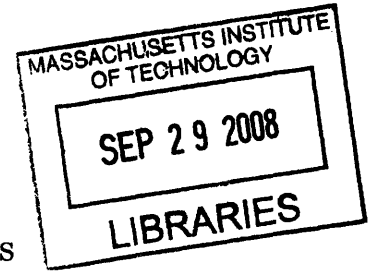


Analogues of Kähler Geometry on Sasakian Manifolds

by

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A.B., Harvard University (2003)



Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

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Abstract

A Sasakian manifold S is equipped with a unit-length, Killing vector field ξ which generates a one-dimensional foliation with a transverse Kähler structure. A differential form α on S is called *basic* with respect to the foliation if it satisfies

$$\iota_{\xi}\alpha = \iota_{\xi}d\alpha = 0.$$

If a compact Sasakian manifold S is *regular*, i.e. a circle bundle over a compact Kähler manifold, the results of Hodge theory in the Kähler case apply to basic forms on S . Even in the absence of a Kähler base, there is a basic version of Hodge theory due to El Kacimi-Alaoui. These results are useful in trying to imitate Kähler geometry on Sasakian manifolds; however, they have limitations. In the first part of this thesis, we will develop a “transverse Hodge theory” on a broader class of forms on S . When we restrict to basic forms, this will give us a simpler proof of some of El Kacimi-Alaoui’s results, including the basic $\partial\bar{\partial}$ -lemma. In the second part, we will apply the basic $\partial\bar{\partial}$ -lemma and some results from our transverse Hodge theory to conclude (in the manner of Deligne, Griffiths, and Morgan) that the real homotopy type of a compact Sasakian manifold is a formal consequence of its basic cohomology ring and basic Kähler class.

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Chapter 1

Introduction

Let M be a $(2m + 1)$ -dimensional manifold with contact form η . A contact metric structure consists of additional data (g, Φ, ξ) satisfying the following:

- ξ is the unique vector field such that $\iota_\xi \eta = 1$ and $\iota_\xi d\eta = 0$.
- Φ is an endomorphism of TM which is zero on ξ and is an almost complex structure on the contact distribution $\mathcal{D} := \text{Ker}(\eta) \subset TM$.
- g is a Riemannian metric satisfying $\eta = g(\xi, \cdot)$, $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$, and $g(\Phi X, Y) = d\eta(X, Y)$.¹

Any (strict) contact manifold (M, η) admits a contact metric structure [YK].

A contact metric structure is called *normal* if Φ also satisfies the integrability condition

$$N_\Phi + d\eta \otimes \xi = 0,$$

where N_Φ is the Nijenhuis tensor of Φ . A normal contact metric structure is also called a *Sasakian* structure. This definition is due to Sasaki and Hatakeyama [SH].

A Sasakian structure may also be characterized in terms of the metric cone on M , which is the Riemannian manifold $(X = M \times \mathbb{R}_{>0}, g_X)$, where

$$g_X = dr^2 + r^2 g_M$$

and r is the coordinate on $\mathbb{R}_{>0}$. We define an almost complex structure J on X by $J(r \frac{\partial}{\partial r}) = \xi$ and $J = \Phi$ on \mathcal{D} . Then (M, η, ξ, g, Φ) is Sasakian if and only if (X, g_X, J) is Kähler.

¹Some references, including [FOW], replace $d\eta$ with $\frac{1}{2}d\eta$. This scaling is insignificant.

In recent years, physicists have become increasingly interested in Sasakian manifolds for their role in the AdS/CFT correspondence of type IIB string theory [MSY]. In particular, a conformal field theory is dual to $AdS_5 \times M^5$, where AdS_5 is anti-de Sitter space and M is a 5-dimensional Sasaki-Einstein manifold (i.e., Sasakian with Ricci curvature proportional to the metric). For any Sasaki-Einstein manifold M^{2m+1} , the requirement that $X = M \times \mathbb{R}_{>0}$ is Kähler forces the Einstein constant to be $2m$ and forces X to be Calabi-Yau. So Sasaki-Einstein manifolds are of interest in the study of noncompact Calabi-Yau manifolds.

Charles Boyer and the late Krzysztof Galicki have published prodigiously on Sasakian geometry, culminating in the recent publication of their book [BG], which is an up-to-date exposition of the field.

The so-called *Reeb field* ξ on a Sasakian manifold M is unit-length and Killing. Let \mathcal{F}_ξ be the one-dimensional foliation generated by ξ . Around any point $p \in M$ there exist coordinates (x, y^1, \dots, y^{2m}) such that $\xi = \frac{\partial}{\partial x}$. Such a coordinate chart is called a *foliated coordinate chart*. If there exists a positive integer k such that around every point in M there is a foliated chart where each leaf passes through at most k times, ξ generates an effective S^1 -action on M . If $k = 1$ (i.e. S^1 acts freely), M is called *regular* and it is the total space of an S^1 -bundle over a base Kähler manifold B . In this case η is a connection one-form for the bundle, and the Kähler form on the base is $d\eta$. If $k > 1$ and the action is only locally free, M is called *quasi-regular* and it is the total space of an S^1 -bundle over a Kähler orbifold. If no such k exists, M is called *irregular*.

In the regular case, many questions about Sasakian manifolds may be expressed in terms of the base. For example, a Sasakian metric g on M is Einstein if and only if the transverse metric $d\eta(\cdot, \Phi\cdot)$ on the base B is Einstein with constant $m+1$. Generalizing to the quasi-regular case is not difficult, but in the irregular case there is no Kähler base. There are many interesting irregular Sasakian manifolds (including an infinite family of irregular Sasaki-Einstein metrics on $S^2 \times S^3$ [MS]), so it is important not to assume quasi-regularity.

Even in the irregular case, we can often act as if there is a Kähler base by considering so-called *basic forms* on M . A differential form α on M is called basic if

$$\iota_\xi \alpha = 0 \quad \text{and} \quad \iota_\xi d\alpha = 0.$$

The exterior derivative d takes basic forms to basic forms. Defining the operator d_B by the restriction of d to basic forms, we define the *basic cohomology* $H_B^*(M)$ to be the cohomology of d_B . Using the complex structure Φ , we may write $d_B = \partial_B + \bar{\partial}_B$. In the more general setting of a Riemannian foliation with transverse Kähler structure, El Kacimi-Alaoui has worked out a “basic Hodge theory.” Among his results are a basic version of the $\partial\bar{\partial}$ -lemma:

Proposition 1.1 (Basic $\partial\bar{\partial}$ -lemma of [EKA]). Let α be a basic form which is ∂_B -

and $\bar{\partial}_B$ -closed and $\bar{\partial}_B$ - (or ∂_B - or d_B -)exact. Then $\alpha = \partial_B \bar{\partial}_B \beta$ for some β .

El Kacimi Alaoui's work on basic Hodge theory uses a complicated argument in which one must work on the space of closures of leaves of a lifted foliation on a principal $SO(2m)$ -bundle over M . In section 3.1 we will outline this argument in greater detail.

As a strategy for doing geometry on M , one may attempt to express a problem in terms of basic functions and forms. Using El Kacimi-Alaoui's results, this is very similar to solving a problem on a compact Kähler manifold. One illustration of this strategy is recent work in which Futaki, Ono, and Wang [FOW] are able to prove the existence of a Sasaki-Einstein metric on a compact toric Sasakian manifold with $c_1(\mathcal{D}) = 0$ and positive "basic first Chern class"².

The strategy in [FOW] is to look for a function f such that the deformed transverse metric $d\tilde{\eta} := d\eta + 2\sqrt{-1}\partial_B \bar{\partial}_B f$ is a *transverse Kähler-Ricci soliton*. Here the deformed Sasakian structure is given by $\tilde{\eta} = \eta + d_B^c f$, $\tilde{\xi} = \xi$, and $\tilde{\Phi} = \Phi - (\xi \otimes d_B^c f) \circ \Phi$. Letting $\tilde{\rho}^T$ denote the transverse Ricci form (see footnote 2), the deformed metric is a soliton if

$$\tilde{\rho}^T - (m+1)d\eta = L_Z d\tilde{\eta}$$

for some Hamiltonian holomorphic vector field Z . (Cf. [FOW] for definition.) One uses the condition $c_1^B > 0$ and the basic $\partial\bar{\partial}$ -lemma to write $\rho^T - (m+1)d\eta = \sqrt{-1}\partial_B \bar{\partial}_B g$ for some g . Letting θ_Z denote the Hamiltonian function for Z , we obtain a Monge-Ampère equation

$$\frac{\det(g_{i\bar{j}}^T + f_{i\bar{j}})}{\det(g_{i\bar{j}}^T)} = \exp(-(m+1)f - \theta_Z - Zf + g). \quad (1.1)$$

The problem is now essentially a Kähler problem. To show solvability of (1.1), Futaki et al may use the Kähler-Ricci flow arguments in [WZ].³

The above suggests a general program of extending results on compact Kähler manifolds to the Sasakian setting. A key question is the following: to what extent can we act as if there is a Kähler base without assuming quasi-regularity? As Futaki et al's result shows, El Kacimi-Alaoui's basic Hodge theory (and in particular the basic $\partial\bar{\partial}$ -lemma) is an important tool for such a program.

Basic Hodge theory has a major limitation, however: all forms and functions are assumed to be basic. This is especially problematic for irregular Sasakian structures. If the Reeb foliation has a leaf which is dense in M , there are no basic functions. Then

²Using the transverse metric $d\eta$, one defines a "transverse Ricci form," which is a global basic form whose d_B -cohomology class (up to a constant) is called the basic first Chern class, $c_1^B(M)$.

³They do have to have to prove a C^0 estimate for f , however, which does not follow immediately from [WZ].

the basic $\partial\bar{\partial}$ -lemma is vacuous on 2-forms, and assuming $c_1^B(M) > 0$ is the same as assuming the existence of a Sasaki-Einstein metric.⁴ Ideally one would obtain Hodge theory-like results under less rigid assumptions on differential forms. This would require defining ∂ and $\bar{\partial}$ on a larger class of differential forms. Such results might also be useful in describing more general ways to deform Sasakian structures.

Let us call a differential form α on M *transverse* if $\iota_\xi\alpha = 0$. In the first part of this thesis, we will derive a “transverse” Hodge theory which reduces to the basic Hodge theory on basic forms. We will define operators $d_T, \partial_T,$ and $\bar{\partial}_T$ which act on transverse forms. In fact, $\bar{\partial}_T$ is the usual tangential Cauchy-Riemann operator restricted to transverse forms. However, the fact that we have a Sasakian structure allows us to do more with this operator than in the general CR case. In our version of Hodge theory, the ∂_T - and $\bar{\partial}_T$ -Laplacians are not the same and they are not elliptic; however, they differ by a simple expression involving the self-adjoint operator $\sqrt{-1}\iota_\xi d,$ and they can be made elliptic by subtracting $\frac{1}{2}(\iota_\xi d)^2$. We then use these operators to obtain a simpler, more direct proof of the basic Hodge and Lefschetz decompositions as well as the basic $\partial\bar{\partial}$ -lemma without invoking a space of closures of leaves.

In the second part, we will use the basic $\partial\bar{\partial}$ -lemma to conclude something about the real homotopy type of a compact Sasakian manifold. The rational homotopy type of a simply-connected CW complex is encoded in its rational Postnikov tower, which tells us the rational homotopy groups of the space as well as the Whitehead products $(\pi_i \otimes \mathbb{Q}) \times (\pi_j \otimes \mathbb{Q}) \rightarrow \pi_{i+j-1} \otimes \mathbb{Q}$. On a manifold M , much of the information in the Postnikov tower may be determined from a so-called *minimal model* of the de Rham algebra of differential forms on M . Any two differential graded algebras (DGA’s) which are quasi-isomorphic have the same minimal model up to isomorphism. A manifold is called *formal* if its de Rham algebra is quasi-isomorphic to its de Rham cohomology (a DGA with zero differential). In this case the real homotopy type is entirely determined by the cohomology ring of M .

In [DGM], Deligne, Griffiths, and Morgan use the $\partial\bar{\partial}$ -lemma to prove the formality of compact Kähler manifolds. We will imitate their proof, using some results from the transverse Hodge theory of the first part of the thesis. Ultimately we will fall a little short of formality for compact Sasakian manifolds. What we can show is that the real homotopy type of a compact Sasakian manifold is a formal consequence of its basic cohomology and basic Kähler class. We then use this result to give an example of odd-dimensional compact manifolds which do not admit a Sasakian structure.

We will conclude by mentioning another possible application of transverse Hodge theory: deformations of Sasakian structure using the operators $d_T, \partial_T,$ and $\bar{\partial}_T$. Such deformations will hopefully allow us to rely less on basic forms as we try to imitate

⁴Note that this is not a problem for [FOW]. The Reeb vector ξ of a toric Sasakian manifold M^{2m+1} is by definition contained in the Lie algebra of the T^{m+1} acting on M . So the closures of the leaves are at most $(m+1)$ -dimensional in the toric case.

Kähler geometry in the Sasakian setting. We will present some progress toward more general deformations, which we hope will be a fruitful area of study in the future.

Let us begin by discussing some basic facts about Sasakian geometry.

Chapter 2

Sasakian structures

2.1 Examples

The simplest compact Sasakian manifolds are the regular ones, which are principal S^1 -bundles over compact Kähler manifolds. In fact, given any projective manifold with Kähler form ω (representing an integral cohomology class) it is well-known that there exists a circle bundle $\pi : P \rightarrow M$ with connection form η such that $d\eta = \pi^*\omega$ [Bl]. In fact, η is a contact form. The obvious contact metric structure is Sasakian since $d\eta$ is a Kähler form.

The most basic example of such a circle bundle is the odd-dimensional sphere, which is the total space of the Hopf fibration $S^{2m+1} \rightarrow \mathbb{C}P^m$. Let us explicitly describe the Sasakian structure of S^3 in local coordinates. Let (x_1, x_2) be coordinates for \mathbb{C}^2 . Then on a dense open set we may write S^3 as

$$(x_1, x_2) = \frac{e^{i\theta}(z, 1)}{\sqrt{1 + z\bar{z}}}.$$

θ is the coordinate on the S^1 fiber and $z = x_1/x_2$ is a coordinate for $\mathbb{C}P^1$. Under the change of coordinates

$$(x_1, x_2) = \frac{e^{i\theta'}(1, z')}{\sqrt{1 + z'\bar{z}'}},$$

we have $z' = 1/z$ and $\theta' = \theta - i \log \sqrt{\frac{z}{\bar{z}}}$. The Sasakian structure is given by

$$\begin{aligned} \xi &= \frac{\partial}{\partial \theta}, \\ \eta &= d\theta - \frac{i\bar{z}}{2(1 + z\bar{z})}dz + \frac{iz}{2(1 + z\bar{z})}d\bar{z}, \quad \text{and} \end{aligned}$$

$$d\eta = \frac{idz \wedge d\bar{z}}{(1+z\bar{z})^2}. \quad (2.1)$$

As (2.1) suggests, the transverse Kähler metric $d\eta$ on S^{2m+1} is (up to a constant) the Fubini-Study metric on $\mathbb{C}P^m$.

A good source of examples of Sasakian manifolds are the *toric* ones. A toric Sasakian manifold Y^{2m+1} has the property that its cone $X = Y \times \mathbb{R}_{>0}$ is a toric Kähler manifold (as in [Gu] and [Ab]) whose metric is conelike. More specifically, we require an effective, Hamiltonian action of the real torus T^{m+1} on X preserving the Kähler form [MSY]. We also require that ξ is in the Lie algebra of T^{m+1} . Let $\frac{\partial}{\partial\phi^i}$, $i = 1, \dots, m+1$, be vector fields generating the torus action. The existence of a moment map gives us “symplectic coordinates”

$$y^i = -\frac{1}{2}r^2\eta\left(\frac{\partial}{\partial\phi^i}\right).$$

The image of X in \mathbb{R}^{m+1} is then a rational polyhedral cone \mathcal{C} , and one can show that the Reeb vector ξ is given by $\sum b_i \frac{\partial}{\partial\phi^i}$, where the b_i are constants [MSY]. The image of Y in \mathbb{R}^{m+1} is an m -dimensional polytope formed by the intersection of the hyperplane $\sum b_i y^i = \frac{1}{2}$ and \mathcal{C} .

