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L. J. CHU AND D. JACKSON

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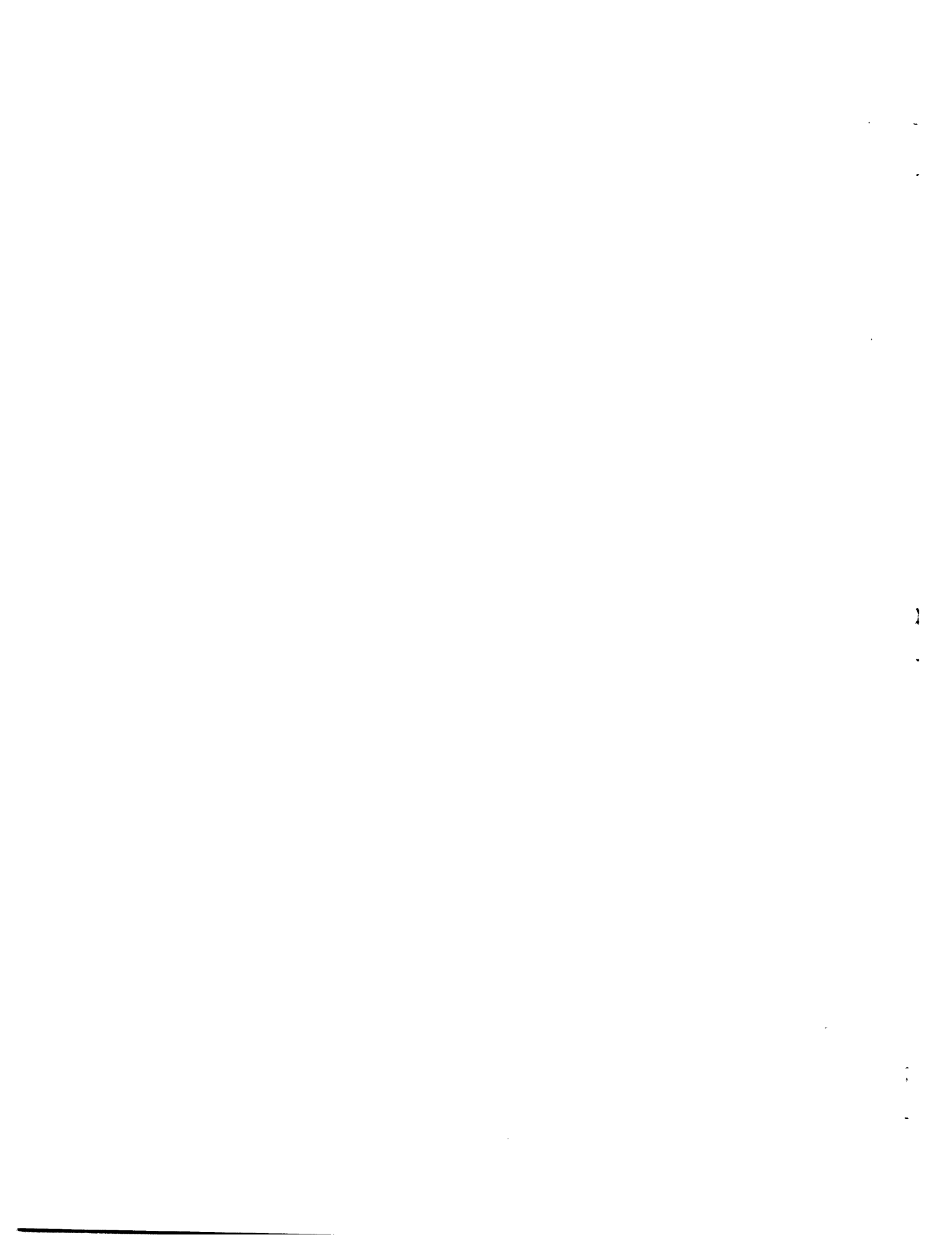
FIELD THEORY OF TRAVELING-WAVE TUBES

by

L. J. Chu and J. D. Jackson

Abstract

The problem of a helix-type traveling-wave amplifier tube, under certain simplifying assumptions, is solved as a boundary-value problem. The results indicate that the presence of the beam in the helix causes the normal mode to break up into three modes with different propagation characteristics. Over a finite range of electron velocities one of the three waves has a negative attenuation, and is thus amplified as it travels along the helix. If the electron velocity is too high or too low for net energy interaction, all three waves have purely imaginary propagation constants; no amplification occurs. Consideration of the beam admittance functions shows that during amplification the electron beam behaves like a generator, with negative conductance, supplying power to the fields through a net loss of kinetic energy by the electrons. Curves are shown for a typical tube, and the effects of beam current and beam radius are indicated. The initial conditions are investigated, as are the conditions of signal level and limiting efficiency.



FIELD THEORY OF TRAVELING-WAVE TUBES

1. Introduction

The analysis of traveling-wave tubes as amplifiers has been carried out by J. R. Pierce^{1,2} of Bell Telephone Laboratories and R. Kompfner³ of the Clarendon Laboratory. In Pierce's paper², the action of the field on the electron beam and the reaction of the beam back on the field were formulated. A cubic equation was obtained which yielded three distinct propagation constants corresponding to the three dominant modes of propagation. Kompfner followed a different line of attack and arrived at essentially the same results.

The interaction between the electrons and the waves in a traveling-wave tube can, with reasonable assumptions, be formulated completely by the field theory. Such an approach to this problem can bring out many physical phenomena useful in the design of such a tube that are not apparent from the circuit theory.

The present analysis follows the procedure which Hahn^{4,5} and Ramo^{6,7} used in dealing with the velocity-modulated tubes. The problem of the traveling-wave tube is idealized, and such approximations are introduced that the field theory can be used throughout to correlate the important factors in the problem. Numerical examples are given for a specific tube to illustrate the effects of various parameters upon the characteristics of the tube.

In this report, only the helix type of traveling-wave tube will be considered. It consists of a cylindrical helical coil which, in the absence of an electron beam, is capable of supporting a wave along the axis of the helix with a phase velocity substantially less than the light velocity. When an electron beam is shot through the helix, the electrons are accelerated or decelerated by the field of the wave, especially the longitudinal electric field. As a result, the electrons will be bunched. The bunched beam travels substantially with the initial velocity of electrons which is usually different from the phase velocity of the wave. Because of the bunching action, there will be, in time, more electrons decelerated than those accelerated over any cross section of the helix or vice versa. As a result, there will be a net transfer of energy from the electron beam to the wave or from the wave to the beam. The bunching of the electrons produces an alternating space charge force or field which modifies the field structure of the wave and consequently its phase velocity. The average energy of the electron beam must change as it moves along on account of the energy transfer. The process is continuous and a rigorous solution to the problem is probably impossible. The procedure of analysis is therefore to find the modes of propagation which can have exponential variation along the tube in the presence of the electron beam. We are interested in those modes which will either disappear or degenerate into the dominant mode when the beam is removed. By studying the properties of these modes and combining them properly, we hope to present a picture of some of the physical aspects of the helix-type traveling-wave tube.

2. Formulation and Solution of the Problem

In order to obtain some theoretical understanding about the behavior of the traveling-wave tube, we have to simplify the problem by making numerous assumptions. Instead of a physical helix, we shall use a lossless helical sheath of radius a and an infinitesimal thickness. The current flow along the sheath is constrained to a direction which makes a constant angle $(90^\circ - \theta)$ with the axis of the helix. The tangential component of the electric field is zero along the direction of current flow, and finite and continuous through the sheath along the direction perpendicular to the current flow. The force acting on the electrons is restricted to that associated with the longitudinal electric field only; and the electrons are assumed to have no initial transverse motion. We shall further assume that the electrons are confined within a cylinder of radius b concentric with the helical sheath. The time-average beam current density is assumed constant over the cross section.

The validity of the above assumptions is subject to question for a practical traveling tube. However, we should be able to reveal some of the physical phenomena associated with the tube at least qualitatively. As in all other waveguide problems, the standard procedure is to find the natural modes of oscillation or propagation along the tube without bothering about the boundary conditions at the input and output terminals of the tube. The problem is further simplified by considering small signals only.

