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NEW SYNTHESIS PROCEDURES FOR REALIZING
TRANSFER FUNCTIONS OF RLC AND RC NETWORKS

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Abstract

Synthesis procedures that realize practical RLC and RC networks are presented. The RLC networks are practical in that they contain no mutual inductance and no perfect coils, i.e. every inductance has an associated series resistance. New techniques employed in the procedures are first discussed in detail and then applied to various synthesis problems. Included among the procedures for synthesizing RLC networks are those for realizing unbalanced structures and lattices whose arms possess identical poles. Reduction of the lattices to unbalanced forms is considered, and it is shown that if real transformers are allowed, i.e. transformers with winding resistance, magnetizing inductance and a coupling coefficient smaller than one, then the lattice realization for a transfer admittance is always reducible.

Unbalanced networks are realized by the RC synthesis procedure. The number of elements required is smaller than that required by the Guillemin method of RC synthesis.

Finally, two new synthesis procedures are presented for realizing a Darlington network without any ideal or unity-coupled transformers, where a Darlington network is considered to be composed of lossless elements plus only one resistance. In one of the two procedures the single resistance appears not as a termination but within the network.

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NEW SYNTHESIS PROCEDURES FOR REALIZING TRANSFER FUNCTIONS OF RLC AND RC NETWORKS

I. Introduction

An important problem in network synthesis is the realization of a linear passive network for a prescribed transfer characteristic (1, 2, 4, 5). There are various procedures available for achieving this realization; a few are only of academic interest. These procedures suffer from a lack of practicality because of the requirement of ideal transformers, unity-coupled transformers, perfect coils (that is, inductances without associated series resistance) or a combination of these. Other procedures are restricted in generality in that the zeros or poles of the transfer function must lie only in specified portions of the complex plane or along specified lines in the plane. To remove these restrictions to practicality and generality by the development of a new synthesis procedure is the main purpose of this research. An auxiliary purpose is the development of new methods for the realization of transfer functions restricted in the location or number of poles and zeros.

A few of the well-known synthesis procedures contain ideas that are useful or suggestive in the development of new techniques. The procedures of interest will be discussed, but it is not intended in this Introduction to present them in all the detail necessary for accomplishing network designs. For this the reader may consult the references. One purpose of this section is to emphasize the underlying ideas or shortcomings of the existing procedures so that possible extensions or improvements suggest themselves.

I.1 The Synthesis Problem

Throughout the major part of this report, consideration is given to the problem of synthesizing the network shown in Fig. 1 which can be characterized by the equations

$$\begin{aligned} I_1 &= y_{11}E_1 + y_{12}E_2 \\ I_2 &= y_{21}E_1 + y_{22}E_2 \end{aligned} \tag{1}$$

or by the inverse set

$$\begin{aligned} E_1 &= z_{11}I_1 + z_{12}I_2 \\ E_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned} \tag{2}$$

where $z_{12} = z_{21}$ and $y_{12} = y_{21}$ by reciprocity, and the y 's and z 's are the familiar short-circuit admittances and open-circuit impedances, respectively. The load in most cases is a pure resistance, but other terminations are also considered. Useful relations may be derived from the above basic equations. For example, for a pure resistance load of one ohm, since numerically $E_2 = -I_2$, the second of Eqs. 1 yields

$$Y_{12} = \frac{I_2}{E_1} = \frac{y_{12}}{1 + y_{22}} \quad (3)$$

and an analogous relationship is derived for Z_{12} from the second of Eqs. 2.

For the load, an open circuit, that is, with $I_2 = 0$, Eqs. 2 yield

$$\frac{E_2}{E_1} = \frac{z_{12}}{z_{11}} \quad (4)$$

while for a short-circuit load

$$\frac{I_2}{I_1} = \frac{y_{12}}{y_{11}} \quad (5)$$

may be obtained from Eqs. 1.

It can be shown (3) that the transfer function, whether an admittance, impedance, or a dimensionless ratio, must be of the form of a quotient of polynomials of the complex variable $s = \sigma + j\omega$. Thus Y_{12} may be written as

$$Y_{12} = \frac{p(s)}{q(s)} \quad (6)$$

where p and q are polynomials the positions of whose zeros are fixed by conditions of physical realizability and by the type of network to be synthesized. For example, for

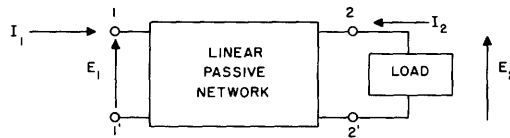


Fig. 1

Two terminal-pair network.

physical realizability of Y_{12} , q is required to be a Hurwitz polynomial, that is, to have zeros only in the left half-plane.

I.2 Bower-Ordung Synthesis of RC Lattices (2)

Bower and Ordung realize a transfer function in the form of an RC lattice network shown in Fig. 2. In their lattice the series and cross arms are of such a form relative to each other that reduction to an unbalanced network is possible in most cases.

If a network contains only R's and C's, then its transfer function has poles restricted to the negative real axis. This is an extremely useful characteristic, for the residue in a real pole cannot be complex; thus by a proper distribution of the positive or negative real residues to the Z_a and Z_b arms, a synthesis procedure can be developed. This is exactly what the Bower-Ordung procedure accomplishes.

How then can the procedure be adapted for RLC networks, when its very essence

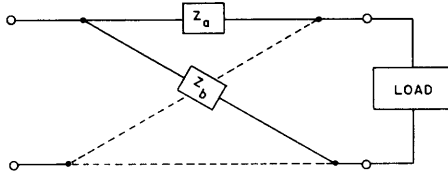


Fig. 2
RC lattice network.

demands that the residues be real? It is evident that residues in complex poles are in general complex and, furthermore, it is not possible to obtain for any positive real RLC function a partial fraction expansion each of whose component fractions (taking complex conjugate poles in pairs, of course) is positive real. Therefore before synthesis of RLC lattice networks with lossy coils can be

accomplished, two difficulties must be resolved. They are:

a. The development of a method for the appropriate breakdown of the Hurwitz denominator q of Eq. 6 into a sum of two polynomials so that the subsequent division by one of them yields a particular type of positive real RLC function. This RLC function must have partial fraction components that are positive real.

b. The determination of the relationship that complex residues must satisfy in order to yield a realizable partial fraction component of Z_a and Z_b .

These two problems are considered in section II, where most of the techniques to be used in the synthesis procedures are developed and discussed. Section III is concerned with the synthesis procedures for the development of RLC lattice networks that realize completely unrestricted transfer functions.

I.3 Guillemin Synthesis of Unbalanced RC Networks (4)

An ingenious method of synthesis of an unbalanced form of RC network is due to Guillemin. Given a transfer admittance with poles only on the negative real axis

$$Y_{12} = \frac{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0}{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}$$

$$= \frac{p}{q} \tag{7}$$

q is decomposed into the sum of two polynomials, $q_1 + q_2$, of the same degree and with alternating zeros, so that the quotient q_2/q_1 is a positive real RC function. Simple algebra along with the use of Eq. 3 then yields

$$Y_{12} = \frac{p}{q} = \frac{p}{q_1 + q_2} = \frac{\frac{p}{q_1}}{1 + \frac{q_2}{q_1}}$$

$$= \frac{y_{12}}{1 + y_{22}} \tag{8}$$

where the identifications $y_{12} = p/q_1$ and $y_{22} = q_2/q_1$ may be made. If the polynomial p is now thought of as explicitly written out, it is possible to make a further subdivision

of y_{12} in the form

$$\begin{aligned}
 y_{12} &= \frac{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}{q_1} \\
 &= \frac{a_n s^n + a_{n-1} s^{n-1}}{q_1} + \frac{a_{n-2} s^{n-2} + a_{n-3} s^{n-3}}{q_1} + \dots + \frac{a_0}{q_1} \\
 &= y_{12}^{(1)} + y_{12}^{(2)} + \dots + y_{12}^{(\frac{n+2}{2})}
 \end{aligned} \tag{9}$$

where n is assumed even.

It is well known that it is possible to realize a given y_{22} and a y_{12} that has zeros only on the negative real axis, including the origin, by means of one RC ladder network. In the synthesis of y_{22} by a Cauer-like procedure the poles of the associated y_{12} are provided for automatically while the negative real zeros are inserted by means of zero-shifting procedures. Since each of the component y_{12} 's in Eq. 9 has been constructed to have only nonpositive real zeros, zero shifting is possible. Each $y_{12}^{(l)}$ can therefore characterize a ladder structure whose y_{22} is given, except for a constant multiplier, by q_2/q_1 . It is finally evident that a network consisting of a one-ohm termination of the component ladder structures paralleled at input and output terminals, as shown in Fig. 3, may be characterized, except for a constant multiplier, by the given transfer function

$$Y_{12} = \frac{y_{12}}{1 + y_{22}} \tag{10}$$

so that the complete synthesis can be achieved.

We desire to analyze the above procedure from two points of view: one with the purpose of improvement by reduction of the number of ladders and number of elements in each ladder, and the other with a view toward extension of the techniques to RLC synthesis.

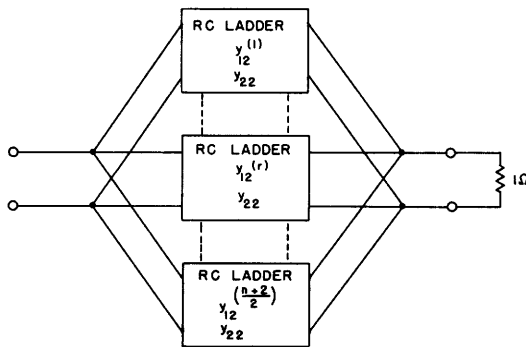


Fig. 3
Final composite network.

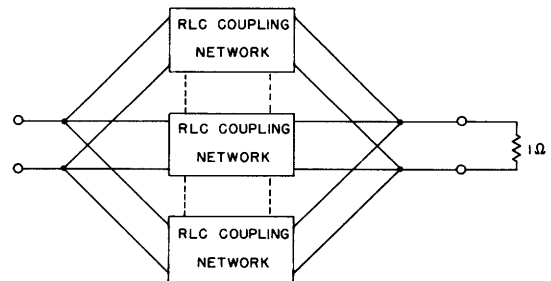


Fig. 4
Desired form of final network.

If the degree of the numerator $p(s)$ is n , the number of ladders is fixed by the method of breakdown at the smallest integer satisfying the relationship

$$l \geq \frac{n+1}{2}. \quad (11)$$

The number of elements in each ladder is determined by the degree of the denominator q , since q is decomposed into two polynomials of the same degree and the total y_{22} is associated with each ladder. Therefore, for improvement of the RC synthesis procedure, it is necessary:

a. to find a method for determining in a straightforward manner whether a smaller number of ladders than that given by Eq. 11 is possible in a particular problem;

b. to demonstrate that it is more economical of elements to break q into two polynomials not of the same degree and that it is possible to associate each component ladder with only a portion of y_{22} .

A synthesis procedure embodying the above ideas is treated fully in section IV.

With regard to an extension to RLC synthesis, it is required that techniques analogous to those used in RC synthesis be used. Analysis of the Guillemin article shows that the three basic steps of the procedure are:

a. breakdown of Hurwitz denominator q into a sum of polynomials, $q_1 + q_2$, so that a subsequent division by q_1 yields 1 plus a positive real RC function;

b. breakdown of the numerator polynomial p into components each of which has zeros only on the negative real axis, including the origin;

c. zero shifting to produce zeros along the negative real axis.

Therefore, to achieve the synthesis of RLC coupling networks in the form shown in Fig. 4, it is necessary:

a. to develop a method for the appropriate breakdown of the Hurwitz denominator q into a sum of two polynomials; a subsequent division by one of these polynomials must yield the sum of 1 plus a positive real RLC function. (This is essentially the same requirement as in section I.2a.)

b. to develop a method of zero shifting not only along a line but in the entire left half-plane.

c. to develop a method for the appropriate breakdown of the numerator polynomial $p(s)$ into a sum of components.

We have not been able to find a method of zero shifting with a general positive real function of arbitrary degree. However, a method has been discovered of zero shifting with one pair of complex poles at a time, to place zeros anywhere in the left half-plane. This allows a restricted method of synthesis to be developed. The discussion of the above techniques is left to section II, while the complete synthesis procedure is given in section V.

I.4 Synthesis with Vacuum Tubes

Where ruggedness and permanence are not essential, vacuum tubes may be used in

a design procedure, not as frequency-shaping elements but merely for isolation and gain. Indeed, as Guillemin points out in relation to RC synthesis (4), in any synthesis of a function of high degree "decomposition into stages, while in some cases absolutely necessary to prevent the signal from being attenuated below the noise level, turns out to be effective in reducing computational work and also the over-all loss, which is usually smaller than it would be if the entire characteristic were obtained with a single complex network." This is true also for RLC networks. Thus the designer may often divide a transfer function into factors of simpler component functions each of which is realized between vacuum tubes connected in cascade.

Guillemin (1) discusses a method of using surplus factors to obtain component transfer functions that are positive real and of the form

$$\frac{s+a}{s^2+bs+c} \quad (12)$$

or

$$\frac{s^2+ds+e}{s+a} \quad (13)$$

Each has a simple network realization as a driving-point impedance and may therefore be used as the plate load of a pentode stage.

If the positive real requirement was removed, then the need for surplus factors, which increases the number of elements, would be eliminated, and, second, if "chunks" of higher degree could be synthesized, fewer vacuum tubes would be needed.

A method is described in section VI for realizing a general minimum-phase transfer function with a numerator of third degree or less and a denominator of fourth degree. This can always be done. Very often, moreover, rational functions of higher degree may be realized in one fell swoop so that the need for a tube is eliminated.

I.5 The Darlington Problem (1, 5)

One aspect of the Darlington problem is the synthesis for a prescribed squared magnitude of transfer function of a lossless network terminated in resistance. In general the lossless network contains ideal transformers.

For the particular case of a transfer admittance

$$Y_{12} = \frac{p}{q} \quad (14)$$

where p is an even function of the complex variable s , it is always possible by use of the lattice synthesis procedures discussed in section III to achieve an unbalanced form of Darlington network (that is, lossless coupling network terminated in resistance) that has no ideal transformers but only real transformers (with a coupling coefficient $k < 1$ and finite magnetizing inductance). Of course, the Darlington network without ideal transformers can also be achieved by means of zero-shifting procedures and paralleling

of ladder networks (more than one ladder is necessary when the zeros of p are not all on the j axis).

These two forms of network synthesis are considered in section VII.

II. Special Techniques Used in Synthesis Procedures

II.1 Partial Fraction Expansion of a Positive Real RLC Function

For the driving-point functions of two-element kind networks the Foster method of synthesis can be successfully applied, but for general RLC functions the method unfortunately breaks down. However, the method may work for a particular RLC function. In this section we determine the necessary conditions on the residues for the success of the method, and, the conditions necessary so that a series resistance can be associated with every inductance; finally we indicate the application of the above in the synthesis of transfer functions.

The partial fraction expansion of a positive real RLC impedance

$$Z = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2} + \frac{k_3}{s - s_3} + \frac{k_4}{s - s_4} + \dots \quad (15)$$

can be written more explicitly as

$$\begin{aligned} Z &= \frac{\alpha + j\beta}{s + \sigma_1 - j\omega_1} + \frac{\alpha - j\beta}{s + \sigma_1 + j\omega_1} + \frac{R}{s + a} + \dots \\ &= \frac{2\alpha \left(s + \sigma_1 - \frac{\beta\omega_1}{\alpha} \right)}{s^2 + 2\sigma_1 s + \omega_0^2} + \frac{R}{s + a} + \dots \\ &= z_1 + z_2 + \dots \end{aligned} \quad (16)$$

so that the typical terms z_1 and z_2 become apparent. To simplify exposition, we have assumed only simple poles. In Eq. 16a, σ_1 , and ω_1 are real constants and positive, the constants R , α and β are real, and $\omega_0^2 \equiv \sigma_1^2 + \omega_1^2$. It is obvious that for terms like z_2 to be positive real and hence separately realizable as a simple driving-point impedance R must be positive. However, the condition on the residue of the partial fraction component z_1 containing complex poles is not so obvious, except that it is necessary for α to be positive. By application to z_1 of the well-known test for positive real character, the additional condition (6) is found to be

$$\frac{|\beta|}{\alpha} \leq \frac{\sigma_1}{\omega_1}. \quad (17)$$

In words, the condition for the existence of a positive real partial fraction component z_1 is that α be positive and the angle of the pole to the imaginary axis be greater than the angle of its residue. This is illustrated in Fig. 5, where angle $\phi \equiv \tan^{-1}(\beta/\alpha)$ must be less than or equal to the angle $\psi = \tan^{-1}(\sigma_1/\omega_1)$, or the residue may lie anywhere in

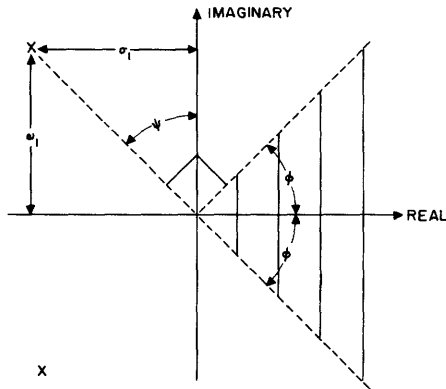


Fig. 5

Graphical illustration of pole and residue relationship.

satisfaction with the inequality sign allows some dissipation to be associated with the coil. Thus, in both cases ($\beta > 0$, $\beta < 0$), use of the inequality sign calls for an impedance containing a lossy coil; and since, as will be shown in section III, the lattice synthesis procedure requires that the lattice arms Z_a and Z_b have β 's which are equal numerically but of opposite sign, it is mandatory that the inequality sign be used if it is desired to obtain lossy coils in both arms.

A similar discussion can of course be made on the admittance basis.

Now suppose that after a partial fraction expansion of Z has been made it is discovered that the above conditions are not satisfied in each of the components. The Foster method of synthesis is then not possible. However, as first pointed out by Bode (7), it is possible by the addition of enough positive resistance to make each term separately realizable. Unfortunately this reservoir of resistance is not always available in Z ; yet the idea is important in its application to transfer functions. We recall that the transfer impedance for an open-circuited lattice is given in terms of its arms by

$$Z_{12} \equiv z_{12} = \frac{1}{2} (Z_b - Z_a). \quad (18)$$

As first pointed out by Guillemin, we may write

$$Z_{12} = \frac{1}{2} [(Z_b + R) - (Z_a + R)] \quad (19)$$

with R chosen so large that it is possible to add enough positive resistance to each component fraction to make it positive real. The modified arms may now be realized simply. This incidentally proves that any transfer impedance may be realized as the difference between two driving-point impedances. Thus if we use a network or a device that performs a subtraction, for example, the passive lattice or two tubes, each with one impedance as a plate load and driven in phase but with out-of-phase outputs, we can realize the transfer function simply. It is furthermore believed, in regard to the lattice, that

the crosshatched portion of the plane.

It is desired that the inductance used in the synthesis of z_1 have a series resistance associated with it. Since β may be positive or negative, two cases arise. For $\beta > 0$, if Eq. 17 is satisfied with the equality sign, then $(\beta\omega_1/a) = \sigma_1$ and the quantity $\sigma_1 - (\beta\omega_1/a)$ appearing in Eq. 16 equals zero, so that a perfect coil is required. On the other hand, if the inequality sign is used, $\sigma_1 - (\beta\omega_1/a)$ is greater than zero which guarantees a lossy coil. For the case of a negative β , it is possible to associate all of the dissipation with the coil by making $\sigma_1 - (\beta\omega_1/a) = 2\sigma_1$ with satisfaction of Eq. 17 by the equality sign. Again,

by suitably choosing an additional driving-point impedance Z_1 in

$$Z_{12} = \frac{1}{2} \left[(Z_b + R + Z_1) - (Z_a + R + Z_1) \right] \quad (20)$$

it is possible to guarantee the reduction of this open-circuited lattice to an unbalanced form, but we have been unable to prove this.

We are now led to a consideration of a special type of positive real RLC function whose partial fraction residues are all real and positive, that is, lie on the center line of the crosshatched area of Fig. 5; thus not only is it positive real, but each of its partial fraction components (complex poles taken in pairs) is positive real.

II.2 Breakdown of a Hurwitz Polynomial

In sections I.2 and I.3 a need was established for a specified breakdown of a Hurwitz polynomial. The method explained here satisfies that need.

We wish to appropriately divide the Hurwitz denominator q of the transfer function

$$Y_{12} = \frac{p}{q}. \quad (21)$$

A desirable breakdown is

$$q \equiv q_1 + Aq_1' \quad (22)$$

where q_1 is a Hurwitz polynomial, q_1' is the derivative of q_1 , and A is a positive real constant. The proof that this decomposition can always be achieved follows simply from a theorem in algebra (8) which states that the roots of a polynomial are continuous functions of the coefficients. If we choose the limiting value of zero for A , then q_1 equals q and is surely Hurwitz. Because of the theorem given above, the Hurwitz character is not lost as we increase A slightly. An upper bound on A exists beyond which all the roots of q_1 are no longer in the left half-plane.

The procedure for finding a suitable q_1 is quite straightforward. We have in general as the denominator of a given transfer function

$$q = s^n + \gamma_{n-1} s^{n-1} + \gamma_{n-2} s^{n-2} + \dots + \gamma_0 \quad (23)$$

where all the γ coefficients are known. We necessarily choose q_1 of the same degree as q and write

$$q_1 = s^n + \delta_{n-1} s^{n-1} + \delta_{n-2} s^{n-2} + \dots + \delta_0 \quad (24)$$

where the δ coefficients are to be determined. It follows that

$$Aq_1' = A \left[ns^{n-1} + (n-1) \delta_{n-1} s^{n-2} + (n-2) \delta_{n-2} s^{n-3} + \dots + \delta_1 \right]. \quad (25)$$

Equating coefficients in accordance with Eq. 22 gives the set of simple equations

$$\begin{aligned}
\delta_{n-1} + nA &= \gamma_{n-1} \\
\delta_{n-2} + (n-1)\delta_{n-1}A &= \gamma_{n-2} \\
\delta_{n-3} + (n-2)\delta_{n-2}A &= \gamma_{n-3} \\
\vdots & \\
\delta_0 + \delta_1A &= \gamma_0
\end{aligned} \tag{26}$$

which can be solved one at a time successively for the unknown δ 's once a suitable A is assumed. Since A in many applications of this technique has a direct influence on the gain of the final network (as we shall see, for example, in the lattice synthesis procedure of section III), it is advisable from the point of view of gain to choose A as large as possible. As A increases, however, the complex roots of q_1 move closer to the j axis, a circumstance that calls for high- Q coils, so that a compromise value of A may have to be chosen. It is not to be assumed that the value of A always affects the gain; for example, in the synthesis procedure of section VI, A fixes the admittance level but has no effect on the gain of the given transfer voltage ratio.

A numerical example will help to clarify the preceding discussion. The q chosen is the one associated with the use of the Tschebyscheff function

$$V_n(x) = \cos(n \cos^{-1} x) \tag{27}$$

in the squared magnitude of the transfer admittance

$$\left| Y_{12} \right|_{s=j\omega}^2 = \frac{1}{|q|_{s=j\omega}^2} = \frac{1}{1 + \epsilon^2 V_n^2(\omega)} \tag{28}$$

with $n = 4$ and $\epsilon = 0.5$. The denominator $q(s)$ is found to be

$$q = s^4 + 0.963679s^3 + 1.464337s^2 + 0.7540305s + 0.279496. \tag{29}$$

Writing

$$\left. \begin{aligned}
q_1 &= s^4 + \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0 \\
Aq_1' &= A(4s^3 + 3\delta_3 s^2 + 2\delta_2 s + \delta_1)
\end{aligned} \right\} \tag{30}$$

we obtain by use of Eqs. 26

$$\left. \begin{aligned}
\delta_3 + 4A &= 0.963679 \\
\delta_2 + 3\delta_3 A &= 1.464337 \\
\delta_1 + 2\delta_2 A &= 0.7540305 \\
\delta_0 + \delta_1 A &= 0.279496
\end{aligned} \right\} \tag{31}$$

as the equations to be solved for the δ 's. Since a necessary (but not sufficient) condition for q_1 to be a Hurwitz polynomial is that its coefficients be positive, A must surely

not be greater than the value found by setting δ_3 equal to zero in the first of Eqs. 31, which value thus provides a rough upper bound on A. In accordance with this reasoning A must be chosen not greater than 0.240920. Generally a good first choice is approximately half the value yielded in this manner; therefore, using $A = 0.1$, we obtain

$$\left. \begin{aligned} \delta_3 &= 0.563679 \\ \delta_2 &= 1.295233 \\ \delta_1 &= 0.494984 \\ \delta_0 &= 0.229998 \end{aligned} \right\} \quad (32)$$

so that

$$\left. \begin{aligned} q_1 &= s^4 + 0.563679s^3 + 1.295233s^2 + 0.494984s + 0.229998 \\ Aq_1' &= 0.1(4s^3 + 1.691037s^2 + 2.590466s + 0.494984). \end{aligned} \right\} \quad (33)$$

It is finally necessary to apply the Hurwitz test (9) to q_1 . (As was shown in sections I.2 and I.3, we require that q_1'/q_1 be positive real, but this is guaranteed, as we shall see below, by the Hurwitz character of q_1 .) For the test, to simplify computation, all the significant figures generally need not be used, so that the test yields

$$\begin{array}{r} 0.564s^3 + 0.495s \quad \left| \begin{array}{l} \frac{s}{0.564} \\ s^4 + 1.295s^2 + 0.230 \\ s^4 + 0.878s^2 \\ \hline 0.417s^2 + 0.230 \end{array} \right. \quad \left| \begin{array}{l} 1.353s \\ 0.564s^3 + 0.495s \\ 0.564s^3 + 0.311s \\ \hline 0.184s \end{array} \right. \quad \left| \begin{array}{l} 2.266s \\ 0.417s^2 + 0.230 \\ 0.417s^2 \\ \hline 0.230 \end{array} \right. \quad \left| \begin{array}{l} 0.800s \\ 0.184s \\ 0.184s \end{array} \right. \end{array}$$

which, because of the positive quotients, guarantees q_1 Hurwitz. This completes the example.

It is now necessary to examine the positive real function that results for y_{22} when we carry out the manipulations that are similar to those in RC synthesis

$$\begin{aligned} Y_{12} &= \frac{y_{12}}{1 + y_{22}} \\ &= \frac{p}{q} = \frac{p}{q_1 + Aq_1'} = \frac{\frac{p}{q_1}}{1 + \frac{Aq_1'}{q_1}} \end{aligned} \quad (34)$$

This function is Aq'_1/q_1 .

If we assume that q_1 has only simple zeros, a condition that can always be achieved by an appropriate choice of A , and further assume that q_1 is of third degree with two complex roots, then there is no loss in generality in the following discussion. The partial fraction expansion of y_{22}

$$\begin{aligned}
 y_{22} &= \frac{Aq'_1}{q_1} = \frac{Aq'_1}{(s - s_1)(s - s_2)(s - \bar{s}_2)} \\
 &= \frac{A}{s - s_1} + \frac{A}{s - s_2} + \frac{A}{s - \bar{s}_2} \\
 &= \frac{A}{s+d} + \frac{A}{s+a-j\omega_d} + \frac{A}{s+a+j\omega_d} \\
 &= \frac{A}{s+d} + \frac{2A(s+a)}{s^2 + 2as + \omega_0^2} \\
 &= y_1 + y_2 \tag{35}
 \end{aligned}$$

yields two typical positive real terms y_1 and y_2 whose network realizations are shown in Figs. 6 and 7, respectively. In obtaining the residues all equal to A , we have made use of the fact that the residue in a simple pole is the numerator divided by the derivative of the denominator, the resulting function being evaluated at the pole. It is of incidental interest to note that the immediately preceding discussion showing that Aq'_1/q_1 is positive real proves that the derivative of a Hurwitz polynomial is itself Hurwitz.

It is, of course, possible and often useful in Eqs. 34 to divide numerator and denominator by Aq'_1 instead of q_1 so that the $z = 1/y_{22}$ has component networks which are the duals of those given in Figs. 6 and 7.

In examples for which this step leads to a possible synthesis procedure, the resulting substitution of capacitances for inductances at the negative real poles is certainly desirable.

It is clear that the purpose of the breakdown of q is to arrive at the positive real RLC function discussed above. This is a particularly simple function that lends itself to the zero-shifting procedure discovered by the author. In some cases, however, the above breakdown is not necessary, for it is possible to achieve the same type of function but one that generally yields a final network with a greater gain and coils of lower Q . We manipulate the transfer function as follows

$$\begin{aligned}
 Y_{12} &= \frac{p}{q} \\
 &= \frac{\frac{p}{(-1)}}{\frac{q}{(-1)}} \tag{36}
 \end{aligned}$$

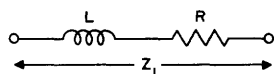


Fig. 6
RL network with
 $y_1 = [1/L(s+R/L)]$.

where $q^{(-1)}$ represents the integral of q . Now $q/q^{(-1)}$ is obviously the same type of function as discussed previously if $q^{(-1)}$ is a Hurwitz polynomial. However, unlike the derivative the integral of a Hurwitz polynomial is not necessarily Hurwitz. This represents a drawback to the use of the method, but it is a simple matter in any particular problem to form the integral, choosing the arbitrary constant conveniently, and then test for Hurwitz character. The increase in the degree of the denominator of the positive real function necessitates an increase in the number of elements used in the realization of Y_{12} , but this is generally a small price to pay for the Q-reduction and for the increase in gain. Thus an alternate, sometimes useful procedure is provided.

Finally, let us discuss one more breakdown after which we will list a table of numerical examples of the procedure with denominator polynomials that have useful filter characteristics. The polynomial q can always be divided into

$$q = q_1 + Aq_1' + Bq_1'' \quad (37)$$

by the same procedure as used for the breakdown given in Eq. 22. Again, q_1 , q_1' and A have the same significance as in Eq. 33, and B is a positive real constant while q_1'' is the derivative of q_1' . This breakdown allows the manipulation

$$\begin{aligned} Y_{12} = \frac{p}{q} &= \frac{p}{q_1 + Aq_1' + Bq_1''} \\ &= \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'}} \end{aligned} \quad (38)$$

where now Bq_1''/Aq_1' and the reciprocal of q_1/Aq_1' are of the same form as the functions already discussed. This form of breakdown is also useful for RLC networks as will be seen more fully in the sections that treat the synthesis procedure.