As in the case of a compact toric Kähler manifold, in symplectic coordinates one can parametrize the complex structure Φ of a toric Sasakian manifold with a “symplectic potential.” This is a function defined on the interior of \mathcal{C} with a certain kind of singular behavior on the boundary. Martelli, Sparks, and Yau [MSY] have shown that one may construct a symplectic potential corresponding to any Reeb vector field $\sum b_i \frac{\partial}{\partial\phi^i}$ for (b_1, \dots, b_{m+1}) in \mathcal{C}_0^* , the dual to the interior of \mathcal{C} . This gives a nice description of the moduli space of symplectic potentials for smooth Sasakian metrics on Y as

$$\mathcal{C}_0^* \times \mathcal{H}(1),$$

where $g \in \mathcal{H}(1)$ is a smooth homogeneous degree one function on \mathcal{C} which may be added to the symplectic potential without affecting smoothness of the metric or the Reeb field.

Unfortunately if we do not make the toric assumption, we do not have such a nice description of the moduli space of Sasakian structures on a Sasakian manifold. In particular, if we make a non-toric deformation of a Sasakian structure it is difficult to ensure that the integrability condition on Φ holds. We will revisit this problem in chapter 5.

2.2 Local transverse Kähler structure of Sasakian manifolds

Let (g, ξ, η, Φ) be a contact metric structure on a manifold M^{2m+1} , and let $\mathcal{D} = \text{Ker } \eta$ be the contact distribution. Φ restricts to an almost complex structure on \mathcal{D} , so $\mathcal{D} \otimes \mathbb{C} \cong \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. If M is Sasakian, we have

$$N_\Phi + d\eta \otimes \xi = 0. \quad (2.2)$$

Since N_Φ is given by

$$N_\Phi(V, W) = \Phi^2[V, W] + [\Phi V, \Phi W] - \Phi[\Phi V, W] - \Phi[V, \Phi W],$$

(2.2) becomes

$$-[V, W] + V\eta(W)\xi - W\eta(V)\xi + [\Phi V, \Phi W] - \Phi[\Phi V, W] - \Phi[V, \Phi W] = 0. \quad (2.3)$$

Consider (2.3) in a local frame $\{\xi, Z_i, \bar{Z}_i\}$ of $TS \otimes \mathbb{C}$, where $Z_i \in \mathcal{D}^{1,0}$. First, let $V = \xi$ and $W = Z_i$. We obtain

$$-[\xi, Z_i] - \Phi[\xi, \Phi Z_i] = 0,$$

i.e. $[\xi, Z_i] \in \mathcal{D}$ is type $(1, 0)$. Letting $V = Z_i$ and $W = \bar{Z}_i$, the left-hand side of (2.3) vanishes as one would expect. If $V = Z_i$ and $W = Z_j$, we obtain that $[Z_i, Z_j]$ must be type $(1, 0)$, also as one would expect. It is then clear that the Sasakian condition is equivalent to the requirement that

$$[\xi, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0} \text{ and } [\mathcal{D}^{1,0}, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0}. \quad (2.4)$$

Definition 2.1. If we only assume the first half of (2.4), $[\xi, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0}$, the contact metric structure is called *K-contact*. This is equivalent to the condition $L_\xi \Phi = 0$, which is also equivalent to the fact that ξ is Killing.

Since ξ generates a foliation \mathcal{F}_ξ , around any $p \in M$ we have coordinates (x, y^1, \dots, y^{2m}) on a neighborhood U such that $\xi = \frac{\partial}{\partial x}$, i.e. y^i are coordinates on the local leaf space B . We will call such coordinates *real foliated coordinates* with respect to the foliation \mathcal{F}_ξ . If (x', y'^i) are foliated coordinates on another chart V , $\xi = \frac{\partial}{\partial x'}$, we also have that on $U \cap V$

$$\frac{\partial y'^i}{\partial x} = 0,$$

i.e. a leaf in one chart matches up with a leaf in the other.

Let $p : U \rightarrow B$ be the projection map. Consider the automorphism I of TB given by $I(\frac{\partial}{\partial y^i}) := p_*(\Phi(\frac{\partial}{\partial y^i}))$. $I(\frac{\partial}{\partial y^i}) = \Phi(\frac{\partial}{\partial y^i}) + f\xi$ for some f , and since $\Phi(\xi) = 0$ we have $I^2(\frac{\partial}{\partial y^i}) = I(\Phi(\frac{\partial}{\partial y^i}) + f\xi) = \Phi^2(\frac{\partial}{\partial y^i}) + g\xi$ for some g . So $I^2(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial y^i} + \tilde{g}\xi$ for some \tilde{g} , but clearly $\tilde{g} = 0$ since I is an automorphism of TB . So I defines an almost

complex structure on B .

Claim 2.1. If the contact metric structure on M is normal, I is integrable.

Proof. Let Y and Z be vector fields on B which are type $(1, 0)$ with respect to I . We wish to show $I[Y, Z] = \sqrt{-1}[Y, Z]$. Let $\tilde{Y} = Y - \eta(Y)\xi$ and $\tilde{Z} = Z - \eta(Z)\xi$. Then \tilde{Y} and \tilde{Z} are both in $\mathcal{D}^{1,0}$, so by the Sasakian condition we have $\Phi[\tilde{Y}, \tilde{Z}] = \sqrt{-1}[\tilde{Y}, \tilde{Z}]$. Moreover we may write

$$\begin{aligned} [\tilde{Y}, \tilde{Z}] &= [Y - \eta(Y)\xi, Z - \eta(Z)\xi] \\ &= [Y, Z] - \eta(Z)[Y, \xi] - \eta(Y)[\xi, Z] + f\xi \\ &= [Y, Z] - \eta(Z)[\tilde{Y}, \xi] + \eta(Y)[\tilde{Z}, \xi] + F\xi \end{aligned}$$

for some f and F . So $\Phi[\tilde{Y}, \tilde{Z}] = \sqrt{-1}[\tilde{Y}, \tilde{Z}]$ becomes

$$\Phi[Y, Z] - \eta(Z)\Phi[\tilde{Y}, \xi] + \eta(Y)\Phi[\tilde{Z}, \xi] = \sqrt{-1}([Y, Z] - \eta(Z)[\tilde{Y}, \xi] + \eta(Y)[\tilde{Z}, \xi] + F\xi).$$

The Sasakian condition also tells us that $\Phi[\tilde{Y}, \xi] = \sqrt{-1}[\tilde{Y}, \xi]$ and likewise for $[\tilde{Z}, \xi]$, and so $\Phi[Y, Z] = \sqrt{-1}([Y, Z] + F\xi)$. Since $[Y, Z]$ is tangent to B , we may then conclude that

$$I[Y, Z] = \sqrt{-1}[Y, Z].$$

□

Again let (x, y^1, \dots, y^{2m}) be real foliated coordinates on a neighborhood U of $p \in M$. From the integrability of I , there exist local coordinates $(x, z^1, \bar{z}^1, \dots, z^m, \bar{z}^m)$ such that the z^i are holomorphic coordinates on the local leaf space B and $\xi = \frac{\partial}{\partial x}$. We call such coordinates *complex foliated coordinates*. Coordinate changes between two such charts must satisfy

$$\frac{\partial z^i}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z^i}{\partial \bar{z}^j} = 0. \quad (2.5)$$

In the example of S^3 above, (θ, z, \bar{z}) and (θ', z', \bar{z}') are complex foliated coordinate charts.

The symplectic form $d\eta$ on B is compatible with I as an immediate consequence of its compatibility with Φ , and so it is a Kähler form on B . So locally the Reeb foliation has a Kähler leaf space. In fact, the existence of complex foliated coordinates compatible with $d\eta$ is equivalent to the Sasakian condition on a contact metric structure. We will use these coordinates in what follows.

Chapter 3

Transverse Hodge theory

3.1 El Kacimi-Alaoui's approach

In this section we outline El Kacimi-Alaoui's approach to basic Hodge theory.

Let (M, \mathcal{F}) be a foliated manifold, where the codimension of the foliation is q . Then locally there is a space of leaves homeomorphic to an open subset of \mathbb{R}^q . Let $B_T^1(M, \mathcal{F})$ denote the principal bundle of frames of TM/\mathcal{F} , and let $B^1(\mathbb{R}^q)$ be the principal bundle of frames of $T\mathbb{R}^q$. A *transverse G -structure* on (M, \mathcal{F}) is a principal G sub-bundle of $B_T^1(M, \mathcal{F})$ which is locally given by the pullback of a subbundle of $B^1(\mathbb{R}^q)$ [BG]. A *Riemannian foliation* (M, \mathcal{F}) is one with a transverse $O(q, \mathbb{R})$ -structure. A Sasakian manifold has such a structure, which is given by the transverse metric $d\eta(\cdot, \Phi\cdot)$. That the resulting $O(q, \mathbb{R})$ -bundle comes from a bundle on the local leaf space is a consequence of the fact that a Sasakian structure is K-contact. If $G = \{e\}$, a transverse G -structure is just a collection of q global sections of TM/\mathcal{F} which are linearly independent at each point. A foliation admitting such a structure is called *transversely parallelizable* (T.P.). On a T.P. foliation (M, \mathcal{F}) we have the following result:

Proposition 3.1 ([Mo]). The closures of leaves of a T.P. foliated manifold (M, \mathcal{F}) are fibers of a locally trivial fibration $\pi : M \rightarrow W$, where W has a manifold structure and π is a submersion. $M \rightarrow W$ is called the *basic fibration*.

There is no reason why the Reeb foliation on a Sasakian manifold should be T.P. However, one may lift any Riemannian foliation (M, \mathcal{F}) to a foliation $(E_T^1(M, \mathcal{F}), \mathcal{F}^\#)$, where $E_T^1(M, \mathcal{F})$ is the (oriented) orthonormal transverse frame bundle, and the lifted foliation is in fact T.P. (cf. [Mo] ch. 3).¹

¹Here we assume (M, \mathcal{F}) is transversely orientable. This condition may always be achieved by

Let E be a Hermitian \mathcal{F} -vector bundle² on a T.P. foliated manifold (M, \mathcal{F}) . For $u \in W$, let F_u denote a fiber of the basic fibration. Let E_u and \mathcal{F}_u denote the restrictions of E and \mathcal{F} to F_u , and let $\bar{E}_u = C^\infty(E_u/\mathcal{F}_u)$. This notation means sections of E_u which are basic with respect to the foliation \mathcal{F}_u . (A section $\alpha \in C^\infty(E_u)$ is basic if $\iota_X \alpha = \iota_X d\alpha = 0$ for all X tangent to \mathcal{F}_u .) The \bar{E}_u glue together to give a Hermitian vector bundle \bar{E} over W [EKA]. There is an isomorphism between basic sections of E and sections of \bar{E} . A “transversely elliptic” operator D on E induces an elliptic operator on \bar{E} . This enables us to conclude that $\mathcal{H}(E/\mathcal{F})$, the space of basic D -harmonic sections of E , is finite dimensional. Therefore we have a decomposition of basic sections of E :

$$C^\infty(E/\mathcal{F}) = \mathcal{H}(E/\mathcal{F}) \oplus \text{Im } D^*. \quad (3.1)$$

In the case where (M, \mathcal{F}) is a Riemannian (but not necessarily T.P.) foliation, we may lift E to a bundle E^\sharp on $E_T^1(M, \mathcal{F})$, and there is an isomorphism between basic sections $C^\infty(E/\mathcal{F})$ and $SO(n)$ -invariant basic sections of E^\sharp with respect to the foliation \mathcal{F}^\sharp . By the above, there is a basic fibration $E_T^1(M, \mathcal{F}) \rightarrow W^\sharp$ and a bundle \bar{E}^\sharp over W^\sharp such that basic sections of E correspond to $SO(n)$ -invariant sections of \bar{E}^\sharp , which we denote by $C_{SO(n)}^\infty(\bar{E}^\sharp)$. To a transversely elliptic operator D on E we may associate an $SO(n)$ -equivariant elliptic operator D^\sharp on $C^\infty(\bar{E}^\sharp)$. It follows that $\text{Ker } D^\sharp \cap C_{SO(n)}^\infty(\bar{E}^\sharp)$ is finite dimensional. This gives us the decomposition (3.1) for an arbitrary Riemannian foliation.

El Kacimi-Alaoui goes on to define an L^2 inner product on basic forms using the transverse metric. (He does not assume any specific metric on M .) Let us assume that (M, \mathcal{F}) has a transverse Kähler structure. Defining d_B to be d restricted to basic forms, we may use the transverse complex structure to write $d_B = \partial_B + \bar{\partial}_B$. Each of the operators d_B , ∂_B , and $\bar{\partial}_B$ defines a Laplacian. El Kacimi-Alaoui shows that the Laplacians satisfy

$$\Delta_{d_B} = 2\Delta_{\partial_B} = 2\Delta_{\bar{\partial}_B}$$

and are all transversely elliptic. He is therefore able to prove basic versions of the Hodge and Lefschetz decompositions, the $\partial\bar{\partial}$ -lemma, and even the Calabi-Yau theorem.

El Kacimi-Alaoui’s reliance upon a space W^\sharp of leaf closures is necessary because basic forms on (M, \mathcal{F}) are not sections of any bundle over M (nor are they G -invariant sections for a Lie group G). To obtain the finite dimensionality of the kernel of an elliptic operator, he needs to pass to $SO(n)$ -invariant sections of some bundle over W^\sharp . In what follows we will develop a simpler transverse Hodge theory and recover many of El Kacimi-Alaoui’s results without reference to a space of closures of leaves. The idea is to broaden the class of forms under consideration so that they are sections

passing to a two-sheeted cover of M .

²Cf. [EKA] for definitions of Hermitian \mathcal{F} -vector bundles and transversely elliptic operators.

of a vector bundle over M . There are two main advantages to our approach. First is the simplicity of the argument. Second, it allows us to say something about transverse forms which are not necessarily basic.

3.2 Definitions of the transverse operators

Let (S, g, ξ, η, Φ) be a Sasakian manifold. The contact distribution \mathcal{D} is isomorphic to TS/L_ξ , where L_ξ is the C^∞ line bundle spanned by ξ . For $X \in T_p S$, $[X]$ in $(TS/L_\xi)_p$ is mapped to $X - \eta_p(X)\xi_p$ in \mathcal{D}_p . $(TS/L_\xi)^*$ may be identified with $\text{Ker } \iota_\xi \subset T^*S$.

Moreover, $\bigwedge^r(\text{Ker } \iota_\xi) = \text{Ker } \iota_\xi \subset \bigwedge^r T^*S$. We can see this by considering a local frame $\{\eta, \theta_i\}$ for T^*S , where $\iota_\xi \theta_i = 0$. Any r -form in $\text{Ker } \iota_\xi$ must be of the form $\sum a_{i_1, \dots, i_r} \theta_{i_1} \wedge \dots \wedge \theta_{i_r}$, i.e. it must be a section of $\bigwedge^r(\text{Ker } \iota_\xi)$.

We will say a differential r -form on S is *transverse* if it is a section of the subbundle $\text{Ker } \iota_\xi \subset \bigwedge^r T^*S$. By the above, any transverse form admits a decomposition into forms of type (p, q) under the complex structure Φ . In complex foliated coordinates (x, z^i, \bar{z}^i) , a transverse form has no dx term. The holomorphic type of a transverse form $a_{I\bar{J}} dz^I \wedge d\bar{z}^J$ is $(|I|, |J|)$, where I and J are multi-indices. (The conditions in (2.5) ensure that the type is preserved under a change of coordinates.)

In what follows we will be concerned with operators that take transverse forms to transverse forms. While the exterior derivative is not such an operator, we may adjust it by following d with a projection.

Definition 3.1. Let α be a transverse form. Define $d_T \alpha := d\alpha - \eta \wedge \iota_\xi d\alpha$. Let β be a transverse form of type (p, q) . Define $\partial_T \beta := (d_T \beta)^{p+1, q}$. Define $\bar{\partial}_T \beta := (d_T \beta)^{p, q+1}$. We extend the operators ∂_T and $\bar{\partial}_T$ linearly to act on all transverse forms.

Claim 3.1. Let α be a transverse form. Then $d_T \alpha = \partial_T \alpha + \bar{\partial}_T \alpha$.

