File Copy

SEPARATION OF LAPLACE'S EQUATION

R. M. REDHEFFER

TECHNICAL REPORT NO. 68

JUNE 2 1948

RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

The research reported in this document was made possible through support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the Army Signal Corps, the Navy Department (Office of Naval Research), and the Air Force (Air Materiel Command), under the Signal Corps Contract No. W-36-039 sc-32037.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Research Laboratory of Electronics

Technical Report No. 68 June 2, 1948

SEPARATION OF LAPLACE'S EQUATION*

Raymond Moos Redheffer

* This report is a copy of a thesis with the same title submitted by the author in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics at the Massachusetts Institute of Technology, May 1948.

ABSTRACT

For the Laplace equation $\nabla^2\Theta=0$ in curvilinear Euclindean co-ordinates (u,v,w) to be directly separable into two equations, one for S and one for z, when the solution is $\Theta=R(u,v,w)S(u,v)Z(w)$ with fixed R, it is necessary and sufficient that the surfaces w=constant be orthogonal to the other surfaces and be parallel planes, planes with a common axis, concentric spheres, spheres tangent at a common point, spheres through a fixed circle, or the spheres and plane generated by rotating bipolar co-ordinates about the line joining the poles. We have R=1 always and only in the first three cases; here, but only here, the wave equation separates in the sense RSZ, and hence for the wave equation R=1 automatically.

For further separation of the equation found above for S, when $S=X(u)Y(v)$ so that the solution is now RXYZ, it is necessary and sufficient that the co-ordinates be toroidal, or one of those where the wave equation so separates, or any inversions of these. The co-ordinates where the wave equation so separates, moreover, are only the well-known cases where this happens with R=1, namely, degenerate ellipsoidal or paraboloidal co-ordinates; in these cases, but only these, R=1 for the Laplace equation too. Co-ordinates for RXYZ separation of the Laplace equation have the group property under inversion.

The above assumes separation of the equation, which is more restrictive than mere separation of the solution; for example the solution separates with R=1 in all confocal co-ordinates, while the equation separates with R=1 in the degenerate cases only. Separation of the equation in one step is equivalent to existence of a two-parameter family $RX_{ab}Y_aZ_b$ together with $\partial X/\partial a=\partial X/\partial b$ and a certain auxiliary assumption (Theorem VI in Sec. 9.1). Separation in two steps as above is equivalent to $RX_{ab}Y_{ab}Z_b$ plus VI, but for the general case $RX_{ab}Y_{ab}Z_b$ the equation does not separate. If VI is satisfied, this last case is equivalent to existence of three solutions independent in a certain sense. For each type of separation we have obtained the necessary and sufficient conditions on R and the linear element of the space. These conditions show how to determine R by inspection.

TABLE OF CONTENTS

CONCLUDING REMARKS

PART II - SOLUTIONS $R(u,v,w)X_{ab}(u)Y_{ab}(v)Z_b(w)$

FURTHER SEPARATION OF EQUATION

DETERMINATION OF α AND β

DETERMINATION OF f

CONCLUDING REMARKS

PART III - THE MEANING OF SEPARATION

SIMPLE EQUIVALENCES

THE GENERAL CASE $RX_{ab}Y_{ab}Z_{ab}$

OTHER QUESTIONS

BIBLIOGRAPHY

SEPARATION OF LAPLACE'S EQUATION

ORIENTATION

<u>0.0 The nature of the problem</u>--Separation of variables is one of the most frequently used procedures for solving partial differential equations with prescribed boundary conditions. The basis of the method is to assume a form for the solution that involves functions of a single co-ordinate only; for example, one may assume solutions of the type XYZ or X+Y+Z, with X,Y and Z functions of a single variable: $X = X(u)$, $Y = Y(v)$, $Z = Z(w)$. Here as elsewhere we take u,v,w as general curvilinear co-ordinates. Upon substituting this form into the partial differential equation we obtain an ordinary differential equation for X, for example, by regarding v and w as constant. If the equation actually admits solutions of the prescribed form, matters can be so arranged that this ordinary differential equation for X will continue to be satisfied by the same X even when v and w are not constant. These remarks illustrate the first objective of separation methods, namely, to obtain ordinary differential equations rather than partial ones for the unknown functions.

A second objective, which applies particularly when the form is XYZ, is to facilitate the construction of solutions that satisfy prescribed boundary conditions. For this it is not enough to have just one solution XYZ, but rather one must have a complete family of solutions $X_{ab}Y_{ab}Z_{ab}$, with a and b independent parameters. Completeness of the set means that a sufficiently well behaved but otherwise arbitrary function Θ of u and v (for example) may be expanded in the form $\Theta = \sum A_{ab}X_{ab}Y_{ab}$. Considering the method in detail as applied to boundary-value problems, one finds it to be convenient and useful only when the surface over which the values are specified is a co-ordinate surface. If this is the case and the surface is w=c (for example), we form the series

(0.1)
$$\sum \frac{A_{ab}}{Z_{ab}(c)} X_{ab}(u)\, Y_{ab}(v)\, Z_{ab}(w)$$

The sum reduces to the desired function Θ on the surface w=c, and it is a solution for all values of w, being a linear combination of solutions. Here we use the fact that the partial differential equation is linear, which is the case for those equations of mathematical physics to which the method is applied.

Since the method applies only when one of the boundary surfaces is a co-ordinate surface, any limitation on the co-ordinates which can be used is a limitation on the physical situations that can be successfully treated. Now it turns out that the method of separation actually does restrict the coordinate system; in only a few cases will the equation admit solutions of the required form. Hence it is natural to inquire, How shall we determine all the co-ordinate systems in which the equation may be solved by separation of variables? Once this is answered, we know all the physical configurations for which the general boundary value problems can be solved by the method in question. Such is the type of problem with which the present thesis is concerned.

Investigations of this sort are divided into two clearly defined parts. First we write down the fact that the equation separates, a condition which gives restrictions on the co-ordinate system. Next, as the second part of the investigation, we use the fact that the system is imbedded in Euclidean space to obtain certain additional conditions. Those co-ordinate systems that satisfy both requirements are the ones sought.

0.1 Historical remarks-- Equations which have been systematically investigated with regard to separation in one sense or another are the Hamilton-Jacobi equation

$$\sum \frac{1}{H_i^2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 + K^2(E-V)\phi = 0 \tag{0.2}$$

and the scalar wave equation, one form of which is

$$\sum H \frac{\partial}{\partial x_i}\left(\frac{H}{H_i^2}\frac{\partial \phi}{\partial x_i}\right) + K^2(E-V)\phi = 0 \tag{0.3}$$

with $H=\prod H_i$. It has been shown by Stäckel [5] that all co-ordinate systems in which the former equation separates must have the linear element in a certain special form, the so-called Stäckel form. Here the separated solution was assumed to be a sum of functions X+Y+Z, with the maximum possible number of independent parameters. By a somewhat similar method Robertson [12] [1] showed that separation of the

1) Numbers in square brackets refer to the Bibliography; those in round brackets, to equations.

wave equation leads again to the Stäckel linear element provided certain additional assumptions are made, and to an auxiliary condition which Eisenhart [3] proved equivalent to the condition $R_{ij} = 0$ in the Riemannian space with metric ds^2 equal to that of the co-ordinate system. Here, however, the form assumed is a product XYZ rather than a sum, the number of parameters again being maximum.

The results just described, which were carried out for n dimensions, are concerned with the first aspect of the problem, viz., separation of the equation. To complete the investigation one must find those co-ordinate systems which have the Stäckel form and are at the same time imbedded in Euclidean space. Weinacht [4] and Blaschke [1] have shown that these conditions are met only by the various confocal systems and their degenerates, in three dimensions. Eisenhart [3] gives an independent proof and considers the case of higher dimensionality.

These results are for the wave equation. It might be supposed that the question of separation for the Laplace equation would be included as a special case, since the Laplace equation itself is a special case of the wave equation (set $k=0$ in (0.2)). Actually, however, the reverse situation prevails, and the Laplace equation leads to the more general problems. The co-ordinate systems in which the Laplace equation separates include those in which the wave equation separates, and once the larger class of co-ordinates is known it is a simple matter to pick out the smaller subclass by separate examination. In this sense, then, the Laplace equation includes the other. The two cases are so related because k of (0.2) is not regarded as a constant, but as an independent parameter; separation is required not only for one but for all values. These observations on the relation of the Laplace and wave equations apply for separation of any type, as well as for the case XYZ hitherto considered.

As far as the author has been able to ascertain, the present status of separation problems in three dimensions is as follows. The result is known for the Hamilton-Jacobi equation; with a complete family $X+Y+Z$ and no other condition, one obtains the Stäckel form, which in turn leads to confocal co-ordinates in Euclidean space. For the wave equation, separation of the type XYZ has been shown to give the same configuration, but in the course of the proof it appears that certain statements were used which are not really consequences of separation. These extraneous assumptions are considered in Sec. 9, where they are shown to be valid in rather general circumstances. There is still a possibility, however, that the equation may separate in more than the 11 co-ordinate systems found in the classical works. Finally, for the Laplace equation no investigation seems to have been given. It is known that it separates in the 11 co-ordinates for which the wave equation separates, at least; but there may be more. This incompleteness persists even when we make the auxiliary assumptions noted above.

0.2 Scope of this report--Partly because it includes the wave equation and partly because the results are still unknown, the emphasis here is on separation of variables in Laplace's equation. We confine the discussion to three dimensions as the case of greatest practical interest. Also this loss of generality, which is believed to be inconsequential, enables us to put the derivation in a form readily followed by the general reader whose interests are not primarily mathematical. Such considerations have weight because the problems treated are of interest to physicists as well as mathematicians.

Turning now to the question of the mode of separation, we observe that a more general form than the XYZ hitherto considered will achieve the objectives noted in Sec. 0.0. Specifically we may assume the form RXYZ, where R is a single fixed function of the three co-ordinates: $R = R(u,v,w)$. If X,Y and Z form a complete set and R is known, the operations of Sec. 0.0 can be carried out with no increase in complexity. Instead of expanding θ in (0.01) we expand θR, for example. It might be thought that a partial differential equation would have to be solved to get R, but actually we shall see that this is not the case.

Assuming the form XYZR rather than XYZ, we have a slightly milder restriction on the solutions and we might expect, therefore, to find a larger class of permissible co-ordinates. Such is indeed the case; with toroidal co-ordinates, for example, the equation is known to separate in the extended sense but not in the restricted sense [11], and the same is true for the so-called Dupin cyclides [7]. The cases considered in Sec. 0.1 correspond to those cases in the new theory for which $R \equiv 1$.

A discussion of separation may consider the method of obtaining the solution, as well as its form. Nothing has yet been assumed about this aspect of the question, the procedure of Sec. 0.0 being operable for all cases. In most circumstances, however, the solution is obtained by separating the equation in the literal sense; that is, we multiply through by a fixed function of the co-ordinates to obtain two sets of terms, one set involving w only, for example, and the other involving u and v only. The same procedure is then used for the u,v terms. In most of the present work we assume, not only that the solution has the form XYZR, but also that the equation can be separated in this way. Such an assumption is restrictive; in particular we do not obtain the Dupin cyclides even though the solutions have the prescribed form. One of the catagories of co-ordinates in [3] is also omitted. After completing the investigation on that basis we consider the general case, with its relation to this and other special modes of obtaining the solutions.

0.3 Results obtained-- We consider first the case in which only one variable separates, so that the solution has the form S(u,v)Z(w)R(u,v,w) with S and Z a family of functions for fixed R. Next we consider the cases (necessarily included in these) for which the solution separates still further to give XYZR. Our results are summarized in the following theorems.

I. With fixed R, if solutions R(u,v,w)S(u,v)Z(w) satisfy Laplace's equation, and if separate differential equations for S and Z can be found by multiplying the equation by a suitably chosen function, then the co-ordinate surfaces w = constant must be orthogonal to the other two co-ordinate surfaces, and must consist of parallel planes, planes with a common line of intersection, spheres tangent at a common point, concentric spheres, the plane and set of spheres obtained when one set of bipolar co-ordinates is revolved about the line joining the poles, or a set of spheres all passing through a single fixed circle.

II. If the surfaces w = constant have any of the forms in Theorem I, and if the u and v surfaces are orthogonal to them, then Laplace's equation can always be separated in the prescribed manner.

III. In I and II we may assume $R = 1$ always and only when the surfaces w = constant consist of parallel planes, planes with a common line of intersection, or concentric spheres.

IV. If the procedure of Theorem I can be applied to the resulting differential equation for S to get further separation, so that the solutions have form $R(u,v,w)\ X(u)\ Y(v)\ Z(w)$, then the co-ordinates must all be orthogonal and must be toroidal co-ordinates, or the well known cases (I--III but not IV in [3]) giving separation of this type with $R = 1$, or else the co-ordinates obtained by inversion of these in a sphere. All inversions are permissible.

V. The wave equation separates in the sense of Theorem I when, and only when, we have the cases in III, so that R=1. Similarly the wave equation separates in the sense IV only when R = 1.

Besides these results we have those of Part III, which show the relation of various modes of separation to each other and to the linear element of the co-ordinate system.

PART I - SOLUTIONS $R(u,v,w)S_a(u,v)Z_a(w)$

SEPARATION OF THE EQUATION

1.0 Laplace's equation, and the second derivative terms-- Consider the Laplace operator under a transformation of co-ordinates from (x,y,z) in Euclidean three-dimensional space to curvilinear co-ordinates (u,v,w). Since the appearance of cross derivatives involving w makes separability impossible we consider only the case of orthogonal co-ordinates at first. (see Sec. 4.1). In the curvilinear co-ordinates (u,v,w) let the linear element be

$$ds^2 = f^2du^2 + g^2dv^2 + h^2dw^2 \tag{1.0}$$

where f, g, and h are functions of u, v, and w. Then [11]

$$\nabla^2\theta = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\theta_u\right)+\frac{\partial}{\partial v}\left(\frac{hf}{g}\theta_v\right)+\frac{\partial}{\partial w}\left(\frac{fg}{h}\theta_w\right)\right]$$

which becomes, if $\theta = R(u,v,w)S(u,v)Z(w)$,

$$(1.1) \qquad \nabla^2\theta = SZR\left[\frac{S_{uu}}{f^2S}+\frac{S_{vv}}{g^2S}+\frac{Z''}{h^2Z}+F_1\frac{S_u}{S}+G_1\frac{S_v}{S}+H_1\frac{Z'}{Z}+F_2+G_2+H_2\right]$$

where

$$(1.2) \qquad F_1 = \frac{2R_u}{f^2R}+\frac{1}{fgh}\left(\frac{gh}{f}\right)_u$$

and similarly for G_1 and H_1. Also

$$(1.3) \qquad F_2 = \frac{R_{uu}}{f^2R}+\frac{R_u}{fghR}\left(\frac{gh}{f}\right)_u$$

and similarly for G_2 and H_2.

