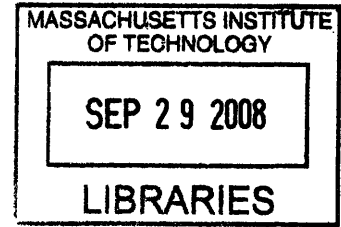


Open Strings in the Cotangent Bundle and Morse

Homotopy

by

Vito Iacovino



Bachelor of Arts, Pisa University, June 2000

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

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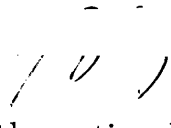
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Author

Department of Mathematics

August 4, 2008

Certified by

 Shing-Tung Yau

Professor of Mathematics, Harvard University

Thesis Supervisor

Accepted by



David Jerison

Chairman, Department Committee on Graduate Theses

ARCHIVES

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Abstract

Let M be a riemannian manifold of dimension 3. We study the genus zero open rigid J -holomorphic curves in T^*M with boundaries mapped in perturbations of the zero section. The perturbations of the zero section is defined fixing a set of functions on M . We consider the graphs of the differential of the functions rescaled by an $\varepsilon \geq 0$. For a generic choice of the functions, we prove that, for ε small enough, there exists a one to one correspondence between the J holomorphic curves and the planar Morse graphs of the functions.

Thesis Supervisor: Shing-Tung Yau

Title: Professor of Mathematics, Harvard University

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Chapter 1

Introduction

Let M be a riemannian manifold. In [2] Floer construct a chain level isomorphism between the Lagrangian Floer complex of the zero of T^*M and the Morse complex of M . Floer studied the following problem. Let f be a Morse function on M and let ε be a positive real number. Considerer the Hamiltonian perturbation of the zero section given by the graph of εdf . For ε small enough, the pseudo-holomorphic strips that are bounded by the zero section and the graph of εdf are in correspondence with the gradient flow lines of f . Actually, there exists a complex structure for which it is possible to describe explicitly all these strips.

In [7], Witten argued the equivalence between topological string theory on T^*M (with branes in the zero section) and Chern-Simons theory on M , for a three manifold M . This statement can be seen as a correspondence between J -holomorphic curves bounding the zero section of T^*M and Feynman graphs on M . Since the Lagrangian is exact in this case there are no honest J -holomorphic curves (instantons of the physical model), but in some sense there are contributions from degenerate instantons (the boundary of the Teichmuller space). The degenerate instantons can be seen as fat graphs on M corresponding to Feynman graphs of the Chern-Simons theory.

The idea of Witten was made mathematically more precise by Fukaya and Oh in [3]. Fukaya and Oh considered perturbations of the zero section given by graphs $\{\varepsilon\Gamma_{df_i}\}$ of a set of functions $\{f_i\}$ that satisfy suitable transversality conditions. For $\varepsilon \rightarrow 0$, J -holomorphic disks bounding the Lagrangians $\{\varepsilon\Gamma_{df_i}\}$ degenerates to the

Morse trees of the functions $\{f_i\}$ on M .

An important consequence of [3] is the equivalence between the A_∞ structure associated to the zero section of T^*M and the A_∞ Morse structure of M . This has lead to important applications to Mirror Symmetry (see Kontsevich-Soibelman [4]) and to the study of the Fukaya category of T^*M (see Nadler-Zaslow [5]).

One main complication of [3] with respect to [2] is that it is not possible to construct an explicit solution corresponding to a graph. Therefore, in order to construct a curve from a given Morse graph it is necessary to find an approximative solution and use an iterative process.

In the argument of [3] a rescaled Sobolev norm is used in order to prove transversality and the uniform bound of the right inverse. The gluing argument used in [3] is incorrect. Lemma 6.1 of [3] does not hold. It is necessary to use cutoff functions with derivative that goes to zero faster than ε . Moreover, the required quadratic estimate does not hold using the norm of [3]. The argument of [3] used to prove the converges of disks to Morse trees is not clear.

We propose a different argument and generalize the result of [3].

Theorem 1. *Fix $d \geq 2$. Let $f = (f_0, \dots, f_d)$ be a generic collection of functions on M , and let $\Gamma = (\Gamma_{df_0}, \dots, \Gamma_{df_d})$ be the graphs of their differentials. For sufficiently small $\varepsilon > 0$ there is a bijection between the space of rigid Morse gradient trees (with $d + 1$ external edges) of f and the space of rigid pseudo holomorphic Riemannian disks with $d + 1$ punctures bounding the Lagrangians $\varepsilon\Gamma$.*

The approximative solution can be constructed by gluing the strips of [2]. In order to make the iterative process working it is useful to use norms (depending on ε) for which the initial error becomes small with ε and the norm of the right inverse of the linearization and the constant of the quadratic estimate are bounded independently of ε .

In order to prove the bound of the right inverse for the rescaled norm we solve the linearized equation on any strip. This allows us to reduce the problem to functions supported near vertices. In this case it is possible get a good estimation using an

exponential weighted norm. A gluing argument and the Morse transversality imply the existence and the bound of the right inverse for the weighted norm. In particular we do not need use gluing in the case of the cup product (at difference of [3]). A careful analysis yields the required bound.

As usual, the constant of the quadratic estimate will depend on the constant of the Sobolev embedding. We prove the independence with respect to ε of the constant of the Sobolev embedding for our norm.

In the other direction we need to prove that, for ε small, any J -holomorphic disk degenerates in some sense to a graph.

A punctured disk can be decomposed by flat strips and "vertex regions". This is a reminiscence of the thick-thin decomposition.

A vertex region can be seen as a pointed disk with strip like ends removed. These disks are allowed to vary only in a bounded region of their moduli.

A J holomorphic strip can be seen as an open string that propagate. For ε small the open string can be approximate to a point. In this approximation, the J -holomorphic equation reduces to the Morse flow equation.

Chapter 2

Morse trees

In this chapter defines the Morse trees and we study the transversality. In the last section we introduced the concept of quasi Morse tree that will be used in Chapter 4.

2.1 Definition

Consider the space of trees. We assume that all the internal vertices are at least trivalent. For a tree Γ let $E(\Gamma)$, $E^{ex}(\Gamma)$ and $E^{tot}(\Gamma)$ be respectively the sets of the edges, internal edges and external edges of Γ . Let $V(\Gamma)$ be the set of internal vertices of Γ . Some time we will indicate with E^{tot} the set $E(\Gamma)$. Let $H(\Gamma)$ be the set of half edges or edges with starting point. In $H(\Gamma)$ there are two elements for any internal edge and one for any external edge. Some time we will omit Γ .

For any vertex $v \in V(\Gamma)$ let $H(v)$ be the set of half edges starting in v . A ribbon structure on Γ assigns to any vertex v a cyclic order of $H(v)$.

A riemannian structure assigns a length to any internal edge, that is a function $L : E^{in}(\Gamma) \rightarrow \mathbb{R}^+$. We indicate by \mathcal{L} the moduli of riemannian structures.

Suppose that a function is assigned to any boundary components of the ribbon tree. Let $\{f_i\}$ be the set of functions. We assume that $f_i - f_j$ is a Morse function for any $i \neq j$. To any oriented edge $e \in H$ there are associated two functions f_e^R and f_e^L . Define $f_e = f_e^R - f_e^L$.

A Morse gradient tree is a collection of maps $\{\gamma_e\}_{e \in H(\Gamma)}$, where $\gamma_e : [0, L(e)] \rightarrow M$

for any internal edge e and $\gamma_e : (-\infty, 0] \rightarrow M$ for any external edge e , such that

$$\dot{\gamma}_e - \nabla f_e(\gamma_e) = 0 \quad (2.1)$$

$\gamma_e(0) = \gamma_{e'}(0)$ if e and e' start at the same vertex and $\gamma_e(t) = \gamma_{e'}(L(e) - t)$ if e and e' are the same edge but with opposite orientation.

We fix for any external edge e the external vertex $p_e \in M$. p_e is a critical point of f_e . Observe that for close choice of the functions f_i the critical points can be identified.