Proof. It suffices to consider α of type (p, q) . Let X_0, \dots, X_{p+q} be vectors in \mathcal{D} of which m are in $\mathcal{D}^{1,0}$ and $n = p + q + 1 - m$ are in $\mathcal{D}^{0,1}$. $d\alpha(X_0, \dots, X_{p+q})$ may be written in terms of expressions of the form $X_i(\alpha(\dots, \hat{X}_i, \dots))$ and terms of the form $\alpha([X_i, X_j], \dots, \hat{X}_i, \hat{X}_j, \dots)$. Terms like the former are clearly only nonzero when $m = p$ or $p+1$. For terms like the latter, this is also true because of the integrability condition (2.4) for Φ . The expression $(\eta \wedge \iota_\xi d\alpha)(X_0, \dots, X_{p+q})$ vanishes since $X_i \in \text{Ker } \eta$. We conclude that $d_T \alpha(X_0, \dots, X_{p+q}) = d\alpha(X_0, \dots, X_{p+q}) - (\eta \wedge \iota_\xi d\alpha)(X_0, \dots, X_{p+q})$ is only nonzero when $m = p$ or $p+1$. \square

Claim 3.2. $\partial_T^2 = \bar{\partial}_T^2 = 0$. $\{\partial_T, \bar{\partial}_T\} = d_T^2 = -d\eta \wedge \iota_\xi d$.

Proof. Let α be a transverse form of type (p, q) .

$$\begin{aligned}
d_T^2 \alpha &= d(d\alpha - \eta \wedge \iota_\xi d\alpha) - \eta \wedge \iota_\xi d(d\alpha - \eta \wedge \iota_\xi d\alpha) \\
&= -d\eta \wedge \iota_\xi d\alpha + \eta \wedge d\iota_\xi d\alpha - \eta \wedge \iota_\xi (-d\eta \wedge \iota_\xi d\alpha + \eta \wedge d\iota_\xi d\alpha) \\
&= -d\eta \wedge \iota_\xi d\alpha + \eta \wedge d\iota_\xi d\alpha - \eta \wedge d\iota_\xi d\alpha \\
&= -d\eta \wedge \iota_\xi d\alpha.
\end{aligned}$$

In complex foliated coordinates (x, z^i, \bar{z}^i) , the operator $\iota_\xi d$ takes $\sum a_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}$ to $\sum \frac{\partial a_{I\bar{J}}}{\partial x} dz^I \wedge d\bar{z}^{\bar{J}}$, so it preserves type. Moreover, since $d\eta$ is a transverse Kähler form it is type $(1, 1)$. The claim then follows. \square

3.3 Comparison to $\bar{\partial}_b$

In fact the operator $\bar{\partial}_T$ is the well-known “tangential Cauchy-Riemann operator,” usually denoted by $\bar{\partial}_b$, restricted to transverse forms. Let us recall some basic facts of CR geometry.

Definition 3.2. A *CR structure* on a C^∞ manifold M is a subbundle L of the complexified tangent bundle $T^{\mathbb{C}}M$ satisfying

1. $L_p \cap \bar{L}_p = \{0\}$ at every $p \in M$ and
2. $[L, L] \subset L$.

We may consider L to be the $(-i)$ -eigenspace of a complex structure J . The *CR codimension* of (M, L) is $\dim_{\mathbb{C}}(T^{\mathbb{C}}M/(L \oplus \bar{L}))$. A Sasakian manifold (S, g, η, ξ, Φ) is therefore equipped with a codimension one CR structure, $\mathcal{D}^{0,1}$.

Let π_p denote the projection $T_p^{\mathbb{C}}(M) \rightarrow T_p^{\mathbb{C}}(M)/(L_p \oplus \bar{L}_p)$. The *Levi form* \mathcal{L} on M is defined by $\mathcal{L}_p(X_p, \bar{Y}_p) = \frac{1}{2i} \pi_p([X, \bar{Y}]_p)$, where X and \bar{Y} are any sections of L and \bar{L} (respectively) which are equal to X_p and \bar{Y}_p (respectively) at p . A CR manifold (M, L) is called *strictly pseudoconvex* if its Levi form is definite. On a Sasakian manifold, the Levi form is simply $-\frac{1}{2}d\eta(\cdot, \Phi\cdot)$, which is (negative) definite.

On a CR-manifold (M, L) with a Hermitian metric, one defines the tangential Cauchy-Riemann operator as follows [Bo]. Let X be the orthogonal complement of $L \oplus \bar{L} \subset T^{\mathbb{C}}$. We set

$$\begin{aligned}
T^{0,1}(M) &= L \quad \text{and} \\
T^{1,0}(M) &= \bar{L} \oplus X.
\end{aligned} \tag{3.2}$$

In the Sasakian case, X is the trivial complex C^∞ line bundle spanned by ξ . So essentially we are treating ξ as a type $(1, 0)$ vector field. Using the decomposition

$T^{\mathbb{C}}M = T^{0,1}(M) \oplus T^{1,0}(M)$, we let $\pi^{p,q}$ be the projection from $(p+q)$ -forms to forms of type (p,q) . For a (p,q) -form α , one defines $\bar{\partial}_b$ in the obvious manner as $\pi^{p,q+1}d\alpha$. Including X in $T^{1,0}$ is necessary to ensure that $\bar{\partial}_b^2 = 0$. This follows from the fact that for α type (p,q) we have $\pi^{p-i,q+i+1}d\alpha = 0$ for $i > 0$ (and so $\bar{\partial}_b^2\alpha = \pi^{p,q+2}d^2\alpha = 0$). We can see this from the expression $\alpha([X_0, X_1], X_2, \dots, X_{p+q})$. The idea is that $T^{0,1}$ is involutive even though $T^{1,0}$ need not be. So if the X_i are all either in $T^{1,0}$ or $T^{0,1}$, at most $q+1$ of them can be type $(1,0)$ for the expression to be nonzero. Note that in the general CR setting we do not have a nice operator $\bar{\partial}_b$.

Here we see a key difference between a Sasakian structure and an arbitrary codimension one CR structure: $[X, \bar{L}] \subset \bar{L}$ in the Sasakian case. In particular, $T^{1,0}$ is involutive for a Sasakian manifold. So the operator $\bar{\partial}_b$ could be interesting. However, we will not use this operator because of the asymmetry introduced by (3.2). In light of claims 3.1 and 3.2, our approach seems better.

The operator $\bar{\partial}_b$ has been extensively studied, but it is most tractable on strictly pseudoconvex domains in \mathbb{C}^n [CS]. Under this assumption one can show L^2 -existence and certain regularity results for the $\bar{\partial}$ -Neumann problem. Clearly in the Sasakian setting we cannot make such a strong assumption. For $(M, \mathcal{D}^{0,1})$ any compact, oriented, strictly pseudoconvex CR manifold with $\dim_{\mathbb{C}}\mathcal{D}^{0,1} = m$, the $\bar{\partial}_b$ -Laplacian Δ_b is hypoelliptic [CS] and we obtain the following decomposition on L^2 forms of type (p,q) for $p \leq m, q \leq m-1$:

Proposition 3.2 ([CS] Prop. 8.4.9).

$$L^2_{p,q} = \mathcal{R}(\Delta_b) \oplus \mathcal{H}^b_{p,q}(M), \quad (3.3)$$

where $\mathcal{R}(\Delta_b)$ denotes the range of Δ_b and $\mathcal{H}^b_{p,q}(M)$ denotes Δ_b -harmonic forms of type (p,q) .

When $q = m$, the decomposition (3.3) does not hold. This is the result of a certain condition on the number of positive eigenvalues of the Levi form which is required to prove subelliptic estimates for the $\bar{\partial}_b$ -complex. An advantage of our approach will be that we do not run into this problem.

Let us return to the discussion of the transverse operators d_T, ∂_T , and $\bar{\partial}_T$ on a Sasakian manifold S .

3.4 Further properties of the transverse operators

Claim 3.3. Let α and β be transverse forms, $\alpha \in \Omega^p(S)$. Then $d_T(\alpha \wedge \beta) = d_T\alpha \wedge \beta + (-1)^p\alpha \wedge d_T\beta$ and similarly for ∂_T and $\bar{\partial}_T$.

Proof. We will prove the claim for d_T . First, for any form γ we denote $\tilde{\gamma} := \gamma - \eta \wedge \iota_\xi \gamma$. Then $d_T(\alpha \wedge \beta) = d(\widetilde{\alpha \wedge \beta})$.

$$\begin{aligned} d_T(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta - \eta \wedge \iota_\xi(d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta) \\ &= \widetilde{d\alpha} \wedge \beta + (-1)^p \alpha \wedge \widetilde{d\beta} + (-1)^{p+1} \alpha \wedge \eta \wedge \iota_\xi d\beta \\ &= \widetilde{d\alpha} \wedge \beta + (-1)^p \alpha \wedge \widetilde{d\beta}. \end{aligned}$$

Here we have used $\iota_\xi \alpha = \iota_\xi \beta = 0$. It is clear that ∂_T and $\bar{\partial}_T$ are also graded derivations. \square

Let $*_T := \iota_\xi *$, acting on transverse forms, where $*$ is the Hodge star operator on the Riemannian manifold (S, g) . Extend $*_T$ \mathbb{C} -linearly to operate on complex-valued transverse forms.

Claim 3.4. Let $\alpha \in \text{Ker } \iota_\xi \subset \Omega^p(S)$. $*_T \alpha = (-1)^p *(\eta \wedge \alpha)$. $*_T^2 \alpha = (-1)^p \alpha$.

Proof. In real foliated coordinates (x, y^1, \dots, y^{2m}) for the Reeb foliation, we have $\xi(y^i) = 0$, so $(\eta, dy^i)_g = 0$. Also we have $(\eta, \eta)_g = 1$. Consider the frame $\{\eta, dy^1, \dots, dy^{2m}\}$ for T^*S . Note that $\eta \wedge *(\eta \wedge dy^I) = (-1)^{|I|} *dy^I$. Taking ι_ξ of both sides, we get

$$*(\eta \wedge dy^I) - \eta \wedge \iota_\xi *(\eta \wedge dy^I) = (-1)^{|I|} *_T dy^I. \quad (3.4)$$

If we choose an orthonormal frame $\{\theta^1, \dots, \theta^{2m}\}$ for the cotangent bundle of the local leaf space, we have that $\{\eta, \theta^1, \dots, \theta^{2m}\}$ is an orthonormal frame for T^*S . So $*(\eta \wedge dy^I)$ must be transverse. Therefore the undesired term in (3.4) vanishes.

For the second part of the claim, we have $*_T^2 \alpha = (-1)^p \iota_\xi *(\eta \wedge \alpha) = (-1)^p \iota_\xi (\eta \wedge \alpha) = (-1)^p \alpha$ since S is odd-dimensional and the operators are acting only on transverse forms. \square

Let α and β be two complex-valued transverse p -forms. Then $\langle \alpha, \beta \rangle := \int_S \alpha \wedge \overline{*\beta}$ defines an L^2 inner product on the bundle $\text{Ker } \iota_\xi \subset \bigwedge^p T^*S$.

Proposition 3.3. The adjoint of d_T is given by $d_T^* = - *_T d_T *_T$. Likewise, $\partial_T^* = - *_T \bar{\partial}_T *_T$ and $\bar{\partial}_T^* = - *_T \partial_T *_T$.

Proof. We wish to show that $\int_S \bar{\partial}_T \alpha \wedge \overline{*\beta} = \int_S \alpha \wedge \overline{*\bar{\partial}_T^* \beta}$ for $\bar{\partial}_T^*$ as given in the statement of the proposition. Assume α and β are p -forms. From the proof of claim 3.4, $*\beta = \eta \wedge \gamma$ for some transverse γ . Taking ι_ξ of both sides, $\gamma = \iota_\xi * \beta$. So $\int_S \bar{\partial}_T \alpha \wedge \overline{*\beta} = (-1)^{p+1} \int_S \eta \wedge \bar{\partial}_T \alpha \wedge \overline{\iota_\xi * \beta}$.

Claim 3.5. For any transverse form γ on S , $\int_S \eta \wedge \bar{\partial}_T \gamma = 0$.

Proof. It suffices to consider γ of holomorphic type $(m, m-1)$ because only these terms contribute to the integral. Then $\partial_T \gamma = 0$, so $\bar{\partial}_T \gamma = d_T \gamma$, and $\eta \wedge \bar{\partial}_T \gamma = \eta \wedge d_T \gamma = \eta \wedge d\gamma$. Integrating by parts, we have $\int \eta \wedge \bar{\partial}_T \gamma = -\int d\eta \wedge \gamma$. But the integrand here is transverse, so the integral is zero. \square

Returning to the proof of Proposition 3.3, $\int \eta \wedge \bar{\partial}_T(\alpha \wedge \overline{\iota_\xi * \beta}) = 0 = \int \eta \wedge \bar{\partial}_T \alpha \wedge \overline{\iota_\xi * \beta} + (-1)^p \int \eta \wedge \alpha \wedge \bar{\partial}_T \overline{\iota_\xi * \beta}$. So

$$\begin{aligned} \int \bar{\partial}_T \alpha \wedge \overline{\iota_\xi * \beta} &= \int \eta \wedge \alpha \wedge \bar{\partial}_T \overline{\iota_\xi * \beta} \\ &= \int \eta \wedge \alpha \wedge \overline{\partial_T \iota_\xi * \beta} \\ &= (-1)^p \int \alpha \wedge \eta \wedge \overline{\partial_T \iota_\xi * \beta} \\ &= (-1)^p \int \alpha \wedge * * (\eta \wedge \partial_T \iota_\xi * \beta) \\ &= -\int \alpha \wedge * * \partial_T * \beta, \end{aligned}$$

and so $\bar{\partial}_T^* = - * \partial_T *$. The proof for ∂_T^* is the same, and adding the two adjoints gives d_T^* . \square

Definition 3.3. Let $\Delta_T = \Delta_{d_T} = d_T^* d_T + d_T d_T^*$. Define Δ_{∂_T} and $\Delta_{\bar{\partial}_T}$ in the obvious manner. Define L on transverse forms by $L = d\eta \wedge \cdot$, and let Λ be its adjoint. Define $M := \iota_\xi d$, acting on transverse forms.

Note that since $\bar{\partial}_T$ is only defined on transverse forms, $\Delta_{\bar{\partial}_T}$ is not the same as Δ_b , the Laplacian discussed in section 3.3.

Claim 3.6. $M^* = -M$.

Proof. Clearly $\int_S \iota_\xi \gamma = 0$ for any γ . Let $\alpha, \beta \in \Omega^p S$ be transverse forms. We have

$$\begin{aligned} \int \iota_\xi d\alpha \wedge * \beta &= (-1)^p \int d\alpha \wedge \iota_\xi * \beta \\ &= -\int \alpha \wedge d\iota_\xi * \beta \\ &= -\int \alpha \wedge * (d\iota_\xi * \beta). \end{aligned}$$

Since $\int \alpha \wedge * (\eta \wedge \gamma) = 0$ for any transverse α and γ , we have

$$\int \iota_\xi d\alpha \wedge * \beta = -\int \alpha \wedge * (*d\iota_\xi * \beta - \eta \wedge \iota_\xi (*d\iota_\xi * \beta)).$$

So $M^* = -(*d\iota_\xi * - \eta \wedge \iota_\xi * d\iota_\xi *)$. (Note that this operator takes transverse forms to transverse forms, whereas $*d\iota_\xi *$ does not.)

To show that M^* is actually the same as $-M$, it suffices to do so at each point in real foliated coordinates (x, y^1, \dots, y^{2m}) . Note that $g^{-1}(\eta, \eta) = 1$ and $g^{-1}(\eta, dy^i) = \iota_\xi dy^i = 0$. Define $\tilde{Y}^i := g^{-1}(dy^i, \cdot)$. Let $U \subset \mathbb{R}^{2m}$ be the local leaf space, and let g_T denote the transverse metric $d\eta(\cdot, I\cdot) = d\eta(\cdot, \Phi\cdot)$ on U . (Recall the notation from section 2.2.) Let $Y^i := g_T^{-1}(dy^i, \cdot)$. In fact, $\tilde{Y}^i = Y^i - \eta(Y^i)\xi$. This follows since $g(Y^i - \eta(Y^i)\xi, Z) = d\eta(Y^i, \Phi Z)$ for any Z tangent to U , and $g(Y^i - \eta(Y^i)\xi, \xi) = 0$. So $g^{-1}(dy^i, dy^j) = g(\tilde{Y}^i, \tilde{Y}^j) = g_T(Y^i, Y^j) = (g_T)^{-1}(dy^i, dy^j)$.

Let us change to normal coordinates on U , so that g_T^{-1} is Euclidean up to second order. M and M^* are first order differential operators, and so they only involve first derivatives of the metric. Hence in what follows we may assume g_T^{-1} is Euclidean.