For separability of $\nabla^2\theta=0$ in the form (1.1) we require that the function Z separate off into an ordinary differential equation. That is, there must exist a function $A^2(u,v,w)$ such that when $\nabla^2\theta$ is multiplied by A^2, only the variable w appears in the coefficients of Z''/Z and Z'/Z and w does not appear in the coefficient of any other differentiated terms. Moreover $A^2(F_2+G_2+H_2)$ must break up into the sum of a function of w only and a function of u,v only.

Using the condition on the coefficients of the terms involving second derivatives of S and Z, and bearing in mind the definition (1.0) of f, g, and h, we see that the linear element must have the form

$$(1.4) \qquad ds^2 = A^2\left[F^2du^2 + G^2dv^2 + H^2dw^2\right]$$

where F, G are functions of u,v only and H is a function of w only.

1.1 Permissible changes of variable-- If we replace w by a function of w, and the u, v co-ordinates by new co-ordinates which still do not involve w, we may put (1.4) in the form

$$(1.5) \qquad ds^2 = A^2\left[F^2(du^2 + dv^2) + dw^2\right]$$

where F is a function of u and v alone while H is a function of w alone. Here we have written A for the new value of A and u,v,w for the new values of u, v and w.

The w transformation is certainly permissible, as it amounts merely to a change of scale for the w co-ordinate. Hence it does not alter the geometrical configuration of the co-ordinate surfaces. When we say that a change of variable is permissible, in this connection, we mean that the geometric properties relevant to separation of Laplace's equation are essentially unchanged. Hence proof of the existence of a certain geometric property in the transformed system shows that the same property was present in the original system.

That the change of u, v co-ordinates is permissible in this sense follows from the fact that the new linear element willhave the form (1.4). Hence the condition that the equation should separate is still satisfied (note that Eq. (1.8) will also persist in form). Moreover, since every linear element of the form (1.4) corresponds to a triply orthogonal system, we see that the equations used later, (2.0) and (2.1), will continue to be valid in the new system. From these and similar remarks concerning changes of coordinates we condlude finally that it suffices to specify the surfaces $w =$ constant. If the equation separates in our sense for a particular system of u and v surfaces orthogonal to these it will separate for every other such system. Conversely, if a change of u and v co-ordinates leads to a certain set of u and v surfaces, and it is then found that the w surfaces must have certain special properties, these properties will in fact persist for every choice of the u or v surfaces. To simplify the analysis these two operations, changing the w scale and changing the u, v co-ordinates, will be repeatedly used in the ensuing discussion.

1.2 The first derivative terms--Turning now to the coefficients of terms involving first derivatives, we use the fact that AH_1 must be a function of w only to find, by virtue of (1.2) and (1.5),

$$A^2 \left[(2/A^2)(R_w/R) + (1/F^2A^3)(F^2A)_w\right] = h_1(w)$$

which simplifies to $2R_w/R + A_w/A = h_1(w)$. We integrate with respect to w, we note that the constant of integration may be an arbitrary function of u and v, and we take the exponential of each side, to find finally

$$R^2A = h_2(w)F_2(u,v) \quad . \tag{1.6}$$

Since it is permitted that R involve u,v, and w we may always modify R in such a way that

$$R^2A=1 \tag{1.7}$$

Thus, the term h_2 in (1.6) may be absorbed in Z and F_2 in S.

It may be verified that the coefficients of S_u/S and S_v/S are already functions of u and v only, without any new condition (actually they are zero). From separation of the equation, therefore, we have only (1.5), (1.7), and an extra condition

$$(1.8) \qquad A^2(F_2+G_2+H_2) = h_4(w)+F_4(u,v) \quad .$$

It will be found that (1.8) is a consequence of the Euclidean character of the space.

EUCLIDEAN SPACE

2.0 The functional form of A--If the space is euclidean then the linear element (1.0) satisfies the relations [2],[6] ,

$$(2.0) \qquad f_{uv} - \frac{g_v f_v}{g} - \frac{h_v f_w}{h} = 0$$

$$(2.1) \qquad \left[\frac{g_u}{f}\right]_u + \left[\frac{f_v}{g}\right]_v + \frac{f_w g_w}{h^2} = 0$$

We also have those relations obtained by simultaneous cyclic permutation of (fgh) and (uvw). These equations, which state that the Riemannian curvature of the space is zero, give a necessary and sufficient condition that the co-ordinate system be imbedded in Euclidean space.

In terms of (1.5), the relation (2.0) containing g_{uw} becomes

$$(2.2) \qquad \frac{A_{uw}}{A_u} = 2\frac{A_w}{A}$$

after simplification. Integrating with respect to w, as in the derivation of (1.6), we find that $(1/A)_u$ is a function of u and v alone. The same is true of $(1/A)_v$, whence it follows that $1/A=f(u,v)+H(w)$, or

(2.3) $$A = 1/\left[f(u,v) + H(w)\right] .$$

The two relations (2.0) used in the derivation are now satisfied, and in terms of the new linear element, the third gives

(2.4) $$f_{uv} = F_v f_u/F + F_u f_v/F .$$

Turning now to the relations (2.1) we find that (2.1), as it stands, leads without detailed calculation to an expression of the form[1]

1) The equation is written explicitly in (2.9)

(2.5) $$H'^2 + c_1H^2 + 2c_2H + c_3 = 0$$

where the c's are functions of u and v only. Differentiating with respect to w, we find

(2.6) $$H'(H'' + c_1H + c_2) = 0 .$$

Since the solution depends on w only while the coefficients depend on u and v only, these coefficients must actually be constant[2]. We have then a linear equation for H with constant

2) This result has been carefully proved by substituting the expressions (2.7) back into the equation.

coefficients, which may be solved to give

$$(2.7) \quad H = 0,\ w,\ w^2,\ e^w,\ \sinh w,\ \cosh w,\ \sin w$$

as the only possible values of H that are essentially distinct. Here the case $H' = 0$ has been reduced to $H = 0$ by absorbing the constant value of H in the functinn f.

In summary, we have found that the linear element must be of the form

$$(2.8) \qquad ds^2 = \frac{F^2(du^2+dv^2) + dw^2}{(f+H)^2}$$

with f and F functions of u and v only, while H is one of the functions (2.7). The only additional conditions are

(1.8), (2.1), and (2.4).

To use the relations (2.1) we substitute for f, g and h their values as given by (2.8). The equation (2.1) as it stands gives

$$(2.9) \qquad \frac{f_{uu}+f_{vv}}{f+H} = \frac{f_u^2 + f_v^2 + F^2H'^2}{(f+H)^2} + \left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v$$

while the sum of the two others found by permutation leads to

$$(2.10) \qquad f_{uu} + f_{vv} = 2\,\frac{f_u^2 + f_v^2 + F^2H'^2}{f+H} - 2F^2H''$$

and their difference gives

$$(2.11) \qquad f_{uu} - f_{vv} = 2f_u\frac{F_u}{F} - 2f_v\frac{F_v}{F}\,.$$

The system (2.9) - (2.11), slightly simpler than (2.1), is equivalent to it. These three equations and (2.4) give the necessary and sufficient condition that the metric (2.8) be for orthogonal co-ordinates in Euclidean space. The fact that ds^2 has the form (2.8), plus the relation (1.8), on the other hand, gives the necessary and sufficient condition that this co-ordinate system be one in which Laplace's equation separates. Systems satisfying both sets of conditions, and only those, are the ones we are seeking.

<u>2.1 The Second fundamental form--</u> It is known from differential geometry [2], [6] that a triply orthogonal system is completely determined, except for its orientation in space, by the linear element ds^2. Hence the second fundamental form (as well as the first) for each of our co-ordinate surfaces is specified by f,g and h of Eq. (1.0) and we might expect to find specific relations from which they could be computed. Such relations actually exist [2], [7]; on the surface w = constant we have

$$(2.12) \qquad \ell = -ff_w/h, \; m = 0, \; n = -g\, g_w/h$$

for the second fundamental form

$$-d\bar{X} \cdot d\bar{N} = \ell\, du^2 + m\,du\,dv + n\,dw^2,$$

with the results in other cases to be obtained by cyclic permutation of u,v,w and f,g,h.

In particular for the linear element (2.8), we obtain

$$(2.13) \qquad -d\mathbf{X}.d\mathbf{N} = \frac{F^2H'}{(f+H)^2} \; (du^2+dv^2) \; .$$

On the other hand setting $dw^2 = 0$ in (2.8) gives

$$(2.14) \qquad d\mathbf{X}.d\mathbf{X} = \frac{F^2}{(f+H)^2} \; (du^2+dv^2)$$

for the first fundamental form. Since the two forms (2.13) and (2.14) are proportional, we known that the surfaces w = constant must be spheres or planes [13]. Computing the Gaussian curvature as the ratio of the two discriminants d^2/D^2 (Ref.[13]) we find the radius of the sphere corresponding to a given value of w:

$$(2.15) \qquad \text{radius} = 1/H'(w)$$

This result will be frequently used in the ensuing investigation.

SEPARATE EXAMINATION OF CASES

3.0 <u>The case H = 0</u> -- For each value of H in (2.7) we obtain a new set of relations for F. These equations are not easy to solve as they stand, and our procedure will be to seek a change of u,v variables that will reduce them to simpler form. By the discussion of Sec. 1.1 we know that any restrictions on the w surfaces obtained after the change must have been valid before it as well.

In case H = 0, as assumed here, we see by (2.13) that λ = m = n = 0 and hence the surfaces w = constant must be planes [13]. On one of these planes let us pick the u, v co-ordinates so that the linear element takes the Cartesian form $du^2 + dv^2$. This is possible, since the surface is a plane. By comparison with (2.8) when $dw^2 = 0$ we see that the present u and v co-ordinates make F/f = 1. Also, since both f and F are independent of w, we now have F/f = 1 for all values of w, not merely for the constant value first selected.

With this procedure equations (2.10) and (2.11) become

$$f_{uu} + f_{vv} = 2\,(f_u^2 + f_v^2)/f$$

$$f_{uu} - f_{vv} = (f_u^2 - f_v^2)/f$$

whence, after adding and dividing by f_u,

$$f_{uu}/f_u = 2f_u/f$$

with a similar result for f_{vv}. Integrating twice, we find

$$1/f = uf_1(v) + f_2(v)$$

and a corresponding result with u and v interchanged. The two together show that 1/f must have the form

$a + bu + cv + duv$, with a,b,c,d constant, and Eq.(2.4) tells us that $d = 0$. The remaining conditions (2.9) and (1.8) are now satisfied, so that there is no other restriction.

If $b = c = 0$, the surfaces w = constant represent parallel planes, as we see by comparison with the linear element for Euclidean co-ordinates. If b or c is not zero, however, the coefficient of dw^2 vanishes for certain values of u and v. At a point corresponding to such a value of u and v we can go from the plane $w = w_0$, say, to the plane $w = w_1$ by passing through zero distance. It follows that these two planes intersect. Along the line of intersection the coefficient of dw^2 must be zero, and hence on this line we can pass not only from $w = w_0$ to $w = w_1$ without an increase of distance, but from the plane $w = w_0$ to any other plane of the family. These planes, then, must all pass through the line of intersection first found; that is, the surfaces w = constant are planes with a common axis.

3.1 <u>The case H = w</u> -- Turning now to the case $H = w$, we find from (2.10)

$$(f_{uu} + f_{vv})(f+w) = 2(f_u^2+f_v^2 + F^2)$$

Since this is an identity in w the coefficient of w must vanish, so that $f_{uu}+f_{vv} = 0$, and hence also $f_u^2+f_v^2+F^2 = 0$.

This is impossible since it makes $F = 0$.

3.2 The case $H = w^2$-- Next if $H = w^2$ we find

(3.1) $\quad (F_u/F)_u + (F_v/F)_v = 0$

(3.2) $\quad f_{uu}+f_{vv} = 4F^2$

(3.3) $\quad f_{uu} + f_{vv} = (f_u{}^2+f_v{}^2)/f$

by equating to zero the coefficients of w^4, w^2 and 1 in (2.9). Nothing new is obtained from (2.10), and hence these with (2.4) and (2.11) are the only conditions. Equations (3.1) and (3.3) are equivalent to the statements that log F and log f are harmonic functions.

By simply writing the Gauss equation for curvature in terms of the first fundamental form, we find that for any F satisfying (3.1), this curvature vanishes for the the surfaces with $ds^2 = F(du^2+dv^2)$. These surfaces therefore are developable, and we may introduce a new set of u--v co-ordinates which will make $F = 1$. When this is the case, Eq.(2.4) tells us that $f_{uv} = 0$, so that $f = \alpha(u) + \beta(v)$. Equation (3.2) now reduces to $\alpha'' + \beta'' = 4$, which gives $\alpha'' = 4 + c$, $\beta'' = 4 - c$ with c a constant. It follows that $\alpha = u^2$, $\beta = v^2$, after making a change of variable, if necessary, to eliminate the arbitrary constants. Our linear element now takes the form

$$(3.4) \qquad ds^2 = \frac{du^2 + dv^2 + dw^2}{(u^2 + v^2 + w^2)^2}$$

after f has been given its value as determined above. We observe from (7.1) and (7.2) that the linear element (3.4) is the one which would be obtained by inversion of Cartesian co-ordinates in the origin. Because the linear element is sufficient to determine the co-ordinate system completely, as noted in Sec. 2.1, it follows that the co-ordinate surfaces, as well as the linear element, will be the same as those which would be obtained by the inversion described. In particular the surfaces w = constant must consist of a plane and a set of spheres all tangent to it at one point. This result, incidentally, can be obtained directly from (3.4), as in the examples considered below.

3.3.--<u>The case $H = e^w$</u>-- Turning now to the case $H = e^w$ we substitute in (2.10) and equate to zero the coefficients of 1 and e^w, respectively, to find

$$f(f_{uu}+f_{vv}) = 2(f_u^2+f_v^2)$$

$$f_{uu} + f_{vv} = -2F^2f$$

which gives

$$(3.5) \qquad f_u^2 + f_v^2 + F^2f^2 = 0.$$

Equation (3.5) implies that f is zero. We have then from (2.8)

$$(3.6) \qquad ds^2 = \frac{F(du^2+dv^2) + dw^2}{e^{2w}}$$

The other relations are now all satisfied, if F is suitably restricted.