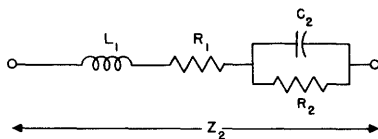


Fig. 7

RLC network with

$$y_2 = \frac{\left(s + \frac{1}{R_2 C_2}\right)}{L_1 \left[s^2 + \left(\frac{R_1}{L_1} + \frac{1}{R_2 C_2}\right) s + \frac{R_1}{L_1 R_2 C_2} + \frac{1}{L_1 C_2} \right]}$$

and $(L_1/R_1) = C_2 R_2$.

There are theorems that touch on the characteristics of the transformation from q to q_1 by means of Eq. 22, e.g. the theorem given as an exercise by Marden (10): "None of the complex zeros of the polynomial $f(z) + cf'(z)$ (this obviously is our $q = q_1 + Aq_1'$) for c real and $f(z)$ a real polynomial lies outside of the set of circles drawn with lines connecting conjugate zeros of $f(z)$ as diameters."

However, most of the theorems are not so illuminating as illustration by actual examples. Therefore tables showing such examples are given as Table 1 and Table 2. A second purpose is served by these tables: they contain q_1 's that arise in the design of normalized filters that approximate ideal lowpass behavior in the Tschebyscheff or in the Butterworth sense.

The Butterworth functions, $B_{2n}(\omega^2) = 1 + \omega^{2n}$, are for use in the representation of a squared magnitude of transfer function, e.g.

$$\left| Y_{12} \right|_{s=j\omega}^2 = \frac{1}{\left| q(s) \right|_{s=j\omega}^2} = \frac{1}{1 + \omega^{2n}}. \quad (39)$$

The coefficients and factors for $n = 1$ to $n = 8$ were obtained from reference 15 which contains a short discussion of both the Butterworth and Tschebyscheff polynomials.

The Tschebyscheff polynomials, $V_n(x) = \cos(n \cos^{-1} x)$, are also used in transfer functions in the form

$$\left| Y_{12} \right|_{s=j\omega}^2 = \frac{1}{\left| q(s) \right|_{s=j\omega}^2} = \frac{1}{1 + \epsilon^2 V_n^2(\omega)}. \quad (40)$$

Table 2 gives the factored form of q and the breakdowns for values of n that the author has used; the ripple factor ϵ was chosen as 0.5. In some cases partial fraction expansions useful in the lattice synthesis procedures are also given. As will be seen in section III, once we have obtained the given partial fraction expansion most of the calculation necessary to realize a lattice has been accomplished.

It is evident from Table 1C, for $n = 4$, $A = 0.25$, 0.1 , and 0.04 , that as A decreases the real part of the zeros of q_1 move farther from the j axis. It is important to note this relationship, for the Q of the coils used in the synthesis is dependent on the distance of the zeros from the j axis.

Table 1 Butterworth Polynomials

$$\left(\text{for use in } \left| Y_{12} \right|_{s=j\omega}^2 = \frac{1}{\left| q(s) \right|_{s=j\omega}^2} = \frac{1}{1 + \omega^{2n}} \right)$$

A. Polynomial Form: $q(s) = \gamma_n s^n + \gamma_{n-1} s^{n-1} + \dots + \gamma_1 s + 1$

n	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8
1	1							
2	$\sqrt{2}$	1						
3	2	2	1					
4	2.613	3.414	2.613	1				
5	3.236	5.236	5.236	3.236	1			
6	3.864	7.464	9.141	7.464	3.864	1		
7	4.494	10.103	14.606	14.606	10.103	4.494	1	
8	5.126	13.138	21.848	25.691	21.848	13.138	5.126	1

B. Factored Form of q

n	q(s)
1	(s+1)
2	$s^2 + 1.4142s + 1$
3	$(s^2 + s + 1)(s+1)$
4	$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$
5	$(s+1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9318s + 1)$
7	$(s+1)(s^2 + 0.4449s + 1)(s^2 + 1.2465s + 1)(s^2 + 1.8022s + 1)$
8	$(s^2 + 0.3896s + 1)(s^2 + 1.1110s + 1)(s^2 + 1.6630s + 1)(s^2 + 1.9622s + 1)$

C. Application of Breakdown $q = q_1 + Aq_1'$ or of Integral $q^{(-1)}$

n	Value of A	Polynomial Obtained From Breakdown	Partial Fraction Expansion
4	0.25 (A = 0.5 gives non-Hurwitz q_1)	$q_1 = (s^2 + 0.47850s + 1.091123)$ $\times (s^2 + 1.13462s + 0.570272)$	
	0.1	$q_1 = (s^2 + 1.61976s + 0.813459)$ $\times (s^2 + 0.59336s + 0.975703)$	$\frac{1}{q_1} = \frac{0.44509 - j1.0854}{s + 0.80988 - j0.39693}$ $+ \frac{-0.44509 + j0.21483}{s + 0.29668 - j0.94217}$ + conjugates
	0.04	$q_1 = (s^2 + 0.68948s + 0.978323)$ $\times (s^2 + 1.76364s + 0.925516)$	
	--	$5q^{(-1)} = (s + 0.282216)$ $\times (s^2 + 2.5174s + 2.089093)$ $\times (s^2 + 0.42676s + 1.695985)$ (for arbitrary constant = 1)	$\frac{1}{5q^{(-1)}} = \frac{0.41430}{s + 0.282216}$ $+ \frac{-0.20335 + j0.075606}{s + 1.2587 - j0.71047}$ $+ \frac{-0.0037996 + j0.11254}{s + 0.21338 - j1.2847}$ + conjugates of last two terms
5	0.25 (A = 0.5 gives non-Hurwitz q_1)	$q_1 = (s + 0.51087)$ $\times (s^2 + 1.06228s + 1.020644)$ $\times (s^2 + 0.41284s + 1.037197)$	
	0.1	$q_1 = (s + 0.87052)$ $\times (s^2 + 0.46292s + 0.997114)$ $\times (s^2 + 1.40256s + 0.870274)$	$\frac{1}{q_1} = \frac{1.8124}{s + 0.87052}$ $+ \frac{-0.10000 + j0.44281}{s + 0.23146 - j0.97136}$ $+ \frac{-0.80619 - j1.0235}{s + 0.70128 - j0.61602}$ + conjugates of last two terms
5	0.04	$q_1 = (s + 1.1561)$ $\times (s^2 + 1.25442s + 0.704399)$ $\times (s^2 + 0.62552s + 1.087864)$	

C. Application of Breakdown $q = q_1 + Aq_1^!$ or of Integral $q^{(-1)}$ (continued)

n	Value of A	Polynomial Obtained From Breakdown	Partial Fraction Expansion
--		$6q^{(-1)} = (s+1.4694)(s+0.23913)$ $\times(s^2+1.98756s+1.972000)$ $\times(s^2+0.187134s+1.443240)$ (for arbitrary constant = 1; arbitrary constant = 5 gives non-Hurwitz $q^{(-1)}$) (for arbitrary constant = 2):	
		$6q^{(-1)} = (s^2+2.2022s+1.224127)$ $\times(s^2+1.42180s+1.262714)$ $\times(s^2+0.25920s+1.191418)$	$\frac{1}{6q^{(-1)}} = \frac{0.4986 - j0.6619}{s+1.1011 - j0.34214}$ $+ \frac{-0.5188 - j0.1642}{s+0.71090 - j0.87025}$ $+ \frac{0.02020 + j0.1721}{s+0.12960 - j1.0838}$ + conjugates
		(for arbitrary constant = 3):	
		$6q^{(-1)} = (s^2+2.7370s+2.175743)$ $\times(s^2+1.08268s+1.622458)$ $\times(s^2+0.063528s+0.849821)$	
6	0.25 (A = 0.5 gives non-Hurwitz q_1)	$q_1 = (s^2+0.272854s+1.050074)$ $\times(s^2+1.291154s+1.522121)$ $\times(s^2+0.799992s+0.333286)$	
	0.1	$q_1 = (s^2+0.376794s+1.008894)$ $\times(s^2+1.224278s+0.928860)$ $\times(s^2+1.662928s+0.770285)$	$\frac{1}{q_1} = \frac{0.2569 + j0.3441}{s+0.18840 - j0.98661}$ $+ \frac{-1.429 + j0.1525}{s+0.61214 - j0.74441}$ $+ \frac{1.172 - j2.137}{s+0.83146 - j0.28134}$ + conjugates
	0.04	$q_1 = (s^2+1.3344s+0.951957)$ $\times(s^2+1.8432s+0.917468)$ $\times(s^2+0.4464s+0.991688)$	
	--	(for arbitrary constant = 1):	
		$7q^{(-1)} = (s+0.207556)$	$\frac{1}{7q^{(-1)}} = \frac{0.31728}{s+0.207556}$

C. Application of Breakdown $q = q_1 + Aq_1'$ or of Integral $q^{(-1)}$ (continued)

n	Value of A	Polynomial Obtained From Breakdown	Partial Fraction Expansion
		$\times(s^2 + 0.072481s + 1.268145)$	$+\frac{0.05860 + j0.04820}{s + 0.036241 - j1.125536}$
		$\times(s^2 + 2.744038s + 2.108976)$	$+\frac{-0.16352 - j0.06164}{s + 1.372019 - j0.475962}$
		$\times(s^2 + 1.483936s + 1.801458)$	$+\frac{-0.05373 + j0.13016}{s + 0.741968 - j1.118455}$
			+ conjugates of last three terms
6		(for arbitrary constant = 2): $7q^{(-1)} = (s + 1.221743)$	$\frac{1}{7q^{(-1)}} = \frac{0.76063}{s + 1.221743}$
		$\times(s^2 + 0.25266s + 1.130052)$	$+\frac{0.15753 + j0.11433}{s + 0.126330 - j1.055506}$
		$\times(s^2 + 1.94566s + 1.341361)$	$+\frac{0.16548 - 0.82072}{s + 0.972830 - j0.628461}$
		$\times(s^2 + 1.08795s + 1.079955)$	$+\frac{-0.70333 + j0.14927}{s + 0.543975 - j0.885464}$
			+ conjugates of last three terms
8	0.25	$q_1 = (s^2 + 0.39913s + 0.161697)$	$\frac{1}{q_1} = \frac{0.15575 - j0.70692}{s + 0.19956 - j0.34910}$
		$\times(s^2 + 1.96778s + 2.579670)$	$+\frac{-0.04298 - j0.01382}{s + 0.98389 - j1.2695}$
		$\times(s^2 + 0.58140s + 1.310155)$	$+\frac{-0.33714 + j0.28874}{s + 0.29070 - j1.10709}$
		$\times(s^2 + 0.177698s + 0.930474)$	$+\frac{0.53588 + 0.071259}{s + 0.08885 - j0.96051}$
			+ conjugates
	0.04	$q_1 = (s^2 + 0.29704s + 1.010353)$	
		$\times(s^2 + 1.988324s + 1.465984)$	
		$\times(s^2 + 1.545970s + 0.669542)$	
		$\times(s^2 + 0.974660s + 0.836514)$	

Table 2 Tschebyscheff Polynomials

$$\left(\text{for use in } \left| Y_{12} \right|_{s=j\omega}^2 = \frac{1}{\left| q(s) \right|_{s=j\omega}^2} = \frac{1}{1 + \epsilon^2 V_n^2(\omega)} \right)$$

$$\text{where } V_n(x) = \cos(n \cos^{-1} x) \text{ and } \epsilon = 0.5$$

A. Factored Form of q

n	q(s)
6	$(s^2 + 0.125749s + 0.992035) (s^2 + 0.343554s + 0.559019)$ $\times (s^2 + 0.469303s + 0.126002)$
8	$(s^2 + 0.070791s + 0.994865) (s^2 + 0.201595s + 0.724259)$ $\times (s^2 + 0.301709s + 0.341575) (s^2 + 0.355891s + 0.070978)$

B. Application of Breakdown $q = q_1 + Aq_1'$

n	Value of A	Polynomial Obtained From Breakdown	Partial Fraction Expansion
6	0.05 (A = 0.1 gives non-Hurwitz q_1)	$q_1 = (s^2 + 0.36248s + 0.104308)$ $\times (s^2 + 0.040040s + 1.016465)$ $\times (s^2 + 0.23608s + 0.550460)$	$\frac{1}{q_1} = \frac{0.6522 - j4.029}{s + 0.18124 - j0.26732}$ $+ \frac{0.6719 - j0.7387}{s + 0.02002 - j1.008}$ $+ \frac{-1.324 + j2.521}{s + 0.11804 - j0.73248}$ $+ \text{conjugates}$
8	0.05	$q_1 = (s^2 + 0.191448s + 0.333072)$ $\times (s^2 + 0.087830s + 0.730835)$ $\times (s^2 + 0.24584s + 0.054782)$ $\times (s^2 + 0.004876s + 1.040202)$	$\frac{1}{q_1} = \frac{-3.954 + j9.409}{s + 0.095724 - j0.56913}$ $+ \frac{3.769 - j5.127}{s + 0.043915 - j0.85376}$ $+ \frac{1.619 - j12.34}{s + 0.12292 - j0.19918}$ $+ \frac{-1.434 + j1.467}{s + 0.0024382 - j1.0199}$ $+ \text{conjugates}$

II. 3 Zero Shifting in the Left Half-Plane

Some of the procedures for RLC synthesis being built up in this report require a method of zero shifting with two or more complex poles. In the synthesis procedure zero shifting with two-element kind driving-point functions to points only on the negative real axis is also used, but since this method has already been worked out in general, it is not treated here. This section proves that zero shifting to any two points in the left half-plane is always possible with one pair of complex poles, and further demonstrates conditions under which zero shifting may be carried on with positive real functions of higher degree.

Let us assume for purposes of illustration that in Eq. 34 the numerator and denominator had been divided by Aq_1' instead of q_1 so that

$$z = \frac{1}{y_{22}} = \frac{Aq_1'}{q_1} \quad (41)$$

has the typical partial fraction expansion given by Eq. 35. Thus $2A(s+a)/s^2 + 2as + \omega_0^2$ represents the typical term with complex poles z_2 . It is desired to illustrate zero shifting with this term.

Since the $2A$ is merely a scale factor, for convenience in manipulation and no loss in generality, let $2A = 1$. Inverting z_2 and performing long division yields

$$\begin{aligned} y &= \frac{1}{z_2} = \frac{s^2 + 2as + \omega_0^2}{s+a} \\ &= s + a + \frac{\omega_0^2 - a^2}{s+a} \\ &= s + a + \frac{\omega_d^2}{s+a}. \end{aligned} \quad (42)$$

Next in the zero-shifting step we remove the admittance

$$y' = (1-a)s + (1-b)a + \frac{(1-c)\omega_d^2}{s+a} \quad (43)$$

where

$$\left. \begin{aligned} 0 &\leq a \leq 1 \\ 0 &\leq b \leq 1 \\ 0 &\leq c \leq 1. \end{aligned} \right\} \quad (44)$$

The inverted remainder, which is used in the zero-producing step, is

$$z' = \frac{1}{y-y'} = \frac{s+a}{a \left[s^2 + \left(1 + \frac{b}{a}\right)as + \frac{b}{a}a^2 + \frac{c}{a}\omega_d^2 \right]}. \quad (45)$$

Suppose that the desired complex conjugate positions for zeros in the left half-plane are at $s = -\beta \pm j\omega_1$, which yield a quadratic factor $s^2 + 2\beta s + \omega_2^2$, where $\omega_2^2 = \beta^2 + \omega_1^2$. Since the poles of z' , which represents a series arm of a ladder network, appear as zeros of the over-all transfer function, it is only necessary to find the values of a , b and c such that the denominator of z' is made to have zeros at the desired positions, that is, to satisfy the identity

$$s^2 + 2\beta s + \omega_2^2 \equiv s^2 + \left(1 + \frac{b}{a}\right)as + \frac{b}{a}a^2 + \frac{c}{a}\omega_d^2. \quad (46)$$

This can always be done provided only that $\beta \geq \alpha/2$. (Since an increase in the value of A will decrease α , i. e. bring the zeros of q_1 closer to the j axis, this is not a severe restriction on the method. Moreover, a modification of the method that removes the restriction $\beta \geq \alpha/2$ but uses more elements is explained after the completion of the proof.)

The proof of the above statement on the range of zero shifting is straightforward. We see immediately from the first of the following identifications derived from Eq. 46

$$\left. \begin{aligned} \left(1 + \frac{b}{a}\right)a &= 2\beta \\ \frac{b}{a}a^2 + \frac{c}{a}\omega_d^2 &= \omega_2^2 \end{aligned} \right\} \quad (47)$$

that the minimum value of β is $\alpha/2$, since b/a can take on any nonnegative value. Since the same restriction applies to c/a , whose value, moreover, is independent of that of b/a , and since $\omega_2^2 > \beta^2$ because of the complex conjugate nature of the roots, to prove that any ω_2^2 can be achieved, we have only to show that

$$\frac{b}{a}a^2 \leq \beta^2. \quad (48)$$

Substituting the value of a from the first of Eqs. 47 into $(b/a)a^2$, we have

$$\begin{aligned} \frac{b}{a}a^2 &= \frac{\frac{b}{a}4\beta^2}{\left(1 + \frac{b}{a}\right)^2} \\ &= \frac{\frac{b}{a}4\beta^2}{\left(\frac{b}{a}\right)^2 + 2\frac{b}{a} + 1} \\ &= \frac{x4\beta^2}{x^2 + 2x + 1}. \end{aligned} \quad (49)$$

This function has a maximum value of β^2 at $x = b/a = 1$ so that the proof is complete.

For the case in which $\beta < \alpha/2$ and we do not wish to decrease α by the method of increasing A , we can use a modification of the preceding method. By means of an appropriate decomposition of the z_2 of Eq. 42 before inversion, we can realize a β as

close to the j axis as we wish. For example, for a β equal to $0.4a$, we write

$$\begin{aligned}
 z_2 &= \frac{s+a}{s^2+2as+\omega_0^2} = \frac{\frac{3}{4}s + \frac{1}{2}a}{s^2+2as+\omega_0^2} + \frac{\frac{1}{4}s + \frac{1}{2}a}{s^2+2as+\omega_0^2} \\
 &= \frac{\frac{3}{4}(s + \frac{2}{3}a)}{s^2+2as+\omega_0^2} + \frac{\frac{1}{4}(s+2a)}{s^2+2as+\omega_0^2} \\
 &= z_2' + z_2'' \tag{50}
 \end{aligned}$$

We remove z_2'' as a series branch and work with z_2' . Performing inversion and long division as before, we obtain the admittance

$$\begin{aligned}
 y &\equiv \frac{1}{z_2'} = \frac{s^2+2as+\omega_0^2}{\frac{3}{4}(s + \frac{2}{3}a)} \\
 &= \frac{4}{3} \left[s + \frac{4}{3}a + \frac{\omega_0^2 + \frac{1}{9}a^2}{s + \frac{2}{3}a} \right] \\
 &= \frac{4}{3} \left[s + \frac{4}{3}a + \frac{\omega_a^2}{s + \frac{2}{3}a} \right] \tag{51}
 \end{aligned}$$

which upon removal of the shunt branch

$$y' = \frac{4}{3} \left[(1-a)s + (1-b)\frac{4}{3}a + \frac{(1-c)\omega_a^2}{s + \frac{2}{3}a} \right] \tag{52}$$

yields an inverted remainder

$$z' \equiv \frac{1}{y-y'} = \frac{4}{3} \left[\frac{s + \frac{2}{3}a}{a \left[s^2 + \left(\frac{2}{3} + \frac{4}{3}\frac{b}{a} \right) as + \frac{8}{9}\frac{b}{a}a^2 + \frac{c}{a}\omega_a^2 \right]} \right] \tag{53}$$

With this remainder it is clear that zeros can have a

$$\beta \geq \frac{1}{3}a \tag{54}$$

so that the given zero positions can be achieved.

Frequently it is useful for reducing the restrictions on the breakdown of the numerator p to be able to zero shift with a positive real function possessing a third or fourth degree denominator. It has been found possible in many cases to place a zero at infinity and two others at arbitrary points by use of the combination of a real pole with two

complex poles and to place two zeros at infinity and two at arbitrary points with a combination of two pairs of complex poles. Though it is cumbersome to formulate the restrictions on the coefficients of the zero-shifting function for either of the above techniques to work, experience with the techniques soon shows one how to use them efficiently. The idea is simply to combine the two partial fractions, invert, take out the pole at infinity and the real part at infinity. This reduces the degree of the numerator by two. In order for these steps to yield a positive real remainder, the real part of the function after the pole at infinity has been removed must have its minimum at infinity. If this does not work, using only a part of the residue of either or both partial fractions for the zero-shifting combination may prove successful.

The procedure is illustrated by a numerical example using one real pole and one pair of complex poles. Suppose

$$\begin{aligned}
 y_{22} &= \frac{Aq_1'}{q_1} = y_1 + y_2 + \dots \\
 &= \frac{A}{s+d} + \frac{2A(s+a)}{s^2 + 2as + \omega_0^2} + \dots \\
 &= \frac{1}{s+4} + \frac{2(s+2)}{s^2 + 4s + 7} + \dots
 \end{aligned} \tag{55}$$

and we wish to zero shift only with the portion of y_{22} containing $y_1 + y_2$. We combine the terms y_1 and y_2 , invert, and perform ordinary long division to find

$$\begin{aligned}
 z &\equiv \frac{1}{y_1 + y_2} = \frac{s^3 + 8s^2 + 23s + 28}{3s^2 + 16s + 23} \\
 &= 0.33s + 0.89 + \frac{1.09(s + 6.91)}{3(s^2 + 5.33s + 23)}.
 \end{aligned} \tag{56}$$

Since $6.91 > 5.33$, the last term is not positive real and the method fails. However, if we divide y so that

$$\begin{aligned}
 y &= y_1 + y_2 = y_1' + y_1'' + y_2 \\
 &= \frac{0.5}{s+4} + \frac{0.5}{s+4} + \frac{2s+4}{s^2 + 4s + 7}
 \end{aligned} \tag{57}$$

and then remove y_1' and perform zero shifting with the remaining sum, we obtain by the same procedure

$$\begin{aligned}
 z &\equiv \frac{1}{y_1'' + y_2} = \frac{s^3 + 8s^2 + 23s + 28}{2.5s^2 + 14s + 19.5} \\
 &= 0.4s + 0.96 + \frac{1.76(s + 5.22)}{2.5(s^2 + 5.6s + 7.8)}
 \end{aligned} \tag{58}$$

whose final term is positive real and of the form that allows zero shifting to any two points by the method explained.

II.4 A Network Theorem Useful in Synthesis

A network theorem, which has been previously derived and called a partitioning theorem (11), but is basically an application of Thevenin's or Norton's Theorem, is herein explained with indications given of its application to synthesis problems.

Consider a network N shown in Fig. 8a, where E_2 represents the open-circuit voltage. After dividing this network into any two smaller networks N_a and N_b connected by a pair of wires, apply Norton's Theorem to the network on the left. Finally, because of convenience for applications to synthesis, we show explicitly the final shunt arm of network N_b . This is indicated in Fig. 8d, where N_b is the total right-hand network comprising N_b' and Z , and E_2 is now the output voltage across the arbitrary termination Z . The complete sequence of steps is shown in Fig. 8.

With reference to the symbols defined in the figure, the following definitions are self-evident

$$\left. \begin{aligned}
 Y_{12} &\equiv \frac{I_2}{E_1} \\
 K &\equiv \frac{I_2 Z}{E_1} = \frac{E_2}{E_1} \\
 Y_{12b} &\equiv \frac{I_2}{E} \\
 K_b &\equiv \frac{I_2 Z}{E} = \frac{E_2}{E} = Z Y_{12b} \\
 y_{12a} &\equiv \frac{I_{sc}}{E_1}
 \end{aligned} \right\} \quad (59)$$

Also let Y_a and Y_b be the respective driving-point admittances of N_a and N_b for the directions indicated in Fig. 8b. From its definition it is apparent that Y_a is equivalent to the short-circuit admittance y_{22a} . We now proceed to the derivation of the useful formulas.

From Fig. 8d we note that I_{sc} , the short-circuit current obtained by the Norton transformation, flows through Y_a plus Y_b to give rise to the voltage E ; thus

$$\begin{aligned}
 E &= \frac{I_{sc}}{Y_a + Y_b} \\
 &= \frac{E_1 y_{12a}}{Y_a + Y_b} \\
 &= \frac{E_1 y_{12a}}{y_{22a} + Y_b}
 \end{aligned} \quad (60)$$

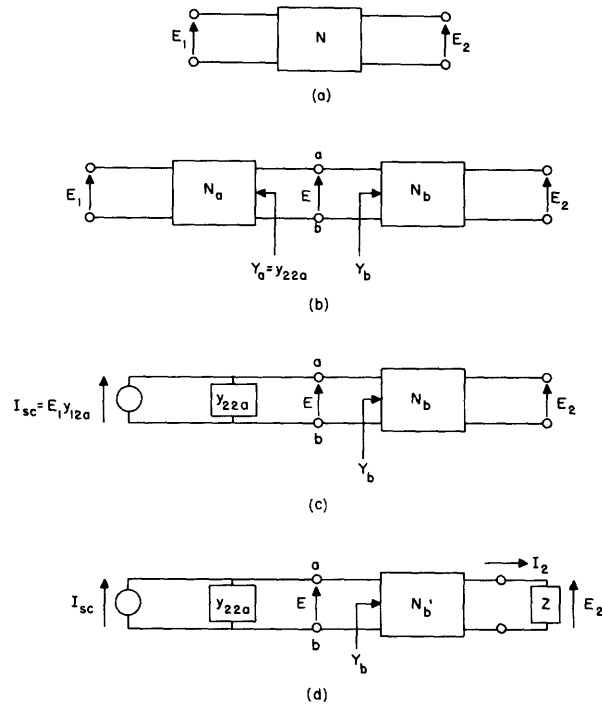


Fig. 8

Simplification of network N; (a) original network, (b) network decomposed into two parts, (c) network after application of Norton's Theorem to N_a , (d) network with final branch shown explicitly.

Furthermore, since

$$I_2 = EY_{12b} \quad (61)$$

then substitution of Eq. 60 into Eq. 61 yields

$$I_2 = \frac{E_1 y_{12a} Y_{12b}}{y_{22a} + Y_b} \quad (61a)$$

or

$$Y_{12} = \frac{I_2}{E_1} = \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} \quad (62)$$

and

$$\begin{aligned} K &= \frac{E_2}{E_1} \\ &= \frac{y_{12a} Y_{12b} Z}{y_{22a} + Y_b} \end{aligned} \quad (62a)$$

which yields the final form

$$K = \frac{E_2}{E_1} = \frac{y_{12a} K_b}{y_{22a} + Y_b} \quad (63)$$

Though up to the present time most of the applications of Eqs. 62 and 63 have been to network analysis, this report will show how they may be exploited for synthesis.

It can easily be shown that for $Z = R = 1$ ohm, Eq. 62 reduces to the familiar Eq. 3. The latter equation is used, for example, in the Guillemin synthesis of RC networks. There is nothing to prevent the use of Eq. 62 for such synthesis, where we zero shift and parallel ladder structures to form network N_a , and zero shift with Y_b to obtain N_b ; that is, we zero shift looking in both directions from within network N . From these remarks it is clear that any advantages obtained by use of Eq. 62 rather than Eq. 3 for synthesis derive from the fact that we have in effect used as a termination a network whose transfer and driving-point admittances are complicated functions of frequency rather than a termination of merely one mho. Similar remarks apply to RLC synthesis.

II.5 Breakdown of Numerator Polynomial for RLC Synthesis

Of the three steps necessary for the extension of the basic ideas of the Guillemin RC synthesis procedure to RLC networks, two have already been considered. These two are the breakdown of the Hurwitz denominator and zero shifting in the left half-plane. Each imposes restrictions on the third step, which is the breakdown of the numerator polynomial. Restrictions are also imposed on this step by the variant of the synthesis procedure used; for example, an acceptable type of breakdown of p may be possible with the use of $q = q_1 + Aq_1' + Bq_1''$ but not with $q = q_1 + Aq_1'$; or it may be possible if we identify y_{22} with q_1/Aq_1' but not if we use $y_{22} \equiv Aq_1'/q_1$; or, again, unless we can zero shift with a combination of a real pole and a pair of complex poles, we may find the breakdown of p impossible.

It is the purpose of this section to examine the acceptable breakdowns of the numerator of a given minimum-phase transfer admittance that is to be realized directly (i. e. not through the intermediary of the lattice structure) by unbalanced RLC networks with lossy coils. This step is not so elegant as the other two in that it is not accomplished like the others by a direct automatic procedure. There are different cases to be considered and, furthermore, the designer must understand in detail the implications of the other steps and of the synthesis procedures themselves. He must know what is and what is not possible by the techniques of this section; for example, to use an idea that comes readily to mind, he must realize in the use of the network theorem of section II.4 that the component ladders of N_a must end in a series branch, N_b in a shunt branch, or that N_a may consist of paralleled ladders while, on the contrary, N_b must be realized as one ladder. These characteristics and others determine the acceptable types of p -breakdown.

If the transfer function is a third degree polynomial over a fourth, or of lower

degree, the polynomial p need not be decomposed by the methods of this section. The synthesis can always be achieved directly, as explained in section VI. In fact, very often functions of higher degree may be synthesized by the direct method of that section. A numerator breakdown, as discussed in this section, is necessary for high degree transfer functions for which the method of section VI does not apply.

The preliminary statement is made that in the decomposition into

$$p = p_1 + p_2 + \dots \quad (64)$$

all the component polynomials of p must, of course, have positive coefficients. The number of components is generally unrestricted, but since each component is associated with a ladder, it is desirable to use as small a number of components as possible. The fact that no negative sign can be allowed in the decomposition is the source of the difficulty of this step.