To find the natural modes of propagation along the tube, it is convenient to divide the space into three physical regions with well-defined boundaries. First we have the region occupied by the electrons. As shown in Fig. 1, this region is cylindrical in shape and extends from $r = 0$ to $r = b$. Then, there is the region between the electron beam and the helical sheath (from $r = b$ to $r = a$). Lastly, the space outside the helix forms the third region. These three regions are separated by two well-defined boundaries. In the following, we shall find the expressions of the field appropriate for the three regions and satisfying the conditions at the boundaries, and an investigation of the properties of the field will follow.

2.1 General Field and Wave Equations. If the fields are circularly symmetric about the coordinate axis, and assumed to vary with $e^{j\omega t - \gamma z}$, then the field equations can be written in the form:

TE Wave

$$\left. \begin{aligned} \gamma E_\phi + j\omega\mu H_r &= 0 \\ \frac{1}{r} \frac{\partial}{\partial r} (rE_\phi) + j\omega\mu H_z &= 0 \\ \frac{\partial H_z}{\partial r} + \gamma H_r + j\omega\epsilon E_\phi &= -J_\phi \end{aligned} \right\} \quad (1)$$

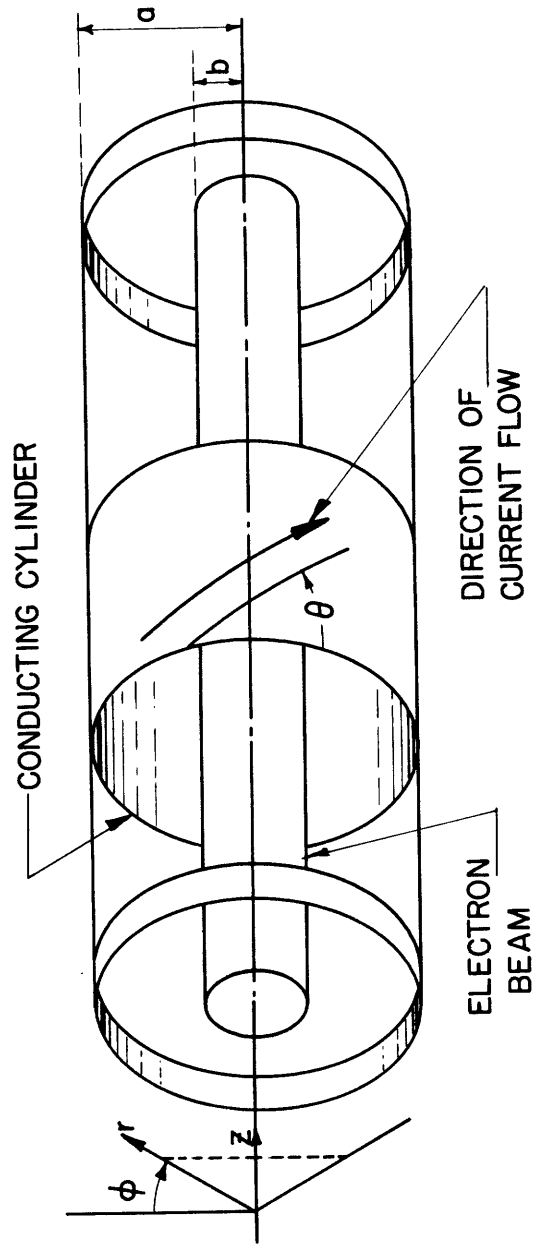


Figure 1. Idealized representation of a section of a helix-type traveling-wave tube.

TM Wave

$$\left. \begin{aligned} \gamma H_{\phi} - j\omega \epsilon E_r &= J_r \\ \frac{1}{r} \frac{\partial}{\partial r} (r H_{\phi}) - j\omega \epsilon E_z &= J_z \\ \frac{\partial E_z}{\partial r} + \gamma E_r - j\omega \mu H_{\phi} &= 0 \end{aligned} \right\} \quad (2)$$

where (z, r, ϕ) are the cylindrical coordinates,

$\gamma = \alpha + j\beta$ is the propagation constant along the z direction,

E_z, E_r, E_{ϕ} are the electric field components,

H_z, H_r, H_{ϕ} are the magnetic field components,

J_z, J_r, J_{ϕ} are the components of the vector current density.

The grouping of field components into TE and TM waves is for mathematical convenience only. All six components are required to satisfy the boundary conditions on the helical sheath. From the field equations the following inhomogeneous wave equations for H_z and E_z can be deduced:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial H_z}{\partial r} \right) + (\gamma^2 + k^2) H_z = -\frac{1}{r} \frac{\partial}{\partial r} (r J_{\phi}) \quad (3)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + (\gamma^2 + k^2) E_z = -\frac{(\gamma^2 + k^2)}{j\omega \epsilon} J_z + \frac{\gamma}{j\omega \epsilon} \frac{1}{r} \frac{\partial}{\partial r} (r J_r) \quad (4)$$

where $k^2 = \omega^2 \mu \epsilon$.

2.2 TE Wave within the Electron Beam. Since the electrons are assumed to have no transverse motion the a-c current density J has only one component, namely J_z . Thus, Eq. (3) for H_z reduces to the form of a homogeneous wave equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial H_z}{\partial r} \right) + (\gamma^2 + k^2) H_z = 0 \quad (5)$$

The other components of the TE wave are given by:

$$E_{\phi} = \frac{j\omega \mu}{\gamma^2 + k^2} \frac{\partial H_z}{\partial r}, \quad H_r = \frac{-\gamma}{\gamma^2 + k^2} \frac{\partial H_z}{\partial r}.$$

The solution of Eq. (5) involves Bessel's functions. It is convenient to write the solution in the form:

$$H_z = [A_1 I_0(pr) + A_2 K_0(pr)] e^{j\omega t - \gamma z} \quad (6)$$

where

$$p^2 = -(\gamma^2 + k^2), \quad (7)$$

and $I_0(x), K_0(x)$ are modified Bessel's functions of the zeroth order, related to the more familiar Bessel's and Hankel's functions through the equations:

$$\left. \begin{aligned} I_\nu(x) &= j^{-\nu} J_\nu(jx) \\ K_\nu(x) &= \frac{\pi}{2} j^{\nu+1} H_\nu^{(1)}(jx). \end{aligned} \right\} \quad (8)$$

Since H_z is continuous and finite at $r = 0$, the constant A_2 in Eq. (6) must be identically zero. The solutions for the components of the TE wave within the electron beam are:

$$\left. \begin{aligned} H_z &= A_1 I_0(pr) e^{j\omega t - \gamma z} \\ H_r &= A_1 \frac{\gamma}{p} I_1(pr) e^{j\omega t - \gamma z} \\ E_\phi &= -A_1 \frac{j\omega \mu}{p} I_1(pr) e^{j\omega t - \gamma z}. \end{aligned} \right\} \quad (9)$$

2.3 Dynamics of the Electron Beam. It will be seen from Eq. (4) that a knowledge of J_z as a function of E_z is necessary in order to obtain the solution of the TM wave within the electron beam. To find such a relationship, the behavior of the electrons under the action of electric and magnetic fields must be considered.

As was indicated earlier, the motion of the electrons is assumed to be confined to the axial direction. In practice this assumption is very nearly realized by means of the focusing action of a strong d-c magnetic field applied parallel to the helix axis. It is also assumed that the a-c components of charge, current, electron velocity vary exponentially with the same propagation constant as the wave traveling in the helix, while the average electron velocity is substantially constant over a finite section of the helix. This last assumption depends on the tacit supposition that the phenomena can be described by a small-signal analysis.

The notation used will be the following:

- ρ = a-c component of charge density
- ρ_0 = average value of charge density
- v = a-c component of electron velocity
- v_0 = average value of electron velocity
- J_z = a-c component of current density
- J_0 = average value of current density
- $\frac{e}{m}$ = ratio of charge to mass of the electron.

Thus

$$\vec{J} = \vec{i}_r J_r + \vec{i}_\phi J_\phi + \vec{i}_z (J_z + J_0),$$

$$\rho_T = \rho_0 + \rho.$$

where the \vec{i} 's are the unit vectors in cylindrical coordinates.