The energy of an internal edge e is given by

$$\int |\dot{\gamma}_e|^2 = f_e(\gamma_e(l_e)) - f_e(\gamma_e(0)).$$

For an external edge

$$\int |\dot{\gamma}_e|^2 = f_e(\gamma_e(0)) - f_e(p_e).$$

The total energy of the tree is obtained summing on all the edges

$$\sum_{e \in E^{tot}(\Gamma)} |\dot{\gamma}_e|^2 = - \sum_{e \in E^{ex}} f_e(p_e). \quad (2.2)$$

In particular the Morse trees without external vertices are constants.

For any external edge e let M_e be the unstable manifold of p_e . For any internal edge e let M_e be the submanifold of M^2 given by

$$M_e = \{(x, \phi_t^e(x)) | x \in M \setminus C_{f_e}, t \in \mathbb{R}^+\}$$

the pairs of the initial and final points of the Morse trajectories of f_e . Here C_{f_e} is the set of critical points of f_e . Observe that M_e is not a closed manifold since we not include the points of the diagonal of M^2 and we not consider broken lines.

We want to define the space \mathcal{G} of Morse trees of Γ as a subset of $M^{H(\Gamma)}$. For any internal edge e let $\pi_e : M^{H(\Gamma)} \rightarrow M^2$ the projection correspond to the vertices attached to e . For an external edge $\pi_e : M^{H(\Gamma)} \rightarrow M$ is the projection to the only

internal vertex attached to e . Define the submanifold $\mathcal{E}(\Gamma)$ of $M^{H(\Gamma)}$ by

$$\mathcal{E}(\Gamma) = \bigcap_{e \in E(\Gamma)} \pi_e^{-1}(M_e).$$

For any vertex v let $\pi_v : M^{H(\Gamma)} \rightarrow M^{H(v)}$ the projection correspond to the edges attached to v . Let

$$\mathcal{V}(\Gamma) = \bigcap_{v \in V(\Gamma)} \pi_v^{-1} \Delta$$

where $\Delta \subset M^{|H(v)|}$ is the diagonal.

The space $\mathcal{G}(\Gamma)$ of Morse gradient flow of the ribbon tree Γ is $\mathcal{V}(\Gamma) \cap \mathcal{E}(\Gamma)$.

2.2 Morse transversality

We say that the set of functions f_i is transversal if \mathcal{V} and \mathcal{E} intersect transversally.

The virtual dimension of moduli space of the Morse trees \mathcal{G} associated to Γ is defined as

$$\dim \mathcal{E} + \dim \mathcal{V} - \dim M^H.$$

We can express the virtual dimension of \mathcal{G} in terms of the Morse index of the external vertices

$$n|V(\Gamma)| - (n-1)|E^{in}(\Gamma)| - \sum_{e \in E^{ex}} \mu_e(p_e) \quad (2.3)$$

where μ_e is the Morse index of f_e in p_e . If the functions f_i are transversal this is the dimension of the manifold \mathcal{G} .

We will be interested only to the space of rigid trees. Therefore we fix the external vertices p_e of the Morse trees such that

$$n - 3 \sum_{e \in E^{ex}} (1 - \mu_e(p_e)).$$

In this case the trivalent graphs have virtual dimension zero.

Lemma 2. *For generic (f_0, \dots, f_d) , for any ribbon tree Γ as above, $\mathcal{V}(\Gamma)$ and $\mathcal{E}(\Gamma)$*

intersect transversally and $\mathcal{G}(\Gamma)$ is finite.

Proof. Fix a tree Γ . Let \mathcal{F} be the space of functions $f = (f_0, \dots, f_d)$ close enough such that the critical points of $f_i - f_j$ can be identified. The universal moduli space \mathcal{UG} of Morse trees is the subset of $M^H \times \mathcal{F}$ given by points (x_H, f) such that x_H represent a Morse tree for f . Analogously to \mathcal{UG} , define \mathcal{UE} and \mathcal{UV} . We have $\mathcal{UG} = \mathcal{UE} \cap \mathcal{UV}$. We want to see that this intersection is transversal outside a baire subset of \mathcal{F} .

In order to prove the transversality in a point of \mathcal{UG} it is enough to see that the projection of tangent space of \mathcal{UE} in the component M^H is surjective. This follows since, for any edge e it is easy to see that the image of the linearization with respect to f_e of (2.1) is all TM^2 .

We can repeat the argument for any Γ . If Γ is not trivalent $\mathcal{G}(\Gamma)$ is empty for a generic choice of the functions f_i since the virtual dimension is less than zero.

In order to conclude that \mathcal{G} is compact and therefore finite we need to deal with the non compactness due to the divergence of the length of some edge. However a sequence of Morse trees with an edge with length that goes to infinity converge (up to subsequence) to a tree with broken edges. The virtual dimension of these trees is less than zero therefore these trees can be eliminated generically. \square

Now, we want to relate the transversality of the Morse tree with the surjectivity of the linearization of the equations of Morse the trajectories.

Let first e be an internal edge and let γ_e be a solution of the Morse equation (2.1). For $\xi_e \in \Gamma(\gamma_e^*TM)$ and $\lambda \in T\mathcal{L}_e$ define

$$(\exp_{\gamma_e}(\xi_e, \lambda_e))(t) = \exp_{\gamma_e(t)}(\xi_e(t))(e^{\lambda_e t}).$$

The linearization of (2.1)

$$D_e : \Gamma(\gamma_e^*TM) \oplus T\mathcal{L}_e \rightarrow \Gamma(\gamma_e^*TM)$$

is given by

$$D_e(\xi_e, \lambda_e) = \frac{d}{dt}\xi_e - \nabla(\nabla f_e)(\xi_e) + \lambda_e \dot{\gamma}_e \tag{2.4}$$

For an external edge e there is not the modular parameter in (2.4).

Now consider the full linearization for the Morse tree

$$D : L_1^2(\gamma^*(TM)) \oplus T\mathcal{L} \rightarrow L^2(\gamma^*(TM)). \quad (2.5)$$

Here $\gamma^*(TM)$ indicate the union of the $\gamma_e^*(TM)$ attached to the vertices. A section in $L_1^2(\gamma^*(TM))$ is defined by a section in $L_1^2(\gamma_e^*(TM))$ for any $e \in E$ that are compatible in the vertices. The restriction of D on an edge e is given by D_e .

Lemma 3.

$$\ker D = T\mathcal{G}.$$

Proof. Observe first that for any $e \in E$

$$\ker D_e \cong TM_e. \quad (2.6)$$

The isomorphism send an element in $\ker D_e$ in his values in the boundaries of the edges.

Let $\xi \in \ker D$. Observe that ξ has to be smooth. Let $v_\xi \in T^*M^H$ the vector which component in $e \in H$ is given by the value of ξ in the starting point of e . It is immediate to see that $v \in T\mathcal{V}$. From (2.6) follows that $v \in T\mathcal{E}$.

□

Lemma 4.

$$\text{coker } D = (T\mathcal{V} + T\mathcal{E})^\perp.$$

In particular the functions f_i are transversal if and only if D is surjective.

Proof. On any $e \in E^{in}(\Gamma)$, the adjoint D_e^* of D_e , , satisfies

$$\langle D_e(\xi, \lambda), \eta \rangle + \langle (\xi, \lambda), D_e^* \eta \rangle = \langle \xi(L(e)), \eta(L(e))(\partial_t) \rangle - \langle \xi(0), \eta(0)(\partial_t) \rangle. \quad (2.7)$$

For $e \in E^{ex}(\Gamma)$ we have

$$\langle D_e \xi, \eta \rangle + \langle \xi, D_e^* \eta \rangle = \langle \xi(0), \eta(0)(\partial_t) \rangle. \quad (2.8)$$

Take $\eta \in L^2(\Omega^1(\gamma^*(TM)))$ in the coker D . that is

$$\langle D(\xi, \lambda), \eta \rangle = 0$$

for any $\xi \in L_1^2(\gamma^*(TM))$ and $\lambda \in T\mathcal{L}$. Using equations (2.7) and (2.8) for ξ compact support, on any edge e , η satisfies $D_e^* \eta = 0$. In particular η is smooth.

For any $e \in H$ evaluate η on the positive unit vector in the starting point of e . This defines a vector $v_\eta \in TM^H$.

Let $v \in TM_e$ for $e \in E$. Equations (2.7) or (2.8) for (ξ, λ) or ξ corresponding to v by (2.6) implies that $\langle v, v_\eta \rangle = 0$. Therefore $v_\eta \in T\mathcal{E}^\perp$.

