



LINEAR PROGRAMMING FOR FINANCIAL PLANNING

UNDER UNCERTAINTY

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The purpose of this paper is to propose, justify and explain the properties of a class of linear programming (hereafter "LP") approaches to long-term, corporate financial planning under uncertainty. The models discussed are novel in the following respects.

1. They are directly based on a theory of market equilibrium under uncertainty. Thus the capabilities of the models to deal with choice among risky assets and liabilities can be rigorously justified, assuming that the firm's objective is to maximize share price. Past linear programming models have been constructed assuming certainty, and have dealt with some aspects of uncertainty through heuristic modifications.¹

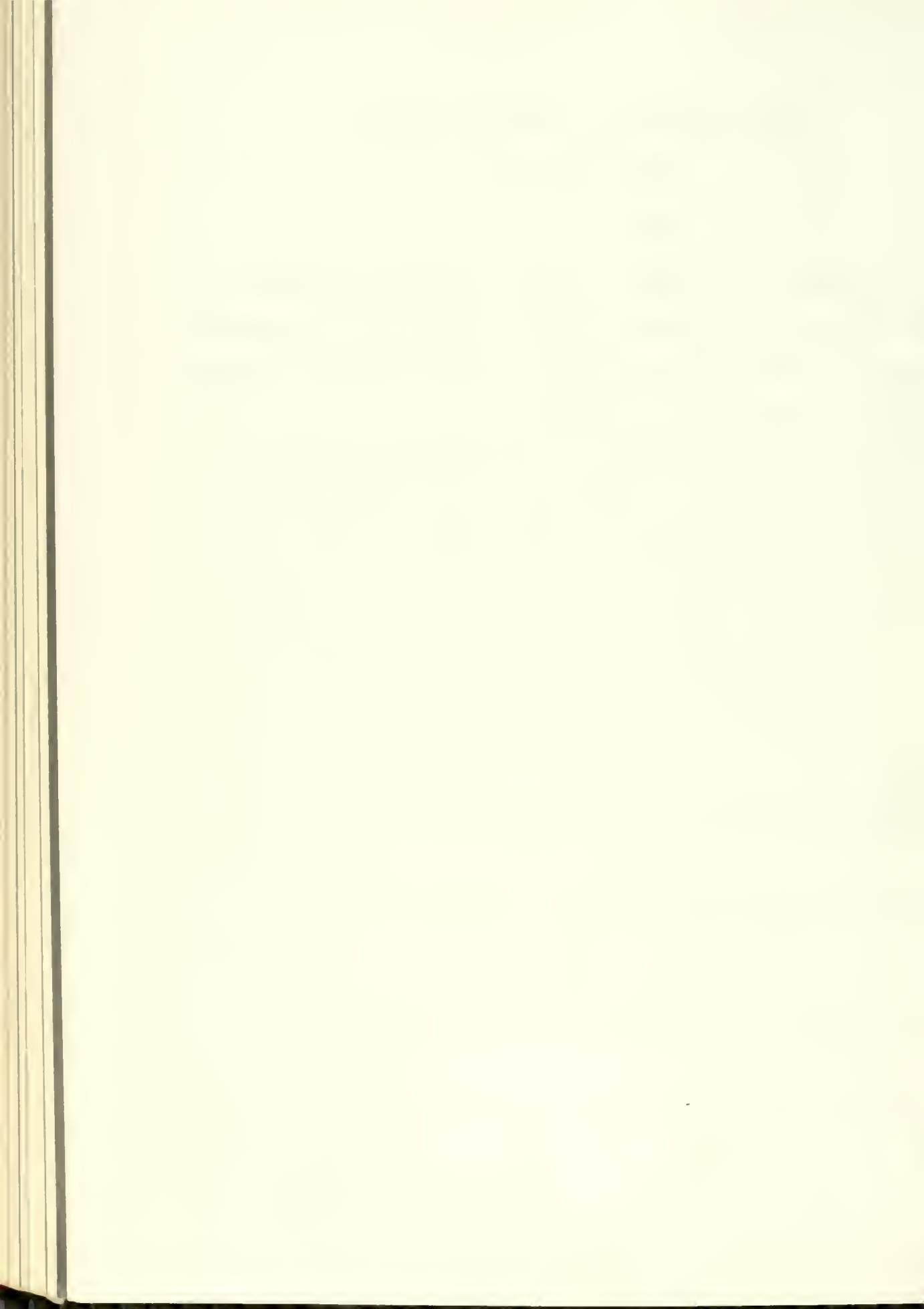
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The paper was greatly improved by comments of my colleagues at the Sloan School, particularly G. A. Pogue, with whom I am working on models for practical application.

The deficiencies of this paper are my own.

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¹Probably the most important contributions are those of Weingartner [24] and Charnes, Cooper and Miller [4]. See also Weingartner's survey article [23].



2. The models yield simultaneous solutions for the firm's optimal financing and investment decisions. The financing decision is not considered "with the investment decision given," nor vice-versa.
3. Some practical difficulties associated with the cost of capital concept are avoided. The traditional weighted average cost of capital does not appear in these LP models.

The first two characteristics should lead to some interest in the models as theory; the third, along with the ease of solution of LP problems, should generate interest in the models as practical decision-making tools. However, this paper does not include a detailed model for practical application.

The paper is organized as follows. The general linear format is explained in the next section. The key assumption justifying it is that the structure of security prices at equilibrium is best described by the class of security valuation models which imply risk-independence of financing and investment options. The following section examines a simple model in detail, and contrasts the LP approach with "traditional" approaches using the cost of capital. Practical implications of the model are discussed in the third section.

I. THE LINEAR FORMAT FOR FINANCIAL PLANNING

Introduction

We will consider the firm's financial planning problem in the following terms. The firm begins with a certain initial package of assets and

liabilities. For a brand-new firm, this may be simply money in the bank and stock outstanding. For a going concern, the package will be much more complicated. Any firm, however, has the opportunity to change the characteristics of its initial package by transactions in real or financial assets. The problem is to determine which set of transactions for the initial period will maximize the firm's stock price.

We will be concerned primarily with long-lived assets and liabilities, so the optimal transactions for the initial period will reflect the firm's opportunities and strategy in subsequent periods. Therefore, I have characterized the firm's problem as long-range financial planning, even though tomorrow's decisions do not have to be made today.

Assuming linearity, the firm's objective function is:

$$(1) \quad \Psi = \sum_{j=1}^n x_j A_j + \sum_{j=1}^m y_j F_j$$

where Ψ = change in stock price

x_j = decision variable for the j^{th} investment project -- i.e., j^{th} real asset option. $x_j = 1$ indicates that the project is accepted.

A_j = change in stock price if project j is accepted; in other words, project j 's present value.

y_j = decision variable for the j^{th} financing option. $y_j = 1$ means that one dollar of financing is obtained from the j^{th} source.

F_j = the change in stock price per dollar of financing obtained from source j .

What is implied by stating the firm's objective in this way? First, clearly, maximization of current share price must be an acceptable objective. Second, we assume that acceptance of option j leads to a definite change

in stock price. In other words, a financing or investment option with uncertain returns does not have an uncertain value; the "market's" preferences are well-defined.²

Third, we assume that the change in stock price due to accepting option j is independent of management's decisions regarding other investment or financing options. Clearly, this assumption is crucial to the argument and requires close examination.

Are the Investment Options Mutually Interdependent?

