time periods until mandatory expediting as a *stage j* order: stage = R-age. In an installation, there can be multiple outstanding orders with different stages. We express the state by stage inventory level and location as

$$(v_0, v_1, \cdots, v_{R-1}, l_1, l_2, \cdots, l_{R-1}),$$

where v_0 is the inventory at the manufacturing facility, v_j, $j \geq 0$, is the amount of the stock at stage j, and l_j is the corresponding physical location. Note that an order at stage j can have at most one location. If there is no order at stage j, then we assume $v_j = 0, l_j = 0$. Therefore, installation l_j contains the order at stage j. We define $l_0 = 0$, $l_R = \bar{K}$, and v_R as

the amount of fresh order. In Figure 5-2, we show physical installations with age bins. On the other hand, in Figure 5-3, we show stage inventory level and location representation. For
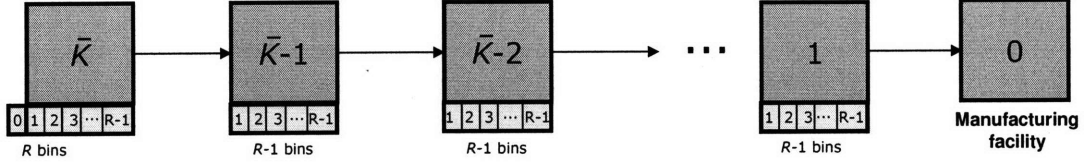

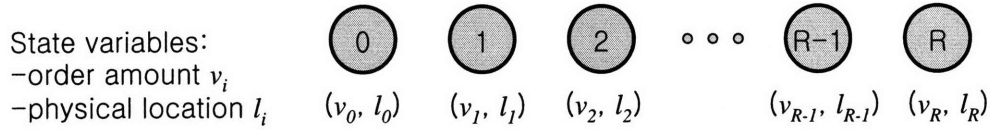Figure 5-2: Physical installations with age bins

State variables:
−order amount v_i
−physical location l_i

Figure 5-3: Stage inventory level and location representation

notational convenience, let $\bar{v}_i = (v_i, v_{i+1}, v_{i+2}, \cdots, v_{R-1})$ and $\bar{l}_i = (l_i, l_{i+1}, l_{i+2}, \cdots, l_{R-1})$. Then, the state can be written compactly as

$$(v_0, \bar{v}_1, \bar{l}_1).$$

Note that if $v_i > 0$ for $i > 0$, then l_i must be positive as well. Let us denote by NS_i the next state from the current state $A_i = (x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i)$ for $1 \leq i \leq R - 1$ under expediting only from stage i.

5.3 Sequential Systems

Let us denote by $l(l_j, W)$ the lead time without mandatory order expediting at installation l_j. Also, let us denote by $\bar{l}(j, l_j, W)$ the lead time with mandatory expediting of the order at stage j, installation l_j. Note that $\bar{l}(j, l_j, W) \leq j$ and $\bar{l}(j, l_j, W) \leq l(l_j, W)$. The following theorem states that expiry dates on outstanding orders work favorably with regard to sequential systems. The added order expiry constraint puts an additional incentive to expedite from the lowest nonempty installation, which is essentially the definition of sequential systems.

Theorem 8. *Sequential systems without the expiry constraint remain sequential after imposing expiry of any length on orders in transit.*

74

Proof. Consider two nonempty stages i and j, $i > j$, which contain $unit_i$ and $unit_j$, respectively. We compare the following two strategies.

- Strategy 1: Expedite $unit_j$ from stage j, and then expedite $unit_i$ after (realized) $\bar{l}(j, l_j, w)$ time periods from the corresponding position. This strategy is feasible by keeping track of the position of a fictitious unit in stage j, since it takes (realized) $\bar{l}(j, l_j, w)$ time periods for the fictitious unit to arrive at the manufacturing facility. The expected cost of strategy 1 is $d_{l_j} + E[d_{M^{\bar{l}(j,l_j,w)}(l_i,W)}]$.

- Strategy 2: Expedite $unit_i$, and then do not do anything on $unit_j$ except mandatory expediting.

  Expected cost of strategy 2 is $d_{l_i} + E[d_{M^{\bar{l}(j,l_j,w)}(l_j,W)}]$.

In a sequential systems, we have

$$d_{l_i} + E[d_{M^{\bar{l}(j,l_j,w)}(l_j,W)}] \geq d_{l_j} + E[d_{M^{\bar{l}(j,l_j,w)}(l_i,W)}]. \tag{5.1}$$

In Assumption 3 of Chapter 3, we require $d_i - E[d_{M(i,W)}] \geq d_{i-1} - E[d_{M(i-1,W)}]$ for all i, where $d_0 = 0$. From Lemma 6 of Chapter 3, we have $d_i - d_j \geq E[d_{M^n(i,W)} - d_{M^n(j,W)}]$, for any i and j, $i \geq j$, and $n \geq 1$. Since $M(\cdot)$ is defined independently from the expiry date of R time periods, the same result as Lemma 6 holds true. The condition (5.1) for sequential systems is obtained by considering $n = \bar{l}(j, l_j, w)$. The proof is completed. $\square$

As a result, for $1 \leq j \leq R$, let $J_t^j(\cdot)$ be the optimal cost-to-go that can be achieved by a restricted control space, in which expediting from *stage* $j + 1, j + 2, \cdots, R$ in time period t is not allowed. For a sequential system, we have

$$
\begin{aligned}
J_t(v_0, \bar{v}_1, \bar{l}_1) = \min\{ & J_t^1(x^0, \bar{v}_1, \bar{l}_1), \\
& d_{l_1}v_1 + J_t^2(x^1, 0, \bar{v}_2, 0, \bar{l}_2), \\
& d_{l_1}v_1 + d_{l_2}v_2 + J_t^3(x^2, \bar{0}^2, \bar{v}_3, \bar{0}^2, \bar{l}_3), \\
& \cdots, \\
& \sum_{i=1}^{R-1} d_{l_i}v_i + J_t^R(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1}) \},
\end{aligned}
\tag{5.2}
$$

where

$$J_t^1(x^0, \bar{v}_1, \bar{l}_1) = \min_{\substack{x^0 \leq y_1 \leq x^1 \\ z \geq x^{R-1}}} \{d_{l_1}(y_1 - x^0) + c(z - x^{R-1}) + L(y_1)$$

$$+ E[d_{M(l_1,W)}](x^1 - y_1) + E[J_{t+1}(NS_1)]\},$$

(5.3)

and

$$J_t^i(x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i) = \min_{\substack{x^{i-1} \leq y_i \leq x^i \\ z \geq x^{R-1}}} \{d_{l_i}(y_i - x^{i-1}) + c(z - x^{R-1}) + L(y_i)$$

$$+ E[J_{t+1}(NS_i)]\},$$

(5.4)

for $i > 1$.

5.4 Optimal Policies for Sequential Systems

In this section, we focus on identifying optimal policies for sequential systems with order expiry.

5.4.1 Preliminaries

To facilitate the analysis of the system dynamics, let us define the following set of probabilities:

- $p(l_i) = prob(M(l_i, W) > 0)$, and

- $p(l_i, l_{i+1}) = prob(M(l_i, W) = 0$ and $M(l_{i+1}, W) > 0)$ for all i.

Note that $p(\cdot)$'s are deterministic functions, and if $l_i = l_{i+1}$, then $p(l_i, l_{i+1}) = 0$. We first present the following lemma.

Lemma 15. *We have $p(l_i) + p(l_i, l_{i+1}) = p(l_{i+1})$, for all i.*

Proof. By assumption, regular orders do not cross in time. Since $p(l_i) = prob(M(l_i, W) > 0) = prob(M(l_i, W) > 0$ and $M(l_{i+1}, W) > 0)$, we have $p(l_i) + p(l_i, l_{i+1}) = prob(M(l_i, W) > 0$ and $M(l_{i+1}, W) > 0) + prob(M(l_i, W) = 0$ and $M(l_{i+1}, W) > 0) = prob(M(l_{i+1}, W) > 0)$ $= p(l_{i+1})$ for all i and any value of l_i. $\qquad\square$

For ease of exposition, let $M(\bar{l}_i, W) = (M(l_i, W), M(l_{i+1}, W), \cdots, M(l_R, W))$.

The future cost in (5.4), which corresponds to the state $(x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i)$ of the current time period, is

$$
\begin{aligned}
E[J_{t+1}(NS_i)] = {} & p(l_i)E[J_{t+1}(y_i - D, \bar{0}^{i-2}, x^i - y_i, \bar{v}_{i+1}, u, \bar{0}^{i-2}, M(\bar{l}_i, W))|M(l_i, W) > 0] \\
& + p(l_i, l_{i+1})E[J_{t+1}(x^i - D, \bar{0}^{i-1}, \bar{v}_{i+1}, u, \bar{0}^{i-1}, M(\bar{l}_{i+1}, W))|M(l_{i+1}, W) > 0] \\
& + p(l_{i+1}, l_{i+2})E[J_{t+1}(x^{i+1} - D, \bar{0}^i, \bar{v}_{i+2}, u, \bar{0}^i, M(\bar{l}_{i+2}, W))|M(l_{i+2}, W) > 0] \\
& \vdots \\
& + p(l_{R-1}, l_R)E[J_{t+1}(x^{R-1} - D, \bar{0}^{R-2}, u, \bar{0}^{R-2}, M(l_R, W))|M(l_R, W) > 0] \\
& + (1 - p(l_R))E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})],
\end{aligned}
$$

$$(5.5)$$

where u is the regular ordering amount: i.e. $u = z - x^{R-1}$. Note the conditional expectations in (5.5). Let us define the following recursive functions:

$$
\begin{aligned}
f_{1,t}(y_1, l_1) &= d_{l_1}y_1 + L(y_1) - p(l_1)E[d_{M(l_1,W)}(y_1 - D)|M(l_1, W) > 0] \\
&= d_{l_1}y_1 + L(y_1) - E[d_{M(l_1,W)}(y_1 - D)], \text{ and} \\
f_{i,t}(y_i, l_i) &= d_{l_i}y_i + L(y_i) + p(l_i)E[S^1_{i-1,t+1}(y_i - D, M(l_i, W))|M(l_i, W) > 0] \\
&= d_{l_i}y_i + L(y_i) + E[S^1_{i-1,t+1}(y_i - D, M(l_i, W))], \text{ for } i > 1.
\end{aligned}
$$

Let us also define

$$
\begin{aligned}
S^0_{i,t}(l_i) &= p(l_i)E[S^0_{i-1,t+1}(M(l_i, W))|M(l_i, W) > 0] + a_{i,t}(l_i) \\
&= E[S^0_{i-1,t+1}(M(l_i, W))] + a_{i,t}(l_i), \\
S^1_{i,t}(x^{i-1}, l_i) &= g_{i,t}(x^{i-1}, l_i) - d_{l_i}x^{i-1} \\
S^2_{1,t}(x^1, l_1) &= h_{1,t}(x^1, l_1) - L(x^1) + p(l_1)E[d_{M(l_1,W)}|M(l_1, W) > 0]x^1 \\
&= h_{1,t}(x^1, l_1) - L(x^1) + E[d_{M(l_1,W)}]x^1, \text{ and} \\
S^2_{i,t}(x^i, l_i) &= h_{i,t}(x^i, l_i) - L(x^i) + p(l_i)E[S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0] \\
&= h_{i,t}(x^i, l_i) - L(x^i) + E[S^2_{i-1,t+1}(x^i - D, M(l_i, W))],
\end{aligned}
$$

for $1 < i \leq R - 1$, and $S_{0,t}^0(\cdot) = S_{0,t}^1(\cdot) = S_{0,t}^2(\cdot) = 0$ for all t, and $S_{i,T+1}^0(\cdot) = S_{i,T+1}^1(\cdot) = S_{i,T+1}^2(\cdot) = 0$ for all i. Here, $a_{i,t}$, $g_{i,t}$, and $h_{i,t}$ are defined according to Lemma 8 with respect to $f_{i,t}$. This lemma is applied for any fixed l_i and thus $g_{i,t}$ and $h_{i,t}$ depend on l_i, while $a_{i,t}$ now becomes a function of l_i. Starting from the last time period T, functions $f_{i,t}$ and $S_{i,t}^j$ can be obtained recursively. It is easy to check for all i and t that $f_{i,t}(y_i, l_i)$ is convex in y_i for a given l_i if the system is sequential. Also, $S_{i,t}^0(l_i) + S_{i,t}^1(x, l_i) + S_{i,t}^2(x, l_i) = 0$ for every x and l_i.

