Design Sensitivity of Highly Damped Structural Systems

by

Miguel José Licona Núñez

Submitted to the Department of Civil and Environmental Engineering
in partial fulfillment of the requirements for the degrees of

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Abstract

In this thesis we explore the effect of damping on the dynamic behavior of structural systems subjected to earthquake loadings. Time step integration and modal analysis techniques are used to solve for the response of the system.

In the case of the single degree of freedom system we use the linear acceleration method. For multi degree of freedom systems we use the state-space formulation. Modal analysis in state-space is carried out in the multi degree of freedom case. We also look at the time history of the different kinds of energy present in the system. This enables us to take an energy approach to the study of structural dynamics, which turns out to be very powerful for establishing how the input energy is converted to stored energy.

The state-space formulation turns out to be very useful, since it enables to understand damping better by looking at the complex representation of the frequencies. It also allows for modal decomposition without restrictions on the damping matrix. Its robustness, generality and ease with which it can handle modern concepts like structural control, make it a very attractive approach.

Thesis Supervisor: Jerome J. Connor
Title: Professor of Civil and Environmental Engineering

I dedicate this thesis to the memory of my grandfather,

José Tomas Núñez Aguilar

Acknowledgments

This work would not have been possible without the constant encouragement and advice of Prof. Jerome J. Connor. He inspired me through his lectures and personal discussions. His ideas have influenced and changed my way of thinking. I am greatly in debt with him for being patient and all his support. I would also like to thank Prof. Eduardo Kausel for the explanation he gave me on modal damping.

I am specially greatful to my parents who have supported me financially through out the years and had to wait more than they would have liked to for their son to finish his thesis. Special thanks go to my brothers and the rest of my family for their continued support.

Also, I would like to thank all of my friends that have encouraged me on my thesis. I am fortunate enough to have many friends, which brings about the difficulty of naming all of those who have helped without forgetting any of them. Great thanks go to all of you, specially to the ones I have met in Boston, those who are here and those who have left also.

Last, but first in my heart and in my mind I give thanks to God for all the opportunities You have given me in life.

Miguel

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Chapter 1

Introduction

A system is an assemblage of components acting as a whole. Here we are interested in systems subjected to dynamic loads; earthquakes, wind, machine vibrations, etc. The systems we consider are linear and time invariant (LTI). We concentrate on structural systems, more specifically buildings, subjected to earthquake loads. This does not limit the generality of the concepts we present, since they are applicable to any dynamic LTI system.

1.1 Motion Control and Damage to Structures

The ground shaking during an earthquake produces considerable motion of the structures nearby. These motions produce stresses in the structural components of the system. If the stresses are high enough, the elastic capacity of the components is exceeded producing inelastic deformation, which results in damage to the structure. If the motions of a structure are excessive, they can make it collapse thus having terrible consequences like the loss of life, injuries and economic loss.

Traditional design approaches have put great emphasis on avoiding collapse during extreme loading, in order to avoid injuries and loss of life. This strategy is acceptable when the goal is to avoid human losses, but it does not take into account the possibility of having economic loss. Past earthquakes, like the 1994 Northridge and 1995 Kobe, have left a substantial amount of damage. The damage for the 1994

Northridge earthquake was estimated at around \$15 billion. The damage during the Kobe earthquake was estimated to be between \$95 to \$147 billion dollars. The Kobe earthquake also produced a drop in the Nikkei Index of 5.6% in one single day. It is evident then, that other than the loss of life and injuries caused by earthquakes, the economic impact it causes is great and can no longer be neglected.

In order to avoid damage a different design approach has to be used. This approach is designing the structure to avoid damage caused by motion. The design methodology is called motion based design, since the idea is to control motion in the system so that there is no damage to the structure. So in order to avoid damage the structural system has to be designed such that its components have no inelastic deformation. In order for this to be possible we have to provide means of dissipating energy, other than inelastic deformation of the structure. To achieve this we can think of using secondary members that will yield during the load, and then we can replace this secondary members. The idea is similar to the shock absorbers in a car, where they are used to avoid damage to the other components of the car due to roadway vibrations, etc, and when they are no longer working properly we replace them. This idea is called passive damping. Passive damping is not the only way of providing energy dissipation in a structure. Other mechanisms we can use are base isolation, tuned mass dampers, and active control. It is difficult to establish which of these is the best to use. We believe the ideal solution lies in a combination of all of these mechanisms. This is a problem we do not look at in this work, the optimal design methodology is then work for further research.

1.2 Energy Approach to Dynamic Analysis

The idea of an energy approach to dynamic analysis is introduced here and the details developed further on for the single degree of freedom (SDOF) and the multi degree of freedom (MDOF) cases.

When doing static analysis we usually solve for the displacements in a structure, then use these to find the forces in the members. In dynamic analysis we tend to do the same procedure, but the difference is that in dynamic analysis we have a solution for each instant of time. Thus what is usually done is take some algebraic combination of the maximum displacement over time at each degree of freedom, defining a displacement envelope. Although this approach gives us some information of the structural response, it will not show important characteristics of the dynamic behavior of the structure because the approach is trying to find a somewhat static type of solution for a dynamic problem.

In the energy approach we calculate the different kinds of energy that are present in the system at each instant of time. For the structural system it means that we calculate the kinetic, dissipated (damping), and strain (spring) energy at each instant of time. By looking at the strain energy at each instant of time we can then identify the instant at which the strain energy is a maximum. This instant is when the system is working at its highest demand, but it does not necessarily mean that it is the instant at which every component has its maximum in strain energy. Thus we have to be careful on how we interpret this. The difficulty arises because the analysis is dynamic and we also have a spatial variation (more than one degree of freedom). So other than looking at the energy in the system, we should also look at the energy in the components of the system. We need to look at both, we can not just look at everything from the component level because this will not show us the behavior of the system, which is what we are interested in the end.

We will see that the energy approach gives better insights into dynamic behavior. The disadvantage is that it provides us with great quantities of information, which can not only be difficult to save, but we can also get lost looking at every detail and forget that the goal is to design the structure and not to analyze it.

1.3 Linear Elastic Analysis

One of the first limitations that might be mentioned about the work done here is that through out it we use linear elastic analysis. We do not even mention anything about plastic or non-linear analysis. Here we present the reasons why we used linear elastic analysis and the applications and limitations of the work done.

Elastic analysis means that the structure does not exceed the elastic limit during its loading, and thus all the deformation is recoverable. We mentioned previously that irrecoverable deformation leads to damage in the structure. We also mentioned that we want to avoid damage in the structure, so if the goal is to have no damage then we do not need to do elasto-plastic, or any type of plastic analysis. In order for our analysis to be valid, we do have to make sure that none of the components exceed their elastic limit. It is really easier to do this, than to carry out plastic analysis, which can be very complicated, time consuming and have convergence problems. The use of elastic analysis comes naturally from the design methodology we chose, which is to avoid damage in the structural system, and not because we wanted to simplify the analysis. The fact that it actually does so is just another advantage of using this design method.

When we talk about non-linear analysis we can have to types of non-linearities. These are material non-linear behavior and geometrically non-linear behavior.

Material non-linear behavior is the one that arises because the material follows a stress-strain law that is not linear. Notice that this does not mean that there has been plastic deformation. Geometrically non-linear behavior is the one in which the usual assumption that the displacements are small enough such that we can enforce equilibrium on the undisplaced structure and force diagram is no longer valid. A typical example of a geometrically non-linear problem is the one of the oscillations of a pendulum. If we consider small oscillations about the stable equilibrium position we can use linear analysis, but in general we need to use non-linear analysis.

Analysis of non-linear problems is quite complex. First, there is really not one single rational approach to solve every non-linear problem. Also before one makes use of a numerical method to solve the non-linear problem, there is usually the need for considerable intuition on the problem behavior. This brings about difficulties with accuracy and convergence. Some times the system might not even be stable. Stability can also be difficult to asses. Because of its complexity and the solutions being particular to a given type of problem, we totally avoid non-linear behavior in this work. Fortunately for us, even though the world is non-linear, we can idealize most systems as being essentially linear, and there are only few exceptions in which non-linear analysis is needed.

This discussion should convince us that the methods and ideas presented here are applicable for most of the problems we might encounter. The decision on whether a more refined analysis technique is needed is left to the engineer.

1.4 General Outline

In the chapters that follow we expand and give the details of the energy approach to dynamic analysis. We first do this by looking at the SDOF case using two earthquake records, and then we look at systems with MDOF using the same records also. The intuition gain on dynamic behavior by this kind of analysis is of great value.

Then we make an introduction to dynamic analysis using the state space formulation. After that we compare the advantages of doing modal superposition in state space with conventional modal superposition analysis.

As we mentioned before the ideas presented here are not necessarily specific to structural systems and earthquake loads. Some of the definitions and discussions are specific for the applications studied, but the approach and concepts are general enough to be applied in other types of problems.

Chapter 2

Response History and Spectra for SDOF Systems

This chapter looks at the response of a single degree of freedom system. First we present the general theory for an arbitrary loading. The theory provides the background for formulating the energy response spectra which is applied here for the case of earthquake loading. After this derivation, we examine the response history of various systems subjected to earthquake loading. Our ultimate goal is to have a better understanding on how to design structures that can survive earthquakes without damage and thus avoid expensive repair costs.

2.1 General Problem Formulation

Figure 2-1 shows a typical single degree of freedom system (SDOF) subjected to an excitation force F_E. Using Newton's second law it can be shown that

$$F_I + F_D + F_S = F_E \tag{2.1}$$

where F_I, F_D and F_S are the inertial, damping and the spring forces respectively. Assuming viscous damping and substituting for the forces, equation 2.1 equals to

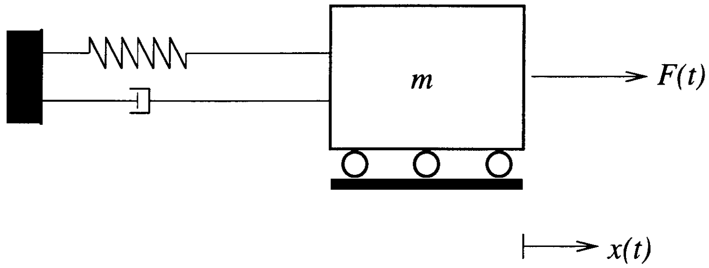

Figure 2-1: Single Degree of Freedom System.

$$m\ddot{x}(t) + c\dot{x}(t) + F_S(x(t)) = F(t) \tag{2.2}$$

where m, c, are the mass and damping coefficients, and $F_S(x(t))$ is the spring force that depends on the system response. For small displacements, one can assume F_S to be linear, and equation 2.2 becomes

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{2.3}$$

where k is the stiffness of the system. Equation 2.3 is the governing differential equation for a linear single degree of freedom system. Note that it is assumed that the system properties are time independent.

We can multiply both sides of Equation 2.3 by $dx = \dot{x}(t)dt$ and integrate from an initial time t_o to a final time t_1. Usually we would have $t_o = 0$. This leaves us with

$$m\int_{t_o}^{t_1} \ddot{x}(t)\dot{x}(t)dt + c\int_{t_o}^{t_1} \dot{x}(t)\dot{x}(t)dt + k\int_{t_o}^{t_1} x(t)\dot{x}(t)dt = \int_{t_o}^{t_1} F(t)\dot{x}(t)dt \tag{2.4}$$

which is really an equation relating the different kinds of energy present in the system. If we assume that the system starts initially from rest ($t_o = 0$) then we can define each of the energy terms present in equation 2.4

Kinetic Energy

$$E_K = m\int_{t_o}^{t_1} \ddot{x}(t)\dot{x}(t)dt = \frac{m}{2}[\dot{x}(t_1)]^2 \tag{2.5}$$

Dissipated Energy

$$E_D = c\int_{t_o}^{t_1} \dot{x}(t)\dot{x}(t)dt \tag{2.6}$$

Strain Energy

$$E_S = k\int_{t_o}^{t_1} x(t)\dot{x}(t)dt = \frac{k}{2}[x(t_1)]^2 \tag{2.7}$$

Total Energy Input

$$E_I = \int_{t_o}^{t_1} F(t)\dot{x}(t)dt \tag{2.8}$$

These definitions allow us to write the energy equation of the the system in the following form

$$E_K(t) + E_D(t) + E_S(t) = E_I(t) \tag{2.9}$$

where it is understood that each term has been integrated from time zero to a time t. Now we have an equation that relates the change of energy through time. This allows us to take our SDOF linear system, excite it with any loading we want and then look at the variation of the different kinds of energy in time.

There is another quantity that we need to define. This is the equivalent velocity V_E, defined as

$$V_E = \sqrt{\frac{2E_I(t_l)}{m}} \tag{2.10}$$

where t_l is the duration of the load and $E_I(t_l)$ represents the total energy input from time zero to t_l. This quantity represents the velocity that a mass m needs to have so that its kinetic energy equals $E_I(t_l)$.

To gain some insight on the behavior of the system, we consider first the undamped case, and thus move on to the damped case.

2.1.1 Undamped SDOF System

In the case of no damping, E_D is zero. Also note that E_K and E_S will only be zero for the case where the velocity and displacement respectively are zero. So at rest, $E_K = E_S = 0$. We suppose that the system is initially at rest and apply a load from $t_0 = 0$ to a given time t_1. The energy equation for the system is

$$E_K(t) + E_S(t) = E_I(t). \tag{2.11}$$

Notice that since initially there is no energy present all the terms are zero for $t = 0$. From $0 < t < t_1$ E_I will be changing and the energy balance is kept by the kinetic and the strain energy. When $t_1 < t$ the E_I term will remain constant (it will not be zero) and the system will go into undamped free vibration with some initial conditions that match $E_I(t_1)$. The absence of damping means that the system will oscillate forever. Since a linear spring does not dissipate energy. It can store energy when displaced but after the load is removed, the displacement and therefore the strain energy, return to zero. This basic idea is important for the damped case.

2.1.2 Damped SDOF System

For the damped case we rearrange equation 2.9 so that the kinetic and the strain energy terms are on the left side

$$E_K(t) + E_S(t) = E_I(t) - E_D(t). \tag{2.12}$$

Again we assume that at time $t_0 = 0$ the system starts from rest. All the energy terms are zero initially. Then we apply a load from $t_0 = 0$ to a given time t_1, and let the system vibrate until it comes to rest again. Let us call t_2 the time at which the system comes to rest.

From $0 < t < t_1$ E_I and E_D are changing. E_D will lag E_I by the sum of the kinetic and the strain energy. When $t_1 < t < t_2$ E_I would have reached a constant value and the system will be in a damped free vibration mode. During this period the energy in the system will be dissipated by the damping. When we reach rest again ($t_2 < t$) the kinetic and the strain energy are zero. Equation 2.12 becomes

$$E_I(t) - E_D(t) = 0 \tag{2.13}$$

The total energy input to the system has been dissipated by the damping mechanism.

This is an important result. However, it is valid only when the spring remains elastic. Therefore, there must be sufficient energy dissipation so that the spring actually remains linear at all time. This does not mean that the spring is a useless component of the system. The spring serves as a temporary storage device that allows for a certain lag between E_I and E_D. If one provides a spring that is able to remain linear (holding the stiffness constant) for larger displacements, the damping needed to dissipate the energy in the system can be reduced.

2.2 Earthquake Response Spectra

An earthquake is a seismic event that generates waves that travel through the earth's crust. These waves generate motion of the ground that will move the support of structures. This support motion will generate forces throughout the structure that can cause damage.

Characterizing an earthquake is a very difficult task. It is clear that an earthquake can not be characterized by a single parameter such as peak ground acceleration (PGA). The reason is that the damage caused by an earthquake not only depends on parameters like PGA but also on the frequency content of the earthquake and on the period of the structure being analyzed. This is the motivation for developing the response spectra.

An earthquake response spectra is generated by taking many single degree of freedom systems, characterized by their period and damping level, and analyzing their response to an earthquake excitation. The maximum response of the system is

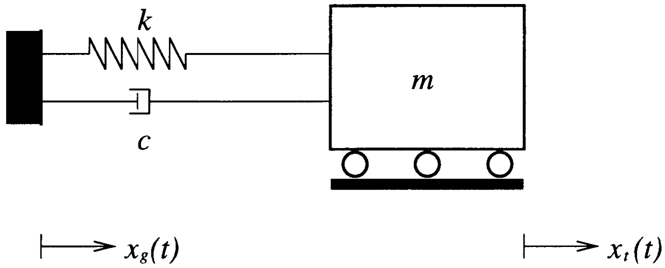

Figure 2-2: Single Degree of Freedom System Subjected to Support Motion.

then plotted against the period of the SDOF system.

Figure 2-2 shows a single degree of freedom system subjected to support motion. Using equation 2.3 and $F(t) = -ma_g(t)$ we can write the governing differential equation of the SDOF system which is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -ma_g(t) \tag{2.14}$$

where $a_g(t)$ is the acceleration of the ground at the site at any given time. Introducing $\omega = \sqrt{k/m}$ and $\xi = c/2m\omega$ we can write equation 2.14 as

$$\ddot{x}(t) + 2\xi\omega\dot{x}(t) + \omega^2 x(t) = -a_g(t) \tag{2.15}$$

which we use to calculate an earthquake response spectra. We usually specify ω, ξ and let the system start from rest. The loading is the recorded acceleration of an earthquake. Figures 2-3 and 2-4 show typical earthquake accelerograms. The peak ground acceleration for these two earthquakes is nearly the same, but we will see that their frequency content is different.

Figures 2-5 and 2-6 show the frequency content for each of these earthquakes. Although they both contain low frequencies, the way in which they will affect structural response is different. We will look at this more closely in the next section.

We noted earlier that a response spectra is a plot of the maximum response against the period of the SDOF system. In the past engineers have concentrated on three quantities, spectral displacement, spectral velocity and spectral acceleration. Spec-