We now consider the different cases that may arise. Each case is first characterized by a variant of the synthesis procedure and by the type of zero shifting possible. Then the restrictions on the p -breakdown are stated, and finally, when necessary, we add some remarks on the characteristics of the synthesis method. The type of zero shifting possible in a particular problem is determined by trial (see section II.3). There are two types considered, designated for each numbered subdivision by a and b, respectively. They are:

- a. zero shifting is possible with all negative real poles (which, of course, is always true) or with only one pair of complex poles at a time;
- b. zero shifting is possible with all negative real poles or with a combination of poles, e.g. one pair of complex poles plus one real pole or two pairs of complex poles.

The various procedures open to the designer follow.

A. Without use of the network theorem in section II.4, a synthesis procedure that realizes an RLC network N terminated in resistance or complex impedance

1. Breakdown of q into $q_1 + Aq_1'$ and a subsequent division of numerator and denominator of Y_{12} by Aq_1' , so that

$$Y_{12} = \frac{y_{12}}{1 + y_{22}} = \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'}} \quad (65)$$

whence we can identify $z \equiv 1/y_{22} = Aq_1'/q_1$.

- a. The requirement on the p -breakdown for this case is that each component of p possess either all the zeros of q_1 except for one complex pair or all the zeros of q_1 except for all the negative real zeros. For the former condition the other zeros of the components may be anywhere in the left half-plane; for the latter they must lie on the negative real axis, including the origin and the point at infinity.

To determine whether the desired breakdown is possible we use a technique that will

be used in all of the divisions: expansion into partial fractions. In this case we expand p/q_1 . Let us assume without loss of generality that q_1 has four complex and two real poles and p is of lower degree than q_1 ; then

$$\begin{aligned} \frac{p}{q_1} &= \frac{k_1}{s+a} + \frac{k_2}{s+b} + \frac{es+f}{s^2+2cs+d} + \frac{gs+h}{s^2+2ns+t} \\ &= \frac{ms+r}{(s+a)(s+b)} + \frac{es+f}{s^2+2cs+d} + \frac{gs+h}{s^2+2ns+t} \end{aligned} \quad (66)$$

where as always s is the complex frequency variable while the other letters are real constants. As shown in the above equation, the real poles are always combined into one fraction. If the coefficients of the numerator of this combined fraction and of each of the numerators of the fractions with complex poles are all positive, then the desired breakdown is possible and the synthesis procedure can be carried out. That the components of p have the desired characteristic can be seen by multiplying both sides of Eq. 66 by q_1 so that

$$\begin{aligned} p &= (ms+r)(s^2+2cs+d)(s^2+2ns+t) + (es+f)(s+a) \\ &\quad \times (s+b)(s^2+2ns+t) + (gs+h)(s+a)(s+b)(s^2+2cs+d). \end{aligned} \quad (67)$$

The requirement of positive coefficients is surely a severe one, but one should not jump to the conclusion that what is requested is that p/q_1 be positive real, or, in fact, the more difficult-to-satisfy requirement that its partial fraction components for complex poles and the combined fraction for the negative real poles be positive real. The first conclusion is not necessary and the second, though sufficient, is also not necessary. Substituting numbers in the fractions of Eq. 66 illustrates the points clearly, for

$$\begin{aligned} \frac{es+f}{s^2+2cs+d} &= \frac{2s+8}{s^2+2s+5} \\ \frac{gs+h}{s^2+2ns+t} &= \frac{3s+12}{s^2+3s+7} \end{aligned} \quad (68)$$

and

$$\begin{aligned} \frac{k_1}{s+a} + \frac{k_2}{s+b} &= \frac{-\frac{3}{2}}{s+3} + \frac{\frac{5}{2}}{s+1} \\ \frac{ms+r}{(s+a)(s+b)} &= \frac{s+6}{s^2+4s+3} \end{aligned} \quad (69)$$

satisfy the conditions of the desired breakdown, but not one of the fractions with quadratic denominators is positive real. It is evident, too, that p must be of the fifth degree.

b. This case places a less stringent requirement on the breakdown of p . Considering again the p/q_1 of Eq. 66, suppose we find that it is possible to zero shift with a combination of the partial fractions of $1/y_{22}$ having denominators $(s+b)$ and $(s^2+2cs+d)$.

Then, if e and f are not positive but k_2 is, we have another avenue opened to us for obtaining the desired positive coefficients. We may use parts of the positive residue k_2 to obtain

$$\begin{aligned} \frac{p}{q_1} &= \frac{k_1}{s+a} + \frac{k_2}{s+b} + \frac{es+f}{s^2+2cs+d} + \frac{gs+h}{s^2+2ns+t} \\ &= \frac{k_1}{s+a} + \frac{k'_2}{s+b} + \frac{k''_2}{s+b} + \frac{es+f}{s^2+2cs+d} + \frac{gs+h}{s^2+2ns+t} \\ &= \frac{us+v}{(s+a)(s+b)} + \frac{ws^2+ys+x}{(s+b)(s^2+2cs+d)} + \frac{gs+h}{s^2+2ns+t} \end{aligned} \quad (70)$$

of which the final equation must have positive coefficients in the numerator if the synthesis procedure is to be carried out successfully.

In words, the requirement on the components of p is that each one have all the zeros of q_1 except for all the negative real zeros or each one must have all the zeros of q_1 except for one complex pair or except for the complex pair and the negative real pole that allow zero shifting in combination. The last part of this statement represents a lessening of the restriction of case a. Thus, if it is found possible to zero shift with a combination of two pairs of complex poles, the same reasoning shows us the necessary requirements on the breakdown.

2. Breakdown of q into $q_1 + Aq'_1$ with a subsequent division by q_1 so that

$$Y_{12} = \frac{\frac{p}{q_1}}{1 + \frac{Aq'_1}{q_1}} \quad (71)$$

and

$$y_{22} = \frac{Aq'_1}{q_1}.$$

a, b. The same conditions as for subdivisions a and b of 1 above apply respectively to a and b for this case. There is also an additional condition placed on the unspecified zeros of the components of p because of the form of y_{22} . Since Aq'_1/q_1 is an admittance (notice that it is an impedance in 1 above) the zero-producing branch for complex poles must be a shunt one; however, it is recalled that the final branch of the ladder structures must be a series one. Therefore complex unspecified zeros in the components of p that have all the zeros of q_1 except for one complex pair are inadmissible; there must be no additional zeros or if there are, they must be negative real.

An advantage of the use of this procedure, when it is possible, is that the zero-pole cancellation in p/q_1 makes for a reduction in the required number of elements.

3. When the integral of q (represented by $q^{(-1)}$) is a Hurwitz polynomial we can

try to satisfy the conditions on the p-components for the realization of an RLC network terminated in Y_L with a transfer voltage ratio

$$K = \frac{E_2}{E_1} = \frac{y_{12}}{Y_L + y_{22}} = \frac{\frac{p}{q^{(-1)}}}{\frac{q}{q^{(-1)}}}. \quad (72)$$

a, b. The conditions on the p-breakdown that must be satisfied for synthesis to be possible are the same as those for 2 except that $q^{(-1)}$ is substituted wherever q_1 appears; that is, the components of p must have all the zeros of $q^{(-1)}$ except ...; we expand $p/q^{(-1)}$ in partial fractions, etc.

4. Breakdown of q into $q_1 + Aq_1' + Bq_1''$ and a subsequent division by Aq_1' so that

$$Y_{12} = \frac{\frac{p}{Aq_1'}}{\frac{q_1}{1 + \frac{Aq_1'}{Aq_1'}} + \frac{Bq_1''}{Aq_1'}} = \frac{y_{12}}{1 + y_{22}}. \quad (73)$$

a, b. The restrictions on the p-breakdown may now take a variety of forms so that this represents a case that is satisfied relatively easily. (At the end of this subdivision there will be a discussion of two synthesis procedures that eliminate the need for any searching for a breakdown of p.) The components of p must be associated with either the zeros of q_1 or of q_1' in the sense that they contain these zeros and satisfy the appropriate conditions. These conditions are those given in 1 for the components associated with q_1 and those in 2 for the components associated with q_1' . At least one p-component must be associated with the zeros of q_1 ; none or all but one of the components may be associated with q_1' . The procedure to achieve this will now be formulated.

First we carry out the partial fraction expansion of p/q_1 as in 1; if the conditions on the components are satisfied, then we need proceed no further, for the synthesis can be achieved. If they are not, we obtain a new polynomial p_a by the operation

$$p_a = kp - p_1 \quad (74)$$

where k is a positive constant large enough so that p_a has all positive coefficients and p_1 is a component that is formed from the zeros of q_1 and meets the necessary conditions of 1. Or we may form p_a from

$$p_a = kp - p_1 - p_2 \quad (75)$$

where p_1 and p_2 are components associated with q_1 . The components may contain no arbitrary zeros or one or two of them. Thus, since the components subtracted may take many forms, p_a may have a number of different forms, one of which may allow the next step to satisfy the necessary conditions of 2. This next step is, of course, forming the partial fraction expansion of p_a/q_1' . If this satisfies the conditions on the components, then the synthesis procedure will succeed.

Lest it appear that the many steps in any of the numbered subdivisions make for an excessive amount of work, a few amplifications are given at this point. (The complete synthesis procedures are given in section V.) First of all, it should be obvious that some of the desired breakdowns can be shown by inspection to be impossible. For example, it is manifestly impossible for

$$\frac{p}{q_1} = \frac{1}{(s+a)(s+b)(s+c)(s^2 + 2ds + e)} \quad (76)$$

to have the desired partial fraction expansion given in case 1 since a negative sign must appear. The impossibility of satisfying Eq. 74 or 75 is again obvious, if the smallest possible degree of p_1 is greater than the degree of p . From what has been said of the conditions on the p -components, we observe that the possible degree of p_1 is directly dependent on the number of complex zeros of q_1 ; if q_1 has only two complex zeros, then the smallest possible degree is two.

The question arises whether all the zeros of q_1 can be negative real even though the zeros of q may lie anywhere in the left half-plane. This is a very important question. On its answer hinges the achievement of an elegant synthesis procedure for RLC networks. The epithet "elegant" is used advisedly since the synthesis procedure for three-element kind networks then has the simplicity of two-element kind synthesis. Another very important condition on the zeros, which again eliminates the need for casting about for an acceptable p -breakdown, is the occurrence of only negative real zeros in q_1' even though q and q_1 have some complex zeros. This circumstance leads to a direct synthesis procedure almost as elegant as the other.

We know that if q has any complex zeros it is impossible to find a q_1 satisfying the conditions of the relationship $q = q_1 + Aq_1'$ and possessing only negative real zeros. For if this were possible, we could then realize a Y_{12} with complex poles by means of RC networks, a manifest impossibility. However, q_1 derived from a q with some complex zeros by the breakdown $q = q_1 + Aq_1' + Bq_1''$ may have all its zeros negative real. If it does, by inspection of

$$Y_{12} = \frac{y_{12}}{1 + y_{22}} = \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'}} \quad (77)$$

we observe that q_1/Aq_1' represents an RC admittance, since by Rolle's Theorem (14) q_1' has all negative real zeros with the zero of smallest magnitude occurring in q_1 ; whereas Bq_1''/Aq_1' by use of similar reasoning corresponds to an RL admittance, that is, the three-element kind network has been divided into two sets of two-element kind networks. Now the synthesis can be the familiar RC synthesis and no p -breakdown need be found. One form of final network may appear as in Fig. 9, where H is a positive constant representing the gain.

An example of this eminently useful breakdown of q is the polynomial without all

negative real zeros

$$q = s^4 + 11s^3 + 54.5s^2 + 127.5s + 106.5 \quad (78)$$

which with

$$\left. \begin{aligned} A &= \frac{1}{4} \\ B &= 1 \end{aligned} \right\} \quad (79)$$

yields

$$\left. \begin{aligned} q_1 &= (s+1)(s+2)(s+3)(s+4) \\ &= s^4 + 10s^3 + 35s^2 + 50s + 24 \\ Aq_1' &= \frac{1}{4}(4s^3 + 30s^2 + 70s + 50) \\ Bq_1'' &= 1(12s^2 + 60s + 70). \end{aligned} \right\} \quad (80)$$

The example, chosen merely to illustrate that the breakdown is possible, was computed by a choice of q_1 , A , and B , which led to a q that can be shown by the Sturm test (14) not to have all negative real zeros. Of course, in actual practice, we are given a q so that our problem is the reverse: we must find an A and a B such that q_1 is Hurwitz with no complex zeros.

If the second happy circumstance occurs, that is, if q_1' has only negative real zeros though q_1 has some complex zeros, we calculate a new polynomial as in Eq. 74, that is

$$p_a = kp - p_1 \quad (81)$$

where as before k is a positive constant large enough so that p_a has all positive coefficients and p_1 is the polynomial made up of the complex zeros of q_1 . Such a polynomial p_a can always be found provided only that p is of high enough degree. For example, if q_1 has only one pair of complex zeros then p must be at least of the second degree.

We thus have

$$kY_{12} = \frac{\frac{p_1}{Aq_1'} + \frac{p_a}{Aq_1'}}{1 + \frac{Bq_1''}{Aq_1'}} \quad (82)$$

It is now possible to synthesize an RLC ladder characterized by $y'_{22} = q_1/Aq_1'$ and by $y'_{12} = p_1/Aq_1'$ within a constant multiplier. Also since q_1' has only negative real roots by Rolle's Theorem (14) so does q_1'' , so that Bq_1''/Aq_1' is an RL admittance, and a two-element kind group of ladders can be synthesized with the short-circuit admittances $y''_{22} = Bq_1''/Aq_1'$ and $y''_{12} = p_a/Aq_1'$, again within a constant multiplier. The complete procedure is discussed in section V but the form of the final network is shown in Fig. 10.

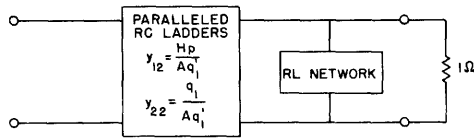


Fig. 9

Simple realization of RLC network with lossy coils.

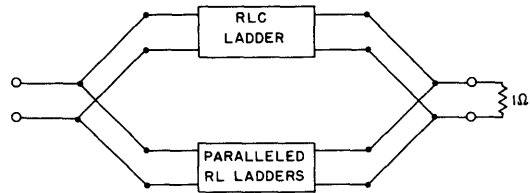


Fig. 10

RLC network for a q_1' with only negative real zeros.

An example which demonstrates the possibility of the desired breakdown is the polynomial with only one negative real root

$$q = s^3 + 12.5s^2 + 35.5s + 62 \quad (83)$$

where we use

$$\left. \begin{aligned} A &= \frac{1}{2} \\ B &= \frac{1}{4} \end{aligned} \right\} \quad (84)$$

so that

$$\left. \begin{aligned} q_1 &= s^3 + 11s^2 + 23s + 45 \\ &= (s+9)(s^2 + 2s + 5) \\ Aq_1' &= \frac{1}{2}(3s^2 + 22s + 23) \\ Bq_1'' &= \frac{1}{4}(6s + 22). \end{aligned} \right\} \quad (85)$$

B. With use of the network theorem of section II. 4, a synthesis procedure that realizes a composite RLC network

1. Breakdown of q into $q_1 + Aq_1'$ so that we obtain

$$Y_{12} = \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} = \frac{\frac{p}{q_1}}{1 + \frac{Aq_1'}{q_1}} \quad (86)$$

by a subsequent division of numerator and denominator of Y_{12} by q_1 .

a, b. It is first pointed out that the form of Y_{12} obtained through division by Aq_1' , as in part A.1 of this section, cannot be used with the network theorem because the admittance obtained in the denominator, not its reciprocal, must be susceptible to expansion in positive real partial fractions.

We may remove from p any factor of degree generally less than or equal to the

second, designated by p_b , so that

$$p_a = \frac{p}{p_b} . \quad (87)$$

This factor p_b is associated with Y_{12b} , whose denominator q_b is chosen as a suitable factor of q_1 given by

$$q_b = \frac{q_1}{q_a} . \quad (88)$$

In order to make Y_{12b} immediately realizable as a single ladder network the q_b must be chosen of degree equal to or greater than the degree of p_b . If p_b has complex roots, so must q_b ; if, on the other hand, p_b has real roots, the roots of q_b may be real or complex. The degree of q_b is generally not greater than the second, but if there is a combination of a real pole and a pair of complex poles of Aq_1'/q_1 with which zero shifting is possible, then the zeros of q_b may be these poles of Aq_1'/q_1 so that q_b is of the third degree.

It is noted that we have said above that p_b is generally not greater than a quadratic. However, if there is a combination of circumstances in a particular problem, namely, a number of real zeros in p and an equal or greater number of real zeros in q_1 , then all or part of this number may be used as p_b and q_b , respectively. The network N_b is then an RL structure and can be realized by the well-known zero-shifting procedures for two-element kind networks.

It is thus observed that a variety of choices for p_b and q_b is possible. The reader will also recall that N_a may consist of a group of paralleled ladders so that, though the problem of numerator polynomial breakdown does not arise for N_b , it does for N_a . Thus we choose p_b and q_b in such a manner that the partial fraction expansion of

$$y_{12a} = \frac{p_a}{q_a} \quad (89)$$

has the characteristics given in part A.2 of this section.

We have spoken of the advantage to be gained from obtaining a q_1 with the smallest possible number of complex zeros. Although for a q with complex zeros it is impossible to find a q_1 with no complex zeros, it is often possible to achieve a q_1 possessing only one pair of complex zeros. When this occurs, no p -breakdown need be found, for the synthesis procedure is always possible without such a breakdown. Network N_b is then a simple RLC network realized by zero shifting with the one complex pair of poles of Aq_1'/q_1 , while N_a is a two-element kind RL network composed of paralleled ladders, each of which is synthesized by known zero-shifting techniques (see section IV on RC synthesis).

2. When the integral of q is a Hurwitz polynomial, we may proceed as shown by the equation

$$Y_{12} = \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} = \frac{\frac{p}{q^{(-1)}}}{\frac{q}{q^{(-1)}}} \quad (90)$$

a, b. In regard to the p -breakdown and the method of synthesis, exactly the same discussion that was given in B.1 can now be applied to this method of synthesis, except that q_1 of the preceding discussion is replaced by $q^{(-1)}$.

3. Breakdown of q into $q_1 + Aq_1' + Bq_1''$ and a subsequent division by Aq_1' so that

$$Y_{12} = \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} = \frac{\frac{p}{Aq_1'}}{\frac{q_1}{1 + \frac{Aq_1'}{Aq_1'}} + \frac{Bq_1''}{Aq_1'}} \quad (91)$$

a, b. It is not possible in this case to factor p/Aq_1' into y_{12a} and Y_{12b} because the total q_1/Aq_1' must be used in the synthesis of N_a (i. e. the reciprocal of q_1/Aq_1' has the useful partial fraction expansion so that we must invert it and use $z = Aq_1'/q_1$ in the synthesis of N_a). The use of the total q_1/Aq_1' necessitates that y_{12a} have as poles all the zeros of Aq_1' so that no factors of the latter can be assigned to Y_{12b} .

However, the network theorem is useful in one particular case: the one in which q_1 is found to have all negative real zeros. When this circumstance occurs, the network has again been divided for us into two sets of two-element kind networks. We can completely remove the RL admittance Bq_1''/Aq_1' as a shunt branch and work on the RC parts by the method of section IV. The form of the final network is shown in Fig. 11.

C. Use of network theorem plus surplus factors to achieve an RLC network

The idea of multiplying the numerator and denominator of Y_{12} by an appropriate linear or quadratic factor in order to achieve an acceptable p -breakdown will be demonstrated for its two different purposes. The first has not yet been worked out to the author's satisfaction but is mentioned here for completeness. If, as before, we have

$$Y_{12} = \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} = \frac{p}{q} \quad (92)$$

for which the procedures of B above have been tried without success in achieving the necessary p -breakdown, it may be possible to make the numerator of the partial fractions positive by division by a factor, that is, adding another appropriate pole

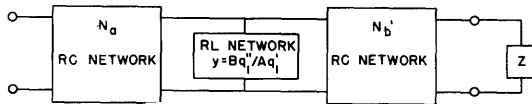


Fig. 11

RLC network realization for a q_1 with only negative real zeros.

in the expansion, say $(s+a)$. Whether this can be done is determined by calculation; if it can, then the numerator and denominator of Y_{12} are multiplied by $(s+b)$ where $b \approx a$. Using the new denominator

$$q_c = (s+b)q \quad (93)$$

we can now perform the $q_1 + Aq_1'$ breakdown. In most examples tried, if A was small, a factor approximating $(s+a)$ appears in q_1 and the other zeros remain approximately equal to the zeros of q_1 obtained from the old denominator of q . Thus $(s+a)$ may be used as a pole of the new partial fraction expansion. The $(s+b)$ factor in the numerator, which has not yet been accounted for, is associated with the numerator of Y_{12b} .

The use of surplus factors has a second purpose of increasing the number of applicable synthesis procedures and p -breakdowns. (This, of course, may be useful in cases that do not use the network theorem.) For example, suppose

$$Y_{12} = \frac{p}{q} = \frac{1}{q} \quad (94)$$

and all the previous methods have been used and none has proved successful (or we can try the following even before the previous methods have been used on p/q). We multiply numerator and denominator by $(s+a)(s+b)$ to obtain

$$Y_{12} = \frac{(s+a)(s+b)}{(s+a)(s+b)q} = \frac{p_c}{q_c} \quad (95)$$

Now we again try the suggested decompositions on the new denominator. The quadratic numerator gives us more possibilities of breakdown, e. g. the methods associated with Eqs. 74 and 75 may now be applicable. Of course, what we would like to achieve, because they give direct synthesis procedures, is a q_1 obtained from $q_1 + Aq_1'$ with only one pair of complex zeros or a q_1 from $q_1 + Aq_1' + Bq_1''$ with only negative real zeros or q_1' with negative real zeros although q_1 has one or more pairs of complex zeros.

D. Recapitulation

From the above it may be concluded that a knowledge of the exact nature of the transformations $q = q_1 + Aq_1'$ and $q = q_1 + Aq_1' + Bq_1''$ would be extraordinarily useful; for we know that the need for finding a p -breakdown is eliminated if q_1 in the first transformation has only one pair of complex zeros or in the second transformation has all negative real zeros or if q_1' of the second has only negative real zeros but q_1 does not. We know that all of these characteristics can be achieved but when or under what conditions cannot be fully stated at the present time. It may be said of the breakdown $q = q_1 + Aq_1' + Bq_1''$, however, that the necessary (but not sufficient) condition $4B > A^2$ must be satisfied for q_1 to have negative real roots if q has all complex roots. This becomes clear when we realize that q_1 may be represented as

$$\int_0^s h(s - uu) q(uu) duu$$

where $h(s)$ represents the impulse response of the equation, that is, the value of q_1 when q is replaced by an impulse. Since q is postulated as having only complex roots, its plot versus s considered as a real variable will be always positive. Now if the plot of $h(s)$ is also positive, then the integral is always positive, so that q_1 cannot have unequal

real roots. To make $h(s)$ go negative, the roots of the characteristic equation $Br^2 + Ar + 1 = 0$ must be complex. Thus we arrive at the condition stated. If in a particular problem we are forced to find a p-breakdown it would be convenient to be able to predict whether acceptable components can or cannot be found. Again, we are not yet in a position to do this. However, it is believed that for those p's for which the breakdown is not manifestly impossible, an acceptable decomposition may be found by the procedures given above.

The reader is again reminded that there is no necessity for finding a p-breakdown for transfer admittances that are not of higher degree than a cubic over a quartic. This synthesis is direct, as shown in section VI. Section V treats the RLC-synthesis procedures that incorporate the methods of this section.

III. Synthesis of RLC Networks in Lattice Form

There is a wide variety of existing synthesis procedures, but much remains to be done. The inadequacy of available procedures shows up particularly in a broad field of communications, namely, synthesis for prescribed transient response (12). In this synthesis both magnitude and phase are important so that the methods for realizing a prescribed magnitude of transfer function are inapplicable. Up to the present time the only procedure that could be used for the realization of both minimum-phase and nonminimum-phase transfer functions has been the one that yields a constant-resistance lattice; since a minimum-phase transfer function (with zeros off the j axis) cannot be realized by a lossless network terminated in resistance, the constant-resistance lattice was generally used for such a function. This type of lattice suffers from many disadvantages. In many cases each of the arms requires either close-coupled or lossless coils. An important disadvantage, too, for those cases in which an unbalanced form of network is definitely preferable, is that the series and cross arms are so complicated relative to each other that without the use of ideal transformers reduction to an unbalanced form of network is virtually impossible.

The new lattice synthesis procedures treated in this section realize a given transfer function within a multiplicative constant. No restriction other than physical realizability is placed on the function to be realized. Among the advantages claimed for the final lattice are that it contains no mutual inductance and all its coils are lossy. In addition, the arms are of so simple a form as to render the lattice amenable to reduction to an unbalanced network. For the case of a transfer admittance, moreover, reduction can always be achieved with at most the use of real transformers, that is, transformers with winding resistance, finite magnetizing inductance, and a coupling coefficient smaller than one.

The dimensions of the transfer function to be realized, i. e. whether it is an admittance, impedance or dimensionless ratio, depend on the type of system in which the synthesized network is to be used. If the driving force, for example, is a pentode which approximates a current source, a transfer impedance is needed. In the first part of

this section, the division into subsections is made on the basis of the form of transfer function under discussion. The subsections toward the end of the section logically follow the discussions of the synthesis procedures. To avoid repetition we mention here that, unless indicated otherwise, the transfer function is always of the form

$$\frac{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0} = \frac{(s-s_1)(s-s_3)\dots}{\frac{b_m}{a_n}(s-s_2)(s-s_4)\dots} \equiv \frac{p}{Hq}, \quad (n \leq m) \quad (96)$$

which we wish to realize with as small a value of the positive real constant H as practical.

III. 1 Open-Circuited Realization for $K = E_2/E_1$

A. Use of $q = q_1 + Aq_1'$ breakdown

We are given

$$K = \frac{E_2}{E_1} = \frac{p}{Hq} \quad (97)$$

a general transfer function which we desire to realize in the form of an open-circuited RLC lattice as shown in Fig. 12.

Since by Eq. 4

$$\frac{E_2}{E_1} = \frac{z_{12}}{z_{11}} \quad (98)$$

and we know that for the lattice

$$\frac{z_{12}}{z_{11}} = \frac{Z_b - Z_a}{Z_b + Z_a} \quad (99)$$

then

$$K = \frac{Z_b - Z_a}{Z_b + Z_a} = \frac{p}{Hq}. \quad (100)$$

The synthesis procedure follows.

Break q into the sum of two polynomials so that

$$q = q_1 + Aq_1' \quad (101)$$

where the symbols and the method are explained in section II.2. This can always be done. After dividing numerator and denominator of the resulting K by q_1 to obtain

$$K = \frac{\frac{p}{q_1}}{H\left(1 + \frac{Aq_1'}{q_1}\right)} \quad (102)$$

expand p/q_1 into partial fractions. Its residues are in general positive or negative real for real poles, complex for complex poles. Also expand Aq_1'/q_1 into partial fractions

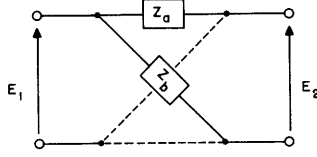


Fig. 12

Lattice network with

$$K = \frac{E_2}{E_1} = \frac{Z_b - Z_a}{Z_b + Z_a}.$$

so that the denominator becomes

$$H \left(1 + \frac{Aq_1'}{q_1} \right) = H \left(k_0^{(d)} + \frac{k_1^{(d)}}{s - s_1} + \frac{k_2^{(d)}}{s - s_2} + \dots \right) \quad (103)$$

where $k_0^{(d)} = 1$. From the discussion in section II.2, we know that all the residues $k_\nu^{(d)}$ for $\nu \neq 0$ are equal to A . If $(Z_b - Z_a)$ and $(Z_b + Z_a)$ are also considered as expanded in partial fractions, the residues of like terms of p/q_1 and $(Z_b - Z_a)$ may be equated as may those of $H(1 + Aq_1'/q_1)$ and $(Z_b + Z_a)$. We thus obtain

$$\begin{aligned} k_\nu^{(b)} - k_\nu^{(a)} &= k_\nu^{(n)} \\ k_\nu^{(b)} + k_\nu^{(a)} &= Hk_\nu^{(d)} \quad (\nu = 0, 1, 2 \dots m, \\ &\quad \text{where } m \text{ is degree of } q) \end{aligned} \quad (104)$$

where the superscripts a , b , n and d refer to Z_a , Z_b , the numerator and the denominator, respectively, while the subscript ν designates the different poles $s_\nu = \sigma_\nu + j\omega_\nu$. Solving Eqs. 104 for the unknown Z_a and Z_b residues as indicated in Eqs. 105

$$\begin{aligned} \text{a) } k_\nu^{(b)} &= \frac{1}{2} \left(Hk_\nu^{(d)} + k_\nu^{(n)} \right) & \text{c) } k_\nu^{(a)} &= \frac{1}{2} \left(Hk_\nu^{(d)} - k_\nu^{(n)} \right) \\ \text{b) } a_\nu^{(b)} + j\beta_\nu^{(b)} &= \frac{1}{2} \left(Hk_\nu^{(d)} + a_\nu^{(n)} + j\beta_\nu^{(n)} \right) & \text{d) } a_\nu^{(a)} + j\beta_\nu^{(a)} &= \frac{1}{2} \left(Hk_\nu^{(d)} - a_\nu^{(n)} - j\beta_\nu^{(n)} \right) \end{aligned} \quad (105)$$

finally yields

$$\begin{aligned} & \nu \neq 0 & & \nu \neq 0 \\ \text{a) } a_\nu^{(b)} &= \frac{1}{2} \left(HA + a_\nu^{(n)} \right) & \text{b) } a_\nu^{(a)} &= \frac{1}{2} \left(HA - a_\nu^{(n)} \right) \\ \beta_\nu^{(b)} &= \frac{1}{2} \beta_\nu^{(n)} & \beta_\nu^{(a)} &= -\frac{1}{2} \beta_\nu^{(n)} \\ & \nu = 0 & & \nu = 0 \\ a_0^{(b)} &= \frac{1}{2} \left(H + a_0^{(n)} \right) & a_0^{(a)} &= \frac{1}{2} \left(H - a_0^{(n)} \right). \end{aligned} \quad (106)$$

where Eqs. 106a and 106b follow respectively from Eqs. 105b and 105d by equating real and imaginary parts.