Let us denote by $y_{i,t}^*(l_i)$ a minimizer of $f_{i,t}(y_i, l_i)$: $y_{i,t}^*(l_i) \in \arg\min_{y_i} f_{i,t}(y_i, l_i)$. The following theorem is an important property of $f_{i,t}(y_i, l_i)$ for sequential systems.

Theorem 9. *For sequential systems, the following holds true.*

a. For any given j, $y_{i,t}^*(j)$ is nonincreasing in i.

b. For any given i, $y_{i,t}^*(j)$ is nonincreasing in j.

Proof. In this proof, we use Lemmas 17, 18, 19, and 20 from Appendix. Let us first consider part (a). We first rewrite

$$f_{1,t}(y_1, j) = d_j y_1 + L(y_1) - E[d_{M(j,W)}] y_1 + E[d_{M(j,W)} D]$$
$$= (d_j - E[d_{M(j,W)}]) y_1 + L(y_1) + E[d_{M(j,W)} D],$$

and

$$f_{i,t}(y_i, j) = d_j y_i + L(y_i) + E[S_{i-1,t+1}^1(y_i - D, M(j,W))]$$
$$= d_j y_i + L(y_i) + E[g_{i-1,t+1}(y_i - D, M(j,W)) - d_{M(j,W)}(y_i - D)]$$
$$= (d_j - E[d_{M(j,W)}]) y_i + L(y_i) + E[g_{i-1,t+1}(y_i - D, M(j,W))]$$
$$+ E[d_{M(j,W)} D],$$

for $i > 1$. Also,

$$f_{i+1,t}(y_{i+1}, j) = (d_j - E[d_{M(j,W)}]) y_{i+1} + L(y_{i+1}) + E[g_{i,t+1}(y_{i+1} - D, M(j,W))]$$
$$+ E[d_{M(j,W)} D].$$

We prove that $\partial f_{i,t}(y, j) \leq \partial f_{i+1,t}(y, j)$ for all j. We use induction on t. The base case is

78

when $t = T$ where $f_{i,T}(y,j) = f_{i+1,T}(y,j) = d_j y + L(y)$ for all i. Assuming $\partial f_{i,t+1}(y,j) \leq \partial f_{i+1,t+1}(y,j)$ for a fixed $t < T$, we obtain $\partial g_{i,t+1}(y,j) \leq \partial g_{i+1,t+1}(y,j)$. Therefore, we have $\partial f_{i,t}(y,j) \leq \partial f_{i+1,t}(y,j)$, and $y^*_{i,t}(j) \geq y^*_{i+1,t}(j)$. The proof of part (a) is completed.

Next, we prove part (b) that $\partial f_{i,t}(y,j) \leq \partial f_{i,t}(y,j+1)$ for all i. We use induction on t. The base case is when $t = T$, where $f_{i,T}(y,j) = d_j y + L(y)$. Therefore, we have $\partial f_{i,T}(y,j) \leq \partial f_{i,T}(y,j+1)$ for all i due to $d_j \leq d_{j+1}$. Now assume $\partial f_{i,t+1}(y,j) \leq \partial f_{i,t+1}(y,j+1)$ for all i and a fixed $t < T$. We have

$$f_{i,t}(y,j) = (d_j - E[d_{M(j,W)}])y + L(y) + E[g_{i-1,t+1}(y - D, M(j,W))] + E[d_{M(j,W)}D],$$

$$f_{i,t}(y,j+1) = (d_{j+1} - E[d_{M(j+1,W)}])y + L(y) + E[g_{i-1,t+1}(y - D, M(j+1,W))]$$
$$+ E[d_{M(j+1,W)}D].$$

Because of the induction hypothesis and the definition of sequential systems, we have $\partial f_{i,t}(y,j) \leq \partial f_{i,t}(y,j+1)$ and thus $y^*_{i,t}(j) \geq y^*_{i,t}(j+1)$. Note that $M(j,w) \leq M(j+1,w)$ for any realization of w of W, and the monotonicity of $\partial g_{i-1,t+1}$ in the second variable follows from the induction hypothesis on $\partial f_{i-1,t+1}$. In other words, if $\partial f_{i-1,t+1}$ is monotone in the second variable then $\partial g_{i-1,t+1}$ is also monotone. The proof is completed. $\qquad\square$

The following lemma is used later in the proof of the optimal policy.

Lemma 16. *For such j and $w \in \mathbf{W}$ that $M(j,w) > 0$, $p(j)S^2_{i-1,t}(x, M(j,w)) + g_{i,t}(x,j)$ is convex in x for all i.*

Proof. We prove the convexity of

$$g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$$
$$= g_{i,t}(x,j) + p(j)\{h_{i-1,t}(x, M(j,w)) - L(x) \qquad\qquad (5.6)$$
$$+ p(M(j,w))E[S^2_{i-2,t+1}(x - D, M^2(j,W))|M^2(j,W) > 0]\},$$

for all w. We use induction. The base case of $t = T + 1$ is obvious. We assume convexity at $t + 1$ as the induction hypothesis. First, we show convexity of (5.6) when $x \leq y^*_{i,t}(M(j,w))$ and $x \geq y^*_{i,t}(j)$.

- When $x \leq y^*_{i,t}(M(j,w))$, we have $h_{i-1,t}(x, M(j,w)) = f_{i-1,t}(x, M(j,w)) - a_{i-1,t}(M(j,w)) = d_{M(j,w)}x + L(x) + p(M(j,w))E[S^1_{i-2,t+1}(x-D, M^2(j,W))|M^2(j,W) > 0] - a_{i-1,t}(M(j,w))$.

79

By using the fact that $S^1(x,j) + S^2(x,j) = -S^0(j)$, it is easy to see that $g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$ is convex in x for $x \le y^*_{i,t}(M(j,w))$.

- When $x \ge y^*_{i,t}(j)$, we have $g_{i,t}(x,j) = f_{i,t}(x,j) - a_{i,t}(j) = d_j x + L(x) + p(j)E[S^1_{i-1,t+1}(x - D, M(j,w))] - a_{i,t}(j)$. Since $S^1_{i-1,t+1}(x - D, M(j,w)) = g_{i-1,t+1}(x - D, M(j,w)) - d_{M(j,w)}(x-D)$, and $p(M(j,w))E[S^2_{i-2,t+1}(x-D, M^2(j,W))|M^2(j,W) > 0] + g_{i-1,t+1}(x - D, M(j,w))$ is convex by induction hypothesis, it is also easy to see convexity for $x \ge y^*_{i,t}(j)$.

Since $y^*_{i,t}(j) \le y^*_{i,t}(M(j,w))$, we consider the following two cases: $y^*_{i,t}(j) < y^*_{i,t}(M(j,w))$ and $y^*_{i,t}(j) = y^*_{i,t}(M(j,w))$. If $y^*_{i,t}(j) < y^*_{i,t}(M(j,w))$, then $g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$ is convex since it is convex for two partially overlapping intervals. Otherwise, if $y^* = y^*_{i,t}(j) = y^*_{i,t}(M(j,w))$, then

$$g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$$
$$= g_{i,t}(x,j) + p(j)\{-L(x) + p(M(j,w))E[S^2_{i-2,t+1}(x - D, M^2(j,W))|M^2(j,W) > 0]\}$$

for $x \ge y^*$, and

$$g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$$
$$= p(j)\{h_{i-1,t}(x, M(j,w)) - L(x) + p(M(j,w))E[S^2_{i-2,t+1}(x - D, M^2(j,W))|M^2(j,W) > 0]\}$$

for $x \le y^*$. Given the convexity of the function for each interval and the convexity of $g_{i,t}(x,j) + p(j)h_{i-1,t}(x, M(j,w))$, we conclude that $g_{i,t}(x,j) + p(j)S^2_{i-1,t}(x, M(j,w))$ is convex also in this case. $\square$

5.4.2 Optimal Policies

For sequential systems we have the following theorem, which is the key result in this chapter.

Theorem 10. a. *The base stock policy with respect to the corresponding echelon stock x^{i-1} is optimal for expediting from stage i. Also, the base stock policy with respect to the inventory position x^{R-1} is optimal for regular ordering.*

b. *Function $p(l_R)E[S^2_{R-1,t}(z - D, M(l_R,w))|M(l_R,w) > 0] + E[J_t(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]$ is convex in z.*

80

c. For $1 \leq i \leq R - 1$, we have $J_t(x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i) - J_k(x^i, \bar{0}^i, \bar{v}_{i+1}, \bar{0}^i, \bar{l}_{i+1}) = S_{i,t}^0(l_i) + S_{i,t}^1(x^{i-1}, l_i) + S_{i,t}^2(x^i, l_i)$.

Proof. We prove parts (a), (b), and, (c) concurrently by induction on t. In the base case $t = T + 1$, the optimal expediting and regular ordering policies are null. Also (b) and (c) hold obviously when $t = T + 1$. Now we proceed to the induction step.

We prove that part (a) holds at time period t. First consider (5.5). By repeatedly applying part (c) with time period $t + 1$, which holds by the induction hypothesis, we have

$$
\begin{aligned}
E[J_{t+1}(NS_i)] = {} & p(l_i)E[S_{i-1,t+1}^0(M(l_i, W)) + S_{i-1,t+1}^1(y_i - D, M(l_i, W)) \\
& + S_{i-1,t+1}^2(x^i - D, M(l_i, W))|M(l_i, W) > 0] \\
& + \cdots \text{ terms of only state variables } \cdots \\
& + p(l_R)E[J_{t+1}(x^{R-1} - D, \bar{0}^{R-2}, u, \bar{0}^{R-2}, M(l_R, W))|M(l_R, W) > 0] \\
& + (1 - p(l_R))E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})],
\end{aligned}
$$

where Lemma 15 is used. This is again rearranged to the following by using part (c):

$$
\begin{aligned}
E[J_{t+1}(NS_i)] = {} & p(l_i)E[S_{i-1,t+1}^0(M(l_i, W)) + S_{i-1,t+1}^1(y_i - D, M(l_i, W)) \\
& + S_{i-1,t+1}^2(x^i - D, M(l_i, W))|M(l_i, W) > 0] \\
& + \cdots \text{ terms of only state variables } \cdots \\
& + p(l_R)E[S_{R-1,t+1}^0(M(l_R, W)) + S_{R-1,t+1}^1(x^{R-1} - D, M(l_R, W)) \\
& + S_{R-1,t+1}^2(z - D, M(l_R, W))|M(l_R, W) > 0] + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})].
\end{aligned}
$$

Let us denote by *OT* the terms that contain only state variables. Then,

$$
\begin{aligned}
E[J_{t+1}(NS_i)] = {} & p(l_i)E[S_{i-1,t+1}^1(y_i - D, M(l_i, W))|M(l_i, W) > 0] \\
& + p(l_R)E[S_{R-1,t+1}^2(z - D, M(l_R, W))|M(l_R, W) > 0] \qquad (5.7) \\
& + E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})] + OT.
\end{aligned}
$$

