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POWER GAIN IN FEEDBACK AMPLIFIERS

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S. J. Mason

Abstract

A linear transistor model (or other linear two-terminal-pair device) is imbedded in a lossless passive network N and the properties of the complete system, as measured at two specified terminal pairs, are described by the open-circuit impedances Z_{11}, Z_{12}, Z_{21}, Z_{22}. The quantity

$$U = \frac{|Z_{21} - Z_{12}|^2}{4(R_{11}R_{22} - R_{12}R_{21})}$$

where R_{jk} is the real part of Z_{jk}, is defined as the unilateral gain of the transistor. Quantity U is independent of the choice of N and is (consequently) invariant under permutations of the three transistor terminals and also under replacement of the open-circuit impedances by short-circuit admittances. If U exceeds unity at a specified frequency, then N can always be chosen to make R_{11} and R_{22} positive and Z_{12} zero at that frequency. Quantity U is identifiable as the available power gain of the resulting unilateral structure.

An arbitrary coupling network may be decomposed into a portion that accomplishes unilateralization and a remaining complementary portion that provides feedback around the unilateralized structure. Such decomposition brings some of the methods of elementary feedback theory to bear upon nonunilateral circuit problems and offers a viewpoint from which signal flow and power flow can be simply related.

POWER GAIN IN FEEDBACK AMPLIFIERS

INTRODUCTION

In the analysis and design of transistor circuits, certain academic problems suddenly become more interesting to engineers. A vacuum tube, very often represented as a simple transconductance driving a passive impedance, may lead to relatively simple amplifier designs in which the input impedance (and hence the power gain) is effectively infinite, the voltage gain is the quantity of interest, and the input circuit is isolated from the load. The transistor, however, usually cannot be characterized so easily. The linear transistor model is a two-port (two-terminal-pair) circuit containing four significant elements. The simplest transistor amplifier designs yield an input impedance dependent upon the load. The impedance levels are such that the power gain becomes an important consideration.

The study of passive filter theory has led to classical design procedures by which the power loss through a given filter can be minimized. Unfortunately, these methods are not generally applicable to the problem of power gain maximization in an active filter, since the question of stability arises. Single-loop feedback theory lends design methods in which the stability is conveniently controlled, but the active device treated is unilateral, the feedback is external to the device and separately adjustable, and voltage gain rather than power gain is the quantity related to stability.

A viewpoint is needed from which the classical filter theory and the single-loop feedback theory can join forces in a more useful manner. This paper offers a possible viewpoint, answers some questions, and, of course, raises other questions.

STATEMENT OF THE PROBLEM

Figure 1 is a schematic of an arbitrary linear two-port amplifier Z, in general not unilateral, connected through a lossless reciprocal (nongyratory) coupling network N to a source and load situated at the exterior ports b1 and b2. For convenience each port-pair is shown with a common terminal. The problem is to adjust N for maximum source-to-load power gain, subject to some allowable margin of stability against oscillations. We assume here that the amplifier is capable of a power gain greater than unity, so that the coupling through N, between ports a1 and a2, may possibly result in unstable feedback. At the verge of instability, of course, the available source-to-load power gain of the complete system approaches infinity.

IMPEDANCE TRANSFORMATIONS

The amplifier Z is characterized by the pair of linear equations

$$V_1 = Z_{11} I_1 + Z_{12} I_2 \qquad (1)$$

$$V_2 = Z_{22} I_1 + Z_{22} I_2 \qquad (2)$$

in which, for simplicity, the port-pair designation a has been omitted. In matrix notation these equations become

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \tag{3}$$

or, in the more abbreviated form,

$$V_a = Z_{aa} I_a \tag{4}$$

Similarly, the properties of the complete system, measured at port-pair b, are described by the matrix equation

$$V_b = Z_{bb} I_b \tag{5}$$

Retaining the abbreviated matrix notation, we write the four equations of the reactive coupling network N as

$$V_a = - jN_{aa} I_a + jN_{ab} I_b \tag{6}$$

$$V_b = - jN_{ba} I_a + jN_{bb} I_b \tag{7}$$

It follows from (4) through (7) that the transformation of Z_{aa} into Z_{bb} is represented by the operations

$$(Z_{bb} - jN_{bb}) = N_{ba}(Z_{aa} + jN_{aa})^{-1} N_{ab} \tag{8}$$

Since the coupling network is reciprocal, matrix N is symmetric and (8) has the general form

$$Z' - jx' = n(Z + jx)^{-1} n_t \tag{9}$$

where x and x' are arbitrary real symmetric matrices, n is an arbitrary real matrix, and subscript t denotes transposition.