If A_j is to be independent of decision variables for other projects, then the cash flows of project j cannot be causally related to what other assets are acquired. If this is true for all projects, $1, 2, \dots, n$, then all are physically independent in the same sense G.M. and Ford shares are independent from the point of view of an investor: although these securities' returns may be statistically related, Ford's actual future prices and dividends are not affected by whether or not the investor buys G.M. stock.

Assuming physical independence means that the linear format cannot deal directly with an important class of capital budgeting problems. Suppose, for example, that investment options 1 and 2 are, respectively, a fleet of new trucks and a computer. If the trucks are purchased, then purchase of the computer will allow management to schedule usage of the trucks more efficiently. For this reason, the change in stock price if both projects 1 and 2 are accepted is greater than the sum of their present values

²This assumption is innocuous, but worth stating because of the common assumption that present value should be regarded as a random variable under uncertainty.

separately considered. An interaction effect exists.

Writing the objective function as Eq. (1) also assumes that projects are risk-independent, in the sense that there are no statistical relationships among projects' returns such that some combinations of projects affect stock price by an amount different than the sum of their present values considered separately. In particular, risk-independence implies that there is no advantage to be gained by corporate diversification.³

I have shown elsewhere [13] that risk-independence is a necessary condition for equilibrium in security markets. Naturally, the proof rests on certain assumptions about the markets -- in particular that, equilibrium security prices conform to the time-state-preference model of security valuation, advanced by me in still another paper [14]. Although there is some controversy, the Sharpe-Lintner model of security valuation under uncertainty also implies risk independence. See [20], [9]. The proof is given in Appendix A.⁴

The meaning of risk-independence may be illustrated by considering a bundle of risky assets, denoted by A, and two additional "projects" B and C. (The projects represent given streams of uncertain, incremental cash flows. Physical independence is assumed. However, no restriction is placed on the distribution of cash flow over time, or on the projects' characteristics.) Risk-independence requires that

³"Diversification" as used here simply means "pooling of risks" in the context of portfolio selection.

⁴Lintner, however, has argued that investment projects are not risk-independent in his model -- specifically, that "the problem of determining the best capital budget of any given size is formally identical to the solution of a security portfolio analysis." [7], p. 65.



$$(2) \quad P_{AB} - P_A = P_{ABC} - P_{AC},$$

where P_A is the price per share if neither B or C is accepted, P_{AB} the price if only B is accepted, and so on. My proof⁵ of Eq. (2) establishes that the change in stock price if B is adopted is independent of whether C is also adopted, and therefore that the projects are risk-independent.

Whether risk-independence is a property of actual security markets is a question that cannot be answered here, although it seems reasonable to expect at very least a tendency toward this result. In any case, the implications of risk-independence are worth considering. Therefore, we shall assume it to exist for purposes of this paper.

Are Financing Options Mutually Independent?

If no restrictions are placed on the risk characteristics or pattern over time of B or C's cash flows, then Eq. (2) applies as well to financial assets as to real ones. Project B can be regarded as a bond issue, and C as a stock issue, without changing the proof in the slightest. Having assumed risk-independence among real assets, it is no further step to assume that financial assets are likewise risk-independent.

Physical independence among financial assets may seem to be another matter. It is commonplace that the interest and principal payments on bonds are affected by the size of the firm's equity base. A highly levered firm may encounter difficulties servicing its debt, and creditors will demand a higher promised yield in compensation. Conversely, returns to equity depend on commitments to creditors. Therefore "debt," if it is

⁵See Myers [13].

regarded as a single financing option, is not physically independent of equity issues.

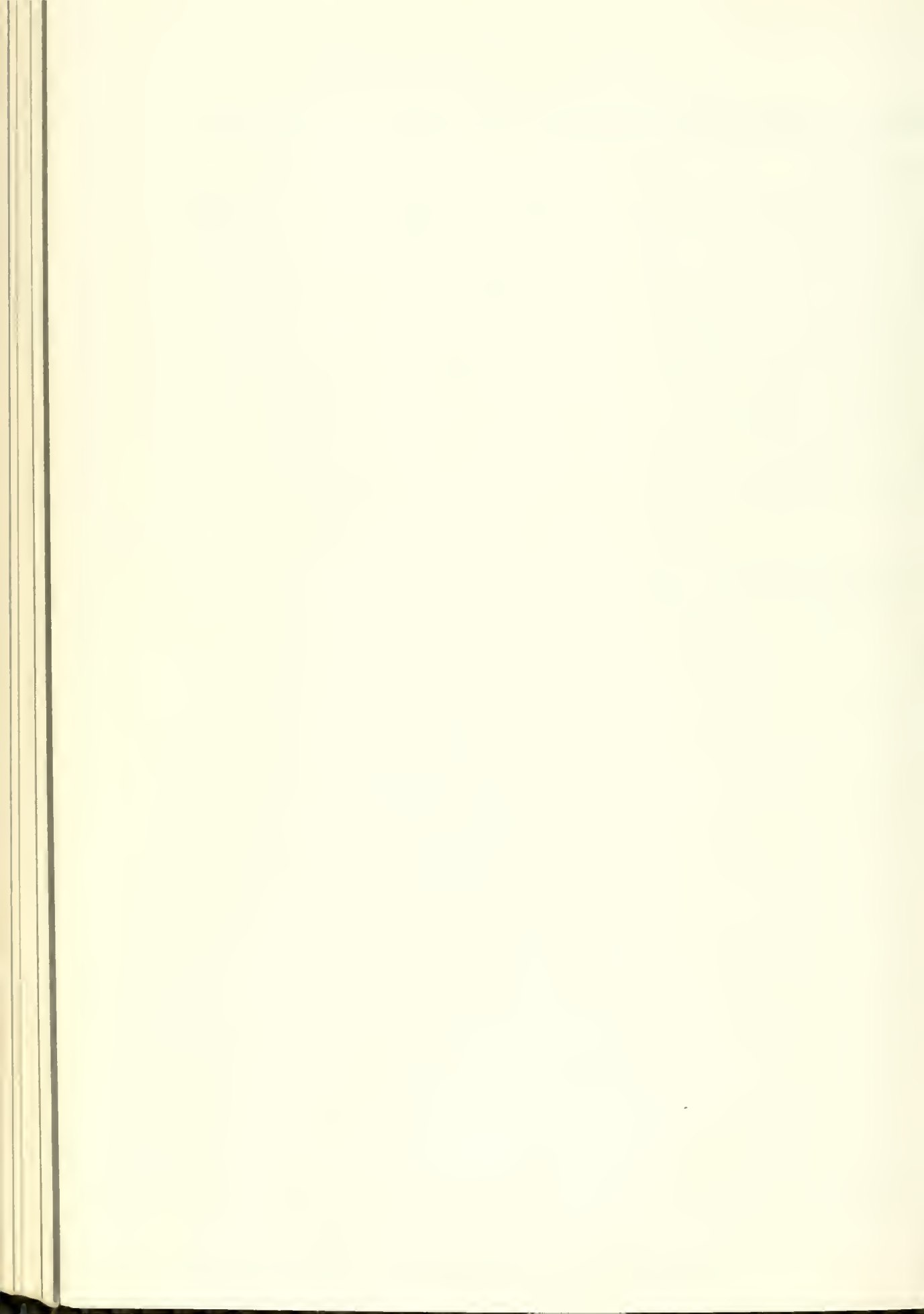
This difficulty disappears if a range of financing options is specified and constraints are added to rule out options made inappropriate by investment or other financing choices. Many different options can be grouped under the heading "long term debt," for example, ranging from practically riskless to highly speculative ones.⁶ Which of these options are feasible depends on other financing and investment decisions. But financing options so defined can be treated as both risk-independent and physically independent.