We may write $\eta = dx + \sum f_i dy^i$, where $f_i = f_i(y)$. This follows from the facts that $\iota_\xi \eta = 1$ and $d\eta$ is transverse. $-M^*(\alpha_I dy^I) = *d\iota_\xi * \alpha_I dy^I - \eta \wedge \iota_\xi * d\iota_\xi * \alpha^I dy^I$. Let the terms on the right-hand side be ① and ②, respectively.

$$\textcircled{1} = (-1)^{|I|} \left(\frac{\partial \alpha_I}{\partial x} * (dx \wedge *(\eta \wedge dy^I)) + \frac{\partial \alpha_I}{\partial y^k} * (dy^k \wedge *(\eta \wedge dy^I)) \right).$$

Here we have used $\iota_\xi * dy^I = (-1)^{|I|} * \eta \wedge dy^I$. The second term above is of the form $\eta \wedge \gamma$ for some γ . The first term is just $(-1)^{|I|} \frac{\partial \alpha_I}{\partial x} * (\eta \wedge *(\eta \wedge dy^I))$, which is transverse, plus a term which is $*$ of something transverse and hence of the form $\eta \wedge \gamma$ for some γ . The $\eta \wedge \gamma$ terms will all be precisely canceled by ②, so we are left with $-M^*(\alpha_I dy^I) = (-1)^{|I|} \frac{\partial \alpha_I}{\partial x} * (\eta \wedge *(\eta \wedge dy^I)) = (-1)^{|I|} \frac{\partial \alpha_I}{\partial x} *_T^2 dy^I = \frac{\partial \alpha_I}{\partial x} dy^I = M(\alpha_I dy^I)$. \square

Claim 3.7 (transverse Kähler identities). $[\Lambda, \partial_T] = \sqrt{-1} \bar{\partial}_T^*$ and $[\Lambda, \bar{\partial}_T] = -\sqrt{-1} \partial_T^*$.

Proof. Let us use complex foliated coordinates (x, z^i, \bar{z}^i) . Since the local leaf space is Kähler, we may perform a holomorphic change of coordinates on the base so that the transverse metric g_T is Euclidean up to second order. Again we are dealing with first order operators, so we may actually take the transverse metric to be Euclidean. Since η is real, we may write $\eta = dx + \sum f_i dz^i + \sum \bar{f}_i d\bar{z}^i$.

Following the proof of the Kähler identities in [GH], let e_k and \bar{e}_k denote the operators $dz^k \wedge \cdot$ and $d\bar{z}^k \wedge \cdot$ respectively, and let ι_k and $\bar{\iota}_k$ denote their adjoints. (In fact $\iota_k = 2\iota_{\frac{\partial}{\partial z^k}}$ and likewise for $\bar{\iota}_k$.) Let $\partial_k(\alpha_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}) = \frac{\partial \alpha_{I\bar{J}}}{\partial z^k} dz^I \wedge d\bar{z}^{\bar{J}}$ and define $\bar{\partial}_k$ similarly. Then $\Lambda = -\frac{\sqrt{-1}}{2} \bar{\iota}_k \iota_k$ and $\partial_T = \sum \partial_k e_k - \sum f_k M e_k$.

As in the case of a Kähler manifold, we have $\{e_k, \iota_k\} = 2$ and $\{e_k, \iota_l\} = 0$ for $k \neq l$. Also, integration by parts easily shows that the adjoint of ∂_k is $-\bar{\partial}_k$ (of course we must use compactly supported forms). As in the Kähler case, $\partial_k, \bar{\partial}_k$, and M commute

with each other and with $\iota_k, \bar{\iota}_k, e_k,$ and \bar{e}_k . We then have

$$\begin{aligned}
\Lambda \partial_T &= -\frac{\sqrt{-1}}{2} \sum \bar{\iota}_k \iota_k (\partial_l e_l - f_m M e_m) \\
&= -\frac{\sqrt{-1}}{2} \sum \partial_l e_l \bar{\iota}_k \iota_k - \sqrt{-1} \sum \partial_k \bar{\iota}_k + \frac{\sqrt{-1}}{2} \sum f_m M e_m \bar{\iota}_k \iota_k \\
&\quad + \sqrt{-1} \sum f_k M \bar{\iota}_k \\
&= \partial_T \Lambda + \sqrt{-1} \left(-\sum \partial_k \bar{\iota}_k + \sum f_k M \bar{\iota}_k \right).
\end{aligned}$$

Since $\bar{\partial}_T = \sum \bar{\partial}_k \bar{e}_k - \sum \bar{f}_k M \bar{e}_k$, $\bar{\partial}_T^* = -\sum \bar{\iota}_k \partial_k + \sum f_k M \bar{\iota}_k$. So we have shown that $[\Lambda, \partial_T] = \sqrt{-1} \bar{\partial}_T^*$. The proof that $[\Lambda, \bar{\partial}_T] = -\sqrt{-1} \partial_T^*$ is the same. \square

We thus obtain $\{\partial_T, \bar{\partial}_T^*\} = \{\bar{\partial}_T, \partial_T^*\} = 0$ as in the Kähler case.

Claim 3.8. $\Delta_T = \Delta_{\partial_T} + \Delta_{\bar{\partial}_T} = 2\Delta_{\bar{\partial}_T} + \sqrt{-1}[L, \Lambda]M$.

Proof. $\Delta_T = \Delta_{\partial_T} + \Delta_{\bar{\partial}_T}$ is the same as in the Kähler case. Moreover, we have

$$\begin{aligned}
-\sqrt{-1}\Delta_{\partial_T} &= -\sqrt{-1}(\partial_T \partial_T^* + \partial_T^* \partial_T) \\
&= \partial_T \Lambda \bar{\partial}_T - \partial_T \bar{\partial}_T \Lambda + \Lambda \bar{\partial}_T \partial_T - \bar{\partial}_T \Lambda \partial_T
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{-1}\Delta_{\bar{\partial}_T} &= \sqrt{-1}(\bar{\partial}_T \bar{\partial}_T^* + \bar{\partial}_T^* \bar{\partial}_T) \\
&= \bar{\partial}_T \Lambda \partial_T - \bar{\partial}_T \partial_T \Lambda + \Lambda \partial_T \bar{\partial}_T - \partial_T \Lambda \bar{\partial}_T \\
&= \sqrt{-1}\Delta_{\partial_T} - (\partial_T \bar{\partial}_T + \bar{\partial}_T \partial_T) \Lambda + \Lambda (\partial_T \bar{\partial}_T + \bar{\partial}_T \partial_T) \\
&= \sqrt{-1}\Delta_{\partial_T} + d\eta \wedge \iota_\xi d\Lambda - \Lambda(d\eta \wedge \iota_\xi d).
\end{aligned}$$

Finally, note that $[M, L] = 0$ and so $[M, \Lambda] = 0$. \square

Observation. If α is a transverse form of type (p, q) and $\dim_{\mathbb{R}}(S) = 2m + 1$, then $[L, \Lambda]\alpha = (p + q - m)\alpha$.

The proof is the same as in the Kähler case [GH].

We now prove the following key result:

Theorem 3.1. $\Delta_T - M^2$, $2\Delta_{\partial_T} - M^2$, and $2\Delta_{\bar{\partial}_T} - M^2$ are all self-adjoint elliptic operators acting on transverse forms.

Proof. First we compute the symbol of Δ_T . Note that second derivatives of the transverse metric do not contribute to the symbol, so we may assume that the transverse metric is Euclidean. We will use real foliated coordinates (x, y^1, \dots, y^{2m}) , where

$\langle \eta, dy^i \rangle = 0$ and $\langle dy^i, dy^j \rangle = \delta^{ij}$. Write $\eta = dx + f_i dy^i$ and recall that $\frac{\partial f_i}{\partial x} = 0$.

$$\begin{aligned} d_T d_T^*(\alpha_I dy^I) &= -d_T *_T \left(\left(\frac{\partial \alpha_I}{\partial y^i} - f_i \frac{\partial \alpha_I}{\partial x} \right) dy^i \wedge *_T dy^I \right) \\ &= - \left(\frac{\partial^2 \alpha_I}{\partial y^i \partial y^j} - \frac{\partial^2 \alpha_I}{\partial x \partial y^j} f_i - \frac{\partial \alpha_I}{\partial x} \frac{\partial f_i}{\partial y^j} - \frac{\partial^2 \alpha_I}{\partial x \partial y^i} f_j + \frac{\partial^2 \alpha_I}{\partial x^2} f_i f_j \right) dy^j \wedge \\ &\quad *_T (dy^i \wedge *_T dy^I) \end{aligned}$$

and

$$\begin{aligned} d_T^* d_T(\alpha_I dy^I) &= - \left(\frac{\partial^2 \alpha_I}{\partial y^i \partial y^j} - \frac{\partial^2 \alpha_I}{\partial x \partial y^j} f_i - \frac{\partial \alpha_I}{\partial x} \frac{\partial f_i}{\partial y^j} - \frac{\partial^2 \alpha_I}{\partial x \partial y^i} f_j + \frac{\partial^2 \alpha_I}{\partial x^2} f_i f_j \right) \\ &\quad *_T (dy^j \wedge *_T (dy^i \wedge dy^I)). \end{aligned}$$

Observe that the term $\frac{\partial \alpha_I}{\partial x} \frac{\partial f_i}{\partial y^j}$ is first order and therefore does not contribute to the symbol. The remaining coefficient terms $-\left(\frac{\partial^2 \alpha_I}{\partial y^i \partial y^j} - \frac{\partial^2 \alpha_I}{\partial x \partial y^j} f_i - \frac{\partial^2 \alpha_I}{\partial x \partial y^i} f_j + \frac{\partial^2 \alpha_I}{\partial x^2} f_i f_j\right)$ are unchanged upon swapping i and j . By the same argument as Voisin uses in the Kähler case [Vo], we are left with

$$\Delta_T(\alpha_I dy^I) = - \sum_i \left(\frac{\partial^2 \alpha_I}{\partial (y^i)^2} - 2 \frac{\partial^2 \alpha_I}{\partial x \partial y^i} f_i + \frac{\partial^2 \alpha_I}{\partial x^2} (f_i)^2 \right) dy^I + \text{lower order terms.}$$

We view the symbol as a map from $T_p^* S$ to endomorphisms of transverse forms. It follows from the above that for any one-form $\zeta = \zeta_0 dx + \sum \zeta_i dy^i$ and any transverse form ω , the symbol takes ζ to the map $\omega \mapsto - \sum_{i>0} (\zeta_i - f_i \zeta_0)^2 \omega$. So the kernel of the symbol is spanned by η . Note that we may in fact make f_i vanish at p by the change of coordinates $x' = x + \sum f_i(p) y^i$, $y'^i = y^i$. (The new coordinates will still be foliated since $\frac{\partial y'^i}{\partial x} = 0$.) So we now assume $f_i = 0$ at p .

$-M^2(\alpha_I dy^I) = -\frac{\partial^2 \alpha_I}{\partial x^2} dy^I$, so its symbol takes ζ to the map $\omega \mapsto -\zeta_0^2 \omega$. The symbol of $\Delta_T - M^2$ then takes ζ to $\omega \mapsto - \sum_{i=0}^{2m} \zeta_i^2 \omega = -\|\zeta\|^2 \omega$. The equality is true because $\eta = dx$ at p and the transverse metric is Euclidean, and so the total metric on S is Euclidean at p . It is then clear that $\Delta_T - M^2$ has injective symbol at p , and so it is elliptic.

The operators $2\Delta_{\partial_T} - M^2$ and $2\Delta_{\bar{\partial}_T} - M^2$ differ from $\Delta_T - M^2$ by a first-order differential operator. Hence they have the same symbols, and all three operators are elliptic. \square

We obtain:

Proposition 3.4. The kernels of $\Delta_T - M^2$, $2\Delta_{\partial_T} - M^2$, and $2\Delta_{\bar{\partial}_T} - M^2$ are finite

dimensional, and we have the usual orthogonal decomposition

$$\Omega^p(S) \supset \text{Ker } \iota_\xi = \text{Ker}(\Delta_T - M^2) \oplus \text{Im}(\Delta_T - M^2),$$

and likewise for the other two operators.

3.5 Basic Hodge and Lefschetz decompositions and the basic $\partial\bar{\partial}$ -lemma

We recall that a transverse form α is *basic* if $M\alpha = 0$. The operators d_B , ∂_B , and $\bar{\partial}_B$ discussed in chapter 1 are the restrictions of d_T , ∂_T , and $\bar{\partial}_T$ to basic forms.

For any basic form γ , $\Delta_T\gamma = 2\Delta_{\partial_T}\gamma = 2\Delta_{\bar{\partial}_T}\gamma$. Define Δ_B to be Δ_T restricted to basic forms. For any transverse form $\alpha \in \text{Ker}(\Delta_T - M^2)$, we have

$$0 = \langle \alpha, (\Delta_T - M^2)\alpha \rangle = \langle d_T\alpha, d_T\alpha \rangle + \langle d_T^*\alpha, d_T^*\alpha \rangle + \langle M\alpha, M\alpha \rangle,$$

and so in particular $M\alpha = 0$. So α is basic and is in the kernel of all three Laplacians. We call such a form *basic harmonic*.

We will need the fact that M commutes with ∂_T , $\bar{\partial}_T$, d_T , and their adjoints. This is clear since for any transverse ω , $Md_T\omega = \iota_\xi d(-\eta \wedge \iota_\xi d\omega) = \iota_\xi(\eta \wedge d\iota_\xi d\omega) = \widetilde{d\iota_\xi d\omega} = d_T M\omega$. (Recall that $\tilde{\gamma} := \gamma - \eta \wedge \iota_\xi \gamma$.) Since M preserves holomorphic type, it must therefore also commute with ∂_T and $\bar{\partial}_T$. M is skew-adjoint, so the commutators with the adjoints also vanish.

For any basic γ , write $\gamma = \mu + (\Delta_T - M^2)\nu$, where μ is basic harmonic. Since $M\gamma = 0$, we have $M(\Delta_T - M^2)\nu = 0$. By the above, M commutes with Δ_T and so $(\Delta_T - M^2)M\nu = 0$, i.e. $M\nu$ is basic harmonic. In particular, $M^2\nu = 0$, which implies that ν is basic since M is skew-adjoint. Hence we may write $\gamma = \mu + \Delta_B\nu$.

Let γ be a d_B -closed basic form. Note that d_T^* takes basic forms to basic forms; this is true since it holds for the operators $*_T$ and d_T . We will write d_B^* for d_T^* restricted to basic forms.³ It is then clear that $d_B\Delta_B = d_B d_B^* d_B = \Delta_B d_B$. Let γ be a d_B -closed form, with $\gamma = \mu + \Delta_B\nu$ for μ basic harmonic. As usual, we obtain $\gamma = \mu + d_B d_B^* \nu$, and so the basic cohomology class $[\gamma]_B$ may be represented by a basic harmonic form. A basic harmonic form which is d_B -exact is in particular d_T -exact and hence zero. So we have the expected isomorphism

$$\mathcal{H}_B(S) \cong H_B^*(S)$$

³This coincides with El Kacimi-Alaoui's d_B^* , but we will not need that here.

between basic harmonic forms and basic cohomology. Hence we have a basic Hodge decomposition:

Proposition 3.5.

$$H_B^r(S; \mathbb{C}) \cong \bigoplus_{p+q=r} H_B^{p,q}(S),$$

$$H_B^{p,q}(S) \cong \overline{H_B^{q,p}(S)}.$$

Let $h = [\Lambda, L]$. As in the Kähler case, Λ , L , and h give an infinite dimensional representation of \mathfrak{sl}_2 on transverse forms. To obtain a finite dimensional representation, we restrict these operators to basic harmonic forms. Since L , Λ , and h commute with Δ_B , we obtain the expected hard Lefschetz theorem:

Proposition 3.6. Let S be a compact Sasakian manifold of dimension $2m + 1$.

$$L^k : H_B^{m-k}(S) \rightarrow H_B^{m+k}(S)$$

is an isomorphism. Setting $P^{m-k}(S) = \text{Ker } L^{k+1} : H_B^{m-k}(S) \rightarrow H_B^{m+k+2}(S)$, we have

$$H_B^i(S) = \bigoplus_k L^k P^{i-2k}(S).$$

Remark. Note that the condition of being basic is necessary to achieve finite dimensionality for both the Hodge and Lefschetz decompositions. This comes from the fact that we needed to subtract M^2 from the transverse Laplacians to make them elliptic. The kernel of $\Delta_{\bar{\partial}_T}$, for example, is certainly *not* finite dimensional. We can see this explicitly for the example of S^3 . Let x_1 and x_2 be the complex coordinates of \mathbb{C}^2 , restricted to S^3 . Let $f = F(x_1, x_2)$, where $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is any smooth function. Then $\bar{\partial}_T f = 0$ and so $\Delta_{\bar{\partial}_T} f = 0$. It is clear that the space of all such f is infinite dimensional. From this discussion we conclude that we should not expect a version of the Hodge or Lefschetz decompositions to hold on a broader class of forms than the basic forms.