The impedance transformation does not always involve inversion of Z. Consider the addition of a second coupling network which carries Z' into Z",

$$Z'' - jy'' = n'(Z' + jy')^{-1} n'_t \tag{10}$$

Choosing

$$x' + y' = 0 \tag{11}$$

we find

$$Z'' - jy'' = n'n_t^{-1}(Z + jx)n^{-1} n'_t \tag{12}$$

which has the general form

$$Z' - jx' = n(Z + jx) n_t \tag{13}$$

Hence lossless reciprocal coupling can accomplish any desired sequence of one or more operations chosen from the set

$$Z' = Z + jx \qquad (x_t = x) \tag{14}$$

$$Z' = Z^{-1} \tag{15}$$

$$Z' = nZn_t \tag{16}$$

where x and n are real.

Physical realizations of the three basic coupling operations are shown in Fig. 2. These are representative but not unique. Real transformation, for example, is also obtainable (at a specified frequency) in the form of properly tuned reactance networks. Tuning is suggested by (11), in which x' and y' represent facing reactance matrices in two adjacent coupling networks.

Observe that real transformation includes permutations of the amplifier terminals. For example, consider the n matrix

$$n = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \tag{17}$$

which yields the coupling shown in Fig. 3. Any other desired permutation of terminals is evidently obtainable in a similar manner.

THE UNILATERAL GAIN

Lossless reciprocal coupling has the property that the parameter

$$U = \frac{|Z_{21} - Z_{12}|^2}{4(R_{11}R_{22} - R_{12}R_{21})} \tag{18}$$

which we shall call the unilateral gain, is invariant. In other words,

$$U(Z') = U(Z) \tag{19}$$

The invariance of U under (14) is apparent, but (15) and (16) require closer examination. For a proof of invariance under inversion (15), first write

$$Z\,Z' = (R + jX)\,(R' + jX') = 1 \tag{20}$$

The imaginary part of (20) is

$$RX' + XR' = 0 \tag{21}$$

Hence

$$R(R' + jX') = (R - jX)R' \tag{22}$$

and

$$R = \bar{Z}\,R'\,Z \tag{23}$$

where the bar denotes the complex conjugate.

The determinant of a product of matrices is the product of their determinants, so that

$$R_{11}R_{22} - R_{12}R_{21} = |Z_{11}Z_{22} - Z_{12}Z_{21}|^2\,(R'_{11}R'_{22} - R'_{12}R'_{21}) \tag{24}$$

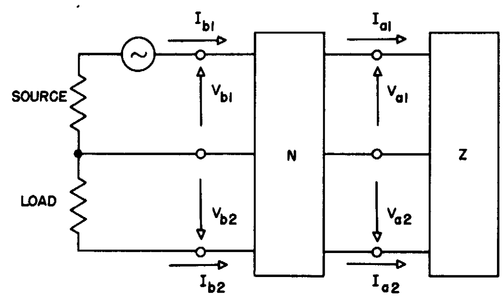

Fig. 1

A two-port amplifier coupled to a source and load.

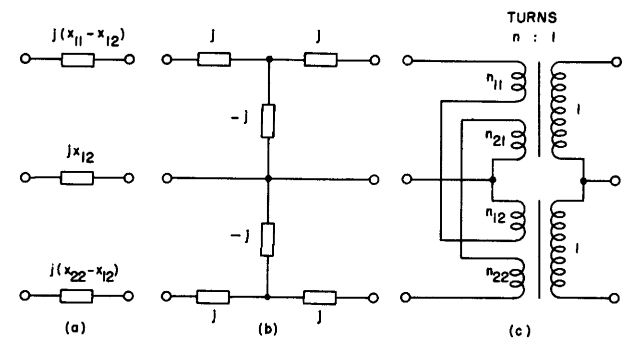

Fig. 2

Realization of elementary couplings: (a) reactance padding, (b) inversion, (c) real transformation.

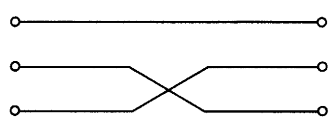

Fig. 3

Permutative coupling.