Are Financing Options Independent of Investment Options, and Vice Versa?

The proof of risk-independence applies when investment and financing projects are considered simultaneously. Certain inter-relationships must nevertheless be allowed for.

1. The firm's choice of assets determines the risk characteristics of its aggregate liabilities. Or, from a different point of view, we can say that the firm potentially can choose among a large number of financial assets, but that many combinations of real and financial options are infeasible -- e.g., Fledgling Electronics Corporation could not enjoy a 2:1 debt-equity ratio and simultaneously issue AAA bonds.
2. The firm's financing strategy can affect the returns produced by its real assets. Most dramatic is the case of bankruptcy due to large debt-servicing requirements. The real costs apparently associated with

⁶In practice, it is sufficient to identify a handful of debt classes to capture the major differences.



bankruptcy⁷ may be attributed to financing decisions, providing bankruptcy could have been avoided by a more conservative financial structure.

Investors will take the likelihood of bankruptcy into account in assessing the value of a firm's securities. They will also consider the possibility that management will incur real costs in avoiding bankruptcy if it seems imminent in some future contingency. Consequently, firms' market values will reflect the "financial risk" it undertakes. This is inconsistent with a linear objective function, since financial risk depends on the firm's overall financing and investment strategy, not simply on the individual options undertaken.

The LP format can be preserved in spite of these difficulties by using constraints to express the interrelationships. The simplest arrangement is to require that total debt not exceed debt capacity, which in turn, is related to the risk characteristics of the firm's real assets and the amount of equity backing provided. This provides a framework for assessing financial risk and imputing the value of bankruptcy costs. Simultaneously, it provides means to rule out inconsistent financing-investment packages, such as Fledgling Electronics' AAA bonds.

The required constraints are discussed in more detail in Sections II and III below.

A Comment on Risk-Independence and the Modigliani-Miller Propositions

The statement that financing and investment decisions are risk-

⁷See Baxter [1], and Robichek and Myers [18], esp. pp. 15-22.



independent is closely related to the well-known Modigliani-Miller (MM) propositions. These require that "the cutoff rate [minimum permissible rate of return] for investment in the firm . . . will be completely unaffected by the type of security used to finance the investment."⁸

As may be expected, quite similar arguments support risk-independence and the MM propositions. However, the two hypotheses are not identical. MM assert not only that financing and investment options are (risk) independent, but also that the present value of debt is zero (in a tax-free world) or equal to the present value of debt-related tax savings (in actuality). Their hypothesis is, therefore, disproved if the present value of debt financing is observed to be different from the present value of the associated tax savings. However, this observation would not necessarily imply that financing and investment options are risk-dependent. In other words, proof of the MM propositions is sufficient, but not necessary to prove risk-independence.⁹

⁸[12], p. 288. MM intend this statement to apply only in a no-tax world. When corporate taxes exist, the cutoff rate in their model depends on financial leverage.

In the LP model MM's statement is true regardless of the tax environment -- true, that is, in terms of the objective function. Eq. (1) implies that the present values of financing and investment options are mutually independent. However, the firm's financing and investment decisions are related through the LP constraints. This will be made more clear in Sections II and III below.

⁹To see this, remember that the proof of risk independence is identical regardless of whether real assets, financial assets or both are considered. The fact that investment proposals are risk-independent says nothing about whether the proposals' present values are large or small. That financing options are risk-independent likewise says nothing about which of these options are most valuable.



Admittedly, disproof of the MM propositions could raise reasonable doubts about the existence of risk-independence, because the assumed market processes on which the two hypotheses are based are similar.

The Meaning of Present Value for Financing Decisions

Before going into further detail it may be helpful to give some concrete meaning to the concept "present value" for financing options. (We usually examine the cost of financing, measured by the expected rates of return required by investors who contribute capital.)

Consider the firm's equilibrium stock price, $P(0)$, at the start of period $t = 0$. $P(0)$ is the present value of the stream of dividends $R(0)$, $R(1)$, . . . , $R(t)$, etc. Adoption of project j changes the dividend stream by j 's cash flows, a_{j1} , a_{j2} , The cash flows are, of course, measured net of corporate income taxes.

Assuming the project is a real asset, the change in $P(0)$, or present value, associated with it is usually computed as¹⁰

$$(3) \quad A_j = \Delta P(0) = \sum_{t=0}^{\infty} \frac{a_{jt}}{(1 + \rho(j))^t},$$

where a_{jt} = the mean of \tilde{a}_{jt} , and

$\rho(j)$ = the discount rate for a stream of cash flows with project j 's risk characteristics.

At equilibrium, $\rho(j)$ is determined by the expected rate of return offered by securities with risk characteristics similar to project j 's.

¹⁰This is the most common format for computing present value. It is adopted here for convenience of exposition. See, however, Robichek and Myers [16].



The rate $\rho(j)$ is not a weighted average cost of capital. The project's cash flows are assumed to affect the firm's dividends directly, without modification by any intervening financing arrangements. The relevant question is, "What is the market value of project j ?" not "What is the market value of project j when financed by the normal proportion of debt?"

Exactly the same procedure can be applied to determine the present value of financing options. They are unusual only in that the initial cash flow (f_{j0}) will usually be positive and future expected cash flows (f_{jt}) negative or zero. As for real assets, the appropriate discount rate is determined by expected equilibrium rates of return on other (financial) assets with similar risk characteristics.

The cash flows of the financing option are also assumed to affect the firm's dividends directly -- there is no presumption that proceeds of the financing are used to finance real assets.

Thus the treatments of real and financial assets are symmetrical.

As a practical matter, computation of the present value per dollar of a financing option is substantially eased by using the following observation as a benchmark: in perfect markets, the present value of all financing options is zero.

The proof of this statement is not at all difficult. By definition, all participants in perfect markets have access to the same trading opportunities at the same prices, and no single participant affects prices by his own actions. Thus a firm wishing to issue a bond, for example, is forced to do so on exactly the same terms as other firms (or individuals). Our

hypothetical firm will be able to issue bonds priced to yield the equilibrium market rate established for bonds with its risk characteristics -- no more, no less. But then the expected yield on the new bonds is exactly equal to the appropriate discount rate and the bonds' present value is zero.

This argument clearly can be applied to any type of generally traded financial asset.

Financial markets are not absolutely perfect, of course, but it is easiest to start with the presumption that $F_j = 0$, and then consider how any imperfections change this figure. Two of the most important effects are:

1. Costs of issue should be subtracted from present value. In practice, this reduces the present value of both bond and stock issues, stock issues by the greater amount.
2. However, the tax advantages of corporate debt increase its present value. Thus F_j should reflect the present value of the tax savings associated with a debt option j . For example, suppose a \$1000 perpetuity is issued at an interest cost of 7.0 percent. The tax-deductibility of interest will generate yearly tax savings of \$35 per year, assuming the corporate income tax rate is .50. Thus, the present value of this financing option is:¹¹

$$1000 + \frac{35}{.07} - \frac{70}{.07} = 500.$$

¹¹The tax savings are discounted at 7 percent on the grounds that the risk characteristics of the tax savings are equivalent to those of the interest payments. Various adjustments could be considered -- e.g., for tax-loss carry forwards, for the difference between promised and expected interest rates, etc. However, the procedure shown is reasonable for most practical purposes.