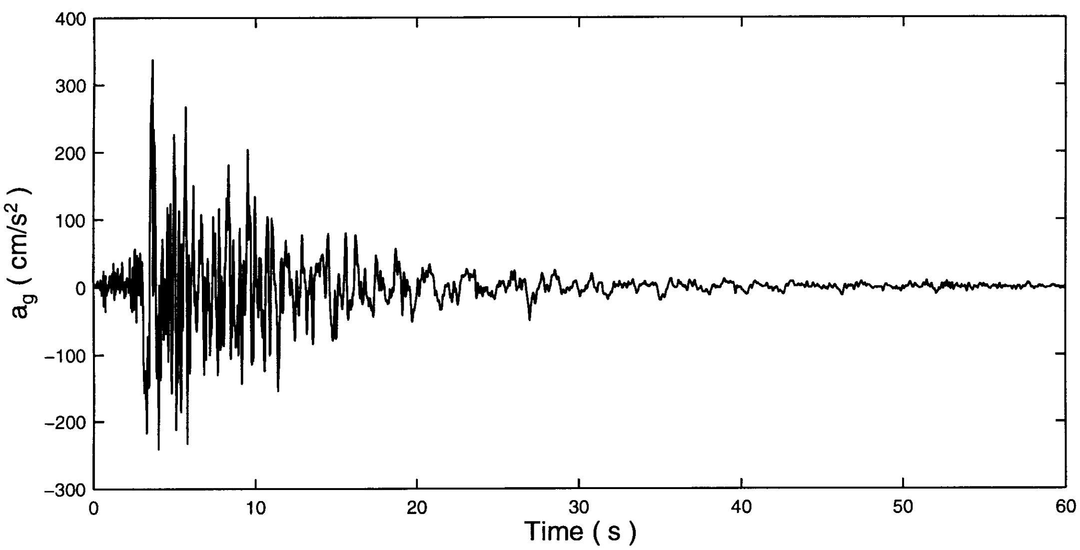

Figure 2-3: Acceleration Record, Arleta Station (90 DEG), Northridge 1994

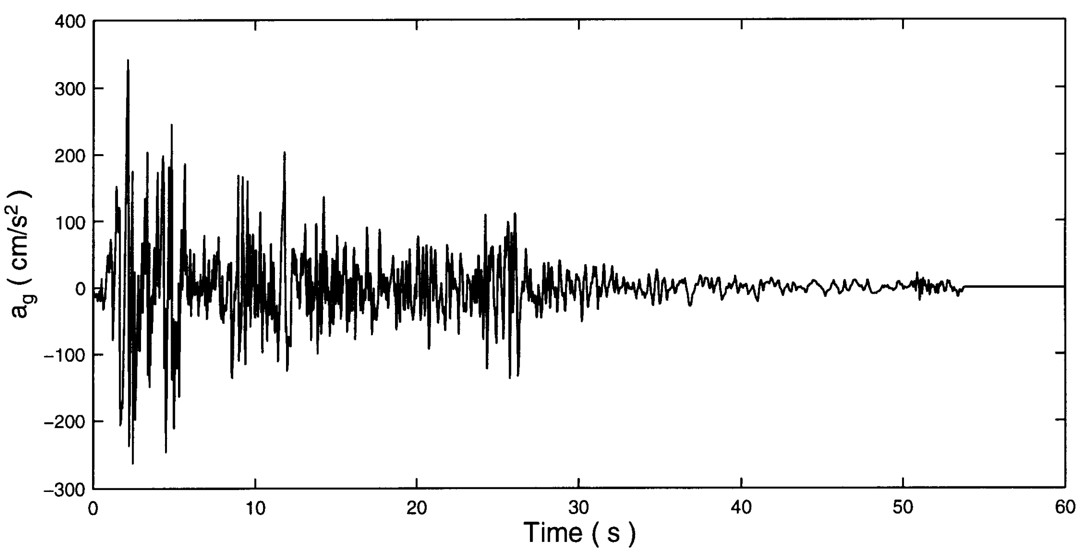

Figure 2-4: Acceleration Record, El Centro (S00E), Imperial Valley 1940

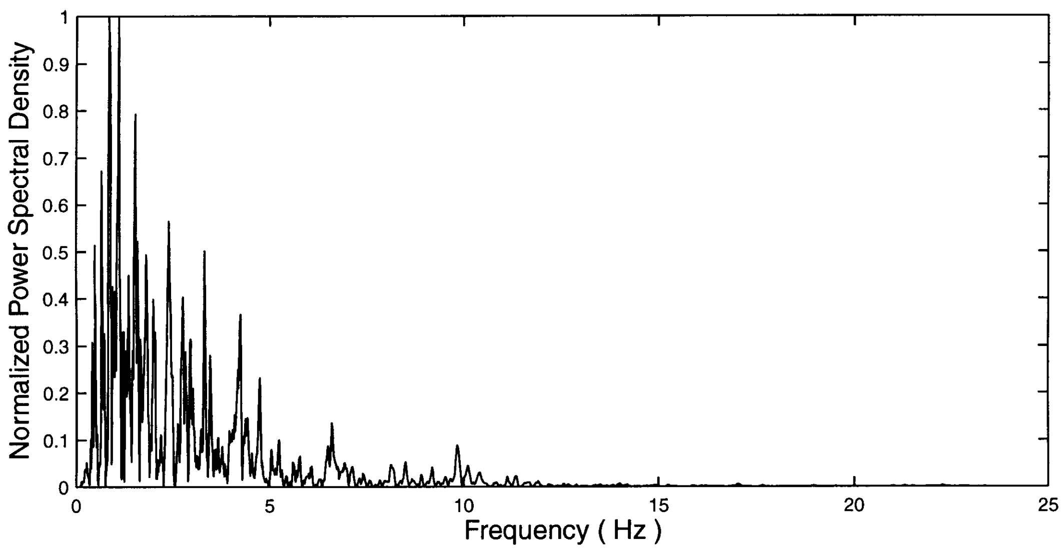

Figure 2-5: Frequency Content, Arleta Station (90 DEG), Northridge 1994

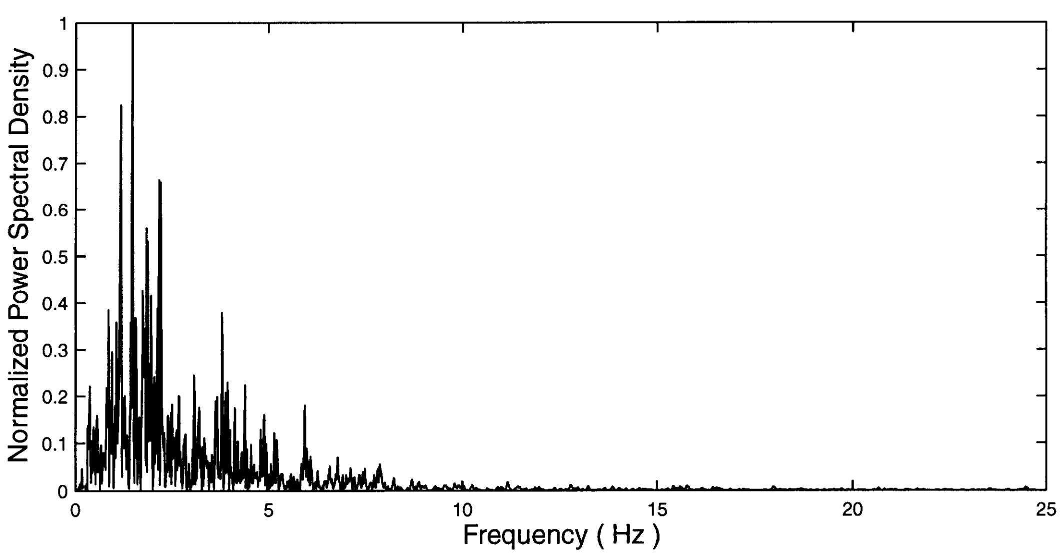

Figure 2-6: Frequency Content, El Centro (S00E), Imperial Valley 1940

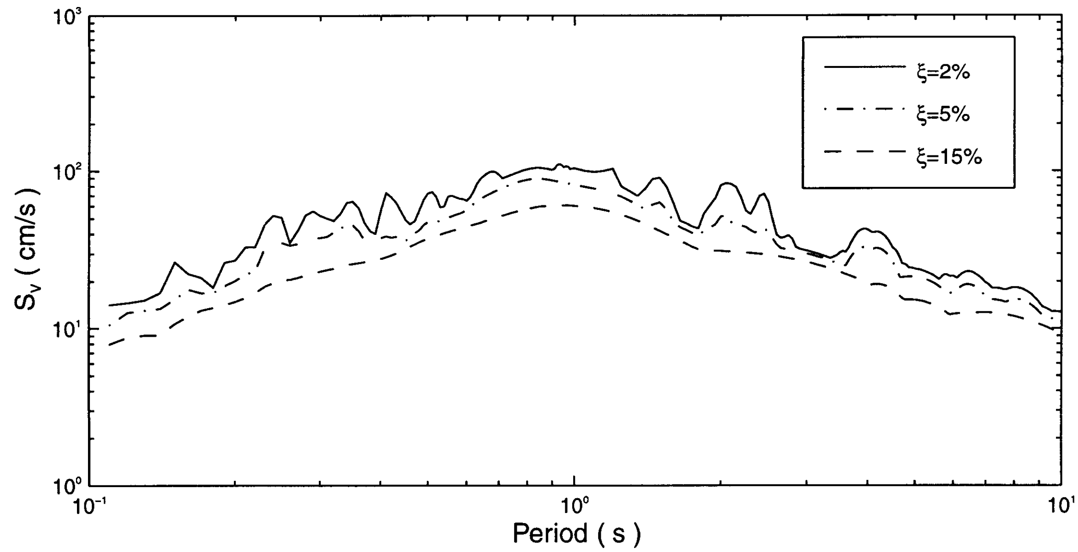

Figure 2-7: Spectral Velocity, Arleta Station (90 DEG), Northridge 1994

tral displacement (S_d) is the maximum displacement during the time interval being analyzed. Spectral Velocity is defined as $S_v = \omega S_d$, spectral acceleration is $S_a = \omega^2 S_d$.

We can also obtain the response spectra for any other quantity we want. For example we could decide to plot the spectra for the equivalent velocity (V_E). This is what we would call an energy response spectra since our calculations are based on the energy input to the system.

Figure 2-7 shows a response spectra for the spectral velocity. We notice that the maximum spectral velocity depends on the damping level and on the period of the system. Now, let us look at figure 2-8 that shows the equivalent velocity spectra. We observe a very interesting result. The equivalent velocity (and therefore the total energy input) is essentially independent of the damping level (not for 2%).

These results could help us characterize earthquakes by looking at the energy spectra. The reasoning behind this is that we really want to associate damage with energy and not a single parameter that does not take frequency content into account like PGA. Thus if we plotted the V_E spectra of many different earthquakes that can occur in a region, we could find an upper bound envelope for V_E. This would give a better idea of the amount of energy that we need to dissipate.

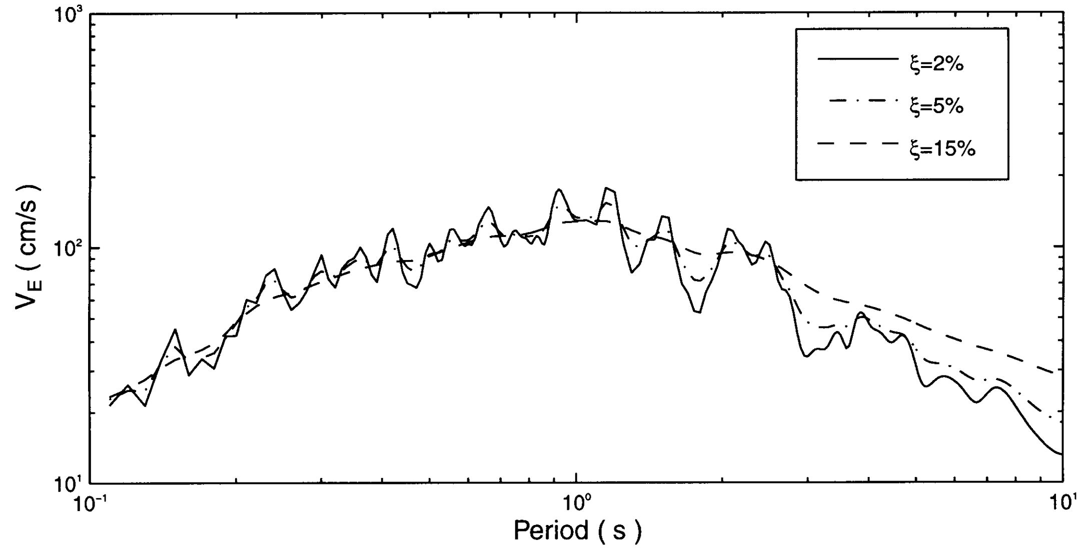

Figure 2-8: Equivalent Velocity Spectra, Arleta Station (90 DEG), Northridge 1994

2.3 Earthquake Response History

When considering a response spectra there is an important variable that we do not take into account, time. It is important to observe how the energy builds up in time, and also how fast we can dissipate it. When the energy builds up in a very short period of time, it is possible to exceed the capacity of our storage device (the spring) and this will produce inelastic deformation in the spring. Ideally we want to avoid this, since once we have inelastic deformation we will have damage. Thus we need to choose the spring and the damping device to obtain a system that can effectively dissipate the energy that is coming in without having inelastic deformation. By looking at the response history and the energy build up in time we can gain some understanding about the system behavior.

To obtain the response history of the system we solve equation 2.14 and we keep track of the system response at all time. We have to make sure that the system goes into a free vibration mode so that it can come to rest again. Here we can keep track of any response variables that are of interest to us. Relative displacement and velocity can serve as checks that the analysis was done properly, since the system has to reach rest again. Our interest is primarily on energy and we are going to focus on that.

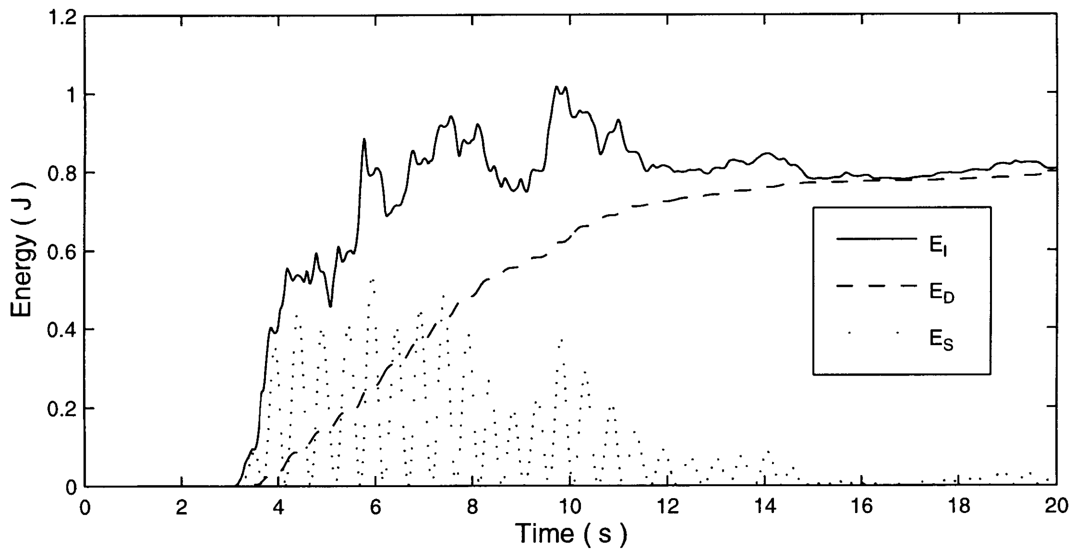

Figure 2-9: Energy Build Up, Arleta Station (90 DEG), Northridge 1994, $\xi = 2\%$

Another check we can use in this case is that the strain and kinetic energy have to return to zero.

In our analysis we took a system with the following properties: $m = 1Kg$, $T = 1$s, $\xi = 2\%$ and subjected it to two different earthquake loadings. We used the El Centro (S00E Component) and the Arleta, Northridge 1994 (90 DEG Component). They have a PGA of $341cm/s^2$ and $337cm/s^2$ respectively. If we just took PGA as a characterization of an earthquake we would expect the response to be similar. This is not true since by doing this we are neglecting frequency content.

We then varied the damping ratio to see how damping affects behavior. The other levels of damping used are $\xi = 5\%$ and $\xi = 10\%$.

Figure 2-9 shows the energy build up to the Arleta excitation, for $\xi = 2\%$. We notice that E_I grows rather quickly. In a matter of about three seconds the system has received about eighty percent of the energy input. Also, notice that E_D builds up slowly, thus we are in trouble since the rate at which we dissipate energy is much slower than the rate at which this energy comes in. This happens only during a short period of time, but this can be enough to produce damage in the system. It is not a surprise then to see that during this rapid build up period, the energy demand on the spring is large. Thus in the time between three and six seconds the response is

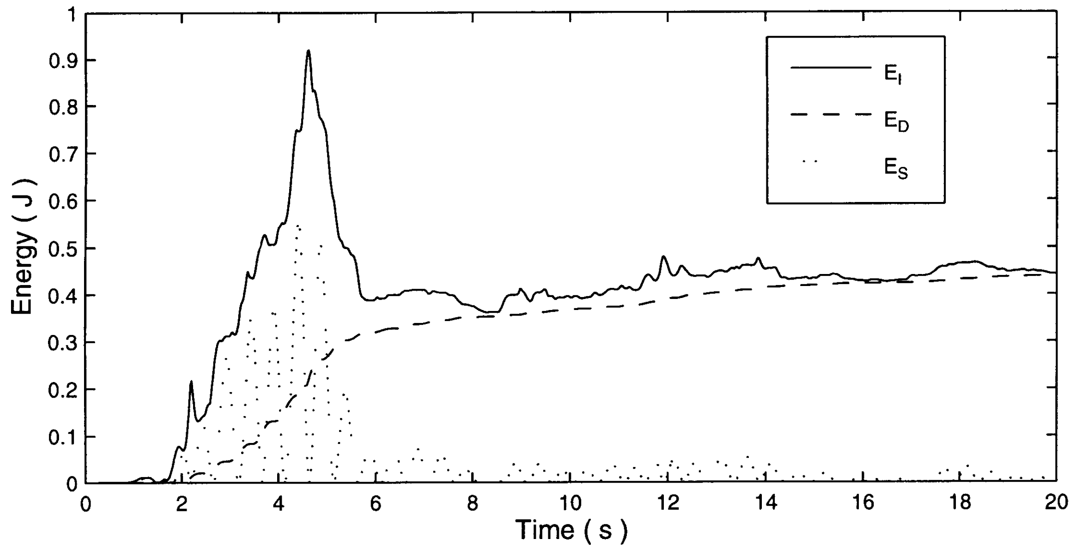

Figure 2-10: Energy Build Up, El Centro (S00E), Imperial Valley 1940, $\xi = 2\%$

dominated by stiffness. By looking at the strain energy demand on the spring, we can design the spring so that it will remain elastic. This procedure depends on being able to guarantee that we will have the damping level specified. The other way of looking at this design problem is to look at other damping levels so that the strain energy demand on the spring is smaller.

Now let us look at Figure 2-10. Here we see the energy build up for the same system, but the loading is the El Centro record. Notice that the energy demand is different from the one for the Arleta record (even though both earthquakes had almost the same PGA). In both cases we see a similar general pattern, i.e, E_D builds rather slowly while E_I grows very quickly. Thus we should think of an earthquake as an impulsive loading and should remember that damping needs some cycles before it starts becoming effective.

It is also interesting to note that the maximum value of E_I occurs somewhere before t_l. Remember that in equation 2.10 we defined V_E as a function of $E_I(t_l)$. Now, if we look at the response history we can see a significant difference between E_{Imax} and $E_I(t_l)$. In the case of our system, the difference is greater (about 50%) for the El Centro excitation. This might lead us to find other ways of getting a response spectra. Thus we could define other quantities that we think might help us

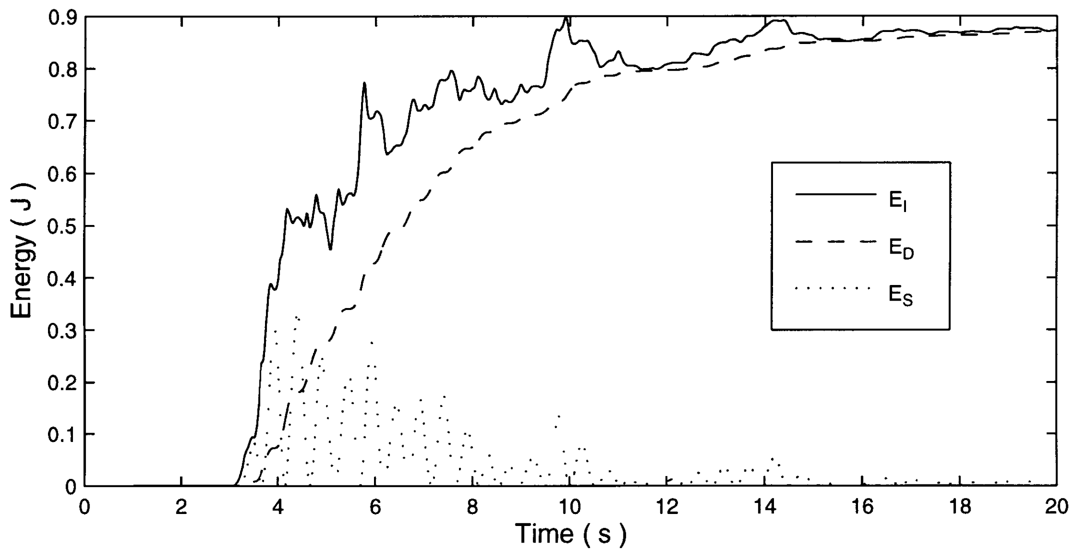

Figure 2-11: Energy Build Up, Arleta Station (90 DEG), Northridge 1994, $\xi = 5\%$

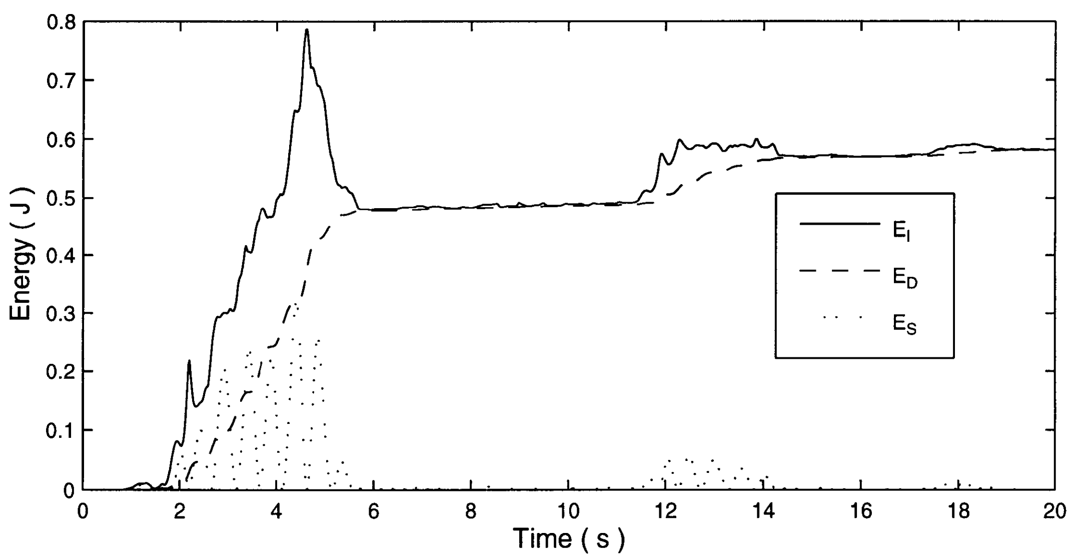

Figure 2-12: Energy Build Up, El Centro (S00E), Imperial Valley 1940, $\xi = 5\%$

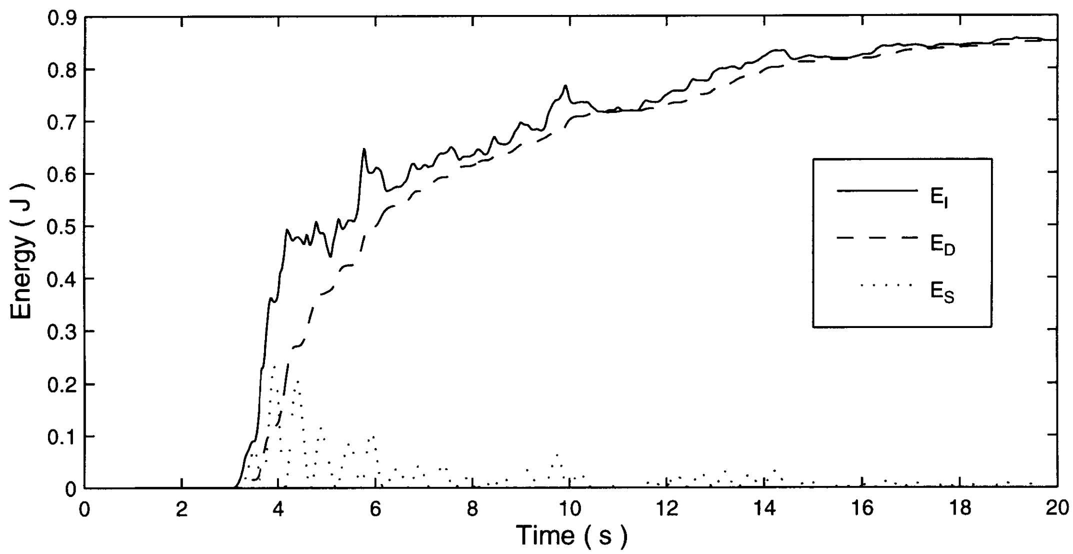

Figure 2-13: Energy Build Up, Arleta Station (90 DEG), Northridge 1994, $\xi = 10\%$

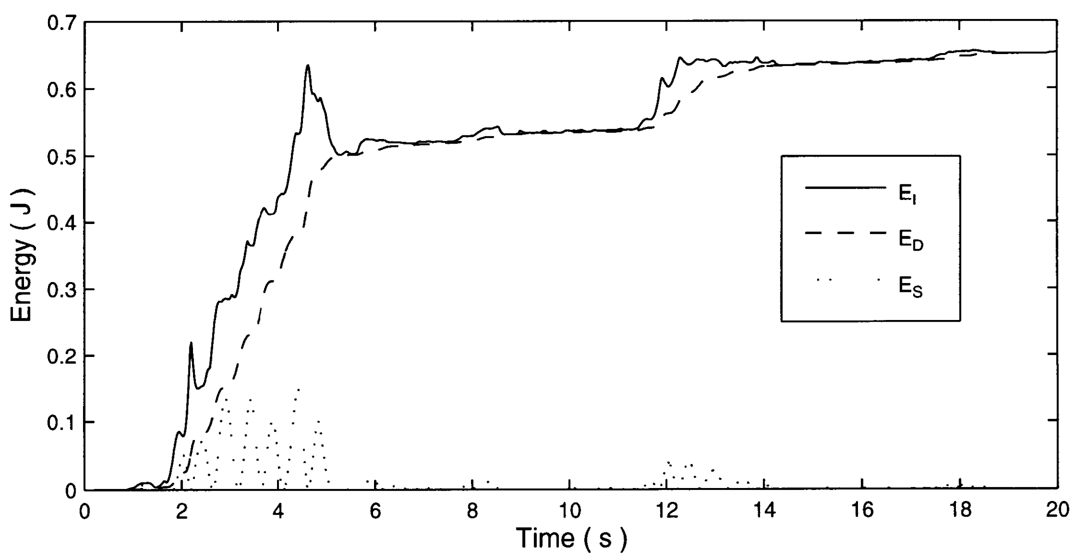

Figure 2-14: Energy Build Up, El Centro (S00E), Imperial Valley 1940, $\xi = 10\%$

in understanding the behavior of the system.