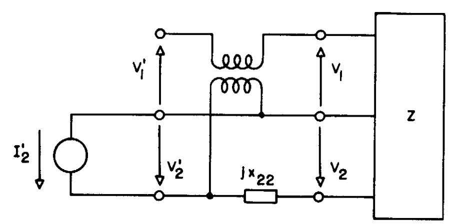

Fig. 4

A unilateralization scheme.

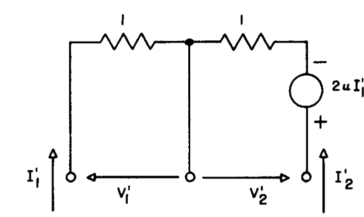

Fig. 5

The normalized unilateral amplifier.

Also write the identity

$$Z - Z_t = Z_t(Z_t' - Z')\, Z \tag{25}$$

which stems from (20). Since $Z - Z_t$ is antisymmetric, the determinant of (25) is

$$(Z_{21} - Z_{12})^2 = (Z_{11}Z_{22} - Z_{12}Z_{21})^2\, (Z_{21}' - Z_{12}')^2 \tag{26}$$

and it follows from (24) and (26) that

$$U(Z^{-1}) = U(Z) \tag{27}$$

Now, to demonstrate invariance under a real transformation, consider the real part of (16),

$$R' = n\, R\, n_t \tag{28}$$

whose determinant is

$$R_{11}'R_{22}' - R_{12}'R_{21}' = (n_{11}n_{22} - n_{12}n_{21})^2\, (R_{11}R_{22} - R_{12}R_{21}) \tag{29}$$

Also transpose (16) and subtract to obtain

$$Z' - Z_t' = n(Z - Z_t)\, n_t \tag{30}$$

which has the determinant

$$(Z_{21}' - Z_{12}')^2 = (n_{11}n_{22} - n_{12}n_{21})^2\, (Z_{21} - Z_{12})^2 \tag{31}$$

Relations (29) and (31) show that

$$U(n\, Z\, n_t) = U(Z) \tag{32}$$

and the proof is complete.

THE UNILATERAL AMPLIFIER

A lossless reciprocal coupling N can always be chosen to make Z' unilateral, that is, $Z_{12}' = 0$. Figure 4 shows one possible scheme which accomplishes this result for any given Z at a specified frequency. A reactance x_{22} is first added to bring V_2' into phase coincidence (or phase opposition) with V_1, after which the ideal transformer is adjusted to cancel V_1 and give zero voltage at V_1'. Since U is invariant and Z_{12}' vanishes, we have

$$U = \frac{|Z_{21}'|^2}{4\, R_{11}'\, R_{22}'} = \frac{|Z_{21} - Z_{12}|^2}{4(R_{11}R_{22} - R_{12}R_{22})} \tag{33}$$

Also by inspection of Fig. 4,

$$R_{22}' = R_{22} \tag{34}$$

It follows from (33) and (34) that for positive U, a unilateralizing network N can always be chosen to give R_{11}' and R_{22}' the same sign as R_{22}.

Several cases need to be considered.

Case 1. $U > 0$, $R_{11} > 0$, $R_{22} > 0$

The input and output resistances, R'_{11} and R'_{22}, are positive and U is identifiable as the available power gain of the unilateral amplifier. By a minor modification, allowing values of n_{11} and n_{22} other than unity and nonzero values of x'_{11} and x'_{22}, it is possible to obtain the normalized impedance

$$Z' = \begin{pmatrix} 1 & 0 \\ 2u & 1 \end{pmatrix} \tag{35}$$

where

$$u = \frac{Z_{21} - Z_{12}}{2(R_{11}R_{22} - R_{12}R_{21})^{1/2}} \tag{36}$$

Case 2. $U > 0$, $R_{11} < 0$, $R_{22} > 0$

The results are the same as in Case 1, since R'_{11} and R'_{22} are both positive

Case 3. $U > 0$, $R_{11} > 0$, $R_{22} < 0$

By permuting the ports of Z we re-enter Case 2.