The present value per dollar of financing from this source would be entered in the objective function of the linear program (Eq. (1))

as $F_j = \$.50$.

II. ANALYSIS OF A SIMPLE LP MODEL

The rudimentary example discussed in this section assumes the firm has open to it only one financing option, simply "debt," and that its financing problem is only to choose the stock of debt outstanding in each period from $t = 0$ to $t = H$, the horizon. However, the planned stock of debt cannot exceed "debt capacity" in any of these periods.

The LP problem is:

$$(4) \quad \begin{array}{l} \text{Max} \\ x_j, y_j \end{array} \quad \Psi = \sum_{j=1}^n x_j A_j + \sum_{t=0}^H y_t F_t$$

subject to:

$$\begin{aligned} \phi_t &= y_t - Z_t \leq 0, & t=0, 1, \dots, H, \\ j &= x_j - 1 \leq 0, & j=1, 2, \dots, n. \end{aligned}$$

Here Z_t is defined as debt capacity for period t .

The Kuhn-Tucker conditions for the optimal solution are as follows.

$$(5) \quad \begin{aligned} \delta\Psi/\delta x_j - \sum_{t=0}^H \lambda_t \frac{\delta\phi_t}{\delta x_j} - \lambda_j \frac{\delta\phi_j}{\delta x_j} &\leq 0, \quad \text{all } j, \\ \delta\Psi/\delta y_t - \lambda_t \frac{\delta\phi_t}{\delta y_t} &\leq 0, \quad \text{all } t. \end{aligned}$$

The variables λ_t and λ_j are the imputed costs associated with the constraints ϕ_t and ϕ_j , respectively. Substituting for the partial derivatives, the conditions are



$$(5a) \quad A_j + \sum_{t=0}^H \lambda_t Z_{jt} - \lambda_j \leq 0,$$

$$F_t - \lambda_t \leq 0,$$

where $\delta Z_t / \delta x_j$ is written more simply as Z_{jt} .

If we assume that corporate income is taxed, then $F_t > 0$ for all t . Obviously the optimal solution will include as much debt as possible in every future period, and the constraints ϕ_t will be binding. The Kuhn-Tucker conditions also require, therefore, that $F_t - \lambda_t = 0$, or $F_t = \lambda_t$. This supports the further simplification

$$(6) \quad A_j + \sum_{t=0}^H Z_{jt} F_t - \lambda_j \leq 0.$$

Eq. (6) implies that the contribution of project j to stock price is measured by A_j , the "intrinsic" value of the project plus the present value of the additional debt the project supports. If $A_j + \sum Z_{jt} F_t > 0$ then the project should be accepted (if so, $\lambda_j > 0$ and Eq. (6) is an equality); if $A_j + \sum Z_{jt} F_t < 0$ then the project should be rejected (if so, $\lambda_j = 0$ and Eq. (6) is an inequality).

Example

Table 1 shows two hypothetical projects, each evaluated according to Eq. (6). Both projects lead to roughly the same total change in stock price, but for different reasons.

After the initial investment is made both projects generate cash flows over a ten-year period. For project 1, $A_j = \$10$. Thus, the project is worthwhile even if debt financing is not available. However, management



estimates that 20 percent of the project's required investment can be financed by debt -- in other words, the project increases the firm's total debt capacity by \$20 in each period from $t = 1$ to $t = 10$. The interest cost of additional debt is 7 percent. Taking the associated tax savings into account,¹² the project's total, or adjusted, value is $\$10.00 + \$4.92 = \$14.92$.

Project 2 is not worthwhile for its own sake, but only because of the additional debt capacity it creates. It is much less "risky" than 1 and can be 50 percent debt-financed. Project 2 is therefore worth $-\$10.00 + \$24.60 = \$14.60$.

Comparison to the Cost of Capital Concept

In a general way, the procedure just illustrated is equivalent to the usual doctrine that the weighted average cost of capital is a declining function of financial leverage, providing that reasonable debt limits are not exceeded. This doctrine implies that leveraged firms can undertake less valuable projects than unleveraged firms, which Eq. (6) also implies.

Nevertheless, there are important differences between even this simple LP model and the cost of capital approach. The cost of capital is usually computed as a single number reflecting (1) the risk characteristics of the firm's existing assets and (2) the firm's existing financial structure, presumably appropriate to existing assets. This figure is used directly as a standard of profitability for new assets with risk characteristics

¹²The Modigliani-Miller Hypothesis is assumed to hold. Thus the present value of debt solely reflects the present value of associated tax savings.

Table 1

Project	Present Value of Project Cash Flows (A_j)	Investment	Debt Capacity (Z_{jt})	Present Value of Debt Supported by Project	Total Contribution to Stock Price ($A_j + \sum_{t=1}^{10} Z_{jt}F_t$)
1	+10.00	100.00	20.00	4.92	+14.92
2	-10.00	200.00	100.00	24.60	+14.60

$$F_t = \frac{.5(.07)}{(1.07)^t} \cdot$$

similar to existing ones. Adjustments are made for assets with dissimilar risk characteristics.

However, it is not easy to arrive at the correct rate for all cases purely by judgment. The adjustment should reflect not only (1) the risk characteristics of the project in question, but also (2) the amount of debt it will support (presumably riskier projects support less debt). The second factor is usually ignored.

Despite wide use of the cost of capital concept. There are only a few attempts to provide a logically complete procedure for arriving at the required adjustments.¹³ In contrast, the LP approach takes projects' risk characteristics and debt capacities into account simultaneously and automatically. This is evident from the conditions for the optimal solution.

Further Comparison of LP and Cost of Capital Approaches

It will be of some interest to give a more precise idea of the range of situations in which the LP and cost of capital approaches are equivalent.

Starting with the simple LP model just described, we make three further assumptions.

1. That all investment projects under consideration are perpetuities.

Thus $A_j = a_j / \rho(j) - I_j$, where I_j is the initial investment, a_j is the expected cash return required by the market for assets with j 's risk characteristics.

2. That projects' debt capacities are the same in all future periods. Thus

$$Z_{jt} = Z_j, \text{ a constant for all } t.$$

¹³See Solomon [19] and Tuttle and Litzenberger [22].

3. That the MM propositions hold. Thus the present value of the dollar's worth of debt outstanding in period t is the tax saving in t discounted to the present; $F_t = \frac{i T_c}{(1+i)^t}$ where T_c is the corporate income tax rate and i is the bondholders' required rate of return.

Under these assumptions, the optimal solution requires

$$\frac{a_j}{\rho(j)} - I_j + Z_j \sum_{t=1}^H \frac{i T_c}{(1+i)^t} - \lambda_j \leq 0.$$

As H approaches infinity, the project's contribution to stock price or "adjusted present value" is

$$(7) \quad APV_j = \frac{a_j}{\rho(j)} - I_j + Z_j T_c$$

The project's APV is positive only if its expected rate of return \bar{a}_j/I_j , is greater than the cutoff rate ρ^* ; that is if:

$$(8) \quad a_j/I_j > \rho^* = \rho(j)(1-d_j T_c),$$

where $d_j = Z_j/I_j$.