Let $v \in TV$ and let $\xi \in L_1^2(\gamma^*(TM))$ such that the value in the vertices is given by v . From (2.7) and (2.8) we have $\langle v, v_\eta \rangle = \langle D(\xi, 0), \eta \rangle = 0$. Therefore $v_\eta \in TV^\perp$. \square

2.3 Quasi trees

Fix $\delta > 0$ small.

Let f be a Morse function on M . A δ -quasi Morse trajectory of f is a finite sequence of Morse trajectories of f , $\{\gamma_i\}_{1 \leq i \leq k}$, such that the image of any γ_i dist more than $\frac{\delta}{2}$ from any critical point of f and, for any i there exist a critical point of f with distance less than δ from the final point of γ_i and the initial point of γ_{i+1} .

As before, let Γ be a ribbon tree with a Morse function associated to any boundary component. A δ -quasi Morse tree assigns to any edge $e \in H$ a δ -quasi Morse trajectories γ_e of f_e such that the distance of the initial points of the trajectories of edges starting to the same vertex is less than δ .

Define the virtual dimension of a Morse δ -quasi tree using (2.3).

The energy of a quasi tree will be still given by (2.2) up to an error of the order of δ .

Lemma 5. *For δ small enough, the δ -quasi Morse trajectories of the edges of any rigid δ quasi Morse tree have one only connected component. In particular any quasi Morse tree is a point of \mathcal{E} with distance of order δ from \mathcal{V} .*

Proof. This follows by contradiction using an elementary compactness argument. Suppose that there exist a sequence of δ_i -quasi trees with at least one edge with more components and $\delta_i \rightarrow 0$. Up to take a subsequence, the Morse quasi tree converges to a Morse tree with some edge broken at some critical point. For a generic choice of the Morse function this tree cannot exist. \square

Consider the space \mathcal{E}' defined as \mathcal{E} except now we allow the length of the edges to be zero. \mathcal{E}' is a (not compact) manifold with corners. We can still think \mathcal{E}' contained in M^H .

Lemma 6. *\mathcal{E}' and \mathcal{V} intersect transversally.*

Proof. This follows from Lemma 2. \square

From the definition of quasi Morse trajectory it is clear that there exist a constant C such that $|\dot{\gamma}_e| \geq \frac{1}{C}$ for any $e \in H$. This has the consequence that the length of the edges are bounded:

$$\int |\dot{\gamma}_e| \leq CE_e^2. \quad (2.9)$$

Lemma 7. *There exists a constant C such that, for δ small enough, for any rigid δ quasi Morse tree there exists a Morse tree which distance less than $C\delta$.*

Proof. This follows immediately by the Lemma before since by (2.9) we are only considering a compact region of \mathcal{E}' . \square

Take an edge $e \in H'$ of the tree and let e' the correspondent edge of the quasi tree in lemma 7. If γ_e and $\gamma_{e'}$ are the corresponding Morse trajectories there exist $\xi_e \in \Gamma(\gamma_e^*TM)$ and $\lambda \in T\mathcal{L}_e$ such that

$$\gamma'_{e'} = \exp_{\gamma_e}(\xi_e, \lambda_e)$$

and

$$\|\xi_e\|_\infty + |\lambda_e| \leq C\delta. \quad (2.10)$$

This is an easy consequence of the fact that the extreme points of γ_e and γ'_e are of distance of order δ and the length of the edge has a fixed bound (2.9).

Chapter 3

J-holomorphic disks

In this chapter, for ε is small enough, to each Morse tree we will associate a J -holomorphic disk. We will indicate by C all the constants that depend only by the geometry (the riemannian manifold and the functions f_i) but are independent by ε . By Lemma 2 we have only a finite number of Morse tree. In the following we fix a Morse tree.

3.1 Approximative solution

Given a Morse tree for ε small we want to construct a surfaces Σ and an approximative J -holomorphic map $u : \Sigma \rightarrow T^*M$ with boundaries mapped in the trees of εdf_i .

The domain $\Sigma = \Sigma_\varepsilon$ is constructed from the metric ribbon tree in the following way. Fix a riemannian disk with three punctures and strip like ends. Define the vertex region as the complementary of the strips like ends. Replace any edge of length l with a flat strip $[0, \frac{l}{\varepsilon}] \times [0, 1]$ of length $\frac{l}{\varepsilon}$ (for external edges the strip has length infinity) and any vertex with the a vertex region, in a way compatible with the Ribbon structure. The boundary components of the ribbon tree are in correspondence with the boundary components of Σ . In particular any boundary component i of Σ is labeled with a Morse function f_i .

3.1.1 Strips

Fix a oriented edge e .

Define $S_e : [0, \frac{l}{\varepsilon}] \times [0, 1] \longrightarrow T^*M$ by

$$S_e(t, s) = \varepsilon[sdf_e^R(\gamma_e(\varepsilon t)) + (1 - s)df_e^L(\gamma_e(\varepsilon t))] \quad (3.1)$$

Then

$$\partial_t S_e(t, s) = (\varepsilon^2[s\nabla(df_e^R)(\gamma_e(\varepsilon t))\dot{\gamma}_e(\varepsilon t) + (1 - s)\nabla(df_e^L)(\gamma_e(\varepsilon t))\dot{\gamma}_e(\varepsilon t)], \varepsilon\dot{\gamma}_e(\varepsilon t))$$

$$\partial_s S_e(t, s) = (\varepsilon df_e(\gamma_e(\varepsilon t)), 0).$$

Using the Morse flow equation we get

$$\bar{\partial} S_e(t, s) = (\varepsilon^2[s\nabla(df_e^R)(\gamma_e(\varepsilon t))\dot{\gamma}_e(\varepsilon t) + (1 - s)\nabla(df_e^L)(\gamma_e(\varepsilon t))\dot{\gamma}_e(\varepsilon t)], 0) \quad (3.2)$$

Therefore

$$|dS_e|(t, s) \leq C\varepsilon(|(df_e^R)(\gamma_e(\varepsilon t))| + |(df_e^L)(\gamma_e(\varepsilon t))|) \quad (3.3)$$

$$|\bar{\partial} S_e|(t, s) \leq C\varepsilon^2(|\nabla(df_e^R)(\gamma_e(\varepsilon t))| + |\nabla(df_e^L)(\gamma_e(\varepsilon t))|). \quad (3.4)$$

3.1.2 Vertices

Now, we want to extend the maps (3.1) in any vertex region. First of all we will extend the "real" part \tilde{u} of u , that is the projection of u on M .

Fix a vertex region. We define \tilde{u} in a neighborhood of the vertex region in the following way. In a strip as in 3.1 starting from the vertex region we have $\tilde{u}(t, s) = \gamma_e(\varepsilon t)$. We want to define \tilde{u} in vertex region constantly equal to the corresponding vertex of the Morse graph. \tilde{u} is not smooth in the start of the strips at $t = 0$. We simply smooth the function $t\chi_{t \geq 0}$ near $t = 0$.

For any boundary component i define a function $\rho_i : \Sigma \rightarrow [0, 1]$ such that $\rho_i = 1$ on the boundary i and 0 in the other boundaries. Moreover we suppose that on a strip with upper label i , $\rho_i(t, s) = s$ in the usual flat coordinates.

Now put

$$u(z) = \varepsilon \Sigma \rho_i(z) df_i(\tilde{u}(z)).$$

It is immediate that

$$|du| \leq C\varepsilon. \quad (3.5)$$

3.2 Rescaled norms

Define the rescaled norms

$$\|\xi\|_{1,p,\varepsilon} = \varepsilon^{1/p} (\|\xi\|_p + \frac{\|\nabla \xi\|_p}{\varepsilon})$$

for $\xi \in \Gamma(u^*(TX))$ and

$$\|\eta\|_{p,\varepsilon} = \varepsilon^{1/p} \frac{\|\eta\|_p}{\varepsilon}$$

for $\eta \in \Omega^1(u^*(TX))$.

From (3.3) and (3.5) follows

$$\|du\|_{p,\varepsilon} \leq C. \quad (3.6)$$

From (3.4) and (3.5) follows

$$\|\bar{\partial}u\|_{p,\varepsilon} \leq C o(\varepsilon). \quad (3.7)$$

3.3 Linearization

In order to define the linearization of the $\bar{\partial}$ equation, we need first of all identify the space of $(0, 1)$ forms for surfaces near to Σ .