Case 4. $U > 1$, $R_{11} < 0$, $R_{22} < 0$

Here R'_{11} and R'_{22} are both negative, so that a new problem arises. First turn to Fig. 5, which pertains to Case 1. If U exceeds unity for the circuit of Fig. 5, then transfer power gain is available and the circuit can certainly be made to oscillate by coupling the two ports together through a properly chosen lossless reciprocal feedback network. A negative resistance R_o must appear at some port of the oscillatory system. Now suppose that the algebraic signs of all impedances in the system are changed. At a specified frequency such a change does not destroy the lossless reciprocal nature of the coupling network. The input and output resistances of the unilateral amplifier both become negative and R_o becomes positive.

Hence the unilateral Z' of Case 4 can be transformed into a new Z, in general not unilateral, whose R_{22} plays the role of R_o and therefore is positive. Thus Case 1 or 2 is safely reached, with the additional restriction that U must exceed unity rather than zero.

Figure 6 offers a simple example. The original amplifier (a) has negative input and output resistances and a U greater than one. The addition of the lossless coupling shown in (b) transforms the original circuit into a unilateral amplifier (c) whose input and output resistances are positive.

Case 5. $0 < U < 1$, $R_{11} < 0$, $R_{22} < 0$

Unilateralization with positive R'_{11} and R'_{22} is not possible by the schemes considered here, since the argument of Case 4 breaks down. Such an amplifier would be of little use, since it would have a power gain less than unity.

Case 6. $U < 0$

The product $R'_{11}R'_{22}$ is necessarily negative.

As a consequence of the various case considerations, we have the theorem:

> At any specified frequency, a device Z whose U is greater than unity can be transformed, by means of lossless reciprocal coupling, into a unilateral amplifier having positive input and output resistances, and the available power gain of the unilateral structure is always equal to U.

FEEDBACK COUPLING

Having reduced the original amplifier to a normalized, unilateral, positive-definite form, let us now consider the effect of an additional lossless coupling network S connected as shown in Fig. 7. It is convenient to visualize the four S-ports as transmission lines of unity characteristic resistance. On this basis the incident waves $\underline{W}$ and reflected waves $\underline{W}'$ are related by the scattering matrix $\underline{S}$.

$$\underline{W}' = \underline{S}\,\underline{W} \tag{37}$$

In the present problem, however, only the waves W_1, W_3, W_2', and W_4' are of direct interest. Since W_2 and W_4 must vanish we may extract the sub-matrix equation

$$\begin{pmatrix} W_2' \\ W_4' \end{pmatrix} = \begin{pmatrix} S_{21} & S_{23} \\ S_{41} & S_{43} \end{pmatrix} \begin{pmatrix} W_1 \\ W_3 \end{pmatrix} \tag{38}$$

which will be denoted by the letters

$$W' = SW \tag{39}$$

Waves W_1' and W_3' represent wasted power which might better be directed into the load. For greatest efficiency let W_1' and W_3' vanish, so that W and W' carry the same power,

$$|W_2'|^2 + |W_4'|^2 = |W_1|^2 + |W_3|^2 \tag{40}$$

In matrix form, (40) becomes

$$W_t'\overline{W}' = W_t\overline{W} \tag{41}$$

From (39) we find

$$W_t'\,\overline{W}' = W_t\,S_t\,\overline{S}\,\overline{W} \tag{42}$$

and in order to satisfy the power conservation condition represented by (41), the S matrix must be unitary,

$$\overline{S}_t = S^{-1} \tag{43}$$

It follows that the determinant of S has unity magnitude and therefore may be written as

$$S_{21}S_{43} - S_{23}S_{41} = \exp(j\theta) \tag{44}$$

Hence, from (43) and (44),

$$\begin{pmatrix} S_{21} & S_{23} \\ S_{41} & S_{43} \end{pmatrix} = \begin{pmatrix} \overline{S}_{43} & -\overline{S}_{41} \\ -\overline{S}_{23} & \overline{S}_{21} \end{pmatrix} \exp(j\theta) \tag{45}$$

We shall need these results shortly.