This is exactly the cutoff rate recommended by MM, assuming project j has risk characteristics similar to the firm's existing assets.¹⁴

¹⁴The MM propositions imply [12, p. 268] that V , the aggregate market value of the firm is

$$V = \bar{a}/\rho + T_c D,$$

where a is the expected after-tax cash flow of the firm, ρ the capitalization rate appropriate to this stream and D the stock of debt (consols) currently outstanding. A small increase dI in the scale of the firm's assets implies

$$\frac{dV}{dI} = \frac{1}{\rho} \cdot \frac{da}{dI} + T_c \frac{dD}{dI}.$$

This action is acceptable if $dV/dI > dI/dI = 1$. Thus the minimum acceptable rate of return da/dI is $\rho^* = \frac{da}{dI} = \rho(1-T_c \frac{dD}{dI})$,

which is equivalent to Eq. (8)



Often the cutoff rate is computed as the weighted average cost of capital:

$$(9) \quad \rho^* = \frac{D}{V} (1-T_c)i + \frac{E}{V} k,$$

where D = market value of debt,
 E = market value of equity,
 $V = D + E$, and
 k = the cost of equity capital.

Equations (8) and (9) are equivalent if $d_j = D/V$, project j 's risk characteristics are similar to those of the firm's existing assets, and k behaves as MM predict.

If the MM propositions do not apply, use of a weighted average cost of capital corresponds to a somewhat different LP model. The only major change in assumptions is that F_t , the present value of debt, would be different than the MM propositions indicate.

Summary: Assumptions of Cost of Capital Approaches

The discussion above shows that the linear programming procedure is more generally applicable than cost of capital approaches. The Modigliani-Miller cost of capital concept assumes:

1. The risk characteristics of the project under consideration are similar to those of the firms' existing assets.
2. The MM propositions hold.
3. The project under consideration is a perpetuity.
4. The projects' debt capacity is the same in all future periods.

Use of the weighted average cost of capital does not necessarily require assumption (2). But it assumes in addition:



5. The ratio of additional debt financing to the projects' adjusted present value/^{is the same}as the ratio of existing debt to existing equity (market value).

This is possibly not the definitive list of assumptions. Assumption (5) may follow from the first four, and assumption (4) from the first three. But the general point holds regardless. The cases in which the MM and/or weighted average cost of capital approaches arrive at the same present value for a project as the LP approach are rather special ones. This does not imply the cost of capital approaches (either MM or weighted average) always lead to wrong decisions when the various special assumptions they require are relaxed. Nevertheless, their use can lead to wrong decisions in situations where the LP approach serves perfectly well.

III. PRACTICAL IMPLICATIONS

Broadly speaking, this paper has two implications for practical decisionmaking. First, the cost-of-capital framework for capital budgeting decisions has been shown to be a special case which may well lead to errors. Second, linear programming techniques are promising tools for long-range financial planning, and deserve further investigation.

The second point does not call for immediate changes in practical decisionmaking procedures. It is impossible to specify a general LP model appropriate for any firm's problems. Further, detailed design and experimentation are prerequisites for actual use.



But it is not necessary to wait for a usable LP model before discarding or supplementing the cost of capital concept for capital budgeting purposes. This will be shown prior to more detailed discussion of LP models.

A Simple Alternative to the Cost of Capital Framework

The following rule of thumb captures the most important capital budgeting implications of the LP framework: a project's total contribution to shareholders' wealth equals its present value considered separately plus the present value of tax savings due to additional debt supported by the project. This is simply a prose statement of Eq. (6) above.

The rule of thumb implies a four-step evaluation of capital budgeting projects:

1. Estimate the project's cash flow after taxes.
2. Assess the risk of the cash flows; choose the appropriate discount rate (ρ_j); compute the project's present value (A_j).
3. Assess the debt capacity of the project for future periods. Compute the present value of the extra debt capacity generated by the project ($\sum Z_{jt}F_t$).
4. Compute APV_j , the sum of A_j and $\sum Z_{jt}F_t$. Accept the project if APV_j is positive.

For more or less routine investments, projects might be assigned to "risk classes." All projects in a risk class would be evaluated at the same discount rate and "debt capacity factor." With four risk classes, the numbers might look like this:



Table 2

<u>Risk Class</u>	<u>Discount Rate</u>	<u>Debt Capacity Factor</u>
1	9%	4
2	12%	3
3	20%	2
4	30%	1

Extra debt capacity generated by the project in period t would be computed by multiplying expected cash flows for that period by the debt capacity factor.

For a numerical example, consider four projects, each requiring an investment of \$100, and offering expected returns of \$30 per year for ten years. The projects differ only in risk: One falls into each of the risk classes shown above.

Table 4 shows how the projects are evaluated.

Table 3

<u>Class</u>	<u>A_j</u>	<u>$\sum_{t=1}^{10} A_j F_t$</u>	<u>APV_j</u>	<u>Approximate Required "Cost of Capital"</u>
1	92.53	41.52	134.05	4.8%
2	69.51	31.14	100.65	8.1%
3	25.78	20.76	46.54	15.7%
4	-7.26	10.38	3.12	26.3%

$$A_j = \sum_{t=1}^{10} \frac{30}{(1+r)^t}$$

$$Z_j = 30(\text{Debt Capacity Factor}); F_t = \frac{.035}{(1.07)^t}$$

The right hand column in Table 3 shows the discount rate which would yield the correct APV_j if applied directly to the projects' cash flows of \$30 per year without separate adjustment for the value of additional debt capacity. This is projects' "cost of capital" as the term is normally used.



Table 4

Major Components of a Practical LP Model

ItemExamplesObjective Function

$$\Psi = \sum x_j A_j + \sum y_j F_j + \sum q_j S_j \quad (10a)$$

Constraints

Project acceptance

$$x_j \leq 1 \quad (10b)$$

$$x_i + x_j \leq 1 \quad (10c)$$

$$x_i - x_j \leq 0 \quad (10d)$$

Expected Sources of Funds =
Expected Uses

$$\sum x_j a_{jt} + \sum y_j f_{jt} = b_t \quad (10e)$$

Debt Capacity

a. Class 1 Debt

$$\sum x_j Z_{jt} - \sum_{\text{Class 1}} y_j D_{jt} \geq \text{constant} \quad (10f)$$

b. Class 2 Debt

$$\sum x_j Z_{jt} - \sum_{\text{Classes 1 and 2}} y_j D_{jt} + S_t \geq \text{Constant} \quad (10g)$$

c. Class 3 Debt

$$\sum x_j Z_{jt} - \sum_{\text{Classes 1, 2 and 3}} y_j D_{jt} + S_t + S_t^* \geq \text{Constant} \quad (10h)$$

Dividend Policy

$$x_t^* - x_{t-1}^* \geq 0 \quad (10i)$$

Variables Used

x_j, y_j = decision variables for investment and financing options, respectively;

A_j = present value of project j ;

F_j = present value per dollar of financing option j ;

a_{jt} = the expected cash flow of the j^{th} project in period t ;

f_{jt} = the expected cash flow per dollar of the j^{th} financing option in period t ;

S_j = slack or "dummy" variables;

q_j = cost or penalty associated with S_j ;

b_t = expected autonomous cash flow in period t .

Z_{jt} = contribution of project j to corporate debt capacity at t .

D_{jt} = expected principal outstanding at t per dollar of borrowing under debt option j ;

x_t^* = cash dividend payments in t .

Outline of a LP Model for Long-Range Financial Planning.