Continuity of charge demands that:

$$\nabla \cdot \vec{J} + \frac{\partial \rho_T}{\partial t} = 0 \quad (10)$$

since

$$J_\phi = J_r = 0$$

and

$$\frac{\partial J_z}{\partial z} = \frac{\partial \rho_0}{\partial t} = 0,$$

the continuity equation becomes:

$$\frac{\partial J_z}{\partial z} = -\frac{\partial \rho}{\partial t}$$

or

$$J_z = \frac{j\omega}{\gamma} \rho. \quad (11)$$

The force equation for non-relativistic motion is:

$$\vec{F} = q \left[\vec{E} + \vec{v} \cdot \vec{B} \right]. \quad (12)$$

Since the velocity of the electrons is a small fraction of the velocity of light, the force due to the magnetic field can be neglected in comparison to the force due to the electric field, so that:

$$\frac{d}{dt} (v_0 + v) = -\frac{e}{m} E_z \quad (13)$$

Now

$$\frac{d}{dt} (v_0 + v) = \frac{dv}{dt} = \frac{dv}{dz} + v_0 \frac{dv}{dz} = v_0 \left(j \frac{\omega}{v_0} - \gamma \right) v. \quad (14)$$

Then Eq. (13) can be written as:

$$v = \frac{\left(-\frac{e}{m} \right) E_z}{v_0 \left(j \frac{\omega}{v_0} - \gamma \right)}. \quad (15)$$

To a first approximation the current density is:

$$J = J_0 + J_z = (v_0 + v)(\rho_0 + \rho) \cong v_0 \rho_0 + \rho v_0 + \rho_0 v. \quad (16)$$

Since

$$J_0 = v_0 \rho_0$$

and

$$\rho = \frac{\gamma}{j\omega} J_z,$$

the a-c current density can be put in the form:

$$J_z = \frac{j\omega J_0}{v_0^2(j\frac{\omega}{v_0} - \gamma)} v. \quad (17)$$

Substituting for v from Eq. (15), the a-c current density as a function of E_z is:

$$J_z = \left[\frac{-j\omega \frac{e}{m} J_0}{v_0^3(j\frac{\omega}{v_0} - \gamma)^2} \right] E_z. \quad (18)$$

2.4 TM Wave within the Electron Beam. The necessary relation between J_z and E_z having been obtained, it is now possible to solve Eq. (4). In view of Eq. (18) and the fact that $J_r = 0$ Eq. (4) becomes:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dE_z}{dr} \right) + (\gamma^2 + k^2) \left[1 - \frac{\frac{e}{m} J_0}{\epsilon v_0^3(j\frac{\omega}{v_0} - \gamma)^2} \right] E_z = 0 \quad (19)$$

or

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dE_z}{dr} \right) - \eta^2 E_z = 0 \quad (20)$$

where

$$\eta^2 = p^2 \left[1 + \frac{\frac{e}{m} I}{\pi b^2 \epsilon v_0^3(j\frac{\omega}{v_0} - \gamma)^2} \right] \quad (21)$$

$$p^2 = -(\gamma^2 + k^2) \quad (7)$$

$$I = -\pi b^2 J_0 = \text{d-c beam current.}$$

Equation (20), being of the same form as Eq. (5), has the solution:

$$E_z = B_1 I_0(\eta r) e^{j\omega t - \gamma z} \quad (22)$$

From Eqs. (2) the other components of the TM wave within the beam can be obtained:

$$\left. \begin{aligned} E_r &= B_1 \frac{\gamma \eta}{p^2} I_1(\eta r) e^{j\omega t - \gamma z} \\ H_\phi &= B_1 \frac{j\omega \epsilon \eta}{p^2} I_1(\eta r) e^{j\omega t - \gamma z} \end{aligned} \right\} \quad (23)$$

2.5 Admittance Functions. The fields within the electron beam will have to be matched to the fields outside the beam at the boundary $r = b$. One method of matching the fields is equating corresponding radial impedance or admittance functions⁸ at the boundary. Normalized radial admittances for both TE and TM waves can be defined by:

$$y_r^{(1)} = \frac{1}{Y_0} \left(-\frac{H_z}{E_\phi} \right) \quad (24)$$

$$y_r^{(2)} = \frac{1}{Y_0} \left(\frac{H_\phi}{E_z} \right) \quad (25)$$

where

$$Y_0 = \sqrt{\frac{\epsilon}{\mu}}.$$

Thus defined, the admittances are to be measured in the direction of decreasing r .

Therefore, the two admittances within the beam are:

$$y_r^{(1)} = \frac{p}{jk} \frac{I_0(pr)}{I_1(pr)} \quad \text{for TE wave} \quad (26)$$

$$y_r^{(2)} = \frac{jk \eta}{p} \frac{I_1(\eta r)}{I_0(\eta r)} \quad \text{for TM wave} \quad (27)$$

2.6 TE Wave in Charge-free Regions. Outside the electron beam, $J = 0$. Thus, Eq. (3) for H_z again reduces to Eq. (5). It is clear therefore, that in the whole region $0 < r < a$ the solution for H_z and the other TE components is independent of the presence of charge and current in the region $0 < r < b$. Thus the TE wave components both inside and outside the electron beam are given by Eqs. (9). The equating of corresponding admittance functions (24) at $r = b$ merely results in an identity. To obtain the TE field components outside the helix, the general solution given by Eq. (6) must be considered. Physically, the field must vanish at $r = \infty$. Since $I_0(pr)$ contradicts this, the constant A_1 must be identically zero. Thus the field components for the TE wave outside the helix are:

$$\left. \begin{aligned} H_z &= A_2 K_0(pr) e^{j\omega t - \gamma z} \\ H_r &= -A_2 \frac{\gamma}{p} K_1(pr) e^{j\omega t - \gamma z} \\ E_\phi &= A_2 \frac{j\omega \mu}{p} K_1(pr) e^{j\omega t - \gamma z} \end{aligned} \right\} \quad r > a \quad (28)$$

2.7 TM Wave in Charge-free Regions. Outside the electron beam Eq. (4) takes the same form as Eq. (5):

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + (\gamma^2 + k^2) E_z = 0 \quad (29)$$

and has the general solution of the same form as Eq. (6). Clearly the same arguments apply to the TM fields outside the helix as were given above for the TE fields. Therefore, outside the helix the TM field components are:

$$\left. \begin{aligned} E_z &= C_1 K_0(pr) e^{j\omega t - \gamma z} \\ E_r &= -C_1 \frac{\gamma}{p} K_1(pr) e^{j\omega t - \gamma z} \\ H_\phi &= -C_1 \frac{j\omega \epsilon}{p} K_1(pr) e^{j\omega t - \gamma z} \end{aligned} \right\} r > a \quad (30)$$

In the charge-free region within the helix ($b < r < a$) we must use both types of the modified Bessel function for completeness:

$$\left. \begin{aligned} E_z &= [C_3 I_0(pr) + C_4 K_0(pr)] e^{j\omega t - \gamma z} \\ E_r &= \frac{\gamma}{p} [C_3 I_1(pr) - C_4 K_1(pr)] e^{j\omega t - \gamma z} \\ H_\phi &= \frac{j\omega \epsilon}{p} [C_3 I_1(pr) - C_4 K_1(pr)] e^{j\omega t - \gamma z} \end{aligned} \right\} b < r < a \quad (31)$$

Since the TM wave is mainly responsible for the interactions between the waves and electrons, we shall discuss briefly here the nature of the field structure. If the phase velocity of the wave is radically less than the velocity of light, from (7) p will essentially be a real quantity. Figure 2 is a sketch of the electric field of the TM wave associated with the $I_0(pr)$ function. The electric lines start and terminate at r equals infinity. Clearly this type of field structure cannot exist outside the helical sheath. In the charge-free region within the helix, all the electric lines start and terminate on charges on the conducting helical sheath. The field intensity increases with the radius. This type of field will exist even in the absence of any electron beam. Consider now a continuous stream of electrons of uniform density passing through the tube at a constant velocity approximately the same as the phase velocity of the wave. Half of the electrons will stay with an accelerating field for a while and the rest will be decelerated at the same time. The result is a slow change of velocity and density as the electrons move on - a bunching process.