Now let us see what happens as we increase the damping in the system. By looking at figures 2-11 thru 2-14 we can see the effect of damping on the response. We see that greater levels of damping reduce the strain energy demand. Also as we increase the damping, E_D builds up faster and follows E_I more closely. This is good for controlling motion and reduces significantly the amount of stiffness needed. We also notice that damping affects the way E_I builds up. Specifically it reduces the difference between E_{Imax} and $E_I(t_l)$. Thus, increasing the damping level is a good strategy for controlling the structural response. However, there are some issues that need to be addressed. The first is that we need to be able to analyze the cost of increasing the damping to see if it is economically feasible. The second, and more important one is: can we guarantee that the structure will have this level of damping? Since the response is quite sensitive at low damping levels, we need to be able to estimate it quite well, otherwise the actual behavior will differ from the one we modeled and this could be dangerous. Also we should remember that a 10% damping level is a high level of damping for a structural system and that it might be hard to obtain it.

2.4 Conclusions

We have investigated the behavior of a SDOF system quite thoroughly. After looking at the response spectra and the response history, we have observed that the response spectra helps us in the design of the system. Thus, the spectra is a valuable design aid, but we should not forget to look at the response history of the system, since this gives us valuable information on to how the response evolves.

The quantities that we looked at and that are important in the response are E_I, E_D and E_S. Strain energy is very important since it will allow us to temporarily store some of the energy that will then be dissipated through damping. Thus by looking at these three quantities we can design a system that will remain elastic and therefore have little or no damage. One quantity we did not look at is the total acceleration ($\ddot{x}_t(t)$). The total acceleration is important because this is what an object that is inside of our structure will feel. If the maximum acceleration is too high we might have objects inside moving too much and could have additional damage to the equipment inside or to the structure.

Further research should try to look at how we could control total acceleration so that it is below a given value and at the same time still have the system in the elastic range during the earthquake.

Chapter 3

Response History for MDOF Systems

This chapter expands upon the discussion of the SDOF system presented in chapter 2. We start with a general formulation for arbitrary loading, and extend the energy concepts to MDOF systems. Then we look more closely at the case in which the system is excited by an earthquake. This will help us gain more insight on the behavior of MDOF systems to dynamic loads. Damage is still one of the most important variables we want to minimize. We look at it by trying to understand the strain energy demand.

3.1 General Theory on MDOF Systems

The governing differential equations for a MDOF discrete-linear-system with time independent properties are:

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{F}(t) \tag{3.1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, damping and stiffness matrices of the system, and $\mathbf{F}$ is the load vector. The system matrices ($\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$) are symmetric positive definite matrices. This condition is essential for stability of the system.

In the discussion that follows we consider the system to have N degrees of freedom. This means that $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are square matrices of dimension $N \times N$ and $\ddot{\mathbf{X}}(t)$, $\dot{\mathbf{X}}(t)$, $\mathbf{X}$, and $\mathbf{F}$ are vectors of size $N \times 1$.

Now, let us take equation 3.1 and premultiply it by $\dot{\mathbf{X}}^T(t)dt$,

$$\dot{\mathbf{X}}^T(t)dt\mathbf{M}\ddot{\mathbf{X}}(t) + \dot{\mathbf{X}}^T(t)dt\mathbf{C}\dot{\mathbf{X}}(t) + \dot{\mathbf{X}}^T(t)dt\mathbf{K}\mathbf{X}(t) = \dot{\mathbf{X}}^T(t)dt\mathbf{F}(t) \tag{3.2}$$

Interchanging dt leads to

$$\dot{\mathbf{X}}^T(t)\mathbf{M}\ddot{\mathbf{X}}(t)dt + \dot{\mathbf{X}}^T(t)\mathbf{C}\dot{\mathbf{X}}(t)dt + \dot{\mathbf{X}}^T(t)\mathbf{K}\mathbf{X}(t)dt = \dot{\mathbf{X}}^T(t)\mathbf{F}(t)dt \tag{3.3}$$

Integrating from time zero to some arbitrary time t, we obtain

$$\int_0^t \dot{\mathbf{X}}^T(t)\mathbf{M}\ddot{\mathbf{X}}(t)dt + \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{C}\dot{\mathbf{X}}(t)dt + \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{K}\mathbf{X}(t)dt = \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{F}(t)dt \tag{3.4}$$

which is the energy equation for the MDOF system. Assuming that at time zero the system starts from rest, the scalar energy functions for the MDOF system have the following form:

Kinetic Energy

$$E_K(t) = \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{M}\ddot{\mathbf{X}}(t)dt = \frac{1}{2}\dot{\mathbf{X}}^T(t)\mathbf{M}\dot{\mathbf{X}}(t) \tag{3.5}$$

Dissipated Energy

$$E_D(t) = \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{C}\dot{\mathbf{X}}(t)dt \tag{3.6}$$

Strain Energy

$$E_S(t) = \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{K}\mathbf{X}(t)dt = \frac{1}{2}\mathbf{X}^T(t)\mathbf{K}\mathbf{X}(t) \tag{3.7}$$

Total Energy Input

$$E_I(t) = \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{F}(t)dt \tag{3.8}$$

Using the above definitions the energy balance equation is

$$E_K(t) + E_D(t) + E_S(t) = E_I(t). \tag{3.9}$$

Notice that equation 3.9 is identical to equation 2.9. The reason is that energy is a scalar, so it does not matter whether the system is a SDOF or a MDOF system. It is also important to observe that nothing has been said about the properties of the damping matrix. Thus, the formulation applies for an arbitrary $\mathbf{C}$ matrix. The only restrictions are that the system is linear and its properties are time independent.

In chapter 2 we looked at some very fundamental ideas about the energy balance for a SDOF system. These ideas are basically the same for the MDOF case, but we will go over them again to ensure clarity of the concepts.

3.1.1 Undamped MDOF System

In the case of no damping E_D is zero. Also note that E_K and E_S will only be zero for the case where the velocity and displacement vectors respectively are zero. So at rest, $E_K = E_S = 0$. We suppose that the system is initially at rest and apply a load from time zero to a given time t_1. The energy equation for the system is

$$E_K(t) + E_S(t) = E_I(t). \tag{3.10}$$

Notice that since initially there is no energy present all the terms are zero for $t = 0$. From $0 < t < t_1$ E_I will be changing and the energy balance is kept by the kinetic and the strain energy. When $t_1 < t$ the E_I term will remain constant (it will not be zero) and the system will go into undamped free vibration with some initial conditions that match $E_I(t_1)$. The absence of damping means that the system will oscillate forever since linear springs do not dissipate energy. They store energy when displaced but after the load is removed, the displacement vector and therefore the strain energy, return to zero. This basic idea is important for the damped case.

3.1.2 Damped MDOF System

For the damped case we rearrange equation 3.9 so that the kinetic and the strain energy terms are on the left side

$$E_K(t) + E_S(t) = E_I(t) - E_D(t). \tag{3.11}$$

Again we assume that at time zero the system starts from rest. All the energy terms are zero initially. We apply a load from $t = 0$ to a given time t_1 and then let the system vibrate until it comes to rest again. Let us call t_2 the time at which the system comes to rest.

From $0 < t < t_1$, E_I and E_D are changing. E_D will lag E_I by the sum of the kinetic and the strain energy. When $t_1 < t < t_2$, E_I would have reached a constant value and the system will be in a damped free vibration mode. During this period the energy in the system will be dissipated by the damping mechanisms. When we reach rest again ($t_2 < t$) the kinetic and the strain energy are zero. Equation 3.11 becomes

$$E_I(t) - E_D(t) = 0 \tag{3.12}$$

The total energy input to the system has been dissipated by the damping mechanism.

This is an important result. However, it is valid only when the springs remain elastic. Therefore, there must be sufficient energy dissipated so that the springs remain linear at all time. The springs serve as temporary storage devices that allow for a certain lag between E_I and E_D. If one provides springs that are able to remain linear for larger displacements, the damping needed to dissipate the energy in the system can be reduced. This means that we have to provide enough stiffness to temporarily store energy that we will later eliminate through damping.

The difficulty of achieving this is greater with a MDOF than with a SDOF system. The reason is that in a MDOF we have more components that can have their elastic capacity being exceeded. If one of the components goes inelastic, it will result in damage that will change the properties of the system. This not only results on a complication in the analysis, because the properties (and therefore the periods) of the system are being modified, but it also results in damage to the structure, which we want to avoid.

3.2 Earthquake Loading

The derivation of the equations of motion for the case of earthquake loading is done by considering the effects of support motion in a MDOF system. Let us think of a MDOF discrete system and subject the support to some arbitrary displacement. This leads to

$$\mathbf{M}\ddot{\mathbf{X}}_{\mathbf{t}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{0} \tag{3.13}$$

which is analogous to the classical derivation for the SDOF system. In this equation $\ddot{\mathbf{X}}_{\mathbf{t}}(t)$ represents the total acceleration vector of the system. In order to determine the earthquake force, we need to express the total displacement as the sum of relative and support motions. For a MDOF system, we write the total displacement as

$$\mathbf{X}_{\mathbf{t}}(t) = \mathbf{X}(t) + \mathbf{E}x_g(t) \tag{3.14}$$

where $\mathbf{E}$, usually called a locator vector, which for the case of a shear beam is a column of ones. This vector multiplied by the displacement of the support expresses the rigid body motion undertaken by each degree of freedom when the support is displaced. In order to make this clearer, let us think that we take the system and displace the support very slowly by a unit displacement. This will lead to a total displacement at each degree of freedom to be one unit. If we repeat our experiment but this time we move the support rapidly, the total displacement at each degree of freedom will be the sum of the unit displacement (rigid body motion) due to the support motion, plus the motions due to vibration of the system because the support and therefore the system was moved rapidly. The case of the displacements induced by ground motions is analogous to this. The only complication is that during an earthquake the ground (and therefore the support of the structure) is being displaced throughout the duration of the earthquake. This is taken care in the formulation by having the time history of the displacement of the ground $x_g(t)$.

Differentiating equation 3.14 twice with respect to time,

$$\ddot{\mathbf{X}}_{\mathbf{t}}(t) = \ddot{\mathbf{X}}(t) + \mathbf{E}\ddot{x}_g(t) \tag{3.15}$$

and substituting in equation 3.13 leads to the governing differential equation for seismic excitation

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = -\mathbf{M}\mathbf{E}\ddot{x}_g(t) \tag{3.16}$$

Equation 3.16 is similar to equation 3.1 except that now the $\mathbf{F}(t)$ term is substituted by the effective load $-\mathbf{M}\mathbf{E}\ddot{x}_g(t)$.

Like before, we premultiply equation 3.16 by $\dot{\mathbf{X}}^T(t)dt$, rearrange terms and integrate from time zero to some arbitrary time t,

$$\int_0^t \dot{\mathbf{X}}^T(t)\mathbf{M}\ddot{\mathbf{X}}(t)dt + \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{C}\dot{\mathbf{X}}(t)dt + \int_0^t \dot{\mathbf{X}}^T(t)\mathbf{K}\mathbf{X}(t)dt = -\int_0^t \dot{\mathbf{X}}^T(t)\mathbf{M}\mathbf{E}\ddot{x}_g(t)dt \tag{3.17}$$

Equation 3.17 represents the energy formulation for the case of seismic excitation. The terms on the left hand side are the kinetic, dissipation and strain energy terms defined previously in equations 3.5, 3.6, and 3.7. The term on the right hand side represents the total energy input for seismic excitation, and is denoted as $E_I(t)$.

$$E_I(t) = -\int_0^t \dot{\mathbf{X}}^T(t)\mathbf{M}\mathbf{E}\ddot{x}_g(t)dt \tag{3.18}$$

Note that $E_I(t)$ represents the amount of energy that needs to be dissipated by damping to avoid damage to the structure.

3.3 Examples of MDOF System Response to Earthquake Loading

Here we look at the response of three MDOF systems to an earthquake load. For comparison purposes the three systems have the same stiffness and mass distribution. Different damping configurations are provided to evaluate how it affects the build up of strain energy in the structure. Table 3.1 contains the properties of these systems.

The system consists of a four degree of freedom discrete system. The stiffness matrix is tridiagonal and the properties for each spring have been set to be distributed in a parabolic way, with the highest stiffness near the support ($k_1 = 1260N/m, k_2 = 1100N/m, k_3 = 780N/m, k_4 = 300N/m$). The mass is constant except at the top, where it is a half of the value of the others($m_1 = m_2 = m_3 = 2Kg, m_4 = 1Kg$). Damping is provided in two different ways. The first way uses modal damping, specifying the same damping ratio for each mode. The second approach places individual dampers between the masses, which leads to a damping matrix that has the same structure as the stiffness matrix. The second damping model provides an accurate and convenient way of modeling the physical properties of the system. The arbitrary damping coefficients for Building 3 were chosen to provide approximately the same amount of damping that Building 2 has. This result can be seen by looking at the comparison between the frequencies for these systems[1]. Although the damped frequencies are close to each other, we realize that there is a greater difference between the frequencies for the fundamental mode.

These three systems were subjected to the two earthquake loadings used for the SDOF case. Analysis was performed in the time domain choosing a sufficiently small time step to ensure that the energy balance at each time step was within the given tolerance.

Let us now look at the results for each of these systems. Figure 3-1 shows the response of Building 1 to the El Centro earthquake. We see from the figure that for a small interval of time, the strain energy in the system dominates the response, then

[1]We elaborate more on this in chapter 4.

Building Name	Mass and Stiffness Properties	Damping Properties
Building 1	custom mass, custom stiffness	5% constant modal
Building 2	custom mass, custom stiffness	2% constant modal
Building 3	custom mass, custom stiffness	arbitrary damping in $N.s/m$ ($c_1 = 2, c_2 = 1.5, c_3 = 0.9, c_4 = 0.25$)

Table 3.1: MDOF System Properties

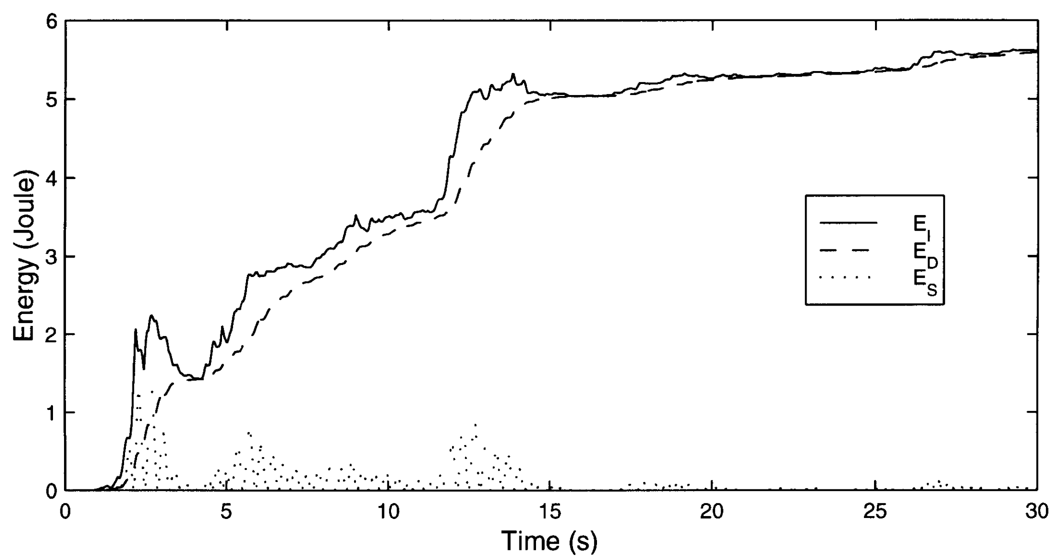

Figure 3-1: Building 1 Energy History, El Centro (S00E), Imperial Valley 1940.[2]

damping takes over and it follows the energy input somewhat closely. We also see two smaller peaks in the strain energy at about 6 and 13 seconds.

Figure 3-2 illustrates what happens when the damping is decreased. We clearly see a greater demand on the strain energy at the early stages of the load, even though the total amount of energy input remains almost unchanged. This is why it is so important to look at the time history of the response. Because looking at the total energy input could lead us to think that both systems have very small differences between each other. But, we see a significant difference in the strain energy demand. The pattern remains similar between the systems. We tend to see the same peaks

[2]See table 3.1

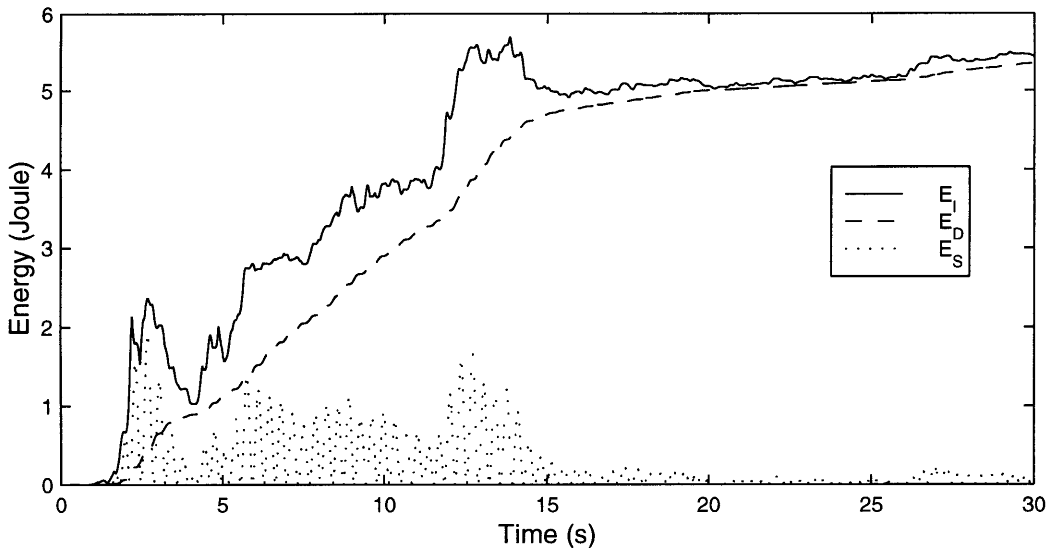

Figure 3-2: Building 2 Energy History, El Centro (S00E), Imperial Valley 1940.[3]

and jumps in the energy in both cases, although we notice an amplification in the strain energy (reduction in the energy dissipated).

The greatest difference and rather surprising result comes when we look at the plot of the response history for Building 3. Figure 3-3 illustrates this. Since we had a damping level comparable to Building 2, we would tend to think that the response would be similar. Instead, we see around a 25% increase in the strain energy demand. The most interesting thing is that this level of demand on the strain energy remains essentially constant for over ten seconds. Thus the springs are working very hard in this case, and it takes many cycles for the damping in the structure to start working. We can see this by looking at the slope of the energy dissipated curve. We notice that it is flatter than in the previous cases. This inability of damping to respond quickly is what puts a greater demand on the elastic components of the system.

Figures 3-4, 3-5 and 3-6 illustrate the energy demand on Building 1, Building 2 and Building 3 when the excitation is the Arleta record from the 1994 Northridge earthquake. The pattern is similar to the trend observed for the El Centro excitation. The strain energy demand is highest for Building 3, and it represents a very high

[3]See table 3.1

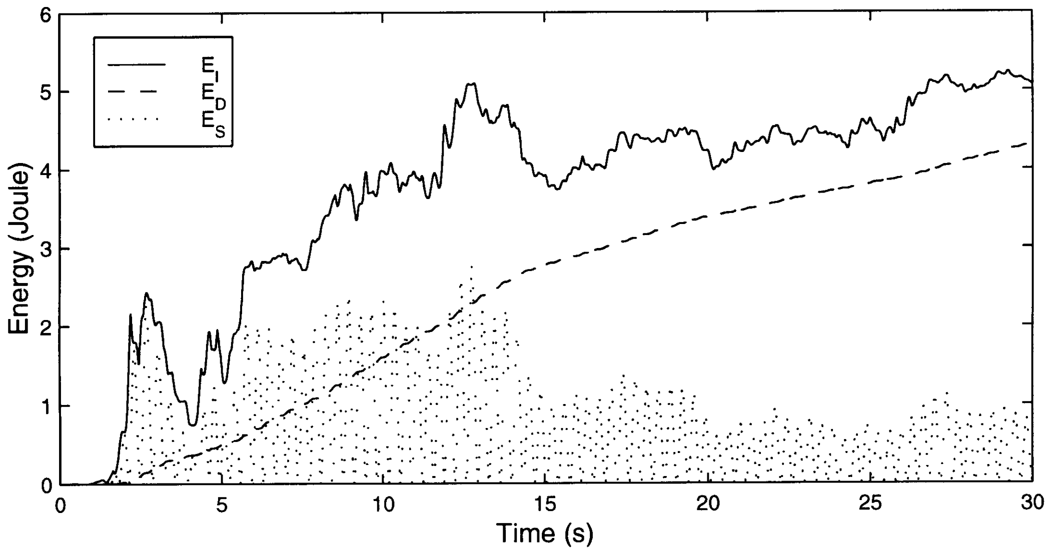

Figure 3-3: Building 3 Energy History, El Centro (S00E), Imperial Valley 1940.[4]

increase, when compared with the response of Building 2. In the case of Building 1, damping works fast and this makes the E_S demand drop in a matter of 3 to 4 seconds. For the less damped systems, the time required to see a significant drop in E_S is longer. In all cases, we see the peak for E_I at around the same instant, but the E_S demand looks different. The difference between the strain energy history for the different buildings is mostly in the magnitude. We see a high amplification, around 100%, between Buildings 3 and 1.