For negative real poles the requirement that the residues, $a_v^{(a)}$ and $a_v^{(b)}$, be real and positive when used in conjunction with Eq. 106 gives as the condition to be satisfied

$$-1 \leq \frac{a_v^{(n)}}{HA} \leq 1 \quad (107)$$

which, of course, is the same as the condition that arises in the Bower-Ordung RC synthesis. The real parts of the residues in the complex poles, on the other hand, must not only be positive but must be equal to or greater than a positive constant c_v . This is seen by the application to the residues of Z_a of the condition for realizability that was derived in section II.1 (where, of course, the same result holds for Z_b)

$$\frac{|\beta_v^{(a)}|}{a_v^{(a)}} \leq \frac{\sigma_v}{\omega_v}$$

or

$$a_v^{(a)} \geq \frac{|\beta_v^{(a)}| \omega_v}{\sigma_v} = c_v \text{ (positive constant)}. \quad (108)$$

When the constant c_v is substituted in those relations of Eqs. 106 for $v \neq 0$, the conditions to be satisfied become

$$1 \geq \frac{2c_v - a_v^{(n)}}{HA} \quad (109)$$

and

$$1 \geq \frac{2c_v + a_v^{(n)}}{HA}. \quad (110)$$

We need only satisfy the stronger of the above two inequalities for any specific complex pole. If $a_v^{(n)}$ is positive Eq. 110 is the stronger and must be used to determine the minimum value of H; if $a_v^{(n)}$ is negative, use Eq. 109. Therefore, to summarize the two steps for the complex poles, we must first determine the c_v for each pole and then determine the value of H necessary to satisfy the stronger of Eqs. 109 and 110.

By satisfaction also of Eq. 107 for the real poles, we may thus tabulate the necessary value of H for each pole. We choose a value of H greater than the largest value required. In determining the values of H necessary for each pole we may use the equality sign in Eqs. 107, 108 and 109; then by finally choosing a value of H greater than the largest required value we automatically guarantee the satisfaction of the condition for each pole with the inequality sign. This guarantees, as we have seen in section II.1, that every inductance appears with an associated series resistance and that each of the partial fraction components (complex conjugate poles taken in pairs) of

$$Z_a = k_0^{(a)} + \frac{k_1^{(a)}}{s - s_1} + \frac{k_2^{(a)}}{s - s_2} + \dots + \frac{k_\nu^{(a)}}{s - s_\nu}$$

and

$$Z_b = k_0^{(b)} + \frac{k_1^{(b)}}{s - s_1} + \frac{k_2^{(b)}}{s - s_2} + \dots + \frac{k_\nu^{(b)}}{s - s_\nu} \quad (111)$$

is positive real so that Z_a and Z_b may be realized simply in the Foster manner.

One final point, useful in subsequent sections, is made regarding the real term in Eqs. 111. As is obvious from Eqs. 106 for $\nu = 0$, we always obtain this real term. Furthermore, if the degree of p is less than that of q

$$a_0^{(b)} = a_0^{(a)} = \frac{1}{2} H \quad (112)$$

while if the degrees of p and q are equal

$$a_0^{(b)} = a_0^{(a)} + 1 = \frac{1}{2} (H + 1). \quad (113)$$

B. Method making use of integral of q

As pointed out in section II.2, use may be made of the integral of q for the development of a synthesis procedure. This can only be done, however, when the integral of q is a Hurwitz polynomial. It is a simple matter to form this integral and check it for Hurwitz character.

The steps in the procedure for synthesizing an open-circuited lattice begin with

$$\begin{aligned} K &= \frac{E_2}{E_1} = \frac{Z_b - Z_a}{Z_b + Z_a} \\ &= \frac{p}{q} = \frac{q^{(-1)}}{q^{(-1)}} \end{aligned} \quad (114)$$

where $q^{(-1)}$ is the integral of q with the arbitrary constant chosen conveniently. Except for a few differences the remaining steps duplicate the procedure in part A of this section. The differences to be noted are that the constant term in the partial fraction expansions, i.e. for $\nu = 0$, is nonexistent and the constant A is equal to one. The latter characteristic represents one of the advantages of this procedure in that the gain is increased. Another advantage, which allows coils of lower Q to be used in the synthesis, is that most of the zeros of $q^{(-1)}$ are farther from the j axis than those of the q_1 polynomial obtained in the breakdown of part A.

For this method the useful equations that correspond to Eqs. 106 are

$$\begin{aligned} \text{a) } \quad & \nu \neq 0 & \quad & \nu \neq 0 \\ & a_\nu^{(b)} = \frac{1}{2} (H + a_\nu^{(n)}) & \quad & \text{b) } a_\nu^{(a)} = \frac{1}{2} (H - a_\nu^{(n)}) \\ & \beta_\nu^{(b)} = \frac{1}{2} \beta_\nu^{(n)} & \quad & \beta_\nu^{(a)} = -\frac{1}{2} \beta_\nu^{(n)} \end{aligned} \quad (115)$$

which yield for the real poles the condition corresponding to Eq. 107

$$-1 \leq \frac{a_v^{(n)}}{H} \leq 1 \quad (116)$$

and for the complex poles yield the inequalities that correspond respectively to Eqs. 109 and 110

$$1 \geq \frac{2c_v - a_v^{(n)}}{H} \quad (117)$$

and

$$1 \geq \frac{2c_v + a_v^{(n)}}{H} \quad (118)$$

The use of the above equations, along with the definition of c_v given in Eq. 108 allows a synthesis to be carried out.

III.2 Realization of Transfer Impedance in Form of Terminated Lattice

A. Resistance termination

We desire in this section that the given rational function be realized as the transfer impedance of the resistance-terminated lattice shown in Fig. 13. Thus we write

$$Z_{12} = \frac{E_2}{I_1} = \frac{p}{Hq} \quad (119)$$

It is possible to use the equation

$$Z_{12} = \frac{z_{12}}{1 + z_{22}} = \frac{\frac{1}{2}(Z_b - Z_a)}{1 + \frac{1}{2}(Z_b + Z_a)} \quad (120)$$

and, by proceeding in a manner similar to that of the preceding section, make the necessary identifications for direct synthesis of the required lattice. The network obtained, however, is the same as the one obtained by application of the reciprocity theorem and well-known lattice equivalents to the open-circuited lattice of the previous section. The latter procedure is believed preferable for demonstration purposes and we will therefore consider the method of synthesis of section III.1 as the basic one from which the other desirable forms of network are easily derived.

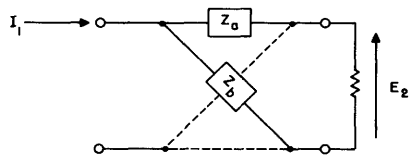
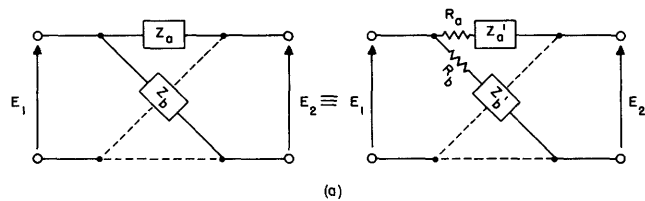


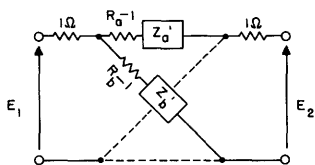
Fig. 13

RLC lattice terminated in resistance.

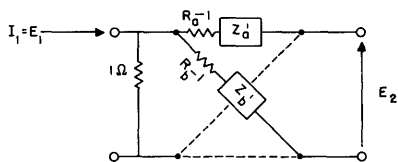
Since, as observed at the conclusion of part A of the preceding section, a series resistance is always present in each arm of the open-circuited lattice, we derive an equivalent lattice (2, 13) by removing one ohm from each arm, then convert to a current source, and finally by use of the reciprocity theorem obtain the desired network. The sequence of steps beginning with the previously realized open-circuited lattice for which



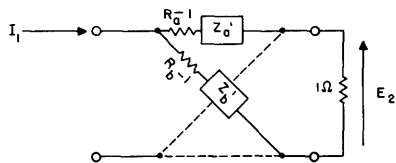
(a)



(b)



(c)

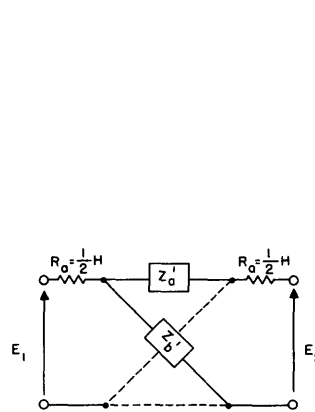


(d)

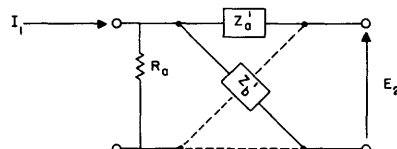
Fig. 14

Steps in the conversion of open-circuited lattice for which $K = E_2/E_1 = p/Hq$ to resistance-terminated lattice with $Z_{12} = E_2/I_1 = p/Hq$:

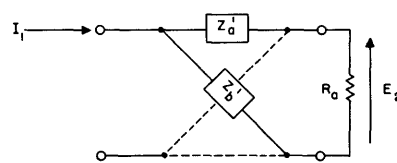
- open-circuited lattice realized in section III. 1 where $K = E_2/E_1 = p/Hq$;
- lattice equivalent to that in a;
- lattice after application of Norton's Theorem;
- lattice given by application of reciprocity theorem, where $Z_{12} = E_2/I_1 = p/Hq$.



(a)



(b)



(c)

Fig. 15

Conversion of open-circuited lattice for which $K = E_2/E_1 = p/Hq$ to lattice terminated in $R_a = 1/2 H$ where $Z_{12} = E_2/I_1 = 1/2(p/q)$:

- lattice equivalent to that of Fig. 14a;
- lattice after application of Norton's Theorem with $I_1 = E_1/R_a = E_1/(1/2 H)$;
- lattice given by application of reciprocity theorem, where $Z_{12} = E_2/I_1 = R_a E_2/E_1 = R_a p/Hq = 1/2(p/q)$.

$$K = \frac{E_2}{E_1} = \frac{p}{Hq} \quad (121)$$

is illustrated by Fig. 14. The one-ohm resistance at the output terminals is omitted in Fig. 14c because the output is open-circuited.

An improvement in gain can be effected by removing more than one ohm from each of the arms. Suppose, for example, that p is of lower degree than q so that, as was noted in section III. 1, $R_a = R_b = 1/2 H$. Then we can remove R_a and follow the same sequence of steps as before to obtain a network terminated in $R_a = 1/2 H$ with the transfer function

$$Z_{12} = \frac{E_2}{I_1} = \frac{1}{2} \frac{p}{q}. \quad (122)$$

This sequence of steps and derivation of the above gain is illustrated by Fig. 15.

It is pointed out, finally, that if we stop at the step given by Fig. 15b, we realize a transfer impedance in the form of an open-circuited lattice, where the R_a is useful in the instrumentation of the system since it may represent the finite internal resistance of the current source. Two additional methods for realizing an open-circuited lattice characterized by a given transfer impedance are discussed in section III. 4.

B. Parallel RC termination

In the instrumentation of a practical circuit it is often useful to have a shunt capacitance at the input or output of a network. We have shown how to obtain a resistance termination for the network obtained by the method employing the $q = q_1 + Aq_1'$ breakdown; we now demonstrate the realization of a parallel RC termination for both the methods of section III. 1.

The following artifice is restricted in that it can only be used for those transfer functions in which the degree of p is less than the degree of q . The procedure will be demonstrated for the synthesis employing the integral of q , but it holds also for the $(q_1 + Aq_1')$ breakdown. Suppose that $q^{(-1)}$ has at least one negative real zero given by $(s+a)$. (If it does not and we wish to employ this technique, we multiply numerator and denominator of the transfer function by a convenient linear term $(s+b)$ to obtain a new denominator from which $q^{(-1)}$ is now determined.) Suppose further that we wish to synthesize a transfer impedance in the form of a lattice with a parallel RC termination, the impedance of the termination being given by $k/(s+a)$. We may then write

$$\begin{aligned} Z_{12} &= \frac{E_2}{I_1} = \frac{p}{Hq} \\ &= \frac{p}{q^{(-1)}} \\ &= \frac{p}{(s+a)q_2} \\ &= \frac{Hq}{q^{(-1)}} \end{aligned} \quad (123)$$

for the lattice shown in Fig. 16. Now if we multiply the transfer function by $(s+a)$ we obtain

$$\begin{aligned}
 K &= (s+a) Z_{12} \\
 &= \frac{(s+a) p}{(s+a) q_2} \\
 &= \frac{Hq}{q^{(-1)}} \\
 &= \frac{p}{q_2} \\
 &= \frac{Hq}{q^{(-1)}} \quad (124)
 \end{aligned}$$

If this function K were synthesized as the voltage ratio for an open-circuited lattice by the method of the preceding section, then, since the numerator residue in the pole $s = -a$ is zero, it is obvious by inspection of Eqs. 125 that the residues for Z_a and Z_b in this pole are each equal to $1/2 H$. The network, which thus has the form shown in Fig. 17a, can be transformed by removal of the RC combination from each arm, after which the successive applications of Norton's Theorem and reciprocity give

$$Z_{12} = \frac{E_2}{I_1} = \frac{1}{2} \frac{p}{q} \quad (125)$$

for the final desired form of network. The pertinent equations and steps are indicated in Fig. 17. Finally, to summarize the procedure, multiply the given transfer impedance by $(s+a)$, where $(s+a)$ is a factor of $q^{(-1)}$ (or of q_1 , if we are using the $q_1 + Aq_1'$ synthesis procedure); then synthesize the resulting function as the voltage ratio for an open-circuited lattice, after which the steps illustrated in Fig. 17 will give the desired transfer impedance for the RC terminated lattice.

III. 3 Realization of Transfer Admittance in Form of Terminated Lattice

The lattice terminated in resistance is now to have a transfer admittance equal to the given quotient of polynomials. As pointed out in section III.2 for an analogous case, it is possible by use of the equations

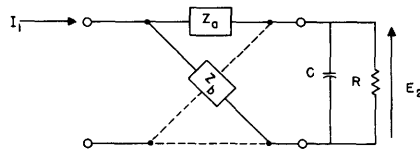


Fig. 16

Lattice with termination $z = k/(s+a)$.

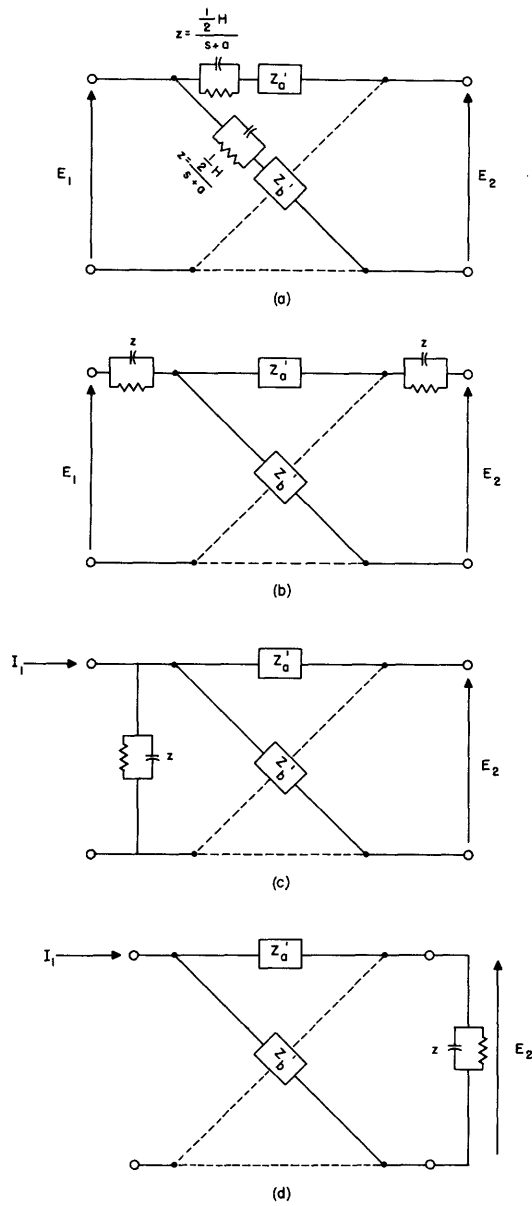


Fig. 17

- Steps in the conversion of an open-circuited lattice for which $K = E_2/E_1 = (s+a) p/Hq$ to an RC terminated lattice for which $Z = E_2/I_1 = 1/2 (p/q)$:
- open-circuited lattice for which $K = (s+a) p/Hq = E_2/E_1$;
 - lattice equivalent to a;
 - lattice after application of Norton's Theorem where $I_1 = (s+a) E_1/(1/2)H$;
 - lattice after application of reciprocity theorem with $Z_{12} = E_2/I_1 = 1/2 HE_2/(s+a) E_1 = 1/2 HK/(s+a) = 1/2 (p/q)$.

$$Y_{12} = \frac{y_{12}}{1 + y_{22}} = \frac{\frac{1}{2}(Y_b - Y_a)}{1 + \frac{1}{2}(Y_b + Y_a)} \quad (126)$$

and

$$\begin{aligned} Y_{12} &= \frac{p}{Hq} = \frac{p}{H(q_1 + Aq_1')} \\ &= \frac{\frac{p}{q_1}}{1 + (H-1) + \frac{HAq_1'}{q_1}} \end{aligned} \quad (127)$$

to make the proper identifications which give rise to inequality conditions on the residues similar to those of section III. 1. The final network obtained by this direct method, however, is the same as the one obtained by the method explained below, except for a change of admittance level.

The simpler approach is to consider the whole procedure of section III. 1 carried over to the dual problem. That is, instead of first synthesizing for a voltage ratio by means of an open-circuited lattice, we now synthesize for

$$K = \frac{I_2}{I_1} = \frac{y_{12}}{y_{11}} \quad (128)$$

in the form of a short-circuited lattice. The dual of the remarks of section III. 2 now applies so that we obtain a final transfer function

$$Y_{12} = \frac{I_2}{E_1} = \frac{1}{2} \frac{p}{q} \quad (129)$$

for the same problem treated in that section. The final lattice with a resistance termination will be of the form shown in Fig. 18.

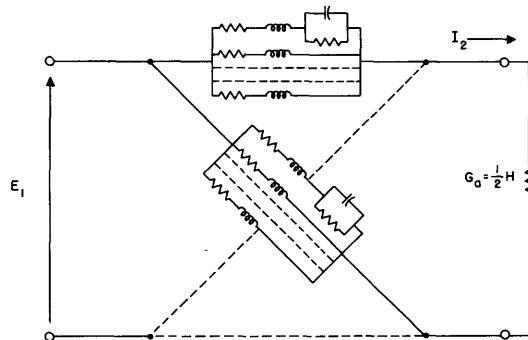


Fig. 18
Lattice for which $Y_{12} = 1/2(p/q)$.

III.4 Realization of Transfer Impedance in Form of an Open-Circuited Lattice

In section III.2 we touched on the problem of realizing an open-circuited lattice whose transfer impedance is a prescribed function of frequency. The lattice obtained there has the advantage over the two methods considered here in that it has a shunt conductance at the input terminals. The second of the methods considered in this section, however, may often be desirable in that the required Q of the coils (since it is associated with the complex zeros of the denominator q) is less than the Q 's required in the method of section III.2 (since they are associated with the complex zeros of q_1 , which are closer to the j axis than those of q). The first method considered, which is due to Guillemin, is included for completeness.

We have seen in section II.1 that the partial fraction expansion of a positive real RLC function does not in general give positive real partial fraction components. If the original function is not positive real, but, like a transfer function, is only restricted in that its poles must lie in the left half-plane, the statement about the partial fraction expansion is, of course, true a fortiori. Bode (7), however, was the first to point out in regard to a positive real function that each of its partial fraction components could be made positive real merely by the addition of resistance. This idea was applied by Guillemin (M.I.T. Network Synthesis Seminar, Spring Term, 1951) to the synthesis of a transfer impedance for an open-circuited lattice; the synthesis is possible in this case, unlike that of the driving-point impedance, because there is, in effect, a reservoir of resistance to draw upon. This is due to the form of the transfer function which is merely the difference of two positive real functions. Therefore, in applying the above remarks to a synthesis procedure, we expand the given

$$Z_{12} = \frac{p}{q} \quad (130)$$

in partial fractions, and then arbitrarily distribute the partial fraction components to the series and cross arms in

$$Z_{12} \equiv z_{12} = \frac{1}{2} (Z_b - Z_a). \quad (131)$$

Enough total R is now added to each of the lattice arms so that the partial fraction components of Z_a and Z_b are separately realizable. Thus we write

$$Z_{12} = \frac{1}{2} [(Z_b + R) - (Z_a + R)] \quad (132)$$

wherein each arm impedance now has partial fraction components that can be realized simply by the Brune method or the Bott and Duffin method (1). In seeking a method for reducing the lattice to an unbalanced form, it may be advantageous to note that we may add and subtract any necessary impedance Z to obtain

$$\begin{aligned} Z_{12} &= \frac{1}{2} [(Z_b + R + Z) - (Z_a + R + Z)] \\ &= \frac{1}{2} (Z'_b - Z'_a). \end{aligned} \quad (133)$$

Of course, as a further amplification of the above method, if we are satisfied with a method of synthesis that requires Z_b to be a complicated driving-point impedance but Z_a a simple resistance, there is no necessity for a decomposition into partial fractions. All we need do is determine the absolute value of the minimum of the real part of Z_{12} for $s = j\omega$, i. e. along the real frequency axis. (Since Z_{12} is not positive real, its real part along the j axis will necessarily go negative.) Then the addition of a resistance R , where R is greater than or equal to this absolute value, will make Z_{12} positive real. We can consider the sum as $1/2 Z_b$. In symbols, then

$$\begin{aligned} Z_{12} &= Z_{12} + R - R \\ &= \frac{1}{2} Z_b - \frac{1}{2} Z_a \\ &= \frac{1}{2} (Z_b - Z_a). \end{aligned} \tag{134}$$

This completes the discussion of the first method and we now focus our attention on a method which demonstrates that it is not at all necessary to add any resistance to each of the partial fraction components in order to realize each one by itself, for we can distribute the residues of Z_{12} so that the partial fraction components of Z_a and Z_b are separately positive real. This is an improvement on the first method in that all coils are lossy and no mutual inductance is required. Though the procedure as applied to RC networks (13) has been well known for a number of years, no one has yet shown its applicability to RLC networks.

If Z_{12} is expanded in partial fractions to give

$$Z_{12} = \frac{p}{q} = \sum_{v=1}^n \frac{k_v}{s - s_v} \tag{135}$$

where n is the degree of q

$$\left. \begin{aligned} k_v &= \alpha_v + j\beta_v \\ s_v &= -\sigma_v + j\omega_v \\ &= |s_v| e^{j\psi_v} \end{aligned} \right\} \tag{136}$$

and if, in addition, we consider the unknown impedance arms as expanded into partial fractions

$$\left. \begin{aligned} \frac{1}{2} Z_b &= \sum_{v=1}^n \frac{k_v^{(b)}}{s - s_v} \\ \frac{1}{2} Z_a &= \sum_{v=1}^n \frac{k_v^{(a)}}{s - s_v} \end{aligned} \right\} \tag{137}$$

where

$$\left. \begin{aligned} k_{\nu}^{(b)} &= \alpha_{\nu}^{(b)} + j\beta_{\nu}^{(b)} = |k_{\nu}^{(b)}| e^{j\phi_{\nu}^{(b)}} \\ k_{\nu}^{(a)} &= \alpha_{\nu}^{(a)} + j\beta_{\nu}^{(a)} = |k_{\nu}^{(a)}| e^{j\phi_{\nu}^{(a)}} \end{aligned} \right\} \quad (138)$$

then by substitution of Eqs. 135 and 137 in Eq. 131 and equating residues in like poles, we obtain

$$k_{\nu} = k_{\nu}^{(b)} - k_{\nu}^{(a)}. \quad (139)$$

Since we now desire each partial fraction term of Eqs. 137 to be positive real, the discussion of section II.1 is applicable. If s_{ν} is real, k_{ν} is positive or negative real and a simple distribution of the residues is

$$\left. \begin{aligned} k_{\nu}^{(b)} &= k_{\nu}; & k_{\nu}^{(a)} &= 0, & \text{if } k_{\nu} > 0 \\ k_{\nu}^{(b)} &= 0; & k_{\nu}^{(a)} &= k_{\nu}, & \text{if } k_{\nu} < 0. \end{aligned} \right\} \quad (140)$$

For complex poles, using the angles defined in Eqs. 136 and 138, we may write Eq. 17, the condition to be satisfied for Z_b and Z_a to have the desired realization, as

$$\left. \begin{aligned} |\phi_{\nu}^{(b)}| &\leq |\psi_{\nu}| - \frac{\pi}{2} \\ |\phi_{\nu}^{(a)}| &\leq |\psi_{\nu}| - \frac{\pi}{2}. \end{aligned} \right\} \quad (141)$$

This condition can always be fulfilled.

Figure 19 shows the geometrical representation in the complex plane of Eq. 139, which makes it clear that what is necessary to satisfy Eqs. 141 is a choice of point M that makes the angles of $k_{\nu}^{(a)}$ and $k_{\nu}^{(b)}$ as small as desired. We can see by inspection that the angles may be made arbitrarily small, but for a rigorous proof (the graphical

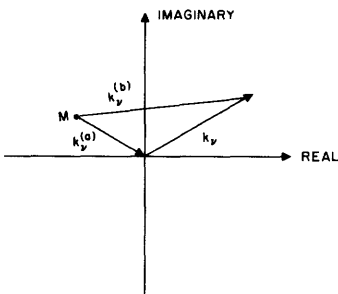


Fig. 19

Geometrical representation of $k_{\nu} = k_{\nu}^{(b)} - k_{\nu}^{(a)}$.

form of proof is due to W. H. Kautz of M.I.T.), consider Fig. 20 for the three cases of α_{ν} positive, α_{ν} negative, and β_{ν} equal to zero. The construction is indicated in the figures where all angles marked with dimension lines are equal to $|\psi_{\nu}| - \pi/2$. Merely using the fact of the equality of alternate interior angles formed by a line intersecting parallel lines, we observe that at the apex of the shaded portion, $|\phi_{\nu}^{(a)}| = |\phi_{\nu}^{(b)}| = |\psi_{\nu}| - \pi/2$, while along the lower boundary line, $\phi_{\nu}^{(b)}$ remains constant and $\phi_{\nu}^{(a)}$ changes, and along the upper boundary line $\phi_{\nu}^{(a)}$ remains constant and $\phi_{\nu}^{(b)}$ changes. Inside the shaded area both angles decrease in magnitude. As for $k_{\nu}^{(a)}$ and $k_{\nu}^{(b)}$, their minimum values occur simultaneously at the

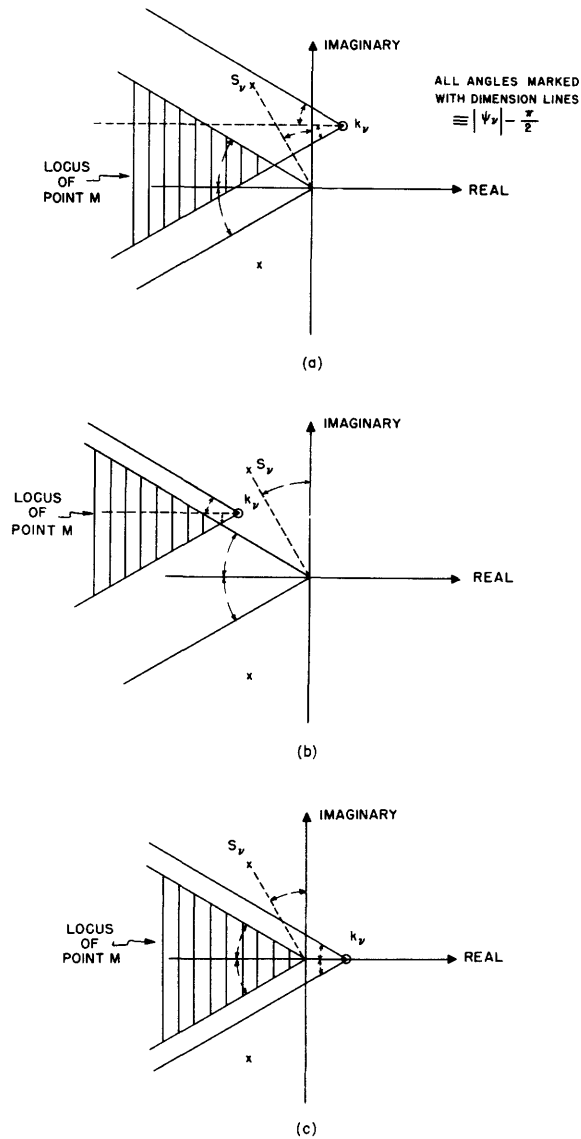


Fig. 20

Locus of point M for satisfaction of Eqs. 141, where all angles marked with dimension lines = $|\psi_v| - \pi/2$:

- a) locus of M (shaded portion) for α_v positive;
- b) locus of M (shaded portion) for α_v negative;
- c) locus of M (shaded portion) for $\beta_v = 0$ and α_v positive.

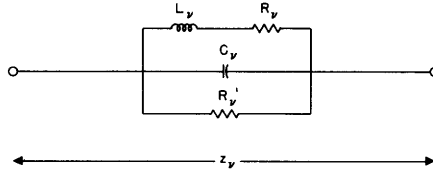


Fig. 21

Network realization of pair of complex poles where z_v is given by Eq. 142.

apex of the shaded area if $|\psi_v| - \pi/2 \leq \pi/4$.