As soon as the electron beam is bunched, the field expressions associated with the I_0 functions only are no longer adequate to describe the electromagnetic phenomena inside the tube. Figure 3 illustrates the field distribution associated with the $K_0(pr)$ function. Here the electric lines start and terminate substantially on the electron beam. There is, of course, the electric field associated with the average or d-c charge distribution superposed on the fluctuation or the a-c charge distribution.

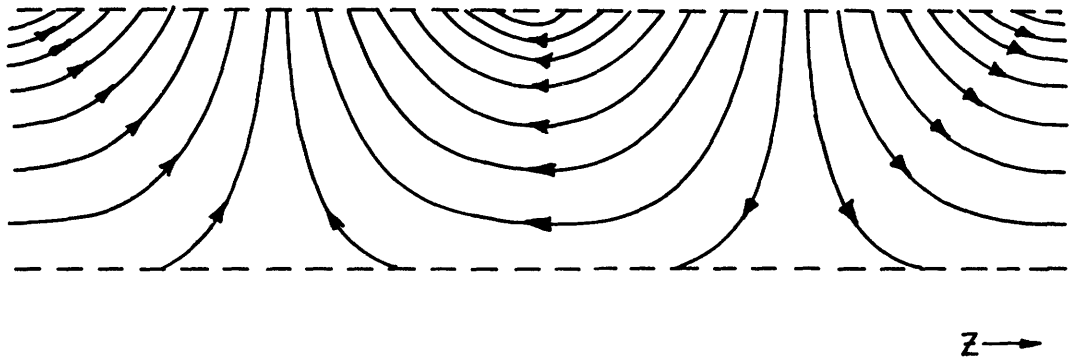


Figure 2. Instantaneous configuration of the electric field associated with the $I_0(pr)$ function (longitudinal section).

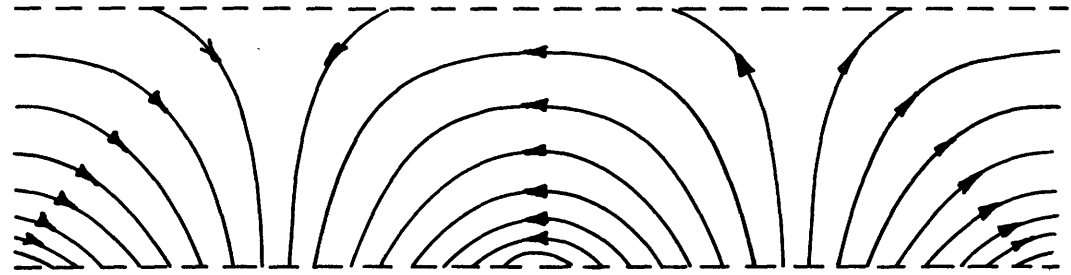


Figure 3. Instantaneous configuration of the electric field associated with the $K_0(pr)$ function (longitudinal section).

Clearly a field of this type is closely related to the space charges and its presence is responsible for the debunching force which will tend to smooth out the fluctuations of the charge distribution.

2.3 Boundary Conditions on the Helix. At the helix wall ($r = a$) the boundary condition is that the current flow be restricted to a direction making a constant angle ($90^\circ - \theta$) with the axis of the helix, θ being the pitch angle of the helix. Also, the electric field directed along the windings shall vanish at $r = a$, the electric field normal to that direction shall be continuous at $r = a$, and the magnetic field parallel to the windings shall be continuous at $r = a$. Employing these conditions, one readily finds that:

$$A_1 = \frac{p}{j\omega\mu} \tan \theta \left[\frac{C_3 I_0(pa) + C_4 K_0(pa)}{I_1(pa)} \right] \quad (32)$$

$$A_2 = -\frac{p}{j\omega\mu} \tan \theta \frac{K_0(pa)}{K_1(pa)} C_1 \quad (33)$$

$$C_1 = \frac{C_3 I_0(pa) + C_4 K_0(pa)}{K_0(pa)} \quad (34)$$

$$\frac{C_4}{C_3} = \left[\frac{\left(\frac{ka}{pa}\right)^2 \cot^2 \theta I_1(pa)K_1(pa) - I_0(pa)K_0(pa)}{K_0^2(pa)} \right] \quad (35)$$

Equations (32) - (35) provide part of the information necessary to determine the fields in the electron beam and helix uniquely except for a constant multiplicative factor throughout.

2.9 Admittance of TM Wave in Charge-free Region. The final step in solving the boundary value problem is the matching of the fields outside and inside the electron beam at the boundary of that beam. As was indicated above, the matching procedure could be accomplished by means of admittance functions defined in Eqs. (24) and (25). It should be recalled that the TE fields are already matched since those fields both inside and outside the beam are given by Eqs. (9). There remains then only the matching of the TM waves at $r = b$.

By using (35), the normalized admittance (25) for the TM wave is as follows:

$$y_r^{(2)} = j \frac{ka}{pa} \left[\frac{I_1(pr) - K_1(pr) \left(\frac{\left(\frac{ka}{pa}\right)^2 \cot^2 \theta I_1(pa)K_1(pa) - I_0(pa)K_0(pa)}{K_0^2(pa)} \right)}{I_0(pr) + K_0(pr) \left(\frac{\left(\frac{ka}{pa}\right)^2 \cot^2 \theta I_1(pa)K_1(pa) - I_0(pa)K_0(pa)}{K_0^2(pa)} \right)} \right] \quad (36)$$

2.10 Matching. At the boundary of the electron beam ($r = b$) the admittance given in Eq. (36) must be equal to that given in Eq. (27). Therefore:

$$j \frac{ka}{pa} \frac{\eta}{p} \frac{I_1(\eta b)}{I_0(\eta b)} = j \frac{ka}{pa} \left[\frac{I_1(pb) - K_1(pb) \left(\frac{(ka)^2}{pa} \cot^2 \theta \frac{I_1(pa)K_1(pa) - I_0(pa)K_0(pa)}{K_0^2(pa)} \right)}{I_0(pb) + K_0(pb) \left(\frac{(ka)^2}{pa} \cot^2 \theta \frac{I_1(pa)K_1(pa) - I_0(pa)K_0(pa)}{K_0^2(pa)} \right)} \right] \quad (37)$$

Equation (37) represents the formal solution of the problem, since, for prescribed conditions of initial current, initial electron velocity, frequency, and physical dimensions, it can be solved, in principle at least, for p and thus the propagation constant γ . Once γ or p is found, the whole behavior of the fields both inside and outside the helix is known because Eqs. (32) - (35) allow the determination of all the field components (9), (28), (30), (31) in terms of a single constant factor that depends upon the power being transmitted along the helix.

2.11 Charge-free Helix Waveguide. Before considering the general form of Eq. (37) and the associated admittances (27) and (36), it is of interest to investigate the special case in which the charge and current densities are zero everywhere within the helix. It is clear from Eq. (21) that in such circumstances $\eta = p$ so that Eq. (37) is valid only if:

$$\frac{I_1(pa)K_1(pa)}{I_0(pa)K_0(pa)} \left(\frac{ka}{pa} \right)^2 \cot^2 \theta = 1. \quad (38)$$

Equation (38) can also be obtained by placing $C_4 = 0$ in Eq. (31) since E_z is finite at $r = 0$. Then Eq. (35) gives the desired relation immediately. From Eq. (38) a relationship can be obtained between the phase velocity $v_p (= \frac{\omega}{\beta_0})$ of the wave in the helix and the parameter $ka (= \frac{2\pi a}{\lambda})$ where β_0 is the phase constant associated with the charge-free helix and λ is the free-space wavelength. Curves indicating this relationship for various angles of pitch are shown in Fig. 4. For large values of ka , Eq. (38) approaches the asymptotic form:

$$\left(\frac{pa}{ka} \right)^2 \tan^2 \theta = 1$$

which gives:

$$\frac{v_p}{c} = \sin \theta.$$

This is the value to be expected from elementary considerations of Fig. 1, neglecting the proximity effects due to a small ratio of diameter to wavelength.