We need to define as "exponential" on element $\lambda \in T\mathcal{L}$ changes the complex structure in any internal edge.

We use a cutoff function

$$\rho : \Sigma_\varepsilon \rightarrow \mathbb{R}. \quad (3.8)$$

The function ρ is 0 in the vertex regions and equal 1 inside the edge regions. We

assume that ρ connects the functions 0 and 1 in the same way near each end of the strips.

Fix a strip $[0, \frac{l}{c}] \times [0, 1]$ with flat coordinates (t, s) . In the complex structure given by the action of λ the flat coordinates are $(\phi_\lambda(t), s)$ where

$$\phi_\lambda(t) = \int_0^t e^{\lambda\rho(\tau)} d\tau.$$

These coordinates give an identification between the $(0, 1)$ forms on any strip:

$$[fd\bar{z}] \circ \phi_\lambda = (f \circ \phi_\lambda)d\bar{z}.$$

Now, define

$$\mathcal{F}_u : \Gamma(u^*(TX)) \oplus T\mathcal{L} \rightarrow \Omega^{(0,1)}(u^*(TX))$$

with

$$\mathcal{F}_u(\xi, \lambda) = \Phi_u(\xi)^{-1} \bar{\partial}(\exp_u \xi \circ \phi_\lambda^{-1}) \circ \phi_\lambda \quad (3.9)$$

where remember that Φ is the parallel transport with respect to the hermitian connection and \exp is the exponential map with respect to a geodesic connection.

We can rewrite (3.9) on a strip as:

$$\begin{aligned} \mathcal{F}_u(\xi, \lambda) &= \frac{1}{2} \Phi_u(\xi)^{-1} [\partial_t(\exp_u \xi) \cdot ((\phi_\lambda^{-1})' \circ \phi_\lambda) + J\partial_s(\exp_u \xi)] \\ &= \frac{1}{2} \Phi_u(\xi)^{-1} [\partial_t(\exp_u \xi) e^{-\lambda\rho} + J\partial_s(\exp_u \xi)]. \end{aligned} \quad (3.10)$$

The linearization of (3.9) becomes:

$$d\mathcal{F}_u(0)(\xi, \lambda) = \bar{\partial}\xi + T_u(du, \xi)^{(0,1)} - \frac{1}{2}\rho(\partial_t u)\lambda.$$

In our case $|u|, |du| \leq C\varepsilon$ so by (A.1) we have $|T_u(du, \xi)| \leq C\varepsilon^2|\xi|$. Therefore up

to negligible terms in the rescaled norm we have on a strip

$$D_\varepsilon(\xi, \lambda) = \bar{\partial}\xi - \frac{1}{2}\varepsilon\lambda\rho\dot{\gamma}(\varepsilon t).$$

3.3.1 Frame

We would like to trivialize u^*TX such that the boundary condition on ξ becomes real.

For any boundary component i , define a function $\rho_i : \Sigma \rightarrow [0, 1]$ such that $\rho_i = 1$ on the boundary i and 0 in the other boundaries. Moreover we suppose that on a strip with upper label i , $\rho_i(t, s) = s$ in the usual flat coordinates.

As in section 3.1.2, let \tilde{u} be the projection on M of u . First we fix a parallel trivialization of γ^*TM (remember the notation of (2.5)). This induces a trivialization of \tilde{u}^*TM . We extend this to $u^*(TX)$ using

$$(v + i\omega) \rightarrow (v, \omega + \varepsilon\Sigma\rho_i\nabla(df_i)(\tilde{u})v)$$

for $v, \omega \in \tilde{u}^*TM$. Here we have used the splitting of TX as in Appendix A. Moreover this linear isomorphism change the rescaled norm in an equivalent norm.

On the strip e , the isomorphism is

$$(v + i\omega) \rightarrow (v, \omega + s\varepsilon\nabla(df_e^R)(\gamma(\varepsilon t))v + (1 - s)\varepsilon\nabla(df_e^L)(\gamma(\varepsilon t))v)$$

Define

$$A_e(t)(v + i\omega) = \nabla(df_e)v.$$

On the strip the linearization becomes

$$D_\varepsilon(\xi, \lambda) = \bar{\partial}\xi + \frac{1}{2}\varepsilon A_e(\varepsilon t)\xi - \frac{1}{2}\varepsilon\lambda_e\rho\dot{\gamma}(\varepsilon t) \tag{3.11}$$

up to negligible terms in the rescaled norm. As usual in a external strip there is not modular term (put $\lambda_e = 0$).

3.4 Floer transversality

Remember we have fixed a disk with strip like ends. Let Σ_V the disjoint union of copies of the disk labeled by V . Fix a correspondence between the strip like ends and H compatible with the ribbon structure. Observe that Σ_ε can be constructed gluing the components of Σ_V .

In the following we use the frame of section 3.3.1. Therefore we will think vectors fields as functions on \mathbb{C}^n .

A vector $v \in T\mathcal{E}$ assigns to any $e \in H$ a vector v_e in tangent space (of M) of the starting point of e . We can think to it as a local constant function defined on the strip regions of Σ_V . On the strip e it has value v_e .

As in 3.8, let $\rho : \Sigma_V \rightarrow \mathbb{R}$ be a cutoff function 0 in the vertex regions and equal 1 inside the edge regions. Given a vector $v \in T\mathcal{E}$, $(\bar{\partial}\rho)v$ is a well defined $(0,1)$ form on Σ_V .

Now fix a weight $\delta > 0$ small.

Define

$$\tilde{D}_0 : L_1^{p,\delta}(\Sigma_V, \mathbb{C}^n) \oplus T\mathcal{E} \rightarrow L^{p,\delta}(\Omega^{(0,1)}(\Sigma_V, \mathbb{C}^n))$$

as

$$\tilde{D}_0(\xi, v) = \bar{\partial}\xi + (\bar{\partial}\rho)v \tag{3.12}$$

Lemma 8. \tilde{D}_0 is surjective.

Proof. Suppose $\eta \in L^{p',-\delta}(\Omega^{(1,0)}(\Sigma_V, \mathbb{C}^n))$ be in coker of \tilde{D}_0 . Since $\langle \eta, \bar{\partial}\xi \rangle = 0$ for any $\xi \in L_1^{p,\delta}(\Sigma_V, \mathbb{C}^n)$ we have that $\bar{\partial}\eta = 0$ and η has real boundary conditions. The residues (that is the asymptotic values in the strip ends) of η define a vector $v_\eta \in T^*M^H$. Since the sum of the residues on any component of Σ_V is zero $v_\eta \in (T\mathcal{V})^\perp$. Since $\langle v_\eta, v \rangle = \langle \eta, (\bar{\partial}\rho)v \rangle = 0$ for any $v \in T\mathcal{E}$ we also have $v_\eta \in (T\mathcal{E})^\perp$. By Morse transversality $v_\eta = 0$ and therefore $\eta = 0$. \square

For $\varepsilon > 0$ small, consider Σ_ε as gluing of Σ_V . The gluing of the operator (3.12) is an operator

$$L_1^{p,\delta}(\Sigma_\varepsilon, \mathbb{C}^n) \oplus T\mathcal{E} \rightarrow L^{p,\delta}(\Omega^{(0,1)}(\Sigma_\varepsilon, \mathbb{C}^n))$$

given again by (3.12) with the obvious change of notation. We will keep to indicate this operator as \tilde{D}_0 .

Since $\delta > 0$ the gluing argument (see section 4.4.1 of [1]) provide a uniform bound for the right inverse in the weighted norm:

$$\| \tilde{Q}_0(\eta) \|_{1,p,\delta} \leq C \| \eta \|_{p,\delta} .$$

Let ξ_v the vector associated to v by (2.6). Put $\xi_v^\varepsilon(t) = \xi_v(\varepsilon t)$. Consider the operator

$$\tilde{D}_\varepsilon(\xi, v) = D_\varepsilon(\xi, 0) + (\tilde{\partial}\rho)\xi_v^\varepsilon \tag{3.13}$$

In the weighed norm the operator (3.13) is a small perturbation of \tilde{D}_0 therefore is still surjective. Moreover his right inverse \tilde{Q} is uniformly bounded.