Writing down the equations of the geodesics in the Riemannian space V_3 with linear element (3.6), and noting the coefficient of dw^2 is independent of u and v, we find that the curve U = a, v = b satisfies these equations and must therefore be a straight line, for any constants a and b. It is seen from (2.15) or from (3.6) itself that $w = \infty$ corresponds to a point. Along the straight line u = a, v = b the distance from this point to the point with co-ordinates (a,b,w) is

$$\int_{\infty}^{w} dw/e^{w}$$

independently of a and b. The point $w = \infty$, then, must be the center of the sphere with parameter w, and since w is arbitrary we conclude that the spheres w = constant are all concentric. Thus the surfaces are completely specified. We remark in passing that the other co-ordinate surfaces must be developable, as is seen from the second fundamental forms.

3.4 <u>The case H = sinh w</u> -- When H = sinh w, we substitute into (2.10) as usual, then put everything in terms of sinh w by using $\cosh^2 w = 1 + \sinh^2 w$, and finally equate to zero the coefficients of 1 and sinhw to obtain

$$(3.7) \qquad f(f_{uu}+f_{vv}) = 2(f_u^2+f_v^2 + F^2)$$

$$(3.8) \qquad f_{uu} + f_{vv} = -2F^2 f \ .$$

Together these relations show that

$$(3.9) \qquad f_u^2 + f_v^2 + F^2 + F^2 f^2 = 0$$

which is impossible, since it makes F = 0.

3.5--<u>The case H = cosh w</u> -- If H = csoh w we proceed similarly to obtain, from (2.9) and (2.10),

$$(3.10) \qquad f_{uu} + f_{vv} + 2fF^2 = 0$$

$$(3.11) \qquad f_u^2 + f_v^2 \quad F^2(f^2-1) = 0$$

$$(3.12) \qquad F^2 + (F_u/F)_u + (F_v/F)_v = 0$$

after slight simplification. Let us notice by (2.15) that the surface w = 0 is a plane, so that reasoning as before, we may assume F = f + 1.

From (2.4) we have

$$f_{uv} = 2f_u f_v/(f+1)$$

which may be integrated, then divide by $(f+1)^2$ and integrated again, to give finally

(3.13) $$1/(f+1) = \alpha(u) + \beta(v) \quad .$$

After divistion by $(f+1)^4$ Equation (3.11) becomes

(3.14) $$\alpha'^2 + \beta'^2 + 1 = 2(\alpha + \beta)$$

which leads to the linear equation

$$\alpha'(\alpha'' - 1) = 0$$

when differentiated with respect to u, v being constant. We have a similar relation for α , whence we conclude

(3.15) $$\begin{aligned} \alpha &= a \quad \text{or} \quad (1/2)u^2 + au + b \\ \beta &= A \text{ or} \quad (1/2)v^2 + Av + B \quad . \end{aligned}$$

That both α and β cannot be constant is seen by (3.13), (3.10), and (3.11); the equations require F = 0, which is not permissible. If α alone is constant, moreover, the relation (3.14) tells us that $A^2 + 1 = 2(\alpha + \beta)$. The value of F thus obtained does not satisfy (3.12). Hence neither α nor β is constant, and we have then

$$f = 2/(u^2+v^2+1) - 1$$

after substituting (3.15) into (3.14) to find b+B = 1/2, making a change of variable to get a = A = 0, and using (3.13). This value of f, which satisfies all relations, leads to

$$ds^2 = \frac{4(du^2+dv^2) + (1+u^2+v^2)^2 dw^2}{[2+(\cosh w-1)(1+u^2+v^2)]^2}$$

as the linear element when $H = \cosh w$. Using $\cosh w - 1 = 2\sinh^2 w/2$, writing w for $w/2$, and taking $\cosh^2 w = 1+\sinh^2 w$ in the denominator, we find

$$(3.16) \qquad ds^2 = \frac{du^2+dv^2+(1+u^2+v^2)^2 dw^2}{[\cosh^2 w+(u^2+v^2)\sinh^2 w]^2}$$

which assumes the proper form, we note, when $w = 0$.

The change of variable $u = r\cos\theta$, $v = r\sin\theta$ leads to

$$ds^2 = \frac{dr^2 + r^2 d\theta^2 + (1+r^2)^2 dw^2}{[1+(1+r^2)\sinh^2 w]^2}$$

and in this case the surfaces θ = constant are all planes, as we see by computing the second fundamental form. On these surfaces θ = constant we have

$$ds^2 = \frac{dt^2 + dw^2}{[\sin^2 t + \sinh^2 w]^2}$$

after making the change of variable $r = \cot t$. The linear element last obtained coincides with that for bipolar co-ordinates, whence we conclude that the spheres w = constant must be one of the sets of surfaces generated when bipolar co-ordinates are revolved about the line joining the two poles. The fact that $w = 0$ is plane shows that the co-ordinates generating our spheres must be those which include the perpindicular bisector of this line.

3.6 <u>The case H = sinw--</u> The final case is H = sin w, which gives

(3.17) $$f_u^2 + f_v^2 = F^2(f^2-1)$$

(3.18) $$f_{uu} + f_{vv} = 2F^2 f$$

(3.19) $$(F_u/F)_u + (F_v/F)_v = F^2$$

when we substitute in (2.9) and (2.10), replace $\cos^2 w$ by $1 - \sin^2 w$, and equate to zero the coefficients of 1, sin w and $\sin^2 w$. By (2.15) the surface $w = \pi/2$ is a plane and hence we may assume F = f+1 to obtain the result (3.13) as before. Equatinn (2.4) is now satisfied; the only ones remaining are (2.11) and those just obtained.

By (3.13) and (3.17)

$$\alpha'^2 + \beta'^2 + 2(\alpha+\beta) = 1.$$

Proceeding as in the discussion of (3.15), we find that neither α' nor β' is zero, and that

$$f = 2/(1-u^2-v^2) \quad -1 ,$$

so that the linear element becomes

$$ds^2 = \frac{4(du^2+dv^2) + (1-u^2-v^2)dw^2}{[2+(\sin w-1)(1-u^2-v^2)]^2} .$$

Using $1 - \sin w = 2\sin^2(w/2 - \pi/4)$, replacing $(w/2)-(\pi/4)$ by a new variable, and simplifying slightly, we find

$$ds^2 = \frac{du^2+dv^2+(1-u^2-v^2)^2 dw^2}{[\cos^2 w + \sin^2 w(u^2+v^2)]^2} .$$

As before the surface w = 0, which we know to be plane, has the assumed linear element.

Let us notice that the coefficient of dw^2 is zero whenever $u^2+v^2 = 1$, and hence the surfaces w = constant must all intersect in this set of points. On the surface w = 0, however, the u-v co-ordinates are simply the usual cartesian co-ordinates for a plane, as we see by the linear element. Hence the set of points $u^2+v^2 = 1$ is a circle on w = 0, and we conclude that the surfaces w = constant are spheres passing through a single fixed circle.

CONCLUDING REMARKS

4.0 Dismissal of an auxiliary condition-- To complete the discussion we must show that (1.8) is satisfied in each of the cases considered. With this end in view we use (1.7), (1.5) and (1.3) to obtain, for the left member of (1.8),

$$(4.1) \qquad \frac{1}{4F^2}\left[\frac{A_u^2+A_v^2+F^2A_w^2}{A^2} - 2\,\frac{A_{uu}+A_{vv}+F^2A_{ww}}{A}\right].$$

Replacing A by its proper value (2.3) and simplifying, we find

$$(4.2) \qquad \frac{1}{4F^2}\left[2\,\frac{f_{uu}+f_{vv}+H''F^2}{f+H} - 3\,\frac{f_u^2+f_v^2+F^2H'^2}{(f+H)^2}\right]$$

which becomes

$$(4.3) \qquad (f_{uu}+f_{vv})/8F^2(f+H) \quad - \quad H''/4(f+H)$$

if we use (2.12).

When H = 0, the expression in (4.3) is a function of u and v alone, so that (1.8) is certainly satisfied. Similarly, if $H = w^2$ we may replace $f_{uu} + f_{vv}$ by its value (3.2), whence it is seen again that (4.3) has the proper form as prescribed by (1.8). If $H = e^w$ the result is again true, since f is zero; for H = cosh w it is a consequence of (3.10); for H = sin w it follows from (3.18). Thus Equation (1.8) is satisfied automatically in all cases, and the co-ordinate systems hitherto obtained will actually lead to separation. What we have shown is that (1.8) is a consequence of the fact that the space is Euclidean.

4.1 <u>Non-orthogonal co-ordinates</u>-- As noted, the w-surfaces are orthogonal to the others, since there must be no cross derivative terms involving the variables to be separated. It is really not necessary, however, that the u and v surfaces be orthogonal to each other, although up to now we have assumed this to be the case. Thus, orthogonality of the u and v surfaces is essential to our derivations, because the relations of Sec. 2. presuppose that the w = constant surfaces are imbedded, or at any rate can be imbedded, in a triply orthogonal system. Our aim now is to discard this condition of orthogonality as an initial assumption, and to show that the equations dependent thereon will be satisfied anyway as a consequence of separation.

To this end we assume

(4.4) $ds^2 = edu^2 + 2fdudv + gdv^2 + hdw^2$

rather than (1.0) and obtain [12]

(4.5) $$\frac{\partial}{\partial u}\left(\frac{g\sqrt{h}}{d}\theta_u - \frac{f\sqrt{h}}{d}\theta_v\right) + \frac{\partial}{\partial v}\left(\frac{e\sqrt{h}}{d}\theta_v - \frac{f\sqrt{h}}{d}\theta_u\right) + \frac{\partial}{\partial w}\left(\frac{eg}{\sqrt{h}\,d}\theta_w\right) = 0$$

in place of (1.1). Upon assuming a solution RSZ as in Sec. 1.0 we find

(4.6) $$\frac{g}{d^2}\frac{S_{uu}}{S} - 2\frac{f}{d^2}\frac{S_{uv}}{S} + \frac{e}{d^2}\frac{S_{vv}}{S} + \ldots = 0$$

where the terms not written involve first derivatives of the unknown functions only, besides e,f,g,h and R. In (4.5) and (4.6) d^2 is the discriminant of the u--v quadratic form,

(4.7) $$d^2 = eg - f^2 \quad .$$

Since the equation separates, there exists a function A(u,v,w) such, that when the equation is multiplied by A, the coefficients of terms involving S are functinns of u and v only, while the coefficients of terms with Z are functions of w only (cf. Sec.1). Also the term free of unknowns must break up into a function of u, v plus a function of w. For our present purposes we need only the coefficients of S_{uu}, S_{uv} and S_{vv}, which tell us that

(4.8) Ag/d^2, Af/d^2, Ae/d^2

are functions of u and v alone. It follows that any combination of these expressions is also a function of u and v alone, and hence in particular

$$(4.9) \qquad \left(\frac{Ag}{d^2}\right)\left(\frac{Ae}{d^2}\right) - \left(\frac{Af}{d^2}\right)^2 = \frac{A^2}{d^2}$$

has this property. Multiplying (4.8) by A and using (4.9) we find that e/A, f/A and g/A are functions of u and v alone, so that the linear element (4.4) has the form

$$(4.10) \quad ds^2 = A[e_1(u,v)du^2 + f_1(u,v)dudv + g_1(u,v)dv^2] + hdw^2.$$

Confining our attention to the terms in brackets, we see that it is always possible to make a change of variables, replacing u by $\bar{u}(u,v)$ and v by $\bar{v}(u,v)$, so that in the new variables we will have $f_1 = 0$. This is an analytic expression of the well-known geometrical fact that every surface admits a set of parametric curves which are orthogonal. When such a change of variables is made, the linear element (4.10) reduces to the form (1.0).

What we have shown is that there exists a change of u and v parameters which will make the new u and v surfaces orthogonal to each other, if they were not originally. Such a change is permissible in the sense of Sec.1.1 , and may therefore be carried out at the beginning of the investigation. We now have a linear element of the form

(1.0), and the foregoing derivation proceeds without further change. Thus we have completed the proof of Theorems I and II.

4.2 Cases for which $R \equiv 1$ -- Let us inquire when we may assume $R \equiv 1$. A necessary and sufficient condition is that R have the form $F(u,v)G(w)$, since in that case, but not otherwise, R may be absorbed in the solution $S(u,v)Z(w)$. From the equation $R^2A \equiv 1$ we see that this condition, and hence the possibility of $R \equiv 1$, is satisfied when, and only when, A also has the form $F(u,v)G(w)$. We know, however, that A must have the form (2.3), whence we conclude that f, H, or both must be constant for $R \equiv 1$. These considerations lead to Theorem III.

4.3 The wave equation-- Next let us consider the possibility of separating the wave equation. As noted, the theory is simpler than the preceding and included in it. Writing the equation in the form

$$\nabla^2\Phi + k\Phi = 0 \tag{4.11}$$

we see that $F_2 + G_2 + H_2$ of (1.1) becomes $F_2 + G_2 + H_2 + k$, and this is the only change. All results previously obtained are valid here too, then, except for (1.8), which becomes

(4.12) $$A^2(F_2 + G_2 + H_2 + K) = F_3(u,v) + H_3(w)$$

But in the course of the foregoing investigation it was shown that (1.8), originally postulated independently as a result of the separation, is actually not an independent relation, but follows automatically from the others (Sec.4.0). Though not required in itself for separation of the wave equation, therefore, this relation must nevertheless hold, in view of the other conditions. Combining it with (4.12) we see that

(4.13) $$A^2 = F_4(u,v) + H_4(w) .$$

If f of (2.3) is not constant it depends on u, say, and the same is true of F_4 in view of (1.8) and (4.13). Equating the two expressions for A, we differentiate with respect to u, solve for $(f+H)^3$, and differentiate the result with respect to w to get

$$3(f+H)^2 H' = 0 ,$$

which shows that H is constant. Thus either H or f must be constant, and we therefore have $R = 1$ immediately for the case of partial separation. In this way we obtain the first half of Theorem V.

PART II -- SOLUTIONS $R(u,v,w)X_{ab}(u)Y_{ab}(v)Z_a(w)$

****

FURTHER SEPARATION OF EQUATION

5.0 General--It has been supposed hitherto that the solution is only partially separated, being of the form $R(u,v,w)S(u,v)Z(w)$ with a fixed function R. If we assume that the solution separates further to give the form $R(u,v,w)X(u)Y(v)Z(w)$, so that $S(u,v) = X(u)Y(v)$ for each function of the family, then we obtain new conditions on the co-ordinates. Because the cross derivative terms make separability impossible, for example, it is known at the outset that all three co-ordinate surfaces must now be orthogonal.