Let us now look at the influence of the loading in the energy build up of the systems. Comparing the response of the same building to the two different excitations, we realize that the record for the Northridge earthquake is much more impulsive than for El Centro. The responses are very different, even though the total energy demand is not much different. In fact it is interesting to note that the El Centro excitation has a higher total E_I, but the maximum E_S is much higher, about a 75% increase, for the Northridge record. This is where the nature of the loading, and the physical properties of the system play an important role. In the Northridge case, the energy is received in a smaller amount of time, even though it is a smaller amount of total

[4]See table 3.1

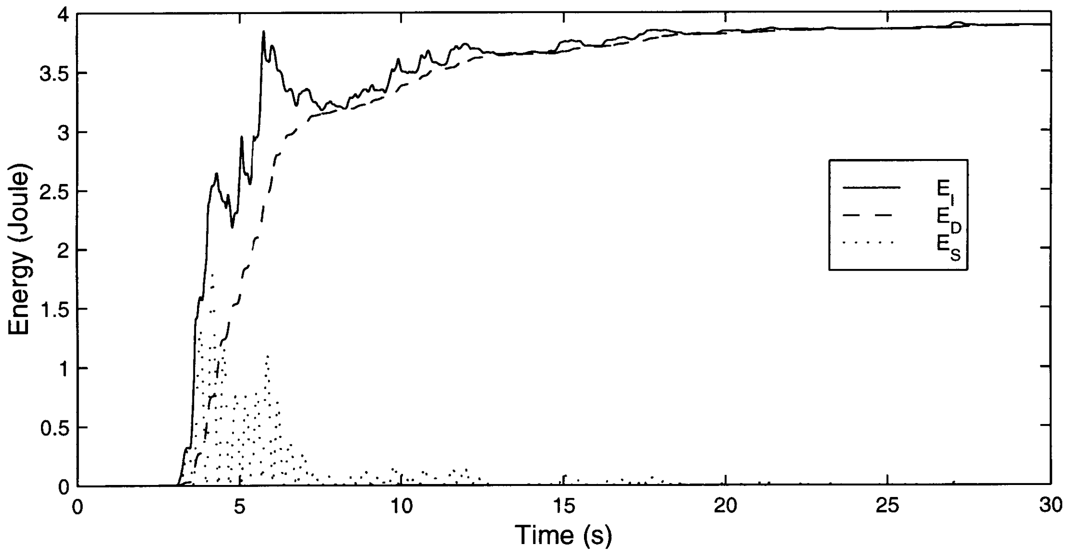

Figure 3-4: Building 1 Energy History, Arleta Station (90 DEG), Northridge 1994.[5]

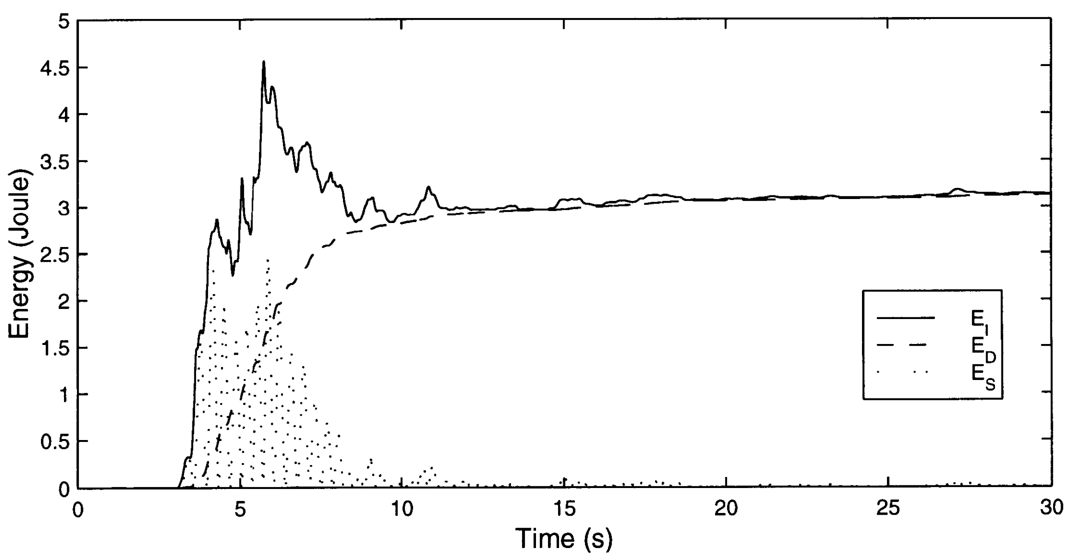

Figure 3-5: Building 2 Energy History, Arleta Station (90 DEG), Northridge 1994.[5]

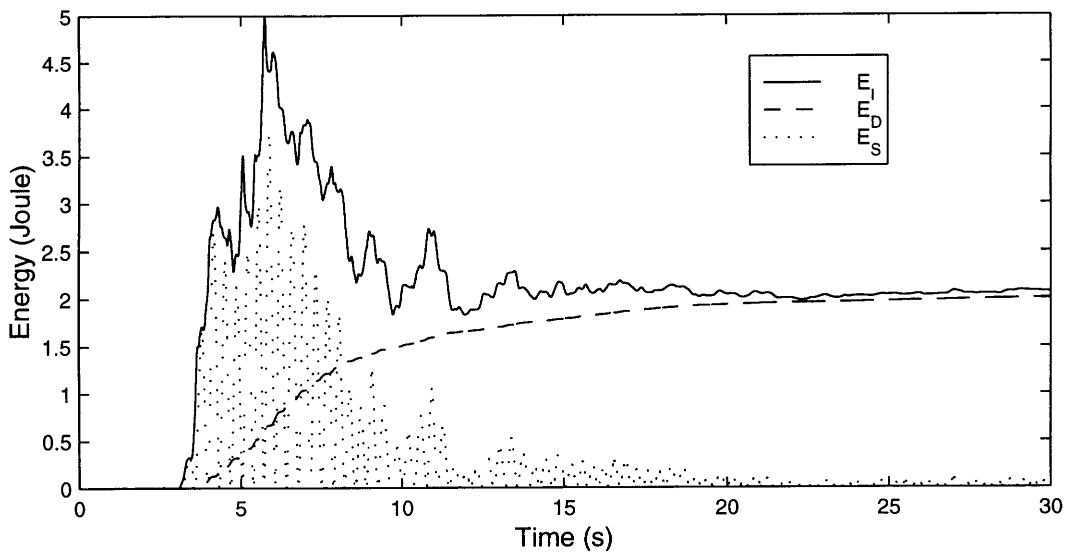

Figure 3-6: Building 3 Energy History, Arleta Station (90 DEG), Northridge 1994.[5]

energy. Also, if we look at figure 3-6, we see that E_{Smax} is about 80% of E_{Imax}. This is an extremely high increase in strain energy when compared to the other systems and loads. Thus in this case, we need to find ways of putting more damping into the structure to reduce the strain energy demand.

We realize that in the MDOF case the difference between energy histories for different loads are clearer than for the SDOF system. Also, the response for the Arleta station makes it more difficult to use the classical concept of response spectra, since the total amount of energy dissipated can vary substantially from E_{Imax}. Thus the MDOF case is quite complex and there are many variables that can play an important role on the system behavior.

3.4 Conclusions

It is difficult to state with certainty the reasons for the high increase in the strain energy demand, when going from Building 2 to Building 3. Nevertheless we discuss some important factors that influence the response. One of the most important reasons is that modal damping is the most effective type of damping we can put in

[5] See table 3.1

a structure. This is of course, after assuming that the response is separable in time and space. If the response is separable in space and time, then we can think of the response as a linear combination of the system mode shapes. Then, if a given mode shape is dominant in the response, we provide damping in that mode to reduce its contribution to the total system response. This is relatively easy to do, from the mathematical point of view, but in reality providing efficient modal damping is easier said than done. Specially when we want to provide high damping to the lower modes which tend to be dominant in the response. Unfortunately for us, arbitrary damping, which is a very realistic model, is not as efficient as modal damping. Nevertheless, in practice, high modal damping can be difficult to attain. To us the answer would be to find the most efficient way of providing arbitrary damping, at a reasonable cost.

Another reason for this increment has to do with the fundamental mode usually being the dominant one in the response. We will see later on, that Building 3 has very low damping in the fundamental mode, and that the damping properties between buildings 2 and 3 have their greatest difference for this mode. This means that we are providing damping in an inefficient way, since ideally we want the arbitrary damping matrix to be such that the lower modes get damped the most. So again we go back to the same question. How can we provide an optimal damping distribution ?

The difference in the energy build up for the two earthquakes suggests that the power of the excitation plays a key role in the system response. We arrive to this conclusion by looking at the impulsive nature of the Arleta record. Thus power can be more important than total energy input. Further study looking at the power of the input could be beneficial.

We have seen that there are many factors contributing to the response of multi degree of freedom systems. This makes it difficult to isolate parts of the system and make very strong conclusions about them. Thus we should always remember that the response of a system depends on all of its components, and then we should try to do a sensitivity analysis on specific components that we are interested in. Unfortunately, analytical solutions are practically impossible, so simulation is our best option. This can be a very time consuming task and requires good intuition on the system behavior that could be expected. After carrying out simulations, the best way to verify the models is to look at real structures. This can be even more difficult, since establishing the properties of actual buildings within a small tolerance of error is hard. But we have to keep in mind that structural performance during earthquakes is the real test for our systems.

Chapter 4

Mode Superposition for MDOF Systems

In this chapter we first look at classical modal superposition. We present it here in order to be able to compare the advantages and disadvantages of this approach with modal superposition in state space. The state space formulation is very powerful, since it imposes no restriction on the form of the damping matrix.

4.1 Classical Mode Superposition

To look at mode superposition, we first consider the undamped free vibration problem

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{0} \tag{4.1}$$

It is convenient to have solutions of the form

$$\mathbf{X}(t) = \boldsymbol{\phi} q(t) \tag{4.2}$$

which are separable in time and space. This means that the amplitude ratio between degrees of freedom is independent of time. Thus the system has a given shape that does not change with time but will only change in amplitude. Now, let us assume that we have

$$q(t) = \exp i\omega t \tag{4.3}$$

Substituting this and equation 4.2 into 4.1, we get

$$(-\omega^2 \mathbf{M}\boldsymbol{\phi} + \mathbf{K}\boldsymbol{\phi}) \exp i\omega t = \mathbf{0} \tag{4.4}$$

that can be simplified further to

$$\mathbf{K}\boldsymbol{\phi} = \lambda \mathbf{M}\boldsymbol{\phi} \tag{4.5}$$

where $\lambda = \omega^2$. In general when the mass and stiffness matrices are positive definite, there will be N (number of degrees of freedom) distinct real and positive eigenvalues λ, with one eigenvector $\boldsymbol{\phi}$ for each eigenvalue. This result is telling us that a MDOF system with N degrees of freedom can respond at N frequencies of vibration with each one of them being associated with one mode shape (eigenvector).

Since there are N eigenvectors and eigenvalues, equation 4.5 can be written as

$$\mathbf{K}\boldsymbol{\phi}_r = \lambda_r \mathbf{M}\boldsymbol{\phi}_r \qquad r = 1, \ldots, N \tag{4.6}$$

The eigenvectors have the important property of being orthogonal. They are also undefined in terms of magnitude, so we need to normalize them in some way. It is usually convenient to normalize the eigenvectors with respect to the mass matrix such that

$$\boldsymbol{\phi}_r^T \mathbf{M}\boldsymbol{\phi}_s = \delta_{rs} \tag{4.7}$$

where δ_{rs} is the Kronecker delta. It is convenient to switch to matrix notation, so we define what is known as the modal matrix

$$\boldsymbol{\Phi} = \lfloor \; \boldsymbol{\phi}_1 \;\; \boldsymbol{\phi}_2 \;\; \ldots \;\; \boldsymbol{\phi}_N \; \rfloor \tag{4.8}$$

where the mode shapes $\boldsymbol{\phi}_r$ have been normalized with respect to the mass matrix. Also we define a matrix $\boldsymbol{\Lambda}$ that stores the eigenvalues on its diagonal and has zeros every where else; i.e.,

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_N^2 \end{bmatrix} \tag{4.9}$$

These equations allow us to write the eigenvalue problem as

$$\mathbf{K\Phi} = \mathbf{M\Phi\Lambda}. \tag{4.10}$$

The orthogonality of the eigenvectors allows us to represent any vector representing the motion of the system in N-dimensional space as a linear combination of the eigenvectors. In fact we could prove that if the eigenvectors are normalized such that $|\boldsymbol{\phi}_r| = 1$, then they form a basis for N-dimensional space. Using this property it becomes convenient to use the linear transformation

$$\mathbf{X}(t) = \mathbf{\Phi q}(t) \tag{4.11}$$

Substituting in equation 4.1,

$$\mathbf{M\Phi\ddot{q}}(t) + \mathbf{K\Phi q}(t) = \mathbf{0} \tag{4.12}$$

and premultiplying by $\mathbf{\Phi}^T$ leads to

$$\mathbf{\Phi}^T\mathbf{M\Phi\ddot{q}}(t) + \mathbf{\Phi}^T\mathbf{K\Phi q}(t) = \mathbf{0} \tag{4.13}$$

Taking into account the normalization procedure that was used, equation 4.13 reduces to

$$\mathbf{\ddot{q}}(t) + \mathbf{\Lambda q}(t) = \mathbf{0} \tag{4.14}$$

Notice that equation 4.13 is a set of N uncoupled differential equations,

$$\ddot{q}_r(t) + \omega_r^2 q_r(t) = 0 \qquad r = 1, \ldots, N \tag{4.15}$$

One of the advantages of modal superposition is that the analytic solution to this generic equation is known. The other advantage is related to the frequencies of the system that are dominant in the response. We will explore this advantage when we deal with the forced-vibration case below.

4.1.1 Undamped Forced Vibration

For the undamped forced vibration case the equations of motion for the system are

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{F}(t) \tag{4.16}$$

which after using the linear transformation 4.11 becomes

$$\ddot{\mathbf{q}}(t) + \mathbf{\Lambda}\mathbf{q}(t) = \mathbf{\Phi}^T\mathbf{F}(t) \tag{4.17}$$

meaning that we have N individual equations of the form

$$\ddot{q}_r(t) + \omega_r^2 q_r(t) = f_r(t) \qquad\qquad r = 1, \ldots, N \tag{4.18}$$

where $f_r(t) = \boldsymbol{\phi}_r{}^T\mathbf{F}(t)$. Again, the equations have been decoupled and we can get the solutions $q_r(t)$ by using the techniques for the SDOF case. Once we obtain the solutions we can find the solution to 4.16 by using the linear transformation again,

$$\mathbf{X}(t) = \sum_{r=1}^{N} \boldsymbol{\phi}_r q_r(t) \tag{4.19}$$

and the process is now complete. With the advantage that we have solved N SDOF equations instead of solving a system of N equations which is, in general, more computationally intensive.

Now, Suppose that the loading $\mathbf{F}(t)$ is such that the response is dominated by a limited number of frequencies, say the first M natural frequencies, where $M \ll N$, and remember that the natural frequencies are such that $\omega_1 < \omega_2 < \ldots < \omega_N$. Using this argument we can obtain an approximate solution by considering only the first M mode shapes

$$\mathbf{X}(t) \approx \sum_{r=1}^{M} \boldsymbol{\phi}_r q_r(t) \tag{4.20}$$

thus saving considerable computation. This savings makes mode superposition very attractive, especially for systems with many degrees of freedom (thousands).

Everything we have said until now about mode superposition can lead us to think that it is the analysis technique to use. The problem is that the real disadvantage shows up when we consider damping, which in reality all systems have at least a small amount. We consider this case in the following section.

4.1.2 Damped Forced Vibration

Before we write the equations of motion for the damped system, let us assume that damping is proportional to velocity. This is not true, but it is convenient mathematically. So we have already set one restriction on damping. Mode superposition puts even greater restrictions on the damping used, as we will see later on.

The equations of motion for the damped system subjected to forced vibration are

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{F}(t) \tag{4.21}$$

to which we can again apply the linear transformation 4.11 and obtain

$$\ddot{\mathbf{q}}(t) + \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}\dot{\mathbf{q}}(t) + \mathbf{\Lambda}\mathbf{q}(t) = \mathbf{\Phi}^T\mathbf{F}(t) \tag{4.22}$$

that is similar to what we had for the undamped case except that we do not know what the product $\mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}$ transforms to. Ideally we would want this transformation to be such that $\mathbf{D} = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}$ is a diagonal matrix. This would cause equation 4.21 to be decoupled and in the form of

$$\ddot{q}_r(t) + d_r\dot{q}_r(t) + \omega_r^2 q_r(t) = f_r(t) \qquad r = 1, \ldots, N \tag{4.23}$$

where d_r is a constant, and it is convenient to have $d_r = 2\xi_r\omega_r$ in order to have everything in the same form as in the SDOF case. With d_r defined in this way, ξ_r would represent the damping ratio per mode.

The conditions on $\mathbf{C}$ such that it diagonalizes this way are very restrictive. Rayleigh showed that a damping matrix of the form

$$\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K} \tag{4.24}$$

in which a_0, a_1 are arbitrary constants, satisfy the orthogonality condition

$$\boldsymbol{\phi}_r^T\mathbf{C}\boldsymbol{\phi}_s = d_{rs}\delta_{rs} \tag{4.25}$$

necessary to decouple the damped equations of motion. This result is evident by taking equation 4.24 premultiplying it by $\mathbf{\Phi}^T$, postmultiplying by $\mathbf{\Phi}$ and then applying the orthogonality conditions of the $\mathbf{M}$ and $\mathbf{K}$ matrices. It turns out that Rayleigh damping (or proportional damping) is a special condition of a more general set of damping matrices that decouple the equations of motion, the matrix obtained this way is a *nonphysical* damping matrix. The more general condition is

$$\mathbf{C} = \mathbf{M} \sum_b a_b (\mathbf{M}^{-1}\mathbf{K})^b \qquad -\infty < b < \infty \tag{4.26}$$

where we can have as many b terms as we want. For example if we choose $b = 0, 1$ we get equation 4.24. The use of equation 4.26 allows us to have any amount of damping that we desire in each mode.

Assuming that $\mathbf{C}$ is such that the orthogonality condition 4.25 is satisfied, the procedure for solving the equations of motion is the same as before. Once we have solved 4.23 we can use it to find the solution

$$\mathbf{X}(t) = \sum_{r=1}^{N} \boldsymbol{\phi}_r q_r(t). \tag{4.27}$$

Again we see a major advantage if the damping of the structure is such that the equations of motion are decoupled when transforming them to modal coordinates. We had mentioned before that the other advantage of modal superposition was that we could consider only a few modes, since the loading will generally excite only the lower modes. If the damping matrix does not satisfy 4.25, the equations of motion will still be coupled in modal coordinates. What is usually done in this case is to carry out the solution in the time domain in modal coordinates, considering only the first M modes. The reason for not being able to uncouple the equations of motion is because classical mode superposition places a severe restriction on the form of the damping matrix.

The state space formulation which we describe in the following section does not pose such a restriction on the damping matrix, and thus we can use any arbitrary damping matrix. The disadvantage is that it increases the size of the problem, making it $2N \times 2N$, but the order of the differential equation is reduced.

4.2 Mode Superposition in State Space

State Space formulation is a modern analysis formulation that enjoys great popularity in control theory. It is a suitable formulation for many types of dynamic problems. The advantage is that in state space we take a higher order equation and reduce it to a first order equation. Then we apply the techniques used for first order differential equations to obtain the solution.

This section first introduces the state space formulation. Then we look at the eigenvalue problem in state space. These will enable us to explore the idea of mode superposition in state space. We look first at the free vibration problem, and then at the forced vibration problem. The undamped case is not treated separately, since damping does not pose a mathematical difficulty in state space. This is one of the other advantages of the method, namely that arbitrary damping can be consider without any computational impediment.

4.2.1 Introduction to the State Space Formulation

In order to introduce the state space formulation we start by looking at the governing differential equation of a MDOF system

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{F}(t) \tag{4.28}$$

which is the same as equation 3.1. We can think of this equation as an input-output relation, where $\mathbf{F}(t)$ is the input to the system and $\mathbf{X}(t)$ is the output. This input-output relationship is multi-valued in the sense that the same input can produce many different outputs (since we have not specified initial conditions). This uniqueness problem can be solved by specifying the displacement and velocity vectors (initial conditions), which we call the state vector (or state). In other words, for us to be able to know uniquely the response of the system $\mathbf{X}(t)$ to an input $\mathbf{F}(t)$ for $t \geq a$, we need to know the state of the system at $t = a$, that is we need to know $\dot{\mathbf{X}}(a)$ and $\mathbf{X}(a)$.

Now that we have defined what the state of the system is and have looked at the differential equation for the system as an input-output relationship, we can transform equation 4.28 into state space form. In order to transform the equation to state space, the first thing we need to do is define the state vector

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \dot{\mathbf{X}}(t) \end{bmatrix} \tag{4.29}$$

and its derivative with respect to time

$$\dot{\mathbf{U}}(t) = \begin{bmatrix} \dot{\mathbf{X}}(t) \\ \ddot{\mathbf{X}}(t) \end{bmatrix} \tag{4.30}$$

Premultiplying equation 4.28 by $\mathbf{M}^{-1}$

$$\ddot{\mathbf{X}}(t) + \mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{M}^{-1}\mathbf{K}\mathbf{X}(t) = \mathbf{M}^{-1}\mathbf{F}(t) \tag{4.31}$$

We can write this as a matrix multiplication such that

$$\begin{bmatrix} \dot{\mathbf{X}}(t) \\ \ddot{\mathbf{X}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{X}(t) \\ \dot{\mathbf{X}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{F}(t) \tag{4.32}$$

which allows for the governing differential equation of the MDOF system (equation 4.28) to be written as the the linear transformation

$$\dot{\mathbf{U}}(t) = \mathbf{A}\mathbf{U}(t) + \mathbf{B}\mathbf{F}(t) \tag{4.33}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \tag{4.34}$$

and the size of matrix $\mathbf{A}$ is $2N \times 2N$, and $\mathbf{B}$ is of size $2N \times N$. Equation 4.33 is the state space representation of the dynamic system. We have done a convenient mathematical transformation, that reduced the second order differential equation to a first order one, at the expense of having a $2N \times 2N$ system.