The wave amplification of the normalized unilateral amplifier is equal to u, that is,

$$W_3 = u\, W_2' \tag{46}$$

Equations (38) and (46) may be interpreted as the signal-flow graph of Fig. 8. By inspection of the graph, the loop transmission around u is

$$T = S_{23}\, u \tag{47}$$

and the source-to-load transmission is given by

$$g = \frac{W_4'}{W_1} = S_{41} + \frac{S_{21}\, u\, S_{43}}{1 - S_{23}\, u} \tag{48}$$

With the aid of (44) and (45) this may be recast in the form

$$g = \left(\frac{u - \bar{S}_{23}}{1 - S_{23}\, u}\right) \exp(j\theta) \tag{49}$$

Elimination of S_{23} between (47) and (49) yields

$$g = \left(\frac{1 - \dfrac{\bar{T}}{|u|^2}}{1 - T}\right) u \exp(j\theta) \tag{50}$$

POWER GAIN AND STABILITY MARGIN

It follows from the definitions of u and U that

$$|u|^2 = U \tag{51}$$

and since the source and load waves have a common characteristic-resistance level, the source-to-load power gain is given by

$$G = |g|^2 = \left|\frac{1 - \dfrac{T}{U}}{1 - T}\right|^2 U \tag{52}$$

Passivity of the scattering network insures that

$$|S_{23}| \le 1 \tag{53}$$

so that

$$|T|^2 \le U \tag{54}$$

For a high-gain amplifier, in which

$$U >> 1 \tag{55}$$

we have, therefore,

$$\left|\frac{T}{U}\right| << 1 \tag{56}$$

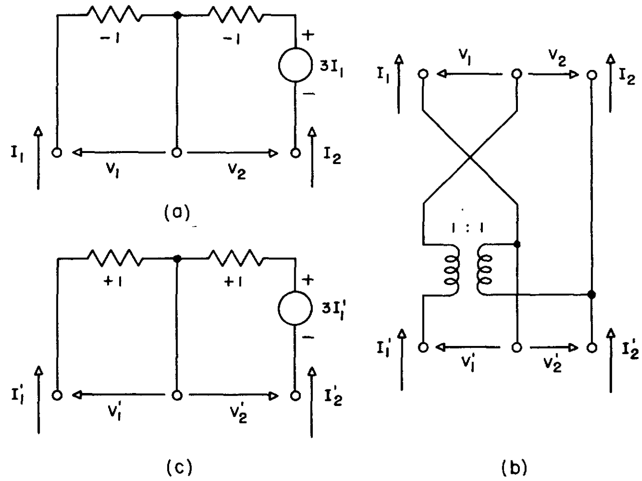

Fig. 6
Elimination of negative terminal resistance.

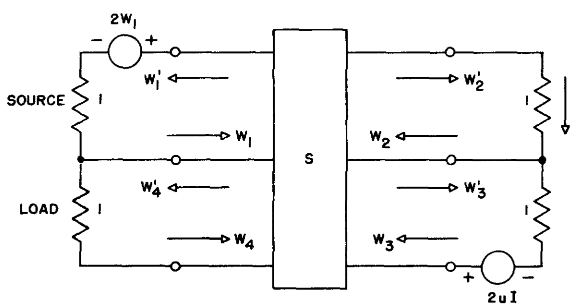

Fig. 7
Feedback coupling.

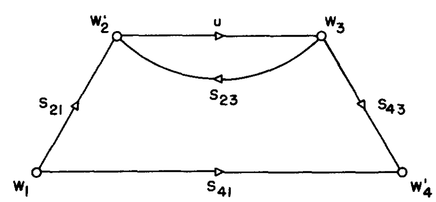

Fig. 8
The signal-flow graph for the unilateral amplifier with feedback.

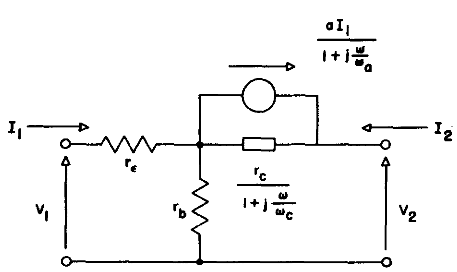

Fig. 9
An amplifier.