The initial stage of the separation led to an ordinary differential equation for $Z(w)$ and, with m constant, to

$$(5.1) \qquad \frac{A^2}{f^2}\frac{S_{uu}}{S} + \frac{A^2}{g^2}\frac{S_{vv}}{S} + A^2F_1\frac{S_u}{S} + A^2G_1\frac{S_v}{S} + F_4(u,v) = m$$

for $S(u,v)$, as we see by using (1.8) and noting that each of the separated groups of terms must be constant. This is true because the sum must be zero, and one expression involves w only while the other involves u and v only. When S has the assumed form XY (5.1) becomes

$$(5.2) \qquad \frac{A^2}{f^2}\frac{X''}{X} + \frac{A^2}{g^2}\frac{Y''}{Y} + A^2F_1\frac{X'}{X} + A^2G_1\frac{Y'}{Y} + F_4 = m$$

For separation there must be a function $J(u,v)$ such that (5.2) separates when multiplied by this. In particular the term mJ must separate, since the only other term it can combine with, $F_4 J$, is independent of m. If separation is to occur for as few as two distinct values of m, it already implies that $F_4 J$ and mJ must separate into the sum of a function of u only and a function of v only. In this connection it should be noted that if we try to pick a new J for each m, the ratio of the J's will have to be a function of u only and also a function of v only, in view of the condition (see below) on the coefficients of X" and Y". Thus the function J could depend on m alone, and the above considerations apply.

5.1 The differentiated terms--It has been seen that our multiplier J(u,v) is of the form $\alpha(u)+\beta(v)$. When (5.2) is multiplied by this separation must occur, and hence the coefficients of X"/X and X'/X must depend on u alone, while those of Y"/Y and Y'/Y depend on u alone. The condition for the second derivatives tells us that

$$ds^2 = A^2\left\{\left[\alpha(u)+\beta(v)\right]\left[du^2+dv^2\right]+dw^2\right\}$$

after a change of scale in the u and v co-ordinates. The condition on coefficients of the first derivatives allows us to assume $R^2A=1$, as we see by following the derivation of (1.7). Also we know from the first separation that A has the form (2.3), so that we obtain finally

$$(5.3) \qquad ds^2 = \frac{[\alpha(u)+\beta(v)](du^2+dv^2)+dw^2}{[f(u,v)+H(w)]^2} .$$

From the derivation it is clear that (5.3) is sufficient as well as necessary for separation.

In connection with (5.3) we note that du^2 and dv^2 have the same coefficients, just as in (1.5). The former result was obtained merely by introducing a change of parameters; it was not a consequence of separability. In the present case, on the contrary, a change of parameters is not permissible. The most we can do is make a change of scale such as replacing u by a function of u. That the linear element has the same coefficients for du^2 and dv^2 is a result which had to be proved from separability of the equation. Also since the u and v co-ordinates cannot be changed at will, the methods formerly used to simplify the equations are not available here; but there is some compensation in that the form of F is now known.

5.2 The term free of unknowns-- The foregoing results follow from consideration of the coefficients of the differentiated terms. From the other terms we obtain an equation analogous to (1.8), namely

$$(5.4) \qquad (\alpha+\beta) F_4 = \alpha_1(u) + \beta_1(v) .$$

In combination with (1.8), which we have seen is always satisfied, (5.4) gives

$$(5.5) \qquad (\alpha+\beta)\, A^2(F_2+G_2+H_2) = (\alpha+\beta)\, h_4(w) + \alpha_1(u) + \beta_1(v).$$

Using (4.3) for $A^2(F_2+G_2+H_2)$ and taking $F^2 = \alpha+\beta$, we find from (5.5)

$$(5.6) \qquad -\frac{f_{uu}+f_{vv}}{8(f+H)} + \frac{H''(\alpha+\beta)}{4(f+H)} = (-)\text{ same},$$

It may be shown that this relation, like (1.8), is a consequence of the others, and we omit the details. The proof is closely analogous to that presented in full in Sec.4.6.

5.8 Transformations leaving equations invariant-- It is convenient to note the changes in functions or variables which leave the equations essentially unaltered. As before, use of such properties permits simplification of the methods used. By inspection of the linear element (5.3), or of the equations themselves, we see that the following transformations will not lead to any essential change, if the k_i are constant:

(i) $\quad f \to k_0 f$

(ii) $\quad \alpha \to k_1\alpha \qquad \beta \to k_1\beta$

(iii) $\quad \alpha \to \alpha + k_2 \qquad \beta \to \beta - k_2$

(iv) $\quad u \to f(u) \qquad v \to g(v)$

(v) $\quad \alpha \to \beta\ , \quad u \to v$

DETERMINATION OF α AND β

6.0 <u>An ordinary differential equation when $H' \neq 0$</u>-- Starting from (2.9), following through the derivation of (2.6) in detail, and using $F = \sqrt{\alpha+\beta}$, we find

$$(6.1) \qquad c_1 = \frac{\alpha'' + \beta''}{2(\alpha+\beta)^2} - \frac{\alpha'^2 + \beta'^2}{2(\alpha+\beta)^3}$$

where c_1 is constant. This result assumes $H' \neq 0$. In (6.1) we hold v constant and write α for $\alpha+\beta$ to obtain

$$2c_1\alpha^3 = \alpha(\alpha'' + c_2) - (\alpha'^2 + c_3)$$

which becomes

$$2c_1\alpha^3 = \alpha\left(p\frac{dp}{d\alpha} + c_2\right) - (p^2 + c_3)$$

when we take α instead of u as the independent variable, $p = \alpha'$ instead of α as unknown, and use $\alpha'' = p\frac{dp}{d\alpha}$. This in turn gives

$$(6.2) \qquad \frac{ds}{d\alpha} - \frac{2s}{\alpha} - 4c_1\alpha^2 + 2c_2 - \frac{2c_3}{\alpha} = 0$$

upon introduction of a new variable $s = p^2 = \alpha'^2$. The integrating factor is $1/\alpha^2$ and leads to

$$(6.3) \qquad s = \alpha'^2 = 4c_1\alpha^3 + 2c_2\alpha - c_3$$

The symmetry of the equations noted in item (v) of Sec.5.3 shows that we have a relation of the same type for β. Equation (6.1) becomes a polynomial in α and β , as we see by computing α'' and β'' from (6.3). Multiplying (6.1) by $(\alpha+\beta)^3$ we find that $\alpha'^2+\beta'^2$ must be zero when $\alpha = -\beta$. It follows, then, that the equation corresponding to (6.3) for β is

$$\beta'^2 = 4c_1\beta^3 + 2c_1\beta + c_3 \,. \tag{6.4}$$

Substitution in (6.1) leads to an identity, and hence there is no additional condition. Because the differential equation (6.3) also arises in the case H' = 0, as we shall see below, discussion of its solutions has been deferred to Sec. 6.4.

6.1 A partial differential equation when H'= 0-- Turning now to the case in which H is constant, we absorb in into the function f, to obtain a simplified form of Eqs. (2.10) and (2.9), equations (2.4) and (2.11) remaining the same. If we take new independent variables $\phi = \log f$ and $\theta = \log F^2 = \log(\alpha+\beta)$ these relations assume a form involving derivatives only, not the unknown functions. Specifically from (2.4), (2.10), (2.11), (2.9) in that order we find the following:

$$2\phi_{uv} + 2\phi_u\phi_v = \theta_u\phi_v + \phi_u\theta_v \tag{6.5}$$

$$\phi_u^2 + \phi_v^2 = \phi_{uu} + \phi_{vv} \tag{6.6}$$

$$\phi_{uu} - \phi_{vv} + \phi_u^2 - \phi_v^2 = \theta_u\phi_u - \theta_v\phi_v \tag{6.7}$$

$$\theta_{uu} + \theta_{vv} = 2(\phi_{uu} + \phi_{vv}) \tag{6.8}$$

Because of the form of θ we also have

(6.9) $$\theta_{uv} + \theta_u \theta_v = 0 .$$

Let us differentiate the relation

(6.10) $$2(\phi_u^2 + \phi_v^2) = \theta_{uu} + \theta_{vv},$$

obtained from (6.6) and (6.8), with respect to u to find

(6.11) $$4(\phi_u \phi_{uu} + \phi_v \phi_{uv}) = \theta_{uuu} + \theta_{uvv} .$$

Upon eliminating the second derivatives ϕ_{uu} and ϕ_{uv} by use of the other relations, one finds that the terms involving ϕ all cancel, leaving the equation in θ

(6.12) $$\theta_u (\theta_{uu} + \theta_{vv}) = \theta_{uuu} + \theta_{uvv} .$$

It will be seen that α and β can be determined from this.

6.2 <u>Complete solution of the case $H' = \beta' = 0$ ---</u> If β does not depend on v, we may replace α by $\alpha + \beta$, as suggested in Sec. 5.3 item (iii), and β by $\beta - \beta = 0$. We have then

(6.13) $$\theta_u \theta_{uu} = \theta_{uuu}$$

from (6.12), with θ equal to log α. Letting $\psi = \theta_u = \alpha'/\alpha$ substituting and integrating we get

$$\psi^2/2 = \psi' - c^2/2$$

with c constant. If $c \neq 0$ we have $\psi = c\tan(u/2 + c')$ but if $c = 0$ then $\psi = -2/(u + c')$. As the general solutions we thus find

$$(6.14) \qquad \alpha = \frac{c''}{\cos^2(uc + c')} \qquad \text{or} \qquad \alpha = \frac{c'}{(u+c)^2}.$$

In view of the equivalence (iv) of Sec. 5.3 the only essentially distinct cases are $\alpha = \operatorname{csch}^2 u$, $\csc^2 u$, or $1/u^2$.

6.3---<u>An ordinary differential equation for the case $H' = 0$ but $\alpha'\beta' \neq 0$</u>-- Suppose now that $\alpha'\beta' \neq 0$, but retain the assumption $H' = 0$. With θ replaced by its value $\log(\alpha + \beta)$, Eq. (6.12) now gives

$$(6.15) \qquad \alpha'''(\alpha+\beta) - 4\alpha''\alpha' - 2\alpha'\beta'' + 3\alpha' \frac{\alpha'^2 + \beta'^2}{\alpha + \beta} = 0,$$

Proceeding as in the derivation of (6.2) we let u be constant to find

$$(6.16) \qquad \frac{ds}{d\beta} - \frac{3s}{\beta} + c_1\beta + c_2 + \frac{c_3}{\beta} = 0$$

where $s = \beta'^2$ and β has been written for $\alpha + \beta$. The integrating factor is $1/\beta^3$ and it gives, after integration and multiplication by β^3,

$$(6.17) \qquad s = \beta'^2 = A + B\beta + C\beta^2 + D\beta^3.$$

By the symmetry noted in (v) we have also

$$\alpha'^2 = a + b\alpha + c\alpha^2 + d\alpha^3 \tag{6.18}$$

where in both cases the coefficients are constant.

Inspection of (6.15) tells us that $\alpha'^2 + \beta'^2$ must be zero whenever $\alpha = -\beta$, and hence if α'^2 is given by (6.18), then β'^2 is uniquely determined as

$$\beta'^2 = -a + b\beta - c\beta^2 + d\beta^3 . \tag{6.19}$$

It may be verified that (6.15) is now satisfied, and hence that there is no additional condition. Since these equations are equivalent to (6.3) and (6.4), obtained for the case $H' = 0$ (cf. Sec. 6.3), their solution completes the determination of α and β. It is seen, incidentally, that if $\alpha(u)$ satisfied (6.18), then the function $-\alpha(iv)$ will satisfy (6.19).

6.4 Canonical forms of the equation-- Let us make use of Sec. 5.3 to simplify (6.18) and (6.19). Writing $-\alpha$ for α will make $d > 0$, if originally $d < 0$; and replacing α by $A\alpha + B$, u by u/C, we get

$$A^2\alpha'^2 C^2 = a + b(A\alpha + B) + c(A\alpha + B)^2 + d(A\alpha + B)^3 . \tag{6.20}$$

Supposing that d has been made > 0 we set

$$a + bB + cB^2 + dB^3 = 0 \tag{6.21}$$

$$b + 2cB + 3dB^3 = dA^2 \tag{6.22}$$

and choose $C = \sqrt{Ad}$ to obtain the canonical form

$$\alpha'^2 = \alpha^3 + \lambda\alpha^2 + \alpha \, . \tag{6.23}$$

The corresponding substitution for β must be $\beta \rightarrow A\beta - B$ to be permissible; it gives (6.23) with α replaced by β and λ by $-\lambda$.

We see that A,B, and C may be assumed real (this is important), and that $A = 0$ is necessary only when there is a triple root of the original equation. These observations follow from the facts that the left hand side of (6.22) is the derivative of (6.21) and that $d > 0$. For a triple root the canonical form is

$$\alpha'^2 = \alpha^3, \qquad \beta'^2 = \beta^3 . \tag{6.24}$$

If we suppose now that $d = 0$ but $c \neq 0$ we find, in a similar way,

$$\alpha'^2 = \alpha^2 + \lambda\alpha + 1, \quad \beta'^2 = -\beta^2 + \lambda\beta + 1 . \tag{6.25}$$

This form includes the double root case, which can be shown, however, to be impossible anyway. One would at first expect a ± in front of α'^2 but this is accounted for by interchanging α and β .

Next c = d = 0 gives

(6.26) $$\alpha'^2 = \alpha \quad , \quad \beta'^2 = \beta$$

and $\alpha' = \beta' = 0$ is the last possibility. In Eq.(6.23) we must distinguish the three cases $|\lambda| < 2$, $|\lambda| = 2$, $|\lambda| > 2$ but in (6.25) we have only the one case $|\lambda| > 2$, since α' and β' must be real simultaneously.

6.5 Solution of the ordinary differential equations for α and β -- We have seen that whenever H' or $\alpha'\beta' \neq 0$, the functions α and β must be solutions of one or another of the canonical differential equations in Sec. 6.4. These equations, though non-linear, may all be solved by elementary methods. The work is rather tedious, particularly since one must be careful to keep all possible solutions, and we therefore content ourselves with a single example. From (6.23) we find

$$du = \frac{d\alpha}{\sqrt{\alpha(\alpha^2 + \lambda\alpha + 1)}}$$

which leads (among other possible expressions) to [14]

$$u + c = \sqrt{\pm \frac{4}{r}}\, F\left[\frac{\sqrt{r - r'}}{r}, \phi\right] , \quad \sin^2\phi = \frac{\alpha - r}{r' - r}$$

when the roots r, r', 0 are real and unequal, that is, when $|\lambda| > 2$. Here F(x,p) is the elliptic function of the first kind. Since we want α as a function of u rather than the converse we introduce the Jacobi elliptic functions to find

$$\frac{\alpha - r}{r' - r} = sn^2\left(\frac{\sqrt{r}}{2}(u+c), \sqrt{\frac{r-r'}{r}}\right)$$

with a similar expression for β. For r and r' we substitute the proper values in terms of λ to obtain, with a new u, a one parameter family. This is for the case $0 < r' < \alpha < r$. Other cases are similarly treated, and lead to two additional expressions when $|\lambda| > 2$. The cases $|\lambda| < 2$, which gives conjugate complex roots, and $|\lambda| = 2$, which gives equal roots, are dealt with in the same way; the latter and all others in (6.24)-(6.26) give elementary functions. The results are tabulated in Sec.7.