It is of interest to control the Q 's of the coils used in the synthesis of the partial fraction components. Each component, that is, a pair of complex poles, can be realized in general by the network shown in Fig. 21 so that we may write

$$\begin{aligned}
 z_v &= \frac{2a_v \left(s + \sigma_v - \frac{\beta_v \omega_v}{a_v} \right)}{s^2 + 2\sigma_v s + |s_v|^2} \\
 &= \frac{2a_v (s + d_v)}{s^2 + 2\sigma_v s + |s_v|^2} \\
 &= \frac{\left(s + \frac{R_v}{L_v} \right)}{C_v \left[s^2 + \left(\frac{R_v}{L_v} + \frac{1}{R_v' C_v} \right) s + \frac{R_v}{L_v R_v' C_v} + \frac{1}{C_v L_v} \right]}. \tag{142}
 \end{aligned}$$

The pole closest to the j axis will determine the minimum value of Q_v that will satisfy every partial fraction realization, where

$$Q_v \equiv \frac{\omega_v}{d_v} = \frac{\omega_v L_v}{R_v}. \tag{143}$$

In order to use coils with as low Q as possible, it is necessary that the value of d_v be as close as possible to its maximum value $2\sigma_v$. This is brought about by causing $\phi_v^{(a)}$ and $\phi_v^{(b)}$ to be negative and of large magnitude. Since by definition

$$d_v = \sigma_v - \frac{\beta_v \omega_v}{a_v} \tag{144}$$

then

$$\begin{aligned}
 \frac{1}{Q_v} &= \frac{d_v}{\omega_v} = \frac{\sigma_v}{\omega_v} - \frac{\beta_v}{a_v} \\
 &= \tan \left(|\psi_v| - \frac{\pi}{2} \right) - \tan \phi_v. \tag{145}
 \end{aligned}$$

Therefore, if we desire all Q_ν to be less than a given value Q_{\max} , then we must make $\phi_\nu^{(a)}$ and $\phi_\nu^{(b)}$ satisfy the relation for ϕ_ν given by

$$\tan \phi_\nu < \tan \left(\left| \psi_\nu \right| - \frac{\pi}{2} \right) - \frac{1}{Q_{\max}}. \quad (146)$$

The above equation concludes the discussion of the synthesis procedure. We have thus seen that any transfer ratio, whether it has the dimensions of an impedance, admittance, or is dimensionless, may be realized in lattice form. The lattice may be unloaded or loaded with a resistance or complex impedance.

III.5 Lattice Reduction to Unbalanced Networks

Lattices may be transformed to unbalanced networks in the following ways (2, 13):

- a. A series impedance may be removed from both Z_a and Z_b and placed in series with both the input and output terminals (see Fig. 22a).
- b. A shunt impedance may be removed from both Z_a and Z_b and placed in shunt with both the input and output terminals (see Fig. 22b).
- c. A shunt impedance may be removed from the series arm Z_a and considered as a bridge across the remainder of the lattice (see Fig. 22c). The ideal transformer that is necessary may be removed when the remainder of the lattice has been transformed to an unbalanced network.
- d. A series impedance may be removed from the cross arm as shown in Fig. 22d. The ideal transformer becomes unnecessary when the remainder of the lattice has been transformed to an unbalanced network.
- e. A lattice may be broken into a group of parallel lattices (see Fig. 22e).

The application of a succession of the above methods may be necessary in any practical problem, or it may first be necessary to resynthesize a lattice arm before one of the methods may be successfully applied. Thus considerable ingenuity may be required. However, the types of lattices that are realized, since they contain the same poles, are reducible in a large number of problems. We mention below a few of the forms which can be recognized as reducible in general.

If all the residues in the real poles of one arm are larger than the corresponding residues in the other arm and, in addition, the coefficients of the numerators of the pairs of complex poles in the first arm are larger than the corresponding coefficients of the poles in the other arm, then the lattice is immediately reducible to a "voltage-divider" network. Since this is a very restricted form of network, the residues will rarely have this desired distribution.

A completely general form of unbalanced network is given, if, at any stage of the lattice reduction process, the lattice arms can be resynthesized into the ladder forms shown in Fig. 23. A two-element kind network is used for simplicity of illustration. How to bring this about, in general, is a matter requiring further

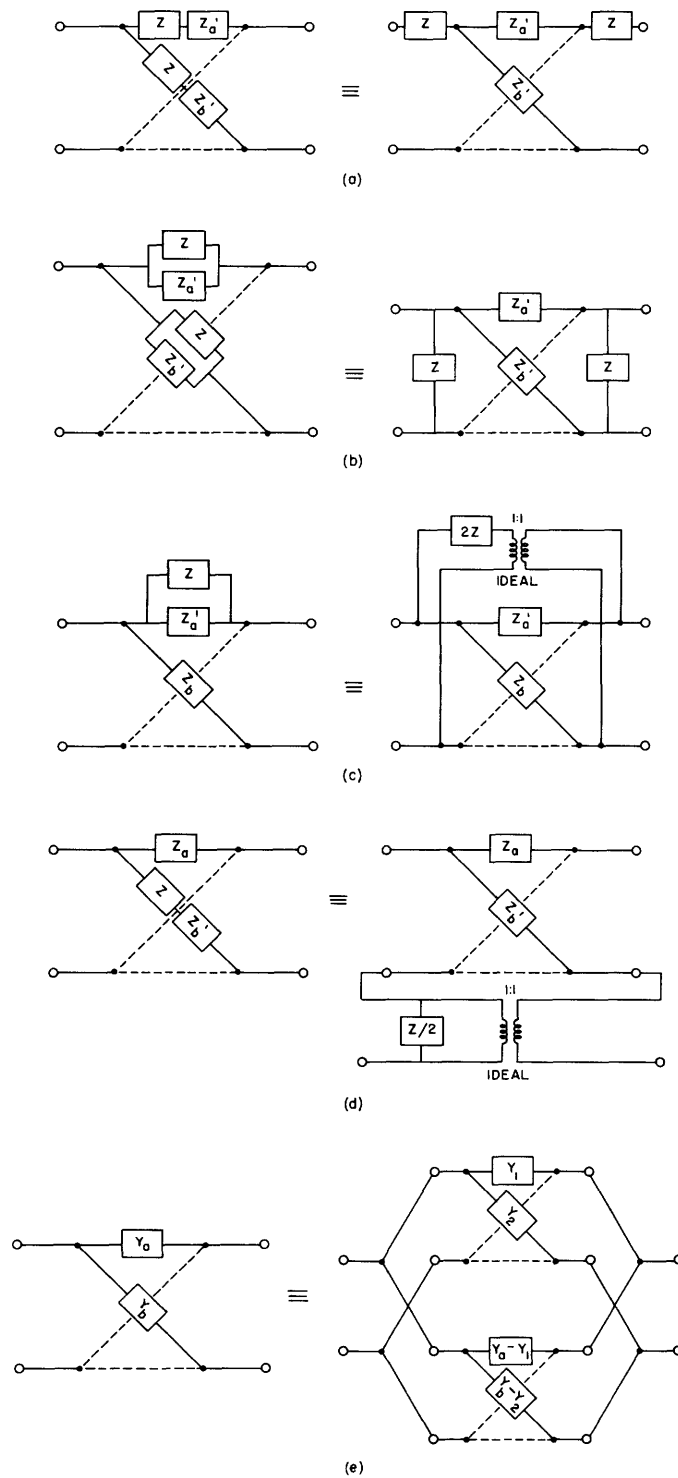


Fig. 22

Methods for the conversion of lattices to unbalanced networks:

- removal of a series impedance from each arm;
- removal of a shunt impedance from each arm;
- removal of a shunt impedance from series arm;
- removal of a series impedance from the cross arm;
- a single lattice decomposed into two lattices in parallel.

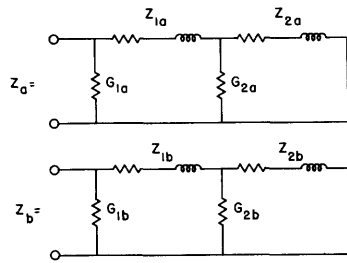


Fig. 23

Desirable form of lattice arm impedances, where $G_{na} \geq G_{nb}$ and $Z_{nb} \geq Z_{na}$.

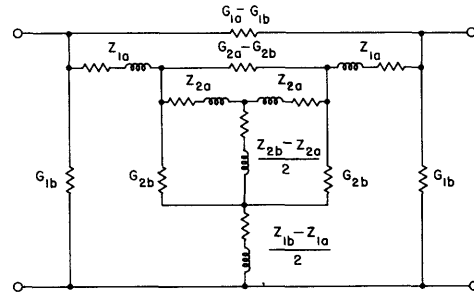


Fig. 24

Unbalanced network corresponding to lattice with arms given in Fig. 22.

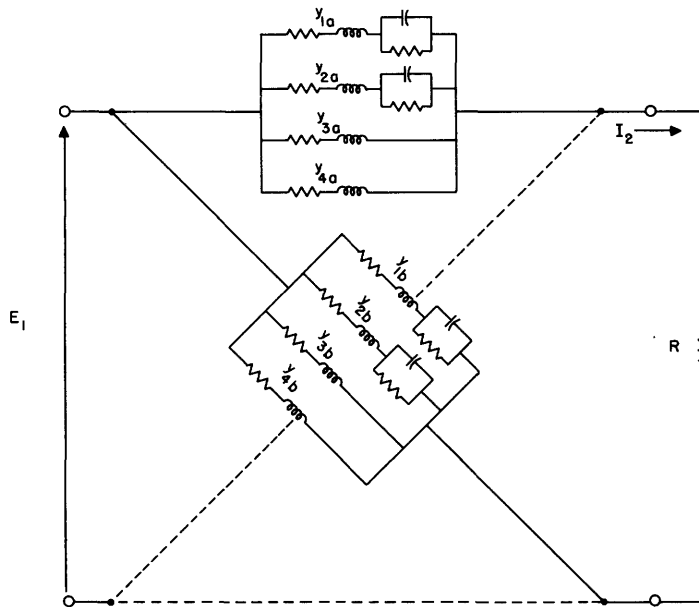


Fig. 25

Six-pole lattice to be reduced to unbalanced network.

investigation* but when it is possible the lattice containing these arms can be reduced to the form shown in Fig. 24.

For the lattice obtained in the synthesis of Y_{12} it is always possible to effect a reduction to an unbalanced network if one of the residues in a real pole is very large, specifically, large enough for the method to be applied. Again, discovering a method for guaranteeing that one residue will be very large is a very difficult problem. As an example of the reduction procedure when a large residue is present, consider the

*After this had been written, it came to the writer's attention that O. Aberth is conducting an investigation into this problem for RC networks as his Master's thesis research in the Dept. of Electrical Engineering, M.I.T.

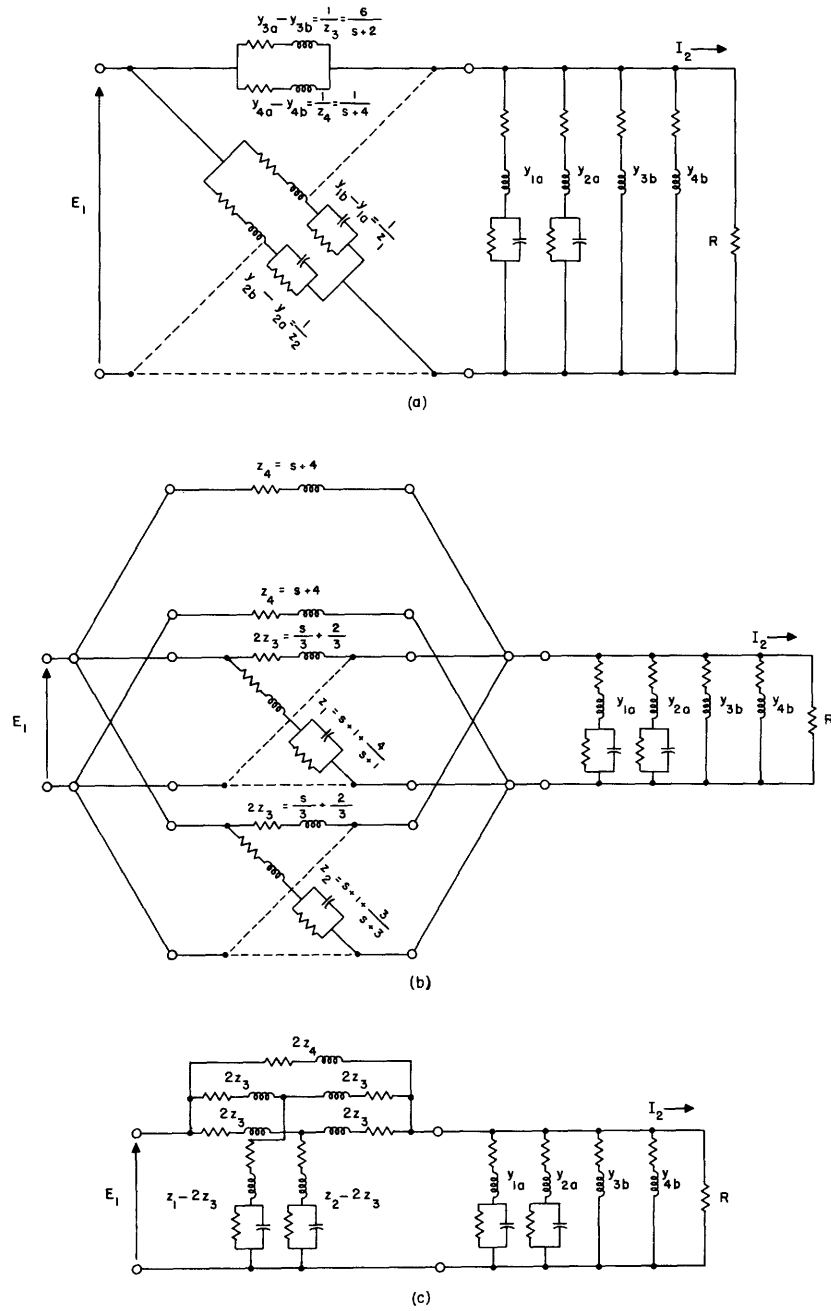


Fig. 26

Steps in reduction of lattice that contains a sufficiently large residue.

six-pole lattice shown in Fig. 25 whose arms are given by

$$\begin{aligned}
 Y_a &= y_{1a} + y_{2a} + y_{3a} + y_{4a} \\
 &= \frac{2s + 1}{s^2 + 2s + 5} + \frac{3s + 4}{s^2 + 4s + 6} + \frac{8}{s + 2} + \frac{2}{s + 4} \quad (147)
 \end{aligned}$$

and

$$\begin{aligned}
 Y_b &= y_{1b} + y_{2b} + y_{3b} + y_{4b} \\
 &= \frac{3s + 2}{s^2 + 2s + 5} + \frac{4s + 7}{s^2 + 4s + 6} + \frac{2}{s + 2} + \frac{1}{s + 4}. \quad (148)
 \end{aligned}$$

Because of the large residue in the admittance y_{3a} the lattice can be reduced. First we remove from each arm the shunt branches y_{1a} , y_{2a} , y_{3b} , and y_{4b} (see Fig. 26a). Since the drive is a voltage source the shunt branches may be omitted from the input terminals. Then we split the remaining lattice into two parallel lattices with a bridging branch, as shown in Fig. 26b. Finally these may be transformed, as shown in Fig. 26c, to obtain a bridged twin-T network with a complicated load.

As a last resort, if all other methods fail, it is always possible by the use of real transformers to reduce any lattice obtained in the synthesis of Y_{12} by the methods presented in this section.

III.6 Use of Mutual Inductance for General Reduction Of Lattice Realization of a Transfer Admittance

The use of real transformers, i. e. transformers possessing leakage inductance, winding resistance, finite magnetizing inductance and core loss, guarantees the reduction to an unbalanced form of any lattice synthesized to realize a transfer admittance by the methods given in this section. As is well known, any lattice may be reduced to an unbalanced form by the use of an ideal transformer. The process calls for a rotation of the output terminals so that the cross arms become series arms. In

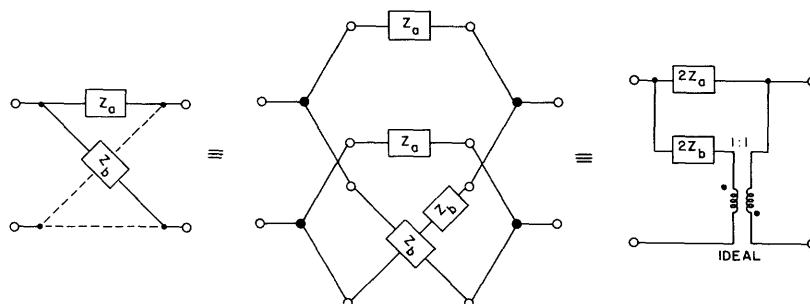
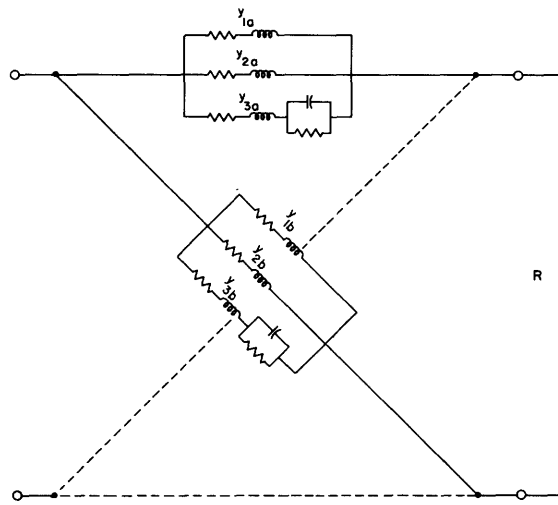
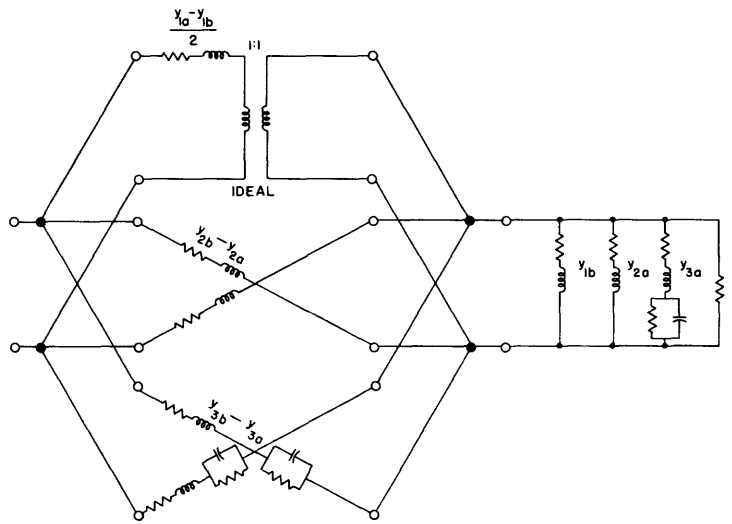


Fig. 27

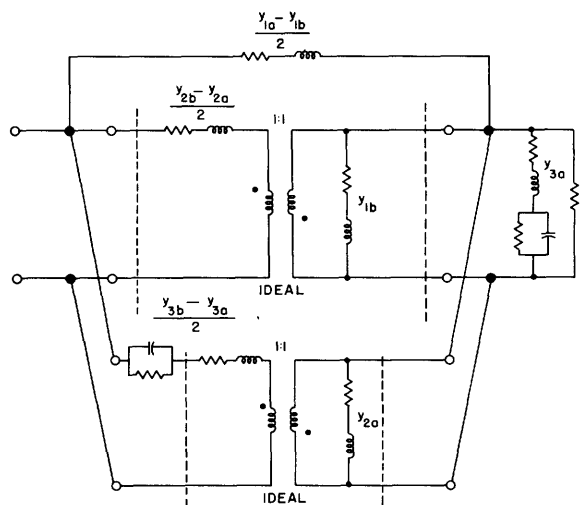
Reduction of a lattice by use of ideal transformer.



(a)



(b)



(c)

Fig. 28

Reduction of a lattice by use of real transformers.

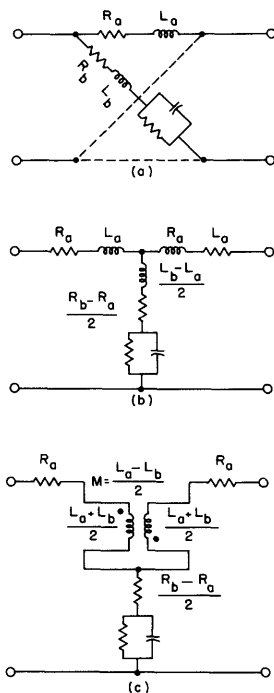


Fig. 29

Reduction of a lattice by use of mutual inductance.

order to compensate for this rotation an ideal transformer providing a phase reversal is necessary. The procedure is illustrated in Fig. 27.

The equivalent circuit of a real transformer is an ideal one with a series resistance and inductance and a shunt resistance and inductance. The form of network realized in the synthesis of a transfer admittance automatically provides the necessary series and shunt branches.

The general procedure is best explained by a simple example. Suppose we wish to reduce the lattice shown in Fig. 28a where the numerator of y_{1a} is larger than the numerator of the same pole in the cross arm, but the numerators of y_{2b} and y_{3b} are larger than the corresponding numerators of y_{2a} and y_{3a} . We may remove y_{1b} , y_{2a} and y_{3a} from both arms and divide the lattice into a group of lattices as shown in Fig. 28b. Then the final step of rotation and use of ideal transformers, which now have associated series and shunt inductances, may be carried out as in Fig. 28c. The networks within the broken lines represent the equivalent circuit of a real transformer.

If it is desired to use physically realizable mutual inductance without a core loss resistance, we may use another method that is often applicable. The lattices that realize a transfer admittance may be divided into components similar to that shown in Fig. 29a. It is obvious that the only requirement for the reduction of this component lattice is that R_b be greater than R_a . For, by removing R_a and L_a from each arm, we then obtain the network shown in Fig. 29b. If $L_b - L_a$ is negative, we can use the mutual inductance form of network shown in Fig. 29c for practical realization. All that remains is to show how to realize a small enough R_a . This happy circumstance often comes about naturally in the realization of a transfer admittance. If it does not, it can often be brought about by the pre-multiplication of numerator and denominator of Y_{12} by $(s+a)$ as the first step in the synthesis procedure, where a is a sufficiently small positive constant.

To conclude, we see that we may often arrive at an unbalanced form of network by the methods presented in this section. If we allow realizable mutual inductance in the Is it possible to obtain an unbalanced network directly, that is, without going through the intermediate step of the lattice structure? This problem will be discussed in section V.

IV. Synthesis of Unbalanced RC Networks

This section concerns itself with the problem of the synthesis of a minimum-phase transfer admittance with negative real poles by means of unbalanced RC networks. It is assumed that the reader is thoroughly familiar with the Guillemin method of synthesis (4) and the general method of zero shifting (1), since the problems of zero shifting to points on the negative real axis and decomposition of numerator and denominator polynomials are also characteristic of the synthesis procedure presented here. (In ref. 1 zero shifting is explained for the reactive case, but the RC problem is analogous.)

For convenience of analysis, the steps in the Guillemin procedure are summarized.

1. The polynomials q_1 and q_2 are chosen to satisfy the relationship $q = q_1 + q_2$. Each of these polynomials is of the same degree as q with the zero of smallest magnitude occurring in q_2 .

2. Both numerator and denominator of

$$Y_{12} = \frac{p}{q} = \frac{p}{q_1 + q_2} \quad (149)$$

are divided by q_1 to yield

$$Y_{12} = \frac{\frac{p}{q_1}}{1 + \frac{q_2}{q_1}} \quad (150)$$

Since

$$Y_{12} = \frac{y_{12}}{1 + y_{22}} \quad (151)$$

for a network terminated in one ohm, y_{12} may be identified with p/q_1 and y_{22} with q_2/q_1 .

3. The numerator rational function is divided into components, each containing two successive terms of p as numerator and q_1 as denominator. Each component is identified with the $y_{12}^{(\ell)}$ of a network whose $y_{22} = q_2/q_1$. The division of y_{12} in such a fashion requires that ℓ ladders be used, where ℓ is the smallest integer satisfying the relationship

$$\ell \geq \frac{n+1}{2} \quad (152)$$

and n is the degree of p .

4. Each ladder is realized by developing y_{22} by the Cauer method, simultaneously inserting the necessary zeros of $y_{12}^{(\ell)}$. The latter is achieved within a constant multiplier.

5. The constant multiplier obtained is generally not the one desired. Thus suppose for the first ladder we desire a transfer function $a_1 y_{12}^{(1)}$ but have achieved $A_1 y_{12}^{(1)}$. What is required is obviously a change in admittance level, that is, a multiplication of the level by a_1/A_1 to give the desired transfer admittance. This multiplication, however,

would also multiply the y_{22} of the ladder. Guillemin therefore defines

$$G = \frac{a_1}{A_1} + \frac{a_2}{A_2} + \dots + \frac{a_\ell}{A_\ell} \quad (153)$$

and multiplies the respective ladders by $a_\ell/A_\ell G$, which, since the same y_{22} has been used for each ladder, allows the achievement of a

$$\begin{aligned} \text{total } y &= \frac{a_1 y_{22}}{A_1 G} + \frac{a_2 y_{22}}{A_2 G} + \dots + \frac{a_\ell y_{22}}{A_\ell G} \\ &= \frac{y_{22}}{G} \left(\frac{a_1}{A_1} + \frac{a_2}{A_2} + \dots + \frac{a_\ell}{A_\ell} \right) \\ &= \frac{y_{22}}{G} (G) \\ &= y_{22} \end{aligned} \quad (154)$$

and an over-all transfer function

$$\frac{Y_{12}}{G} = \frac{\frac{y_{12}}{G}}{1 + y_{22}} = \frac{p}{Gq}. \quad (155)$$

It would appear from the reasoning in this step, specifically, the requirement that a y_{22} must be factored out in Eqs. 154, that the same total y_{22} must be used for each ladder, but, as will be shown, it is possible to circumvent this restriction.

In any practical problem worked out by the above procedure, it is generally found that a very large number of elements is required. Reducing the required number of ladders would reduce the number of elements. If, also, the degree of the denominator of y_{22} could be chosen as one less than that of q , then fewer elements would be required in each ladder. Finally, it is observed in step 3 above that each component $y_{12}^{(\ell)}$ has the total q_1 as its denominator; this requires that to insert only one or two zeros a network of high degree (i. e. the degree of q) be used. If $y_{12}^{(\ell)}$ could be devised to have only some of the factors of q_1 as a denominator and if only some of the partial fractions of y_{22} need be associated with each ladder, then the number of elements required would be substantially decreased.

First we consider the problem of reducing the number of ladders and the number of elements in each ladder; then we will explain the new synthesis procedure which makes use of the network theorem discussed in section II. 4.

It is well known that only one ladder is needed to realize Y_{12} when the zeros of p are all real and nonpositive. However, though we take advantage of the zeros existing in p when all of them are negative real, we neglect to take similar advantage when only some of the zeros are negative real. The direct method explained here corrects this oversight. If p is given in factored form, we note the number of negative real roots;

if it is not given in factored form, we apply the Sturm test (14) to p to determine the number of negative real roots. Let this number be r . Then the number of ladders necessary, rather than being given by Eq. 152, is equal to the smallest integer satisfying

$$\ell \geq \frac{n - r + 1}{2} . \quad (156)$$

For example, suppose p is of the eighth degree and is given by

$$p = (s^2 + 2s + 4) (s^2 + 3s + 5) (s+6) (s+7) (s+8) (s+9). \quad (157)$$

By the Guillemin procedure five ladders are required; by use of Eq. 156, ℓ is given as only three. All we need do is divide the quartic into three components and then multiply each of them by the four negative real roots. By use of the Guillemin procedure to obtain the three components, the breakdown is

$$\begin{aligned} p &= (s^4 + 5s^3 + 15s^2 + 22s + 20) (s+6) (s+7) (s+8) (s+9) \\ &= (s^4 + 5s^3) (s+6) (s+7) (s+8) (s+9) + (15s^2 + 22s) (s+6) \\ &\quad \times (s+7) (s+8) (s+9) + 20 (s+6) (s+7) (s+8) (s+9). \end{aligned} \quad (158)$$

Thus, as is evident from the procedure, the additional computations are the use of the Sturm test and, if the number of real roots is sufficiently large, solving for the roots of the polynomial. The Sturm test may show that the roots should not be found; for example, a polynomial of the seventh degree with only one negative real root will require four ladders by both procedures. It is not claimed that the new method gives the minimum possible number of ladders; it is claimed, however, that it gives a direct procedure for determining whether fewer ladders than the ℓ given by Eq. 152 can be used.

As a first step in the reduction of the number of elements in each ladder, it is suggested that the quartic factor containing the complex roots in Eq. 157 should not be broken down in the Guillemin manner. When it is possible to make the roots of the p components coincide with the chosen roots of q_1 , we avail ourselves of this resource. The subsequent division by q_1 will cause a zero-pole cancellation in the $y_{12}^{(\ell)}$ components so that, because the number of poles of the $y_{12}^{(\ell)}$ in which cancellation occurred is reduced, the number of partial fraction components of y_{22} that need be associated with the $y_{12}^{(\ell)}$ is smaller. As an example, suppose that for the p given in Eq. 157 the q_1 has zeros of interest at $s = -1$ and $s = -3$. We break down p so that the first component has a root at $s = -3$ and the second component one at $s = -1$. Thus, instead of Eq. 158, we obtain

$$\begin{aligned}
p &= s^2(s^2 + 5s + 6)(s+6)(s+7)(s+8)(s+9) + 9(s^2 + 2.44s + 1.44) \\
&\quad \times (s+6)(s+7)(s+8)(s+9) + 7.04(s+6)(s+7)(s+8)(s+9) \\
&= s^2(s+3)(s+2)(s+6)(s+7)(s+8)(s+9) + 9(s+1) \\
&\quad \times (s+1.44)(s+6)(s+7)(s+8)(s+9) + 7.04(s+6)(s+7) \\
&\quad \times (s+8)(s+9). \tag{159}
\end{aligned}$$

This breakdown leads to a network with fewer elements. Of course, following the same reasoning when choosing the roots of q_1 , we should if possible place them at the positions of the original real roots of p , i.e. at -6 , -7 , -8 , or -9 .