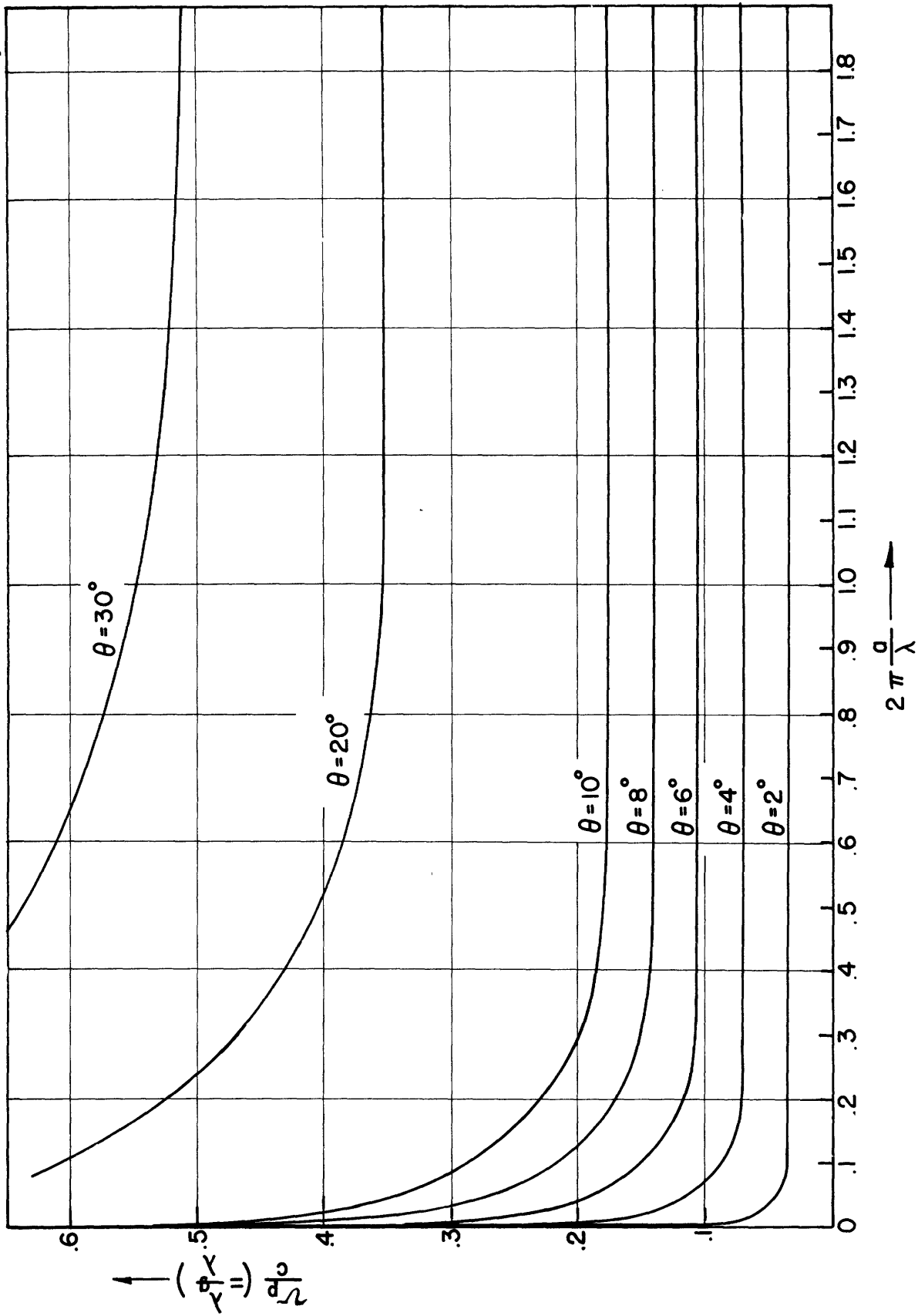


Figure 4. Phase velocity $v_p (= \frac{\omega}{\beta_0})$ in the charge-free helix as a function of $ka (= \frac{2\pi a}{\lambda})$ for various pitch angles (θ).

2.12 Discussion of the Admittance Functions. The form of Eq. (37) makes it practically impossible to determine the behavior of the system under various operating conditions. In order to facilitate calculation of results, it would be desirable to replace the admittance functions (27) and (36) by reasonably simple algebraic forms which approximate as closely as possible the actual admittances as functions of p or η . With this purpose in mind the behavior of the radial admittance (36) (evaluated at $r = b$) as a function of p must be examined. For real values of p , (no energy interaction between the electrons and the field), $y_r^{(2)}$ given by Eq. (36) has the form shown in Fig. 5. Since there is no radial flow of energy, the admittance is purely imaginary for all real values of p . The exact shape of the curve is governed by the choice of b/a , ka , and θ ; but the general features are the same irrespective of the values of these parameters. Along the real axis of p the function has a zero at $p = p'$, and an infinite value at $p = 0$, and $p = p''$. There is every reason to believe that these three points are the only poles and zeros of the function in the complex p -plane.

From a physical viewpoint the nature of the radial admittance makes it necessary that its poles and zeros occur when p is real since an imaginary or complex value of p means that there is radial flow of energy either from or into the electron beam, since the helix itself is assumed to be non-radiative and non-dissipative. In either case the radial admittance must have a finite real part to account for such an energy flow. Clearly then the admittance cannot vanish anywhere in the complex p -plane except on the real axis.

An infinite value of the admittance means that E_z vanishes within the electron beam. Since the bunching and production of a-c current density is conditional upon the presence of E_z in the beam, it is obvious that no energy interchange can take place if the admittance is infinite. But a complex value of p implies an energy exchange between the field and the electrons. Therefore, the admittance can become infinite only for real values of p . It is evident then that the admittance has two infinities and one zero all lying on the real axis of p .

Before investigating the admittance function further, it is of some interest to note the physical meaning of the poles and zero of the function. The infinite value at $p = 0$ is the degenerate case in which $\gamma^2 = -k^2$ or $\beta = k$, and is of little interest. The infinite value at $p = p''$ means that E_z vanishes within the region $0 \leq r \leq b$. Therefore, the electron beam appears as a perfect conductor, and the system behaves as a coaxial transmission line with a helical outer conductor. The zero point at $p = p'$ indicates that H_ϕ vanishes everywhere within the beam. In other terms, the total current density (the sum of the electron current and the displacement current) is zero in the beam.

Returning to the discussion of the admittance function and the search for a simple algebraic form approximating it, we note that Eq. (36) can be, in view of the Weierstrass' factor theorem, approximated by an expression of the following type:

$$y_r^{(2)} = j ka C \left(\frac{p - p'}{p - p''} \right) \quad (39)$$

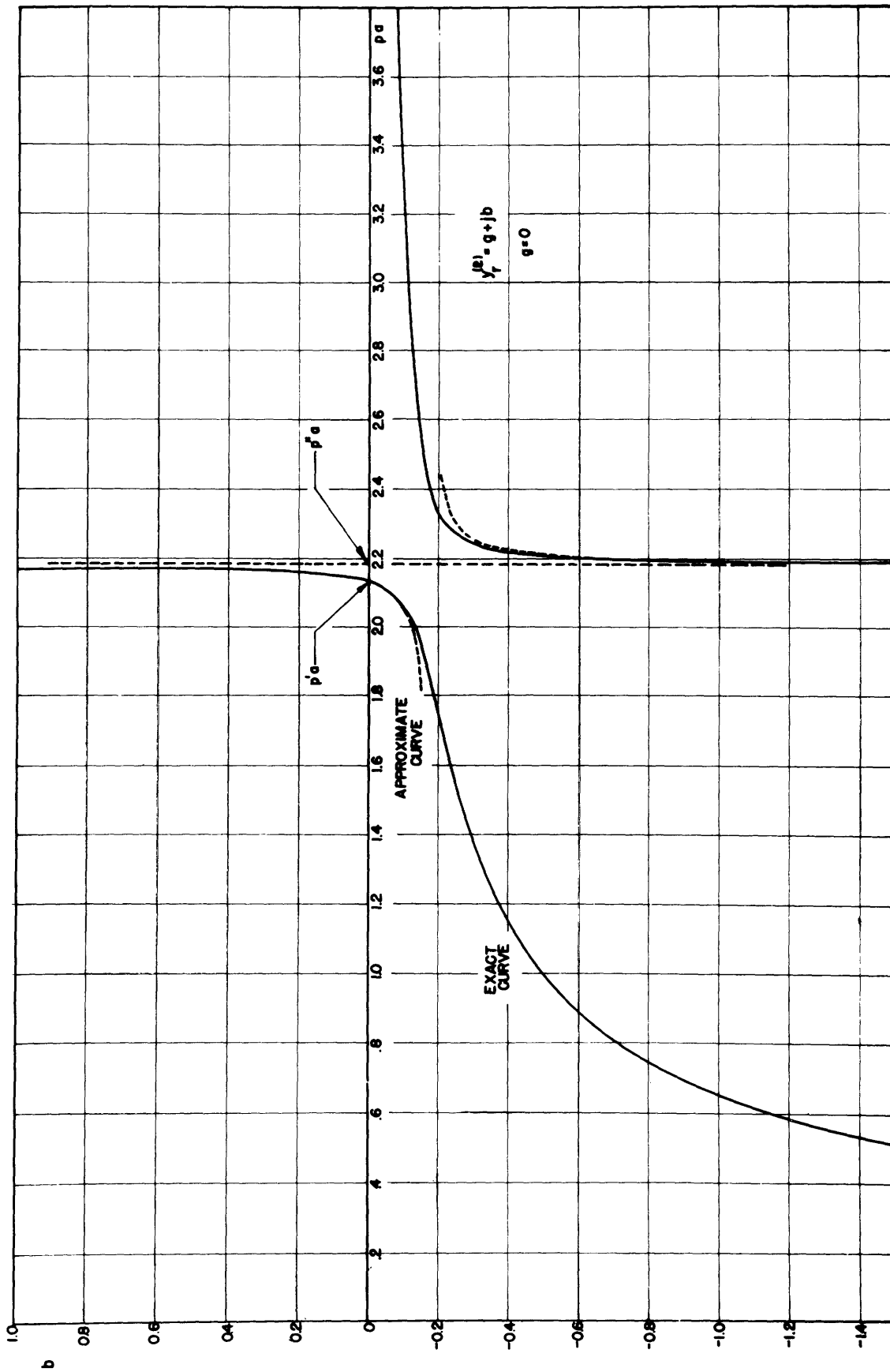


Figure 5. Radial admittance $y_r^{(2)}$ ($= g + jb$) as a function of p/a for real values of p/a . $ka = 0.2$, $\theta = 5^\circ$, $b/a = 0.2$.

where

$$j ka C = - (p'' - p') \left. \frac{\partial y_r^{(2)}}{\partial p} \right|_{p=p'}$$

The singularity of Eq. (36) at $p = 0$ is overlooked, because we are interested in the function only in the neighborhood of p' and p'' , and, as can be seen from Fig. 5, Eq. (39) approximates the exact admittance very closely in this region. Furthermore, since the phase velocities of the waves are very much less than that of light it is evident from Eq. (7) that $jp \approx \gamma$. Upon making this substitution, Eq. (39) becomes:

$$y_r^{(2)} = j ka C \left(\frac{\gamma - j\beta'}{\gamma - j\beta''} \right) \quad (40)$$

where $j\beta'$, $j\beta''$ correspond to p' and p'' .

The admittance (27) derived from the fields within the electron beam will be approximated by:

$$y_r^{(2)} = j \frac{ka}{a} \frac{\eta^2}{p^2} \frac{b}{2} \quad (41)$$

This approximation means physically that E_z is assumed to be constant over the cross section of the electron beam, while H_ϕ increases linearly with r in the same region.

2.13 Approximate Matching Equation. After substituting Eqs. (40) and (41) into Eq. (37) the matching equation becomes:

$$\frac{\eta^2}{2p^2} \left(\frac{b}{a} \right) = C \left(\frac{\gamma - j\beta'}{\gamma - j\beta''} \right) \quad (42)$$

or, substituting for η^2 from Eq. (21):

$$\left(\frac{b}{a} \right) \left[1 + \frac{\frac{e}{m} I}{\pi b^2 \epsilon v_0^3 \left(j \frac{\omega}{v_0} - \gamma \right)^2} \right] = 2C \left(\frac{\gamma - j\beta'}{\gamma - j\beta''} \right). \quad (43)$$

Equation (43) is a cubic equation in γ ; and thus for a fixed set of parameters yields three values of the propagation constant. These three values of γ , since they determine different field configurations as well as different propagation characteristics, can be thought of as indicating three different modes co-existing in the guide. The three modes represent an approximation to the infinite series of modes necessary to describe exactly the fields in the helix.

2.14 Characteristics of the Three Modes. After various simplifications, we finally arrived at Eq. (43) which is a cubic equation for the longitudinal propagation constant involving many parameters. Physically, the normal mode of propagation in a helical waveguide splits into three independent modes in the presence of the electron beam, as characterized by the three independent roots of the cubic equation. Within the limitations of the approximations, these three modes can exist simultaneously within the

waveguide depending upon the initial conditions at the input terminal (to be discussed later). We shall now study the behavior of the three modes of propagation for a specific waveguide at a specific wavelength.

Helix diameter = 1.0 cm.
 Angle of pitch = 5°
 Wavelength (free space) = 16 cms.
 Electron beam diameter = 0.2 cms.
 Beam current $I = 10^{-3}$ amp.

With the conventional notation $\gamma = \alpha + j\beta$, the real and imaginary parts of the three propagation constants are plotted in Fig. 6 as solid lines. In addition, the "phase constant" (β_e) of the d-c electron beam, defined as ω/v_0 , is also plotted. The phase constants β_0 , β' , and β'' are indicated on the curve as horizontal lines. The independent variable in Fig. 6 is the ratio of the d-c electron velocity to the velocity of light.

The cubic equation has alternately real and imaginary coefficients. We expect to have three independent imaginary roots, or a pair of complex roots and one independent one. When the d-c electron velocity is too high or too low, we have three independent waves neither amplified nor attenuated. Over a finite range of the electron velocity, two of the waves have the same phase constant and the same absolute value of the attenuation. The third wave has an imaginary propagation constant.

Over the ranges at which the electron velocity is either too high or too low for a net transfer of energy to or from the electron beam, the three waves seem to follow a pattern. Over each range, one of the waves has a phase constant approximately equal to the phase constant β_0 of the electron-free helical guide. The other two waves seem to travel at a velocity close to that of the electron beam. Over the range of amplification or attenuation, the distinction between these types of waves becomes less marked. This is reasonable since the electron velocity is fairly close to the phase velocity of the unperturbed wave.

2.15 Admittance Function. The normalized admittance $y_r^{(2)}$ as defined in Eq. (25) is one looking into the electron beam at the boundary of the beam. A positive conductance indicates a transfer of power from the field to the beam and vice versa. The admittance functions for the three waves are plotted in Fig. 7 for the above case. The first wave, which has $\alpha = 0$ for all velocities, possesses a susceptance varying slowly with the electron velocity. The susceptance is zero at a certain low velocity point at which the phase constant corresponds to β' . At this point the total a-c current (sum of the displacement current and the a-c electronic current) produced by that wave is zero within the beam. The conductance curve for the other two waves increases in amplitude with the decrease of the electron velocity. It finally drops down to zero at a point where the electron velocity is too low for amplification. The magnetic field at the surface of the electron beam is due to the displacement current as well as the electron current within the beam. Since p is essentially a real quantity, the displacement current is 90° out of phase with the electric field. The conductance is contributed solely by the a-c electron current. From Fig. 7, we can draw the following conclusion.