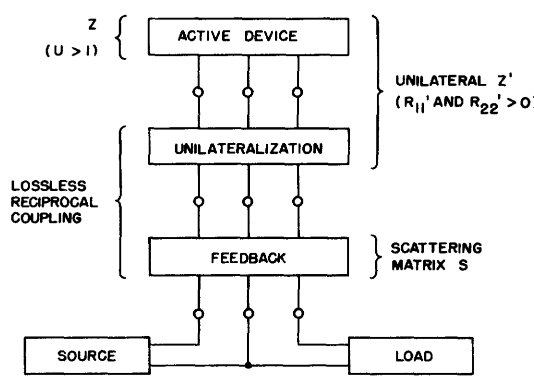

Fig. 10
Decomposition of an arbitrary coupling.

and

$$G \approx \frac{U}{|1 - T|^2} \tag{57}$$

Hence the approximate power gain is given by the decibel sum of the unilateral gain and the feedback,

$$G_{db} \approx U_{db} + F_{db} \tag{58}$$

where the decibel feedback is defined as

$$F_{db} = 10 \log_{10} \left| \frac{1}{1 - T} \right|^2 \tag{59}$$

The maximum permissible gain is limited by stability considerations. A margin of stability may be imposed in the usual way by choosing some convenient or desirable region of the complex T-plane which contains the critical point (+1) and then excluding the complex locus of $T(j\omega)$ from that region.

AN ILLUSTRATIVE UNILATERAL GAIN FUNCTION

The linear amplifier circuit shown in Fig. 9 is sometimes taken as a model for the junction transistor. By inspection of the circuit,

$$Z = \begin{pmatrix} r_b + r_\epsilon & r_b \\ r_b + \dfrac{a r_c}{\left(1 + j\dfrac{\omega}{\omega_a}\right)\left(1 + j\dfrac{\omega}{\omega_c}\right)} & r_b + \dfrac{r_c}{1 + j\dfrac{\omega}{\omega_c}} \end{pmatrix} \tag{60}$$

Computation of the unilateral gain yields

$$U = \frac{\dfrac{a^2}{4}\left(\dfrac{r_c}{r_b}\right)\left(\dfrac{r_c}{r_\epsilon}\right)}{\left[1 + \dfrac{r_c}{r_b} + (1-a)\dfrac{r_c}{r_\epsilon}\right] + \left[1 + a\left(\dfrac{\omega_c}{\omega_a}\right)\dfrac{r_c}{r_\epsilon} + \left(\dfrac{\omega_c}{\omega_a}\right)^2 \left(1 + \dfrac{r_c}{r_b} + \dfrac{r_c}{r_\epsilon}\right)\right]\left(\dfrac{\omega}{\omega_c}\right)^2 + \left(\dfrac{\omega_c}{\omega_a}\right)^2 \left(\dfrac{\omega}{\omega_c}\right)^4} \tag{61}$$

For the element values commonly associated with a junction transistor, certain terms are negligible over the frequency range of interest, and (61) may be approximated by the expression

$$U \approx \frac{\dfrac{a^2}{4}\left(\dfrac{r_c}{r_b}\right)\left(\dfrac{r_c}{r_\epsilon}\right)}{\left[\dfrac{r_c}{r_b} + \dfrac{(1-a) r_c}{r_\epsilon}\right] + \left[\dfrac{a r_c}{\omega_a \omega_c r_\epsilon}\right]\omega^2} \tag{62}$$

Rearrangement yields

$$U \approx \frac{U_o}{1 + \left(\frac{\omega}{\omega_o}\right)^2} \tag{63}$$

wherein

$$U_o \approx \frac{a^2 r_c}{4\left[r_\epsilon + (1-a) r_b\right]} \tag{64}$$

$$\omega_o^2 \approx \left[\frac{r_\epsilon}{r_b} + (1-a)\right] \omega_a \omega_c \tag{65}$$

Hence the approximate product of power gain and squared bandwidth is

$$U_o \omega_o^2 \approx \frac{a^2 r_c \omega_a \omega_c}{4 r_b} = \frac{a^2 \omega_a}{4 r_b C_c} \tag{66}$$

where C_c is the apparent collector capacitance. The gain function (63) drops to unity at a frequency

$$\omega_1 \approx \omega_o (U_o)^{1/2} \approx \frac{a}{2} \left[\left(\frac{r_c}{r_b}\right) \omega_a \omega_c\right]^{1/2} = \frac{a}{2} \left(\frac{\omega_a}{r_b C_c}\right)^{1/2} \tag{67}$$

Above ω_1 the device is no longer a transfer power amplifier since passive coupling can never make G greater than unity unless U is larger than one.

DISCUSSION

At any specified frequency, an arbitrary coupling arrangement N can always be resolved into two components as shown in Fig. 10. The artificial separation places in evidence a unilateral circuit to which the methods of single-loop feedback amplifier design are applicable. The synthesis of an actual coupling N to approximate the desired Z' and S matrices over a range of frequencies is a problem which will not be taken up here.