With the above preliminary remarks, which will be applied in the synthesis of network N_a , we are now ready to make use of the network theorem discussed in section II.4. For convenience, we repeat below the final results of Eqs. 62 and 63 for the partitioned network shown in Fig. 30

$$Y_{12} = \frac{I_2}{E_1} = \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} \tag{160}$$

and

$$K = \frac{E_2}{E_1} = \frac{y_{12a} K_b}{y_{22a} + Y_b} \tag{161}$$

where $K_b = E_2/E$.

One may question why we use both equations given above since the more general Eq. 161 would suffice for both cases. The transfer admittance equation is retained because we are accustomed to thinking of synthesis in terms of it. It may be used when we intend the Z to be a pure resistance. We realize that what we are almost always finally interested in is the output voltage E_2 , not the current I_2 . When the load Z is a pure resistance, obtaining the E_2/E_1 from the synthesized I_2/E_1 represents merely a scale change with consequently no change in frequency behavior. However, when Z is complex (for example, it may be desirable to have a parallel RC network as the load) then E_2/E_1 represents a change in frequency behavior over I_2/E_1 , so that we must use Eq. 161 for the synthesis. It is pointed out that in the final frequency characteristic the

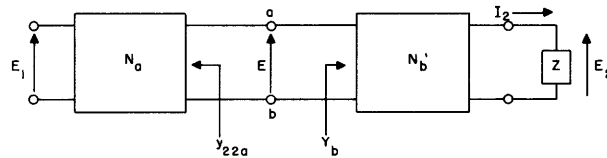


Fig. 30
Partitioned RC network.

zeros of Z appear as zeros of the over-all transfer function, but the poles of Z , as we would expect, cancel out and thus do not appear in K . Thus in the zero-shifting procedure we may use the load Z as a zero-producing branch.

Each partitioned network is worked on independently. For realizing N_a we observe, since it is characterized by short-circuit driving-point and transfer admittances, that we may parallel ladder structures, each ladder having been obtained by zero-shifting techniques, starting from the right and working toward the left. For N_b , on the contrary, we work from left to right and we can use only one ladder. Therefore we must associate only negative real zeros (including the origin and infinity, of course) with the transfer function of N_b . In any particular problem there will naturally be a variety of possible distributions to N_a and N_b of the factors of the numerator of the over-all transfer function. The distribution of poles to the transfer functions of N_a and N_b determines the distribution of the partial fractions of the positive real denominator. One final point may be obvious: each ladder structure of N_a must end on the left with a series branch while the ladder for N_b must end on the right with a shunt branch.

The procedure is spelled out in great detail because many of the ideas seem strange; for example, zero shifting in opposite directions from within a network N .

The steps in the synthesis procedure of a given minimum phase $Y_{12} = p/q$ that makes use of Eq. 160 and its associated network may now be presented as follows.

1. Choose a suitable q_1 whose degree is one less than that of q . The zeros of q_1 are chosen so that they alternate with the zeros of q and the lowest magnitude zero occurs in q . If the factored form of p shows that it has negative real zeros that occur within the intervals formed by the zero positions of q , then, for the purpose of reducing the number of elements in the final network, choose zeros of q_1 at these points. Thus, for a simple illustration, if $q = (s+1)(s+3)(s+5)(s+7)(s+9)(s+11)$ and p is given by $(s+2)(s+4)(s^2+4s+9)$, then an acceptable q_1 is $(s+2)(s+4)(s+6)(s+8)(s+10)$.

2. Divide numerator and denominator of Y_{12} by q_1 to obtain

$$Y_{12} = \frac{\frac{p}{q_1}}{\frac{q}{q_1}} \quad (162)$$

The function q/q_1 can be identified as a positive real RC admittance.

3. Distribute the poles and zeros of p/q_1 to the y_{12a} and Y_{12b} appearing in Eq. 160. The distribution of the factors should be made so that the transfer functions are realizable, a distribution which is always possible, and a small number of ladders and elements is required. For example, if $p = (s^2+2s+3)(s^2+3s+5)$ and q_1 is of the eighth degree, we would not choose

$$\left. \begin{aligned} y_{12a} &= \frac{(s^2+2s+3)(s^2+3s+5)}{\text{7th degree}} \\ Y_{12b} &= \frac{1}{\text{1st degree}} \end{aligned} \right\} \quad (163)$$

for these identifications do not make use of the full power of the method. The synthesis would require three ladders with seventh degree denominators as N_a and one ladder with a first degree denominator for N_b . A better distribution would be

$$\left. \begin{aligned} y_{12a} &= \frac{(s^2 + 2s + 3)(s^2 + 3s + 5)}{\text{4th degree}} \\ Y_{12b} &= \frac{1}{\text{4th degree}} \end{aligned} \right\} \quad (164)$$

for this requires four ladders with fourth degree denominators. Another good distribution is

$$\left. \begin{aligned} y_{12a} &= \frac{(s^2 + 2s + 3)(s^2 + 3s + 5)}{\text{3rd degree}} \\ Y_{12b} &= \frac{1}{\text{5th degree}} \end{aligned} \right\} \quad (165)$$

4. We now need the q/q_1 broken into fractions, each of which is positive real. As is well known the partial fractions of q/q_1 , since this function represents an RC admittance, will not have the required form. The partial fractions of q/sq_1 , however, will have positive residues. Therefore decompose q/sq_1 into partial fractions and then multiply both sides by s to obtain q/q_1 as the sum of positive real fractions. For example, if

$$\frac{q}{q_1} = \frac{(s+2)(s+4)}{s+3} \quad (166)$$

then

$$\begin{aligned} \frac{q}{sq_1} &= \frac{(s+2)(s+4)}{s(s+3)} \\ &= 1 + \frac{8}{3s} + \frac{1}{s+3} \end{aligned} \quad (167)$$

whence

$$\frac{q}{q_1} = s + \frac{8}{3} + \frac{1}{s+3} \cdot \quad (168)$$

5. Identify as y_{22a} the sum of the fractions of q/q_1 that contain the poles of the previously identified y_{12a} , and perform a similar identification for Y_b and Y_{12b} . The constant term may be associated with either y_{22a} or Y_b . Similarly, if a pole at infinity does not occur in y_{12a} or Y_{12b} , then the first degree term representing the pole at infinity may be identified as part of either y_{22a} or Y_b .

6. Develop Y_b as the single ladder network N_b with the final branch a shunt resistance, simultaneously inserting the zeros of Y_{12b} (all of which are, of course, negative real) by zero shifting. If for various reasons we must have a complex impedance as the final shunt branch, we can consider that we are synthesizing K of Eq. 161 and not Y_{12} .

This represents a mere shift in point of view rather than a change in procedure or underlying philosophy. In the development of the ladder we achieve Y_{12b} within a constant multiplier, but no steps need be taken to modify this constant since it appears as a factor of the over-all transfer function.

7. Now develop y_{22a} as a parallel group of ladders, the composite network giving N_a . The Guillemin procedure with the modifications already suggested in this section may be used in the synthesis of N_a . However, if we wish to obtain a final N_a with a reduced number of elements brought about by the pole-zero cancellations in the component ladder transfer functions and by the use of only a part of y_{22a} for each ladder, then a further modification in steps 4 and 5 of the Guillemin procedure is necessary. First, in regard to step 4, associate with each component transfer function only those poles of y_{22a} which it contains after cancellation. We do not associate the total fraction containing the pole but only part of it so that we can satisfy the relation

$$\sum_{m=1}^{\ell} y_{22a}^{(m)} = y_{22a}. \quad (169)$$

Thus for an N_a consisting of three paralleled ladders if

$$y_{22a} = \frac{s}{s+2} + \frac{2s}{s+3} + \frac{3s}{s+4} \quad (170)$$

and if the pole $s = -2$ occurs only in $y_{12}^{(2)}$, the pole $s = -3$ occurs in all three $y_{12}^{(\ell)}$, and the $s = -4$ occurs only in $y_{12}^{(1)}$, then acceptable driving-point functions for the respective ladders are

$$\left. \begin{aligned} y_{22a}^{(1)} &= \frac{2}{3} \frac{s}{s+3} + \frac{3s}{s+4} \\ y_{22a}^{(2)} &= \frac{s}{s+2} + \frac{2}{3} \frac{s}{s+3} \\ y_{22a}^{(3)} &= \frac{2}{3} \frac{s}{s+3} \end{aligned} \right\} \quad (171)$$

This artifice guarantees that the total driving-point admittance is the desired y_{22a} , but we have still to guarantee that the constant multipliers of the $y_{12}^{(\ell)}$ are the desired ones.

Suppose that $a_{\ell} y_{12}^{(\ell)}$ is desired but $A_{\ell} y_{12}^{(\ell)}$ is achieved so that

$$y_{12a} = A_1 y_{12a}^{(1)} + A_2 y_{12a}^{(2)} + \dots + A_{\ell} y_{12a}^{(\ell)} \quad (172)$$

instead of the desired

$$y_{12a} = a_1 y_{12a}^{(1)} + a_2 y_{12a}^{(2)} + \dots + a_{\ell} y_{12a}^{(\ell)}. \quad (173)$$

To correct this we compute each of the A_{ℓ}/a_{ℓ} to find the minimum ratio A_r/a_r .

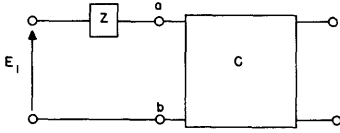


Fig. 31

Representation of a typical ladder of N_a .

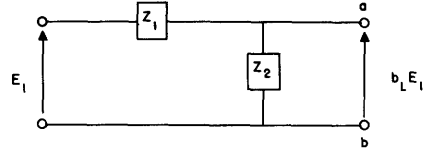


Fig. 32

Voltage division to obtain reduced gain.

Factoring this ratio from the terms on the right side of Eq. 172

$$y_{12a} = \frac{A_r}{a_r} \left(\frac{A_1 a_r}{A_r} y_{12a}^{(1)} + \frac{A_2 a_r}{A_r} y_{12a}^{(2)} + \dots + a_r y_{12a}^{(r)} + \dots + \frac{A_\ell a_r}{A_r} y_{12a}^{(\ell)} \right) \quad (174)$$

yields coefficients of which all except one are greater than the respective coefficients in Eq. 172, the r th coefficient being equal to the desired one. The factors which are necessary to reduce the coefficients are computed as

$$b_\ell = \frac{a_\ell}{A_\ell} \frac{A_r}{a_r} \quad \ell = 1, 2, \dots \quad (175)$$

and the gain of the respective ladders must be reduced by this factor. This is easily done. Recalling that each ladder ends on the left in a series branch, suppose we represent the last branch by Z and the rest of the network by box C , as shown in Fig. 31. If we open circuit terminals a - b , we see looking to the left an impedance Z and a voltage E_1 ; we wish to see the same impedance Z but a reduced voltage $b_\ell E_1$. This can be achieved by the voltage division shown in Fig. 32, where

$$\left. \begin{aligned} \frac{Z_1 Z_2}{Z_1 + Z_2} &= Z \\ \frac{Z_2}{Z_1 + Z_2} &= b_\ell \end{aligned} \right\} \quad (176)$$

which yields in terms of the known b_ℓ and Z

$$\left. \begin{aligned} Z_1 &= \frac{Z}{b_\ell} \\ Z_2 &= \frac{Z}{1 - b_\ell} \end{aligned} \right\} \quad (177)$$

Most often the last branch Z is a resistance so that the procedure adds only one more element to each ladder.

Thus the final transfer function for N_a is

$$y_{12a} = \frac{A_r}{a_r} \left(a_1 y_{12a}^{(1)} + a_2 y_{12a}^{(2)} + \dots + a_\ell y_{12a}^{(\ell)} \right) \quad (178)$$

and in the networks N_a and N_b , as shown in Fig. 26, we have achieved a composite network N with the desired Y_{12} within a constant multiplier. This constant is given by A_r/a_r times the constant multiplier of Y_{12b} .

The theorem has been shown useful in RC synthesis; in succeeding sections it is applied to the synthesis of RLC networks.

V. RLC Synthesis by Means of Unbalanced Networks

In this section we are concerned with the development of a synthesis procedure that yields unbalanced RLC networks with a given minimum-phase transfer admittance characteristic. The given characteristic is achieved in both magnitude and phase. This is accomplished by the combined use of the techniques of sections II.2 through II.5. As was shown there, the necessary techniques of complex zero shifting and the breakdown of the Hurwitz denominator can always be carried out. However, whether an acceptable numerator polynomial breakdown can be found must be investigated for each problem. The type of p -breakdown possible determines which one of the variants of the synthesis procedure to use, as was shown in section II.5. The reader will recall, too, that no numerator breakdown need be found for the following cases.

a. Transfer functions of low degree, specifically of degree not higher than a third divided by a fourth (and in some cases of higher degree). The direct procedure for achieving the realization of such functions is given in section VI, where it is also shown that a transfer function of any degree may be achieved if we allow the use of a few vacuum tubes for isolation and gain.

b. Problems for which the breakdown $q = q_1 + Aq_1'$ yields a q_1 with only one pair of complex zeros.

c. Problems for which the breakdown $q = q_1 + Aq_1' + Bq_1''$ yields a q_1 with all its zeros negative real.

d. Problems for which the breakdown $q = q_1 + Aq_1' + Bq_1''$ yields a q_1 with all negative real zeros though q_1 has one or more pairs of complex zeros. This case is included in the no p -breakdown category, because, although we do decompose p in a way unlike that for the two-element kind networks, the decomposition is direct and its possibility can be determined by inspection.

e. Problems for which $q^{(-1)}$, the integral of q , is Hurwitz and has only one pair of complex zeros. This, of course, requires as a necessary but not sufficient condition that q have only one pair of complex zeros.

The primary desire is to obtain a q -breakdown that simplifies the ensuing synthesis procedure, i. e. eliminates the necessity of finding a p -breakdown. At present no precise mathematical statement can be made about how to guarantee the achievement of one of the desirable breakdowns of the numerator. We must try different decompositions, and this therefore represents the first step in the synthesis. In these trials the tools used again and again are the Hurwitz test for roots in the left half-plane (9) and Sturm's theorem (14). In general the application of the two tests is tedious but the

labor-saving form of the Hurwitz test given in reference 9 and a few short cuts in the use of Sturm's theorem can relieve the tedium. The q-breakdown and the tests are mechanical and may be performed by a computer or an assistant. The importance of this preliminary work that determines the variant of the synthesis procedure to be used cannot be overstressed.

The procedure for the first step is to find a number of Hurwitz q_1 's derived from $q = q_1 + Aq_1'$ by the use of different values of A. Also find a number of q_1 's and q_1' 's by use of different constants in $q = q_1 + Aq_1' + Bq_1''$. A suggested trial is the use among others of a very small value of A with different large values of B. If one wants to increase the chances of a desirable q-breakdown he can also try the above decomposition on $(s+a)q$ and $(s+a)(s+b)q$, where a and b are suitably chosen positive constants. Finally we compute $q^{(-1)}$ and test it for Hurwitz character.

Then, to finish this step, the Sturm test is applied. We test the q_1 's derived from $q = q_1 + Aq_1'$ for the number of negative real roots between minus infinity and zero. Each q_1' from $q = q_1 + Aq_1' + Bq_1''$ is also tested. Since q_1 of this second breakdown cannot have all negative real roots unless q_1' does, we do not test the q_1 unless the q_1' has all its roots negative real. If $q^{(-1)}$ is Hurwitz and q has only one pair of complex zeros, we also test $q^{(-1)}$ for the number of its negative real roots, choosing the arbitrary constant, if possible, to bring about the desirable result of only one pair of complex roots.

If the procedure above demonstrates the existence of a q-breakdown that eliminates the need for finding a p-breakdown, we proceed with the corresponding synthesis procedure given below. If such breakdowns are not found, the partial fraction tests are made, as indicated in section II.5.

The procedures are catalogued under characteristics of polynomials derived from q. Each procedure also applies for polynomials derived from q multiplied by surplus factors, e.g. $(s+a)q$ or $(s+a)(s+b)q$, for which the original numerator p is naturally changed to p_c where $p_c = (s+a)p$ or $p_c = (s+a)(s+b)p$, respectively.

The variants of the synthesis procedure may now be described as follows.

A. The polynomial q_1 derived from $q = q_1 + Aq_1'$ has only one pair of complex zeros

Making use of the network theorem of section II.4, we write, as in section II.5B(1)

$$Y_{12} = \frac{y_{12a}Y_{12b}}{y_{22a} + Y_b} = \frac{\frac{p}{q_1}}{1 + \frac{Aq_1'}{q_1}} \quad (179)$$

We then define

$$\left. \begin{aligned} q_1 &= q_b q_a = (s^2 + 2cs + d) q_a \\ p &= p_a p_b \end{aligned} \right\} \quad (180)$$

where p_b may be any factor of p of the second degree or less, and $q_b = (s^2 + 2cs + d)$ is the factor with two complex zeros so that q_a has only negative real zeros. It is now possible to identify

$$\left. \begin{aligned} y_{12a} &= \frac{p_a}{q_a} \\ y_{22a} &= 1 + \frac{Aq'_a}{q_a} \end{aligned} \right\} \quad (181)$$

and

$$\left. \begin{aligned} Y_{12b} &= \frac{p_b}{q_b} \\ Y_b &= \frac{Aq'_b}{q_b} \end{aligned} \right\} \quad (182)$$

The network N_b is realized by the complex zero-shifting techniques of section II.3 as a single ladder RLC structure with a final shunt arm. As in the RC synthesis treated in section IV, the constant multiplier of Y_{12b} which is achieved is retained as a factor of the over-all transfer function. Remarks made there about termination of N_b in a complex impedance also apply here. Since q_a has only negative real zeros, N_b is an RL network. Its synthesis is analogous to that for an RC network so that the Guillemin procedure, modified by the applicable techniques of section IV, is used for the realization of N_b . The final network has the form shown in Fig. 33, where N_b consists of N'_b plus the termination Z .

B. The polynomial $q^{(-1)}$ is Hurwitz and has only one pair of complex zeros

As in (A) we make use of the network theorem to write

$$Y_{12} = \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} = \frac{\frac{p}{q^{(-1)}}}{\frac{q}{q^{(-1)}}} \quad (183)$$

whence we obtain the identifications

$$\left. \begin{aligned} y_{12a} &= \frac{p_a}{q_a^{(-1)}} \\ y_{22a} &= \frac{q_a}{q_a^{(-1)}} \end{aligned} \right\} \quad (184)$$

and

$$\left. \begin{aligned} Y_{12b} &= \frac{p_b}{q_b^{(-1)}} \\ Y_b &= \frac{q_b}{q_b^{(-1)}} \end{aligned} \right\} \quad (185)$$

where, by definition

$$\left. \begin{aligned} q^{(-1)} &= q_b^{(-1)} q_a^{(-1)} = (s^2 + 2cs + d) q_a^{(-1)} \\ p &= p_a p_b \end{aligned} \right\} \quad (186)$$

The synthesis of N_a and N_b is completed exactly as in (A) above.

C. The polynomial q_1 derived from $q = q_1 + Aq_1' + Bq_1''$ has all negative real zeros. (See sec. II. 5b(3).)

The use of the network theorem allows us to write

$$Y_{12} = \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} = \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'}} \quad (187)$$

where, by virtue of the fact that q_1 has only negative real zeros, Bq_1''/Aq_1' represents an RL admittance whereas q_1/Aq_1' corresponds to an RC admittance.

If we neglect the RL admittance, we can synthesize the remaining RC network by use of the complete procedure given in section IV. Neglecting the RL admittance is equivalent to removing it as a shunt branch between the N_a and N_b structures, i.e. we realize it by the Foster method for two-element kind admittances. The form of network obtained is shown in Fig. 34. RLC synthesis has thus been achieved almost completely by a procedure that applied to two-element kind networks.

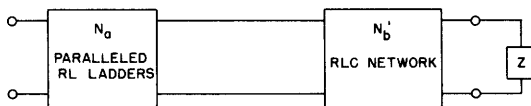


Fig. 33

Form of final network for a q_1 with only one pair of complex zeros.

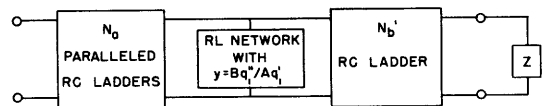


Fig. 34

Network realized for a q_1 with only negative real zeros.

D. The polynomial q_1' derived from $q = q_1 + Aq_1' + Bq_1''$ has all negative real zeros though q_1 has one or more pairs of complex zeros (see sec. II.5A(4)).

For an RLC network terminated in one ohm we write

$$Y_{12} = \frac{y_{12}}{1 + y_{22}} = \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'}} \quad (188)$$

The procedure will always work provided that p is of sufficiently high degree; that is, the degree of p must be equal to or greater than the number of complex zeros in q_1 . If this is true we form a new polynomial p_a by use of

$$p_a = kp - p_1 \quad (189)$$

where k is a positive constant large enough so that p_a has only positive coefficients and p_1 has the complex zeros of q_1 as factors. We can now write

$$kY_{12} = \frac{\frac{p_1}{Aq_1'} + \frac{p_a}{Aq_1'}}{1 + \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'}} \quad (190)$$

and realize the desired network by a group of paralleled ladder structures terminated in one ohm.

The first ladder will contain all three kinds of elements with a $y_{22}^{(1)} = q_1/Aq_1'$ and a $y_{12}^{(1)} = p_1/Aq_1'$, except for a constant multiplier. It is obtained by zero shifting. We expand $z = 1/y_{22}^{(1)} = Aq_1'/q_1$ into partial fractions and then remove all the complex poles as series arms to leave a remainder z' . The removed poles place the zeros of p_1 in $y_{12}^{(1)}$; the remaining zeros of $y_{12}^{(1)}$ are at infinity. Since z' has only negative real poles, it corresponds to an RC structure. It is thus possible to realize it and at the same time place all the remaining zeros of $y_{12}^{(1)}$ at infinity. The ladder will have the form shown in Fig. 35.

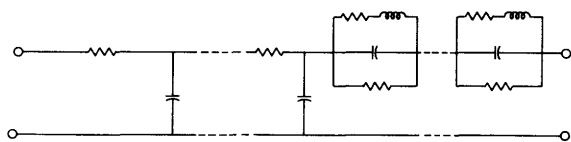


Fig. 35

Form of first ladder network.

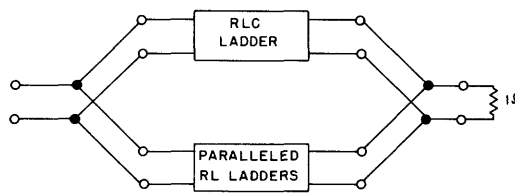


Fig. 36

Final composite network where the RLC ladder has the form shown in Fig. 35.

With the remaining ladders we realize

$$\left. \begin{aligned} y_{12a} &= \frac{p_a}{Aq_1'} \\ y_{22a} &= \frac{Bq_1''}{Aq_1'} \end{aligned} \right\} \quad (191)$$

Since y_{22a} is an RL admittance this synthesis can be achieved by the Guillemin procedure for two-element kind networks, modified by the appropriate changes of section IV, e.g. those yielding fewer ladders and fewer elements per ladder. (The network theorem, of course, is not used.)

There is the problem of obtaining acceptable constant multipliers for each of the ladder networks. This is accomplished as in step (7) of the RC synthesis procedure discussed in section IV. The form of final network is shown in Fig. 36.

E. The components of p possess either all the zeros of $q_1 (q^{(-1)})$ except for one complex pair or all the zeros of $q_1 (q^{(-1)})$ except for all the negative real zeros (see section II.5A (1a), (2a), (3a), (4a).)

Suppose that we consider the procedure

$$Y_{12} = \frac{y_{12}}{1 + y_{22}} = \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'}} \quad (192)$$

where without loss of generality p has the three components p_1 , p_2 , and p_3 , which are given respectively by the terms on the right-hand side of the equation

$$\begin{aligned} p &= (s^2 + 2fs + g) (s+c) (s+d) (s+e) (s+k) \\ &+ (s^2 + 2hs + t) (s+c) (s+d) (s+e) (s+m) \\ &+ (s^2 + 2fs + g) (s^2 + 2hs + t) (s+n) (s+r) \end{aligned} \quad (193)$$

and q_1 is a polynomial of the seventh degree given by

$$q_1 = (s^2 + 2fs + g) (s^2 + 2hs + t) (s+c) (s+d) (s+e) \quad (194)$$

where the quadratic factors possess complex zeros. Then we may identify

$$\left. \begin{aligned} y_{12}^{(1)} &= \frac{p_1}{Aq_1'} \\ y_{12}^{(2)} &= \frac{p_2}{Aq_1'} \\ y_{12}^{(3)} &= \frac{p_3}{Aq_1'} \\ y_{22} &= \frac{q_1}{Aq_1'} \end{aligned} \right\} \quad (195)$$

The complete structure is realized as a set of paralleled ladders just as in the Guillemin RC synthesis. For the first ladder we remove as series arms from $z = 1/y_{22} = Aq_1'/q_1$ the partial fractions with poles equal to all the zeros of p_1 except the factor $(s+k)$. With the single remaining partial fraction with poles given by $(s^2 + 2hs + t)$, we can zero shift to $s = -k$ by the method explained in section II. 3. In a similar manner we can realize the second ladder. The third ladder differs only in that zero shifting is accomplished with an RC remainder function. As in the Guillemin RC synthesis (sec. IV), we adjust the constant multipliers to realize Y_{12} within a multiplicative constant. For a final network we have three paralleled RLC ladders terminated in one ohm.

The same zero-shifting procedure is followed for

$$K = \frac{E_2}{E_1} = \frac{y_{12}}{Y_L + y_{22}} = \frac{\frac{p}{q^{(-1)}}}{\frac{q}{q^{(-1)}}} \quad (196)$$

except that part of the residues in one or more poles of $q/q^{(-1)}$ is first extracted and identified as the termination Y_L . It is also followed for

$$Y_{12} = \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'}} \quad (197)$$

except that Bq_1''/Aq_1' is first removed as a simple (i. e. realized directly by Foster method) RLC shunt branch.

For the procedure

$$Y_{12} = \frac{\frac{p}{q_1}}{1 + \frac{Aq_1'}{q_1}} \quad (198)$$

because of the zero-pole cancellation in $y_{12}^{(l)}$ only part of y_{22} is associated with each ladder. For example, for the first ladder, where

$$y_{12}^{(1)} = \frac{p_1}{q_1} = \frac{s + k}{s^2 + 2hs + t} \quad (199)$$

we use

$$y_{22}^{(1)} = \frac{s + a}{s^2 + 2hs + t} \quad (200)$$

that is, only one appropriate partial fraction component of y_{22} . The method of adjusting the constant multipliers, since only part of y_{22} is associated with each ladder, must be that given in step (7) of section IV.

F. The components of p possess either all the zeros of q_1 ($q_1^{(-1)}$) except for the one complex pair plus a real zero whose combination of partial fractions permits zero shifting, or all the zeros of q_1 ($q_1^{(-1)}$) except for negative real zeros (see sec. II. 5A (1b), (2b), (3b), (4b).)

The procedures are almost the same as those given in (E) above; the only difference is that after the initial removal of the series branch zero shifting for one or more ladders is accomplished with a combination of two partial fractions at least a pair of whose poles is complex. This zero-shifting variation was treated in section II. 3.

G. One or more components of p possess either all the zeros of q_1 except for one complex pair (or one complex pair plus a real zero when zero shifting with the combination is possible) or all the zeros of q_1 except for the negative real zeros. The other components are similarly related to the zeros of q_1' (see sec. II. 5A(4).)

From the equation

$$Y_{12} = \frac{y_{12}}{1 + y_{22}} = \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'}} \quad (201)$$

we obtain the identifications

$$\left. \begin{aligned} y_{22} &= \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'} \\ y_{12} &= \frac{p}{Aq_1'} \end{aligned} \right\} \quad (202)$$

Equation 74 shows the derivation of a new polynomial p_a so that

$$kp = p_a + p_1. \quad (203)$$

Suppose that p_a is equal to the sum of two polynomials, $p_2 + p_3$, each of which has one of the desired relations to the zeros of q_1' .

Then by synthesis with zero shifting, already described in (E) above, we realize $y_{22}^{(1)} = q_1/Aq_1'$ and its associated transfer function $y_{12}^{(1)} = p_1/Aq_1'$. The second and third ladders are realized as the driving-point admittances formed from the appropriate components of the partial fraction expansion of $y_{22}^{(2)} + y_{22}^{(3)} = Bq_1''/Aq_1'$, as explained in (E) after Eq. 198. Thus we achieve the complete network, composed of three paralleled ladders terminated in one ohm.

H. The components of p_a , where $p = p_a p_b$ have all the zeros of q_a , where $q_1 = q_a q_b$ or $q^{(-1)} = q_a q_b$ (see sec. II.5B(1) and (2)), except for one complex pair (or one complex pair plus a real zero when zero shifting with the combination is possible) or all the zeros of q_a except for the negative real zeros.

As explained in section II.5b(1), after writing

$$\begin{aligned}
 Y_{12} &= \frac{y_{12a} Y_{12b}}{y_{22a} + Y_b} = \frac{\frac{p}{q_1}}{1 + \frac{Aq_1'}{q_1}} \\
 &= \frac{\frac{p_a p_b}{q_a q_b}}{1 + \frac{Aq_a'}{q_a} + \frac{Aq_b'}{q_b}} \tag{204}
 \end{aligned}$$

we may make the identifications

$$\left. \begin{aligned}
 y_{12a} &= \frac{p_a}{q_a} \\
 y_{22a} &= 1 + \frac{Aq_a'}{q_a}
 \end{aligned} \right\} \tag{205}$$

and

$$\left. \begin{aligned}
 Y_{12b} &= \frac{p_b}{q_b} \\
 Y_b &= \frac{Aq_b'}{q_b}
 \end{aligned} \right\} \tag{206}$$

The method of (E) above is used to synthesize N_a as paralleled RLC ladders and N_b as a single ladder.