Plugging (5.7) into (5.3) and (5.4) yields

$$J_t^i(x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i) = \min_{x^{i-1} \leq y_i \leq x^i} f_{i,t}(y_i, l_i)$$

$$+ \min_{z \geq x^{R-1}} \{cz + p(l_R) E[S_{R-1,t+1}^2(z - D, M(l_R, W)) | M(l_R, W) > 0]$$

$$+ E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\} + OT,$$

$$(5.8)$$

for $i < R$. For $i = R$, we have

$$J_t^R(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1}) = \min_{x^{R-1} \leq y_R \leq z} \{f_{R,t}(y_R, l_R) + cz$$

$$+ p(l_R) E[S_{R-1,t+1}^2(z - D, M(l_R, W)) | M(l_R, W) > 0]$$

$$+ E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\} \qquad (5.9)$$

$$+ p(l_R) E[S_{R-1,t+1}^0(M(l_R, W)) | M(l_R, W) > 0]$$

$$- d_{l_R} x^{R-1} - c x^{R-1}.$$

Note that $l_R = \bar{K}$. By applying Lemma 9 from Chapter 3, we have

$$J_t^R(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1}) = \min_{x^{R-1} \leq z} \{h_{R,t}(z, l_R) + cz$$

$$+ p(l_R) E[S_{R-1,t+1}^2(z - D, M(l_R, W)) | M(l_R, W) > 0]$$

$$+ E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\} + a_{R,t}(l_R) \qquad (5.10)$$

$$+ p(l_R) E[S_{R-1,t+1}^0(M(l_R, W)) | M(l_R, W) > 0]$$

$$+ g_{R,t}(x^{R-1}, l_R) - d_{l_R} x^{R-1} - c x^{R-1},$$

which is equal to

$$J_t^R(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1}) = \min_{x^{R-1} \leq z} \{h_{R,t}(z, l_R) + cz$$

$$+ p(l_R) E[S_{R-1,t+1}^2(z - D, M(l_R, W)) | M(l_R, W) > 0] \qquad (5.11)$$

$$+ E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\} - S_{R,t}^2(x^{R-1}, l_R) - c x^{R-1}.$$

Therefore, optimal expediting follows the base stock policy from (5.9) with the base stock level given by

$$\min_{x^{i-1} \leq y_i \leq x^i} f_{i,t}(y_i, l_i).$$

The optimal regular ordering policy is the base stock policy with the base stock level z^* determined from (5.11) by

$$\min_{z \geq x^{R-1}} \{h_{R,t}(z, l_R) + cz + p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R, W))|M(l_R, W) > 0]$$

$$+ E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\},$$

for any i. Since $p(l_R)E[S^2_{R-1,t+1}(z-D, M(l_R, W))|M(l_R, W) > 0]+E[J_{t+1}(z-D, \bar{0}^{R-1}, \bar{0}^{R-1})]$ is convex by the induction hypothesis of part (b), the optimal regular ordering policy is well defined. Note that we use (5.11) instead of (5.8) in determining the optimal regular ordering quantity by the same reason as in Chapter 3. This completes the induction step of part (a).

Next, we prove that part (b) holds at time period t. Adding

$$p(l_R)E[S^2_{R-1,t}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0]$$

to both sides of (5.10), we have

$$p(l_R)E[S^2_{R-1,t}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0] + J^R_t(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1})$$

$$= \min_{x^{R-1} \leq z} \{h_{R,t}(z, l_R) + cz + p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R, W))|M(l_R, W) > 0]$$

$$+ J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\} + p(l_R)E[S^0_{R-1,t+1}(M(l_R, W))|M(l_R, W) > 0]$$

$$+ g_{R,t}(x^{R-1}, l_R) + p(l_R)E[S^2_{R-1,t}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0]$$

$$+ a_{R,t}(l_R) - d_{l_R}x^{R-1} - cx^{R-1}.$$

By the induction hypothesis, $p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R, w))|M(l_R, w) > 0]+E[J_{t+1}(z-D, \bar{0}^{R-1}, \bar{0}^{R-1})]$ is convex in z. Therefore,

$$\min_{x^{R-1} \leq z} \{h_{R,t}(z, l_R) + cz + p(l_R)E[S^2_{R-1,t+1}(z - D, M(l_R, W))|M(l_R, W) > 0]$$

$$+ J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]\}$$

is convex in x^{R-1}. Also, $g_{R,t}(x^{R-1}, l_R) + p(l_R)E[S^2_{R-1,t}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0]$ is convex in x^{R-1} by Lemma 16. All the other terms are either constant or linear in x^{R-1}.

Therefore,

$$p(l_R)E[S^2_{R-1,t+1}(x^{R-1}, M(l_R, W))|M(l_R, W)) > 0] + J^R_t(x^{R-1}, \bar{0}^{R-1}, \bar{0}^{R-1})$$

is convex in x^{R-1}, and the proof of part (b) is completed.

It remains to prove part (c). Consider the following two states: $A_i = (x^{i-1}, \bar{0}^{i-1}, \bar{v}_i, \bar{0}^{i-1}, \bar{l}_i)$ and $A_{i+1} = (x^i, \bar{0}^i, \bar{v}_{i+1}, \bar{0}^i, \bar{l}_{i+1})$. Because of the sequential property of the system, we have to first expedite from stage i according to the base stock policy of part (a).

We now examine the following three cases for $i = 1$. Note that NS_1 and NS_2 are the same when we only expedite from stage 1, because of mandatory expediting.

Case 1. Let first $y^*_1(l_1) < x^0$. In this case, no expediting is necessary for both A_1 and A_2. Because of mandatory expediting with probability $p(l_1)$, we have

$$J_t(A_1) = L(x^0) + p(l_1)E[d_{M(l_1,W)}|M(l_1, W) > 0](x^1 - x^0)$$
$$+ \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_1)]\}$$

and

$$J_t(A_2) = L(x^1) + \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_2)]\}.$$

Thus, $J_t(A_1) - J_t(A_2) = L(x^0) + p(l_1)E[d_{M(l_1,W)}|M(l_1, W) > 0](x^1 - x^0) - L(x^1)$.

Case 2. If $x^0 \leq y^*_1(l_1) < x^1$, we have

$$J_t(A_1) = d_{l_1}(y^*_1(l_1) - x^0) + L(y^*_1(l_1)) + p(l_1)E[d_{M(l_1,W)}|M(l_1, W) > 0](x^1 - y^*_1(l_1))$$
$$+ \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_1)]\}$$

and

$$J_t(A_2) = L(x^1) + \min_{z \geq x^{R-1}} \{c(z - x^{R-1}) + E[J_{t+1}(NS_2)]\}.$$

Thus,

$$J_t(A_1) - J_t(A_2) = d_{l_1}(y^*_1(l_1) - x^0) + L(y^*_1(l_1))$$
$$+ p(l_1)E[d_{M(l_1,W)}|M(l_1, W) > 0](x^1 - y^*_1(l_1)) - L(x^1).$$

Case 3. Finally, let $y_1^*(l_1) \geq x^1$. In this case, we have to expedite everything in stage 1, thus $J_t(A_1) - J_t(A_2) = d_{l_1}(x^1 - x^0)$.

The three cases can be summarized as

$$J_t(A_1) - J_t(A_2) = a_{1,t}(l_1) + g_{1,t}(x^0, l_1) + h_{1,t}(x^1, l_1)$$

$$- d_{l_1}x^0 + p(l_1)E[d_{M(l_1,W)}|M(l_1,W) > 0]x^1 - L(x^1)$$

$$= S_{1,t}^0(l_1) + S_{1,t}^1(x^0, l_1) + S_{1,t}^2(x^1, l_1).$$

Next, consider the following three cases for $i > 1$.

Case 1. Let $y_i^*(l_i) < x^{i-1}$, and thus no expediting is necessary for both A_i and A_{i+1}. Therefore $J_t(A_i) = L(x^{i-1}) + \min_{z \geq x^{R-1}}\{c(z - x^{R-1}) + E[J_{t+1}(NS_i)]\}$ and $J_t(A_{i+1}) = L(x^i) + \min_{z \geq x^{R-1}}\{c(z - x^{R-1}) + E[J_{t+1}(NS_{i+1})]\}$. Let us examine NS_i by replacing y_i with x^{i-1} in (5.5). We obtain

$$E[J_{t+1}(NS_i)]$$

$$= p(l_i)E[S_{i-1,t+1}^0(M(l_i,W)) + S_{i-1,t+1}^1(x^{i-1} - D, M(l_i,W))$$

$$+ S_{i-1,t+1}^2(x^i - D, M(l_i,W))|M(l_i,W) > 0]$$

$$+ (p(l_i) + p(l_i, l_{i+1}))E[J_{t+1}(x^i - D, \bar{0}^{i-1}, \bar{v}_{i+1}, u, \bar{0}^{i-1}, M(\bar{l}_{i+1}, W))|M(l_{i+1}, W) > 0]$$

$$+ p(l_{i+1}, l_{i+2})E[J_{t+1}(x^{i+1} - D, \bar{0}^i, \bar{v}_{i+2}, u, \bar{0}^i, M(\bar{l}_{i+2}, W))|M(l_{i+2}, W) > 0]$$

$$\vdots$$

$$+ (1 - p(l_R))E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})]$$

$$= p(l_i)E[S_{i-1,t+1}^0(M(l_i,W)) + S_{i-1,t+1}^1(x^{i-1} - D, M(l_i,W))$$

$$+ S_{i-1,t+1}^2(x^i - D, M(l_i,W))|M(l_i,W) > 0]$$

$$+ p(l_{i+1})E[J_{t+1}(x^i - D, \bar{0}^{i-1}, \bar{v}_{i+1}, u, \bar{0}^{i-1}, M(\bar{l}_{i+1}, W))|M(l_{i+1}, W) > 0]$$

$$+ p(l_{i+1}, l_{i+2})E[J_{t+1}(x^{i+1} - D, \bar{0}^i, \bar{v}_{i+2}, u, \bar{0}^i, M(\bar{l}_{i+2}, W))|M(l_{i+2}, W) > 0]$$

$$\vdots$$

$$+ (1 - p(l_R))E[J_{t+1}(z - D, \bar{0}^{R-1}, \bar{0}^{R-1})],$$

where we applied Lemma 15.

85

Therefore, we have

$$E[J_{t+1}(NS_i)] - E[J_{t+1}(NS_{i+1})] = p(l_i)E[S^0_{i-1,t+1}(M(l_i, W))$$
$$+ S^1_{i-1,t+1}(x^{i-1} - D, M(l_i, W)) + S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0].$$

Thus,

$$J_t(A_i) - J_t(A_{i+1}) = L(x^{i-1}) - L(x^i) + p(l_i)E[S^0_{i-1,t+1}(M(l_i, W))$$
$$+ S^1_{i-1,t+1}(x^{i-1} - D, M(l_i, W)) + S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0].$$

Case 2. If $x^{i-1} \leq y_i^*(l_i) < x^i$, we obtain

$$J_t(A_i) = d_{l_i}(y_i^*(l_i) - x^{i-1}) + L(y_i^*(l_i)) + \min_{z \geq x^{R-1}}\{c(z - x^{R-1}) + E[J_{t+1}(NS_i)]\}$$

and

$$J_t(A_{i+1}) = L(x^i) + \min_{z \geq x^{R-1}}\{c(z - x^{R-1}) + E[J_{t+1}(NS_{i+1})]\}.$$

Similarly to the previous case, we have

$$E[J_{t+1}(NS_i)] - E[J_{t+1}(NS_{i+1})] = p(l_i)E[S^0_{i-1,t+1}(M(l_i, W))$$
$$+ S^1_{i-1,t+1}(y_i(l_i) - D, M(l_i, W)) + S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0].$$

Therefore,

$$J_t(A_1) - J_t(A_2) = d_{l_i}(y_i^*(l_i) - x^{i-1}) + L(y_i^*(l_i)) - L(x^i) + p(l_i)E[S^0_{i-1,t+1}(M(l_i, W))$$
$$+ S^1_{i-1,t+1}(y_i(l_i) - D, M(l_i, W)) + S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0].$$

Case 3. If $y_1^*(l_i) \geq x^i$, then we simply have $J_t(A_1) - J_t(A_2) = d_{l_i}(x^i - x^{i-1})$.