4.2.2 Eigenvalue Problem in State Space

In order to look at the eigenvalue problem in state space, we consider the free vibration equation

$$\dot{\mathbf{U}}(t) = \mathbf{A}\mathbf{U}(t) \tag{4.35}$$

As in the case of classical mode superposition, we consider solutions that are separable in time and space. The solution to equation 4.35 has the form

$$\mathbf{U}(t) = \mathbf{v}g(t) \tag{4.36}$$

For the free vibration case we can take $g(t) = \exp \lambda t$ so that

$$\mathbf{U}(t) = \mathbf{v} \exp \lambda t \tag{4.37}$$

which we substitute into equation 4.35 and obtain

$$\lambda \mathbf{v} \exp \lambda t = \mathbf{A}\mathbf{v} \exp \lambda t \tag{4.38}$$

Simplifying,

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{4.39}$$

which is a standard eigenvalue problem, with at most $2N$ distinct eigenvalues. If $\mathbf{A}$ is real and λ is a complex eigenvalue then $\bar{\lambda}$ (conjugate of λ) is also an eigenvalue, because the characteristic polynomial that results from equation 4.39 has real coefficients. For each eigenvalue λ there is a corresponding eigenvector $\boldsymbol{\psi}$ which also appears in conjugate pairs. These conditions occur because $\mathbf{A}$ is a non-symmetric matrix.

It should also be noted that we are dealing here with a matrix $\mathbf{A}$ for which there are $2N$ linearly independent eigenvectors. Of course, if all the eigenvalues of $\mathbf{A}$ are distinct then the respective eigenvectors are linearly independent, but we are setting this condition, for the case in which there are repeated eigenvalues. If the matrix $\mathbf{A}$ has $2N$ linearly independent eigenvectors then the matrix can be diagonalized. In order to show this, let us first define a matrix $\mathbf{V}$ that contains the eigenvectors in each of its columns,

$$\mathbf{V} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_{2N}} \end{bmatrix} \tag{4.40}$$

we also define a matrix $\mathbf{\Lambda}$ that stores the eigenvalues in the diagonal and has zeros in the off diagonal entries,

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{2N} \end{bmatrix} \tag{4.41}$$

which allow us to write the eigenvalue problem as

$$\mathbf{AV} = \mathbf{V\Lambda} \tag{4.42}$$

and since the columns of $\mathbf{V}$ are linearly independent, its inverse exists. So we can premultiply the above equation by $\mathbf{V}^{-1}$ and get

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{\Lambda} \tag{4.43}$$

showing that we can diagonalize $\mathbf{A}$.

Like in the classical mode superposition case, we need to define some sort of normalization for the eigenvectors. It is convenient to have them normalized such that

$$\mathbf{V}^{-1}\mathbf{V} = \mathbf{I} \tag{4.44}$$

where $\mathbf{I}$ is the $2N \times 2N$ identity matrix. We could do this by computing the inverse of $\mathbf{V}$, but this is a very inefficient task from a computational point of view. What is usually done is to calculate the left eigenvectors of $\mathbf{A}$. The left eigenvectors $\mathbf{w}_s \neq \mathbf{0}$ of $\mathbf{A}$ are those vectors that satisfy

$$\mathbf{A}^T\mathbf{w} = \lambda^*\mathbf{w} \tag{4.45}$$

which we can also write as

$$\mathbf{w}^T\mathbf{A} = \lambda^*\mathbf{w}^T \tag{4.46}$$

We can prove that the eigenvalues λ^* of $\mathbf{A}^T$ are the same as the eigenvalues λ of $\mathbf{A}$. Also, we define a matrix containing the left eigenvectors, such that

$$\mathbf{W} = \lfloor \; \mathbf{w_1} \quad \mathbf{w_2} \quad \ldots \quad \mathbf{w_{2N}} \; \rfloor \tag{4.47}$$

these eigenvectors also diagonalize $\mathbf{A}$. We can write the left eigenvalue problem as

$$\mathbf{A}^T\mathbf{W} = \mathbf{W}\mathbf{\Lambda} \tag{4.48}$$

or as

$$\mathbf{W}^T\mathbf{A} = \mathbf{\Lambda}\mathbf{W}^T \tag{4.49}$$

Postmultiplying by $(\mathbf{W}^T)^{-1}$ (inverse of $\mathbf{W}^T$) we get

$$\mathbf{W}^T\mathbf{A}(\mathbf{W}^T)^{-1} = \mathbf{\Lambda} \tag{4.50}$$

Comparing this result with that of equation 4.43 we realize that in order for our normalization procedure to work, it would be convenient to have $\mathbf{W}^T = \mathbf{V}^{-1}$. This means that we will have the eigenvectors normalized such that

$$\mathbf{W}^T\mathbf{V} = \mathbf{I} \tag{4.51}$$

which means that

$$\mathbf{W}^T\mathbf{A}\mathbf{V} = \mathbf{\Lambda} \tag{4.52}$$

or in tensor notation

$$\mathbf{w}_r^T\mathbf{v}_s = \delta_{rs}. \tag{4.53}$$

This is a very convenient result that allows us to decouple the equations of motion in state space.

4.2.3 Free Vibration

Here we look at the free vibration case in state space using mode superposition. If we were not doing mode superposition, but just time step integration in state space, the analysis procedure is straight forward and can be carried out by solving equation 4.35. For state space mode superposition, we use the linear transformation

$$\mathbf{U}(t) = \mathbf{V}\mathbf{g}(t) \tag{4.54}$$

to decouple equation 4.35,

$$\mathbf{V}\dot{\mathbf{g}}(t) = \mathbf{A}\mathbf{V}\mathbf{g}(t) \tag{4.55}$$

Premultiplying by $\mathbf{W}^T$

$$\mathbf{W}^T\mathbf{V}\dot{\mathbf{g}}(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{g}(t) \tag{4.56}$$

and using equation 4.52 we get

$$\dot{\mathbf{g}}(t) = \mathbf{\Lambda}\mathbf{g}(t) \tag{4.57}$$

which represent the decoupled equations of motion in state space form. We can write this equation in scalar form as

$$\dot{g}_r(t) = \lambda_r g_r(t) \qquad\qquad r = 1, \ldots, 2N \tag{4.58}$$

the analytic solution to this free vibration case is quite simple. All we need are the initial conditions in the modal coordinates g_r. In order to get this, we first need to go from the natural coordinates $\mathbf{X}$ to the state space coordinates $\mathbf{U}$ and then to the modal coordinates g_r. Specifying the initial conditions for the MDOF system is the same as specifying the initial state, so

$$\mathbf{U}(0) = \begin{bmatrix} \mathbf{X}(0) \\ \dot{\mathbf{X}}(0) \end{bmatrix} \tag{4.59}$$

we can also write the initial state of the system as

$$\mathbf{U}(0) = \mathbf{V}\mathbf{g}(0) \tag{4.60}$$

where we already know $\mathbf{U}(0)$ from equation 4.59, and the unknowns are $\mathbf{g}(0)$. We premultiply this equation by $\mathbf{V}^{-1}$ and get

$$\mathbf{g}(0) = \mathbf{V}^{-1}\mathbf{U}(0) \tag{4.61}$$

but in practice we would compute $\mathbf{g}(0)$ using

$$\mathbf{g}(0) = \mathbf{W}^T\mathbf{U}(0). \tag{4.62}$$

which are the initial conditions in the modal coordinates.

Now that we have stated what the initial conditions to equation 4.58 are. We can see that functions of the form

$$g_r(t) = b_r \exp \lambda_r t \tag{4.63}$$

are the solution to the differential equation. Requiring that the initial conditions $g_r(0)$ are satisfied leaves us with

$$b_r = g_r(0) = \mathbf{w}_r^T \mathbf{U}(0) \qquad r = 1, \ldots, 2N \tag{4.64}$$

where it is important to note that $b_r \in \mathbb{C}$, and that the eigenvectors appear in conjugate pairs. Using these conditions and the properties of complex numbers we have that given

$$b_r = \mathbf{w}_r^T \mathbf{U}(0) \tag{4.65}$$

then

$$\bar{b}_r = \bar{\mathbf{w}}_r^T \mathbf{U}(0) \tag{4.66}$$

which means that once we have the initial condition for an eigenvector, the initial condition for its conjugate can be calculated just by taking the conjugate of the initial condition, instead of having to compute the dot product $\bar{\mathbf{w}}_r^T \mathbf{U}(0)$.

Using these properties, we can write the response of the system in state space coordinates as

$$\mathbf{U}(t) = \sum_{r=1}^{2N} b_r \mathbf{v}_r \exp \lambda_r t = \sum_{r=1}^{N} (b_r \mathbf{v}_r \exp \lambda_r t + \bar{b}_r \bar{\mathbf{v}}_r \exp \bar{\lambda}_r t) \tag{4.67}$$

where we can apply the advantage of mode superposition by approximating the response by taking the summation over M modes, where $M \ll N$. Also we should point out that the solution $\mathbf{U}(t) \in \mathbb{R}$, as is expected since the solution to the governing differential equation of the linear system is real.

Real Expansion of the Free Vibration Solution

Let us look more closely at the real expansion of the free vibration solution. This is done by taking a mode shape and its conjugate and adding their contribution to the solution, such that in the end we express the response of the system as a combination of their real and imaginary parts. The easiest way to start is to look at equation 4.67 and write each complex number as a combination of their real and imaginary parts

$$\mathbf{U}(t) = \sum_{r=1}^{N} [(b_{R,r} + i b_{I,r})(\mathbf{v}_{R,r} + i \mathbf{v}_{I,r}) e^{(\lambda_{R,r} + i\lambda_{I,r})t} + (b_{R,r} - i b_{I,r})(\mathbf{v}_{R,r} - i \mathbf{v}_{I,r}) e^{(\lambda_{R,r} - i\lambda_{I,r})t}] \tag{4.68}$$

which we can also write as

$$\mathbf{U}(t) = \sum_{r=1}^{N} e^{\lambda_{R,r}t}[(b_{R,r} + ib_{I,r})(\mathbf{v}_{R,r} + i\mathbf{v}_{I,r})e^{i\lambda_{I,r}t} + (b_{R,r} - ib_{I,r})(\mathbf{v}_{R,r} - i\mathbf{v}_{I,r})e^{-i\lambda_{I,r}t}] \tag{4.69}$$

where the $e^{\lambda_{R,r}t}$ term represents the exponential part of the solution. In order for the response to be bounded, all $\lambda_{R,r}$ have to be negative, if they are positive the solution grows in time. Thus the stability condition for the linear system is that the real part of the eigenvalues has to be negative. The terms in the bracket represent the harmonic part of the solution. We can write this part as a combination of sines and cosines using Euler's formula. After expanding, and simplifying we get

$$\mathbf{U}(t) = \sum_{r=1}^{N} 2e^{\lambda_{R,r}t}[\mathbf{v}_{R,r}h_1(t) - \mathbf{v}_{I,r}h_2(t)] \tag{4.70}$$

where

$$h_1(t) = b_{R,r}\cos\lambda_{I,r}t - b_{I,r}\sin\lambda_{I,r}t \tag{4.71}$$

$$h_2(t) = b_{I,r}\cos\lambda_{I,r}t - b_{R,r}\sin\lambda_{I,r}t \tag{4.72}$$

So we have shown that we have $2N$ modes responding at different frequencies. This approach is convenient since we are dealing only with real quantities and thus we know that the solution will be real. When the system is undamped $\mathbf{v}_{I,r} = \mathbf{0}$, which means that we will have the undamped mode shapes that we compute for the undamped case in classical mode superposition.

Examples of Damped and Undamped Systems

The best way to illustrate the ideas shown in the previous sections is to look at a MDOF system and provide different damping characteristics. We use the systems in chapter 3 and two other buildings to help us show this. The stiffness and mass distribution of the systems are kept constant and we vary the damping properties only. In order to provide a fair comparison with the undamped mode shapes we will use proportional damping in most cases. This neglects the advantage of using the state space formulation for arbitrary damping, but we do it to gain a better understanding

Building Name	Mass and Stiffness Properties	Damping Properties
Building 1	custom mass, custom stiffness	5% constant modal
Building 2	custom mass, custom stiffness	2% constant modal
Building 3	custom mass, custom stiffness	arbitrary damping in $N.s/m$ ($c_1 = 2, c_2 = 1.5, c_3 = 0.9, c_4 = 0.25$)
Building 4	custom mass, custom stiffness	0.1% constant modal
Building 5	custom mass, custom stiffness	50% constant modal

Table 4.1: MDOF System Properties

of the behavior of the system. Nevertheless, we do use arbitrary damping for one of the systems. Table 4.1 shows the properties for each of the buildings.

Figure 4-1 shows the damped and undamped frequencies for a system with 5% damping on each mode. The smallest frequency is the fundamental frequency of the system, and in general it will be the one that dominates the response. As we mentioned earlier, the real part of the eigenvalue is negative, so the system is stable and the response will be damped out. We should also notice that the higher modes have more damping than the lower modes. Here we can ask ourselves, how can this be possible if we specified $\xi = 5\%$ in each mode. The answer is that we are using frequency dependent damping, and thus the modes with a higher frequency will have more damping even though ξ is constant.

Figure 4-2 is a plot of the damped and undamped mode shapes of the same system. D_R represents the damped real mode shape. Meaning that it is the real part of the eigenvector. D_I and U represent the damped imaginary and the undamped mode shapes. Normalization was not done according to the theory presented in the earlier sections. Instead we normalized the undamped mode shapes with respect to 0.45, in order to have a nice plot, and the damped mode shapes were normalized such that the sum of their maximum values gives that of the undamped mode. This was done to be able to look at the damped mode shapes as contributions to the undamped one.

Of course, there are different ways to look at them, but we found this useful specially for the case in which damping is very small.

Figures 4-3 and 4-4 show the same plots as before, but for a system with very little damping ($\xi = 0.1\%$). It is interesting to see that from the plot we can not distinguish between the damped and undamped frequencies. Still, there is a small amount of damping present that is enough to bring the system to rest after the loading period by dissipating energy. The only problem is that more than being able to dissipate the energy, we are interested in the rate at which we can dissipate it, and a very low damping level means that the energy will be dissipated slowly. By looking at the mode shapes, we can observe that the damped imaginary part of the mode shape is very small. This goes according to what we mentioned in the previous section. As we reduce the amount of damping in the system, the damped real mode shape tends to the undamped one, and the contribution of the damped imaginary mode tends to zero.

Figures 4-5 and 4-6 are again the same type of plots as before, but for a system with $\xi = 50\%$. This is a very high amount of damping, that we do not expect to have in a building. In fact if we were able to get $\xi = 20\%$ for a building we would think of this as a very high level of damping. We only consider this case for comparison purposes. Specially to show that in this case the difference between the damped and undamped frequencies is substantial. Here it also clearer that the higher frequencies have greater amounts of damping. So, we can give some arguments as to why higher modes usually do not contribute to the system response. First, the typical loads in a building have small dominant frequencies, meaning that they tend to excite the lower modes. Second, the amount of energy needed to excite the higher modes is more than that to excite the lower ones. Where we should remember that structures tend to respond in those modes that need less amount of energy to be excited. Also, as we mentioned before, the higher modes have more damping, because we are using frequency dependent damping, thus when we look at all of these statements it becomes quite difficult to excite the higher modes of a structure.

Now we compare the properties of Buildings 2 and 3. The plots for these buildings

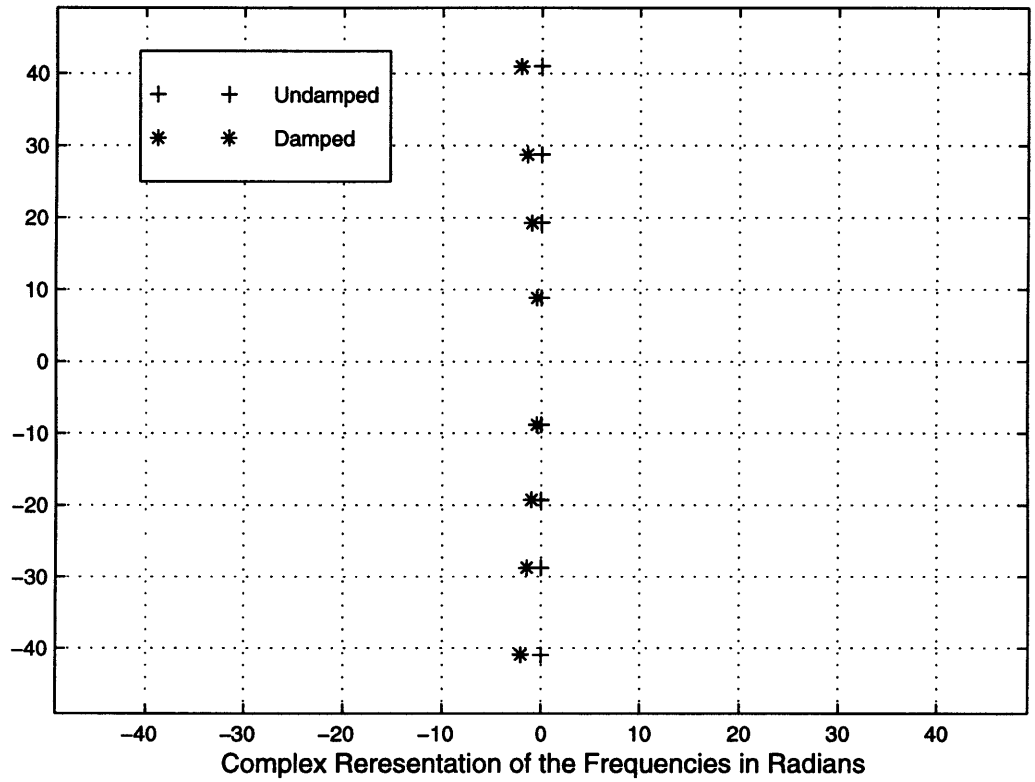

Figure 4-1: Damped and Undamped Frequencies, Building 1.[1]

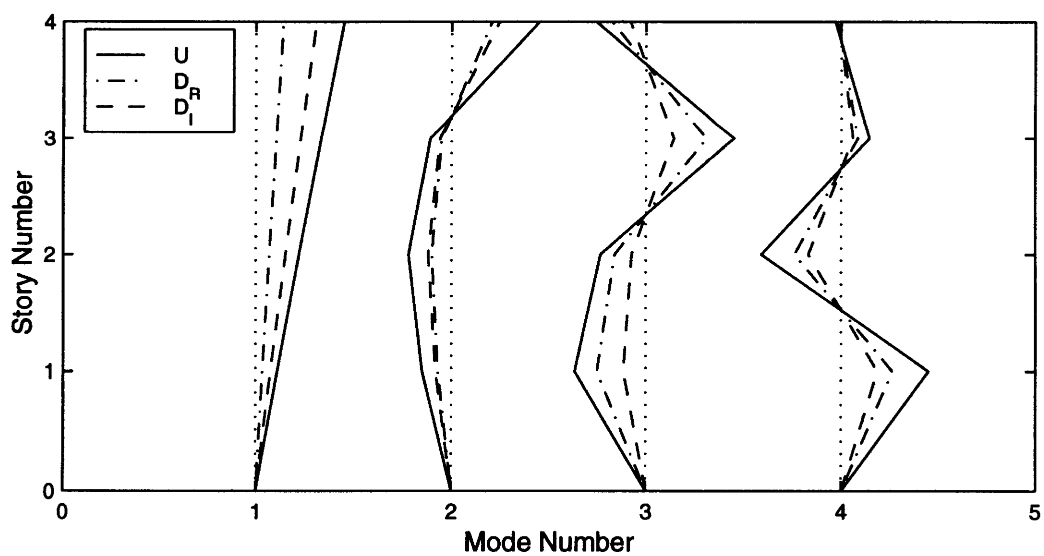

Figure 4-2: Damped and Undamped Mode Shapes, Building 1.[1]

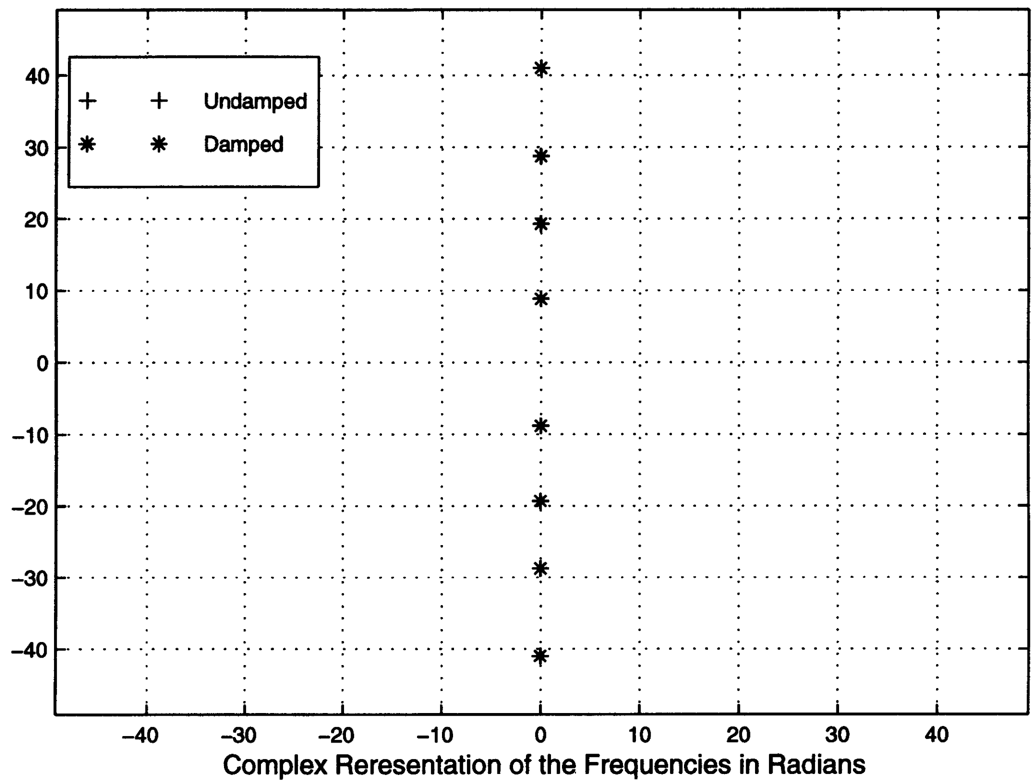

Figure 4-3: Damped and Undamped Frequencies, Building 4.[1]