The probable main features of practical LP models are discussed in this section. My purpose here is not to specify the exact form of such a model, but to illustrate further how the ideas presented in this paper can be applied to actual problems.

The objective function and each major class of constraint are illustrated in Table 4.

Objective Function. -- Compared to Eq. (4), the objective function has been expanded in two ways. First, a large number of distinct financing options is allowed for. In practice, financing options must be distinguished not only by the time when funds are obtained, but also by the instrument used -- e.g., stock issue vs. commercial paper vs. term loan. Second, dummy and slack variables are introduced. For example, penalty costs are assessed when the LP solution violates certain limits of safety, convenience or practicality. Other dummy variables do not contribute directly to the aggregate present value of the financial plan, but do so indirectly by their role in the LP constraints.



Project Acceptance. -- Equations like (10b) are intended to require the program either to accept or reject each project. It does not do this with complete reliability, of course. The decisionmaker using LP will be forced to accept the possibility that the program will recommend accepting, say, 29.1 percent of project j.

Means of dealing with mutually exclusive or contingent options have been fully discussed by Weingartner.¹⁵ Equations (10c) and (10d) convey the essence of the approach. The former applies if options i and j are mutually exclusive, the latter if feasibility of i is contingent on acceptance of j.

Although these types of restrictions usually apply to investment options, financing options may also be so restrained. For example, option j could be purchase of real estate and option i a mortgage. It makes perfect sense to say that i is contingent on j.

Again, these constraints do not prevent the potential problem of fractional projects accepted in the LP solution. For this, integer programming -- or some other method of similar capabilities -- is required.¹⁶

Equality of Sources and Uses of Funds. -- This is simply a requirement of consistency on the plan. Note, however, that Eq. (10e) requires

¹⁵[24], pp. 32-34.

¹⁶Integer programming would also allow the decisionmaker to confront computational problems posed by economies of scale more directly. There are substantial economies of scale in new stock issues, for example, which cannot be treated in a linear program.

consistency only in terms of expected values; it does not establish the plan's feasibility unless expectations are confirmed.

Debt Capacity. -- In the most general sense, the constraints on borrowing shown in Table 3 simply require that aggregate debt issued is less than debt capacity. This is the same tack taken in Eq. (4). However, there are a number of specific changes and assumptions.

1. A constant term is introduced to reflect any unused debt capacity in the existing investment-financing package.
2. It is emphasized earlier in this paper that, although a firm potentially can issue a large number of financial assets, many combinations of real and financial assets are infeasible.

The most common problem is that a newly issued bond can be anything from a blue-chip to a highly speculative security, depending on the characteristics of the firm's assets and the aggregate amount of debt outstanding. Since the present values of blue chips and speculative bonds are likely to differ, the LP program should insure that each is issued only in appropriate circumstances.

One way to accomplish this is as follows. For each type of debt financing, define three classes of financing options. These are associated with low, medium and high-risk borrowing from the viewpoint of potential creditors. Thus, there will be low, medium and high-risk term loans; low, medium and high-risk bond issues, and so on.

We now consider all low-risk debt options (Class 1) separately. Equation (10f) is intended to restrict the amount of low-risk debt issued at t to that amount which can be supported



with negligible chance of financial embarrassment. Aggregate debt capacity, $\sum x_j Z_{jt}$, is defined with this stringent limit in mind.

Equation (10g) in turn restricts medium risk-debt. Aggregate debt capacity is increased to $\sum x_j Z_{jt} + S_t$. The dummy variable S_t is itself constrained -- for example, as a proportion of $\sum x_j Z_{jt}$.

Similarly, Class 3 debt is limited to $\sum x_j Z_{jt} + S_t + S_t^*$ by Eq. (10h).

3. Tax advantages will usually justify borrowing even at the expense of some likelihood of bankruptcy or insolvency. This is why "risky" debt options (Classes 2 and 3) are included in the program.

The factors determining the optimal debt level are shown in Figure 2a, which plots V , the total market value of the firm, against the amount of debt issued. The firm's choice of real assets is taken as given, and the MM propositions assumed to hold. At low levels of debt (Class 1 debt only) there is no significant chance of bankruptcy, and V increases by the full present value of expected tax savings. As Class 1 debt is exhausted, additional tax savings are somewhat offset by bankruptcy costs, which eventually increase rapidly with additional borrowing. The optimum is reached at point Y.

Figure 2b shows how bankruptcy costs are treated in the LP program. As Class 2 debt is issued S_t increases at a cost per dollar of q_t . A cost per