The three cases can be summarized as

$$J_t(A_i) - J_t(A_{i+1}) = a_{i,t}(l_i) + g_{i,t}(x^{i-1}, l_i) + h_{i,t}(x^i, l_i) - d_{l_i}x^{i-1} - L(x^i)$$
$$+ p(l_i)E[S^0_{i-1,t+1}(M(l_i, W))$$
$$+ S^2_{i-1,t+1}(x^i - D, M(l_i, W))|M(l_i, W) > 0]$$
$$= S^0_{i,t}(l_i) + S^1_{i,t}(x^{i-1}, l_i) + S^2_{1,t}(x^i, l_i).$$

The proof of part (c) is thus completed. □

5.4.3 Illustration of the Optimal Policy

Part (a) of Theorem 10 says that the optimal regular ordering policy follows the base stock policy with respect to the inventory position. Compared to the result in Chapter 3, though the base stock level is different, the optimal regular ordering policy remains the same regardless of mandatory expediting. However, the optimal expediting policy is quite different with the introduction of mandatory expediting. We explain the optimal expediting policy described in Theorems 9 and 10 through the following illustration.

Part (a) of Theorem 9 states that there is monotonicity of expediting base stock levels across installations for the same age bins; see Figure 5-4.
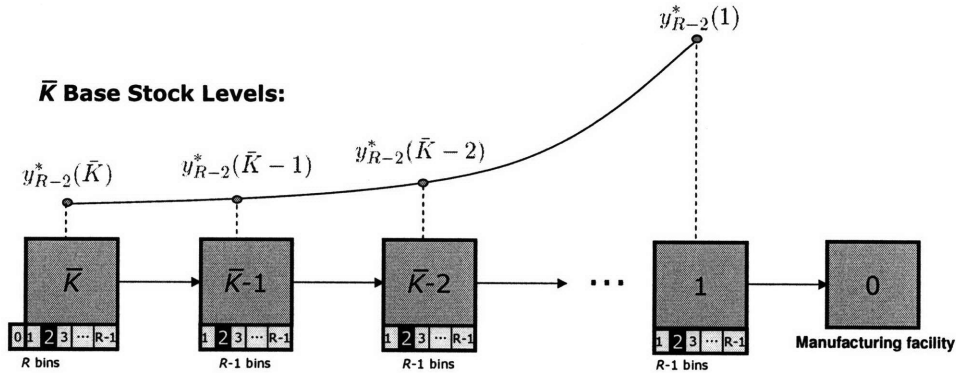

Figure 5-4: Monotonicity across installations of the same age

On the other hand, part (b) of Theorem 9 states that there is monotonicity of expediting base stock levels across all age bins in an installation; see Figure 5-5.

Considering both parts of Theorem 9, we do not have overall monotonic base stock levels for expediting as shown in Figure 5-6. At first, it appears that this nonmonotonicity
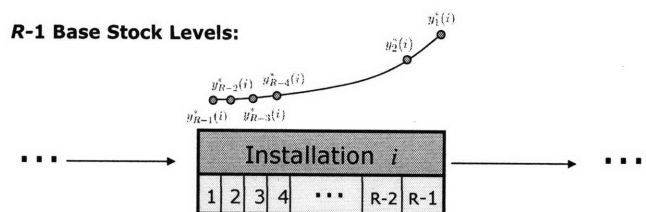
Figure 5-5: Monotonicity within an installation

contradicts the definition of sequential systems or Theorem 8, since order crossing in time might happen for expedited orders. However, if we consider only the expediting base stock
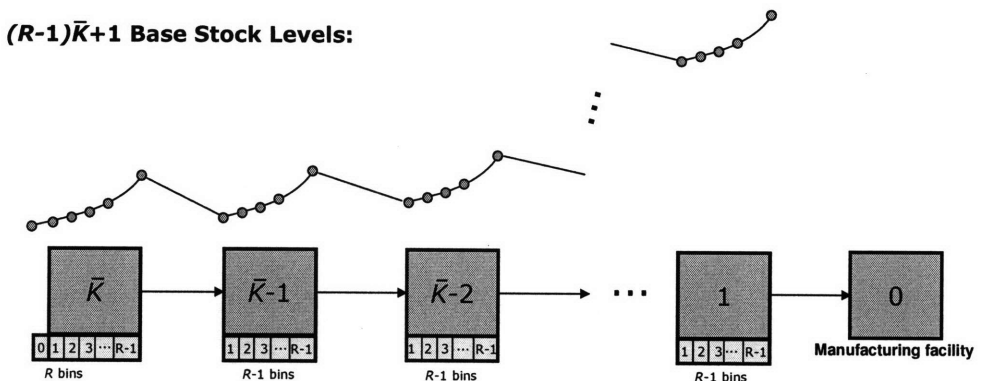

Figure 5-6: Non-monotonicity considering all age bins in all installations

levels for nonempty age bins in all installations, then it becomes obvious that order crossing does not happen for sequential systems, and hence there is no contradiction. The reason is the following. Since Assumption 1 guarantees that regular movements do not cross in time, an order that is placed earlier should be closer to the manufacturing facility. Therefore, parts (a) and (b) of Theorem 9 combined indicate that there is monotonicity of the expediting base stock levels for nonempty age bins at any moment, as shown in Figure 5-7.

Finally, monotonicity of the base stock levels for nonempty age bins and part (a) of Theorem 10 reveal the simple structure of the optimal expediting policy for sequential systems as follows. The echelon stock is nondecreasing as the age bin gets farther away from the manufacturing facility, since it is the sum of nonnegative numbers. At the same time we have monotonicity of the expediting base stock levels. Therefore, there can be at most one intersection point between the echelon stock and the base stock level profiles. Part (a) of Theorem 10 implies to expedite everything up to this intersection; see Figure 5-8.

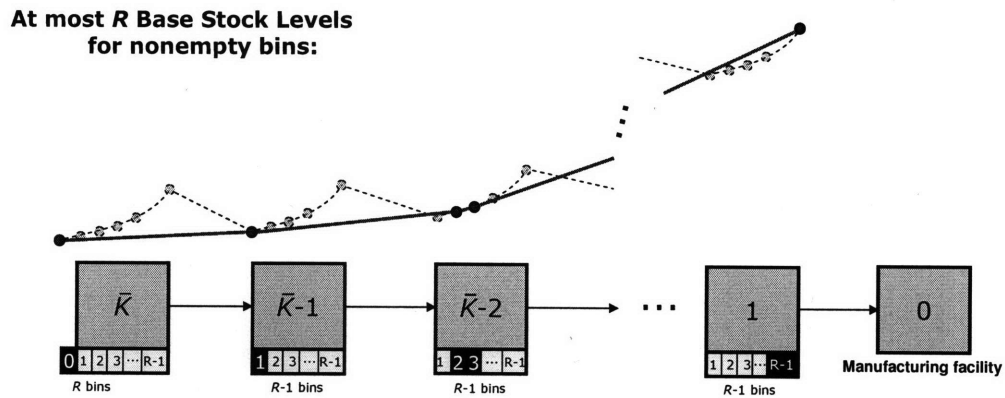**At most *R* Base Stock Levels for nonempty bins:**

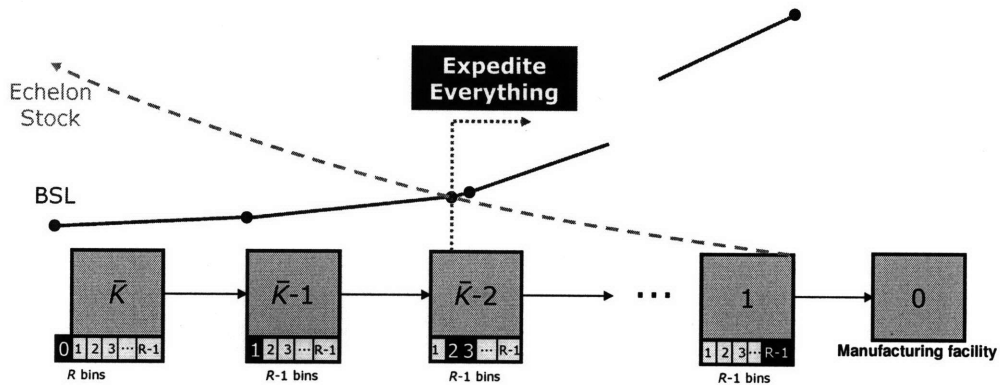Figure 5-7: Monotonicity of nonempty age bins in all installations

Figure 5-8: The simple structure of the optimal expediting policy

As the expiry date increases to infinity, the optimal expediting policy just illustrated converges to the optimal expediting policy in Chapter 3. This is due to the fact that the expediting base stock levels are getting closer to each other as the expiry date increases, and they eventually converge to a single value for each installation. After all, we have only the same number of unique expediting base stock levels as the number of installations. Since they are monotonic, we have the optimal expediting policy as described in Chapter 3. In this sense, the model with expiry dates is the most general model among those discussed in this thesis.

Chapter 6

Conclusion

In order to have competitive advantage, a supply chain must be agile in responding to short-term changes in demand or supply. In this research, we focus at an operational process to increase the agility of complex supply chains. In particular, we utilize a premium transportation method, i.e. by air, in addition to a normal transportation method, i.e. by ground. When high uncertainties in lead time and demand occur, expediting outstanding orders by air can be a viable option for many companies. Through expediting, a firm can reduce excessive backlogging costs as well as excessive holding costs in facing uncertainties associated with demand and lead time. However, a lower operational cost can be achieved only by wisely utilizing the expediting option. Thus the main goal of this research is to find the optimal policy of expediting and regular ordering.

Deterministic Lead Time Model

In Chapter 2, we study the optimal regular ordering and expediting policy for a single-item, periodic-review inventory system with a deterministic lead time. We consider the deterministic lead time model since it forms a good starting point to study the stochastic lead time model presented in later chapters. We introduce the concept of sequential systems, and find an optimal policy of expediting and regular ordering for such systems. The analysis is possible since sequential systems do not allow order crossing in time under optimal control. Sequential systems are easily identifiable using the expediting costs.

The optimal regular ordering policy for sequential systems is the base stock policy with respect to the echelon inventory x^{L-1}, and the structure of the optimal expediting policy

is to expedite everything up to a certain installation, partially from the next installation according to the corresponding base stock level, and nothing beyond. This optimal policy is simple and thus easy to implement. The corresponding base stock levels are defined recursively and are easily computable. Our mathematical approach is novel, and it shows decomposability of the optimal cost-to-go.

Stochastic Lead Time Model

We extend the deterministic lead time model in Chapter 3. In particular, we derive the optimal policy for expediting and regular ordering of a stochastic lead time model with multiple intermediate installations. Since in general the model exhibits complex and nonintuitive policies, we again confine our interest to a class of sequential systems defined by conditions on expediting costs and movement patterns of regular orders. In sequential systems, regular as well as expedited orders do not cross in time. The concept of sequential systems is more general than the corresponding concept in Chapter 2. For sequential systems, the optimal policy for regular ordering is the base stock policy with respect to the inventory position, and the optimal policy for expediting from an installation is the base stock policy with respect to the echelon stock of the downstream installations.

Song and Zipkin (1996) find that the optimal regular ordering policy does not require any state variable information, and that the only relevant information is inventory position and the lead time distribution. Our results indicate that the optimal regular ordering policy as well as expediting requires the state information, since expediting and regular ordering have to be considered concurrently. In other words, the stochastic movement of regular orders in our model requires new information systems to capture the state information to enable optimal expediting decision making. RFID due to its relatively low deployment and maintenance cost can be used for this purpose. Without expediting, RFID, according to the result by Song and Zipkin (1996), does not bring additional value to inventory control. This is a clear confirmation that the value of RFID (or information in a broader sense) comes with new processes such as expediting. We need to actively use new information to unveil additional benefits, and this should be done through quantitative analysis as Lee and Özer (2007) also assert.