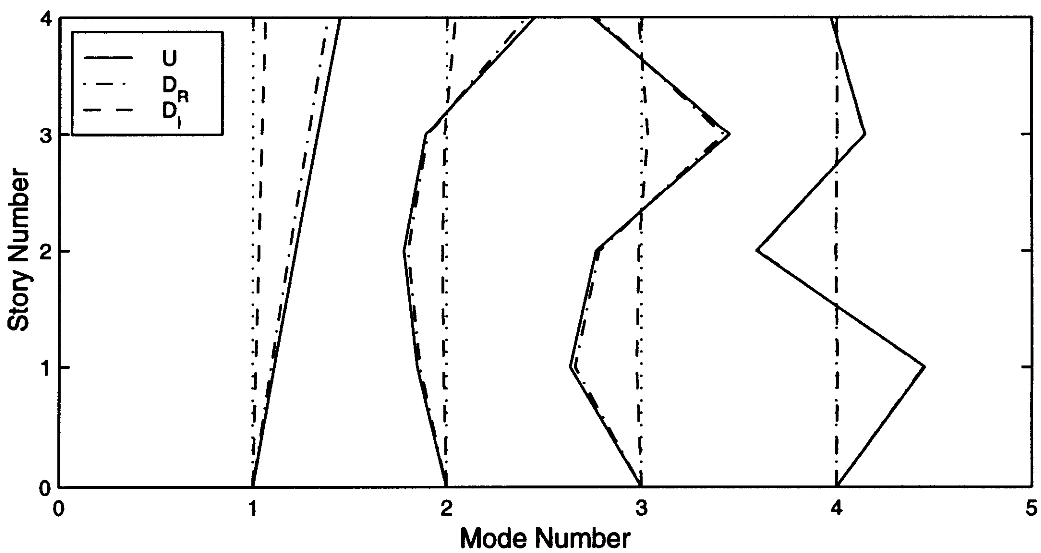

Figure 4-4: Damped and Undamped Mode Shapes, Building 4.[1]

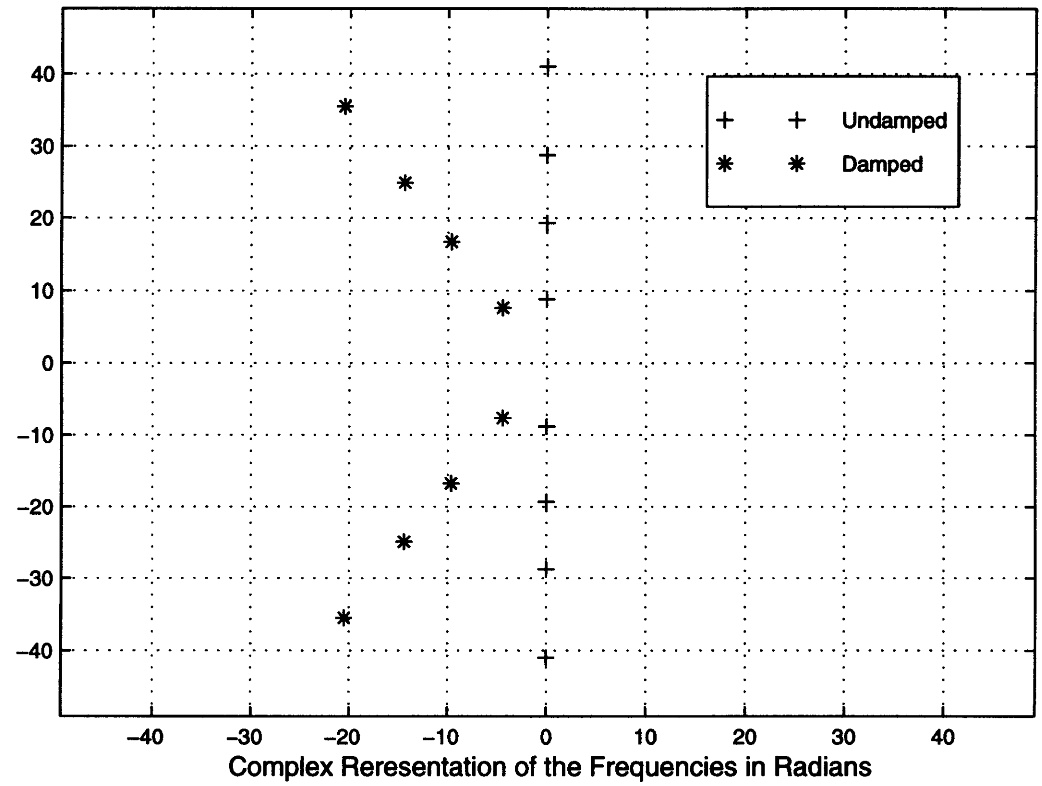

Figure 4-5: Damped and Undamped Frequencies, Building 5.[1]

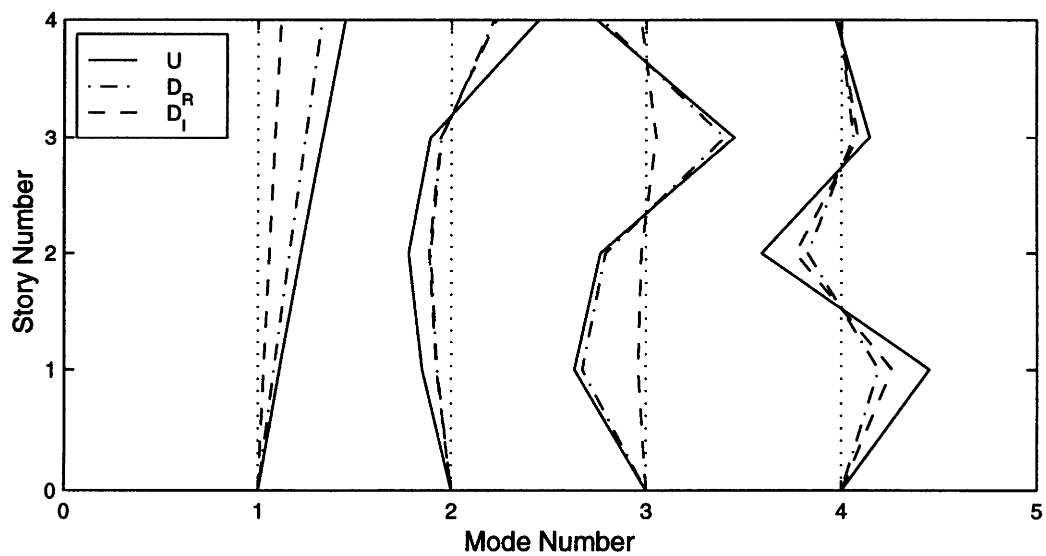

Figure 4-6: Damped and Undamped Mode Shapes, Building 5.[1]

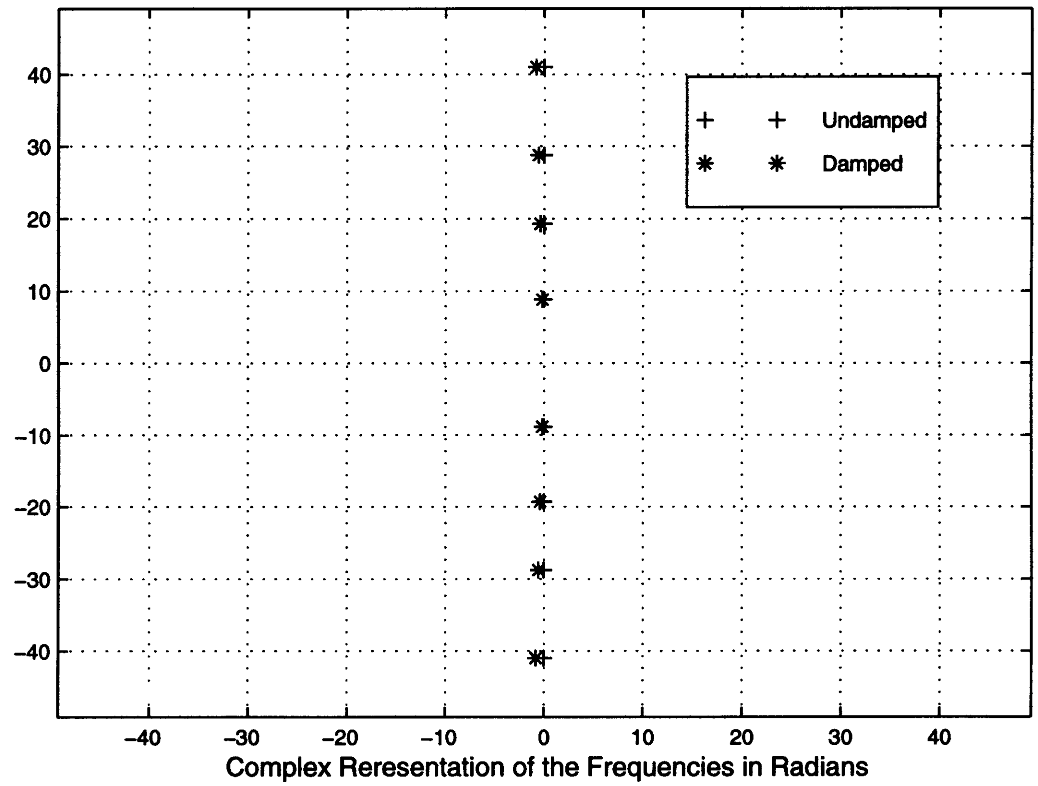

Figure 4-7: Damped and Undamped Frequencies, Building 2.[1]

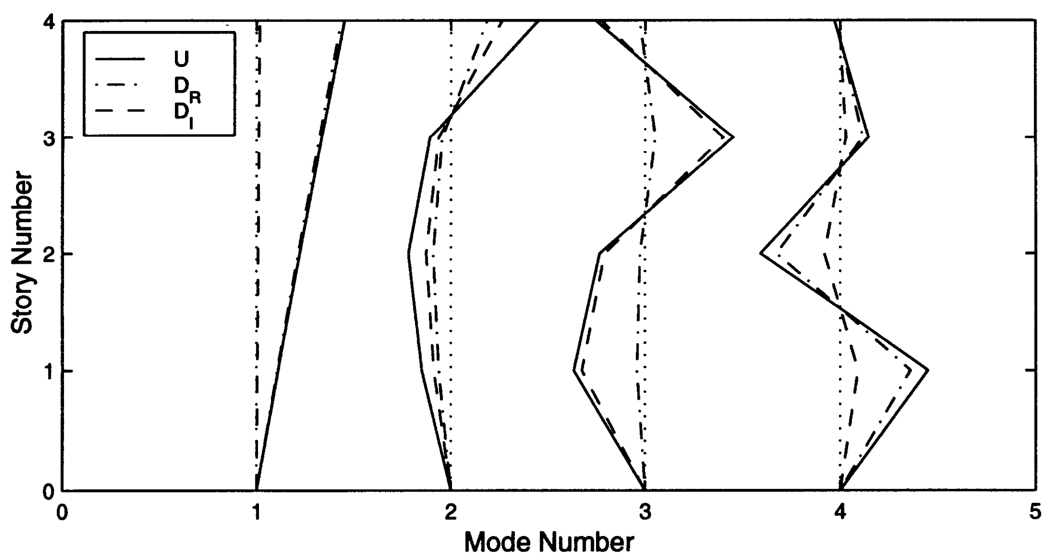

Figure 4-8: Damped and Undamped Mode Shapes, Building 2.[1]

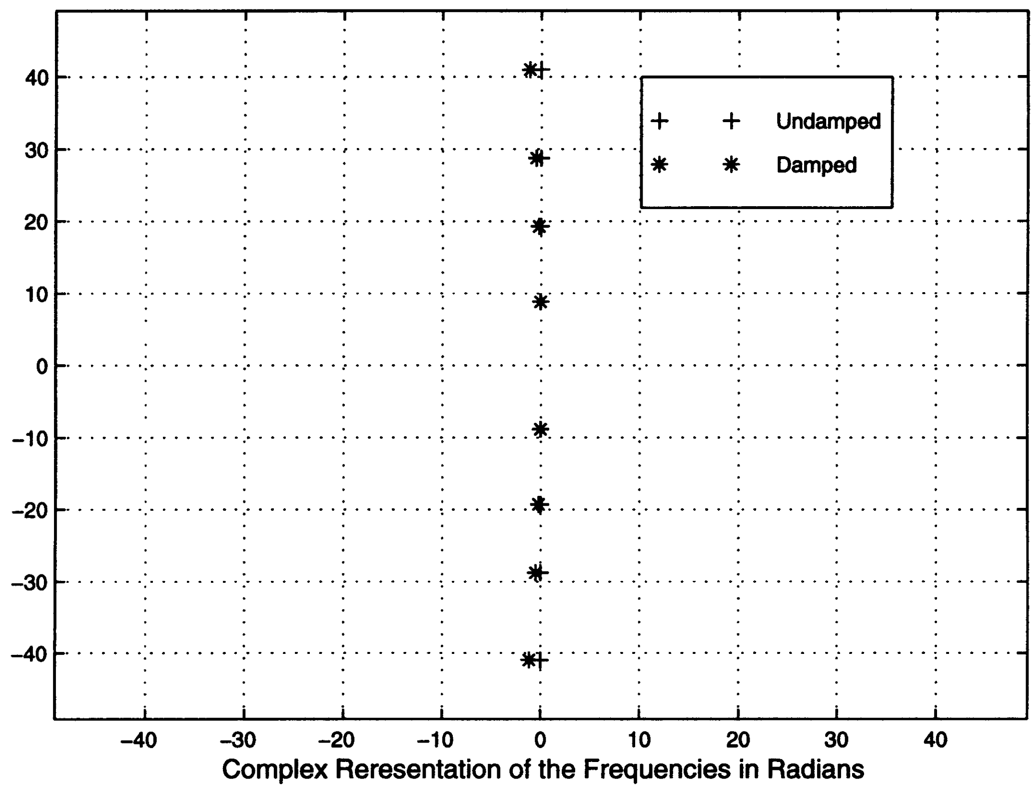

Figure 4-9: Damped and Undamped Frequencies, Building 3.[1]

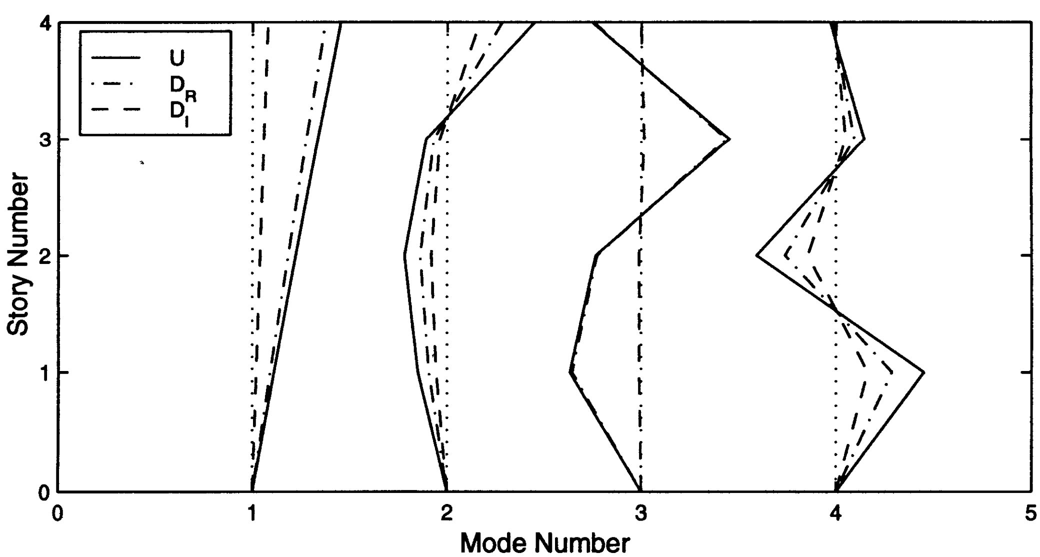

Figure 4-10: Damped and Undamped Mode Shapes, Building 3.[1]

are in figures 4-7, 4-8, 4-9 and 4-10. By comparing the plots of the frequency representation of these two buildings, we can see very little difference between the damped frequencies of the 2% damping case and the specified arbitrary damping matrix one, except for the fundamental frequency. Where we see a greater difference, although from the scale of the plot it looks like the difference is small.

It is this difference in the damped fundamental frequencies that we think is responsible for the great difference on the behavior of these systems in chapter 3. Since, usually the response is dominated by the lower modes.

4.2.4 Forced Vibration

Now, we look at the forced vibration case in state space. Again we are going to look at it using mode superposition, since analysis in the time domain without decoupling should be straight forward. Equation 4.33 is the forced vibration equation in state space, applying the linear transformation 4.54 we obtain

$$\mathbf{V}\dot{\mathbf{g}}(t) = \mathbf{A}\mathbf{V}\mathbf{g}(t) + \mathbf{B}\mathbf{F}(t) \tag{4.73}$$

Premultiplying by $\mathbf{W}^T$ leads to

$$\mathbf{W}^T\mathbf{V}\dot{\mathbf{g}}(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{g}(t) + \mathbf{W}^T\mathbf{B}\mathbf{F}(t) \tag{4.74}$$

and after simplifying we have

$$\dot{\mathbf{g}}(t) = \mathbf{\Lambda}\mathbf{g}(t) + \mathbf{W}^T\mathbf{B}\mathbf{F}(t) \tag{4.75}$$

These equations are decoupled and are very similar to the ones for the free vibration case, except that we now have an extra term on the right hand side. This term represents the loading in the modal coordinates. We can write the equations in scalar form so that

$$\dot{g}_r(t) = \lambda_r g_r(t) + f_r(t) \qquad\qquad r = 1, \ldots, 2N \tag{4.76}$$

where $f_r(t) = \mathbf{w}_r^T\mathbf{B}\mathbf{F}(t)$. The initial conditions are treated in the same way we did for the free vibration case. Notice that in general $g_r(t) \in \mathbb{C}$.

[1]See table 4.1

In order to avoid having to work in the complex domain, we look at a way of modifying the differential equations so that the coefficients of the differential equations are real. We start by looking at the decoupled problem in $\mathbb{C}$, with two sets of N equations

$$\dot{\mathbf{g}}(t) = \mathbf{\Lambda}\mathbf{g}(t) + \mathbf{W}^T\mathbf{B}\mathbf{F}(t) \qquad (N\ equations) \tag{4.77}$$

$$\dot{\bar{\mathbf{g}}}(t) = \bar{\mathbf{\Lambda}}\bar{\mathbf{g}}(t) + \bar{\mathbf{W}}^T\mathbf{B}\mathbf{F}(t) \qquad (N\ equations) \tag{4.78}$$

where $\mathbf{W}^T$ and its conjugate are of size $N \times 2N$. Using the cartesian representation of complex numbers, we can write these equations as

$$\dot{\mathbf{g}}_R(t) + i\dot{\mathbf{g}}_\mathbf{I}(t) = (\mathbf{\Lambda}_R + i\mathbf{\Lambda}_I)[\mathbf{g}_R(t) + i\mathbf{g}_I(t)] + (\mathbf{W}_R^T + i\mathbf{W}_I^T)\mathbf{B}\mathbf{F}(t) \tag{4.79}$$

$$\dot{\mathbf{g}}_R(t) - i\dot{\mathbf{g}}_\mathbf{I}(t) = (\mathbf{\Lambda}_R - i\mathbf{\Lambda}_I)[\mathbf{g}_R(t) - i\mathbf{g}_I(t)] + (\mathbf{W}_R^T - i\mathbf{W}_I^T)\mathbf{B}\mathbf{F}(t) \tag{4.80}$$

and we now proceed to first add these equations and then subtract them. This gives us two coupled differential equations

$$\dot{\mathbf{g}}_R(t) = \mathbf{\Lambda}_R\mathbf{g}_R(t) - \mathbf{\Lambda}_I\mathbf{g}_I(t) + \mathbf{W}_R^T\mathbf{B}\mathbf{F}(t) \qquad (N\ equations) \tag{4.81}$$

$$\dot{\mathbf{g}}_I(t) = \mathbf{\Lambda}_I\mathbf{g}_R(t) + \mathbf{\Lambda}_R\mathbf{g}_I(t) + \mathbf{W}_I^T\mathbf{B}\mathbf{F}(t) \qquad (N\ equations) \tag{4.82}$$

We conveniently write them in matrix notation and state space form as

$$\dot{\mathbf{G}}(t) = \mathbf{A}^*\mathbf{G}(t) + \mathbf{B}^*\mathbf{F}(t) \tag{4.83}$$

where

$$\mathbf{A}^* = \begin{bmatrix} \mathbf{\Lambda}_R & -\mathbf{\Lambda}_I \\ \mathbf{\Lambda}_I & \mathbf{\Lambda}_R \end{bmatrix}; \qquad \mathbf{B}^* = \begin{bmatrix} \mathbf{W}_R^T\mathbf{B} \\ \mathbf{W}_I^T\mathbf{B} \end{bmatrix}; \qquad \mathbf{G}(t) = \begin{bmatrix} \mathbf{g}_R(t) \\ \mathbf{g}_I(t) \end{bmatrix} \tag{4.84}$$

with initial conditions

$$\mathbf{G}(0) = \begin{bmatrix} \mathbf{g}_R(0) \\ \mathbf{g}_I(0) \end{bmatrix} \tag{4.85}$$

where $\mathbf{g}_R(0)$ and $\mathbf{g}_I(0)$ are $N \times 1$ vectors denoting the real and complex part of $\mathbf{g}(0)$ which we calculate using equation 4.62. Notice that this equation contains the modal initial condition and its conjugate, so we only need to calculate one of them to get $\mathbf{g}_R(0)$ and $\mathbf{g}_I(0)$. After having the initial conditions we solve equation 4.83

using any state space algorithm. This gives us the solution in the real and complex modal coordinates. The procedure is then to form the $\mathbf{g}(t)$ vector and then use the linear transformation 4.54 to obtain the response in the state space coordinates. Then from state space we can easily go to the original coordinates in which the differential equations were established.

If we want to include less than N modes (notice we say N and not $2N$, since we are including the real and imaginary parts) in the analysis we can do this. The procedure is to first make up the $\mathbf{A}^*$ and $\mathbf{B}^*$ matrices by including only the modes that we are interested in. Then the steps are the same, except that the matrix $\mathbf{V}$ in the linear transformation 4.54 should only have the modes that we selected. This is because the multiplication of a matrix by a vector is the same as doing a summation, where in this case, the columns of $\mathbf{V}$ represent the modes whose contribution to the response we are interested in.

4.3 Response History Using Mode Superposition in State Space

In this section we look at the response history of the systems that we used in chapter 3 using mode superposition in state space. The idea is not only to compare the solution obtained by using mode superposition, but also to see what modes are important in the response of each of the systems. The earthquakes used are the same that have been used in the previous chapters. This allows to isolate the influence of using mode superposition when compared with the system response calculated by including all the modes in the analysis.