We mention in passing that some of these solutions have been altered by a change of variable, in accord with Sec.1.3. The differential equations which they satisfy may not correspond exactly, then, to the forms here taken as cononical. No significance need be attached to this, however; these canonical forms give a simple method of getting a solution which is general but depends on at most one parameter. Once found, this general solution can then be put into optimum form by inspection. A less efficient procedure, carried out in detail for many equations, is to work directly with the cubic (6.18), and then use Sec.1.3 in connection with known properties of the solution functions to simplify the results. The calculations lead finally to the same simplified solutions as those obtained here.

DETERMINATION OF f

7.0 <u>The partial differential equations</u> --Up to this point we have shown that the linear element of orthogonal co-ordinates in Euclidean space must be of the form (5.3) if Laplace's equation is to separate. In addition H is one of the values (2.7) and α and β are each one of the values obtained in Secs.6.0--6.4. It remains only to determine f of (5.3), and we proceed to this question forthwith.

Treating first the case H' = 0, our aim is to compute the expression involving F on the right of (2.9). Assuming that $\alpha'\beta'$ or $H' \neq 0$ we have (6.18) and (6.19). Differentiating both sides with respect to u and dividing by $2\alpha'$ we get

$$(7.1) \qquad \alpha'' = \frac{3}{2}a\alpha^2 + b\alpha + \frac{c}{2}$$

with a similar expression for β''. Since $F = \sqrt{\alpha+\beta}$ we have, for the expression desired,

$$\left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v = \frac{1}{2}\left[\frac{\alpha''+\beta''}{\alpha+\beta} - \frac{\alpha'^2+\beta'^2}{(\alpha+\beta)^2}\right]$$

which reduces to

$$\left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v = (\alpha+\beta)\,a/4 \tag{7.2}$$

in view of (6.18), (6.19) and (7.1). Here a is the coefficient of α^3 in (6.18), and hence $a=1$ for the cases with canonical equations (6.23), (6.24) but $a=0$ for (6.25), (6.26).

The above assumes (6.18), which is valid only when H' or $\alpha'\beta' \neq 0$. Assuming now that $H'=\beta' = 0$ and writing α for $\alpha+\beta$ we find that the expression on the left of (7.2) becomes $\frac{1}{2}(\alpha'/\alpha)_u$. For the case $H'=\beta'=0$ it has been shown that α must be one of the three functions mentioned at the end of Sec.6.2. By direct calculation we find $\frac{1}{2}(\alpha'/\alpha)_u = \alpha$ every time, and hence (7.2) is valid with $a = 4$. Thus the expression is known in all cases.

When $H=0$ we find

$$f_{uu} + f_{vv} = \frac{f_u^2 + f_v^2}{f} + \frac{fa}{4}(\alpha+\beta) \tag{7.3}$$

$$f_{uu} + f_{vv} = 2\,\frac{f_u^2 + f_v^2}{f} \tag{7.4}$$

from (2.9) and (2.10), after making use of (7.2) and replacing F^2 by $\alpha+\beta$. These relations tell us that $f_u^2+f_v^2=0$ whenever $a=0$, so that in every non-cubic case if H is constant f must be constant also. The only remaining relations are (2.4) and (2.11), which become

(7.5) $$2f_{uv} = \frac{\beta' f_u + \alpha' f_v}{\alpha+\beta}$$

(7.6) $$f_{uu} - f_{vv} = \frac{\alpha' f_u - \beta f_v}{\alpha+\beta}$$

for the present situation $F^2 = \alpha+\beta$. Equations (7.5) and (7.6) are valid for all H.

When H is not zero the corresponding form of (2.9) and (2.10) is obtined by simply setting $F^2 = \alpha+\beta$ in the relations of Secs. 3.1 - - 3.6. Specifically, if $H = w^2$ we have

(7.7) $$f_{uu} + f_{vv} = 4(\alpha+\beta)$$

(7.8) $$f(f_{uu} + f_{vv}) = f_u^2 + f_v^2 ;$$

if $H = e^w$ then $f = 0$; if $H = \cosh(w)$ then

(7.9) $$f_{uu} + f_{vv} + 2f(\alpha+\beta) = 0$$

(7.10) $$f_u^2 + f_v^2 + (\alpha+\beta)(f^2-1) = 0 ;$$

and if $H = \sin(w)$ then

(7.11) $$f_u^2 + f_v^2 = (\alpha+\beta)(f^2-1)$$

(7.12) $$f_{uu} + f_{vv} = 2(\alpha+\beta)f .$$

It may be verified that both (2.9) and (2.10) are satisfied identically when, for a given H, the function f satisfies the appropriate pair of equations from (7.3), (7.4) or (7.7)--(7.12). These relations plus (7.5) and (7.6), then, give the necessary and sufficient condition for separation in Euclidean space.

7.1 Attempts to solve the equations directly-- It was hoped that f could be found analytically from the above equations, without reference to the theorems of differential geometry. To that end the relations for $H' = 0$, thought to be the most intractable, were extensively investigated by methods which may be summarized as follows.

First, a number of substitutions were tried, with the hope of simplifying the equations. Besides the transformation $\phi = \log f$, $\Theta = \log(\alpha+\beta)$ considered previouly, for example, we used $T = 1/\sqrt{\alpha+\beta}$, $F = Tf$ to obtain the symmetric forms

log F is harmonic

T/F is harmonic

$$\frac{F_{uu} - F_{vv}}{F} = \frac{T_{uu} - T_{vv}}{T} \quad ; \quad \frac{F_{uv}}{F} = \frac{T_{uv}}{T}$$

These are the only conditions on f when $H' = 0$. The function log F has also been taken as dependent variable, with results mentioned below.

Besides changing the unknown one may change the independent varuables u and v. If neither α nor β is constant, for example, one may use α and β as independent variables to find, after rearrangements,

$$4\,\frac{f_\alpha^2\alpha'^2 + f_\beta^2\beta'^2}{f^2} = a(\alpha+\beta)$$

$$4\,\frac{f_{\alpha\alpha}\alpha'^2 + f_\alpha\alpha''}{f} = a(\alpha+\beta) + 2\,\frac{\alpha'^2 f_\alpha - \beta'^2 f_\beta}{(\alpha+\beta)f}$$

$$4\,\frac{f_{\beta\beta}\beta'^2 + f_\beta\beta''}{f} = a(\alpha+\beta) - 2\,\frac{\alpha'^2 f_\alpha - \beta'^2 f_\beta}{(\alpha+\beta)f}$$

$$2f_{\alpha\beta} = \frac{f_\alpha + f_\beta}{\alpha+\beta}$$

as the new form of Eqs.(7.3)- - (7.6). The advantages are that the coefficients are simple rational functions, now, rather than trigonometric or elliptic functions, and that the equations remain unchanged in form for all α and β. Of course α'^2, β'^2, α'' and β'' are to be replaced by their values as found from (6.23) -- (6.26). It is rather remarkable that the equations involve only the squares of α' and β' , not the first powers, so that the final result is free of radicals.

To solve these equations, one method which was tried was successive differentiation. The system is over-determined and it becomes more so as we add relations obtained by differentiating successively with respect to u or v. It might be thought, then , that elimination of the higher derivatives would lead to a new independent relation among the first and second derivatives. One such relation, it has been proved, would allow us to obtain f or a derivative directly in terms of α and β . Since α and β are functions of a single variable, the functional form of f would thus be known, the partial differential equations would become ordinary equations, and the whole problem (perhaps) would be solved. But it has been shown, by means of the above subst-

substitution log $f/\sqrt{\alpha+\beta}$, that fewer than five differentiations will not suffice for this program, and that any number is probably insufficient. In other words so many of the new equations are dependent that the number of unknowns keeps pace with the number of equations, contrary to expectation, and no new relation is found.

As a second method we have obtained ordinary differential equations by finding relations involving only those derivatives of an unknown with respect a single variable, e.g., $W(f_u, f_{uv}, f_{uvv}, \ldots) = 0$. When the other variable-- u in our example -- is constant, this is an ordinary differential equation. The program was successful, in that such equations were actually found. But they were always too complicated to solve, being non-linear and at bese of the first order and second degree, or of the second order and first degree.

Still another procedure is to use the general solution of Laplace's equation in two dimensions. Since log F is harmonic, for example, we have

$$F = \xi(u+iv)\,\eta(u-iv)$$

and thus the other relations may be expected to give ordinary differential equations for ξ or η , if one variable is regarded as constant. By subsituting and eliminating one finds that such is indeed the case. We have, for example,

$$\frac{\eta''}{\eta} = \frac{3}{16}\frac{\alpha'^2}{(\alpha+\beta)^2} - \frac{1}{8}\frac{\alpha''-\beta''}{\alpha+\beta} \qquad \text{(v constant)}$$

which is an equation of the Mathieu type.

To obtain this result and others of a similar kind one must use the following artifice. There is always a value of v which makes β equal to zero, as we see by inspection of the functions Use of an appropriate canonical form for the differential equation -- such as (6.23), for example -- allows us to assume $\beta'=0$ as well. There exists a particular v then, for which $\beta=\beta'=0$. If we pick this particular value for the constant value in our equations, we find that certain awkward terms involving square roots drop out, and thus the desired result is found. Since the argument of η still involves the variable u, and since η is really a function of only one variable, we lose no essential information by confining our attention to this special value of v.

These devices were tried in all systems obtained by the above changes of independent or dependent variables, and the investigation consumed much time-- more, in view of later developments, that the importance of the problem warrents. It is felt, therefore, that this brief summary should be included even though the results are inconclusive.

7.2--<u>An indirect method of solving the equations for f--</u> To obtain all possible linear elements we must have all possible values of f; in other words we must have the general solution of the above differential equations. The preceding discussion suggests that this general solution is not easy to obtain directly, although perhaps a particular solution can be found by various artifices. In view of these considerations we use an indirect procedure, which depends on the fact that α and β are already known for every case.

Given a particular linear element ds^2 we note from (5.3) that all others using the same α and β are obtained from this one by means of a conformal transformation, with ds^2 now regarded as the metric of a Riemannian V_3. Thus, since α and β are to be the same in both cases we can change f and H only, and hence the two elements are proportional. By a theorem of Liouville [15], the most general conformal transformation which preserves the Euclidean character of the space--i.e., preserves the relations of Sec. 2-- is a rigid motion, a reflection, or an inversion. Only the latter is of interest here. Hence given α and β , if we can somehow discover just one admissible f and H, then all others can be found from this by inversion.[1]

1) A similar use of Liouville's theorem is made in Ref. 8.

To find a single value of f one may use the differential equations. Actually, however, by putting the linear elements of [3] into our form (5.3) we find that most of our values for α and β are there represented. Since the linear element is given in full in Ref. [3], we simply read off f and H to obtain one permissible pair of values. Finally, the co-ordinate transformation giving this linear element is likewise found in[3] , whence the new result obtained by inversion in a general sphere can be written down by inspection. Specifically, if

$$\bar{x} = c + \frac{r^2}{|x-c|^2}(x-c) \tag{7.13}$$

gives the new vector $\bar{x}$ from the origin to the co-ordinate surface, in terms of the old vector x and the vector c to the center of the sphere used for inversion, then the new linear element $\overline{ds}^2$ is related to the old one by

$$\overline{ds}^2 = \frac{r^4}{|x-c|^4}\, ds^2 \tag{7.14}$$

with r as the radius of the sphere. In our use of (7.13) and (7.14), which are well known and easy to verify, we may take $r=1$ with no loss of generality.

7.3 Illustrations--As an example, consider the case

$$\alpha = \operatorname{csch}^2 u, \quad \beta = \csc^2 v$$

which arises from (6.23) with $\lambda=-2$. This is identified with III_2 in [3] after renaming the variables and dividing by sinh u and sin v. We thus find

$$f = \csc(v)\operatorname{csch}(u), \qquad H = 0$$

and these satisfy the appropriate equations. After inversion one finds, by means of (7.14) and the co-ordinate transformation in [3],

$$\bar{ds}^2 = \frac{ds^2}{\left[(\sinh(u)\sin(v)\cos(w)-a)^2+(\sinh(u)\sin(v)\sin(w)-b)^2+(\cosh(u)\cos(v)-c)^2\right]^2}$$

This has the proper form, with the new f and H given by

$$f = \frac{1+a^2+b^2+c^2}{\sinh(u)\sin(v)} + \frac{\sinh(u)}{\sin(v)} - \frac{\sin(v)}{\sinh(u)} - 2c\,\frac{\cosh(u)\cos(v)}{\sinh(u)\sin(v)}$$

$$H = -2a\cos(w) - 2b\sin(w)$$

The values are then substituted in the equations appropriate to the case in question, which are found to be satisfied for all values of a, b, and c. The calculations are very laborious and we omit them.

As a second illustration let us take the elliptic case (6.23) with $|\lambda|>2$, which by virtue of $cn^2+sn^2=1$ and item (iii) of Sec. 5.3 is found to correspond to the case II_4 in [3], when we introduce a new variable e^w for one of the variables in [3]. We have then

$$\alpha = k^2 cn^2(u,k)$$

$$\beta = k'^2 sn^2(v,k')$$

$$f = 0 \qquad H = e^{-w} \qquad (k^2+k'^2) = 1$$

The appropriate equations are all satisfied. Using the co-ordinate transformation given in [3] we find

$$\bar{ds}^2 = \frac{ds^2}{\left\{[e^w dn(u,k)sn(v,k')-a]^2+[e^w sn(u,k)dn(v,k')-b]^2+[e^w cn(u,k)cn(v,k')-c]^2\right\}^2}$$

By means of the identities

$$\mathrm{sn}^2 + \mathrm{cn}^2 = 1, \quad \mathrm{dn}^2 + k^2\mathrm{sn} = 1$$

one finds that this too is in the correct form, with

$$f = -2a\,dn(u,k)\,sn(v,k') - 2b\,sn(u,k)\,dn(v,k') - 2c\,cn(u,k)\,cn(v,k')$$

$$H = e^{W} + (a^2+b^2+c^2)\,e^{-W} = A\cosh(W+B) \rightarrow \cosh(w)$$

Procedure similar to the above has been carried out for each value of $\alpha+\beta$. First the linear element is identified with one of those in [3], which identification involves nothing more elaborate than renaming the variables, division by a function, or an occasional transformation of the type $w \rightarrow e^{w}$, with the resulting modifications. For the case XII of the table in Sec. 8.0, to be sure, the procedure breaks down, since the linear element does not occur in [3] ; but we recognize the co-ordinates as toroidal, and for these the linear element is given in [11] .