Using the equation

$$\frac{d}{dt}\xi_v + A_\varepsilon\xi_v + \lambda_v\dot{\gamma} = 0$$

it is easy to check

$$D_\varepsilon(\xi + \rho\xi_v^\varepsilon, \lambda_v) = \tilde{D}_\varepsilon(\xi, v). \tag{3.14}$$

The right inverse for the operator $D = D_\varepsilon$ is therefore given by

$$Q(\eta) = (\xi + \rho\xi_v^\varepsilon, \lambda_v)$$

where $(\xi, v) = \tilde{Q}(\eta)$. The bound of \tilde{Q} gives

$$|v| + \| \xi \|_{1,p,\delta} \leq C \| \eta \|_{p,\delta} \tag{3.15}$$

with C independent by ε .

3.5 Right inverse bound

In this section we want to prove the following

Lemma 9. *The following estimate holds*

$$\| Q(\eta) \|_{1,p,\varepsilon} \leq C \| \eta \|_{p,\varepsilon}$$

for C independent by ε .

Let $Q(\eta) = (\xi, \lambda)$.

Using (B.6) we have

$$\| \xi \|_{1,p,\varepsilon} \leq C\varepsilon^{1/p} (\| \xi \|_p + \frac{\| \bar{\partial}\xi \|_p}{\varepsilon}) \leq C\varepsilon^{1/p} (\| \xi \|_p + \frac{\| D\xi \|_p}{\varepsilon}). \quad (3.16)$$

From (3.16) it is easy to see that the lemma is equivalent to the estimates

$$\| \xi \|_p \leq \frac{C}{\varepsilon} \| \eta \|_p \quad (3.17)$$

$$|\lambda| \leq \frac{C}{\varepsilon^{1/p'}} \| \eta \|_p \quad (3.18)$$

with C independent by ε .

3.5.1 Strip solution

Fix an edge of the Morse graph. Let $S = (0, \frac{1}{\varepsilon}) \times [0, 1]$ or $S = (-\infty, 0) \times [0, 1]$ be the correspondent strip in Σ_ε . As before we can use the frame of section 3.3.1 and think the sections as functions with values in \mathbb{C}^n .

Lemma 10. *There exists a constant C such that the following holds. For any $\eta \in \Omega^{(0,1)}(S, \mathbb{C}^n)$ there exist $\xi : S \rightarrow \mathbb{C}^n$ with boundary conditions $\xi(t, 0), \xi(t, 1) \in \mathbb{R}^n$ such that $D\xi = \eta$,*

$$\| \xi \|_p \leq \frac{C}{\varepsilon} \| \eta \|_p \quad (3.19)$$

and

$$\| \xi \|_\infty \leq \frac{C}{\varepsilon^{1/p'}} \| \eta \|_p. \quad (3.20)$$

Proof. Write

$$\eta(t, s) = \sum \eta_n(t) e^{ins}$$

$$\xi(t, s) = \sum \xi_n(t) e^{ins}$$

with $\eta_n = \eta_{-n}$ and $\xi_n = \xi_{-n}$. The equation $D\xi = \eta$ split in the equations

$$\dot{\xi}_n - n\xi_n + \varepsilon A(\varepsilon t)\xi_n = \eta_n.$$

Consider first the problem on the space of function with $\xi_0 = 0, \eta_0 = 0$. Using the analysis of Chapter 3 of [1], on this subspace the operator $\bar{\partial} : L^{1,p}(\mathbb{R} \times [0, 1]) \rightarrow L^p(\mathbb{R} \times [0, 1])$ with real boundary conditions is invertible. In this norm D is close in norm to $\bar{\partial}$ and therefore D is invertible with inverse Q of norm bounded independently by ε . Define $\xi = Q(\eta)$. We have $\|\xi\|_{1,p} \leq C \|\eta\|_p$, that is stronger than (3.19) and (3.20).

Now consider the problem for ξ_0 and η_0 . Put $\tilde{\xi}_0(t) = \xi_0(\frac{t}{\varepsilon})$ and $\tilde{\eta}_0(t) = \frac{1}{\varepsilon}\eta_0(\frac{t}{\varepsilon})$. We have

$$\frac{d}{dt}\tilde{\xi} + A(t)\tilde{\xi} = \tilde{\eta}. \quad (3.21)$$

This is nothing else that the equation for the Morse problem. This equation can be easily solved with

$$\|\tilde{\xi}\|_{1,p} \leq C \|\tilde{\eta}\|_p. \quad (3.22)$$

For an internal edge (finite length) it is easy to integrate equation (3.21). In an external edge the integration can be still done respecting the asymptotic condition and (3.22). However it is also possible get $\tilde{\xi}$ applying the right inverse of the Morse linearization (2.5) to $\tilde{\eta}$ extended by zero on the others edges. From (3.22) we have

$$\|\xi\|_p = \frac{1}{\varepsilon^{1/p}} \|\tilde{\xi}\|_p \leq C \frac{1}{\varepsilon^{1/p}} \|\tilde{\eta}\|_p = \frac{1}{\varepsilon} \|\eta\|_p$$

and

$$\|\xi\|_\infty = \|\tilde{\xi}\|_\infty \leq C \|\tilde{\xi}\|_{1,p} \leq C \|\tilde{\eta}\|_p = \frac{1}{\varepsilon^{1/p}} \|\eta\|_p.$$

□

3.5.2 Proof of the bound

Applying Lemma 10 on any strip we can find a $\tilde{\xi}$ in the strip region such that $D_\varepsilon \tilde{\xi} = \eta$.

By (3.19) we have

$$\|\tilde{\xi}\|_p \leq \frac{C}{\varepsilon} \|\eta\|_p.$$

Note that $D(\rho\tilde{\xi}) = (\bar{\partial}\rho)\tilde{\xi} + \rho\eta$, therefore using (3.20) we have

$$\|D(\rho\tilde{\xi})\|_p \leq \frac{C}{\varepsilon^{1/p'}} \|\eta\|_p.$$

Now put $\eta' = \eta - D(\rho\tilde{\xi})$. It satisfies

$$\|\eta'\|_p \leq \frac{C}{\varepsilon^{1/p'}} \|\eta\|_p.$$

Let $(\xi', v') = \tilde{Q}(\eta')$. Since η' has support in the vertex region (3.15) gives

$$\|\xi'\|_p + |v'| \leq C \|\eta'\|_p \leq \frac{C}{\varepsilon^{1/p'}} \|\eta\|_p.$$

Observe that

$$\|\rho\xi_{v'}^\varepsilon\|_p \leq \frac{C}{\varepsilon^{1/p}} |v'| \leq \frac{C}{\varepsilon} \|\eta\|_p.$$

Finally put $\xi = \rho\tilde{\xi} + \rho v' + \xi'$ and $\lambda = \lambda_{v'}$. According to (3.14) we have $D(\xi, \lambda) = \eta$. The previous estimates lead to $\|\xi\|_p \leq \frac{C}{\varepsilon} \|\eta\|_p$ and $|\lambda| \leq C \|\eta\|_{p,\varepsilon}$.

3.6 Quadratic estimates

Remember the definition of the map \mathcal{F}_u given in 3.9.

Lemma 11. *There exists a constant C such that, if $\|\xi\|_\infty \leq 1$ and $\lambda \leq 1$ we have*

$$\|d\mathcal{F}_u(\xi, \lambda) - D_u\| \leq C(\|\xi\|_{1,p,\varepsilon} + |\lambda|) \quad (3.23)$$

Proof. The lemma follows applying (3.6), (3.7), (3.24) and the following computation.