A similar procedure is used for the synthesis of section II.5B(2), except that $q^{(-1)}$ is used in place of q_1 .

VI. Synthesis of a Restricted Transfer Voltage Ratio Without the Use of Vacuum Tubes or of a General Transfer Voltage Ratio with the Use of Tubes

Many synthesis designs allow the use of vacuum tubes, not for the purpose of shaping the frequency characteristic but for gain. In fact, when a transfer function of high degree is to be realized as an unbalanced RLC network with lossy coils, gain most probably will have to be provided somewhere in the system. Advantages may be obtained in reduction of complexity of computation and decrease of loss introduced by the frequency-shaping network if the necessary gain is introduced between networks representing segments of the over-all transfer function, that is, if the over-all function is divided into factors each of which is realized separately and isolated from the others by use of an amplifier stage. Thus we may write a given transfer voltage ratio K as

$$K = \frac{E_2}{E_1} = K^{(1)}K^{(2)} \dots \quad (207)$$

each $K^{(n)}$ being synthesized separately. In any practical example it is desirable to use a small number of vacuum tubes so that the component transfer functions should, if possible, have degrees higher than the second.

In this section we apply the network theorem of section II. 4, particularly the final result given by Eq. 63, to realize a given minimum-phase transfer voltage ratio with a numerator and denominator of degree not greater than the third and fourth, respectively. This constitutes the restricted transfer function that can be synthesized directly without the use of vacuum tubes. In those cases for which zero shifting is possible with a combination of a pair of complex poles and a real pole, the denominator may be of fifth or higher degree. If we are given a general minimum-phase function of arbitrary degree, it is possible to realize it by component structures with fourth-degree denominators, vacuum-tube stages simultaneously isolating each structure and providing gain. Since a voltage drive is required, each stage consists of an amplifier plus a cathode follower.

Suppose that it is desired to realize a given transfer voltage ratio

$$K = \frac{E_2}{E_1} = \frac{p}{q} \quad (208)$$

where p is a third degree polynomial and q is a fourth. We derive q_1 by use of the breakdown $q = q_1 + Aq_1'$. Without loss of generality we may write

$$\left. \begin{aligned} p &= (s+a)(s^2 + 2bs + c) \\ q_1 &= (s^2 + 2ds + e)(s^2 + 2gs + h) \end{aligned} \right\} \quad (209)$$

that is, p has two and q_1 has four complex zeros.

Now by use of Eq. 63

$$K = \frac{y_{12a} K_b}{y_{22a} + Y_b} \quad (210)$$

and of the given rational function in the form

$$K = \frac{\frac{p}{q_1}}{1 + \frac{Aq'_1}{q_1}} \quad (211)$$

we may obtain the proper identifications. First we expand Aq'_1/q_1 into two partial fractions, each of which has a pair of complex poles. Then the identifications can be made

$$\left. \begin{aligned} y_{12a} &= \frac{p_a}{q_a} = \frac{s+a}{s^2+2ds+e} \\ y_{22a} &= 1 + \frac{2A(s+d)}{s^2+2ds+e} \end{aligned} \right\} \quad (212)$$

and

$$\left. \begin{aligned} K_b &= \frac{p_b}{q_b} = \frac{(s^2+2bs+c)}{s^2+2gs+h} \\ Y_b &= \frac{2A(s+g)}{s^2+2gs+h} \end{aligned} \right\} \quad (213)$$

The network N_a is realized as a single ladder by a development of y_{22a} that inserts the necessary zero of y_{12a} at $s = -a$. Similarly, we develop Y_b as a single ladder, but N_b , unlike network N_a , has a final shunt branch. The zero-shifting technique was explained in detail in section II. 3.

A numerical example is given to illustrate the method and to demonstrate that the constant A plays no role in the final gain achieved for this method of synthesis.

Suppose we desire to realize

$$\begin{aligned} K &= \frac{E_2}{E_1} = \frac{p}{q} \\ &= \frac{(s+1)(s^2+2s+4)}{s^4+8.5s^3+28.6s^2+47.5s+31.5} \end{aligned} \quad (214)$$

Using the relationship $q = q_1 + Aq'_1$, we obtain

$$\left. \begin{aligned} q_1 &= s^4 + 4.5s^3 + 15.1s^2 + 17.3s + 14.2 \\ &= (s^2 + 1.3358s + 1.5177)(s^2 + 3.164s + 9.3567) \\ Aq'_1 &= 1(4s^3 + 13.5s^2 + 30.2s + 17.3) \end{aligned} \right\} \quad (215)$$

The function Aq'_1/q_1 is simple to expand in partial fractions with pairs of complex poles combined into single fractions; it is

$$\frac{Aq'_1}{q_1} = \frac{2(s + 0.6679)}{s^2 + 1.3358s + 1.5177} + \frac{2(s + 1.582)}{s^2 + 3.164s + 9.3567} \quad (216)$$

By use of the equation

$$\begin{aligned} K &= \frac{y_{12a} K_b}{y_{22a} + Y_b} = \frac{\frac{p}{q_1}}{1 + \frac{Aq'_1}{q_1}} \\ &= \frac{\frac{p_a p_b}{q_a q_b}}{1 + \frac{Aq'_a}{q_a} + \frac{Aq'_b}{q_b}} \end{aligned} \quad (217)$$

we can identify

$$\left. \begin{aligned} y_{12a} &= \frac{s + 1}{s^2 + 3.164s + 9.3567} \equiv \frac{s + 1}{D_1} \\ y_{22a} &= 1 + \frac{2(s + 1.582)}{s^2 + 3.164s + 9.3567} \equiv 1 + \frac{2(s + 1.582)}{D_1} \end{aligned} \right\} \quad (218)$$

and

$$\left. \begin{aligned} K_b &= \frac{s^2 + 2s + 4}{s^2 + 1.3358s + 1.5177} = \frac{s^2 + 2s + 4}{D_2} \\ Y_b &= \frac{2(s + 0.6679)}{s^2 + 1.3358s + 1.5177} = \frac{2(s + 0.6679)}{D_2} \end{aligned} \right\} \quad (219)$$

We first develop network N_a . Split the fractional part of y_{22a} into two admittances

$$y_{22a} = 1 + y_1 + y_2 = 1 + \frac{s + 1}{D_1} + \frac{s + 2.164}{D_1} \quad (220)$$

and remove $1 + y_2$ as a shunt branch. Invert the remainder to obtain $z_1 = 1/y_1$, which when totally removed as a series arm will insert the necessary zeros at $s = -1$ and at infinity in y_{12a} .

The network N_a is shown in Fig. 37. The constant multiplier achieved for y_{12a} is, of course, unity, the coefficient of the series branch admittance y_1 .

For the synthesis of network N_b it is necessary to invert Y_b and remove from this impedance the necessary components to produce zeros at the desired points, as explained in section II.3. Thus, from

$$z_3 = \frac{1}{Y_b} = \frac{D_2}{2(s + 0.6679)}$$

$$= \frac{1}{2} \left(s + 0.6679 + \frac{1.0716}{s + 0.6679} \right) \quad (221)$$

we remove

$$\left. \begin{aligned} & \frac{1}{2} (1-a) s \\ & \frac{1}{2} (1-b) (0.6679) \\ & \frac{1}{2} (1-c) \frac{1.0716}{s + 0.6679} \end{aligned} \right\} \quad (222)$$

to obtain a remainder

$$z_4 = \frac{1}{2} \frac{a \left[s^2 + 1 + \frac{b}{a} (0.6679)s + \frac{b}{a} (0.6679)^2 + \frac{c}{a} (1.0716) \right]}{s + 0.6679} \quad (223)$$

Since we invert this impedance to obtain the final shunt branch, the coefficient a is a factor in the gain so that we try to make it as nearly equal to unity as possible. To achieve the zeros of K_b we need

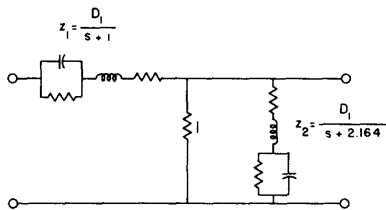


Fig. 37

Network N_a obtained from y_{22a} .

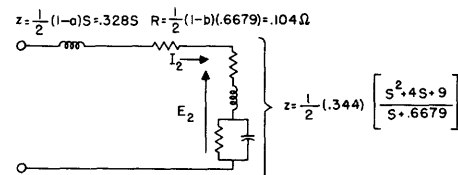


Fig. 38

Network N_b developed from Y_b .

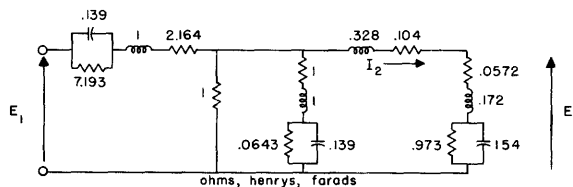


Fig. 39

Final network that realizes 0.344 times given K (see Eq. 214).

$$\left(1 + \frac{b}{a}\right)(0.6679) = 2$$

$$\therefore \frac{b}{a} = 1.995 \quad (224)$$

and

$$\frac{b}{a} (0.6679)^2 + \frac{c}{a} (1.0716) = 4$$

$$\therefore \frac{c}{a} = 2.905. \quad (225)$$

Let $c = 1$ so that

$$\left. \begin{aligned} a &= 0.344 \\ b &= 0.687. \end{aligned} \right\} \quad (226)$$

The network N_b then appears as shown in Fig. 38. The gain of K_b is obviously equal to $a = 0.344$. The final composite network shown in Fig. 39 thus realizes 0.344 times the given K .

If we associated the Y_b with the other pair of poles, could we achieve a larger gain? This is easy to check. We would need

$$\left. \begin{aligned} \left(1 + \frac{b}{a}\right) 1.582 &= 2 \\ \frac{b}{a} (1.582) + \frac{c}{a} (6.857) &= 4 \end{aligned} \right\} \quad (227)$$

or

$$\left. \begin{aligned} \frac{b}{a} &= 0.264 \\ \frac{c}{a} &= 0.487 \end{aligned} \right\} \quad (228)$$

which allows the choice of $a = 1$, so that a greater gain is obtained with the use of one more RC combination and one less inductance.

Another application of this method of synthesis is discussed briefly. Linvill, in designing amplifiers for prescribed frequency characteristics and arbitrary bandwidth (16), has need for the realization of a transfer impedance, whose denominator is of degree not greater than the fourth, by means of a two-terminal pair interstage terminated by a shunt capacitance at input and output terminals. He accomplishes this with simple forms of networks, but the transfer function is restricted in that its poles or its poles and zeros must satisfy fixed relationships; for example, one type of network requires that all four poles have the same real part. If one has no objection to the use of slightly more complicated unsymmetrical networks, still, however, possessing input and output shunt capacitances, no relationship need exist among the critical frequencies of the transfer function. This obviously is desirable. For instance, a Z_{12} with two negative real zeros and four complex zeros can always be realized by use of the dual

of Eq. 62

$$Z_{12} = \frac{z_{12a} Z_{12b}}{z_{22a} + Z_b} \quad (229)$$

Each of the partitioned networks, N_a and N_b , must now be realized as a single ladder by a method similar to that used for the synthesis of network N_a in the example given above. If the desired negative real zero is larger than the maximum zero that can possibly be obtained by the simple method of splitting $2A(s+d)/(s^2 + 2ds + e)$, then, before splitting is carried out, an additional step is necessary, consisting of inversion and zero shifting to obtain complex zeros with sufficiently large real parts. The form of network obtained when no additional step is needed is shown in Fig. 40.

Finally, for the realization of a high-degree transfer function we can realize appropriate factors of the given function as component networks similar to that in Fig. 35 and place them between vacuum tubes.

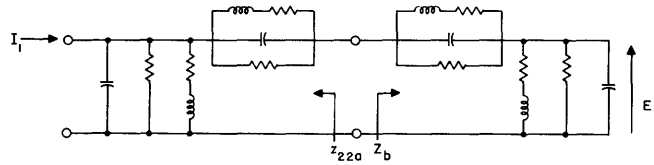


Fig. 40

Interstage network with input and output shunt capacitances.

VII. The Darlington Problem

Darlington made the significant contribution (1, 5) of demonstrating that any positive-real function is realizable as the input impedance of a lossless two-terminal pair network terminated in resistance. This proof forms the basis of a synthesis procedure for transfer functions of lossless two-terminal pair networks terminated in resistance. The Darlington procedure, however, requires ideal transformers in general and the synthesis of the transfer characteristic is for magnitude alone, not for magnitude and phase. Another way of specifying this restriction is by stating that the transfer function can be realized for both magnitude and phase provided that the numerator is either an even or an odd function of the complex variable.

This section concerns itself with two different procedures. One realizes a transfer voltage ratio in the form of a Darlington network without any mutual inductance. The synthesis procedure (1) analogous to that used in the Guillemin RC synthesis can already realize this, but here we do for LC synthesis what was accomplished in section IV for RC synthesis, i. e. we demonstrate a procedure analogous to that in section IV that reduces the required number of ladders and elements. As generally used, the term Darlington network means a lossless coupling network terminated in a single resistance; in the present section, specifically for the first synthesis procedure, the term is used in a more general sense to mean a network of lossless elements plus one resistance; the resistance is necessary to make the transfer function have poles off the j axis, but it may be within the network or may be a termination.

The second synthesis procedure realizes only those transfer admittances with numerators that are even functions of frequency (the first may be used for even or odd numerators). The synthesized network is an unbalanced lossless one terminated in resistance. Mutual inductance may be present in the form of real transformers, i. e. transformers with finite magnetizing inductance and a coupling coefficient of value less than one. The network is realized through the lattice synthesis procedures of section III. The method of reduction to an unbalanced form, a method that is always successful, will be demonstrated.

The first procedure employs a method of synthesis analogous to the one in section IV, e. g. method of reduction of number of required ladders, zero-pole cancellation in the transfer function of each ladder, use of the network theorem of section II.4, etc.

Just as only one RC ladder is necessary to realize a transfer voltage ratio with all negative-real zeros, so only one ladder is needed for the Darlington network when all the zeros of the even or odd numerator are on the j axis. However, when the transfer function contains at least one quadruplet of zeros, i. e. a pair of complex conjugate zeros in the left half-plane plus a pair in the right half-plane symmetrically placed with respect to the origin, more than one ladder is needed.

To decrease the required number of ladders from that obtained by using two successive terms of p as the numerator of one ladder's transfer function, we proceed as

follows. If the numerator is originally odd, factor out an s to obtain an even polynomial $f(s^2)$; if the numerator is even, consider the total numerator as $f(s^2)$. The substitution $y = s^2$ will give $f(y)$, a polynomial with no missing terms and of degree n . Apply the Sturm test (14) to determine the number of negative real roots of $f(y)$ and designate this number by r . Finally, with these new definitions, the formula for the required number of ladders is the same as that given by Eq. 156 for RC synthesis, namely, the smallest integer l satisfying the relationship

$$l \geq \frac{n - r + 1}{2}. \quad (230)$$

For an example, consider

$$p = s^8 + 17s^6 + 121s^4 + 455s^2 + 750 \quad (231)$$

which yields

$$f(y) = y^4 + 17y^3 + 121y^2 + 455y + 750. \quad (232)$$

The Sturm test shows this to have two negative real roots, so that $f(y)$ in factored form is

$$f(y) = (y^2 + 6y + 25)(s+5)(s+6). \quad (233)$$

These roots will be needed for the synthesis. Since $n = 4$ and $r = 2$, the required number of ladders l is two.

Since in the complete synthesis we use the theorem of section II.4, which gives only N_a as a group of paralleled ladders, the above determination of l is applied in its realization, not for that of N_b . We know by Eq. 63

$$K = \frac{E_2}{E_1} = \frac{y_{12a} K_b}{y_{22a} + Y_b} = \frac{p}{q}. \quad (234)$$

Let

$$q = m_2 + n_2 \quad (235)$$

where m_2 is the even and n_2 the odd part of q . If p is even, we divide numerator and denominator of K by n_2 ; if it is odd, we divide by m_2 . Suppose without loss of generality that p is an even polynomial m_1 . Therefore

$$K = \frac{m_1}{m_2 + n_2} = \frac{\frac{m_1}{n_2}}{1 + \frac{m_2}{n_2}}. \quad (236)$$

We now make identifications that require a small number of ladders in N_a , each ladder characterized by a transfer function with a low-degree denominator. These are given by

$$\left. \begin{aligned} y_{12a} &= \frac{m_{1a}}{n_{2a}} \\ y_{22a} &= 1 + \frac{m_{2a}}{n_{2a}} \end{aligned} \right\} \quad (237)$$

and

$$\left. \begin{aligned} K_b &= \frac{m_{1b}}{m_{3b}} \\ Y_b &= \frac{m_{2b}}{m_{3b}} \end{aligned} \right\} \quad (238)$$

where by definition

$$\left. \begin{aligned} m_1 &= m_{1a} m_{1b} \\ n_2 &= n_{2a} m_{3b} \\ \frac{m_2}{n_2} &= \frac{m_{2a}}{n_{2a}} + \frac{m_{2b}}{m_{3b}} \end{aligned} \right\} \quad (239)$$

It is important to note that n_{2a} is chosen odd whereas m_{3b} is a constant or has zeros only on the j axis (since N_b must be a single ladder). The s factor of n_2 is associated not with K_b but with y_{12a} because y_{12a} , a transfer admittance, must be an even polynomial over an odd (or an odd over even) while K_b , a dimensionless voltage ratio, must be an even over an even polynomial. Also since q is Hurwitz, m_2/n_2 is a positive real reactance function (9) whose partial fractions are also positive real and hence separately realizable. The functions m_{2a}/n_{2a} and m_{2b}/m_{3b} are positive real sums of the partial fraction components of m_2/n_2 .

In the synthesis of N_a the constant term is removed immediately from y_{22a} as a shunt branch so that the remaining synthesis embodying paralleling of ladders and zero shifting must be done with only a two-element kind network (1) and is analogous to that in section IV. The final network has the form shown in Fig. 41.

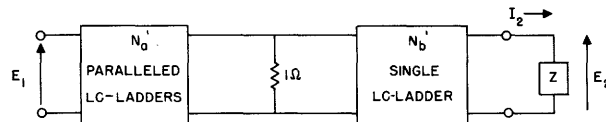


Fig. 41

Final form of Darlington network where the termination Z is composed of lossless elements.

The second procedure is the same as that of section III. 3 which realizes a transfer admittance in the form of a lattice terminated in $1/2(H)$ ohms. The bulk of the procedure is not repeated here, but something is said about the manipulation of the given transfer admittance. We are given

$$Y_{12} = \frac{p}{q} = \frac{m_1}{m_2 + n_2} \quad (240)$$

where we observe that the numerator is even. The even numerator is required in order to permit the reduction of the lattice to an unbalanced form with the use of only real transformers. In addition, in order that only one resistance be present in the network, the denominator must be of higher degree than the numerator. We proceed as before with the division by n_2 to obtain

$$Y_{12} = \frac{\frac{m_1}{n_2}}{1 + \frac{m_2}{n_2}} \quad (241)$$

The residues of m_1/n_2 are all real but may be positive or negative; those for m_2/n_2 are real and positive. Thus we do not have to work with the conditions of section III. 3. that arise because of complex residues, but have as the only condition to be satisfied

$$-1 \leq \frac{Hk^{(n)}_v}{k^{(d)}_v} \leq 1 \quad (242)$$

which is exactly the same as the Bower-Ordung condition for RC synthesis. This is not surprising since up to this point the LC case and the RC one are analogous.

The general form of network obtained has arms composed of parallel branches of series LC structures plus a pure inductance. Since there is a pole at $s = 0$, this shunt

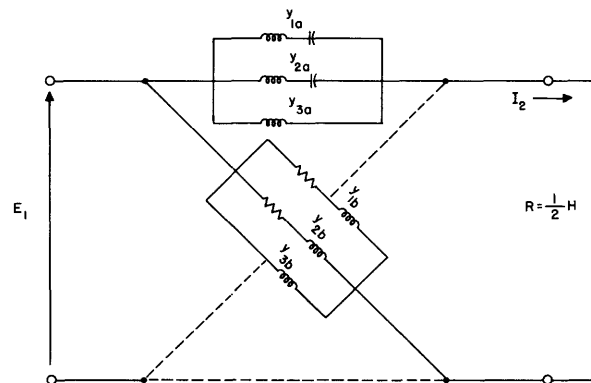


Fig. 42

Lossless lattice network terminated in a resistance.

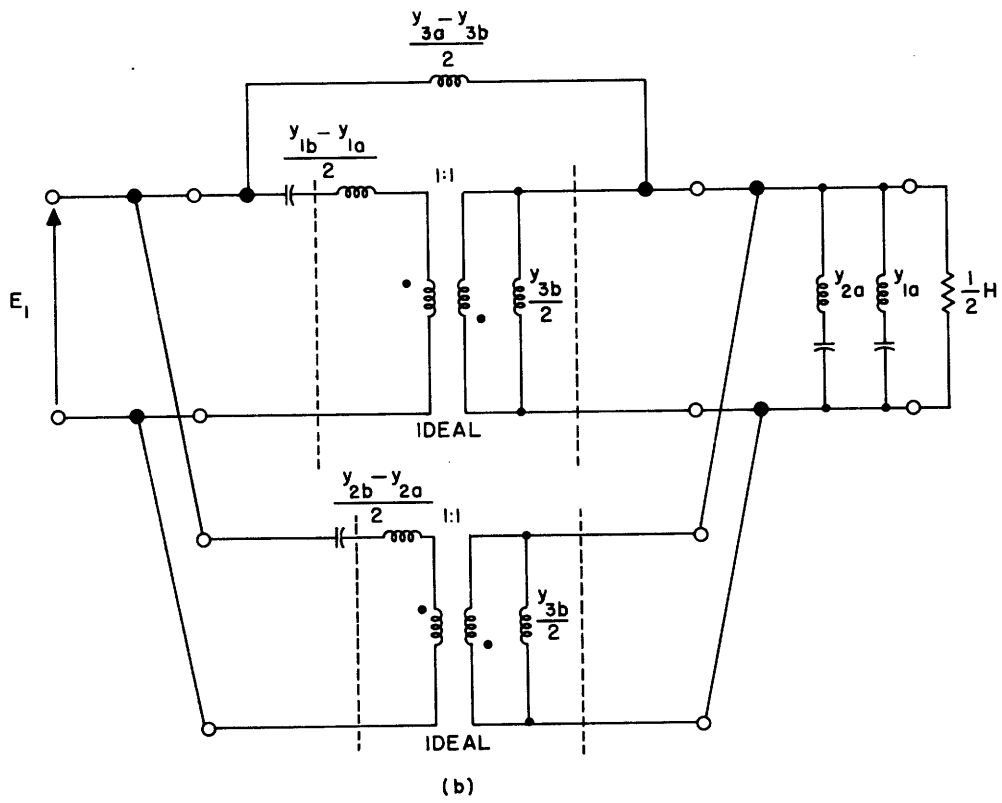
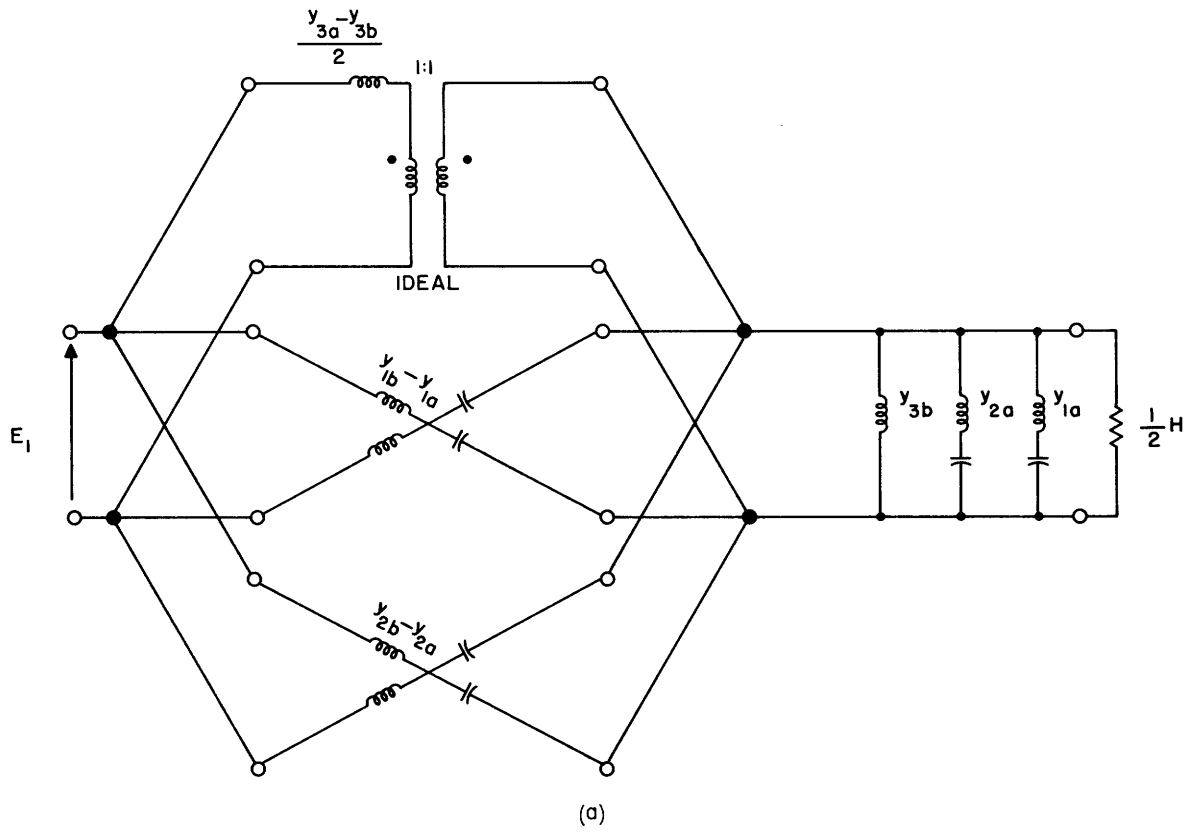


Fig. 43
 Reduction of lattice to an unbalanced form with use of
 only real transformers.

inductance is always present. Now for demonstration of the general method of reduction to an unbalanced form, consider the network of Fig. 42, where like numerical subscripts designate like poles. Suppose that the residues in y_{1b} and y_{2b} are greater than those in y_{1a} and y_{2a} , respectively, but the residue in y_{3a} is greater than that in y_{3b} . By the methods of lattice reduction of section III. 5, we may remove y_{1a} , y_{2a} , and y_{3b} from each arm and break the ladder into a group of parallel ladders, as shown in Fig. 43a. Then by a rotation of the terminals of the second and third lattices, as in Fig. 43b, we obtain an unbalanced form where the elements within the broken lines represent a real transformer, that is a transformer with coupling coefficient less than one and finite magnetizing inductance.

The above completes the discussion of the synthesis procedures.

VIII. Illustrative Examples

1. Example illustrating procedure of section III. 3

We desire to synthesize within a constant multiplier the given

$$Y_{12} = \frac{p}{q} = \frac{1}{(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)}$$

as a resistance-terminated lattice. The given Y_{12} represents the Butterworth approximation to the normalized lowpass filter with $n = 4$ (Table 1). We first synthesize for

$$\frac{I_2}{I_1} = \frac{p}{q} = \frac{Y_b - Y_a}{Y_b + Y_a}$$

as a short-circuited lattice and then by transformations obtain the desired Y_{12} .

From Table 1C for $A = 0.1$ we find

$$q_1 = (s^2 + 1.61976s + 0.813459)(s^2 + 0.59336s + 0.975703)$$

and

$$\frac{1}{q_1} = \frac{0.44509 - j1.0854}{s + 0.80988 - j0.39693} + \frac{-0.44509 + j0.21483}{s + 0.29668 - j0.94217} + \text{conjugates}$$

so that

$$\begin{aligned} \alpha_1^{(n)} &= 0.44509 & \beta_1^{(n)} &= -1.0854 \\ \alpha_2^{(n)} &= -0.44509 & \beta_2^{(n)} &= 0.21483 \\ \sigma_1 &= 0.80988 & \omega_1 &= 0.39693 \\ \sigma_2 &= 0.29668 & \omega_2 &= 0.94217. \end{aligned}$$

We find from

$$c_v = \frac{\frac{1}{2} |\beta_v^{(n)}| \omega_v}{\sigma_v}$$

that

$$c_1 = 0.2660$$

$$c_2 = 0.3411.$$

By use of Eqs. 109 and 110 we find

$$H_1 \geq 9.8$$

$$H_2 \geq 10.3.$$

Choose $H = 25$. Therefore

$a_0^{(b)} = 12.5$	$a_0^{(a)} = 12.5$
$a_1^{(b)} = 1.47255$	$a_1^{(a)} = 1.02746$
$a_2^{(b)} = 1.02746$	$a_2^{(a)} = 1.47255$
$\beta_1^{(b)} = -0.5427$	$\beta_1^{(a)} = 0.5427$
$\beta_2^{(b)} = 0.10742$	$\beta_2^{(a)} = -0.10742.$

The lattice arms are then given by

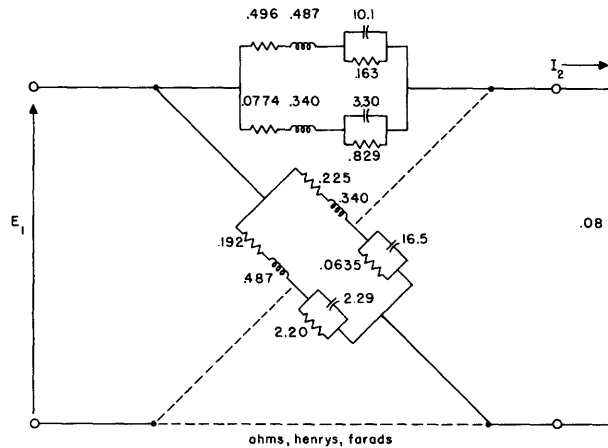


Fig. 44

Network realization for Example 1 where

$$Y_{12} = \frac{1}{2} \frac{p}{q}.$$

$$Y_a = 12.5 + \frac{2.05492 (s + 0.60022)}{s^2 + 1.61976s + 0.813459} + \frac{2.94510 (s + 0.36541)}{s^2 + 0.59336s + 0.975703}$$

$$Y_b = 12.5 + \frac{2.94510 (s + 0.95617)}{s^2 + 1.61976s + 0.813459} + \frac{2.05492 (s + 0.19818)}{s^2 + 0.59336s + 0.975703}$$

Applying the necessary transformations gives the lattice shown in Fig. 44 for which

$$Y_{12} = \frac{1}{2} \frac{p}{q} = \frac{1}{2} \frac{1}{(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)}$$

2. Example illustrating the synthesis procedure of section IV

An RC network is desired to realize the transfer voltage ratio

$$K = \frac{E_2}{E_1} = \frac{(s^2 + 2s + 2)(s + 5.5)(s + 10.5)}{(s+1)(s+2)(s+3)\dots(s+10)}$$

A network will be found using the procedure of section IV.