The values of f and H thus obtained are checked by substitution into the appropriate equations. Next it is verified that the co-ordinate transformation in [3] actually gives the linear element, and the new element obtained after inversion is written down, the information for case XII being found as before from [11]. In every case the result after inversion turned out to be of the form (5.3) with no restriction on the center (a,b,c). The new values of f and H are finally substituted into the appropriate equations, an operation which is tedious but elementary. The equations turn out always to be satisfied, and we have thus completed the proof of Theorems III and IV.

CONCLUDING REMARKS

8.0 Permissible linear elements tabulated-- As remarked previously, the co-ordinate system is completely determined by its linear element. In the following table we present the linear elements for all Euclidean co-ordinate systems in which the Laplace equation separates. The second entries for f and H in a given case were found by inversion.

So far nothing has been said about the entries II--IV, which are apparently not to be found in [3] . We shall see later, however, that all conditions for separation of the wave equation are satisfied by these linear elements. Also it will be found that $R = 1$. It follows, therefore, that II--IV must actually be contained in [3] after all, that is, there must exist a suitable change of scale which will lead to one of the forms there given. The other entries in the table, however, include every linear element of [3] which has the form (5.3). We conclude in this way that the entries II--IV must be equivalent to one of the others in the table and it is unnecessary, therefore, to compute the inversions.

The above reasoning depends strongly on the assumption that Ref. [3] is free of error, and for this reason we have retained the forms II--IV in the table. It would be desirable to have a direct geometric proof of the equivalence, a question which is reserved for subsequent investigation. Throughout the present discussion, however, we write as though the equivalence is established.

Table giving the linear elements ds^2 of all Euclidean co-ordinate systems in which Laplace's equation separates when the solution has the form

$$R(u,v,w)X(u)Y(v)Z(w)$$

The second values of f and H for given α and β were found by inversion of the first. The linear element is

$$ds^2 = \frac{[\alpha(u)+\beta(v)][du^2+dv^2]+dw^2}{[f(u,v)+H(w)]^2}$$

CASE	CORRESPONDING CASE IN [B]	α, β AND λ IN CANONICAL EQUATION	H	f
		$\alpha'^2 = \alpha(\alpha^2+\lambda\alpha+1)$; $\beta'^2 = \beta(\beta^2-\lambda\beta+1)$		
I	II_4	$\alpha = k^2 cn^2(k,u)$ $\beta = k'^2 cn^2(k',v)$ $\lvert\lambda\rvert > 2$	e^w	0
			$\cosh(w)$	$\left[\begin{array}{l} a\,dn(u,k)\,sn(v,k') + \\ + b\,sn(u,k)\,dn(v,k') + \\ + c\,cn(u,k)\,cn(v,k') \end{array}\right] \frac{2}{\sqrt{a^2+b^2+c^2}}$
II	—	$\alpha = k^2 sn^2(k,u)$ $\beta = cs^2(k',v)$ $\lvert\lambda\rvert > 2$	e^w	0
			—	—
III	—	$\alpha = ns^2(k,u)$ $\beta = cs^2(k',v)$ $\lvert\lambda\rvert > 2$	e^w	0
			—	—
IV	—	$\alpha = [ns(k,u) \pm cs(k,u)]^2$ $\beta = [ns(k',v) \pm cs(k',v)]^2$ $\lvert\lambda\rvert < 2$	e^w	0
			—	—
V	III_2	$\alpha = \csc^2(u)$ $\beta = \mathrm{csch}^2(v)$ $\lvert\lambda\rvert = 2$	0	$\mathrm{csch}(u)\csc(v)$
			$\sin(w)$	$\frac{a^2+b^2+\sinh^2 u+\cos^2 v-2b\cosh u\cos v}{2a\sinh(u)\sin(v)}$

VI	III_2	$\alpha = \sec^2(u)$ $\beta = -\operatorname{sech}^2(v)$ $\lvert\lambda\rvert = 2$	0	$\operatorname{sech}(v)\sec(u)$
			$\sin(w)$	$\left[\begin{array}{l} a^2+b^2+ \\ +\cosh^2(u)+\sin^2(v)- \\ -2b\sinh(u)\sin(v) \end{array}\right]\dfrac{1}{2a\cosh(u)\cos(v)}$
VII	$\alpha'^2 = \alpha^3$			$\beta'^2 = \beta^3$
	III_1	$\alpha = \frac{1}{u^2}$ $\beta = \frac{1}{v^2}$	0	$(uv)^{-1}$
			$\sin(w)$	$\dfrac{(u^2+v^2)^2+a^2+b^2+2b(v^2-u^2)}{4auv}$
VIII	$\alpha'^2 = \alpha^2+\lambda\alpha+1$			$\beta'^2 = -\beta^2+\lambda\beta-1$
	II_2	$\alpha = \frac{1}{2}\cosh(2u)$ $\beta = -\frac{1}{2}\cos(2v)$ $\lvert\lambda\rvert > 2$	0	1
			w^2	$(\cosh(u)\cos(v)-a)^2+(\sinh(u)\sin(v)-b)^2$
IX	$\alpha'^2 = \alpha$			$\beta'^2 = \beta$
	II_3	$\alpha = u^2$ $\beta = v^2$	0	1
			w^2	$\frac{1}{4}(u^2-v^2-a)^2+(uv-b)^2$
X	$\alpha'^2 = 0$			$\beta'^2 = 0$
	I_1	$\alpha = 1$ $\beta = 1$	0	1
			w^2	u^2+v^2

	$\alpha'^2=\alpha(\alpha^2+\lambda\alpha+1)$		$\beta'^2=0$	
XI	II_1	$\alpha=\csc^2(u)$	$\odot$	$e^{-v}\csc(u)$
		$\beta=0$, $\lambda=2$	$\sin(w)$	$\dfrac{a\cosh(v)+b\cos(u)}{\sin(u)}$
XII	—	$\alpha=\operatorname{csch}^2(u)$	$\odot$	$[\cosh(u)-\cos(v)][\sinh(u)]^{-1}$
		$\beta=0$, $\lambda=2$	$\sin(w)$	$\begin{bmatrix}\sinh(u)-2c\sin(v)+\sin^2(v)-\\-(a^2+b^2+c^2)(\cosh(u)-\cos(v))^2\end{bmatrix}\times$ $\times\left[2\sqrt{a^2+b^2}\,\sinh(u)(\cosh(u)-\cos(v))\right]^{-1}$
	$\alpha'^2=\alpha^3$		$\beta'^2=0$	
XIII	I_2	$\alpha=\dfrac{1}{u^2}$	$\odot$	$\dfrac{1}{u}$
		$\beta=0$	$\sin(w)$	$\dfrac{a^2+u^2+v^2}{2au}$

8.1 Concerning inversion--In the foregoing work we found by actual trial that all inversions of permissible co-ordinates were themselves permissible, if by permissible we mean that the space is Euclidean and the equation separates. The new linear element had the form (5.3) and the other conditions (7.3)--(7.12) were always satisfied, independently of the center of inversion. We propose now to give a direct proof that such must necessarily be the case.

First, since the original linear element was permissible it had the form (1.5) with $F^2 = \alpha+\beta$. Hence, because an inversion is a comformal transformation, the new linear element will also have this form with a different A, say $\bar{A}$. The equation will spearate if $\bar{R}$, the new value of R, is given by $\bar{R}^2\bar{A} = 1$. This is true provided only that the relations (1.8) and (5.5) continue to hold in the new system, and we shall see that such is the case.

Since the original linear element was permissible it was obtained from a co-ordinate system in Euclidean space. Hence the new one, having been found by inversion, will have the same property. Now the relations (1.8) and (5.5), which are the only remaining conditions for seperation of the equation, have been proved to be satisfied automatically whenever the space is Euclidean. They are therefore satisfied by the new linear element, and hence the equation separates. Moreover, the fact that A has the form (2.3) was also deduced from the Euclidean relations, and hence will persist for $\bar{A}$; and the same is true of all the differential equations for f and H. Thus the linear element obtained by inversion satisfies every one of our conditions if the original linear element does. These remarks apply to partial as well as complete separation.

What we have shown is that the set of all co-ordinates in which we optain separation RSZ or RXYZ, as the case may be, is closed under inversion. If a given co-ordinate system is in the set then the system obtined by inverting it in any sphere will likewise be in the set. This behaviour of course is not found when we confine ourselves to the case $R = 1$.

It must be mentioned that these results on inversion, though interesting mathematically, are of slight practical importance. The way one would actually solve Laplace's equation in co-ordinates which are inversions of standard cases would be to solve the standard case and then invert the solution. The fact that the equation could be solved directly by separation of variables in the new co-ordinate system, though true, would not be used in paractice.

8.2. Conditions for which $R = 1$-- We shall have $R = 1$ when and only when it can be absorbed in the product XYZ, that is, when $R = p(u)q(v)r(w)$. In view of the condition $R^2A=1$ in Eq.(1.7) we see that A too must have this form. Such a condition, when combined with (2.3), makes H constant and f of the form $r(u)s(v)$, or else it makes f constant. Referring now to the table we see that these conditions hold only in cases which are found in [3]. Thus Laplaces equation separates in the sense XYZ when and only when the wave equation so separates. The "when" part of this result is well known, but the "only when" part is believed to be new.

8.3 The wave equation--In Sec.(4.3) we considered the conditions under which the wave equation would separate partially, to give solutions of the type RSZ. Turning now to the case in which there is complete separation RXYZ we encounter a difficulty when we try to show that $F = \alpha(u)+\beta(v)$. The previous argument depended on the fact that one term of (5.2) involved the separation constant m while the other did not. For the wave equation, however, we may have k depending on m, and this method therefore cannot be used. Confining our attention to the case $H=0$, which we have seen is the only case in which F need not be constant in the wave equation, we shall show that the linear element must have the form (5.3) even if the second stage of the separation goes through for but a single value of m.

To obtain this result, observe first that the condition on coefficients of the leading terms in (5.2) insure that $F=G$ in (1.4), so that the linear element has the form (1.5) Note that this does not require a change of u-v co-ordinates. Since H=0 we have the equations (6.5)--(6.8) but not (6.9), with $\theta=\log F^2$; and in addition we have (5.5) with $\alpha+\beta = F^2$, which becomes

(8.1) $$2(\phi_{uu}+\phi_{vv})-3(\phi_u^2+\phi_v^2)= \alpha_2(u) + \beta_2(v)$$

when we note that $h_4 = 0$ is permissible, in view of the fact that all quantities involve u and v only.

As an algebraic consequence of these relations we find

$$(8.2) \qquad \theta_{uu} + \theta_{vv} = 2(\phi_{uu} + \phi_{vv})$$

$$(8.3) \qquad \phi_u^2 + \phi_v^2 = \phi_{uu} + \phi_{vv}$$

$$(8.4) \qquad 2(\phi_{uv} + \phi_u \phi_v) = \theta_u \phi_v + \theta_v \phi_u$$

$$(8.5) \qquad \phi_u^2 + \phi_v^2 = \alpha_2(u) + \beta_2(v)$$

and differentiation of (8.2),(8.3) leads to

$$(8.6) \qquad 4(\phi_u \phi_{uu} + \phi_v \phi_{uv}) = \theta_{uuu} + \theta_{uvv}$$

$$(8.7) \qquad 4(\phi_u \phi_{uv} + \phi_v \phi_{vv}) = \theta_{uuv} + \theta_{vvv}$$

Differentiating (8.5) with respect to u and the result with respect to v we find

$$(8.8) \qquad \phi_u \phi_{uuv} + \phi_{uu} \phi_{uv} + \phi_v \phi_{uvv} + \phi_{uv} \phi_{vv} = 0$$

If for ϕ_{uuv} and ϕ_{uvv} we substitute the values obtained by differentiation of (8.4), then use (8.6) and (8.7) to eliminate appropriate terms involving ϕ, we find that all the terms involving ϕ disappear, to give

$$\theta_u(\theta_{uuv}+\theta_{vvv}) + 2\theta_{uv}(\theta_{uu}+\theta_{vv}) + \theta_v(\theta_{uuu}+\theta_{uvv}) = 0$$

which becomes

$$(\theta_{uu}+\theta_{vv})(\theta_u\theta_v+\theta_{uv}) = 0$$

if we use (6.6),(6.7) and (8.4) in (8.6) and (8.7). From (8.2) and (8.3) we see that

$$2(\phi_u^2+\phi_v^2) = \theta_{uu}+\theta_{vv}$$

so that $\theta_{uu}+\theta_{vv}$ vanishes only when f is constant. Unless f is constant, therefore, we conclude that

$$\theta_u\theta_v + \theta_{uv} = 0 \tag{8.9}$$

Integrating twice, and noting that the constant of integration is an arbitrary function, we find from (8.9)

$$\theta = \log[\alpha(u)+\beta(v)]$$

which tells us that F^2 has the form $\alpha(u)+\beta(v)$. If f is constant we have R constant too, in view of $R^2A = 1$, and hence the above case is the only one of interest.

Now that we have obtained the proper form for F we can use all the results of the preceding sections. Inspection of the table, which certainly represents a necessary condition on the co-ordinates for separation of the wave equation, tells us that we have f or H constant only in the well known cases giving $R = 1$, and Theorem V follows.

PART III--THE MEANING OF SEPARATION

****

SIMPLE EQUIVALENCES

9.0 Introduction--What do we mean when we say that the Laplace equation separates to give solutions XYZ? Do we mean that there exists a single non-constant solution of this form? Do we mean that there is an infinite number of such solutions? Or do we mean that the differential equation itself can be separated into two equations, and one of these further separated, as in the present text? Similar questions arise for separation of the type RXYZ, and it is our purpose now to investigate the relation of these various definitions. We already know that they are not equivalent; to ask separation of the equation as well as of the solution is a non-trivial restriction on the permissible co-ordinate systems. At least two whole catagories of co-ordinates are ruled out, namely, the cases IV_1 and IV_2 of [3], and the Dupin cyclides [7].