For $x \in M$ and $v, w \in T_x M$, put $E_x(v)w = \frac{d}{dt} \exp_x(v + tw)|_{t=0}$ and $\Psi_x(v, w) =$

$\frac{d}{dt}\Phi_x(v+tw)|_{t=0}$. Derive the identity $\Phi_u(\xi_t)\mathcal{F}_u(\xi_t, \lambda_t) = \bar{\partial}(\exp_u \xi_t \circ \phi_\lambda^{-1}) \circ \phi_\lambda$ using the hermitian connection and evaluate it in $t = 0$:

$$\Psi_u(\xi, \xi')\mathcal{F}_u(\xi, \lambda) + \Phi_u(\xi)d\mathcal{F}_u(\xi, \lambda)(\xi', \lambda') = \frac{d}{dt}\bar{\partial}(\exp_u \xi_t)|_{t=0} + \dots$$

Observe that $\Psi_x(v, w)$ is linear in w and zero for $v = 0$. So $|\Psi_x(v, w)| \leq C|v||w|$ and therefore

$$|\Psi_u(\xi, \xi')| \leq C|\xi||\xi'|.$$

Using a switch of derivative we have

$$\frac{d}{dt}\bar{\partial}(\exp_u \xi_t) = \bar{\partial}[E_u(\xi)\xi'] + T_{\exp_u \xi}(d \exp_u \xi, E_u(\xi)\xi')^{(0,1)}.$$

Observe that

$$\begin{aligned} & |T_{\exp_u \xi}(d \exp_u \xi, E_u(\xi)\xi') - \Phi_u(\xi)T_u(du, \xi')| \\ & \leq |T_{\exp_u \xi}(d \exp_u \xi, E_u(\xi)\xi') - T_{\exp_u \xi}(\Phi_u(\xi)du, \Phi_u(\xi)\xi')| \\ & \quad + |T_{\exp_u \xi}(\Phi_u(\xi)du, \Phi_u(\xi)\xi') - \Phi_u(\xi)T_u(du, \xi')| \\ & \leq C(|du||\xi| + |\nabla\xi|)|\xi'| + C|du||\xi||\xi'| \end{aligned}$$

and

$$\begin{aligned} |\nabla[E_u(\xi)\xi'] - \Phi_u(\xi)\nabla\xi'| & \leq |[E_u(\xi) - \Phi_u(\xi)]\nabla\xi'| + |\nabla[E_u(\xi)]\xi'| \\ & \leq |\xi||\nabla\xi'| + C(|du||\xi| + |\nabla\xi|)|\xi'|. \end{aligned}$$

Finally, in the strips we have also

$$|\partial_t(\exp_u \xi)e^{\lambda\rho\lambda'} - \rho\partial_t u\lambda'| \leq C(|du| + |\nabla\xi|)|\lambda||\lambda'|.$$

□

3.6.1 Sobolev embedding

Lemma 12. *There exist a constant C such that we have*

$$\| f \|_{\infty} \leq C \| f \|_{1,p,\varepsilon} \quad (3.24)$$

for any $f : \Sigma \rightarrow \mathbb{C}$ with real value in the boundary.

Proof. Consider first the case of f supported on a strip $(0, \frac{l}{\varepsilon}) \times [0, 1]$. Define $\tilde{f}(t, s) = f(\frac{t}{\varepsilon}, s)$ for $(t, s) \in (0, l) \times [0, 1]$. From $|\nabla \tilde{f}| \leq \frac{1}{\varepsilon} |\nabla f|$ and a changing of variable of integration we have

$$\| f \|_{\infty} = \| \tilde{f} \|_{\infty} \leq C \| \tilde{f} \|_{1,p} \leq C \| f \|_{1,p,\varepsilon} .$$

Suppose that l is the minimal length of the edges of the Morse tree. Consider the case of f supported in the set points of distance less than $\frac{l}{4\varepsilon}$ from a vertex region. Let f_V and f_S the restriction of f on the vertex region and the strips respectively. Attach to any end of the vertex region a strip of length $\frac{l}{4}$. On it we can define the function $\tilde{f} = f_V + f_S$. We have the Poincare inequality:

$$\begin{aligned} \| f \|_{\infty} &= \| \tilde{f} \|_{\infty} \leq C \| \nabla \tilde{f} \|_p \leq C (\| \nabla f_V \|_p + \| \nabla f_S \|_p) \\ &\leq \frac{C}{\varepsilon^{1/p}} \| \nabla f \|_p \leq C \| f \|_{1,p,\varepsilon} . \end{aligned}$$

Now fix $z \in \Sigma_{\varepsilon}$. We can find a function ρ supported on a region of the type considered before such that $\rho(z) = 1$ and $|\nabla \rho| \leq 4\frac{\varepsilon}{l}$. The previous estimates lead to

$$|f(z)| \leq \| \rho f \|_{\infty} \leq C \| \rho f \|_{1,p,\varepsilon} \leq C \| f \|_{1,p,\varepsilon} .$$

Taking the sup in z we get the lemma. □

3.7 Exact solution

In this section we omit the modular parameter in the notation since the split of $\Gamma(u^*(TX)) \oplus T\mathcal{L}$ will not play any role.

Let ξ, ξ' be vectors in the image of Q . Put $\xi_t = t\xi + (1-t)\xi'$. We compute

$$\begin{aligned} [\xi - Q\mathcal{F}_u(\xi)] - [\xi' - Q\mathcal{F}_u(\xi')] &= \xi - \xi' - \int_0^1 Qd\mathcal{F}_u(\xi_t)(\xi - \xi')dt \\ &= \int_0^1 Q[d\mathcal{F}_u(0) - d\mathcal{F}_u(\xi_t)](\xi - \xi')dt. \end{aligned}$$

where in the last step we have used $Qd\mathcal{F}_u(0) = Id$ on the image of Q . Now suppose that $\|\xi\|_\infty, \|\xi'\|_\infty \leq \frac{1}{2}$ so $\|\xi_t\|_\infty \leq 1$ and we can apply (3.23) to get

$$\|[\xi - Q\mathcal{F}_u(\xi)] - [\xi' - Q\mathcal{F}_u(\xi')]\| \leq C \|Q\| (\|\xi\| + \|\xi'\|) \|\xi - \xi'\|. \quad (3.25)$$

Define by induction

$$\xi_{n+1} = \xi_n - Q\mathcal{F}_u(\xi_n)$$

with $\xi_0 = 0$. Fix $\delta > 0$ small such that $\delta \leq \frac{1}{4C\|Q\|}$ and $\delta \leq \frac{1}{2C}$ where C is bigger of the constant in (3.23) and (3.24). We want to prove by induction that

$$\|\xi_i - \xi_{i-1}\| \leq \frac{\delta}{2^{i-1}}. \quad (3.26)$$

Since $\xi_1 = -Q\mathcal{F}_u(0)$ the starting induction is satisfied for ε small enough. Now, suppose that (3.26) hold for $i \leq n$. In particular we have $\|\xi_i\| \leq \delta$ for $i \leq n$. Since $\delta \leq \frac{1}{2C}$ using (3.24) we can apply (3.25) and get

$$\|\xi_{n+1} - \xi_n\| \leq C \|Q\| (2\delta) \|\xi_n - \xi_{n-1}\| \leq \frac{1}{2} \|\xi_n - \xi_{n-1}\|.$$

Chapter 4

Convergence to trees

Let $u : \Sigma \rightarrow X$ a J holomorphic disk bounding $\Gamma_{\varepsilon df}$. In this chapter we prove that, for ε small enough, there exist a Morse graph of the functions f_i such that u is constructed as in the preview chapter.

4.1 Decomposition of a disk

Fix on the space of punctured disks a consistent universal choice of strip-like ends (see section 9g of [6]). An easy consequence of Lemma 9.2 of [6] is the following

Lemma 13. *Fix $d \geq 2$. For any $k \leq d$ there exists a bounded region \mathcal{B}_k of the Teichmüller space of the riemannian disks with $k + 1$ punctures such that any $d + 1$ punctured riemannian disk can be obtained in unique way gluing disks in $\cup_k \mathcal{B}_k$.*

Lemma 13 decompose any punctured disk Σ in vertex regions and edges regions. The vertex regions are the complementary of the strip like ends of the disks in \mathcal{B}_k . The edges regions are the strips in Σ complementary to the vertices regions.

Choose a metric for any in disk \mathcal{B}_k compatible with the the flat metric of the strip like ends. The decomposition of Lemma 13 induces a metric on any disk.

This decomposition associate to any punctured disk a ribbon tree. The boundary components of the disk correspond to the boundary components of the ribbon tree.

Now we can apply these construction to our J holomorphic disk $u : \Sigma \rightarrow X$ and

get a ribbon tree Γ . The boundary conditions label any boundary component of Γ with a function on M .

In the next sections we will prove that, for ε small enough, there exists a rigid Morse tree of Γ for which u is the solution constructed in the chapter before. In particular Γ has to be trivalent.