Non-Sequential Systems

The optimal policy of expediting and regular ordering are derived in Chapters 2 and 3 for sequential systems. However, in practice there are many supply chains that are not sequential. In Chapter 4, we study how to operate such nonsequential systems. For nonsequential system, we argue that the optimal policy is complex even for a simple system with lead time of two. The optimal regular ordering quantity as well as the optimal expediting quantity are functions of the state variables.

In view of this, we propose the extended heuristic, which is a natural extension of the optimal policy of sequential systems to nonsequential systems. The numerical study using the derivative method for three-installation systems reveals that this heuristic exhibits good performance for a much wider class of systems than the set of all sequential systems. More specifically, the extended heuristic achieves a local optimum for systems with nondecreasing expediting costs, which includes several practical systems. At the least, the extended heuristic gives us a valuable guide on operating a nonsequential supply chain with expediting options.

Raw Materials with an Expiry date

An extension to the stochastic lead time model is presented in Chapter 5. We again study supply chains with a supplier and a manufacturing facility. However, the outstanding orders are perishable and thus have to be expedited within finite time periods. In order words, the outstanding orders have deterministic expiry dates. The introduction of expiry dates imposes an additional constraint, *mandatory expediting*, on the control space that the outstanding orders close to the expiry date have to be expedited in order to avoid scrapping, i.e., spoilage.

We derive the optimal policy for expediting and regular ordering for sequential systems. The mandatory expediting brings several changes to the optimal expediting policy, while the optimal regular ordering policy remains similar. The optimal expediting policy identifies a number of expediting base stock levels, which are monotonic only for nonempty age bins across all installations. Because of this monotonicity, the optimal expediting policy is again simple and well-structured. We note that the optimal expediting policy with mandatory expediting converges to the optimal expediting policy without mandatory expediting as the

expiry date increases to infinity.

Future Research

An important but unaddressed situation in our research is the expiry constraint within the manufacturing facility. In the literature, this is also called *shelf life*. There is literature on the shelf life of various models that have deterministic lead times. Our research focuses on the stochastic lead time, and thus it is note overlapped with any previous work. We suggest that one of the most important tasks in the future is to extend our results to include the shelf life of the delivered orders in the manufacturing facility. Figure 6-1 summarized the previous research and our future direction.

		Expiry on Delivery	Shelf Life	
			At most 2 time periods	Multiple time periods
Stochastic Lead Time		Our research ⇨	No previous literature	No previous literature
Deterministic Lead Time	Nonzero	Not meaningful	Williams and Patuwo (1999) - Computational	No previous literature
	Zero	Not meaningful	Not meaningful	Nahmias(1975), Fries (1975) - Analytical

Figure 6-1: Future research

We do not think that this extension will be immediate, since the previous literature on the shelf life suggests the complexity of the optimal policy. However, we believe that we can have theoretical or practical solutions for this extension with the advancement of our understanding in complex supply chain systems.

Appendix A

Proofs and Additional Lemmas

Proof of Lemma 1. By adding

$$d_i - d_{i-1} \geq d_{i-1} - d_{i-2}$$

$$d_{i-1} - d_{i-2} \geq d_{i-2} - d_{i-3}$$

$$\vdots$$

$$d_{i-j+1} - d_{i-j} \geq d_{i-j} - d_{i-j-1} \, ,$$

we get $d_i - d_{i-j} \geq d_{i-1} - d_{i-j-1}$. In turn, we obtain $d_i - d_{i-j} \geq d_{i-1} - d_{i-j-1} \geq d_{i-2} - d_{i-j-2} \geq \cdots \geq d_j - d_0$. Therefore $d_i - d_{i-j} \geq d_j$, because $d_0 = 0$. $\qquad\square$

Proof of Lemma 3. Part (a): We prove this by induction on i. $f_{1,k}(x)$ is convex for all k, so is $S^1_{1,k}(x)$. Assume now that $f_{i,k}(x)$ and $S^1_{i,k}(x)$ are convex for a fixed i and all k. Then $f_{i+1,k}(x)$ and $S^1_{i+1,k}(x)$ are convex for all k because each one of them is a sum of convex functions.

Part (b): The proof is by induction on i. We have $S^0_{0,k} + S^1_{0,k}(x) + S^2_{0,k}(x) = 0$ for all k by definition. Assume for a fixed $i \geq 1$ and for all k that $S^0_{i-1,k} + S^1_{i-1,k}(x) + S^2_{i-1,k}(x) = 0$. Then, by definitions

$$
\begin{aligned}
S^0_{i,k} + S^1_{i,k}(x) + S^2_{i,k}(x) &= a_{i,k} + S^0_{i-1,k+1} + g_{i,k}(x) - d_i x + h_{i,k}(x) - L(x) + E[S^2_{i-1,k+1}(x - D)] \\
&= f_{i,k}(x) - d_i x - L(x) + S^0_{i-1,k+1} + E[S^2_{i-1,k+1}(x - D)] \qquad \text{(A.1)} \\
&= E[S^0_{i-1,k+1} + S^1_{i-1,k+1}(x - D) + S^2_{i-1,k+1}(x - D)] = 0.
\end{aligned}
$$

In (A.1), we use $a_{i,k} + g_{i,k}(x) + h_{i,k}(x) = \min_{x \leq y \leq x} f_{i,k}(y) = f_{i,k}(x)$. This completes the proof of this part.

Part (c): We first fix y and minimize over x as a function of y, then minimize over y. We obtain

$$
\begin{aligned}
\min_{b \leq x \leq y} \{f_1(x) + f_2(y)\} &= \min_{b \leq y} \{\{\min_{b \leq x \leq y} f_1(x)\} + f_2(y)\} \\
&= \min_{b \leq y} \{a_1 + g_1(b) + h_1(y) + f_2(y)\} \qquad \text{(A.2)} \\
&= a_1 + g_1(b) + \min_{b \leq y} \{h_1(y) + f_2(y)\},
\end{aligned}
$$

where, in (A.2), we use Lemma 2. $\qquad\qquad\square$

To prove Theorem 2, we first provide the following preliminary results. For a convex function $f : \mathbb{R} \to \mathbb{R}$, let $\partial f(x)$ be its subdifferential at x, which is a set. For two sets S_1 and S_2, we denote $S_1 \leq S_2$ if there exists $s_2 \in S_2$ such that $s_1 \leq s_2$ for any $s_1 \in S_1$, and there exists $s_1 \in S_1$ such that $s_1 \leq s_2$ for any $s_2 \in S_2$. The following lemmas can be proved by using elementary techniques.

Lemma 17. *Let f_1 and f_2 be convex functions. If $\partial f_1(x) \leq \partial f_2(x)$ for all $x \in \mathbb{R}$, then*

$$
\arg\min_x f_1(x) \geq \arg\min_x f_2(x).
$$

Lemma 18. *Let f_1 and f_2 be convex functions, and let g_1 and g_2 be their penalty functions as in Lemma 2. If $\partial f_1(x) \leq \partial f_2(x)$, then $\partial g_1(x) \leq \partial g_2(x)$.*

Lemma 19. *Let f_1, f_2, $\tilde{f}_1$, and $\tilde{f}_2$ be convex functions. If $\partial f_1(x) \leq \partial f_2(x)$ and $\partial \tilde{f}_1(x) \leq \partial \tilde{f}_2(x)$, then $\partial \{f_1 + \tilde{f}_2\}(x) \leq \partial \{f_2 + \tilde{f}_2\}(x)$.*

Lemma 20. *Let f_1 and f_2 be convex functions, and let $F_1(x) = E[f_1(x - D)]$ and $F_2(x) = E[f_2(x - D)]$. If $\partial f_1(x) \leq \partial f_2(x)$, then $\partial F_1(x) \leq \partial F_2(x)$.*

Proof of Theorem 2. We prove part (a) of Theorem 2 by induction on k. We need to show $\partial f_{i,k}(y) \leq \partial f_{i+1,k}(y)$ for every y and i. Then the statement follows from the definition of $y_{i,k}^*$ and Lemma 17. Note that from part (a) of Lemma 3, we know that $f_{i,k}$ is convex. For the base case $(k = T)$, we have $\partial f_{i,T}(y) \leq \partial f_{i+1,T}(y)$ for all i, because $f_{i,T}(y) = d_i y + L(y)$, and d_i is increasing in i by Lemma 1. In the induction step, for a fixed $k + 1 \leq T$, assume

95

that $\partial f_{i-1,k+1}(y) \leq \partial f_{i,k+1}(y)$ for all i and y. We have

$$f_{i,k}(y) = d_i y + L(y) + E[S^1_{i-1,k+1}(y - D)]$$

$$= d_i y + L(y) + E[g_{i-1,k+1}(y - D) - d_{i-1}(y - D)]$$

$$= (d_i - d_{i-1})y + L(y) + E[g_{i-1,k+1}(y - D)] + d_{i-1} E[D],$$

and $f_{i+1,k}(y) = (d_{i+1} - d_i)y + L(y) + E[g_{i,k+1}(y - D)] + d_i E[D]$.

Let us define $M_i(y) = (d_{i+1} - d_i)y + L(y)$ and $G_{i,k}(y) = E[g_{i,k}(y - D)]$. Then $f_{i,k}(y) = M_{i-1}(y) + G_{i-1,k+1}(y) + d_{i-1}E[D]$ and $f_{i+1,k}(y) = M_i(y) + G_{i,k+1}(y) + d_iE[D]$. From the definition of sequential systems, Lemma 18, and Lemma 20, we obtain $\partial M_{i-1}(y) \leq \partial M_i(y)$, and $\partial G_{i-1,k+1}(y) \leq \partial G_{i,k+1}(y)$ for all i. Therefore, from Lemma 19, we get $\partial f_{i,k}(y) \leq \partial f_{i+1,k}(y)$ for all i. This completes the proof of part (a).

Now let us proceed to prove part (b) of Theorem 2. We prove by induction on i. When $i = 1$, $g_{1,k}(x) + S^2_{0,k}(x) = g_{1,k}(x)$ is convex for all k, which corresponds to the base case. Assume now that for a fixed $i \geq 2$, $g_{i-1,k}(x) + S^2_{i-2,k}(x)$ is convex for all k. Note that $y^*_{i,k} \leq y^*_{i-1,k}$ by part (a), and

$$g_{i,k}(x) + S^2_{i-1,k}(x) = g_{i,k}(x) + h_{i-1,k}(x) - L(x) + E[S^2_{i-2,k+1}(x - D)].$$

Recall that if $x \leq y^*_{i,k}$, then $g_{i,k}(x) = 0$, and if $x \geq y^*_{i-1,k}$, then $h_{i-1,k}(x) = 0$. If $x \leq y^*_{i-1,k}$, then $h_{i-1,k}(x) = f_{i-1,k}(x) - a_{i-1,k}$, thus

$$h_{i-1,k}(x) - L(x) + E[S^2_{i-2,k+1}(x - D)]$$
$$= d_{i-1}x + E[S^1_{i-2,k+1}(x - D)] - a_{i-1,k} + E[S^2_{i-2,k+1}(x - D)] \qquad (A.3)$$
$$= d_{i-1}x - a_{i-1,k} + E[S^1_{i-2,k+1}(x - D) + S^2_{i-2,k+1}(x - D)]$$
$$= d_{i-1}x - a_{i-1,k} - S^0_{i-2,k+1},$$

which is clearly a convex function. On the other hand, if $x \geq y^*_{i,k}$, then $g_{i,k}(x) = f_{i,k}(x) - a_{i,k}$. Thus,

$$g_{i,k}(x) - L(x) + E[S^2_{i-2,k+1}(x - D)]$$
$$= d_i x + E[S^1_{i-1,k+1}(x - D)] - a_{i,k} + E[S^2_{i-2,k+1}(x - D)] \qquad (A.4)$$
$$= d_i x - a_{i,k} + E[S^1_{i-1,k+1}(x - D) + S^2_{i-2,k+1}(x - D)]$$

96

is convex, because $S^1_{i-1,k+1}(x) + S^2_{i-2,k+1}(x) = g_{i-1,k+1}(x) - d_{i-1}x + S^2_{i-2,k+1}(x)$ is convex by the induction hypothesis.