Considering the co-ordinates of [3] in more detail, we observe that all the others are degenerate cases of the last two [11] , i.e., of ellipsoidal and paraboloidal co-ordinates. With $\mathcal{R}=1$ we obtain separation in our sense--separation of the equation as well as the solution--for every case except the last two, whereas separation of the solution alone, in the sense of [12], is found also for the last two. The matter may be thus expressed: The solution separates to give a complete family (v. inf.) XYZ in all confocal co-ordinate systems; the equation separates in the degenerate cases only.

9.1. An auxiliary assumption--By rearranging terms we find that Eq. (1.1) can be put in the form

$$\left.\begin{aligned} &\frac{1}{f^2}\left[\frac{X''}{X}+\left\{\frac{2R_u}{R}+\frac{f}{gh}\left(\frac{gh}{f}\right)_u\right\}\frac{X'}{X}\right]+ \\ &+\frac{1}{g^2}\left[\frac{Y''}{Y}+\left\{\frac{2R_v}{R}+\frac{g}{fh}\left(\frac{fh}{g}\right)_v\right\}\frac{Y'}{Y}\right]+ \\ &+\frac{1}{h^2}\left[\frac{Z''}{Z}+\left\{\frac{2R_w}{R}+\frac{h}{fg}\left(\frac{fg}{h}\right)_w\right\}\frac{Z'}{Z}\right] \end{aligned}\right\} = -\frac{\nabla^2 R}{R} \qquad 9.1$$

where ∇^2 R is the Laplacian of R in the co-ordinates (u,v,w). Here we have assumed complete separation RXYZ, so that $S = X(u)Y(v)$. The form (9.1) resembles that used in [12] for the wave equation.

In the classical works it is assumed that there is a two parameter family of solutions R $X_{ab}Y_{ab}Z_{ab}$, but separation of the equation itself is not required. Nevertheless it is stated that the coefficients of $1/f^2$, $1/g^2$, and $1/h^2$ in (9.1) must be functions of u alone, v alone, and w alone, respectively, as a consequence of separability. Now this statement is easily proved when the equation separates--a proof is given in Sec. 1.0 of the present work--but it is far from evident in the general case. In fact we have been unable to give a proof which assumes only the existence of a two parameter family XYZ. Nevertheless the assumption is useful in a discussion of separation, since it greatly simplifies the analysis, and we shall need it in certain parts of the subsequent discussion. An idea of its scope is given by an equivalence theorem:

VI. The following statements are equivalent:

i. The quantities in brackets in (9.1) are functions of u alone, v alone, and w alone, respectively.

ii. There exist two non-proportional solutions, $RXYZ$ and $R\bar{X}\bar{Y}\bar{Z}$, which are independent in the sense that expressions like $R\bar{X}Y\bar{Z}$ are also solutions.

iii. There exists a family of solutions $(X+a\bar{X})(Y+b\bar{Y})(Z+c\bar{Z})R$ with none of the expressions $\bar{X}/X$, $\bar{Y}/Y$, $\bar{Z}/Z$ constant, and with a,b,c independent parameters

iv. By a change of scale in the u,v, and w co-ordinates the equation (9.1) can be put in the form

$$\frac{1}{f^2}\frac{X''}{X}+\frac{1}{g^2}\frac{Y''}{Y}+\frac{1}{h^2}\frac{Z''}{Z}=-\frac{\nabla^2 R}{R} \quad . \tag{9.2}$$

To prove the equivalence we shall show that i $\rightarrow$ iv $\rightarrow$ iii $\rightarrow$ ii $\rightarrow$ i (read "implies" for the $\rightarrow$). Introducing a notation that will be used throughout Part III, we write $\bar{u}$, $\bar{\bar{u}}$, and $\bar{\bar{\bar{u}}}$ for functions of u alone, and similarly for v and w. Also $\overline{vw}$ will mean a function of v and w only, and likewise for other pairs. Thus, $\bar{u}=f_1(u)$, $\bar{\bar{u}}=f_2(u)$, $\overline{uv}=f_3(u,v)$.

Condition i tells us that

$$(9.3) \qquad \frac{2R_u}{R} + \frac{f}{gh}\left(\frac{gh}{f}\right)_u = \bar{u}$$

in view of the fact that X is a function of u only; and similarly for the Y' and Z' terms. Proceeding as in Sec. 1.2 we integrate and take exponentials to get, with a new $\bar{u}$,

$$(9.4) \qquad R^2 \frac{gh}{f} = \bar{u}\ \overline{vw}$$

By a change in scale we can multiply f by u, so that the new f is $\bar{u}f$. Equation (9.4)now becomes

$$(9.5) \qquad R^2 \frac{gh}{f} = \overline{vw}$$

which tells us that

$$(9.6) \qquad \frac{2R_u}{R} + \frac{f}{gh}\left(\frac{gh}{f}\right)_u = \frac{\partial}{\partial u} \log\left(R^2\frac{gh}{f}\right) = \frac{\partial}{\partial u} \log \overline{vw} = 0$$

and hence in the new variables the coefficient of X' in (9.1) is zero. Also (9.5), like (9.4), persists in form (with a new $\overline{vw}$) if we change the v and w scales. Hence without losing (9.6) we may get an equation like it for v and w, so that i implies iv as was to be proved.

Let us show that iv implies iii. Multiplying (9.2) through by f^2 we get an equation of the form

$$\frac{X''}{X} + \phi(u,v,w) = 0 \tag{9.7}$$

where ϕ involves Y,Z and R. It is assumed that RXYZ is a solution. Hence (9.7) is valid for all v and w, and therefore ϕ must depend on u alone, since X does. The equation now becomes

$$X'' + X\phi(u) = 0 \tag{9.8}$$

which has two linearly independent solutions X and $\bar{X}$, say, with $X + a\bar{X}$ also a solution for any a. Finally, the quantity $(X''+a\bar{X}'')/(X+a\bar{X})$, corresponding to X''/X for the one parameter family, will be independent of a since it equals $\phi(u)$.

Proceeding now to the functon Y we multiply (9.2) through by g^2 to get

$$\frac{Y''}{Y} + \theta(u,v,w,a) = 0 \tag{9.9}$$

if we assume that $X+a\bar{X}$ has been written for X. As before, θ is independent of u and w, and in view of the above it is also independent of a. We complete the discussion for Y as for X above, then proceed similarly for Z to obtain iii.

That iii implies ii is evident, and hence it remains only to show that ii implies i. To this end let us write down (9.1) for XYZ and for $\bar{X}YZ$, subtract, and multiply by

f^2 to obtain

$$\frac{X''}{X} - \frac{\bar{X}''}{\bar{X}} + \left\{ \frac{2R_u}{R} + \frac{f}{gh}\left(\frac{gh}{f}\right)_u \right\}\left(\frac{X'}{X} - \frac{\bar{X}'}{\bar{X}}\right) = 0 \quad .$$

Now the coefficient of the quantity in braces is not zero, since X and $\bar{X}$ are assumed not to be proportional. Hence we can solve for this quantity and conclude, therefore, that it is a function of u alone. Similar procedure gives the result for the other two coefficients in (9.1) and the proof of Theorem VI is thus complete.

9.1 <u>Separation of the type $RX_{ab}Y_{a}Z_{b}$</u>--Suppose that the equation can be separated into three equations by the operations of Sec. 1.0. This is equivalent to assuming that the terms involving u and v are already separated, so that we may take $J = 1$ in Sec. 5.1 For X in the solution RXYZ we obtain a differential equation

$$\frac{X''}{X} + \bar{u}\,\frac{X'}{X} + \bar{\bar{u}} = c$$

and for Y and Z we have the two others obtained by cyclic permutation of the variables (XYZ), (uvw), (abc). Also $a+b+c = 0$.

It follows, then, that we get a two parameter family of the type $X_{-a-b}Y_a Z_b, = X_{ab}Y_a Z_b$ Is it true conversely that such a two parameter family leads to a co-ordinate system for which the equation separates completely in one step, and in what case can either situation arise?

With regard to these questions, which represent the simplest example of the type of question with which we are here concerned, the following theorems have been proved. It must be emphasized that when we speak of a two parameter family, we do not count parameters like the a,b,c of Theorem VI item iii, which leave X"/X.... unaltered.

VII. A necessary and sufficient condition for the existence of a two parameter family of solutions $RX_{ab}Y_aZ_b$, together with a condition in VI, is that

$$ds^2 = \frac{1}{R^4}\left[\bar{u}(du^2+dv^2)+dw^2\right] \tag{9.10}$$

and

$$\frac{\nabla^2 R}{R^5} = \bar{\bar{u}} + \frac{\bar{v}}{\bar{u}} + \bar{w} \; . \tag{9.11}$$

To show neeessity we may take iv as the statement in VI which is satisfied, so that we have (9.2). Differentiating[1]

1) This procedure goes back to Stackel [5] ; it is also used in [12] .

with respect to a and b we find

$$\frac{\bar{\bar{u}}}{f^2} + \frac{\bar{v}}{g^2} = 0 \; ; \qquad \frac{\bar{\bar{u}}}{f^2} + \frac{\bar{\bar{w}}}{h^2} = 0 \tag{9.12}$$

with $\bar{u}.... \neq 0$ since it is assumed that X"/X... depend on a or b. Absorbing $\bar{u}$ in f, $-\bar{v}$ in g, and $\bar{w}$ in h by changes of scale, we get

$$f^2 = g^2 = -\bar{\bar{u}}h^2$$

and hence, writing A^2 for h^2 and $\bar{u}$ for $-\bar{\bar{u}}$,

$$ds^2 = A\left[\bar{u}(du^2 + dv^2) + dw^2\right] \quad . \tag{9.13}$$

By item iv of VI we know that (9.5) is satisfied and hence from (9.13)

$$R^2 \frac{gh}{f} = R^2 A = \overline{vw} \; ; \quad R^2 \frac{hf}{g} = R^2 A = \overline{uw}$$

It follows that $R^2A = \bar{w}$. Absorbing w in Z of the solution RXYZ we find $R^2A=1$ and hence (9.13) becomes (9.10). Equation (9.2) is now

$$\frac{R^4}{\bar{u}} \frac{X''}{X} + \frac{R^4 Y''}{\bar{u} Y} + R^4 \frac{Z''}{Z} = - \frac{\nabla^2 R}{R} \tag{9.14}$$

which gives (9.11) after division by R^4.

For sufficiency of the condition we note that (9.10) gives i in VI, hence every statement in VI. Also (9.1) reduces here to (9.14); dividing by R^4 and using (9.11) we find that the equation separates to give

$$\begin{aligned} &\frac{Z''}{Z} - \overline{W} = b \\ &\frac{X''}{X} + \frac{1}{\bar{u}} \frac{Y''}{Y} + b = \bar{\bar{u}} + \frac{\overline{Y}}{\bar{u}} \end{aligned} \tag{9.15}$$

which leads to the form $RX_{ab}Y_aZ_b$ when we multiply (9.15) by u and separate again.

In the above we did not obtain complete separation at the first step. This can occur only when the linear element is conformal Euclidean, as we see by Theorem VIII.

VIII. The following statements are equivalent:

i. With solution RXYZ, the Laplace equation separates into three equations after being multiplied by a suitable function A^2.

ii. There exists an R and A^2 such that Z separates off as in Sec. 2; for another R and A^2 we can make Y separate off; and for a third we can make X separate off.

iii. The equation admits a two parameter family of the form $RX_{ab}Y_aZ_b$ with $\partial X/\partial a = \partial X/\partial b$ (compare VII), and VI is satisfied.

iv. The linear element is

$$ds^2 = (1/R^4)(du^2+dv^2+dw^2) \tag{9.16}$$

and

$$\nabla^2 R/R^5 = \bar{u}+\bar{v}+\bar{w} \tag{9.17}$$

For the proof we shall show that i implies every statement and conversely. That i implies ii is evident, since we may use the same R and A in each case; that i implies iii follows from actually carrying out the separation, whence it is seen as above that we finally get a family of the type $X_{-a-b}Y_aZ_b$. Also i implies iv, since it makes $\bar{u}$ constant in (9.15), and iv implies i for the same reason. It suffices, then, to show that ii and iii imply i.

Proceeding as in Sec. 2 we get

$$(9.18)\qquad ds^2 = A^2(du^2+dv^2+\overline{uv}dw^2)$$

$$(9.19)\qquad =A_1^2(du^2+\overline{uv}dv^2+dw^2)$$

$$(9.20)\qquad = A_2^2(\overline{vw}du^2+dv^2+dw^2)$$

where we have used the full hypothesis of ii. Here $A=A(u,v,w)$ and similarly for A_1 and A_2. Setting $du^2=dv^2=0$ in (9.18) and (9.19) we get $A/A_1 = \sqrt{\overline{uv}}$, which is constant since u and v are. Thus A/A_1 is constant whenever u and v are constant, and is, therefore, independent of w. Similarly it is independent of u and v; and reasoning in the same way for A/A_2 we conclude finally that $A=A_1=A_2$. Comparing (9.18)--(9.20) we see then that $\overline{uv}$, $\overline{uw}$, and $\overline{vw}$ are all constant. Moreover $R^2A=1$ as in Sec. 2, and hence the linear element has the form (9.16). Also since the A's are equal the R's must be equal, whence we obtain (9.17). Since iv implies i it follows, finally, that ii implies i.

To show that iii implies i we set $\overline{u}=\overline{\overline{u}}$ in (9.12). We thus find that the space is conformal Euclidean, and since VI is satisfied the remainder of the proof causes no difficulty.

9.3 <u>Separation of the type</u> $RX_{ab}Y_{ab}Z_b$--Separation of the equation as carried out in Parts I and II of the present text gives a one-parameter family Z and a differential equation, dependent on this same parameter, for X and Y. Separating the latter equation introduces another parameter for X and the same one for Y, whence we obtain finally a family $RX_{ab}Y_{ab}Z_b$. We inquire now whether the converse is true, that is, whether a family of solutions so parametrized always insures that the equation can be so separated. This and similar questions are answered in Theorem IX.