We define some simple error measures to compare the results obtained by using both methods. Thus we define

Strain Energy Error

$$\epsilon_S = |E_S(t) - E_{Sm}(t)| \tag{4.86}$$

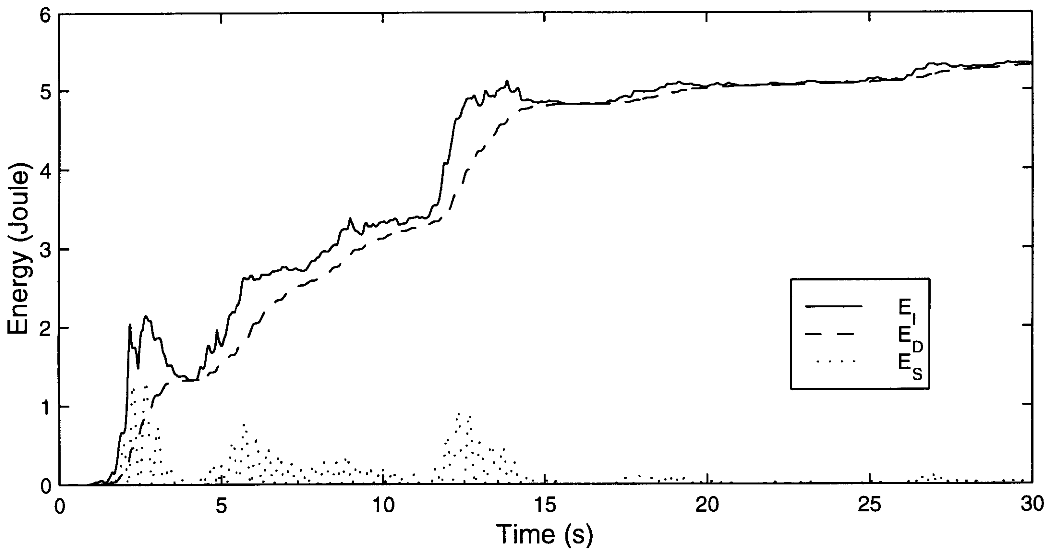

Figure 4-11: Building 1 Energy History (1st mode), El Centro (S00E), Imperial Valley 1940.[2]

Dissipated Energy Error

$$\epsilon_D = |E_D(t) - E_{Dm}(t)| \tag{4.87}$$

Energy Input Error

$$\epsilon_I = |E_I(t) - E_{Im}(t)| \tag{4.88}$$

where the first energy quantity inside the absolute value sign is calculated using analysis in the time domain and the other quantity corresponds to the energy using mode superposition analysis. The use of an error quantity in form of a percentage is not convenient since it is possible for the strain energy to be zero. Also, it is possible to have a small shift in the response history when using an approximate method, and thus a percentage error quantity can mislead us when interpreting the results. It is easier then to look at the error in the form stated above and then realize that we have to compare it with the system response. Then, if the maximum value of the strain energy error represents a significant amount of the elastic capacity of the system, it means that we need to include more modes in the analysis.

Figures 4-11 thru 4-22 show the response of the three systems and the error in each of the energy terms.

[2]See table 4.1

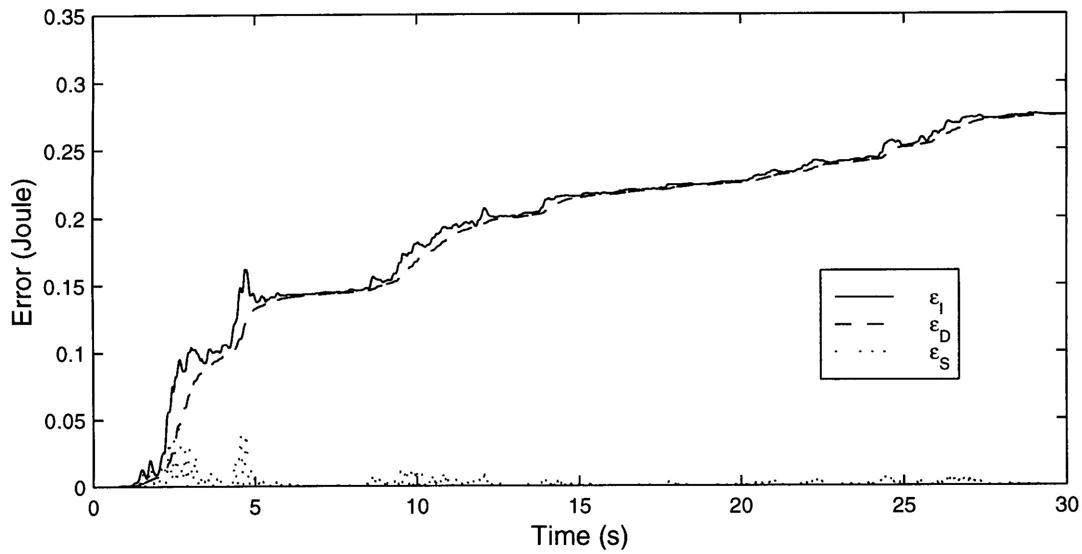

Figure 4-12: Building 1, 1st mode Energy Error, El Centro (S00E), Imperial Valley 1940.[3]

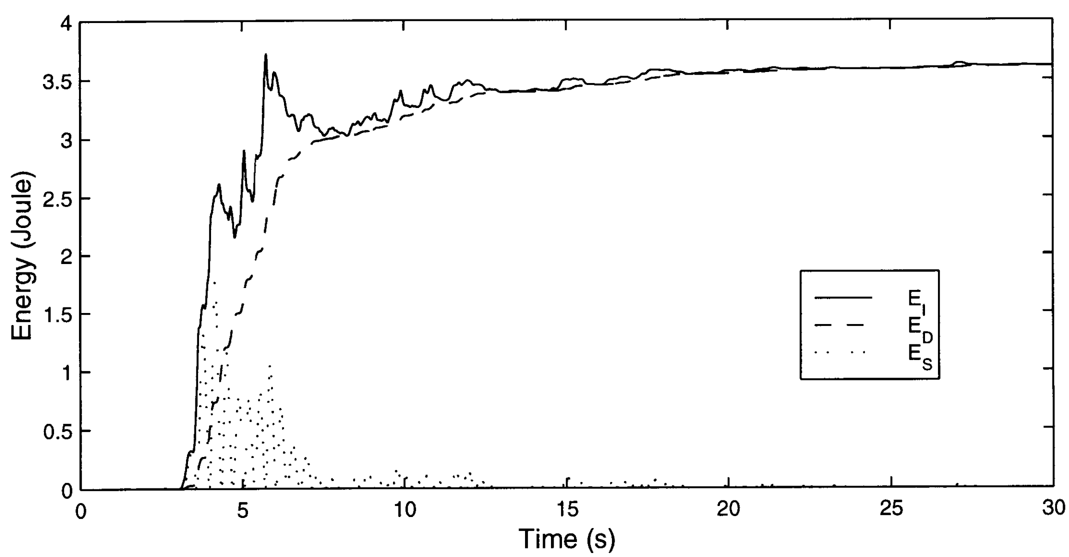

Figure 4-13: Building 1 Energy History (1st mode), Arleta Station (90 DEG), Northridge 1994.[3]

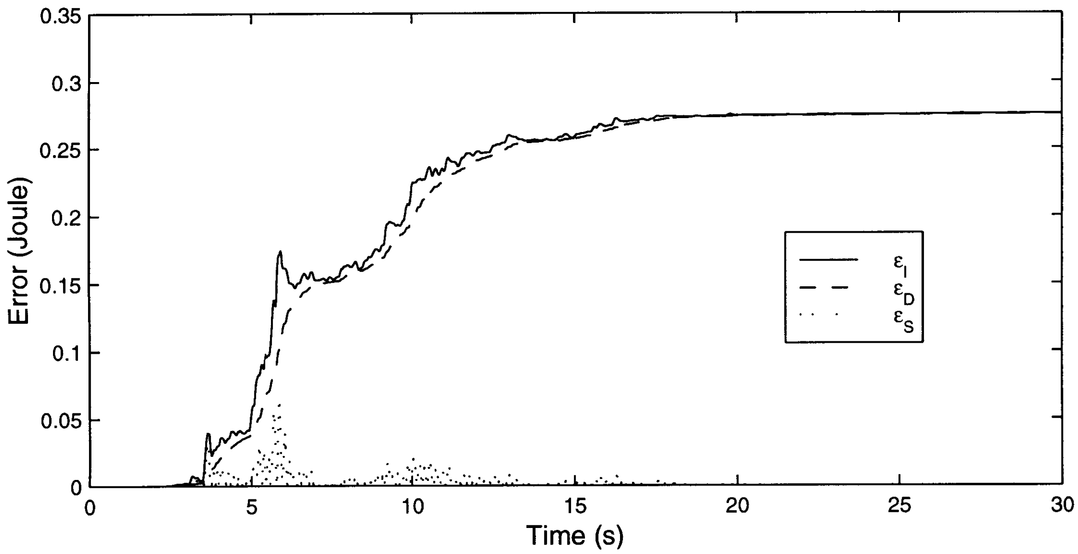

Figure 4-14: Building 1, 1st mode Energy Error, Arleta Station (90 DEG), Northridge 1994.[3]

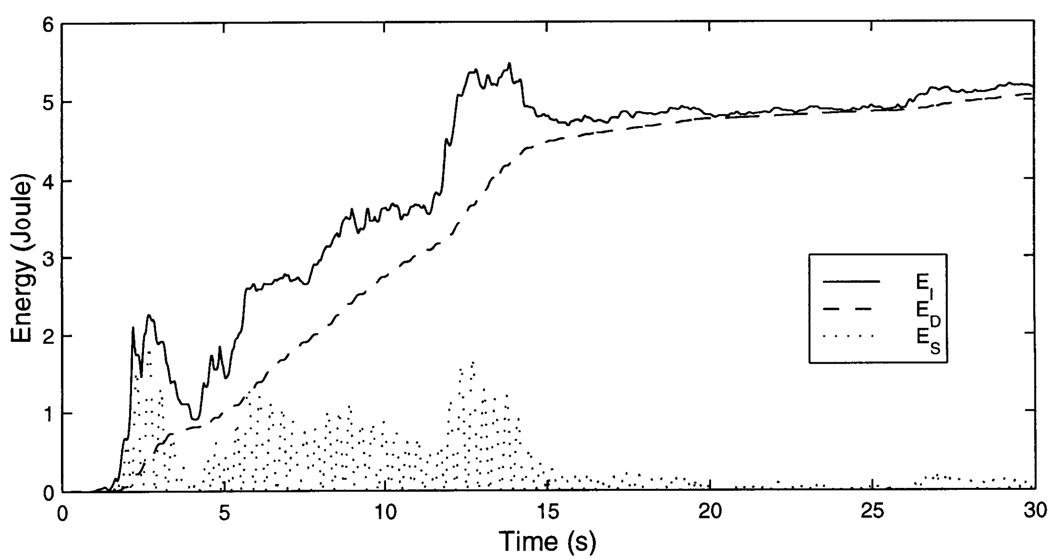

Figure 4-15: Building 2 Energy History (1st mode), El Centro (S00E), Imperial Valley 1940.[3]

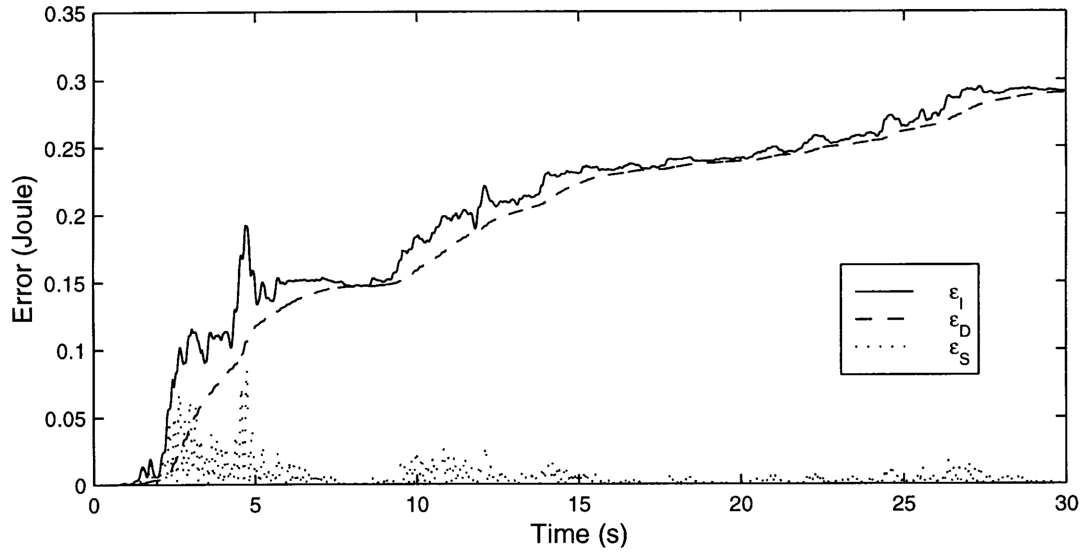

Figure 4-16: Building 2, 1st mode Energy Error, El Centro (S00E), Imperial Valley 1940.[3]

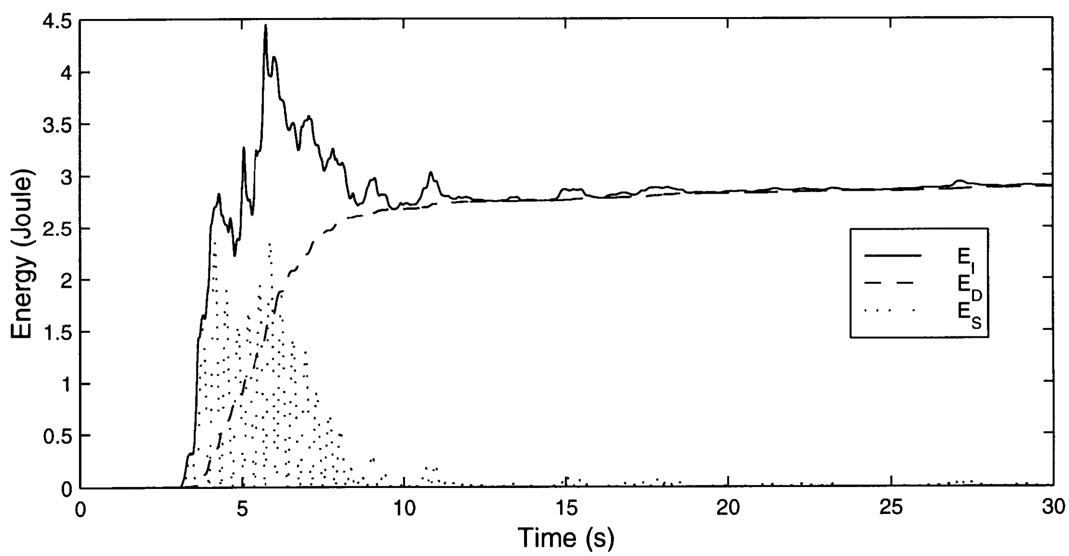

Figure 4-17: Building 2 Energy History (1st mode), Arleta Station (90 DEG), Northridge 1994.[3]

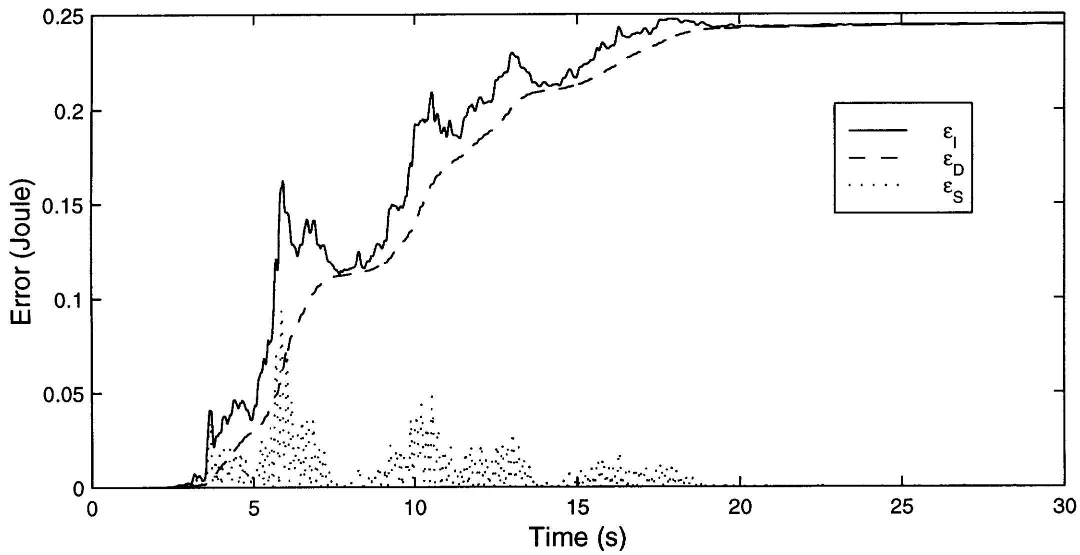

Figure 4-18: Building 2, 1st mode Energy Error, Arleta Station (90 DEG), Northridge 1994.[3]

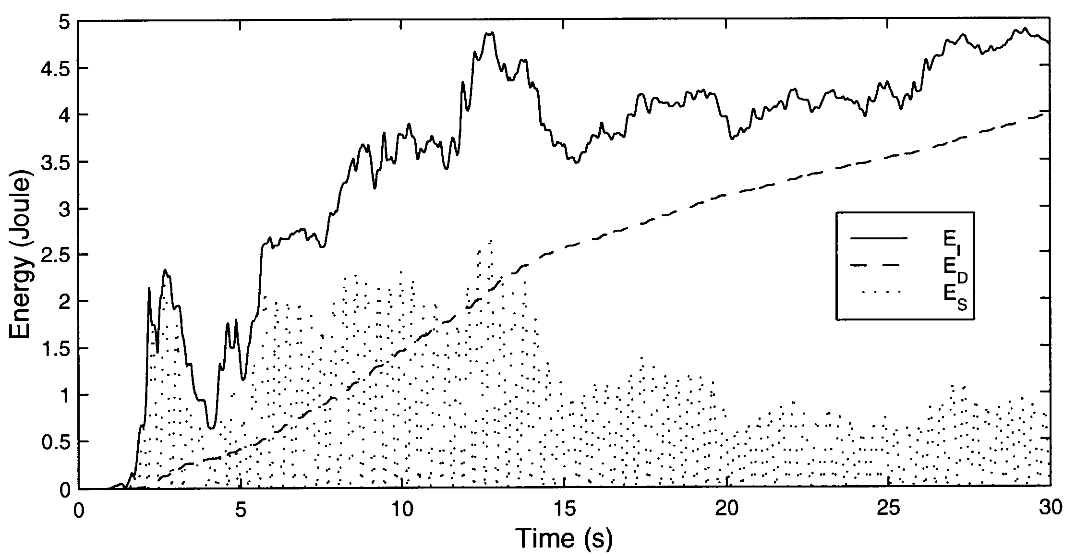

Figure 4-19: Building 3 Energy History (1st mode), El Centro (S00E), Imperial Valley 1940.[3]

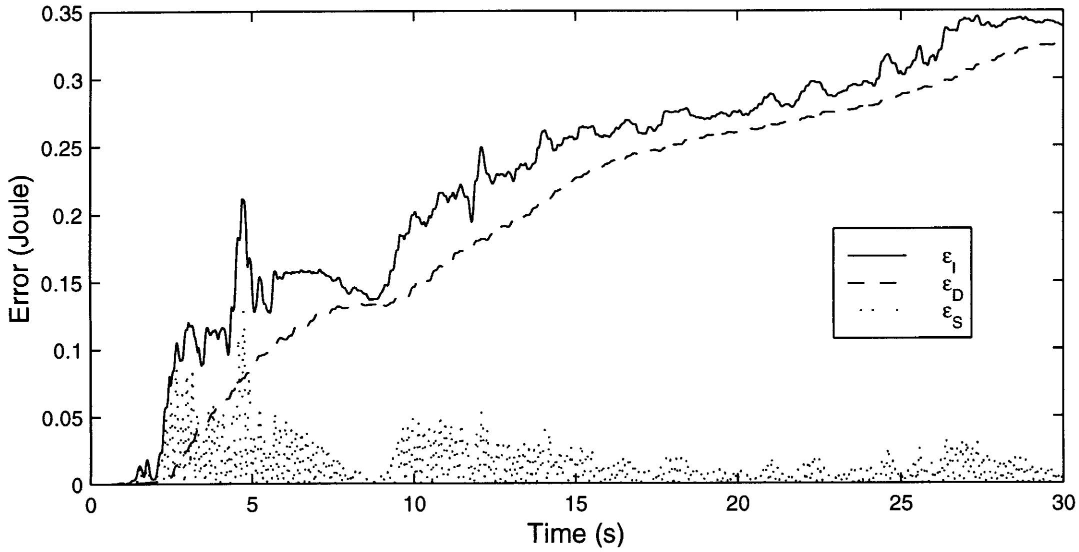

Figure 4-20: Building 3, 1st mode Energy Error, El Centro (S00E), Imperial Valley 1940.[3]

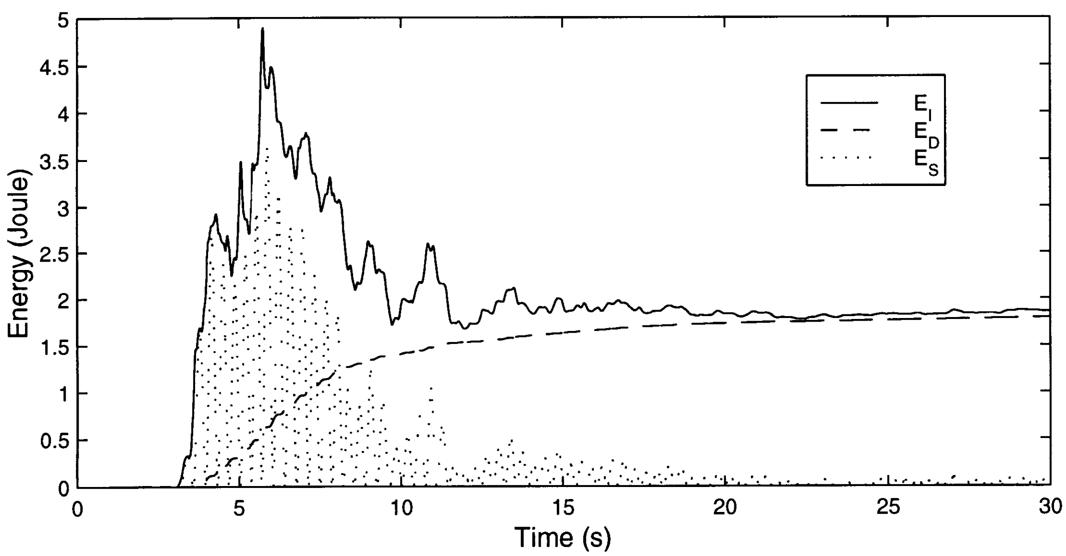

Figure 4-21: Building 3 Energy History (1st mode), Arleta Station (90 DEG), Northridge 1994.[3]

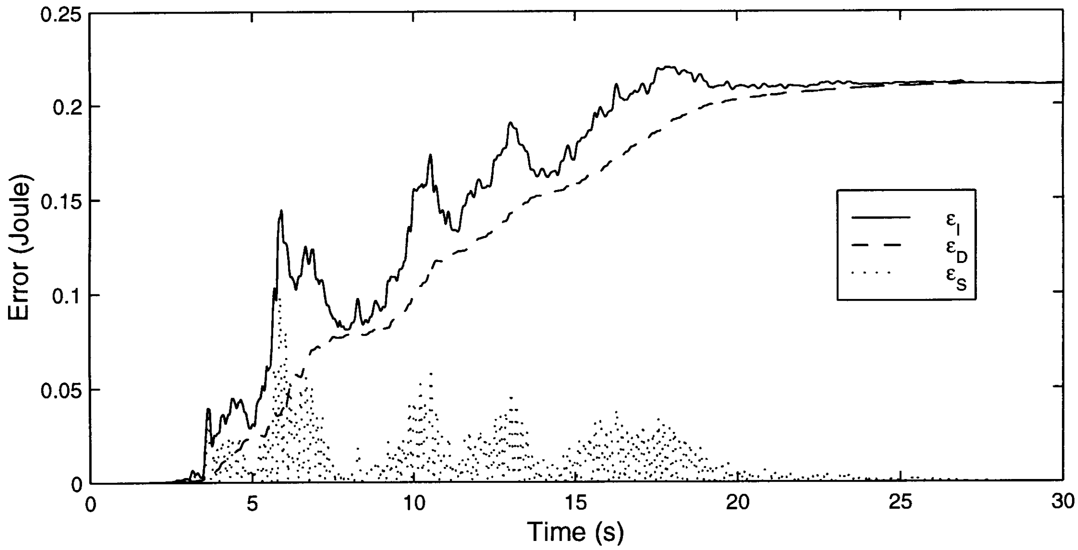

Figure 4-22: Building 3, 1st mode Energy Error, Arleta Station (90 DEG), Northridge 1994.[3]

These figures have to be compared with the figures in chapter 3 in order to see the difference and similarities between them. We see clearly that the response of the first mode is the dominant one. The energy input on the first mode represents more than 80% of the energy input to the structure, and in most cases the contribution of the fundamental mode is greater than 90%. We also realize that even though all the energy quantities are important, we usually are more concerned about the strain energy. Since this is the quantity that we need to monitor closely in order to avoid yielding of the structure.