To summarize, if $x \leq y^*_{i-1,k}$, then $g_{i,k}(x) + S^2_{i-1,k}(x) = g_{i,k}(x) + \{h_{i-1,k}(x) - L(x) + E[S^2_{i-2,k+1}(x - D)]\}$ is convex (see (A.3)). On the other hand, if $x \geq y^*_{i,k}$, then $g_{i,k}(x) + S^2_{i-1,k}(x) = h_{i-1,k}(x) + \{g_{i,k}(x) - L(x) + E[S^2_{i-2,k+1}(x - D)]\}$ is convex (see (A.4)). If $y^*_{i,k} < y^*_{i-1,k}$, then $g_{i,k}(x) + S^2_{i-1,k}(x)$ is globally convex because it is convex for two partially overlapping intervals, which are $x \leq y^*_{i-1,k}$ and $x \geq y^*_{i,k}$.

It remains to prove convexity when $y^*_{i,k} = y^*_{i-1,k}$. In this case, again from (A.3) and (A.4), we get

$$
g_{i,k}(x) + S^2_{i-1,k}(x) = \begin{cases} h_{i-1,k}(x) - L(x) + E[S^2_{i-2,k+1}(x - D)] & x \leq y^*_{i-1,k} = y^*_{i,k} \\ g_{i,k}(x) - L(x) + E[S^2_{i-2,k+1}(x - D)] & x \geq y^*_{i-1,k} = y^*_{i,k}. \end{cases}
$$

We already know that $g_{i,k}(x) + S^2_{i-1,k}(x)$ is convex on $[-\infty, y^*_{i,k}]$ and $[y^*_{i,k}, \infty]$. Since $g_{i,k}$ is nondecreasing at $y^*_{i,k}$, and $h_{i-1,k}$ is nonincreasing at $y^*_{i-1,k} = y^*_{i,k}$, it follows that $\partial h_{i-1,k}(y^*_{i,k}) \leq \partial g_{i,k}(y^*_{i,k})$. In turn we get $\partial\{h_{i-1,k}(\cdot) - L(\cdot) + E[S^2_{i-2,k+1}(\cdot - D)]\}(y^*_{i,k}) \leq \partial\{g_{i,k}(\cdot) - L(\cdot) + E[S^2_{i-2,k+1}(\cdot - D)]\}(y^*_{i,k})$, which means global convexity. This completes the proof. $\square$

Proof of Lemma 5. Because $y^*_{i,k} = \arg\min f_{i,k}(y)$, we instead prove that $f_{i,k}(y)$ and $S^1_{i,k}(x)$ are all equal for $1 \leq i \leq L$ and $k \leq T - i + 1$. We use induction on i. For the base case $i = 1$, $f_{1,k}(y)$ and $S^1_{1,k}(x)$ are independent of k by definition.

Now assume for a fixed $i \geq 1$, that $f_{i,k}(y)$ and $S^1_{i,k}(x)$ are independent of k for $k \leq T - i + 1$. Therefore $S^1_{i,k+1}(x)$ is independent of k for $k + 1 \leq T - i + 1$. Then $f_{i+1,k}(y) = d_{i+1}y + L(y_{i+1}) + E[S^1_{i,k+1}(y - D)]$ is independent of k for $k + 1 \leq T - i + 1 = T - (i + 1) + 2$. In other words, $f_{i+1,k}(y)$ is independent of k for $k \leq T - (i + 1) + 1$. $\square$

Proof of Lemma 6. The statement clearly holds when $i = j$. By Assumption 3, for $i > j$

we have

$$d_i - d_{i-1} \geq E[d_{M(i,W)} - d_{M(i-1,W)}]$$

$$d_{i-1} - d_{i-2} \geq E[d_{M(i-1,W)} - d_{M(i-2,W)}]$$

$$\vdots$$

$$d_{j+1} - d_j \geq E[d_{M(j+1,W)} - d_{M(j,W)}].$$

By summing the above inequalities we obtain $d_i - d_j \geq E[d_{M(i,W)} - d_{M(j,W)}]$. Assumption 1 ensures $M(i,W) \geq M(j,W)$ for $i \geq j$, thus setting $i = M(i,W)$ and $j = M(j,W)$ and taking expectation results in

$$E[d_{M(i,W)} - d_{M(j,W)}] \geq E[d_{M(M(i,W))} - d_{M(M(j,W))}] = E[d_{M^2(i,W)} - d_{M^2(j,W)}].$$

Therefore,

$$d_i - d_j \geq E[d_{M(i,W)} - d_{M(j,W)}] \geq E[d_{M^2(i,W)} - d_{M^2(j,W)}].$$

Note that $M^n(i,W) \geq M^n(j,W)$ for every n, which follows from Assumption 1 and the definition of M^n. By applying the above relation repeatedly, we obtain

$$d_i - d_j \geq E[d_{M^n(i,W)} - d_{M^n(j,W)}],$$

which completes the proof. $\square$

Proof of Lemma 7. Part (a): We have $\{w : M^n(i,w) = 0\} \subseteq \{w : M^{n+1}(i,w) = 0\}$ since an order can stay at installation 0 for one time period. From Assumption 2 it follows

$$1 = Prob[\cup_{n=1}^{\infty}\{w : M^n(i,w) = 0\}] = \lim_{n \to \infty} Prob[M^n(i,W) = 0].$$

Part (b): Clearly $\sum_k Prob[M^n(i,W) = k] = 1$ and by taking the limit we get

$$\sum_k \lim_{n \to \infty} Prob[M^n(i,W) = k] = 1,$$

or equivalently

$$1 = \sum_{k \neq 0} \lim_{n \to \infty} Prob[M^n(i, W) = k] + \lim_{n \to \infty} Prob[M^n(i, W) = 0].$$

Since $\lim_{n \to \infty} Prob[M^n(i, W) = 0] = 1$ by part (a), we conclude that we have $\sum_{k \neq 0} \lim_{n \to \infty} Prob[M^n(i, W) = k] = 0$. $\qquad \square$

Proof of Theorem 5. First, note Lemmas 17, 18, 19, and 20. We prove by induction on t that $\partial f_{i,t}(y) \leq \partial f_{i+1,t}(y)$ for every y and i. For the base case $(t = T)$, we have $\partial f_{i,T}(y) \leq \partial f_{i+1,T}(y)$ for all i because $f_{i,T}(y) = d_i y + L(y)$ and d_i is nondecreasing in i by Proposition 1. In the induction step, for a fixed $t + 1 \leq T$, we assume that $\partial f_{i,t+1}(y) \leq \partial f_{i+1,t+1}(y)$ for all i and y. We have

$$
\begin{aligned}
f_{i,t}(y) &= d_i y + L(y) + E[S^1_{M(i,W),t+1}(y - D)] \\
&= d_i y + L(y) + E[g_{M(i,W),t+1}(y - D) - d_{M(i,W)}(y - D)] \\
&= (d_i - E[d_{M(i,W)}])y + L(y) + E[g_{M(i,W),t+1}(y - D)] + E[d_{M(i,W)}]\,E[D], \text{ and}
\end{aligned}
$$
$$f_{i+1,t}(y) = (d_{i+1} - E[d_{M(i+1,W)}])y + L(y) + E[g_{M(i+1,W),t+1}(y - D)] + E[d_{M(i+1,W)}]\,E[D].$$

Note that $\partial[(d_i - E[d_{M(i,W)}])y] \leq \partial[(d_{i+1} - E[d_{M(i+1,W)}])y]$ by Assumption 3, and

$$\partial\, E[g_{M(i,W),t+1}(y - D)] \leq \partial\, E[g_{M(i+1,W),t+1}(y - D)]$$

for all i since the induction assumption $\partial f_{M(i,W),t+1}(y - D) \leq \partial f_{M(i+1,W),t+1}(y - D)$ is equivalent to $\partial g_{M(i,W),t+1}(y - D) \leq \partial g_{M(i+1,W),t+1}(y - D)$. Therefore we get $\partial f_{i,k}(y) \leq \partial f_{i+1,k}(y)$ for all i. The proof is thus completed. $\qquad \square$

Proof of Lemma 10. The proof is by induction on t. In the base case $t = T$ we have $g_{i,T}(x) + S^2_{M(i,w),T}(x) = g_{i,T}(x) + h_{M(i,w),T}(x) - L(x)$. Consider the following two cases.

(Case 1) If $x \leq y^*_{M(i,w),T}$, then $g_{M(i,w),T}(x) = 0$, thus $g_{i,T}(x) + h_{M(i,w),T}(x) - L(x) = g_{i,T}(x) + f_{M(i,w),T}(x) - a_{M(i,w),T} - L(x) = g_{i,T}(x) + d_{M(i,w)}x - a_{M(i,w),T}$ is convex.

(Case 2) If $x \geq y^*_{i,T}$, then $h_{i,T}(x) = 0$, thus $g_{i,T}(x) + h_{M(i,w),T}(x) - L(x) = f_{i,T}(x) - a_{i,T} + h_{M(i,w),T}(x) - L(x) = d_i x - a_{i,T} + h_{M(i,w),T}(x)$ is convex.

From Theorem 5 it follows $y_{i,T}^* \leq y_{M(i,w),T}^*$ since $i \geq M(i,w)$. If $y_{i,T}^* < y_{M(i,w),T}^*$, then $g_{i,T}(x) + S_{M(i,w),T}^2(x)$ is globally convex because it is convex on two partially overlapping intervals, which are $x \leq y_{M(i,w),T}^*$ and $x \geq y_{i,T}^*$. When $y_{i,T}^* = y_{M(i,w),T}^* = y^*$, then by Proposition 1, we have

$$\partial\{g_{i,T}(y^*) + d_{M(i,w)}y^* - a_{M(i,w),T}\} \leq \partial\{d_i y^* - a_{i,T} + h_{M(i,w),T}(y^*)\}.$$

Since $g_{i,T}(x)$ is nondecreasing and $h_{M(i,w),T}(x)$ is nonincreasing, we obtain $\partial h_{M(i,w),T}(x) \leq \partial g_{i,T}(x)$, hence global convexity. This completes the base case.

Now let us assume that $g_{i,t+1}(x) + S_{M(i,w),t+1}^2(x)$ is convex for all $w \in \mathbf{W}$ and all i, and for some $t + 1 \leq T$. We need to prove that $g_{i,t}(x) + S_{M(i,w),t}^2(x) = g_{i,t}(x) + h_{M(i,w),t}(x) - L(x) + E[S_{M^2(i,w),t+1}^2(x - D)]$ is convex for any $w \in \mathbf{W}$ and for all i. Again consider the following two cases.

(Case 1) If $x \leq y_{M(i,w),t}^*$, then $g_{M(i,w),t}(x) = 0$. Thus

$$\begin{aligned}
g_{i,t}(x) &+ h_{M(i,w),t}(x) - L(x) + E[S_{M^2(i,w),t+1}^2(x - D)] \\
&= g_{i,t}(x) + f_{M(i,w),t}(x) - a_{M(i,w),t} - L(x) + E[S_{M^2(i,w),t+1}^2(x - D)] \\
&= g_{i,t}(x) + d_{M(i,w)}x - a_{M(i,w),t} + E[S_{M^2(i,w),t+1}^1(x - D)] \\
&\quad + E[S_{M^2(i,w),t+1}^2(x - D)] \\
&= g_{i,t}(x) + d_{M(i,w)}x - a_{M(i,w),t} - S_{M^2(i,w),t+1}^0
\end{aligned}$$

is convex.