4.2 gradient bound

Lemma 14. *There exist a constant C such that for any J holomorphic map $u : \Sigma \rightarrow X$ bounding $\Gamma_{\varepsilon df}$ we have*

$$\sup |u| \leq C\varepsilon.$$

Proof. Locally, let (x, y) be an holomorphic coordinate system in Σ . Let $\pi(u(x, y)) = q(x, y) \in M$ and $u(x, y) = p(x, y) \in T_{q(x, y)}^*M$. The J -holomorphic equation $\partial_x u + J\partial_y u = 0$ becomes the system

$$\partial_x q + \nabla_y p = 0$$

$$\partial_y q - \nabla_x p = 0.$$

Therefore the Laplacian of the function $\frac{1}{2}|p|^2$ is positive:

$$\frac{1}{2}\Delta|p|^2 = \partial_x \langle p, \nabla_x p \rangle + \partial_y \langle p, \nabla_y p \rangle = |\nabla_x p|^2 + |\nabla_y p|^2 + \langle p, \nabla_x \nabla_x p + \nabla_y \nabla_y p \rangle \geq 0$$

where in the last step we have used $\nabla_x \nabla_x p + \nabla_y \nabla_y p = -\nabla_x \partial_y q + \nabla_y \partial_x q = 0$. By the maximal principle $|p|^2$ has maximum on the boundary. The Lagrangian boundary conditions of u imply $|u| \leq C\varepsilon$ on the boundary. \square

Lemma 15. *There exists a constant C such that for any J -holomorphic map $u : \Sigma \rightarrow X$ bounding $\Gamma_{\varepsilon df}$ we have*

$$\sup |du| \leq C\varepsilon.$$

Proof. Let $u_k : \Sigma_k \rightarrow X$ a sequence of J -holomorphic disks bounding the trees of $\varepsilon_k df_i$. Suppose that $R_k = \frac{1}{\varepsilon_k} \sup |du_k|$ goes to infinity. Let $z_k \in \Sigma_k$ such that

$|du_k|(z_k) \geq \frac{1}{2}\varepsilon_k R_k$. For any k fix a complex isomorphism between $T_{z_k}X$ and \mathbb{C}^n . Using these isomorphisms we can define a sequence of maps

$$\phi_k : D_k \rightarrow \mathbb{C}^n$$

$$\phi_k(v) = \frac{1}{\varepsilon_k} \exp_{u_k(z_k)}^{-1} u_k\left(z_k + \frac{v}{R_k}\right).$$

Here $D_k = B(0, l_k) \cap \{\operatorname{Im} z \leq m_k\}$ for some sequence l_k and m_k . We can assume that l_k goes to infinity and $\varepsilon_k l_k$ converges to zero when $k \rightarrow \infty$.

By $|du_k|(z_k) \geq \frac{1}{2}\varepsilon_k R_k$ we have

$$|d\phi_k(0)| \geq \frac{1}{2} \tag{4.1}$$

and by Lemma 14

$$|\operatorname{Im} \phi_k| \leq C. \tag{4.2}$$

Since $\sup |\phi_k| \leq l_k \sup |d\phi_k|$ equation (A.2) leads to

$$\sup |\varepsilon_k d\phi_k| \leq \sup \frac{1}{R_k} |du_k| + C\varepsilon_k l_k \sup |d\phi_k| (\varepsilon_k \sup |d\phi_k| + \sup \frac{1}{R_k} |du_k|)$$

and therefore

$$\sup |d\phi_k| \leq 2$$

for k big enough.

Applying equation (A.3) we have $|\bar{\partial}\phi_k| \leq C\varepsilon_k |\phi_k| |d\phi_k|$. Since $|d\phi_k|$ is bounded, $|\phi_k|$ is bounded on bounded sets, we have $|\bar{\partial}\phi_k| \rightarrow 0$ uniformly on bounded sets.

By standard elliptic estimates there exists a subsequence of ϕ_k convergent to a holomorphic map ϕ defined on the complex plane or on the half complex plane depending if m_k diverges or is bounded. By (4.1) ϕ is not constant and by (4.2) the imaginary part of ϕ is bounded. This is impossible.

□

4.3 Open string motion

Fix a J holomorphic disk $u : \Sigma \rightarrow X$ bounding the graphs of εdf_i . Fix a strip $[0, T] \times [0, 1]$ of Σ . Call f_R and f_L the functions associated to the boundaries $[0, T] \times 0$, $[0, T] \times 1$. Define $f = f_R - f_L$.

Fix $\delta > 0$ small compared to the constant in (A.2).

Lemma 16. *There exists a constant C such that the following holds. Suppose that the projection on M of $u([0, T] \times [0, 1])$ is outside the δ neighborhood of the critical points of f . Then for ε small enough there exist a Morse trajectory γ of f such that*

$$u = \exp_S \xi \tag{4.3}$$

such that $|\xi| \leq C\varepsilon$. Here S is the strip associated to γ as in (3.1).

Proof. For any $t \in [0, T]$, the length of the string $u_t(\cdot) = u(t, \cdot)$ is small than $C\varepsilon$. We want to study the trajectory of the string up to an error of this order.

Pick a Morse trajectory γ starting close to the projection of u_0 . Let T_0 such that it is possible to write formula (4.3) for $|\xi| \leq \delta$ in the strip $[0, T_0] \times [0, 1]$. We will prove that we can take $T_0 = T$ and $|\xi| \leq C\varepsilon$.

From (A.2) using $|dS| \leq C\varepsilon$ and the gradient bound of u we have

$$|\nabla \xi| \leq |dS| + |du| + C|\xi|(|\nabla \xi| + |dS|) \leq C\varepsilon + C\delta|\nabla \xi|.$$

For δ small compared to C this leads to $|\nabla \xi| \leq C\varepsilon$.

Also, (A.3) implies

$$|\bar{\partial} \xi| \leq |\bar{\partial} S| + C\varepsilon|\xi| \leq C\varepsilon^2 + C\varepsilon|\xi|. \tag{4.4}$$

We can trivialize the $\gamma^*(TM)$ using the parallel transport and think ξ as a function with values in the complexification of a fixed real vector space. The real and imaginary part correspond respectively to the horizontal and vertical part of ξ .

Let $\xi_t(s) = \xi(t, s)$. We need to estimate the norm of $\int \text{Re } \xi_t ds$. This can be

considered the deviation from γ of the projection on M of the of the barycenter of the string.

Since $|\nabla\xi| \leq C\varepsilon$ we have

$$\sup |\xi| \leq \left| \int \operatorname{Re} \xi_t \right| + C\varepsilon.$$

and we have

$$\frac{d}{dt} \left| \int \operatorname{Re} \xi_t \right| \leq \left| \frac{d}{dt} \int \operatorname{Re} \xi_t \right| \leq \int |\bar{\partial}\xi|(t, s) ds + |\operatorname{Im} (\xi_t(1) - \xi_t(0))|.$$

The boundary condition implies that $|\operatorname{Im} \xi| \leq C\varepsilon|\xi|$ on the boundary. Using this and (4.4)

$$\frac{d}{dt} \left| \int \operatorname{Re} \xi_t \right| \leq C\varepsilon^2 + C\varepsilon \sup |\xi| \leq C\varepsilon^2 + C\varepsilon \left| \int \operatorname{Re} \xi \right|.$$

Integrating this inequality on the interval $[0, T_0]$ we get

$$\left| \int \operatorname{Re} \xi_{T_0} \right| \leq C\varepsilon(e^{C\varepsilon T_0} - 1).$$

Now Lemma 17 implies that

$$\left| \int \operatorname{Re} \xi_t \right| \leq C\varepsilon$$

for any $t \in [0, T_0]$. Together with the gradient bound this implies

$$|\operatorname{Re} \xi| \leq \left| \int \operatorname{Re} \xi \right| + C\varepsilon \leq C\varepsilon.$$

From Lemma 14 we also have $|\operatorname{Im} \xi| \leq C\varepsilon$. Therefore $|\xi| \leq C\varepsilon$.

If we take ε small enough this implies that T_0 has to be equal to T .