It is evident that the chosen example does not represent any desirable frequency characteristic. Because of the positions of the poles we can expect that a totally impractical spread of element values will be required, but this is of no concern to us at the present moment. The example has been chosen merely to concretize the steps of the procedure and to show what is possible in reduction of the number of elements required by the Guillemin procedure.

We desire to identify the components in the equation

$$K = \frac{y_{12a} K_b}{y_{22a} + Y_b}$$

To this end we perform the following manipulations on the given function.

$$\begin{aligned} K &= \frac{(s^2 + 2s + 2)(s + 5.5)(s + 10.5)}{(s + 1.5)(s + 2.5)\dots(s + 9.5)} \cdot \frac{(s+1)(s+2)\dots(s+10)}{(s+1)(s+2)\dots(s+10)} \\ &= \frac{(s^2 + 2s + 0.75 + 1.25)(s + 10.5)}{(s + 1.5)(s + 9.5)} \times \frac{(s + 5.5)}{(s + 2.5)(s + 3.5)\dots(s + 8.5)} \\ &= \frac{(s^2 + 2s + 0.75)(s + 10.5)}{(s + 1.5)(s + 9.5)} + \frac{1.25(s + 10.5)}{(s + 1.5)(s + 9.5)} \times \end{aligned}$$

$$\begin{aligned} & \times \frac{1}{(s+2.5)(s+3.5)(s+4.5)(s+6.5)(s+7.5)(s+8.5)} \\ & = \frac{(s+0.5)(s+10.5)}{s+9.5} + \frac{1.25(s+10.5)}{(s+1.5)(s+9.5)} \\ & \times \frac{1}{(s+2.5)(s+3.5)(s+4.5)(s+6.5)(s+7.5)(s+8.5)} \end{aligned}$$

where the denominator expanded in positive real fractions is

$$\begin{aligned} D = & 2.837732 + s + \frac{0.556412s}{s+1.5} + \frac{0.087855s}{s+9.5} + \frac{0.471313s}{s+2.5} \\ & + \frac{0.392761s}{s+3.5} + \frac{0.328979s}{s+4.5} + \frac{0.275282s}{s+5.5} + \frac{0.227755s}{s+6.5} \\ & + \frac{0.183289s}{s+7.5} + \frac{0.138622s}{s+8.5} . \end{aligned}$$

We now identify

$$\begin{aligned} y_{12a} & = \frac{(s+0.5)(s+10.5)}{s+9.5} + \frac{1.25(s+10.5)}{(s+1.5)(s+9.5)} \\ & = y_{12a}^{(1)} + y_{12a}^{(2)} \end{aligned}$$

and

$$\begin{aligned} y_{22a} & = \left\{ s + \frac{0.047855s}{s+9.5} + c(2.837732) \right\} + \left\{ \frac{0.556412s}{s+1.5} + \frac{0.0400s}{s+9.5} + (1-c)(2.837732) \right\} \\ & = y_{22a}^{(1)} + y_{22a}^{(2)} \end{aligned}$$

where c is a positive constant less than one. Its value will be chosen to make a zero occur in $y_{22a}^{(1)}$ at the desired position $s = -0.5$.

We further identify

$$\begin{aligned} K_b & = \frac{1}{(s+2.5)(s+3.5)(s+4.5)(s+6.5)(s+7.5)(s+8.5)} \\ Y_b & = \frac{0.275282s}{s+5.5} + \frac{0.471313s}{s+2.5} + \frac{0.392761s}{s+3.5} + \frac{0.328979s}{s+4.5} \\ & \quad + \frac{0.227755s}{s+6.5} + \frac{0.183289s}{s+7.5} + \frac{0.138622s}{s+8.5} \\ & = \frac{0.275282s}{s+5.5} + \frac{g}{r} \end{aligned}$$

where

$$\begin{aligned} g = & 1.742719s^6 + 49.443014s^5 + 544.526083s^4 \\ & + 2899.726425s^3 + 7439.756777s^2 + 7336.256602s \end{aligned}$$

and

$$r = s^6 + 33s^5 + 439.750s^4 + 3019.50s^3 + 11,233.93750s^2 + 21,419.0625s + 16,316.015625.$$

The two ladders of network N_a are synthesized first. We determine the constant c .

$$y_{22a}^{(1)} \Big|_{s=-0.5} = \left(s + 2.837732c + \frac{0.047855s}{s+9.5} \right) \Big|_{s=-0.5} = 0$$

$$c = 0.177134$$

$$\therefore 2.837732c = 0.502659$$

$$(1 - c)(2.837732) = 2.335073.$$

Next invert $y_{22a}^{(1)}$ and expand in partial fractions.

$$\frac{1}{y_{22a}^{(1)}} = \frac{0.994418}{s+0.5} + \frac{0.0055813}{s+9.550514}.$$

We remove a series branch containing the total residue in the pole at $s = -0.5$. Then invert the remainder and create a zero at $s = -10.5$ by removing a capacitance equal to 16.2 farads. Invert once more to obtain

$$z = \frac{0.0055813}{0.909573(s+10.5)} = \frac{0.0061361}{s+10.5}$$

which when removed as the final series branch places a zero in $y_{12a}^{(1)}$ at $s = -10.5$.

For the synthesis of the second ladder of N_a we calculate the value at $s = -10.5$ of $1/y_{22a}^{(2)}$. It is 0.293754. We remove a resistance equal to this value.

$$\begin{aligned} z' &= \frac{1}{y_{22a}^{(2)}} - 0.293754 \\ &= \frac{(s+1.5)(s+9.5)}{2.931485(s+1.210769)(s+9.374897)} - 0.293754 \\ &= \frac{0.133875(s+10.5)(s+3.068382)}{2.931485(s+1.210769)(s+9.374897)}. \end{aligned}$$

Next from the inverted z' in the form

$$\frac{1}{z'} = 7.436910 + \frac{2.827242s}{s+10.5} + \frac{10.844648}{s+3.068382}$$

we remove a shunt branch equal to the total residue in the pole at $s = -10.5$. The inverted remainder yields

$$z'' = 0.054700 + \frac{1.820167}{18.281558s + 22.819281}$$

from which the resistance equal to 0.0547 ohms is removed. Again we invert the

remainder and remove the total pole at ∞ . Only a conductance remains which after inversion may be removed as the final series branch.

The transfer function K_b of the network N_b must have all its zeros placed at ∞ and must possess no pole at $s = -5.5$. This is accomplished by removing from Y_b the total residue in the pole at $s = -5.5$ as the initial shunt branch and then expanding the inverted remainder $z''' = r/g$ as the continued fraction

$$\begin{aligned} z''' = & 0.573816 + \frac{1}{0.376494s} + \frac{1}{3.048785} + \frac{1}{0.065548s} \\ & + \frac{1}{30.81115} + \frac{1}{6.715667 \times 10^{-3}s} \\ & + \frac{1}{2.322873 \times 10^2} + \frac{1}{8.285600 \times 10^{-4}s} \\ & + \frac{1}{4.325168 \times 10^3} + \frac{1}{4.529744 \times 10^{-5}s} \\ & + \frac{1}{4.892880 \times 10^4} + \frac{1}{3.718739 \times 10^{-6}s} \end{aligned}$$

where for convenience the notation has been used so that

$$a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} = a_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2}}$$

The values of the elements in the ladder that corresponds to this continued fraction are thus made evident.

The networks that have been achieved thus far are shown in Fig. 45. It is observed that the final termination of Y_b is a capacitance; if a resistance termination is desired, it is necessary to associate part of the constant term 2.837732 of D with Y_b .

There finally remains the step of adjusting the gain of the two component ladders of N_a . We desire

$$y_{12a}^{(1)} = \frac{(s + 0.5)(s + 10.5)}{s + 9.5}$$

but have achieved

$$y_{12a}^{(1)} = \frac{A_1(s + 0.5)(s + 10.5)}{s + 9.5}$$

Allowing $s \rightarrow 0$ in the above equation and by inspection of the network we find that

$$\frac{A_1(0.5)(10.5)}{9.5} = \frac{1}{1.99}$$

$$A_1 = 0.909.$$

Similarly, A_2 is found to be 3.17.

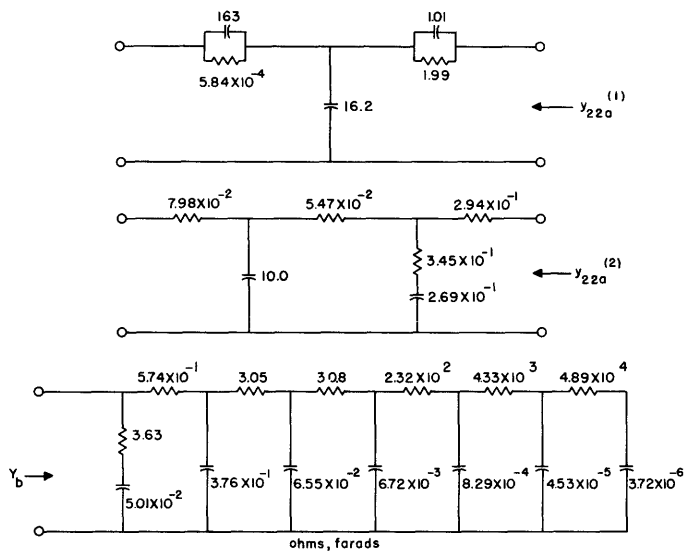


Fig. 45
Component networks derived for Example 2.

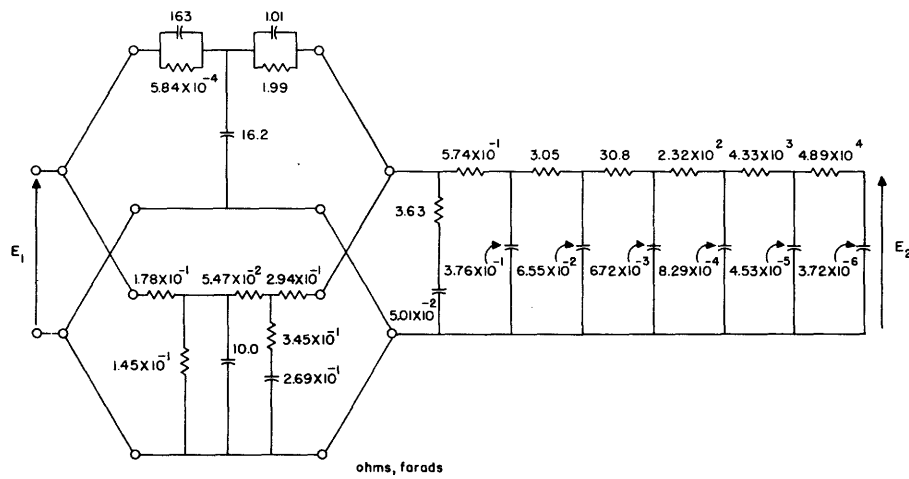


Fig. 46
Network realized for Example 2.

Therefore

$$\frac{A_2}{a_2} = \frac{3.17}{1.25} = 2.54$$

$$\frac{A_1}{a_1} = 0.909$$

and we may write the achieved y_{12a} as

$$\begin{aligned} y_{12a} &= \frac{A_1}{a_1} \left(\frac{a_1}{A_1} A_1 y_{12a}^{(1)} + \frac{a_1}{A_1} A_2 y_{12a}^{(2)} \right) \\ &= 0.909 \left(y_{12a}^{(1)} + 2.79 y_{12a}^{(2)} \right). \end{aligned}$$

It is thus necessary to reduce the gain of the second ladder from 2.79 to 1.25, that is, multiply it by a factor $b = 0.448$. This is accomplished by breaking the final series resistance 7.98×10^{-2} of $y_{22a}^{(2)}$ into a final series R_1 followed by a shunt R_2 , where

$$R_1 = \frac{7.98 \times 10^{-2}}{0.448} = 1.78 \times 10^{-1}$$

$$R_2 = \frac{7.98 \times 10^{-2}}{1 - 0.448} = 1.45 \times 10^{-1}.$$

The gain of the N_b network for K_b may also be determined by inspection. It is the reciprocal of the product of all the resistances and capacitances that realize the continued fraction expansion of z''' . This is calculated to be 1.63115×10^4 . Another way of calculating this constant is by finding the constant term of the denominator polynomial of K_b . The fact that the first four figures obtained by each of these methods are equal constitutes a satisfactory check on the derived element values.

The final network, which is shown in Fig. 42, realizes the given transfer function within the constant multiplier $0.909 \times 1.63155 \times 10^4 = 1.48 \times 10^4$.

If we applied the Guillemin procedure to the same problem, we would require $(2n+2)$ elements for each of two ladders, $(2n+1)$ elements for the final ladder whose transfer function has a constant term as numerator, plus one resistance termination. Since the degree of the denominator polynomial n is 10, the grand total of required elements is 66. The network shown in Fig. 46 uses only 26 elements. There is thus a saving of 40 elements.

3. Example illustrating procedure of section V-A (that is, for a q_1 obtained from $q = q_1 + Aq_1'$ possessing only one pair of complex zeros)

We desire an unbalanced RLC network to realize the transfer voltage ratio

$$K = \frac{E_2}{E_1} = \frac{p}{q} = \frac{(s^2 + 5s + 8)(s + 0.5)}{s^6 + 19s^5 + 134s^4 + 471s^3 + 899s^2 + 900s + 368}$$

Using the breakdown $q = q_1 + Aq_1'$, we find for $A = 1$

$$\begin{aligned} q_1 &= s^6 + 13s^5 + 69s^4 + 195s^3 + 314s^2 + 272s + 96 \\ &= (s^2 + 3s + 4)(s+1)(s+2)(s+3)(s+4) \\ Aq_1' &= 1(6s^5 + 65s^4 + 276s^3 + 585s^2 + 628s + 272). \end{aligned}$$

We then proceed with the familiar steps

$$\begin{aligned} K &= \frac{p}{q} = \frac{p}{q_1 + Aq_1'} \\ &= \frac{\frac{p}{q_1}}{1 + \frac{Aq_1'}{q_1}} \\ &= \frac{\frac{p}{q_1}}{1 + \frac{Aq_{1a}'}{q_{1a}} + \frac{Aq_{1b}'}{q_{1b}}} \end{aligned}$$

where

$$\begin{aligned} q_{1a} &= (s+1)(s+2)(s+3)(s+4) \\ &= s^4 + 10s^3 + 35s^2 + 50s + 24 \\ q_{1b} &= s^2 + 3s + 4. \end{aligned}$$

By use of the formula

$$K = \frac{y_{12a} K_b}{y_{22a} + Y_b}$$

we identify

$$\left. \begin{aligned} y_{12a} &= \frac{(s+0.5)}{(s+1)(s+2)(s+3)(s+4)} \\ y_{22a} &= 1 + \frac{Aq'_{1a}}{q_{1a}} \\ &= 1 + \frac{4s^3 + 30s^2 + 70s + 50}{(s+1)(s+2)(s+3)(s+4)} \\ &= \frac{s^4 + 14s^3 + 65s^2 + 120s + 74}{s^4 + 10s^3 + 35s^2 + 50s + 24} \end{aligned} \right\}$$

and

$$\left. \begin{aligned} K_b &= \frac{s^2 + 5s + 8}{s^2 + 3s + 4} \\ Y_b &= \frac{Aq'_{1b}}{q_{1b}} = \frac{2s + 3}{s^2 + 3s + 4} \end{aligned} \right\}$$

We are now ready to synthesize network N_a as a single RL ladder by zero shifting with y_{22a} to the finite zero of y_{12a} at $s = -0.5$.

Since we find

$$\left. \frac{1}{y_{22a}} \right|_{s=-0.5} = 0.22976$$

we remove this value of resistance from $1/y_{22a}$ as the first series branch. From the inverted remainder

$$y = \frac{4.04245}{s+0.5} + \frac{s^3 + 10.38634s^2 + 33.84224s + 34.89686}{0.77024s^3 + 6.39825s^2 + 16.86652s + 13.99562}$$

we remove the complete residue in the pole at $s = -0.5$.

Placing the remainder of the zeros at infinity can be accomplished by removing series inductance branches, that is, poles of impedance at infinity. Since we desire that each inductance have an associated series resistance, we also remove an appropriate value of real part from the impedance. Therefore, from the remainder we remove as a shunt conductance the minimum of the real part, $1/0.77024 = 1.29830$, to create a zero at infinity. The remaining admittance is

$$y' = \frac{2.07951s^2 + 11.94449s + 16.72639}{0.77024s^3 + 6.39825s^2 + 16.86652s + 13.99562}$$

The reciprocal of y' , shown as a sum of its pole at infinity, a value of real part less

than its minimum, and a third term, is

$$\frac{1}{y'} = 0.37039s + 0.500 + \frac{0.93431s^2 + 4.69890s + 5.63242}{2.07951s^2 + 11.94449s + 16.72639}$$

We remove as a series branch the pole at infinity and the 0.500 ohms and, with the remainder, we repeat the whole process beginning with inversion and removal of the minimum real part of the resulting admittance. This yields a shunt conductance of 2.23 mhos, a series branch impedance of $0.629s + 0.8$ ohms and the remainder

$$z'' = \frac{0.87561s + 2.28020}{1.48612s + 4.19028}$$

The repetition of the procedure on z'' yields a conductance of 1.70 mhos and a final series branch impedance of $2.73s + 7.12$ ohms.

The synthesis of network N_b proceeds simply by the method of complex zero shifting. From the inverted admittance

$$\begin{aligned} z &= \frac{1}{Y_b} = \frac{s^2 + 3s + 4}{2s + 3} \\ &= \frac{s}{2} + \frac{3}{4} + \frac{1.75}{2s + 3} \end{aligned}$$

we remove

$$(1-a)\frac{s}{2} + (1-b)\frac{3}{4} + \frac{(1-c)1.75}{2s+3}$$

to obtain a remainder

$$\begin{aligned} z' &= a\frac{s}{2} + b\left(\frac{3}{4}\right) + \frac{c(1.75)}{2s+3} \\ &= \frac{a\left[s^2 + \left(1+\frac{b}{a}\right)\frac{3}{2}s + \frac{b}{a}\left(\frac{9}{4}\right) + \frac{c}{a}(1.75)\right]}{2s+3} \end{aligned}$$

To obtain the desired zero, we must have

$$\left(1 + \frac{b}{a}\right)\frac{3}{2} = 5$$

$$\therefore \frac{b}{a} = 2.33333$$

and

$$\frac{9}{4}\frac{b}{a} + \frac{c}{a}(1.75) = 8$$

$$\therefore \frac{c}{a} = 1.57143.$$

Since a is the gain of N_b we choose it as large as possible. Therefore, let

$$\begin{aligned}
 b &= 1 \\
 a &= 0.42857 \\
 c &= 0.67347.
 \end{aligned}$$

Now z' is inverted and removed as the final shunt branch to complete the network.

The complete network is shown in Fig. 47. The gain of N_b , as indicated above, is 0.42857, and the constant multiplier of N_a is calculated by inspection of the network that results for $s \rightarrow 0$. It is 0.2877. The total gain is the product of these two constants or 0.123.

4. Example illustrating procedure of section V-C (that is, q_1 derived from the breakdown $q = q_1 + Aq_1' + Bq_1''$ has only negative real zeros)

Given the transfer admittance

$$Y_{12} = \frac{p}{q} = \frac{1}{s^4 + 10.4s^3 + 50s^2 + 117s + 99}$$

we desire its realization in an unbalanced RLC network.

Using the breakdown $q = q_1 + Aq_1' + Bq_1''$, with $A = 0.1$ and $B = 1$, we find

$$\left. \begin{aligned}
 q_1 &= s^4 + 10s^3 + 35s^2 + 50s + 24 \\
 Aq_1' &= 0.1(4s^3 + 30s^2 + 70s + 50) \\
 Bq_1'' &= 1(12s^2 + 60s + 70).
 \end{aligned} \right\}$$

Though q does not have all negative real zeros, q_1 (and hence q_1' and q_1'') do, so that the synthesis employs zero shifting with only a two-element kind function.

The familiar equation

$$Y_{12} = \frac{y_{12}}{1 + y_{22}}$$

for a network terminated in one ohm and the manipulations

$$\begin{aligned}
 Y_{12} &= \frac{p}{q} = \frac{p}{q_1 + Aq_1' + Bq_1''} \\
 &= \frac{\frac{p}{Aq_1'}}{1 + \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'}}
 \end{aligned}$$

lead to the identifications

$$\begin{aligned}
 y_{12} &= \frac{p}{Aq_1'} = \frac{1}{0.1(4s^3 + 30s^2 + 70s + 50)} \\
 y_{22} &= \frac{q_1}{Aq_1'} + \frac{Bq_1''}{Aq_1'} \\
 &= \frac{s^4 + 10s^3 + 35s^2 + 50s + 24}{0.1(4s^3 + 30s^2 + 70s + 50)} + \frac{12s^2 + 60s + 70}{0.1(4s^3 + 30s^2 + 70s + 50)}
 \end{aligned}$$

With the RC function q_1/Aq_1' zero shifting is carried on to place all the zeros of y_{12} at infinity. But first we remove the RL network represented by Bq_1''/Aq_1' in the Foster manner as the first shunt branch. Once the roots of q_1' have been found, the partial fraction expansion of Bq_1''/Aq_1' can be written at sight: this is true because we know by inspection that all the residues are equal to 10. Thus

$$\begin{aligned}
 \frac{10(12s^2 + 60s + 70)}{4s^3 + 30s^2 + 70s + 50} &= \frac{10(12s^2 + 60s + 70)}{4(s + 1.38199)(s + 3.61806)(s + 2.49995)} \\
 &= \frac{10.0}{s + 1.38199} + \frac{10.0}{s + 3.61806} + \frac{10.0}{s + 2.49995}
 \end{aligned}$$

The above representation allows the element values to be found easily.

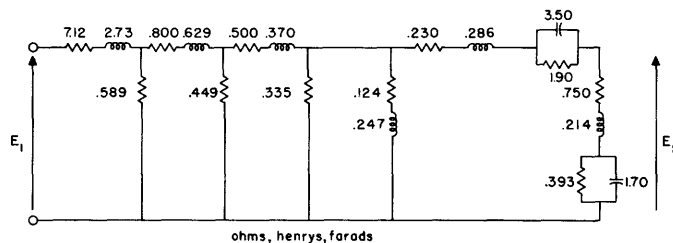


Fig. 47

Network that realizes the transfer function of Example 3.

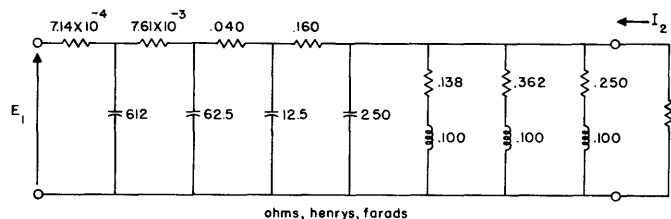


Fig. 48

Network realization for Example 4.

There only remains the evaluation of the continued fraction representation for q_1/Aq_1'

$$\frac{q_1}{Aq_1'} = 2.5s + \frac{1}{0.16} + \frac{1}{12.5s} + \frac{1}{0.04} + \frac{1}{62.5s} \\ + \frac{1}{7.61 \times 10^{-3}} + \frac{1}{612s} + \frac{1}{7.14 \times 10^{-4}} .$$

A ladder network with shunt C's and series R's corresponds to this representation.

The constant multiplier of y_{12} , which is calculated by inspection of the network and y_{12} for $s \rightarrow 0$, is 24.0. The network shown in Fig. 48 thus realizes the given transfer function within a multiplicative factor of 24.

5. Example illustrating the procedure of section VI

The function chosen is the transfer voltage ratio

$$K = \frac{E_2}{E_1} = \frac{p}{q} = \frac{1}{(s^4 + 5s^3 + 11.5s^2 + 13s + 7)(s^4 + 8.5s^3 + 25s^2 + 36.5s + 23.5)} .$$

It is desired to realize this function as the product of the transfer voltage ratios of two networks separated by a vacuum tube stage consisting of an amplifier and a cathode follower.

Letting

$$\left. \begin{aligned} q_a &= (s^4 + 5s^3 + 11.5s^2 + 13s + 7) \\ q_b &= (s^4 + 8.5s^3 + 25s^2 + 36.5s + 23.5) \end{aligned} \right\}$$

we write

$$K = K^{(1)} \times K^{(2)} = \frac{1}{q_a} \times \frac{1}{q_b}$$

where $K^{(1)}$ and $K^{(2)}$ represent the networks that precede and follow the amplifier stage, respectively.

The synthesis of each network proceeds in the same manner. Using the breakdown $q = q_1 + Aq_1'$, we find for q_a

$$q_{1a} = s^4 + 3s^3 + 7s^2 + 6s + 4 \\ = (s^2 + 2s + 4)(s^2 + s + 1) \\ Aq_{1a}' = 0.5(4s^3 + 9s^2 + 14s + 6)$$

so that

$$\frac{Aq_{1a}'}{q_{1a}} = \frac{s+1}{s^2+2s+4} + \frac{s+0.5}{s^2+s+1} .$$

And similarly for q_b

$$\begin{aligned} q_{1b} &= s^4 + 4.5s^3 + 11.5s^2 + 13.5s + 10 \\ &= (s^2 + 3s + 5)(s^2 + 1.5s + 2) \\ A_1 q'_{1b} &= 1(4s^3 + 13.5s^2 + 23s + 13.5) \end{aligned}$$

so that

$$\frac{A_1 q'_{1b}}{q_{1b}} = \frac{2(s + 1.5)}{s^2 + 3s + 5} + \frac{2(s + 0.75)}{s^2 + 1.5s + 2}.$$

Now, synthesizing the network for $K^{(1)}$, we use the last equation in

$$\begin{aligned} K^{(1)} &= \frac{1}{q_a} = \frac{1}{q_{1a} + Aq'_{1a}} \\ &= \frac{1}{q_{1a} + \frac{Aq'_{1a}}{1}} \end{aligned}$$

and the network theorem formula

$$K^{(1)} = \frac{y_{12a} K_b}{y_{22a} + Y_b}$$

to obtain the identifications

$$\left. \begin{aligned} y_{12a} &= \frac{1}{s^2 + 2s + 4} \\ y_{22a} &= 1 + \frac{s + 1}{s^2 + 2s + 4} \\ K_b &= \frac{1}{s^2 + s + 1} \\ Y_b &= \frac{s + 0.5}{s^2 + s + 1} \end{aligned} \right\}$$

In realizing the N_a part of the network we remove the one mho from the y_{22a} and then invert the remainder to obtain

$$z = s + 1 + \frac{3}{s + 1}.$$

Now remove the impedance $s + 1$, the new inverted remainder being

$$y = \frac{s}{3} + \frac{1}{3}$$

The $s/3$ is removed as a shunt capacitance, after which the final series branch becomes a resistance of three ohms.

The admittance for the N_b network, manipulated in the same way, gives

$$\frac{1}{Y_b} = s + 0.5 + \frac{0.75}{s + 0.5}$$

which provides an initial series branch with impedance $(s + 0.5)$ ohms and a final shunt branch of admittance $(4s/3 + 2/3)$ mhos.

This completes the synthesis of the network that precedes the amplifier; for the network that follows the stage of amplification we obtain a similar form of network.

As before we identify

$$\left. \begin{aligned} y_{12a} &= \frac{1}{s^2 + 3s + 5} \\ y_{22a} &= 1 + \frac{2(s + 1.5)}{s^2 + 3s + 5} \\ K_b &= \frac{1}{s^2 + 1.5s + 2} \\ Y_b &= \frac{2(s + 0.75)}{s^2 + 1.5s + 2} \end{aligned} \right\}$$

and exactly the same procedure yields for N_a an initial shunt branch of one mho, a succeeding series branch of $(s/2 + 3/4)$ ohms, a shunt branch of $s/1.375$ mhos, and a final series resistance of 0.917 ohms.

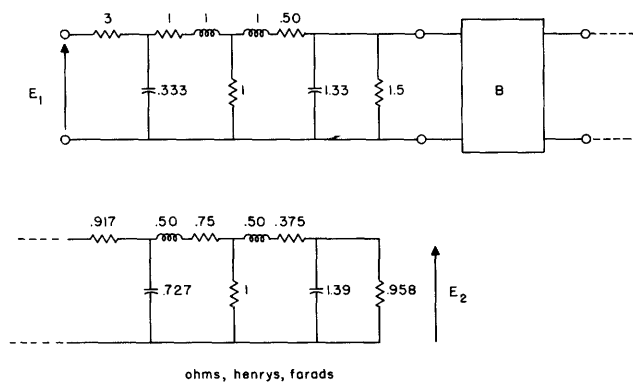


Fig. 49

Network realization for Example 5, where B represents a vacuum tube stage consisting of an amplifier and a cathode follower.

Similarly the first series branch of N_b is the impedance $(s/2 + 0.375)$ and the final shunt branch is the admittance $(1.39s + 1.04)$.

The complete network is shown in Fig. 49. For the network preceding the amplifier the constant multiplier, which is equal to the product of the multipliers for networks N_a and N_b , is 0.75, while that for the second network is 4.31.

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