The standard proof of the $\partial\bar{\partial}$ -lemma [Vo] applies, and we obtain

Proposition 3.7 (Basic $\partial\bar{\partial}$ -lemma). Let α be a basic p -form on S , which is ∂_B - and $\bar{\partial}_B$ -closed. Also assume that α is $\bar{\partial}_B$ -, ∂_B -, d_B -, or d_B^c -exact. Then $\alpha = \partial_B \bar{\partial}_B \beta$ for some β .

Let us attempt to prove a more general version of proposition 3.7; we will see that our hand is essentially forced and we cannot do better than this basic $\partial\bar{\partial}$ -lemma.

We begin with weaker hypotheses: assume that α is ∂_T -closed and $\alpha = \bar{\partial}_T \gamma$.

(Neither α nor γ are assumed to be basic.) Using proposition 3.4, we write

$$\gamma = \mu + 2\Delta_{\bar{\partial}_T}\nu - M^2\nu,$$

where μ is basic harmonic. Then $\bar{\partial}_T\mu = \partial_T\mu = M\mu = 0$.

We obtain

$$\begin{aligned}\alpha &= 2\bar{\partial}_T\Delta_{\bar{\partial}_T}\nu - \bar{\partial}_TM^2\nu \\ &= 2\Delta_{\bar{\partial}_T}\bar{\partial}_T\nu - \bar{\partial}_TM^2\nu \\ &= 2\Delta_{\partial_T}\bar{\partial}_T\nu + (2\sqrt{-1}[\Lambda, L] - M)M\bar{\partial}_T\nu \\ &= -2\partial_T\bar{\partial}_T\partial_T^*\nu + 2\partial_T^*\partial_T\bar{\partial}_T\nu + (2\sqrt{-1}[\Lambda, L] - M)M\bar{\partial}_T\nu.\end{aligned}$$

$\partial_T\alpha = 0$ then yields

$$2\partial_T\partial_T^*\partial_T\bar{\partial}_T\nu + (2\sqrt{-1}[\Lambda, L] - M)M\partial_T\bar{\partial}_T\nu = 0. \quad (3.5)$$

If we can make the second term of the left-hand side of (3.5) vanish, we will obtain $\alpha = -2\partial_T\bar{\partial}_T\partial_T^*\nu$ as desired. However, the clearest and least restrictive way to ensure this is to add the further hypothesis that $M\alpha = 0$. This gives $(2\Delta_{\bar{\partial}_T} - M^2)(M\bar{\partial}_T\nu) = 0$, and so in particular $M^2\bar{\partial}_T\nu = 0$ and therefore $M\bar{\partial}_T\nu = 0$.

It would seem that we have proved something more general than proposition 3.7 since we have assumed that α is merely $\bar{\partial}_T$ -exact (but not necessarily $\bar{\partial}_B$ -exact). However, this does not give a more general version of the basic $\partial\bar{\partial}$ -lemma as a consequence of the following:

Proposition 3.8. A basic form which is $\bar{\partial}_T$ -exact is also $\bar{\partial}_B$ -exact.

Proof. Suppose that $M\bar{\partial}_T\alpha = 0$ for some transverse α . Write $\alpha = \mu + (2\Delta_{\bar{\partial}_T} - M^2)\nu$, where $(2\Delta_{\bar{\partial}_T} - M^2)\mu = 0$. Then as above $\bar{\partial}_T\alpha = (2\Delta_{\bar{\partial}_T} - M^2)(\bar{\partial}_T\nu)$. $M\bar{\partial}_T\alpha = 0$ tells us that $(2\Delta_{\bar{\partial}_T} - M^2)(M\bar{\partial}_T\nu) = 0$. So in particular we have $M^2\bar{\partial}_T\nu = 0$, which yields $M\bar{\partial}_T\nu = 0$ and $\bar{\partial}_T\alpha = 2\bar{\partial}_T\bar{\partial}_T^*\bar{\partial}_T\nu$. But $\bar{\partial}_T\nu$ is basic, and therefore so is $\bar{\partial}_T^*\bar{\partial}_T\nu$. We have

$$\bar{\partial}_T\alpha = 2\bar{\partial}_B\bar{\partial}_B^*\bar{\partial}_T\nu,$$

and so $\bar{\partial}_T\alpha$ is $\bar{\partial}_B$ -exact. \square

Using the isomorphism $H_B^{p,q}(S) \cong \mathcal{H}_B^{p,q} \cong H_{\bar{\partial}_B}^{p,q}$, we then conclude the following result.

Corollary 3.1. There is an injection

$$H_B^{p,q}(S) \hookrightarrow H_{\bar{\partial}_T}^{p,q}.$$

Remark. $H_{\bar{\partial}_T}^{p,q}$ is not the same as the usual Kohn-Rossi cohomology of $\bar{\partial}_b$ since $\bar{\partial}_T$ is only defined on transverse forms. For example, $\bar{\partial}_b \eta = d\eta$, but $d\eta$ is not $\bar{\partial}_T$ -exact.

Chapter 4

Real homotopy type of Sasakian manifolds

The goal in this chapter will be to use the basic $\partial\bar{\partial}$ -lemma and results of transverse Hodge theory to prove that the real homotopy type of a compact Sasakian manifold S is a formal consequence of its basic cohomology ring $H_B^*(S)$ and its basic Kähler class, $[d\eta]_B$.

4.1 Real homotopy type and differential forms

The following exposition follows [GM] and [DGM].

Let X be a simply-connected CW complex, and let X_2 be the Eilenberg-MacLane space $K(\pi_2(X), 2)$. Inductively, one can construct a tower of principal fibrations called the *Postnikov tower*, as shown in figure 4-1. This is a commutative diagram satisfying:

- (i) $\pi_i(X_j) = 0$ for $i > j$,
- (ii) $X_i \rightarrow X_{i-1}$ is a principal fibration with fiber $K(\pi_i(X), i)$, and
- (iii) f_i induces an isomorphism on π_j for all $j \leq i$.

The Postnikov tower of X determines X up to homotopy equivalence. If P_X is the Postnikov tower of X , it is possible to define a “rational Postnikov tower” $P_X \otimes \mathbb{Q}$, where $\pi_i(X_i)$ is replaced everywhere with $\pi_i(X_i) \otimes \mathbb{Q}$. If X is a manifold, there is a

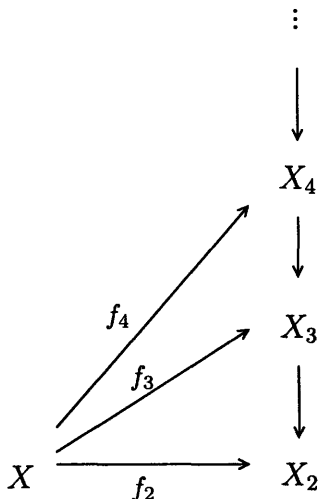


Figure 4-1: The Postnikov tower of a simply-connected CW complex X .

deep relationship between the rational Postnikov tower and the de Rham algebra of X , which we will now describe.

Fix a base field K . A *differential graded algebra* (DGA) over K is a $\mathbb{Z}_{\geq 0}$ -graded K -vector space, equipped with a graded commutative multiplicative structure and a degree 1 differential d which is a graded derivation and satisfies $d^2 = 0$. In what follows we will denote a differential graded algebra by a pair (A, d) , where A is the algebra and d is the differential. A DGA is called “connected” if its zeroth cohomology group (with respect to d) is K . A DGA is called “simply-connected” if in addition its first cohomology group is zero. On a simply-connected DGA (A, d) , we may define homotopy groups $\pi_i(A, d)$. If (A, d) is merely connected, we may only define a “fundamental group” $\pi_1(A, d)$, which is actually a tower of nilpotent Lie groups.

Let V be a K -vector space. Let $\Lambda_n(V)$ denote the free (graded commutative) algebra generated by V in degree n . An *elementary extension* of a DGA (A, d) is a DGA whose underlying algebra is $A \otimes \Lambda_n(V)$ for some V . We also require that the differential restricted to A is d , and the differentials of elements of V must lie in A . To include this in our notation we will follow [DGM] and write the extended algebra as $(A \otimes_d \Lambda_n(V), d)$.

Definition 4.1 ([DGM]). Suppose (A, d) is a DGA which may be written as an increasing union of sub-DGA’s

$$K \subset A_1 \subset A_2 \subset \dots, \quad \text{with} \quad \bigcup_{i \geq 0} A_i = A.$$

Further suppose each $A_i \subset A_{i+1}$ is an elementary extension, and suppose that $d(A) \subset A^+ \wedge A^+$, where A^+ consists of all elements in A of positive degree. Then we will call (A, d) a *minimal* DGA.

Let (A, d) be any DGA.

Definition 4.2. A DGA $(\mathcal{M}, d_{\mathcal{M}})$ along with a map of DGA's $(\mathcal{M}, d_{\mathcal{M}}) \xrightarrow{\rho} (A, d)$ is called a *k-stage minimal model* for (A, d) if $(\mathcal{M}, d_{\mathcal{M}})$ is a minimal algebra satisfying:

- (i) \mathcal{M} is generated in degree $\leq k$.
- (ii) ρ induces an isomorphism on cohomology in degree $\leq k$ and an injection in degree $k + 1$.

If $k = \infty$, we call $(\mathcal{M}, d_{\mathcal{M}})$ a *minimal model* for (A, d) .

Every simply-connected DGA has a minimal model which is unique up to isomorphism, and every connected DGA has a 1-stage minimal model, unique up to isomorphism.

Let $K = \mathbb{Q}$, and let X be a manifold. Instead of using C^∞ differential forms, we may define the algebra (\mathcal{E}_X^*, d) of \mathbb{Q} -polynomial forms on X , whose cohomology is isomorphic to the rational cohomology of X .¹ Assume X is simply-connected. Given a minimal model $(\mathcal{M}, d_{\mathcal{M}})$ for (\mathcal{E}_X^*, d) , we may construct a tower of fibrations. Suppose we have inductively constructed a tower of $n - 1$ CW-complexes, $X_n \rightarrow \cdots \rightarrow X_2$. The idea is that for the $(n + 1)$ st elementary extension $A \otimes_d \Lambda_{n+1}(V)$, the differential on V gives us an element of $H^{n+2}(A, V^*)$, which in turn determines the characteristic class of a fibration over X_n with fiber $K(V^*, n + 1)$. In fact, this construction recovers $P_X \otimes \mathbb{Q}$, the rational Postnikov tower. One can also go the other direction, from $P_X \otimes \mathbb{Q}$ to a minimal algebra. This correspondence gives an isomorphism between $\pi_i(X) \otimes \mathbb{Q}$ and the homotopy groups of (\mathcal{E}_X^*, d) . This isomorphism takes the Whitehead product $\pi_i(X) \otimes \mathbb{Q} \times \pi_j(X) \otimes \mathbb{Q} \rightarrow \pi_{i+j-1}(X) \otimes \mathbb{Q}$ to a simple operation on the homotopy groups of (\mathcal{E}_X^*, d) which comes from d . So $(\mathcal{M}, d_{\mathcal{M}})$ contains all information about rational homotopy and is therefore called the *rational homotopy type* of X .

The de Rham algebra $(\Omega^*(X), d_X)$ has a minimal model $(\mathcal{M}', d_{\mathcal{M}'})$ which is actually isomorphic to $(\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{R}, d_{\mathcal{M}})$. This justifies calling $(\mathcal{M}', d_{\mathcal{M}'})$ the *real homotopy type* of X . $(\mathcal{M}', d_{\mathcal{M}'})$ contains information about the real homotopy groups, their Whitehead products, and other higher-order (Massey) products.

If X is not simply-connected, $(\Omega^*(X), d_X)$ still has a 1-stage minimal model $(\mathcal{M}^{(1)}, d_{\mathcal{M}^{(1)}})$. In this case the DGA fundamental group of $(\mathcal{M}^{(1)}, d_{\mathcal{M}^{(1)}})$ is the real form of the nilpotent completion of $\pi_1(X)$. We call this tower of groups the “de Rham fundamental group” of X .

¹Cf. [DGM] for details.

A *quasi-isomorphism* of two DGA's is a map of DGA's which induces an isomorphism on cohomology.²

Definition 4.3. The real homotopy type of a manifold X is said to be a *formal consequence of its cohomology* if $(\Omega^*(X), d_X)$ is quasi-isomorphic to a DGA with zero differential. (Such a DGA must then be isomorphic to $(H^*(X; \mathbb{R}), 0)$.) In this case we say that X is *formal*.

The idea is that since quasi-isomorphic DGA's have the same minimal model, the real homotopy type of X can be determined by $H^*(X; \mathbb{R})$.³

For the rest of this chapter, we will consider only cohomology with real coefficients and will denote $H^*(X; \mathbb{R})$ by $H^*(X)$.

For any manifold X , let $\alpha \in H^p(X)$, $\beta \in H^q(X)$, and $\gamma \in H^r(X)$, and choose differential forms a , b , and c representing α , β , and γ respectively. Suppose $\alpha \cup \beta = \beta \cup \gamma = 0$. Then we may form a triple product $\langle \alpha, \beta, \gamma \rangle \in H^{p+q+r-1}(X)/(\alpha \cdot H^{q+r-1}(X) + \gamma \cdot H^{p+q-1}(X))$. Writing $a \wedge b = dg$ and $b \wedge c = dh$, the class $\langle \alpha, \beta, \gamma \rangle$ is represented by the form $g \wedge c + (-1)^{p+1} a \wedge h$. One checks that this class is independent of the choices made. This ‘‘Massey product’’ is a structure of the de Rham algebra of X that is not reflected in the cohomology ring alone. However, we have the following well-known result.

Proposition 4.1. On a simply-connected formal manifold, all Massey products vanish.

Proof. We will show the vanishing of the triple product $\langle \alpha, \beta, \gamma \rangle$ described above. Consider the diagram

$$(H^*(X), 0) \xleftarrow{\rho} (\mathcal{M}, d) \xrightarrow{\tilde{\rho}} (\Omega^*(X), d_X),$$

where (\mathcal{M}, d) is the minimal model for both $(H^*(X), 0)$ and $(\Omega^*(X), d_X)$. Since $\tilde{\rho}$ induces an isomorphism on cohomology, there must be a closed $A \in (\mathcal{M}, d)$ such that $a := \tilde{\rho}(A)$ represents α . Likewise there exist closed elements B and C of (\mathcal{M}, d) such that $b := \tilde{\rho}(B)$ and $c := \tilde{\rho}(C)$ represent β and γ respectively. Let $\tilde{\rho}^*$ be the induced map on cohomology of the DGA's. Since $\tilde{\rho}^*[A \wedge B] = 0$ and $\tilde{\rho}^*$ is an isomorphism, $A \wedge B = dG$ for some $G \in (\mathcal{M}, d)$. Likewise, $B \wedge C = dH$. Then $\tilde{\rho}(GC + (-1)^{p+1}AH)$ represents $\langle \alpha, \beta, \gamma \rangle$. Now we consider the map ρ . Since ρ^* is an isomorphism on cohomology, there must exist some *closed* $\hat{G} \in (\mathcal{M}, d)$ such that $\rho(\hat{G}) = \rho(G)$, and likewise with H . Since $\rho(GC + (-1)^{p+1}AH) = \rho(\hat{G}C + (-1)^{p+1}A\hat{H})$,

²Actually, instead of one map we may have a sequence of maps connecting the two DGA's. It does not matter what direction the maps go; all that matters is that they induce isomorphisms on cohomology.