IX. The following statements are equivalent:

i. With solution RXYZ, Laplace's equation can be multiplied by a function to make Z separate off, and the equation so obtained for X and Y can be multiplied by another function to make X and Y separate off.

ii. The equation admits a two-parameter family $RX_{ab}Y_{ab}Z_b$, and VI is satisfied.

iii. The linear element is

$$ds^2 = (1/R^4)\left[(\bar{u}+\bar{v})(du^2+dv^2)+dw^2\right] \tag{9.21}$$

and

$$\nabla^2 R/R^5 = (\bar{\bar{u}}+\bar{\bar{v}})/(\bar{u}+\bar{v}) + \bar{w} \tag{9.22}$$

We shall show that i→ii→iii→i. That iii implies i has been found in the foregoing pages; (9.22) is a consequence of (5.5) and (1.7). We have just seen that i implies ii, and hence it suffices to show that ii implies iii. For this we differentiate (9.1) or (9.2) with respect to a and b to find

$$\bar{\bar{u}}/f^2 + \bar{\bar{v}}/g^2 = 0 \tag{9.23}$$

$$\bar{u}/f^2 + \bar{v}/g^2 + \bar{w}/h^2 = 0 \tag{9.24}$$

After a suitable change of scale for u and v the first equation makes $f=g$. When this is the case the second gives $f^2=(h^2/\bar{w})(\bar{u}+\bar{v})$ which leads to

$$ds^2 = A^2\left[(\bar{u}+\bar{v})(du^2+dv^2) + dw^2\right]$$

when we change the w scale to make $\bar{w}=1$, substitute in (1.0), and write A^2 for $f^2/(\bar{u}+\bar{v})$. Since VI is satisfied we have $R^2A=1$ as usual, and (9.22) is found, finally, by using the values (9.21) in the original equation (9.2).

THE GENERAL CASE $RX_{ab}Y_{ab}Z_{ab}$

10.0. Rank of the equations--For the case in which the mode of parameterization is unrestricted we know at the outset that the equation will not separate in general even though there is a two parameter family of separated solutions. We now have the following results:

X. The following statements are equivalent:

i. Laplace's equation admits a general two parameter family of solutions $RX_{ab}Y_{ab}Z_{ab}$, and VI is satisfied.

ii. If $R \neq 1$ there are three families RXYZ, $R\bar{X}\bar{Y}\bar{Z}$ and $R\bar{\bar{X}}\bar{\bar{Y}}\bar{\bar{Z}}$ such that the 3x3 determinant with elements X''/X... does not vanish identically, and VI is satisfied. If R=1 the determinant has rank 2.

iii. The linear element can be put in the form

$$ds^2 = \frac{UVW}{R^4}\ (du^2/U+dv^2/V+dw^2/W) \tag{10.1}$$

and

$$\nabla^2R/R^5 = \frac{1}{UVW}\begin{vmatrix} \bar{\bar{\bar{u}}} & \bar{\bar{\bar{v}}} & \bar{\bar{\bar{w}}} \\ \bar{u} & \bar{v} & \bar{w} \\ \bar{\bar{u}} & \bar{\bar{v}} & \bar{\bar{w}} \end{vmatrix} \tag{10.2}$$

where U,V, and W are respectively the cofactors of $\bar{\bar{\bar{u}}}, \bar{\bar{\bar{v}}}$, and $\bar{\bar{\bar{w}}}$ in the determinant (10.2).

Let us prove first that i leads to ii. Since VI is satisfied we have (9.2), which gives

(10.3) $\bar{\bar{\bar{u}}}/f^2+\bar{\bar{\bar{v}}}/g^2+\bar{\bar{\bar{w}}}/h^2=-\nabla^2R/R$

(10.4) $\bar{u}/f^2+\bar{v}/g^2+\bar{w}/h^2=0$

(10.5) $\bar{\bar{u}}/f^2+\bar{\bar{v}}/g^2+\bar{\bar{w}}/h^2=0$

when used as it stands and then differentiated with respect to a and b. If we can show that this system has rank 2 for R=1, rank two otherwise, we shall have proved ii. Let us note that independence of the equations (10.4) and (10.5) shows that the rank is 2 when $\nabla^2R=0$; and by considering the augmented matrix of the system we see that the same condition makes the rank 3 when $\nabla^2R\neq0$. If $\nabla^2R=0$, Eq. (9.2) becomes simply the equation for XYZ, and hence the condition $\nabla^2R=0$ is equivalent to R=1 when VI is satisfied. The net result is that to prove ii we need merely prove independence of (10.4) and (10.5).

If the equations are dependent we have

(10.6) $\bar{u}/\bar{\bar{u}} = \bar{v}/\bar{\bar{v}} = \bar{w}/\bar{\bar{w}}$

and since the first term is a function of u alone, the second of v alone and the third of w alone, each term must be constant, independently of u,v, and w. Bearing in mind the method of deriving (10.4) and (10.5), we see that the first term of (10.6) now gives

(10.7) $$\partial/\partial a\ (X''/X) = K\,\partial/\partial b\ (X''/X)$$

with K a function of a and b only, not of u. If we suppose that X''/X is an analytic function of u for each a and b we may write

(10.8) $$X''/X = \sum_{k=0}^{\infty} A_k u^k$$

where the coefficients are functions of a and b. Substituting in (10.7) and equating coefficients of corresponding powers of u we get

$$\partial/\partial a\ A_k = K\,\partial/\partial b\ A_k \qquad (k=0,\ 1,\ 2,\ldots)$$

These relations tell us that the Jacobian of A_0 and A_k is zero,

$$\begin{vmatrix} \frac{\partial A_0}{\partial a} & \frac{\partial A_0}{\partial b} \\ \frac{\partial A_k}{\partial a} & \frac{\partial A_k}{\partial b} \end{vmatrix} = \frac{\partial A_0}{\partial a}\frac{\partial A_k}{\partial b}\begin{vmatrix} 1 & K \\ 1 & K \end{vmatrix} = 0$$

and hence that the functions are not independent. Thus $A_k=\phi_k(A_0)$ and the series (10.8) is a function of one parameter only, contrary to hypothesis. We remark in passing that results on rank similar to this are generally assumed without proof in the literature.

10.1.--<u>The linear element from the solution</u>--Given ii, we write down (9.2) for each solution, then subtract the last two equations from the first to obtain a system similar to (10.3)--(10.5). This time we know by hypothesis that the equations corresponding to (10.4) and (10.5) are independent, and hence we may solve them for the ratios f^2/g^2 and f^2/h^2 to get

(10.9) $$1/f^2=A^2U, \quad 1/g^2=A^2V, \quad 1/h^2=A^2W$$

after defining A^2 by the first relation. Here U, V, and W are given in Theorem X.

Using (10.9) in the square of (9.5) we get $R^4U/A^2VW=\overline{\overline{vw}}$ which gives

(10.10) $$R^4/A^2UVW = \overline{\overline{vw}}$$

when we divide by U^2, noting that U^2 is a function of v and w only. Similarly the expression in (10.10), being unchanged by permutation, must be equal to $\overline{\overline{uv}}$ and to $\overline{\overline{uw}}$. It is therefore a constant, which we may take as 1, and in view of (10.9) the linear element has the form (10.1). By putting a given ds^2 in this form we can determine R by inspection, just as was the case for the simpler situations hitherto considered.

From the Laplace equation (9.2) we now get

(10.11) $$(R^4/UVW)(UX''/X + VY''/Y + WZ''/Z) = -\nabla^2R/R$$

which leads to (10.2) when we give to U,V,W their proper interpretation as cofactors.

10.2 <u>The solution from the linear element</u>--To complete the proof of Theorem X we must show that iii implies i, in other words that a linear element of the form (10.1), plus an auxiliary condition (10.2), will always permit a two parameter family. With this end in view we substitute (10.2) into (10.11), divide by R^4/UVW, collect coefficients of U,V, and W, and make use of the cofactor definition to obtain finally

$$(10.12)\qquad \begin{vmatrix} X''/X-\bar{\bar{\bar{u}}} & Y''/Y-\bar{\bar{\bar{v}}} & Z''/Z-\bar{\bar{\bar{w}}} \\ \bar{u} & \bar{v} & \bar{w} \\ \bar{\bar{u}} & \bar{\bar{v}} & \bar{\bar{w}} \end{vmatrix} = 0$$

for the Laplace equation in our co-ordinate system.

This determinant will vanish if the rows are linearly dependent, that is, if constants a and b can be found so that

$$X''/X - \bar{\bar{\bar{u}}} + a\bar{u} + b\bar{\bar{u}} = 0$$

and similarly for Y and Z, with the same a and b. Solving these differential equations for X, Y, and Z we obtain the required two parameter family. It is clear now why the columns in (10.12) must be functions of only a single variable.

OTHER QUESTIONS

11.0 Euclidean space--To complete the investigation of separation RXYZ one must discuss the general case of Sec. 10, determining those linear elements of the form(10.1) which are also Euclidean. Details of such a program are reserved for later investigation, but one can say in a general way how the work should be done.

First, let us observe that requiring a linear element to be conformal to a certain form, and in Euclidean space, is the same thing as requiring it to be in that form and conformal to Euclidean space. This result is readily proved from the fact that the inverse of a conformal transformation is again conformal. Hence the problem at hand is equivalent to finding those linear elements that have the form

$$ds^2 = du^2/U + dv^2/V + dw^2/W$$

and are conformal to Euclidean space, with the conformality factor UVW/R^4 satisfying (10.2).

Now U, V, and W involve functions of a single variable only. Also the necessary and sufficient conditions that a ds^2 be conformal Euclidean are well known. Using these conditions, then, we should get differential equations for U, V, and W which would be ordinary differential equations in $\bar{u}$... when we regard the other variables as constant. With this approach it is possible that the problem could be readily solved.

11.1 The wave equation--For separation of the wave equation we have the conditions obtained here, generally speaking, plus certain others. It would be desirable to obtain these other conditions and to determine whether they lead to $R = 1$ in the case of Theorem X. That one must have $R = 1$ has been proved in all other cases, but the last question is still open.

11.2 Incompleteness of existing results-- Theorem X depends on VI, which has not been obtained as a consequence of separation. It would be desirable to have a proof that these results are, or are not, necessary, so that we should really known whether Theorem X gives the most general condition. It must be noted that VI has not been proved even for the case $R = 1$ and the wave equation, which is presumably the most restrictive (and therefore easiest) condition. Since the classical results on separation of the wave equation use results analogous to VI, they are, in the author's opininn, not quite conclusive.

Another question which should be considered is the possibility of non-orthogonal co-ordinates when only the solution separates. We know that cross derivatives are not permissible when the equation separates, but they may be permitted in the general case.

11.3 <u>More general types of separation</u>--It has been seen that instead of assuming separation of the type XYZ one may, with nearly equal convenience, assume the more general form RXYZ. It is natural to inquire whether still more general forms might lead to useful results in co-ordinates that are excluded by the present method. Certain forms are seen at once to be of no interest, it is true; for example R(X+Y+Z) or even RX+SY+TZ (with R,S,T fixed functions) are useless because they will never form a complete set. Also the form RXYZ+S leads to no new problems if VI is satisfied, as the term S just changes the right side of (9.2) to $-\nabla^2 R/R-\nabla^2 S/S$, and the derivation of the theorems is essentially unchanged. For any R we could pick S so that (10.2) is satisfied, and hence the necessary and sufficient condition is that the linear element be in the form (11.0) and conformal Euclidean. The case S=0 would arise when and only when R satisfies (10.2). On the other hand the most general linear form RXYZ+SXY+TXZ+UYZ+AX+BY+CZ+D might lead to something new, and in view of the discussion of Sec. 1.0 it would accomplish most of the objectives desired.

Considering these objectives more closely, one concludes that the most general expression which can be regarded as a separated solution is F(X,Y,Z,u,v,w) where F is a single fixed function and X,Y,Z form a two parameter family. The Laplace equation now becomes

$$(gh/f)(F_x X''+F_{xx}X'^2+2F_{xu}X'+F_{uu})+(gh/f)_u(F_x X'+F_u)+\text{cyclic}=0$$

and we want the condition that there exist a non-constant F for which this is satisfied. Such generalizations lead to inconvenience in that the equations to be solved, though ordinary, may be non-linear, and also the solutions will not generally turn out automatically to be an orthogonal set. But the number of possible co-ordinate systems might be greatly increased.

BIBLIOGRAPHY

1. W. Blashke, "Eine Verallgemeinerung der Theorie der Konfokalen F_2", Math. Zeit. 27, 653 (1928).

2. L. P. Eisenhart, "Riemannian Geometry", Princeton, 1926.

3. L. P. Eisenhart, "Separate Systems of Stackel", Annals of Math. 35, 284 (1934).

4. J. Weinacht, "Über die bedingt-periodischen Bewegung eines Massenpunktes", Math. Ann. 91, 279 (1922).

5. P. Stäckel, "Habilitationsschrift", Halle, 1891; also, "Über die Bewegung eines Punktes in einer n-fachen Mannigfaltigkeit", Math. Ann. 42, 537 (1893).

6. A.R. Forsyth, "Lectures on the Differential Geometry of Curves and Surfaces", Cambridge University Press (1912).

7. M. Bôcher, "Über die Reihenentwickelungen der Potentialtheorie", Leipzig (1894).

8. L. P. Eisenhart, "Stäckel Systems in Conformal Euclidean Space", Annals of Math. 36, 57 (1935).

9. E. Haskins, "Stackel System", M.I.T. Math. Dept. Thesis, (1936).

10. C. W. Murchison, Jr., "Separable Stackel Systems", M.I.T. Math. Dept. Thesis (1937).

11. H. Feshbach, Jr., P. M. Morse, "Methods of Theoretical Physics", Technology Press (1936), Chapter I.

12. H. P. Robertson, "Bermerkung über separierbare Systeme in der Wellenmechanik", Math. Ann. 98, 749, (1928).

13. W. C. Graustein, "Differential Geometry", Macmillan (1935).

14. P. Franklin, "Methods of Advanced Calculus", McGraw-Hill (1944).

15. J. Liouville, "Note au sujet de l'article précédent", J. de Mathématiques 12, 265 (1847).

ACKNOWLEDGEMENT

The author is glad to take this opportunity of expressing his gratitude to Professor Norman Levinson for generous assistance throughout the research program. The research arose in connection with a paper of Levinson's on the wave equation, and many of the techniques here used were taken from this. The derivations and results of Sec. 5 concerning separation of the equation, the use of the Euclidean character of the space, the introduction of Liouville's theorem at a crucial stage of the proceedings, and the method of changing variables to simplify the equations, as for example in Sec. 3, are examples of major ideas due to Levinson. Besides major ideas of this sort there were of course many smaller details in which the author has been helped by numerous conversations. Without this positive and active assistance the research could probably not have been completed.

The author wishes next to express his appreciation to Mr. Bruce Bogert, to whom is due the idea of introducing the second fundamental form as a tool in the research. This technique is apparently essential to the success of the enterprise. The author is indebted to Bogert also for the whole of Sec. 4.1, and for a most painstaking verification of the equations in Parts I and II.

Finally, the author is grateful to the Research Laboratory of Electronics and the Mathematics Department for the considerate and co-operative spirit in which they have granted leisure for carrying out the research.