(Case 2) If $x \geq y_{i,t}^*$, then $h_{i,t}(x) = 0$. Thus

$$\begin{aligned}
g_{i,t}(x) &+ h_{M(i,w),t}(x) - L(x) + E[S_{M^2(i,w),t+1}^2(x - D)] \\
&= f_{i,t}(x) - a_{i,t} + h_{M(i,w),t}(x) - L(x) + E[S_{M^2(i,w),t+1}^2(x - D)] \\
&= d_i x - a_{i,t} + h_{M(i,w),t}(x) + E[S_{M(i,w),t+1}^1(x - D)] + E[S_{M^2(i,w),t+1}^2(x - D)] \\
&= d_i x - a_{i,t} + h_{M(i,w),t}(x) + E[g_{M(i,w),t+1}(x - D) - d_{M(i,w)}(x - D) \\
&\quad + S_{M^2(i,w),t+1}^2(x - D)]
\end{aligned}$$

is convex since $g_{M(i,w),t+1}(x - D) + S_{M^2(i,w),t+1}^2(x - D)$ is convex by the induction

100

hypothesis.

Now we apply a similar logic as in the base case. From Theorem 5 we obtain $y^*_{i,t} \leq y^*_{M(i,w),t}$ since $i \geq M(i,w)$. If $y^*_{i,t} < y^*_{M(i,w),t}$, then $g_{i,t}(x) + S^2_{M(i,w),t}(x)$ is globally convex because it is convex for two partially overlapping intervals, which are $x \leq y^*_{M(i,w),t}$ and $x \geq y^*_{i,t}$. If $y^*_{i,t} = y^*_{M(i,w),t} = y^*$, then

$$g_{i,t}(x) + S^2_{M(i,w),t}(x) = \begin{cases} h_{M(i,w),t}(x) - L(x) + E[S^2_{M^2(i,w),t+1}(x - D)] & x \leq y^* \\ g_{i,t}(x) - L(x) + E[S^2_{M^2(i,w),t+1}(x - D)] & x \geq y^*. \end{cases}$$

Since $g_{i,t}(x)$ is nondecreasing and $h_{M(i,w),t}(x)$ is nonincreasing, we have $\partial h_{M(i,w),t}(x) \leq \partial g_{i,t}(x)$, which means global convexity of $g_{i,t}(x) + S^2_{M(i,w),t}(x)$ when $y^*_{i,t} = y^*_{M(i,w),t}$. This completes the proof. □

The remainder of the proof of Theorem 6 and Proposition 3. We show part (c) at time period t by assuming parts (a), (b), and (d) hold on and after time period t and part (c) holds on and after time period $t + 1$. We compare two states $(x^{i-1}, \bar{0}^{i-1}, v_i, v_{i+1}, \cdots, v_{\bar{K}})$ and $(x^{i-1} + e, \bar{0}^{i-1}, v_i - e, v_{i+1}, \cdots, v_{\bar{K}})$.

For convenience in the remainder of the proof, let A denote $(x^{i-1}, \bar{0}^{i-1}, v_i, v_{i+1}, \cdots, v_{\bar{K}})$ and let B denote $(x^{i-1} + e, \bar{0}^{i-1}, v_i - e, v_{i+1}, \cdots, v_{\bar{K}})$. Also, let A^+ and B^+ denote the next states of A and B under the respective optimal control (they depend on the underlying realization but we do not show this dependency). Let w be the realized value of W at the current time period and let j denote $M(i,w)$. Finally, let A^+_j and B^+_j denote the next states of A and B under respective optimal control given w at the beginning of the next time period. We consider three cases.

Case 1 If $y^*_{i,t} \leq x^{i-1}$, then no expediting is necessary. If $j > 0$, then the two states in the next time period $t + 1$ are $A^+_j = (x^{i-1} - D, \bar{0}^{j-1}, x^{N(j,w)} - x^{N(j-1,w)}, x^{N(j+1,w)} - x^{N(j,w)}, \cdots, x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})$ and $B^+_j = (x^{i-1} + e - D, \bar{0}^{j-1}, x^{N(j,w)} - x^{N(j-1,w)} - e, x^{N(j+1,w)} - x^{N(j,w)}, \cdots, x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})$, where u is the regular ordering quantity, which is the same for both states. For $j > 0$, the induction hypothesis implies

$$J_{t+1}(A^+_j) - J_{t+1}(B^+_j) = S^0_{j,t+1} + S^1_{j,t+1}(x^{i-1} - D) + S^2_{j,t+1}(x^{i-1} + e - D). \tag{A.5}$$

101

On the other hand, if $j = 0$, then the two states at time period $t + 1$ are the same and they are $A_0^+ = B_0^+ = (x^{N(0,w)} - D, x^{N(1,w)} - x^{N(0,w)}, \cdots, x^{N(M(\bar{K},w)-1,w)} - x^{N(M(\bar{K},w)-2,w)},$ $x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})$. Since $S_{0,t+1}^0 = S_{0,t+1}^1(x^{i-1} - D) = S_{0,t+1}^2(x^{i-1} + e - D) = 0$ by definition, (A.5) still holds. Using (A.5) we get

$$
\begin{aligned}
&E[J_{t+1}(A^+) - J_{t+1}(B^+)] \\
&= E[\sum_j Prob[M(i, W) = j]\{J_{t+1}(A^+) - J_{t+1}(B^+)|M(i, W) = j\}] \\
&= E[\sum_j Prob[M(i, W) = j]\{J_{t+1}(A_j^+) - J_{t+1}(B_j^+)\}] \\
&= E[\sum_j Prob[M(i, W) = j]\{S_{j,t+1}^0 + S_{j,t+1}^1(x^{i-1} - D) + S_{j,t+1}^2(x^{i-1} + e - D)\}] \\
&= E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(x^{i-1} - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)].
\end{aligned}
$$

No expediting implies $J_t(A) = L(x^{i-1}) + \min_{z \geq x^{\bar{K}}}\{c(z - x^{\bar{K}}) + E[J_{t+1}(A^+)]\}$, and $J_t(B) = L(x^{i-1}+e) + \min_{z \geq x^{\bar{K}}}\{c(z-x^{\bar{K}})+E[J_{t+1}(B^+)]\}$. Since the minimizations in the above equations have the same optimal control with respect to regular ordering, $J_t(A) - J_t(B) = L(x^{i-1}) - L(x^{i-1} + e) + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(x^{i-1} - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)]$.

Because $y_{i,t}^* \leq x^{i-1}$, we have $h_{i,t}(x^{i-1}) = 0$ and $h_{i,t}(x^{i-1} + e) = 0$. Therefore,

$$
\begin{aligned}
&L(x^{i-1}) - L(x^{i-1} + e) + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(x^{i-1} - D) \\
&\quad + S_{M(i,W),t+1}^2(x^{i-1} + e - D)] \\
&= d_i x^{i-1} + L(x^{i-1}) + E[S_{M(i,W),t+1}^1(x^{i-1} - D)] - d_i x^{i-1} - L(x^{i-1} + e) \\
&\quad + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^2(x^{i-1} + e - D)] \\
&= f_{i,t}(x^{i-1}) - d_i x^{i-1} - L(x^{i-1} + e) + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^2(x^{i-1} + e - D)] \\
&= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1}) - d_i x^{i-1} - L(x^{i-1} + e) \\
&\quad + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^2(x^{i-1} + e - D)] \\
&= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1} + e) - d_i x^{i-1} - L(x^{i-1} + e) \\
&\quad + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^2(x^{i-1} + e - D)].
\end{aligned}
$$

Case 2 If $x^{i-1} < y_{i,t}^* \leq x^{i-1} + e$, then expediting $y_{i,t}^* - x^{i-1}$ from installation i is optimal in state A and no expediting is optimal in state B. We have

$$A_j^+ = (y_{i,t}^* - D, \bar{0}^{j-1}, x^{N(j,w)} - y_{i,t}^*, x^{N(j+1,w)} - x^{N(j,w)}, \cdots,$$
$$x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)}),$$

$$B_j^+ = (x^{i-1} + e - D, \bar{0}^{j-1}, x^{N(j,w)} - x^{N(j-1,w)} - e, x^{N(j+1,w)} - x^{N(j,w)}, \cdots,$$
$$x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})$$

for $j > 0$, and

$$A_0^+ = B_0^+ = (x^{N(0,w)} - D, x^{N(1,w)} - x^{N(0,w)}, \cdots, x^{N(M(\bar{K},w)-1,w)} - x^{N(M(\bar{K},w)-2,w)},$$
$$x^{N(M(\bar{K},w),w)} - x^{N(M(\bar{K},w)-1,w)} + u, \bar{0}^{\bar{K}-M(\bar{K},w)})$$

for $j = 0$. From the induction hypothesis, $J_{t+1}(A_j^+) - J_{t+1}(B_j^+) = S_{j,t+1}^0 + S_{j,t+1}^1(y_{i,t}^* - D) + S_{j,t+1}^2(x^{i-1} + e - D)$ for $j \geq 0$, and therefore

$$E[J_{t+1}(A^+) - J_{t+1}(B^+)]$$
$$= E[\sum_j Prob[M(i,W) = j]\{J_{t+1}(A^+) - J_{t+1}(B^+)|M(i,W) = j\}]$$
$$= E[\sum_j Prob[M(i,W) = j]\{J_{t+1}(A_j^+) - J_{t+1}(B_j^+)\}]$$
$$= E[\sum_j Prob[M(i,W) = j]\{S_{j,t+1}^0 + S_{j,t+1}^1(y_{i,t}^* - D) + S_{j,t+1}^2(x^{i-1} + e - D)\}]$$
$$= E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(y_{i,t}^* - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)].$$

We have $J_t(A) = d_i y_{i,t}^* + L(y_{i,t}^*) - d_i x^{i-1} + \min_{z \geq x^{\bar{K}}} \{c(z - x^{\bar{K}}) + E[J_{t+1}(A^+)]\}$, and $J_t(B) = L(x^{i-1} + e) + \min_{z \geq x^{\bar{K}}} \{c(z - x^{\bar{K}}) + E[J_{t+1}(B^+)]\}$. Therefore, $J_t(A) - J_t(B) = d_i y_{i,t}^* + L(y_{i,t}^*) - d_i x^{i-1} - L(x^{i-1} + e) + E[S_{M(i,W),t+1}^0 + S_{M(i,W),t+1}^1(y_{i,t}^* - D) + S_{M(i,W),t+1}^2(x^{i-1} + e - D)]$.

Because $x^{i-1} < y_{i,t}^* \le x^{i-1} + e$, we have $g_{i,t}(x^{i-1}) = 0$ and $h_{i,t}(x^{i-1} + e) = 0$, and

$$d_i y_{i,t}^* + L(y_{i,t}^*) - d_i x^{i-1} - L(x^{i-1} + e)$$
$$+ E[S^0_{M(i,W),t+1} + S^1_{M(i,W),t+1}(y_{i,t}^* - D) + S^2_{M(i,W),t+1}(x^{i-1} + e - D)].$$
$$= f_{i,t}(y_{i,t}^*) - d_i x^{i-1} - L(x^{i-1} + e) + E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)]$$
$$= a_{i,t} - d_i x^{i-1} - L(x^{i-1} + e) + E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)]$$
$$= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,T}(x^{i-1} + e) - d_i x^{i-1} - L(x^{i-1} + e)$$
$$+ E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)].$$

Case 3   If $y_{i,t}^* > x^{i-1} + e$, then we expedite $\min(v_i, y_{i,t}^* - x^{i-1})$ from installation i in state A and $\min(v_i, y_{i,t}^* - x^{i-1} - e)$ from installation i in state B. Therefore, in the next time period, states A^+ and B^+ are the same and the only cost difference between $J_t(A)$ and $J_t(B)$ is $d_i e = d_i(x^{i-1} + e) - d_i x^{i-1}$. Thus, $J_t(A) - J_t(B) = d_i(x^{i-1} + e) - d_i x^{i-1}$.