When we look at the strain energy, we again tend to see that all of the systems have a first mode dominant response to the loads. We had said in chapter 3 that the reason for the difference between the response histories of Buildings 2 and 3 was that the response of Building 3 was dominated by the first mode, where we had less damping than in Building 2. In fact if we compare the magnitude of the errors in the strain energy for these two buildings, we see that they are reasonably close to each other, even though the maximum demand on the strain energy was substantially higher for Building 3 than Building 2. So we can say, even though we are not using an

[3]See table 4.1

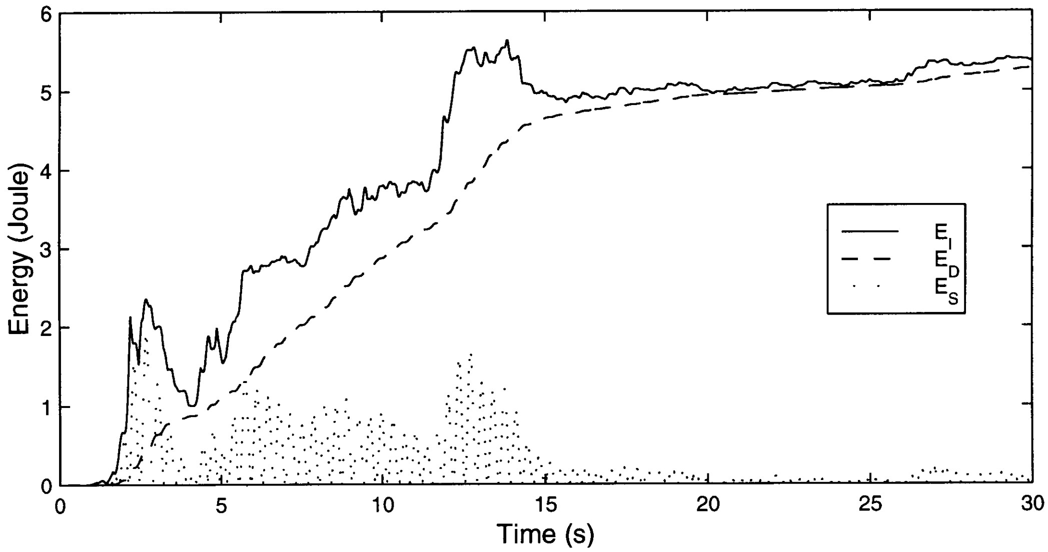

Figure 4-23: Building 2 Energy History (1st 2 modes), El Centro (S00E), Imperial Valley 1940.[4]

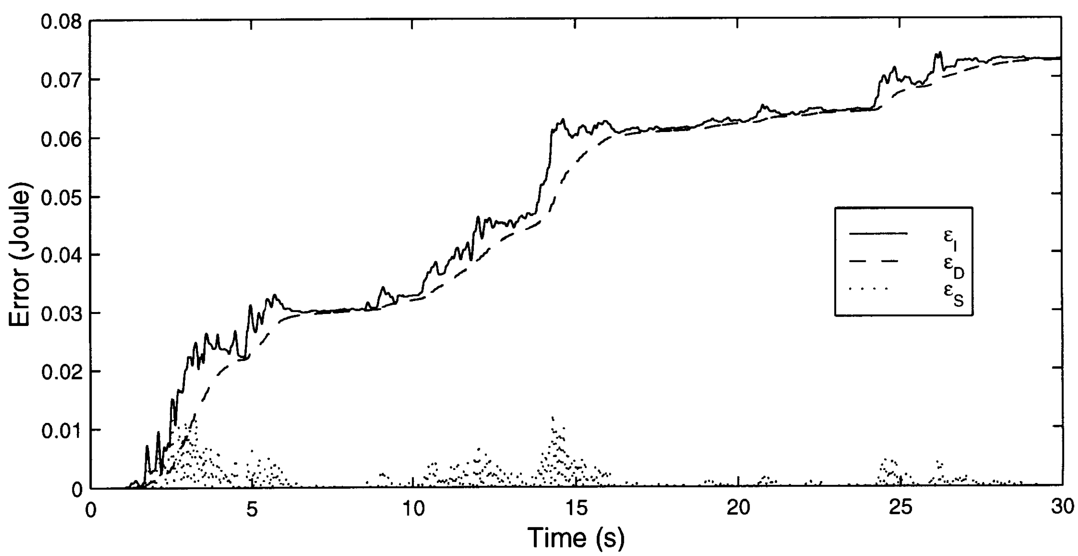

Figure 4-24: Building 2, 1st 2 modes Energy Error, El Centro (S00E), Imperial Valley 1940.[4]

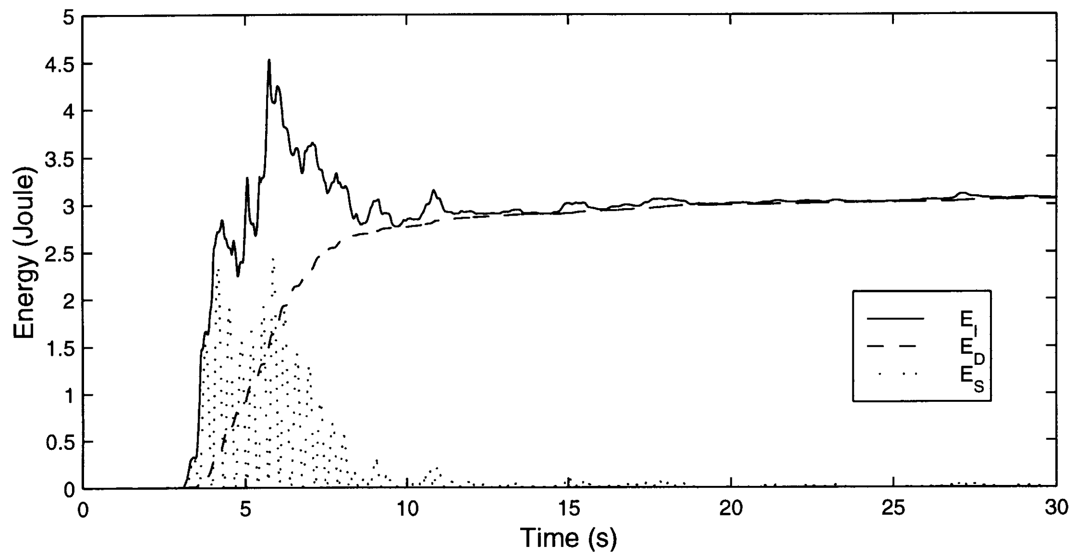

Figure 4-25: Building 2 Energy History (1st 2 modes), Arleta Station (90 DEG), Northridge 1994.[4]

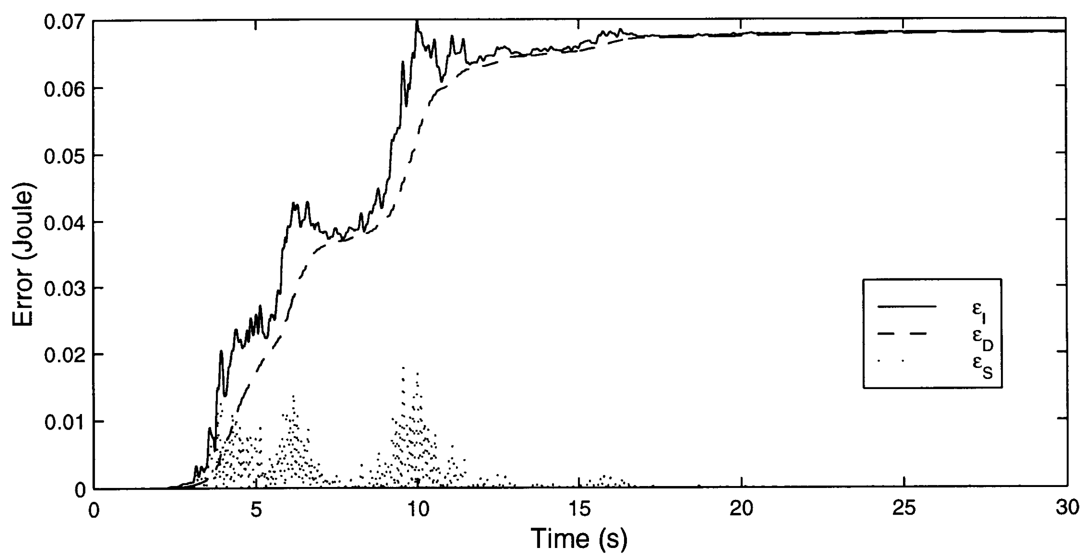

Figure 4-26: Building 2, 1st 2 modes Energy Error, Arleta Station (90 DEG), Northridge 1994.[4]

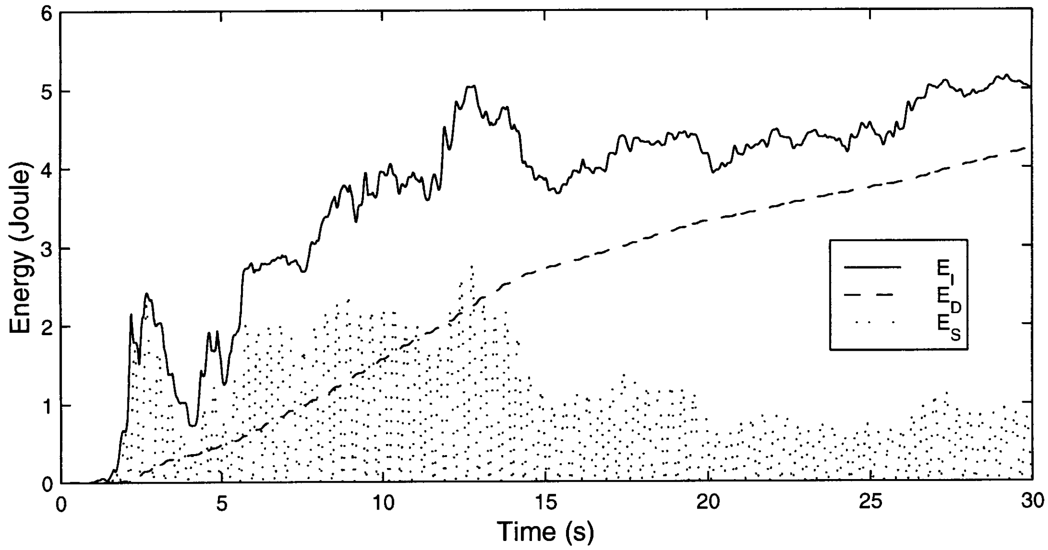

Figure 4-27: Building 3 Energy History (1st 2 modes), El Centro (S00E), Imperial Valley 1940.[4]

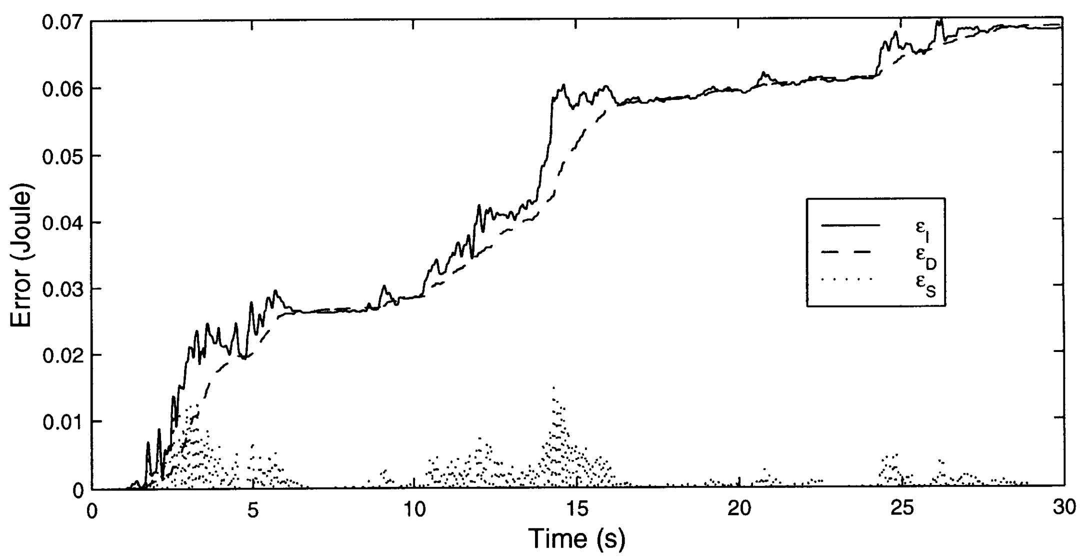

Figure 4-28: Building 3, 1st 2 modes Energy Error, El Centro (S00E), Imperial Valley 1940.[4]

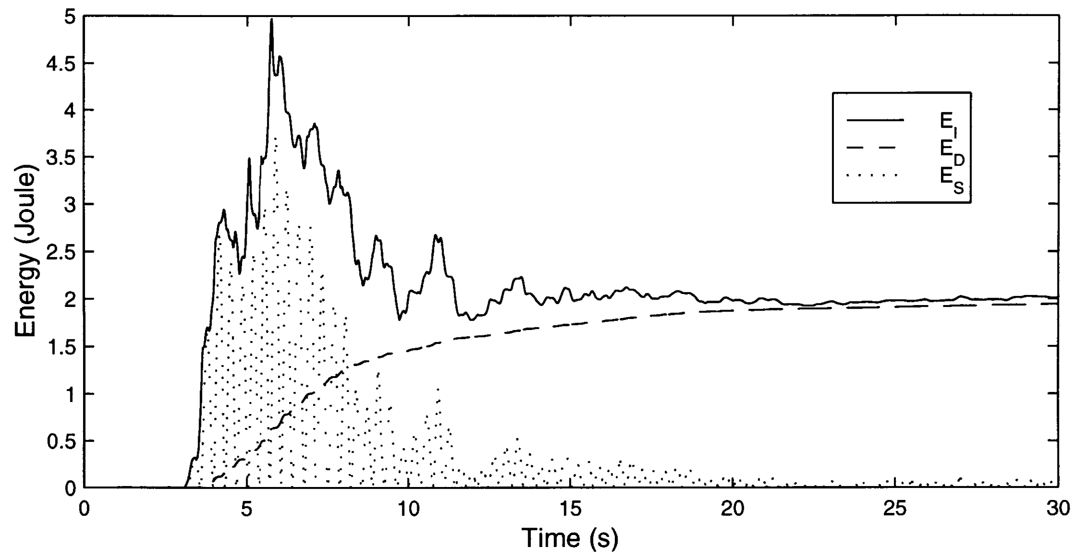

Figure 4-29: Building 3 Energy History (1st 2 modes), Arleta Station (90 DEG), Northridge 1994.[4]

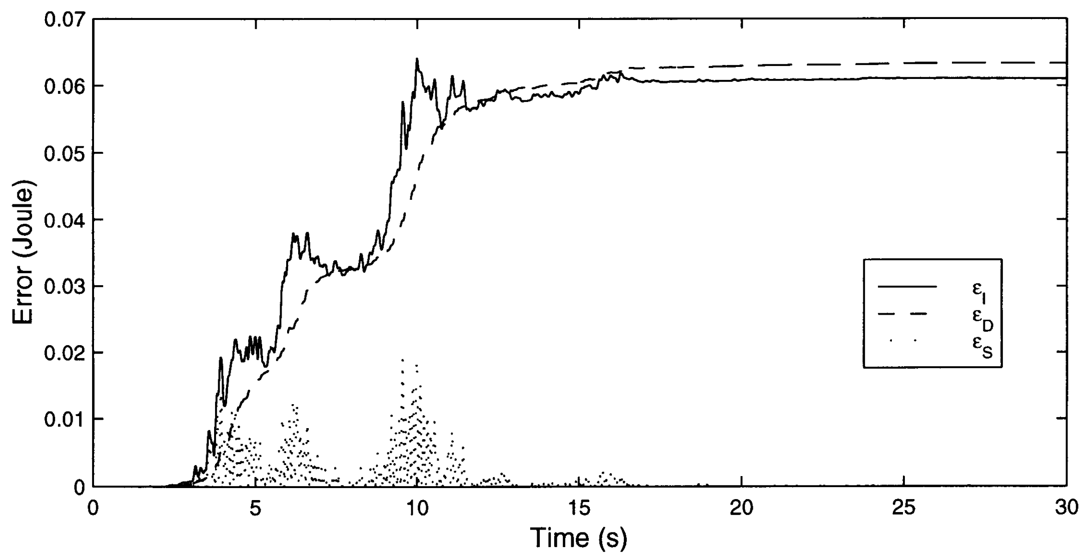

Figure 4-30: Building 3, 1st 2 modes Energy Error, Arleta Station (90 DEG), Northridge 1994.[4]

exact measure to define this, that Building 3 is more dominated by the fundamental mode than Building 2. This is due to the difference in the damping properties of the two buildings. The undamped periods and mode shapes of the two buildings are the same since they have the same mass and stiffness distribution.

Figures 4-23 thru 4-30 show the same type of plots as before for Building 2 and Building 3, but this time we include the first two modes in the response. The error is now very small and practically it is not significant. We realize then that the other modes need not be included in the response to obtain accurate results.

4.4 Conclusions

We have shown that looking at the equations of the dynamic system in state space has physical and conceptual advantages. It is more convenient to use the state space formulation since this allows us to look at the frequencies in the complex domain. The advantage of having the frequencies in a complex plot is that we get a better sense of the amount of damping we have in the structure. This also allows us to compare systems with different damping properties without having to calculate the response of the systems to a given load. Thus it can be used as a preliminary comparison between different damping configurations that we think will be good to explore further.

Another major advantage is that the state space formulation allows for the use of an arbitrary damping matrix when doing mode superposition. This means that we have no need of using a damping matrix that can be diagonalized by the eigenvectors from the linear eigenvalue problem, which leads to a damping matrix that has no physical meaning. In fact we can now look at any damping configuration that models our physical system very closely without having to worry about the mathematical techniques used in the analysis. This is a significant advancement since now we can concentrate in putting any type of damping mechanism or several in the structure and then monitor how the energy builds up so that we can reduce the elastic demand on the system.

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[4]See table 4.1

Conceptually the state space formulation brings us further into understanding damped forced dynamic response, since it allows us to explore more the role of damping in the system. Also this method is more elegant, there are no fuzzy assumptions. The trade off is that it is in a sense more complicated than the classical approach. Another disadvantage is that numerically it makes the problem twice as large. We have not explored the numerical advantages and disadvantages of the algorithms here, but remember that for the non symmetric matrix the eigenvalues and eigenvectors appear in conjugate pairs. This fact can be used to reduce the numerical effort introduce by having a larger system. Also, as in any case in which we do mode superposition we do not need to compute the whole eigenvalue problem, but only those frequencies and mode shapes that we are interested in. In terms of accuracy we see that the method works well. The only difficulty we had was that the time step had to be somewhat reduced in order for the energy calculations to be accurate. This problem is not related to the mode superposition analysis but to the integration schemes used to obtain E_D and E_I. We had the same problem when doing the analysis in the time domain in chapter 3.

Here we have looked only at a very small system subjected to two different loads. Further research should look at large scale systems to investigate the advantages and disadvantages of the numerical procedures. Also another interesting thing to do is to take a smaller system and to work out an analytic solution, to some simple and well known loads. Then there is also the issue of providing more damping in the structure, and how to distribute it optimally.

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