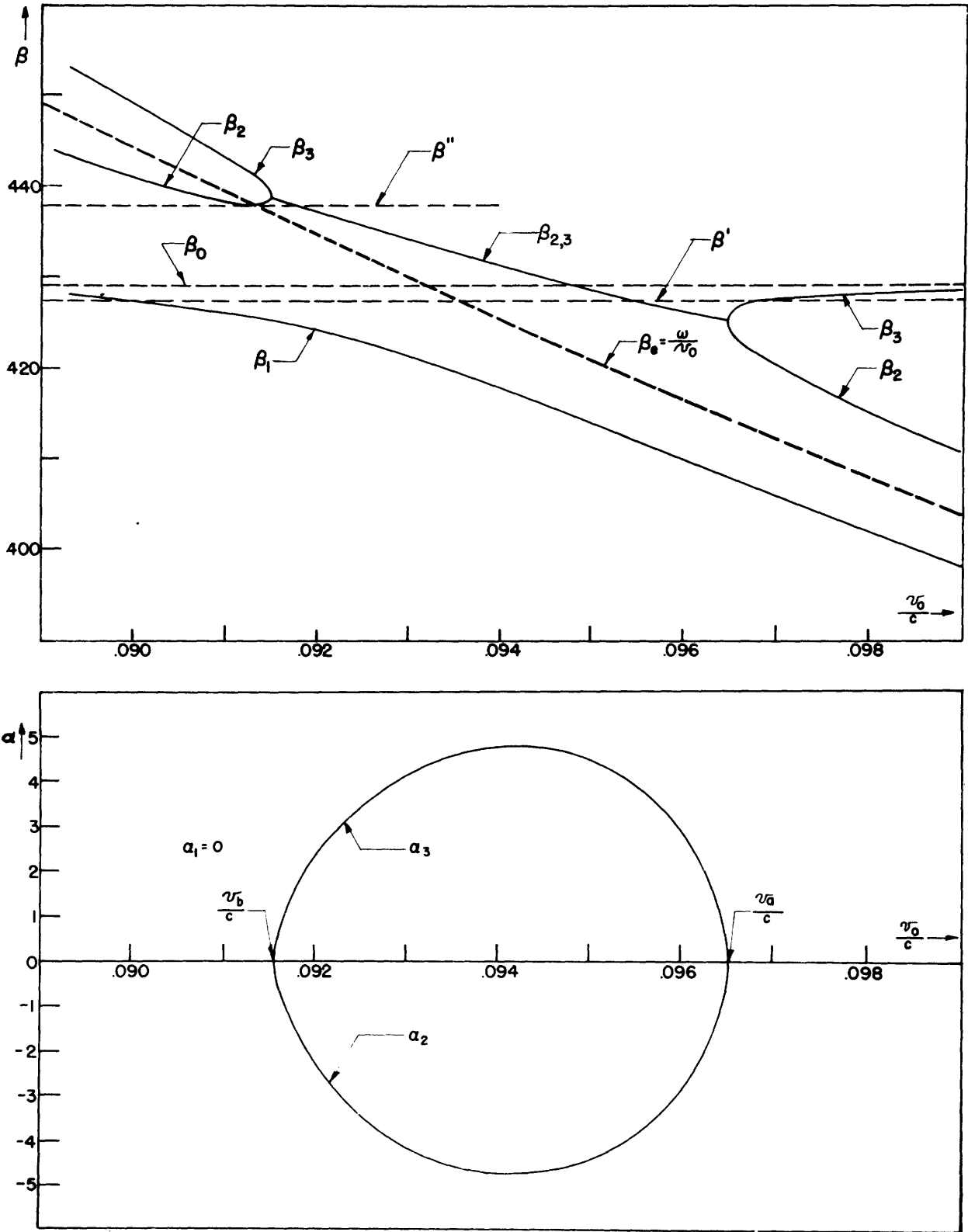


Figure 6. Attenuation and phase constants for the three component waves as functions of the average electron velocity (v_0). $I = 10^{-3}$ amp.

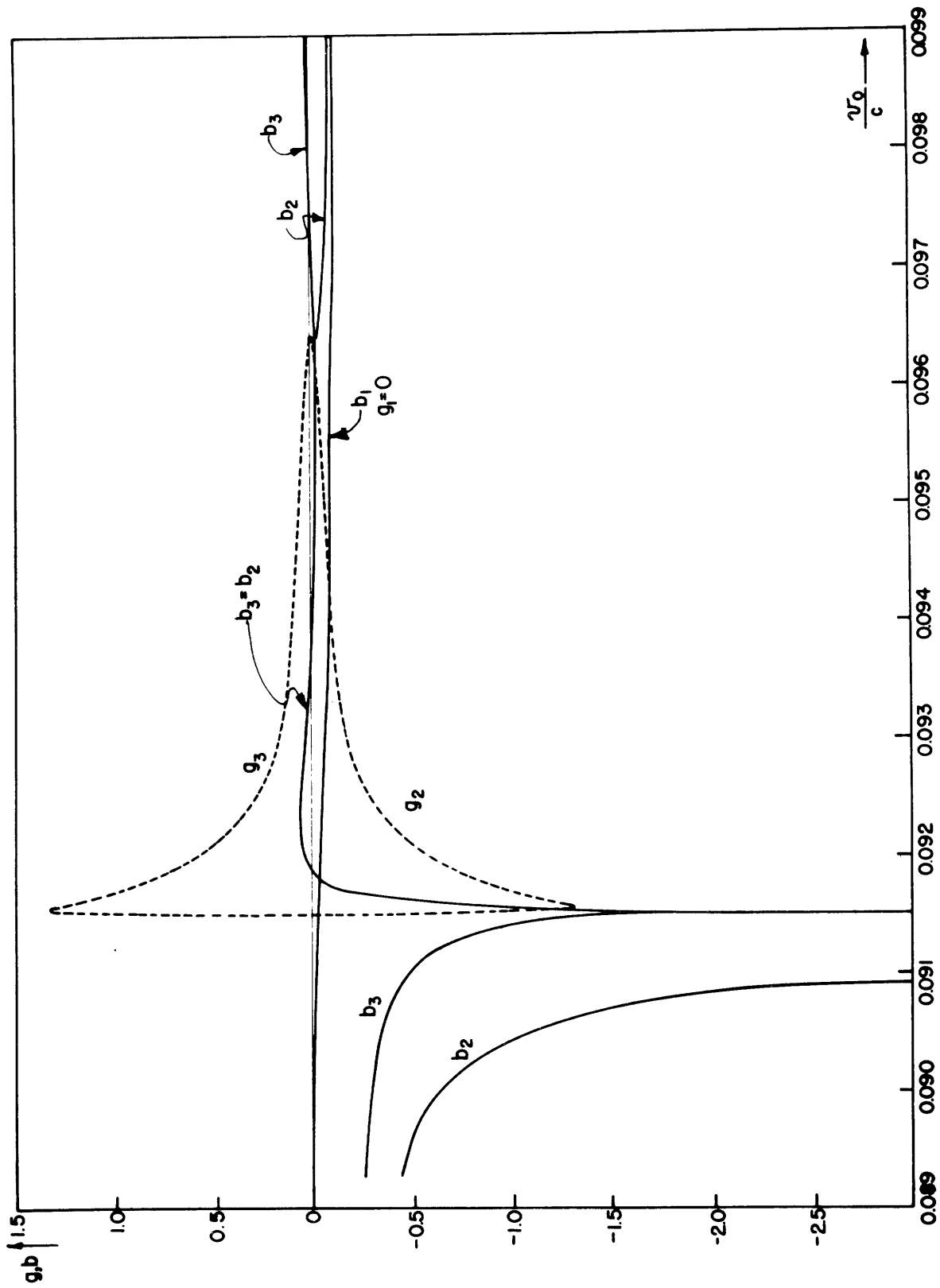


Figure 7. Radial admittances $y_r^{(2)}$ ($= g + jb$) for the three component waves as functions of the average electron velocity (v_0). $I = 10^{-3}$ amp.

Over the amplifying range, the amplified or attenuated wave has a relatively strong a-c electron current for a low d-c electron velocity v_0 and a weak a-c electron current for a high v_0 . The phenomenon of the high a-c electron current is closely connected with the infinity β'' of the admittance function as plotted in Fig. 5. The β'' line in Fig. 6 intercepts the curve for the phase constant β_2 or β_3 where we have high current density.

2.16 Effect of the Beam Current. Figure 8 indicates the behavior to be expected for various values of beam current I . As the beam current is increased the velocity range over which amplification can occur increases, as well as the maximum amplification attainable. It should be noted that both α and β curves for the amplified wave shift toward the right as the beam current increases. High beam current calls for a higher electron velocity to take advantage of the higher amplification. However, with currents ranging up to 10 ma the shift of optimum electron velocity is not appreciable.

If we plot the maximum value of α against the d-c beam current, we obtain an empirical formula:

$$\alpha_{\max} = \text{constant} \times I^{\frac{1}{4}}$$

The maximum amplification is proportional to the fourth root of the d-c beam current. This conclusion was reached also from a different approach to the problem of the traveling-wave tube.