Because $y_{i,t}^* > x^{i-1} + e$, we have $g_{i,t}(x^{i-1}) = 0$ and $g_{i,t}(x^{i-1} + e) = 0$. Note that $S^0_{j,t+1} + S^1_{j,t+1}(x) + S^2_{j,t+1}(x) = 0$, or $S^1_{j,t+1}(x) = -S^0_{j,t+1} - S^2_{j,t+1}(x)$. We conclude that

$$d_i(x^{i-1} + e) - d_i x^{i-1}$$
$$= a_{i,t} - a_{i,t} + g_{i,t}(x^{i-1}) - g_{i,t}(x^{i-1} + e) + h_{i,t}(x^{i-1} + e) - h_{i,t}(x^{i-1} + e)$$
$$+ d_i(x^{i-1} + e) - d_i x^{i-1}$$
$$= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1} + e) - f_{i,t}(x^{i-1} + e) + d_i(x^{i-1} + e) - d_i x^{i-1}$$
$$= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1} + e) - d_i(x^{i-1} + e) - L(x^{i-1} + e)$$
$$- E[S^1_{M(i,W),t+1}(x^{i-1} + e - D)] + d_i(x^{i-1} + e) - d_i x^{i-1}$$
$$= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1} + e) - d_i x^{i-1} - L(x^{i-1} + e)$$
$$- E[S^1_{M(i,W),t+1}(x^{i-1} + e - D)]$$
$$= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1} + e) - d_i x^{i-1} - L(x^{i-1} + e)$$
$$+ E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)].$$

Finally, Cases 1, 2, and 3 can be summarized as

$$J_t(x^{i-1}, \bar{0}^{i-1}, v_i, v_{i+1}, \cdots, v_{\bar{K}}) - J_t(x^{i-1} + e, \bar{0}^{i-1}, v_i - e, v_{i+1}, \cdots, v_{\bar{K}})$$

$$= a_{i,t} + g_{i,t}(x^{i-1}) + h_{i,t}(x^{i-1} + e) - d_i x^{i-1} - L(x^{i-1} + e)$$

$$+ E[S^0_{M(i,W),t+1} + S^2_{M(i,W),t+1}(x^{i-1} + e - D)]$$

$$= S^0_{i,t} + S^1_{i,t}(x^{i-1}) + S^2_{i,t}(x^{i-1} + e).$$

Therefore, part (c) is proved, and this completes the induction step of the entire proof. □

The Rolling Heuristic

This heuristic uses rolling at each time period. The steps in this heuristic are the following:

1. Determine optimal regular ordering quantity by assuming no expediting now and in the future.

2. Determine optimal expediting quantities from intermediate installations by assuming no expediting options in the future.

3. In the next time period, repeat.

We actually expedite orders at all time periods, although the optimal expediting decisions assume no future expediting.

Formal Derivations

Let us denote by u the regular ordering quantity, and by e_i the expediting quantity from installation i. Then the optimality equation reads

$$J_k(x, v_1, \cdots, v_{L-1}) = \min_{\substack{u, e_1, \cdots, e_L \\ u \geq e_L \geq 0 \\ v_i \geq e_i \geq 0 \\ i=1, \cdots, L-1}} \{ \sum_{i=1}^{L} d_i e_i + L(x + \sum_{i=1}^{L} e_i) + cu$$

$$+ E[J_{k+1}(x + v_1 + \sum_{i=2}^{L} e_i - D, v_2 - e_2, \cdots, v_{L-1} - e_{L-1}, u - e_L)] \}.$$

Let I_k is the cost-to-go without any expediting now and in the future at time period k. Then the optimality equation for the heuristic regular ordering quantity u^* is

$$I_k(x, v_1, \cdots, v_{L-1}) = \min_{u^*}\{cu^* + L(x) + E[I_{k+1}(x^1 - D, v_2, \cdots, v_{L-1}, u^*)]\}.$$

It is easy to show that for convex functions p_k^i, we have

$$I_k(x, v_1, \cdots, v_{L-1}) = p_k^1(x) + p_k^2(x^1) + \cdots + p_k^L(x^{L-1}).$$

Furthermore, by defining $q_k^i(x) = E[p_k^i(x - D)]$, we have

$$p_k^1(x) = L(x)$$
$$p_k^2(x^1) = q_{k+1}^1(x^1)$$
$$\vdots$$
$$p_k^{L-1}(x^{L-2}) = q_{k+1}^{L-2}(x^{L-2})$$
$$p_k^L(x^{L-1}) = \min_{x^L \geq x^{L-1}}\{cx^L + q_{k+1}^L(x^L)\} - cx^{L-1} + q_{k+1}^{L-1}(x^{L-1}).$$

The heuristic is based on substituting u^* from I_k for u and I_{k+1} for J_{k+1}. Therefore, the current cost to go J_k' is

$$J_k'(x, v_1, \cdots, v_{L-1}) = \min_{\substack{e_1, \cdots, e_L \\ u^* \geq e_L \geq 0 \\ v_i \geq e_i \geq 0 \\ i=1, \cdots, L-1}} \{\sum_{i=1}^{L} d_i e_i + L(x + \sum_{i=1}^{L} e_i) + cu^*$$
$$+ E[I_{k+1}(x + v_1 + \sum_{i=2}^{L} e_i - D, v_2 - e_2, \cdots, v_{L-1} - e_{L-1}, u^* - e_L)]\}.$$

In order to obtain the heuristic decisions on expediting, we need to solve the following problem.

$$
\begin{aligned}
J'_k(x, v_1, \cdots, v_{L-1}) = \min_{\substack{e_1, \cdots, e_L \\ u^* \geq e_L \geq 0 \\ v_i \geq e_i \geq 0 \\ i=1, \cdots, L-1}} \{ \sum_{i=1}^{L} d_i e_i + L(x + \sum_{i=1}^{L} e_i) + c u^* \\
+ q^1_{k+1}(x + v_1 + \sum_{i=2}^{L} e_i) \\
+ q^2_{k+1}(x + v_1 + v_2 + \sum_{i=3}^{L} e_i) \\
\vdots \\
+ q^{L-1}_{k+1}(x^{L-1} + e_L) \\
+ q^L_{k+1}(x^{L-1} + u^*) \}.
\end{aligned}
$$

The rolling heuristic is easy to understand because it is myopic. Also, this heuristic allows order crossing in time in principle for nondecreasing expediting costs. Therefore, this heuristic also works well for the systems with very high expediting costs at intermediate installations. However, because we do not consider future expediting, the optimal expediting pattern prefers expediting from locations with smaller expediting costs.

Bibliography

Anupindi, R., R. Akella. 1993. Diversification under supply uncertainty. *Management Science* **39** 944–963.

Barankin, E. W. 1961. A delivery-lag inventory model with an emergency provision (the single-period case). *Naval Research Logistics Quarterly* **8** 285–311.

Chiang, C., G. J. Gutierrez. 1996. A periodic review inventory system with two supply modes. *European Journal of Operations Research* **94** 527–547.

Chiang, C., G. J. Gutierrez. 1998. Optimal control policies for a periodic review inventory system with emergency orders. *Naval Research Logistics* **45** 187–204.

Clark, A. J., H. Scarf. 1960. Optimal policies for a multi-echelon inventory problem. *Management Science* **6** 475–490.

Daniel, K. H. 1963. A delivery-lag inventory model with emergency. Herbert Scarf, D. M. Gilford, M. W. Shelly, eds., *Multistage Inventory Models and Techniques*, chap. 2. Stanford University Press, 32–46.

Ehrhardt, R. 1984. (s, S) policies for a dynamic inventory model with stochastic lead times. *Operations Research* **32** 121–132.

Feng, Q., G. Gallego, S. P. Sethi, H. Yan, H. Zhang. 2005. Periodic-review inventory model with three consecutive delivery modes and forecast updates. *Journal of Optimization Theory and Applications* **124** 137–155.

Fries, B. E. 1975. Optimal ordering policy for a perishable commodity with fixed lifetime. *Operations Research* **23** 46–61.

Fukuda, Y. 1964. Optimal policies for the inventory problem with negotiable leadtime. *Management Science* **10** 690–708.

Gaukler, G., Ö. Özer, W. H. Hausman. 2005. RFID and product progress information: Improved dynamic emergency ordering policies. *Working Paper*, Stanford University.

Glasserman, P., S. Tayur. 1995. Sensitivity analysis for base-stock levels in multi-echelon production-inventory systems. *Management Science* **41** 263–281.

Groenevelt, H., N. Rudi. 2003. A base stock inventory model with possibility of rushing part of order. Working Paper, University of Rochester, Rochester, NY.

Huggins, E. L., T. L. Olsen. 2003a. Inventory control with generalized expediting. *Working Paper*, Fort Lewis College, Durango, CO.

Huggins, E. L., T. L. Olsen. 2003b. Supply chain management with guaranteed delivery. *Management Science* **49** 1154–1167.

Kaplan, R. 1970a. A dynamic inventory model with stochastic lead times. *Management Science* **16** 491–507.

Kaplan, R. S. 1970b. A dynamic inventory model with stochastic lead times. *Management Science* **16** 491–507.

Karush, W. 1959. A theorem in convex programming. *Naval Research Logistics Quarterly* **6** 245–260.

Lawson, D. G., E. L. Porteus. 2000. Multistage inventory management with expediting. *Operations Research* **48** 878–893.

Lee, H. 2004. The triple-A supply chain. *Harvard Business Review* .

Lee, H., Ö. Özer. 2007. Unlocking the value of RFID. *Production and Operations Management* **16** 40–64.

Muharremoglu, A., J. N. Tsitsiklis. 2003a. Dynamic leadtime management in supply chains. Tech. rep., Massachusetts Institute of Technology, Cambridge, MA.

Muharremoglu, A., J. N. Tsitsiklis. 2003b. A single-unit decomposition approach to multi-echelon inventory systems. Tech. rep., Massachusetts Institute of Technology, Cambridge, MA.

Nahmias, S. 1975. Optimal ordering policies for perishable inventory-II. *Operations Research* **23** 735–749.

Nahmias, S. 1979. Simple approximations for a variety of dynamic leadtime lost-sales inventory models. *Operations Research* **27** 904–924.

Nahmias, S. 1982. Perishable inventory theory: A review. *Operations Research* **30** 680–708.

Neuts, M. F. 1964. An inventory model with an optional time lag. *Journal of the Society for Industrial and Applied Mathematics* **12** 179–185.

Porteus, E. 2002. *Foundations of Stochastic Inventory Theory*. Stanford Business Books.

Simchi-Levi, D., X. Chen, J. Bramel. 2004. *The Logic of Logistics: Theory, Algorithms, and Applications for Logistics and Supply Chain Management*. 2nd ed. Springer.

Song, J., P. Zipkin. 1996. Inventory control with information about supply conditions. *Management Science* **42** 1409–1419.

Tagaras, G., D. Vlachos. 2001. A periodic review inventory system with emergency replenishments. *Management Science* **47** 415–429.

Veinott, A. F. 1966. The status of mathematical inventory theory. *Management Science* **12** 745–777.

Vlachos, D., G. Tagaras. 2001. An inventory system with two supply modes and capacity constraints. *International Journal of Production Economics* **72** 41–58.

Whittemore, A. S., S. C. Saunders. 1977. Optimal inventory under stochastic demand with two supply options. *Journal of the Society for Industrial and Applied Mathematics* **32** 293–305.

Williams, C. L., B. E. Patuwo. 1999. A perishable inventory model with positive order lead times. *European Journal of Operational Research* **116** 352–373.

Zhang, V. L. 1996. Ordering policies for an inventory system with three supply modes. *Naval Research Logistics* **43** 691–708.