Fig. 2a

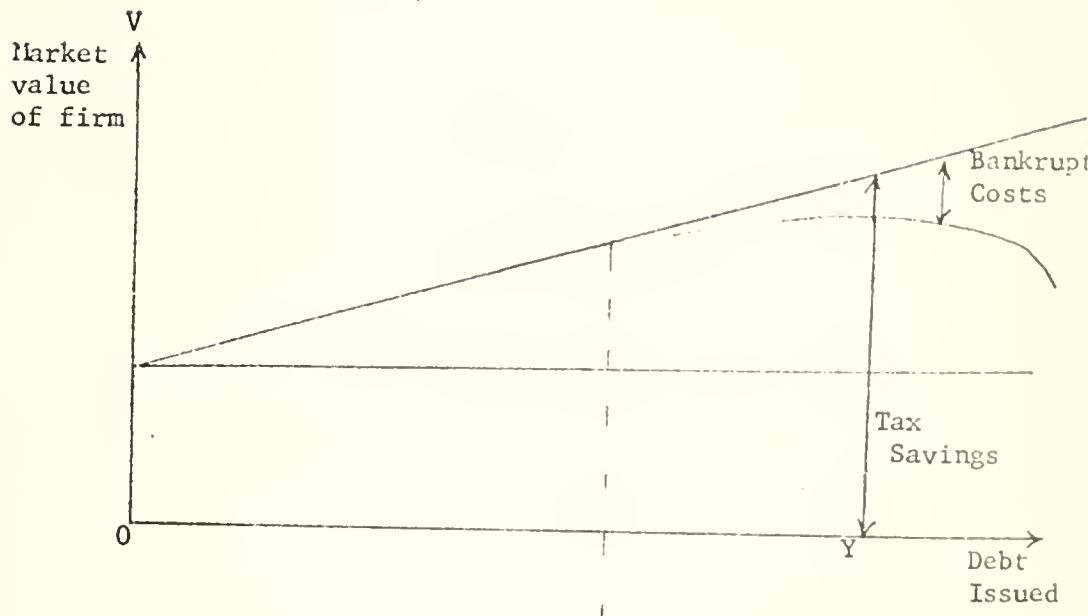


Fig. 2b

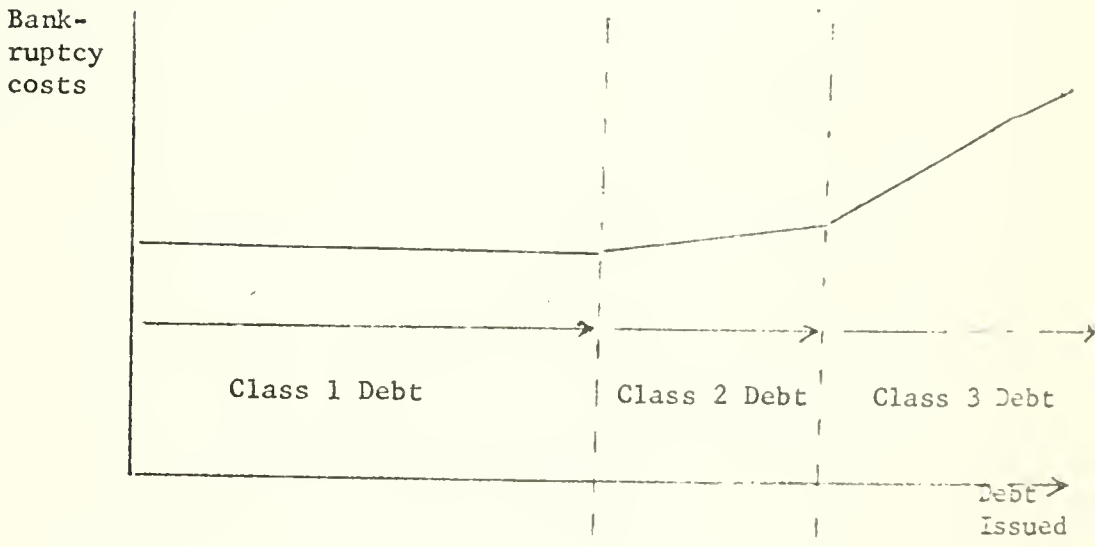
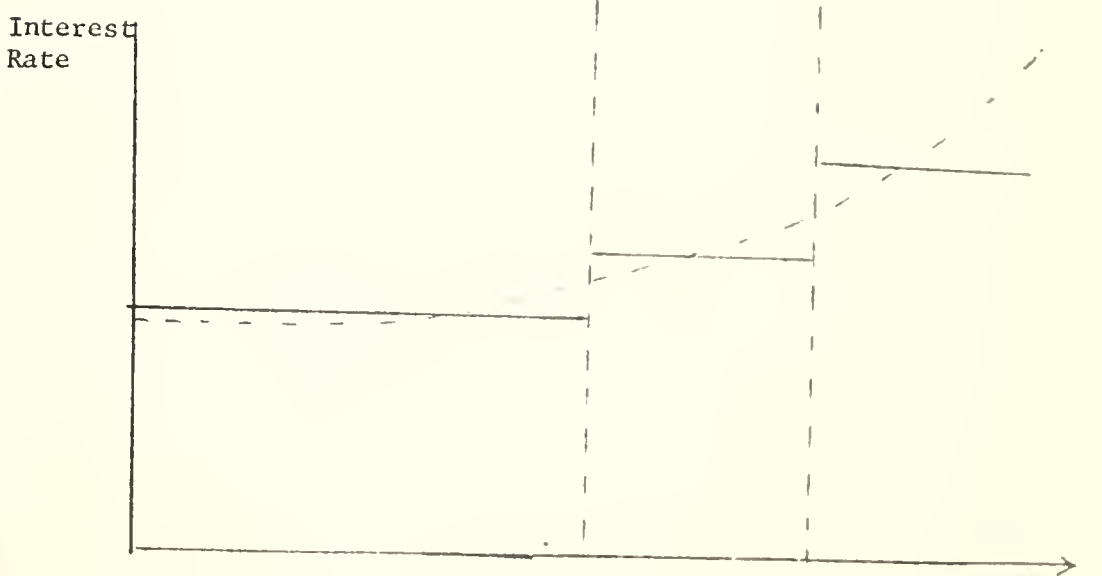


Fig. 2c





dollar q_t^* is applied to Class 3 debt by means of the dummy variable S_t^* . Figure 2b is thus a piecewise linear approximation of the bankruptcy cost component of Figure 2a.

The dashed line in Figure 2c shows as a function of the total amount borrowed, holding debt capacity constant. For purposes of the LP program, a stepwise approximation (solid lines) would be used.¹⁷

4. Expression of the firm's aggregate debt capacity as $\sum x_j z_{jt}$, the sum of each project's debt capacity, may well be a strong assumption. If, for example, debt capacity is determined by the standard deviation of the firm's aggregate cash flow, then it is certainly not correct to say that the whole is equal to the sum of the parts.¹⁸

¹⁷There are several traps in estimating marginal interest rates. If Class 2 debt is fully subordinate to Class 1 debt, then an extra dollar of Class 2 debt should not change the market's evaluation of planned or existing Class 1 debt. (The only possible link is through bankruptcy costs, which are treated separately.) In this case the direct interest cost of Class 2 debt is also the marginal cost of the issue. This convenience does not occur in the absence of subordination.

¹⁸Even if this is true there are promising avenues of approach. For example, the firm could be regarded as a portfolio of projects subject to Sharpe's "diagonal model" of security performance. See [21]. This would postulate one common factor (company sales?) affecting all projects' cash flows. Debt capacity for each project would be inversely related to the project's dependence on the common factor.

Dividend Policy. -- It is now widely accepted that dividend policy is irrelevant in perfect capital markets.¹⁹ If this is taken as a first approximation, then dividend policy is easily handled in the LP format. We treat each period's aggregate dividend payment as an investment which yields no cash returns to the firm, but nevertheless has a net present value of zero. Then the LP program will accept all investments with positive present values and will treat dividend payments as a residual. This is the appropriate strategy when dividend policy is irrelevant.

However, there are a number of reasons why dividend policy may be regarded as relevant. First is the different rates at which investors' capital gains and regular income are taxed. This factor is ignored in the present model.

Second, are the alleged market imperfections which lead investors to prefer high dividend payouts to low ones. Dividends may be assigned positive, rather than zero, present values if this is the case.

Third, is the informational content of dividends. Changes in dividends seem to be regarded as signals of changes in the firm's long-run profitability. Therefore, dividends are cut only when financial difficulties force it, and raised only when it is reasonably clear that the increase can be maintained.

The easiest way to reflect the informational content of dividends is to constrain aggregate dividends in period t to be no less than in period $t - 1$. See Eq. (10i). We may allow the program to violate the constraint, but at a penalty cost.

¹⁹For the original proofs of the proposition, see Miller and Modigliani [10] and Lintner [6].



Unfortunately, this strategy is inappropriate when the firm issues new equity. The informational content of dividends is really associated with dividends per share. Thus, it is not appropriate to constrain aggregate dividends when the number of shares is variable. The number of shares outstanding, given an equity issue of fixed amount, depends on share price, among other things; share price in turn depends on the anticipated value of the firm's financing-investment package. These inter-relationships appear to make this problem irretrievably non-linear. However, the problem is not fatal because of the relative rarity of new stock issues.

Conclusion

The pros and cons of applying linear programming to practical problems in long-range financial planning are about as follows.

Disadvantages. -- Such models will not fit reality exactly. First, economies of scale or physical interdependencies among real projects are clumsily handled at best. Integer constraints alleviate this problem, but they are not easy to work with.

Second, several aspects of the long-range financial planning problem may call for non-linear constraints -- for example, debt capacity and the informational content of dividend policy. It is not clear whether linear constraints on these variables are adequate for practical purposes.

Third, various sequential aspects of the decision problem (not discussed in this paper) are difficult to include in a linear program.

Advantages. -- Balanced against these difficulties are substantial

advantages. The linear programming format is widely understood, easy to work with and capable of handling problems of great complexity. It is applicable to choice among risky assets, and fully consistent with a theory of security valuation under uncertainty. It allows simultaneous consideration of the firm's financing and investment decisions. Finally, the linear format avoids several unfortunate assumptions built into the usual cost of capital approaches.