Much can be said, however, without looking at the enbodiment of a design. For example, suppose that a number of active devices are to be placed in cascade, each being unilateralized with lossless reciprocal coupling in order to avoid the problem of wave reflection back through the chain. Moreover, suppose that the U of each device exceeds unity and that stability considerations happen to prohibit a negative input or output resistance in any stage. The theorem proven previously gives us the immediate answer that the design is possible and that all possible designs yield the same over-all maximum power gain, equal to the product of the unilateral gains $U_1 U_2 \ldots U_n$.

The resolution of a given coupling network N into two components, as in Fig. 10, may be either explicit or implicit. As some parameter of Z undergoes a change, the unilateralizing component may be altered accordingly so as to maintain a unilateral Z'.

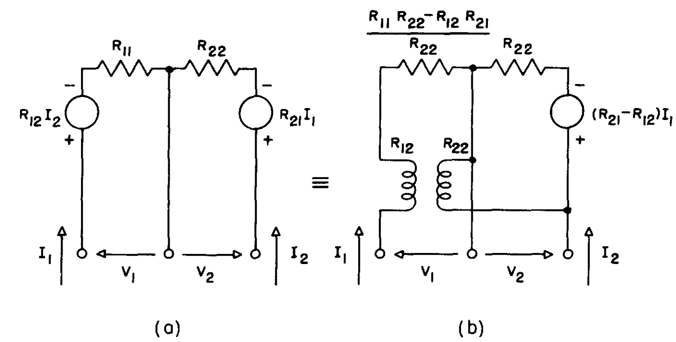

Fig. 11
Resolution of a nonunilateral circuit into a unilateral amplifier with lossless feedback.

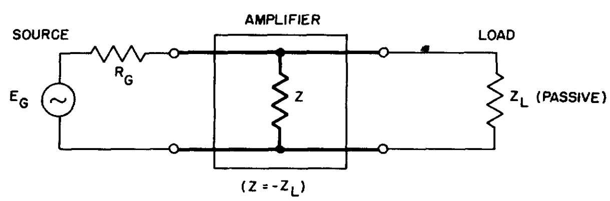

Fig. 12
An elementary negative-resistance power amplifier.

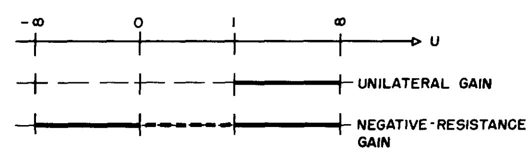

Fig. 13
Regions of unilateral and negative-resistance gain availability.

Hence, for a fixed coupling N, the complementary feedback component S becomes a function of Z. This fact must be kept in mind when the sensitivity of the source-to-sink gain g to changes in Z is considered. Any variation of Z which affects only the transfer element Z'_{21} of Z' leaves S unaffected, of course, and negative feedback discriminates against changes in g due to such variations. Figure 11 shows a simple resistive circuit (a) and an equivalent form (b) exhibiting a unilateral amplifier with lossless feedback. Notice that the turns ratio of the feedback transformer is dependent only upon the ratio of R_{12} to R_{22} and is therefore independent of R_{11} and R_{21}.

A power amplifier can always be constructed from a one-port negative resistance. Figure 12 offers a very elementary example in which the power gain approaches infinity as the source resistance R_g is allowed to become arbitrarily large. For such designs, a stability margin must be imposed upon the product of the amplifier impedance (Z) and the negative of the net admittance which it faces ($-Y_f$), in order to keep the product ($-ZY_f$) at a safe distance from the critical point (+1). Thus an active device whose U exceeds unity may be coupled to produce either unilateral positive-resistance gain or bilateral negative-resistance gain. The possibilities are summarized in Fig. 13. Solid lines show regions in which power gain is available. For values of U from zero to one a negative resistance may or may not be realizable, depending upon the signs of R_{11} and R_{22}.

The next job is to show (if possible) that U and functions of U are the only invariants of Z under lossless reciprocal transformation. Further investigation might then lead to other invariant forms which stem from other representations of the device Z. In particular, it would be of great interest to study the invariant properties of a model Z in which noise and incipient nonlinearity are represented.

ACKNOWLEDGEMENTS

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