³If X is not simply-connected, we still use the terminology ‘‘formal.’’ In this case the implication is that the de Rham fundamental group of X is determined by $H^*(X)$.

it follows that $GC + (-1)^{p+1}AH$ and $\hat{G}C + (-1)^{p+1}A\hat{H}$ are cohomologous. Therefore $\tilde{\rho}(\hat{G})_c + (-1)^{p+1}a\tilde{\rho}(\hat{H})$ also represents $\langle \alpha, \beta, \gamma \rangle$. Since $\tilde{\rho}(\hat{G})$ and $\tilde{\rho}(\hat{H})$ are closed, however, $\langle \alpha, \beta, \gamma \rangle = 0$. \square

4.2 Deligne-Griffiths-Morgan proof of formality for a compact Kähler manifold

Let X be a compact Kähler manifold. In [DGM], Deligne et al use the $\partial\bar{\partial}$ -lemma to show formality of X . Their argument goes as follows. Let $\Omega^c(X)$ denote d^c -closed forms on X , and let $H^c(X)$ denote the d^c -cohomology of $\Omega^*(X)$. Consider the diagram

$$(H^c(X), 0) \xleftarrow{p} (\Omega^c(X), d_X) \xrightarrow{j} (\Omega^*(X), d_X),$$

where j is inclusion and p sends a form α to its d^c -cohomology class $[\alpha]_c$. It is clear that both p and j respect the multiplicative structures of the algebras, and it is clear that j respects the differentials. p also respects the differentials for the following reason. For any $\alpha \in \Omega^c(X)$, $d\alpha$ is d^c -closed and d -exact. So by the $\partial\bar{\partial}$ -lemma, $d\alpha = dd^c\beta$ for some β . Then $p \circ d\alpha = 0$ as needed.

To see that the induced map p^* on cohomology is injective, let $\alpha \in \Omega^c(X)$ be d -closed with $p\alpha = 0$. Then α is d -closed and d^c -exact, and so $\alpha = dd^c\beta$. So the DGA cohomology class of α in $(\Omega^c(X), d_X)$ is 0.

For surjectivity of p^* , consider any d^c -cohomology class $[\alpha]_c$. α is d^c -closed; if we can pick another representative of $[\alpha]_c$ which is d -closed, we will be done. We do this as follows. $d\alpha$ is d -exact and d^c -closed, so we write $d\alpha = dd^c\beta$. Then $\alpha - d^c\beta$ is a d -closed representative of $[\alpha]_c$.

To see that j^* is injective, let $\alpha \in \Omega^c(X)$ be d -closed with $\alpha = d\beta$ for some $\beta \in \Omega^*(X)$. Then we may apply the $\partial\bar{\partial}$ -lemma to α , writing $\alpha = dd^c\gamma$. So the DGA cohomology class of α in $(\Omega^c(X), d_X)$ is 0.

Lastly, let us see that j^* is surjective. For any d -cohomology class $[\alpha]$ on X , we need to find a d^c -closed representative. We achieve this by the same trick as above; we apply the $\partial\bar{\partial}$ -lemma to $d^c\alpha$ by writing $d^c\alpha = dd^c\beta$. Then $\alpha + d\beta$ is the representative we are looking for.

We conclude from the above that any compact Kähler manifold is formal. Proposition 4.1 then shows that the Massey products vanish if X is simply-connected. In the non-simply-connected case this is also true, since the proof of the proposition merely used the fact that $\Omega^*(X)$ is quasi-isomorphic to $H^*(X)$.

We will not be able to show formality for Sasakian manifolds. However, we show the next best thing: on a compact Sasakian manifold S , $(\Omega^*(S), d_S)$ is quasi-isomorphic to an elementary extension of $(H_B^*(S), 0)$ which has only one nonzero differential. If we know $H_B^*(S)$ and a single distinguished element (the basic Kähler class), we know the real homotopy type of S .

4.3 The formality result for compact Sasakian manifolds

Let S be a compact Sasakian manifold. Consider the differential graded algebra $(H_B^*(S) \otimes_d \Lambda_1(y), dy = [d\eta]_B)$ obtained from $(H_B^*(S), 0)$ by adding a free element y in degree 1 and defining its differential dy to be $[d\eta]_B$. (The tensor product is graded commutative, so $y \otimes y = 0$.) Let (Ω_B^c, d_B) denote the algebra of d_B^c -closed (basic) forms in S , with differential d_B . ($d_B^c := \sqrt{-1}(\bar{\partial}_B - \partial_B)$.) Let H_B^c denote the d_B^c -cohomology of basic forms. If S is regular and there is a Kähler base, the argument in [DGM] shows that the differential induced by d_B on H_B^c is 0. In fact, this argument still holds in the irregular case: if a is a d_B^c -closed basic form on S , $d_B a = d_B d_B^c \kappa$ by the $d_B d_B^c$ -lemma, i.e. $d_B a$ is d_B of a d_B^c -closed form. Denoting by $[a]_c$ the class of a in H_B^c , we have $d_B [a]_c = 0$.

Denote by (Ω_S^*, d_S) the de Rham algebra of S . Motivated by the approach in [DGM], we will now construct the following quasi-isomorphisms of differential graded algebras:

$$(H_B^c \otimes_d \Lambda_1(y), dy = [d\eta]_c) \xleftarrow{\tau} (\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta) \xrightarrow{\sigma} (\Omega_S^*, d_S). \quad (4.1)$$

Here the differential on $\Omega_B^c \otimes_d \Lambda_1(y)$ is d_B on Ω_B^c and $d(y) = d\eta$.⁴

Define $\tau : (\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta) \rightarrow (H_B^c \otimes_d \Lambda_1(y), dy = [d\eta]_c)$ by

$$\alpha + \beta \otimes y \mapsto [\alpha]_c + [\beta]_c \otimes y.$$

Since the d_B^c -cohomology class of a d_B -exact and d_B^c -closed form is 0 by the basic $\partial\bar{\partial}$ -lemma, $\tau \circ d(\alpha + \beta \otimes y) = \pm[\beta \wedge d\eta]_c = d \circ \tau(\alpha + \beta \otimes y)$. It is clear that τ respects the product structures of the algebras.

Claim 4.1. τ^* is injective on cohomology.

Proof. Let $x = \alpha + \beta \otimes y$ be a closed element in $(\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta)$ in degree k . Since x is closed we obtain $d_B \beta = 0$ and $d_B \alpha + (-1)^{k+1} d\eta \wedge \beta = 0$. Suppose $\tau(x) = d([\alpha]_c + [\beta]_c \otimes y) = (-1)^{k+1} [b \wedge d\eta]_c$. Then $[\beta]_c = 0$ and $[\alpha]_c = (-1)^{k+1} [b \wedge d\eta]_c$,

⁴Recall that $d\eta$ is not exact in $H_B^*(S)$. Also note that $d_B^c(d\eta) = 0$.

i.e. $\alpha = (-1)^{k+1}b \wedge d\eta + d_B^c \kappa$ for some κ . β is d_B -closed and d_B^c -exact, and so we may write $\beta = d_B d_B^c \nu$. Define $z := \alpha + (-1)^{k+1}d\eta \wedge d_B^c \nu$. Then $d_B z = 0$. Write $z = \zeta + \Delta_B \theta$ for ζ basic harmonic. $d_B z = 0$ implies that $z = \zeta + d_B d_B^* \theta$. z and ζ are both d_B^c -closed, so we may write $d_B d_B^* \theta = d_B d_B^c \epsilon$ and $z = \zeta + d_B d_B^c \epsilon$. We then have

$$\zeta + d_B d_B^c \epsilon = z = (-1)^{k+1}b \wedge d\eta + d_B^c \kappa + (-1)^{k+1}d\eta \wedge d_B^c \nu. \quad (4.2)$$

Let λ be a basic harmonic form in the image of L , i.e. $\lambda = d\eta \wedge \mu$ for some basic harmonic μ . Then $\lambda = d(\pm\mu \otimes y)$ in $(\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta)$. (Note that $d_B^c \lambda = d_B^c \mu = 0$.) Therefore we may replace the α above by α plus any harmonic form in $\text{Im } L$ without affecting the DGA cohomology class of $x = \alpha + \beta \otimes y$. So we may then assume that $\zeta \in (\text{Im } L)^\perp$.

A priori, ζ is only orthogonal to forms which are $d\eta \wedge$ a Δ_B -harmonic form. However any basic form χ may be expressed as $\chi_1 + \Delta_B \chi_2$ where $\Delta_B \chi_1 = 0$. L commutes with Δ_B , and so $L\chi = L\chi_1 + \Delta_B(L\chi_2)$. $\zeta \in (\text{Im } L)^\perp$ tells us that ζ is orthogonal to $L\chi_1$. Also ζ is orthogonal to the image of Δ_B , and so ζ is orthogonal to $L\chi$.

Returning to (4.2), we conclude that ζ is orthogonal to everything else and so it is orthogonal to itself, i.e. $\zeta = 0$. Therefore $z = d_B d_B^c \epsilon$, and letting $w = d_B^c \epsilon + d_B^c \nu \otimes y \in (\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta)$, we have

$$\begin{aligned} dw &= d_B d_B^c \epsilon + (-1)^k d_B^c \nu \wedge d\eta + d_B d_B^c \nu \otimes y \\ &= \alpha + (-1)^{k+1} d\eta \wedge d_B^c \nu + (-1)^k d_B^c \nu \wedge d\eta + \beta \otimes y \\ &= x. \end{aligned}$$

□

Claim 4.2. τ^* is surjective on cohomology.

Proof. Let $x = [\alpha]_c + [\beta]_c \otimes y$ be a closed element of $(H_B^c \otimes_d \Lambda_1(y), dy = [d\eta]_c)$ in degree k . This means that $[\beta \wedge d\eta]_c = 0$. Pick any representatives α and β of these d_B^c -cohomology classes. Following the argument in [DGM], we can find a d_B -closed representative of $[\beta]_c$ as follows. β is d_B^c -closed, so we may use the basic $\partial\bar{\partial}$ -lemma to write $d_B \beta = d_B d_B^c \nu$. Then $\tilde{\beta} := \beta - d_B^c \nu$ is d_B -closed and $[\tilde{\beta}]_c = [\beta]_c$. Similarly we may define $\tilde{\alpha}$ to be a d_B -closed representative of $[\alpha]_c$.

$\tilde{\beta} \wedge d\eta$ is then d_B -closed and d_B^c -exact, so we write $\tilde{\beta} \wedge d\eta = d_B d_B^c \kappa$. Let $\hat{\alpha} = \tilde{\alpha} + (-1)^k d_B^c \kappa$. Then

$$\begin{aligned} d(\hat{\alpha} + \tilde{\beta} \otimes y) &= 0 + (-1)^k \tilde{\beta} \wedge d\eta + 0 + (-1)^{k+1} \tilde{\beta} \wedge d\eta \\ &= 0. \end{aligned}$$

So $w := \hat{\alpha} + \tilde{\beta} \otimes y$ defines a cohomology class in $(\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta)$, and

$$\tau^*([w]) = [[\hat{\alpha}]_c + [\tilde{\beta}]_c \otimes y] = [x],$$

where the outer brackets denote the DGA cohomology class. \square

Next we define a quasi-isomorphism $\sigma : (\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta) \rightarrow (\Omega_S^*, d_S)$ by

$$\alpha + \beta \otimes y \mapsto \alpha + \beta \wedge \eta.$$

It is clear that σ respects the differentials since $\sigma \circ d(\alpha + \beta \otimes y) = d_B \alpha \pm d\eta \wedge \beta + d_B \beta \wedge \eta = d_S \circ \sigma(\alpha + \beta \otimes y)$. Again it is clear that σ respects the product structures of the algebras.

Claim 4.3. σ^* is injective on cohomology.

Proof. Consider any closed element $x = \alpha + \beta \otimes y$ of $(\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta)$ in degree k . x being closed means that $d_B \alpha + (-1)^{k+1} d\eta \wedge \beta = 0$ and $d_B \beta = 0$. Suppose $\sigma(x) = \alpha + \beta \wedge \eta = d_S(a + b \wedge \eta)$. Collecting transverse forms, $d_S(a + b \wedge \eta) = (d_T a + (-1)^k b \wedge d\eta) + ((-1)^{k+1} M a + d_T b) \wedge \eta$. So we may conclude that

$$\alpha = d_T a + (-1)^k b \wedge d\eta \quad \text{and} \quad \beta = (-1)^{k+1} M a + d_T b. \quad (4.3)$$

Write $\beta = \varphi + \Delta_B \psi$, where $\Delta_B \varphi = 0$. $d_B \beta = 0$ tells us that $\beta = \varphi + d_B d_B^* \psi$. β and φ are both d_B^c -closed, so $d_B^c d_B d_B^* \psi = 0$. Then $d_B d_B^* \psi$ is d_B -exact and d_B^c -closed, and we may apply the basic $\partial\bar{\partial}$ -lemma to write $d_B d_B^* \psi = d_B d_B^c \nu$ for some ν . We have

$$\beta = \varphi + d_B d_B^c \nu = (-1)^{k+1} M a + d_T b.$$

Since $\Delta_T \varphi = M \varphi = 0$, φ is orthogonal to the image of d_T and the image of M , where these operators are acting on any transverse forms on S . So φ is actually orthogonal to itself and is therefore zero. We conclude that $\beta = d_B d_B^c \nu$.

Once again we define $z := \alpha + (-1)^{k+1} d\eta \wedge d_B^c \nu$ and write $z = \zeta + \Delta_B \theta$, where $\Delta_B \zeta = 0$. Again we have $d_B z = 0$ and so $z = \zeta + d_B d_B^* \theta$. By (4.3),

$$\zeta + d_B d_B^* \theta = d_T a + (-1)^k b \wedge d\eta \pm d\eta \wedge d^c \nu. \quad (4.4)$$

As noted earlier, we may assume ζ is orthogonal to the image of L (acting on all transverse forms) without affecting the DGA cohomology class of $x = \alpha + \beta \otimes y$. In (4.4) we see that ζ is orthogonal to everything else and so $\zeta = 0$ and $z = d_B d_B^* \theta$. Since $d_B^c z = 0$, we may apply the basic $\partial\bar{\partial}$ -lemma to write $z = d_B d_B^* \theta = d_B d_B^c \epsilon$. As before, $w := d_B^c \epsilon + d_B^c \nu \otimes y$ is an element of $(\Omega_B^c \otimes_d \Lambda_1(y), d_B; dy = d\eta)$, and

$$dw = \alpha + \beta \otimes y = x.$$

□

In their proof of formality for a Kähler manifold, Deligne et al show that if X is a compact Kähler manifold, the d_X^c -cohomology and d_X -cohomology of X are isomorphic as differential graded algebras (with zero differential). In the notation of section 4.2, the isomorphism is $j^* \circ (p^*)^{-1}$. Since a Kähler form ω is d_X^c -closed, we have $j^* \circ (p^*)^{-1}([\omega]_{d_X^c}) = [\omega]_{d_X}$. Deligne et al's argument relies solely on the basic $\partial\bar{\partial}$ -lemma, so we may replace $H^c(X)$ and $H^*(X)$ with the d_B^c - and d_B -cohomologies of a compact Sasakian manifold. The isomorphism between them takes $[d\eta]_c$ to $[d\eta]_B$. So it is clear that $(H_B^c \otimes_d \Lambda_1(y), dy = [d\eta]_c)$ is isomorphic as a DGA to $(H_B^* \otimes_d \Lambda_1(y), dy = [d\eta]_B)$. Letting $\tilde{\tau}$ be the composition of this isomorphism with τ , we rewrite (4.1) as

$$(H_B^* \otimes_d \Lambda(y_1), dy = [d\eta]_B) \xleftarrow{\tilde{\tau}} (\Omega_B^c \otimes_d \Lambda(y_1), d_B; dy = d\eta) \xrightarrow{\sigma} (\Omega_S^*, d_S).$$

From the above we know that the cohomology of $(H_B^* \otimes_d \Lambda(y_1), dy = [d\eta]_B)$ injects into $H^*(S)$, but we have not shown surjectivity. It suffices to show that as graded vector spaces,

$$H^i(S) \cong H_B^i / \text{Im } L \oplus \{y \otimes [\alpha]_B \mid [\alpha]_B \in H_B^{i-1}, [d\eta \wedge \alpha]_B = 0\}. \quad (4.5)$$

For a closed Riemannian manifold (S, g) admitting a unit Killing vector field T , there is a well-known long exact sequence relating basic cohomology to ordinary de Rham cohomology [To]:

$$\dots \rightarrow H_B^{i-2} \xrightarrow{\delta} H_B^i \xrightarrow{\iota^*} H_S^i \xrightarrow{\iota_T^*} H_B^{i-1} \xrightarrow{\delta} H_B^{i+1} \rightarrow \dots \quad (4.6)$$

Here ι is the inclusion of basic forms into $\Omega^*(M)$ and δ is a connecting homomorphism which in our case coincides with $L = d\eta \wedge \cdot$ [BGN]. ι_T is the interior product with T (which is ξ in our case). The subgroup of isometries generated by the flow of T is a torus G . We are making use of the fact that H_S^i is isomorphic to the cohomology of G -invariant forms. One checks that any G -invariant form may be written as $a + \eta \wedge b$, where a and b are both basic [To]. So $\iota_T^*[a + \eta \wedge b]_S := [b]_B$. (This is a well-defined map on cohomology.) The exactness of (4.6) gives us the isomorphisms (4.5).