APPENDIX ARISK-INDEPENDENCE DERIVED FROM
THE SHARPE-LINTNER MODEL

We consider the k^{th} firm which has two or more investment projects ($j=1, 2, \dots$) open to it at $t = 0$.

Definitions

R = Risk-free rate of return;

R_M = Expected rate of return on the "market" portfolio.

R_k = Expected rate of return on firm k 's stock;

x_j = Dollar amount invested by firm k in the j^{th} project.

V_k = Expected value per share of firm k at $t = 1$. V_k reflects dividends and capital gains, and $V_k \equiv (1+R_k)P_k$.

All rates of return apply a single interval -- e.g., from $t=0$ to $t=1$.

Requirements for Risk-Independence

Risk-independence requires that the change in P_k due to adoption of project 1 be independent of the firm's decisions about other projects. In other words, that dP_k/dx_1 must be independent of x_2, x_3, \dots . We assume there are no physical interdependencies.

Since $V_k = (1+R_k)P_k$,

$$\frac{dP_k}{dx_1} = \frac{1}{1+R_k} \left[\frac{dV_k}{dx_1} - P_k \frac{dR_k}{dx_1} \right] . \quad (A1)$$

Thus it is sufficient to show that two terms in brackets are independent of x_2, x_3, \dots .



Proof for a Strict One-Period Model

Let V_{jk} be the expected terminal value of project j considered separately. In a one-period world, V_{jk} is j 's cash payoff. Thus, $V_k = \sum x_j V_{jk}$ and $dV_k/dx_1 = V_{1k}$, which is independent of the other x 's. (We are, of course, assuming physical independence.)

The Sharpe-Lintner Model implies²⁰

$$(R_k - R) = \text{Cov} [\tilde{R}_k, \tilde{R}_M] (\tilde{R}_M - R) / \text{Var}[\tilde{R}_M] \quad (\text{A2})$$

where

$$\begin{aligned} \text{Cov}[R_k, R_M] &= E [(\tilde{R}_k - R_k)(\tilde{R}_M - R_M)]. \\ &= \frac{1}{P_k} E[(\tilde{V}_k - V_k)(\tilde{R}_M - R_M)]. \end{aligned}$$

The tilde (\sim) indicates the random variable rather than its expectation.

$$\begin{aligned} P_k \frac{dR_k}{dx_1} &= (R_M - R) E\left[\left(\frac{d\tilde{V}_k}{dx_1} - \frac{dV_k}{dx_1}\right)(\tilde{R}_M - R_M)\right] / \text{Var}[\tilde{R}_M] \\ &= (R_M - R) E[(\tilde{V}_{1k} - V_{1k})(\tilde{R}_M - R_M)] / \text{Var}[\tilde{R}_M]. \end{aligned}$$

which is independent of x_2, x_3, \dots . Partial equilibrium analysis justifies taking all derivatives involving R_M or R as zero.

Proof When the Model is applied within a Multi-Period World

The proof given above requires that $V_k = \sum x_j V_{jk}$. This is not clearly true when the terminal value V_k is part cash payoff and part residual market value.

²⁰See Sharpe [20] or Lintner [9].



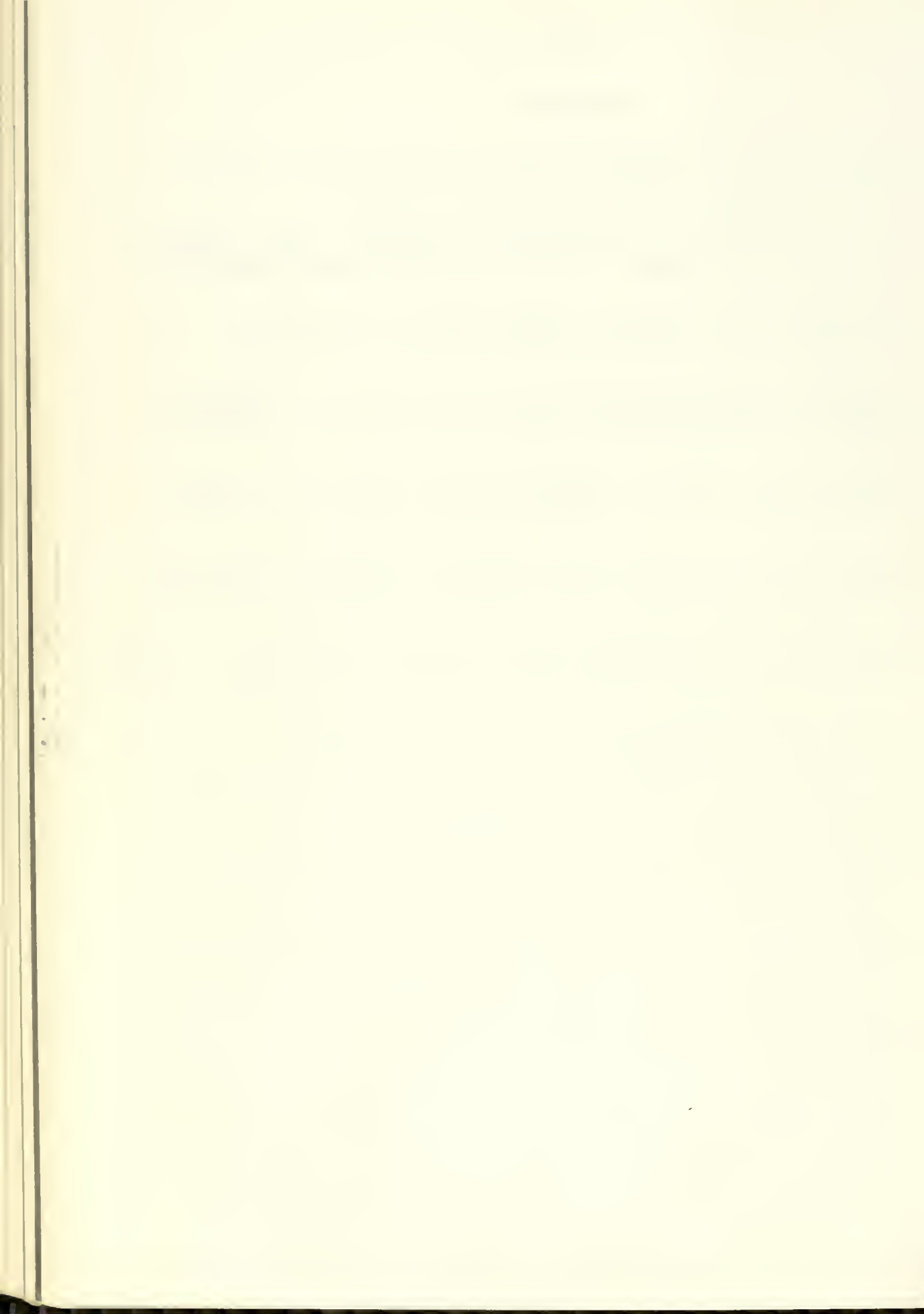
A formal proof for the multiperiod case follows by a sort of backwards induction.

1. Consider a horizon H beyond which all projects can yield no cash flows. By definition V_{jk} for period H can reflect cash flows only.
2. Prove risk-independence relative to P_k at $t = H - 1$. This is sufficient to establish $V_k = \sum x_j V_{jk}$ at $H - 1$.
3. Prove risk-independence at $H - 2$, etc.



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