□

Lemma 17. *Under the same hypotheses of lemma 16 we have*

$$T \leq \frac{C}{\delta^2\varepsilon}.$$

Proof. There exist a constant C depending by the slope of the tree of df , such that

in the region outside the δ neighborhood of the critical points the Lagrangians dist more than $\frac{\delta\varepsilon}{C}$. Therefore

$$\delta\varepsilon \leq C \int_0^1 |\partial_s u|(t, s) ds$$

for any $t \in [0, T]$. From other hand the bound on the energy gives

$$\int |\partial_s u|^2 dt ds \leq C\varepsilon.$$

These inequalities lead to $\delta\varepsilon T \leq C(T\varepsilon)^{\frac{1}{2}}$ and we get the lemma. \square

4.4 Building a Morse tree

In order to apply Lemma 16 we will need the following straightforward Lemma. We use the same notation of Lemma 16.

Lemma 18. *Let $[0, T] \times [0, 1]$ be a strip of Σ . There exist real numbers $\{a_0 = 0 < b_0 < a_1 < b_1 < \dots < b_k = T\}$ such that for any i the image the projection on M of $u([a_i, b_i] \times [0, 1])$ dist more than $\frac{\delta}{2}$ from any critical point of f and the projection of $u([b_i, a_{i+1}] \times [0, 1])$ lives inside a δ neighborhood of a critical point of f .*

Lemma together with Lemma 16 permits to construct a quasi Morse trajectory. Now we will use the analysis of section 2.3 in order to construct a Morse graph of Γ .

Lemma 19. *There exist a Morse tree such that if u' is the approximative solution of the tree*

$$u = \exp_{u'} \xi \circ \phi_\lambda^{-1}$$

for some satisfying

$$\|\xi\|_\infty + |\lambda| \leq C\varepsilon \tag{4.5}$$

Proof. For ε small, the gradient bound implies that the vertex regions of lemma 13 have imagine of diameter small compared to the δ of lemma 7. Therefore the ends of the strips will define a point of M^H close to \mathcal{V} . Lemma 16, Lemma 4.4 and Lemma 5

imply that this point is also close to \mathcal{E} . Therefore we get a δ quasi tree, with $\delta \leq C\varepsilon$.
Apply Lemma 7 gives a Morse tree.

The estimate (4.5) follows from (2.10).

□

Now we want to improve the estimate 4.5 proving that also $\|\xi\|_{1,p,\varepsilon}$ is small. The estimate for $|\lambda|$ implies

$$|\bar{\partial}(\exp_{u'} \xi)| \leq C\varepsilon(|\nabla\xi| + |du'|).$$

From (A.2)

$$|\bar{\partial}\xi| \leq |\bar{\partial}u'| + C\varepsilon(|\nabla\xi| + |du'|).$$

We also have

$$|D\xi| \leq |\bar{\partial}\xi| + C\varepsilon|du'|$$

and therefore

$$\|D\xi\|_{p,\varepsilon} \leq C\|\bar{\partial}u'\|_{p,\varepsilon} + C\varepsilon(\|\nabla\xi\|_{p,\varepsilon} + \|du'\|_{p,\varepsilon}).$$

Since $\xi = QD\xi$ and Q is bounded we have

$$\|\xi\|_{1,p,\varepsilon} \leq \|Q\| \|D\xi\|_{p,\varepsilon} \leq C\|\bar{\partial}u'\|_{p,\varepsilon} + C\varepsilon(\|\nabla\xi\|_{p,\varepsilon} + \|du'\|_{p,\varepsilon}).$$

For ε small this leads to

$$\|\xi\|_{1,p,\varepsilon} \leq C\|\bar{\partial}u'\|_{p,\varepsilon} + C\varepsilon\|du'\|_{p,\varepsilon}.$$

By (3.6) and (3.6), $\|\xi\|_{1,p,\varepsilon} \leq C\varepsilon$. Therefore u is the solution associated to the Morse tree in the preview chapter.

Appendix A

Cotangent bundle

Let $\pi : T^*M \rightarrow M$ be the projection of the cotangent bundle of M . Fix a metric on M . Using the metric on M we will identify T^*M with TM . The Levi-Civita connection induces a connection on T^*M and hence a natural splitting

$$T(TM) = \pi^*(TM) \oplus \pi^*T^*M$$

in horizontal and vertical vectors. The pullback of the Levi-Civita connection defines a connection ∇ on $T(T^*M)$. ∇ is not torsion free. Actually we have

$$\mathbb{T}(Z)(X, Y) = (R(X_h, Y_h)Z, 0) \tag{A.1}$$

where $Z \in TM$ and $X = (X_h, X_v), Y = (Y_h, Y_v) \in T(TM)$.

Let J be the canonical complex structure on $T(T^*M)$ given by $J(v, w) = (-w, v)$. Observe that ∇ is hermitian with respect to J .

Let Φ be the parallel transport with respect to the hermitian connection ∇ and let \exp is the exponential map with respect to some connection (for us this will be applied to a geodesic connection). For any map $u : \Sigma \rightarrow T^*M$ and section $\xi \in \Gamma(u^*(T(T^*M)))$

$$|\Phi_u(\xi)^{-1}d \exp_u \xi - du - \nabla \xi| \leq C|\xi|(|\nabla \xi| + |du|) \tag{A.2}$$

where the constant is uniformly bounded on compact sets of $T(T^*M)$ (for example

$|\xi|$ and $|u| \leq 1$). (A.2) is an easy consequence of the bound of the derivatives of $\exp : T(T^*M) \rightarrow T^*M$ on compact sets. We will need to apply (A.2) for a smooth family of connections on a compact interval, so it is important to observe that it holds uniformly.

Observe that (A.2) in particular imply

$$|\Phi_u(\xi)^{-1} \bar{\partial} \exp_u \xi - \bar{\partial} u - \bar{\partial} \xi| \leq C|\xi|(|\nabla \xi| + |du|) \quad (\text{A.3})$$

Appendix B

elliptic estimate

Lemma 20. *For any pointed disk with strip like ends there exist a constant C such that the following holds. For any complex function f compact support and real restricted to the boundary*

$$\| \nabla f \|_p \leq C \| \bar{\partial} f \|_p \tag{B.1}$$

Proof. We will prove the estimate for $f \in L^{1,p,\delta}$, with δ be a negative weight. We can suppose that the integral of f is zero. $\bar{\partial} : L^{1,p,\delta} \rightarrow L^{p,\delta}$ is an Fredholm operator with kernel the constants. Therefore in the space orthogonal of the constant functions we have

$$\| f \|_{1,p,\delta} \leq C \| \bar{\partial} f \|_{p,\delta} . \tag{B.2}$$

Let ρ be a cutoff function equal to 1 in the strips and 0 in the vertex region. From (B.2) we have

$$\| (\bar{\partial}\rho)f \|_p \leq C \| f \|_{p,\delta} \leq C \| \bar{\partial} f \|_{p,\delta} \leq C \| \bar{\partial} f \|_p . \tag{B.3}$$

Since ρf is a function on the strips, the analysis of Chapter 3 of [1] implies

$$\| \nabla(\rho f) \|_p \leq C \| \bar{\partial}(\rho f) \|_p \tag{B.4}$$

In [1] this is done for cylinders, but it extends easily to strips. Therefore

$$\| \nabla(\rho f) \|_p \leq C \| (\bar{\partial}\rho)f \|_p + C \| \rho \bar{\partial}f \|_p \leq C \| \bar{\partial}f \|_p \quad (\text{B.5})$$

where we have used (B.3).

The estimate (B.2) implies also that (B.1) holds inside the vertex region. This and (B.4) give the lemma. \square

Lemma 21. *For any function $f \in L^{1,p}(\Sigma_\varepsilon, \mathbb{C})$ the following estimate holds*

$$\| \nabla f \|_p \leq C(\| \bar{\partial}f \|_p + \varepsilon \| f \|_p). \quad (\text{B.6})$$

Proof. We can suppose that f is smooth compact support. Pick a set of functions $\rho_i : \Sigma_\varepsilon \rightarrow \mathbb{R}$ with $\sum_i \rho_i = 1$, $|d\rho_i| \leq C\varepsilon$ and such that any ρ_i has support in vertex region with attached the edge regions starting from it. Remember that the length of the internal strips of Σ_ε are at least $\frac{1}{\varepsilon}$ the minimal length of the edges of the Morse graph. We can apply (B.1) to any function $\rho_i f$ and get

$$\| \nabla f \|_p \leq \sum_i \| \nabla(\rho_i f) \|_p \leq C \sum_i \| \bar{\partial}(\rho_i f) \|_p \leq C(\| \bar{\partial}f \|_p + \varepsilon \| f \|_p).$$

\square

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