We may show (4.5) another way, which will tell us something about the harmonic forms on S with respect to the usual Laplacian.

Let $x = [\alpha]_B + [\beta]_B \otimes y$ be a closed element of $H_B^* \otimes_d \Lambda_1(y)$, where α and β are chosen to be basic harmonic representatives of their cohomology classes. Since x is

closed, $\beta \wedge d\eta$ must be d_B -exact. But since L commutes with Δ_B , $\beta \wedge d\eta$ is basic harmonic and therefore must be zero. Basic harmonic forms are d_B^c -closed and d_B -closed, so $\tilde{\tau}^*[\alpha + \beta \otimes y] = [[\alpha]_B + [\beta]_B \otimes y]$, where the outer brackets on each side denote DGA cohomology classes. Thus $\sigma^* \circ (\tilde{\tau}^*)^{-1}[x] = [\alpha + \beta \wedge \eta]_S$. If we can show that any cohomology class in S can be represented by a form $a + \eta \wedge b$ where a and b are both basic harmonic and $d\eta \wedge b = 0$, we will have shown that $\sigma^* \circ (\tilde{\tau}^*)^{-1}$ is an isomorphism.

Any class in $H^*(S)$ may be represented by a unique Δ -harmonic form, where Δ is the usual Laplacian on the Riemannian manifold (S, g) . So surjectivity of $\sigma^* \circ (\tilde{\tau}^*)^{-1}$ will follow from the following lemma.

Proposition 4.2. Let $x = a + \eta \wedge b$ be a Δ -harmonic form on S , where a and b are transverse. Then a and b are in fact basic harmonic and $d\eta \wedge b = 0$.

Proof. $\Delta x = 0$ means that $dx = d^*x = 0$, where the operators d and d^* are the usual operators on S . $dx = 0$ gives

$$d_T a + d\eta \wedge b = 0 \text{ and } M a - d_T b = 0.$$

$d^*x = 0$ tells us that

$$\langle x, d(f + \eta \wedge g) \rangle = 0$$

for any transverse forms f and g . Expanding, we have

$$\begin{aligned} 0 &= \langle a + \eta \wedge b, (d_T f + d\eta \wedge g) + \eta \wedge (M f - d_T g) \rangle \\ &= \langle a, d_T f + d\eta \wedge g \rangle + \langle b, M f - d_T g \rangle. \end{aligned} \tag{4.7}$$

To obtain the last line we have used that $g^{-1}(\eta, \eta) = 1$ and η is orthogonal to any transverse form.

Let us apply (4.7) with $f = -Mb$ and $g = 0$. We obtain $\langle a, -Md_T b \rangle + \langle b, -M^2 b \rangle = 0$. Since M is skew-adjoint on transverse forms and $\langle \cdot, \cdot \rangle$ is the same inner product we have been using all along, we get $\langle Ma, d_T b \rangle + \langle Mb, Mb \rangle = 0$. However, $Ma = d_T b$, and so we conclude that $Ma = d_T b = Mb = 0$, i.e. a and b are both basic and b is d_B -closed.

Next we apply (4.7) with $g = 0$ and f any transverse form. Since b is basic, we may ignore the right summand of (4.7) and we have $\langle a, d_T f \rangle = 0$, i.e. $d_T^* a = d_B^* a = 0$.

We will apply (4.7) once more with the following f and g . Since $d_B b = 0$, we may write $d_B^c b = d_B d_B^c \kappa$. Let $f = d_B^c \kappa$ and $g = -d_T^* b = -d_B^* b$. We obtain

$$\begin{aligned} 0 &= \langle a, d_B d_B^c \kappa \rangle + \langle a, -L d_T^* b \rangle + \langle b, d_B d_B^* b \rangle \\ &= \langle a, d_B d_B^c \kappa \rangle + \langle a, -d_T^* L b \rangle - \langle a, d_T^c b \rangle + \langle b, d_B d_B^* b \rangle \end{aligned} \tag{4.8}$$

$$\begin{aligned}
&= \langle a, -d_T^* Lb \rangle + \langle b, d_B d_B^* b \rangle \\
&= \langle d_T a, -Lb \rangle + \langle d_B^* b, d_B^* b \rangle
\end{aligned}
\tag{4.9}$$

In (4.8) we have used the transverse Kähler identities. Since $d_T a = -Lb$, (4.9) yields $d_B a = Lb = 0$ and $d_B^* b = 0$. So a and b are both basic harmonic and $d\eta \wedge b = 0$. \square

Consider the case where S is regular and simply-connected with Kähler base B . We may then apply the Serre spectral sequence; the $E_2^{p,q}$ terms are shown in figure 4-2. Here y is a generator of $H^1(S^1; \mathbb{R})$ which may be chosen such that $d_2 y = d\eta$. The spectral sequence degenerates at E_3 , and the resulting conclusions about $H^i(S)$ and $H^i(B)$ match (4.5).

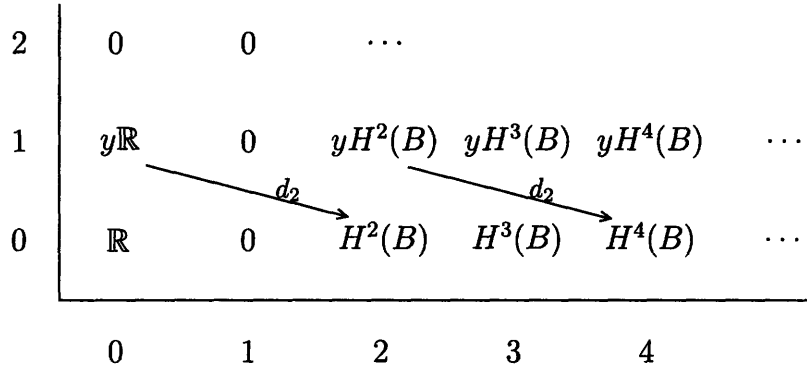


Figure 4-2: $E_2^{p,q} = H^p(B; H^q(S^1; \mathbb{R}))$.

Observation. Even though one may not use the Serre spectral sequence on an irregular (compact) Sasakian manifold, the conclusions derived from it still hold.

Thus we have a quasi-isomorphism between (Ω_S^*, d_S) and $(H_B^* \otimes_d \Lambda_1(y), dy = [d\eta]_B)$, which is an elementary extension of a DGA with 0 differential. So if we know the basic cohomology ring $H_B^*(S)$ and the distinguished element $[d\eta]_B$, we may form this elementary extension and then find the minimal model. This in turn determines the higher real homotopy groups of S , the real Whitehead products, and certain other higher order products. So in this sense, we have shown

Theorem 4.1. *The real homotopy type of S is a formal consequence of its basic cohomology ring and its basic Kähler class.*

We have not shown that the de Rham algebra of S is formal, i.e. quasi-isomorphic to its cohomology. One would like a quasi-isomorphism between $(H_S^*, 0)$ and $(H_B^* \otimes_d \Lambda_1(y), dy = [d\eta]_B)$. As we will see, in general this is difficult (if not impossible) to achieve by a single map σ of DGA's.

It is easy to see that there is no right-to-left quasi-isomorphism $(H_S^*, 0) \xleftarrow{\rho} (H_B^* \otimes_d \Lambda_1(y), dy = [d\eta]_B)$. Since $[d\eta]_B$ is exact in $(H_B^* \otimes_d \Lambda_1(y), dy = [d\eta]_B)$, ρ would have to map $[d\eta]_B \mapsto 0$. Suppose ρ maps $[(d\eta)^{m-1}]_B \otimes y$ to some $a \in H^*(S)$. For ρ to respect the multiplicative structure of the algebras we need $\rho([(d\eta)^m]_B \otimes y) = 0 \cdot a = 0$, but $[(d\eta)^m]_B \otimes y$ represents a nonzero DGA cohomology class by the isomorphism (4.5). ($\eta \wedge (d\eta)^m$ is a volume form on S .)

Let us attempt to construct a left-to-right quasi-isomorphism $(H_S^*, 0) \xrightarrow{\rho} (H_B^* \otimes_d \Lambda_1(y), dy = [d\eta]_B)$. Noting isomorphism (4.5), we represent an element of $H^i(S)$ by $[a] + b \otimes y$, where $a \in H_B^i$, $b \in H_B^{i-1}$, and $Lb = 0$. Here $[a]$ denotes the equivalence class of a in $H_B^i/\text{Im } L$. It is natural to set $\rho(b \otimes y) = b \otimes y$. The most natural way to define $\rho([a])$ would be to use a splitting of the short exact sequence

$$0 \rightarrow \text{Im } L \rightarrow H^i(B) \xrightarrow{f} H^i(B)/\text{Im } L \rightarrow 0. \quad (4.10)$$

Then we would set $\rho([a] + b \otimes y) = f([a]) + b \otimes y$. For ρ to respect the multiplicative structures, we need

$$\rho([a] + b \otimes y)\rho([c] + d \otimes y) = f([a])f([c]) + ([a]d \pm b[c]) \otimes y$$

to equal

$$\rho\left(\left([a] + b \otimes y\right)\left([c] + d \otimes y\right)\right) = f([ac]) + ([a]d \pm b[c]) \otimes y,$$

i.e. f must be multiplicative.⁵ It is tempting to use orthogonal projection to produce such an f , but such a splitting is not in general multiplicative. Below we give an example where no multiplicative f exists.

Let $B = d\mathbb{P}_2$ be the second del Pezzo surface, i.e. $\mathbb{C}P^2$ blown up at two points. A basis of $H^2(B; \mathbb{R})$ is given by ℓ , E_1 , and E_2 , where ℓ is the hyperplane class and E_1 and E_2 are the exceptional divisors. (Of course these are actually integral classes in $H^{1,1}$.) The Mori cone of effective divisors is spanned by E_1 , E_2 , and $\ell - E_1 - E_2$ [DHOR]. Letting ℓ^2 be the generator for $H^4(B)$, we have $E_i \cdot \ell = 0$ and $E_i \cdot E_j = -\delta_{ij}$. It follows that $h := 3\ell - E_1 - E_2$ is an ample divisor satisfying $h \cdot E_i = 1$, $h \cdot (\ell - E_1 - E_2) = 1$, and $h^2 = 7$. So h is a Kähler class on B . Let ω be a Kähler form in this class. By the discussion in section 2.1, there is a regular Sasakian manifold (Y, η, ξ, g, Φ) with base B such that $d\eta$ is the pullback of ω . So we may take h to be our basic Kähler class, and $\text{Im } L$ is the ideal generated by h .

Let us try to define a multiplicative splitting f as in (4.10). Again let $[\alpha]$ denote the equivalence class of $\alpha \in H_B^*$ in $H_B^*/\text{Im } L$. Since $[\alpha] = 0$ for any $\alpha \in H_B^4$, f must be zero on H_B^4 in order for f to be linear. So $f([E_1^2]) = 0$, and therefore $f([E_1])$ must square to zero. Suppose $f([E_1]) = E_1 + ah$ for some a . Then $(E_1 + ah)^2 = 0$ gives $7a^2 + 2a - 1 = 0$, or $a = \frac{1}{7}(-1 \pm 2\sqrt{2})$. Let the two roots be a_+ and a_- . So there are

⁵Of course $[a]d$ and $[b]c$ are well-defined since $b, d \in \text{Ker } L$.

only four possibilities for how to define f on $H_B^2/\text{Im } L$, given by $f([E_1]) = E_1 + ah$ and $f([E_2]) = E_2 + bh$ where $a, b = a_{\pm}$. However we must have $f([E_1]) \cdot f([E_2]) = 0$, and so we need $(E_1 + ah)(E_2 + bh) = a + b + 7ab = 0$. This does not hold for any of the four cases.

Thus there is no easy way to use our results to obtain honest-to-goodness formality of a compact Sasakian manifold S . However, theorem 4.1 is almost as good; the issue of real homotopy type is still determined by cohomological data.

Unfortunately our result does not guarantee the vanishing of Massey products. Let a, b , and c be elements of $(H_B^*(S) \otimes_d \Lambda_1(y), dy = [d\eta]_B)$, and let the degrees of a, b , and c be p, q , and r respectively. If $a \wedge b = [d\eta \wedge g]_B$ and $b \wedge c = [d\eta \wedge h]_B$, then $\langle [a], [b], [c] \rangle$ is represented by $x := y \wedge g \wedge c \pm a \wedge y \wedge h$.⁶ There is no obvious reason for this to be zero in $H^{p+q+r-1}(S)/([a] \cdot H^{q+r-1}(S) + [c] \cdot H^{p+q-1}(S))$.

4.4 A non-example

Deligne et al give the following (simplest) example of a manifold which is not formal [DGM]. Let N be the space of upper triangular 3 by 3 matrices with ones along the diagonal:

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Let Γ be the subgroup of integral matrices, and let $M = N/\Gamma$. M is a circle bundle over T^2 . The 1-stage minimal model $(\mathcal{M}^{(1)}, d)$ is given by:

$$\mathcal{M}_1^{(1)} = \Lambda_1(x, y) \quad d = 0$$

$$\mathcal{M}_2^{(1)} = \Lambda_1(x, y) \otimes_d \Lambda_1(z), \quad dz = x \wedge y.$$

The fact that M is not formal follows from the fact that $x \wedge z$ is closed but not exact. Suppose there is a map $\rho : (\mathcal{M}^{(1)}, d) \rightarrow (H^*(M), 0)$ which induces an isomorphism on H^1 and an injection on H^2 . Set $a = \rho(z)$. Then there must be a closed $w \in \mathcal{M}^{(1)}$ such that $a = \rho(w)$. However, $x \wedge w$ must be zero in cohomology, so $\rho(x \wedge w) = \rho(x \wedge z) = 0$, which is a contradiction.

We now modify the above example by considering the product $Y = M \times X$, where X is any simply connected compact manifold of real dimension $2n$ for $n \in \mathbb{N}$. It is

⁶The argument in the proof of proposition 4.1 shows that it does not matter whether we compute the Massey product in $(\Omega^*(S), d_S)$ or in a quasi-isomorphic DGA.

clear that the composition of maps

$$(\mathcal{M}^{(1)}, d) \xrightarrow{\sigma} (\Omega^*(M), d_M) \hookrightarrow (\Omega^*(Y), d_Y)$$

satisfies the criteria for a 1-stage minimal model. If Y is Sasakian, $(\mathcal{M}^{(1)}, d)$ must also be the 1-stage minimal model for $(H_B^*(Y) \otimes_d \Lambda_1(y), dy = [d\eta]_B)$. Let ρ be the resulting map between $(\mathcal{M}^{(1)}, d)$ and this algebra.

Write $\rho(z) = a + cy$, where c is constant and a is degree one. As before there exists a closed $w \in \mathcal{M}^{(1)}$ such that $\rho(w) = a$. Also as before, $x \wedge w$ must be zero in cohomology. Letting $z' = z - w$, $\rho(z') = cy$ and (since $x \wedge z'$ is nonzero in cohomology) $c \neq 0$. Let $\rho(x) = b + Cy$. x is closed, so $C = 0$. Then $\rho(x \wedge z') = cb \otimes y$. So $b \otimes y$ must be closed, i.e. $Lb = 0$. However, the basic hard Lefschetz theorem tells us that

$$L^n : H_B^1(Y) \rightarrow H_B^{2n+1}(Y)$$

is an isomorphism. In particular, L must be injective on $H_B^1(Y)$. (This is why we needed $\dim_{\mathbb{R}}(Y) \geq 5$.) We then obtain $\rho(x) = b = 0$, but this violates the fact that ρ must be an isomorphism on H^1 . We conclude:

Proposition 4.3. Y does not admit a Sasakian structure.

Note that M is orientable; $da \wedge db \wedge dc$ on N descends to a volume form on M . A result of Geiges and Altschuler-Wu (cf. [Bl] Thm. 3.7) states that the product of a compact orientable 3-manifold with any compact orientable surface carries a contact form. So in particular, $M \times S^2$ admits a contact metric structure but not a Sasakian structure.

Chapter 5

Future direction: Sasakian deformations

We would like to use the operators d_T , ∂_T , and $\bar{\partial}_T$ to define deformations of a Sasakian structure (S, η, ξ, g, Φ) . Such an approach could give us a family of Sasakian metrics parametrized by any (not necessarily basic) function on S . This in turn would allow us to ease strict hypotheses such as the condition $c_1^B > 0$ of [FOW]. In this section, we describe some progress on this problem. As we will see, the essential difficulty is ensuring that the deformed structure is K-contact.

Let (S, η, ξ, g, Φ) be a Sasakian manifold. Let us attempt to construct deformations of the Sasakian structure which are parametrized by some (real) function u on S . Let the deformed Sasakian structure be $(\tilde{\eta}, \tilde{\xi}, \tilde{g}, \tilde{\Phi})$, and write

$$\tilde{\eta} = \eta + \alpha,$$

where $\alpha = \alpha(u)$ is a one-form parametrized by u . Suppose $\tilde{\xi} = \xi$. Then the conditions that $\iota_{\tilde{\xi}}\tilde{\eta} = \iota_{\xi}\eta = 1$ and $\iota_{\tilde{\xi}}d\tilde{\eta} = 0$ give us $\iota_{\xi}\alpha = \iota_{\xi}d\alpha = 0$, i.e. α is basic. It is not clear how to construct a basic one-form from an arbitrary function u . However, if u is basic we have some obvious candidates: $d_B u$, $\partial_B u$, and $\bar{\partial}_B u$. Since we must require $d\tilde{\eta}$ to be a real basic $(1, 1)$ -form, we are essentially forced to choose $\alpha = d_B^c u$. ($\alpha = d_B u$ would not change the transverse metric.) We may take $\tilde{\Phi} = \Phi - (\xi \otimes \alpha) \circ \Phi$. We should understand $\tilde{\Phi}$ as follows. We have canonical isomorphisms between TS/\mathcal{F}_{ξ} and the contact distributions \mathcal{D} and $\tilde{\mathcal{D}}$. We obtain $\tilde{\Phi}$ by asserting an isomorphism of complex vector bundles $(\mathcal{D}, \Phi) \cong (\tilde{\mathcal{D}}, \tilde{\Phi})$. The isomorphism ensures that the deformed contact metric structure is normal.

The deformation described above with $\tilde{\eta} = \eta + d_B^c u$ is what Boyer and Galicki call a “type II deformation” of a Sasakian structure [BG]. The work of Futaki-Ono-Wang [FOW] on toric Sasaki-Einstein metrics and the work of Boyer-Galicki-Simanca [BGS]

on canonical Sasakian metrics make heavy use of these deformations.

If we want α to be parametrized by any (not necessarily basic) function u , we must allow the Reeb field to change. This destroys the canonicity of the isomorphism $\mathcal{D} \cong \tilde{\mathcal{D}}$, so it is not immediately clear how to define $\tilde{\Phi}$ to get a contact metric structure, much less a Sasakian structure. Let us take α to be transverse. (If $\iota_\xi \alpha \neq 0$, the deformation may be achieved by first scaling η , which we will discuss in further detail below, and then adding a transverse form.) In general we do not have a way to parametrize the transverse complex structure Φ . Of course, in the toric case we may use a symplectic potential. However, in order for the deformed structure to be toric we would need α to be T^{m+1} -invariant. In particular we would need $L_\xi \alpha = 0$ since ξ is in the Lie algebra of the torus. Then α would be basic, ξ would not change, and we would return to the type II deformations described above. So the toric assumption is not helpful, and the only clear way to define $\tilde{\Phi}$ is to pick some isomorphism between \mathcal{D} and $\tilde{\mathcal{D}}$ and use it to define $\tilde{\Phi}$ on $\tilde{\mathcal{D}}$. The following isomorphism will at least give a contact metric structure:

$$\begin{aligned} A: \mathcal{D} &\rightarrow \tilde{\mathcal{D}} \\ V &\mapsto W = V - \tilde{\eta}(V)\tilde{\xi}. \end{aligned}$$

Its inverse is given by $W \mapsto V = W - \frac{\eta(W)}{\eta(\tilde{\xi})}\tilde{\xi}$. For $Y \in \tilde{\mathcal{D}}$, we define $\tilde{\Phi}Y := A\Phi A^{-1}Y$ and we set $\tilde{\Phi}\tilde{\xi} = 0$.

We need to verify that $d\tilde{\eta}(\tilde{\Phi}Y, \tilde{\Phi}Z) = d\tilde{\eta}(Y, Z)$ for $Y, Z \in \tilde{\mathcal{D}}$. First, observe that for any $V, W \in \mathcal{D}$,

$$\begin{aligned} d\tilde{\eta}(AV, AW) &= d\tilde{\eta}(V - \tilde{\eta}(V)\tilde{\xi}, W - \tilde{\eta}(W)\tilde{\xi}) \\ &= d\tilde{\eta}(V, W). \end{aligned} \tag{5.1}$$

So for $Y, Z \in \tilde{\mathcal{D}}$,

$$\begin{aligned} d\tilde{\eta}(\tilde{\Phi}Y, \tilde{\Phi}Z) &= d\tilde{\eta}(\Phi A^{-1}Y, \Phi A^{-1}Z) \\ &= d\eta(A^{-1}Y, A^{-1}Z) + d\alpha(\Phi A^{-1}Y, \Phi A^{-1}Z). \end{aligned}$$

If we can show that

$$d\alpha(\Phi V, \Phi W) = d\alpha(V, W) \tag{5.2}$$

for all V and W in \mathcal{D} , we will have $d\tilde{\eta}(\tilde{\Phi}Y, \tilde{\Phi}Z) = d\tilde{\eta}(A^{-1}Y, A^{-1}Z) = d\tilde{\eta}(Y, Z)$ by again applying (5.1).

Two obvious choices for α are $d_T u$ and $d_T^c u$. We combine these by setting $\alpha = d_T u + d_T^c v$. In fact, this choice of α does satisfy (5.2). Since

$$d(d_T u + d_T^c v) = d_T^2 u + d_T d_T^c v + \eta \wedge \iota_\xi d(d_T u + d_T^c v),$$

for $V, W \in \text{Ker } \eta$ we have $d\alpha(V, W) = -(Mu)d\eta(V, W) + \sqrt{-1}(\partial_T \bar{\partial}_T - \bar{\partial}_T \partial_T)u(V, W)$.

Both $d\eta$ and $(\partial_T \bar{\partial}_T - \bar{\partial}_T \partial_T)u$ are type $(1, 1)$, so (5.2) follows.

We set $\tilde{g} = \tilde{\xi} \otimes \tilde{\xi} + d\tilde{\eta}(\cdot, \tilde{\Phi}\cdot)$. So far we have shown that $(S, \tilde{\eta}, \tilde{\xi}, \tilde{g}, \tilde{\Phi})$ is a contact metric structure. As observed in section 2.2, the Sasakian condition is

$$[\tilde{\mathcal{D}}^{1,0}, \tilde{\mathcal{D}}^{1,0}] \subset \tilde{\mathcal{D}}^{1,0} \quad \text{and} \quad [\tilde{\xi}, \tilde{\mathcal{D}}^{1,0}] \subset \tilde{\mathcal{D}}^{1,0}. \quad (5.3)$$

Let (x, z^i, \bar{z}^i) be complex foliated coordinates for the original Sasakian structure. Write $\eta = dx + \sum f_i dz^i + \sum \bar{f}_i d\bar{z}^i$, and let $Z_i = \frac{\partial}{\partial z^i} - f_i \xi$. Then $Z_i \in \mathcal{D}^{1,0}$, and so $\tilde{Z}_i := A(Z_i) \in \tilde{\mathcal{D}}^{1,0}$. We will consider (5.3) in the frame $\{\tilde{Z}_i, \tilde{\bar{Z}}_i\}$ of $\tilde{\mathcal{D}}$. Writing $g_i := \tilde{\eta}(Z_i)$, we expand $[\tilde{Z}_i, \tilde{Z}_j]$:

$$\begin{aligned} [\tilde{Z}_i, \tilde{Z}_j] &= [Z_i - g_i \xi, Z_j - g_j \xi] \\ &= [Z_i, Z_j] + (Z_j(g_i) - Z_i(g_j))\xi + g_j[\xi, Z_i] - g_i[\xi, Z_j]. \end{aligned} \quad (5.4)$$

Note that $[Z_i, Z_j] = [\frac{\partial}{\partial z^i} - f_i \xi, \frac{\partial}{\partial z^j} - f_j \xi]$ is a multiple of ξ but is also in \mathcal{D} , and so it must be zero. Moreover, $[\tilde{Z}_i, \tilde{Z}_j] \in \text{Ker } \tilde{\eta}$ since $d\tilde{\eta}$ is type $(1, 1)$ with respect to $\tilde{\Phi}$. Evaluating $\tilde{\eta}$ on both sides of (5.4), it follows that $-(Z_j(g_i) - Z_i(g_j)) = \tilde{\eta}(g_j[\xi, Z_i] - g_i[\xi, Z_j])$. Therefore the first condition in (5.3) is equivalent to $A(g_j[\xi, Z_i] - g_i[\xi, Z_j]) \in \tilde{\mathcal{D}}^{1,0}$. (This is a slight abuse of notation since the argument of A is not in \mathcal{D} .)

Noting that $\tilde{\mathcal{D}} \ni [\tilde{\xi}, \tilde{Z}_i] = [\tilde{\xi}, Z_i - g_i \xi] = [\tilde{\xi}, Z_i] - \tilde{\xi}(g_i)\xi$, we observe that $[\tilde{\xi}, \tilde{Z}_i] = A([\tilde{\xi}, Z_i])$. So in fact, both conditions in (5.3) are satisfied as long as the second holds. In other words,

Proposition 5.1. $(S, \tilde{\eta}, \tilde{\xi}, \tilde{g}, \tilde{\Phi})$ as defined above is a Sasakian deformation if and only if it is K-contact, i.e. $[\tilde{\xi}, \tilde{\mathcal{D}}^{1,0}] \subset \tilde{\mathcal{D}}^{1,0}$.

Unfortunately, K-contactness puts a messy restriction on the functions u and v which parametrize the deformation. Hopefully in future efforts we can better understand the requirements of this condition.

Boyer and Galicki have also studied another type of Sasakian deformation: one which preserves the contact distribution \mathcal{D} . (They call this a ‘‘type I deformation’’ [BG].) Let f be some (not necessarily basic) real-valued function on S . We define a deformation of the contact structure by $\tilde{\eta} = e^f \eta$. This determines a new Reeb field $\tilde{\xi}$, and we define $\tilde{\Phi}$ to be Φ on $\mathcal{D} = \tilde{\mathcal{D}}$ and $\tilde{\Phi}\tilde{\xi} = 0$. Then for $X, Y \in \tilde{\mathcal{D}}$,

$$\begin{aligned} d\tilde{\eta}(\tilde{\Phi}X, \tilde{\Phi}Y) &= d\tilde{\eta}(\Phi X, \Phi Y) \\ &= e^f d\eta(X, Y) + e^f df \wedge \eta(\Phi X, \Phi Y) \\ &= e^f d\eta(X, Y) \\ &= d\tilde{\eta}(X, Y), \end{aligned}$$

and we conclude that the deformed structure is a contact metric structure. Integrability of the CR structure comes for free since we have not changed $(\mathcal{D}, \Phi|_{\mathcal{D}})$. So

as Boyer and Galicki note, if such a deformation is K-contact, it is automatically Sasakian.

For a fixed Sasakian structure (η, ξ, g, Φ) , a *transversely holomorphic vector bundle* is one whose transition functions are in the kernel of $\bar{\partial}_T = \bar{\partial}_b$. In fact, $\mathcal{D}^{1,0}$ is such a bundle. Using the frame $\{Z_i\}$ (as defined before), under a change of coordinates $(x, z^i, \bar{z}^i) \mapsto (x', z'^i, \bar{z}'^i)$ we have

$$\begin{aligned} Z_i &\mapsto \frac{\partial z'^j}{\partial z^i} \frac{\partial}{\partial z'^j} + \text{a multiple of } \xi \\ &= \frac{\partial z'^j}{\partial z^i} Z'_j + \text{a multiple of } \xi \\ &= \frac{\partial z'^j}{\partial z^i} Z'_j. \end{aligned}$$

Here we have used (2.5) and the fact that $\eta(Z'_j) = 0$ for $Z'_j = \frac{\partial}{\partial z'^j} - \eta(\frac{\partial}{\partial z'^j})\xi$. By (2.5), we also have that $\bar{Z}_k(\frac{\partial z'^j}{\partial z^i}) = 0$ for all k , which is equivalent to $\bar{\partial}_T(\frac{\partial z'^j}{\partial z^i}) = 0$. So $\mathcal{D}^{1,0}$ is in fact transversely holomorphic, and it makes sense to define transversely holomorphic vector fields.

Returning to a type I deformation of Sasakian structure, we may write

$$\tilde{\xi} = e^{-f}\xi + Z + \bar{Z},$$

where $Z \in \mathcal{D}^{1,0}$. We will now assume that Z is transversely holomorphic. $[\tilde{\xi}, Z_i] = [e^{-f}\xi, Z_i] + [Z + \bar{Z}, Z_i]$. Since Z and Z_i are both in $\mathcal{D}^{1,0}$, so is $[Z, Z_i]$. $[e^{-f}\xi, Z_i]$ is the sum of a term of type $(1,0)$ and a multiple of ξ . However, since $\eta([\tilde{\xi}, Z_i]) = \bar{\eta}([\tilde{\xi}, Z_i]) = 0$, the ξ term cancels with the ξ term in $[\bar{Z}, Z_i] = [\bar{Z}, \frac{\partial}{\partial z^i} - f_i\xi]$. Let us write $\bar{Z} = \sum g_j \bar{Z}_j$, where $Z_i(g_j) = 0$ since \bar{Z} is antiholomorphic. Then

$$\begin{aligned} [\bar{Z}, Z_i] &= \left[\sum g_j \left(\frac{\partial}{\partial \bar{z}^j} - \bar{f}_j \xi \right), \frac{\partial}{\partial z^i} - f_i \xi \right] \\ &= \left(\sum g_j Z_i(\bar{f}_j) - \bar{Z}(f_i) \right) \xi, \end{aligned}$$

and so we conclude that the ξ term of $[e^{-f}\xi, Z_i]$ cancels *all* of $[\bar{Z}, Z_i]$. It then follows that $[\tilde{\xi}, Z_i] \in \mathcal{D}^{1,0}$, and so the deformed structure is K-contact and hence Sasakian.

We can say more about the relationship between Z and f (the scaling of the original contact structure). We claim that Z satisfies

$$\iota_Z d\eta = -\bar{\partial}_T(e^{-f}). \quad (5.5)$$

If this is true, then we have $\iota_{\tilde{\xi}} d\eta = -d_T(e^{-f}) = e^{-f} d_T f$ and so $\iota_{\tilde{\xi}} d_T f = 0$. So we

compute

$$\begin{aligned}
\iota_{\tilde{\xi}}d\tilde{\eta} &= \iota_{\tilde{\xi}}(e^f df \wedge \eta + e^f d\eta) \\
&= e^f \left((\iota_{\tilde{\xi}}df)\eta - \eta(\tilde{\xi})df + \iota_{\tilde{\xi}}d\eta \right) \\
&= e^f \left(\iota_{\tilde{\xi}}(\xi(f)\eta)\eta - e^{-f}df + e^{-f}d_Tf \right) \\
&= \xi(f)\eta - df + d_Tf \\
&= 0.
\end{aligned}$$

Since this choice of Z satisfies $\iota_{\tilde{\xi}}\tilde{\eta} = 1$ and $\iota_{\tilde{\xi}}d\tilde{\eta} = 0$ and $\tilde{\xi}$ is uniquely determined by these properties, we conclude that (5.5) must hold. We obtain

Proposition 5.2. If the Hamiltonian $(1, 0)$ -vector field for e^{-f} (as defined in (5.5)) is transversely holomorphic, the type I deformation given by scaling the contact form by e^f is a Sasakian deformation.

This condition is still quite restrictive on f . However, guided by the study of Hamiltonian holomorphic vector fields on Kähler manifolds, we hope we can better understand it in the future.

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