2.17 Effect of Beam Radius on Amplification. Equation (43) has been investigated for various values of b/a in order to obtain information about the effect of the beam radius on the amplification for a fixed beam current. The calculations indicate that the maximum value of α increases approximately as the cube root of the ratio b/a . This is to be expected since the strength of all the field components is weakest on the axis and increases as we approach the boundary of the helix. For a given total beam current more electrons travel in regions of higher electric fields as the beam radius is increased. Therefore, we expect greater interaction and energy exchange, and therefore more amplification, than when the beam is constrained to a small region near the axis.

The cube-root dependence of α on beam radius should not be taken as exact since it was derived from Eq. (43) which has inherent in it the assumption that E_z is constant throughout the beam while H_θ increases linearly with r . However, it does give a qualitative indication of the behavior to be expected.

2.18 Initial Conditions. We have so far discussed in general terms the existence of three waves along electron-filled tubes. Their amplitudes depend not only upon the relative amplification but also upon the initial conditions at the input end. Physically there will be other modes of propagation, which must exist at least at the beginning of the tube. Consideration of these other modes would make the problem of initial conditions complicated. We shall neglect all the higher modes in the present treatment.

We shall deal with the type of initial conditions associated with an unmodulated electron beam and an r-f signal at the input $z = 0$. The electron beam can be idealized as a uniform stream of electrons flowing into the tube at $z = 0$ with uniform

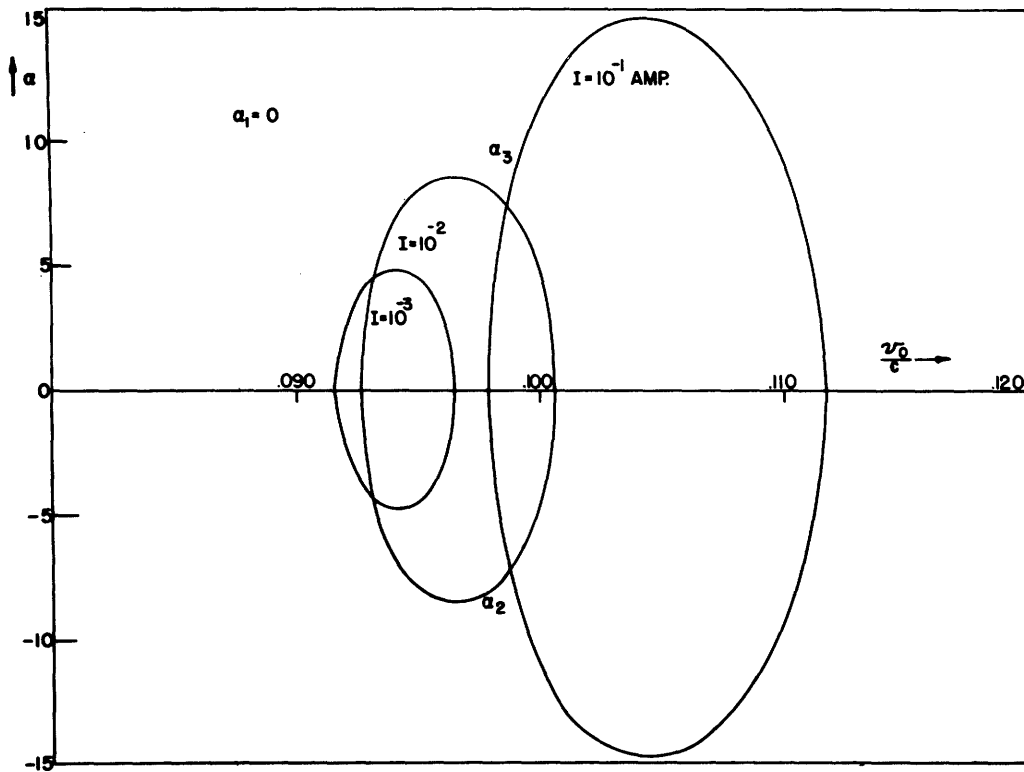
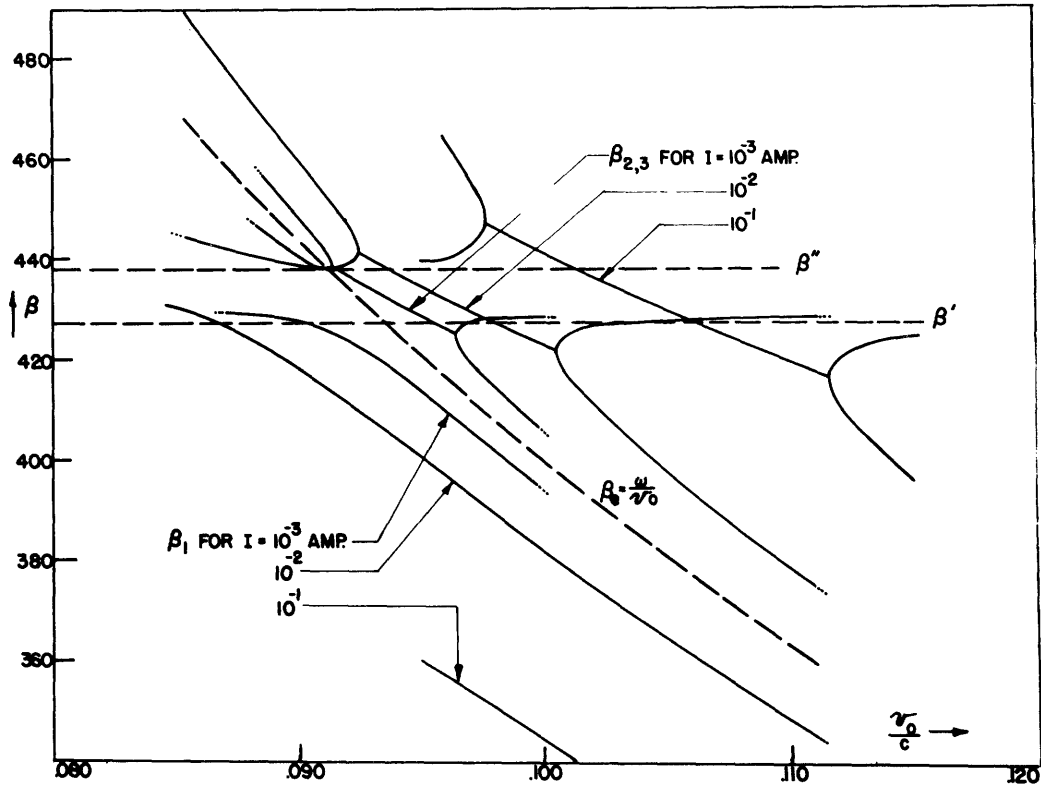


Figure 8. Attenuation and phase constants of the three component waves as functions of the average electron velocity (v_0) for beam currents $I = 10^{-3}$, $I = 10^{-2}$, $I = 10^{-1}$ amp.

density and velocity. It takes time for the electrons to change their velocity and it takes more time for the electron beam to change its density. Consequently, the r-f wave will travel for a short while undisturbed by the electron beam. The phase velocity and the field structure will essentially be the same as those of the charge-free wave along the tube. The proper procedure of solving this problem would be to match the field components of the three perturbed waves to those of the unperturbed wave for a short length of the tube. This turns out to be an extremely difficult procedure. The next best thing to do is to use the initial condition of zero a-c current and charge density (or a-c electron velocity).

Then from equations (15) and (18) it is evident that:

$$\frac{E_{z1}}{j \frac{\omega}{v_0} - \gamma_1} + \frac{E_{z2}}{j \frac{\omega}{v_0} - \gamma_2} + \frac{E_{z3}}{j \frac{\omega}{v_0} - \gamma_3} = 0 \quad (44)$$

$$\frac{E_{z1}}{(j \frac{\omega}{v_0} - \gamma_1)^2} + \frac{E_{z2}}{(j \frac{\omega}{v_0} - \gamma_2)^2} + \frac{E_{z3}}{(j \frac{\omega}{v_0} - \gamma_3)^2} = 0 \quad (45)$$

where E_{z1} , E_{z2} , E_{z3} are the initial values of the longitudinal electric field within the electron beam associated with the propagation constants γ_1 , γ_2 , γ_3 respectively. These two equations can be solved to obtain two of the three components in terms of the third for a fixed value of v_0 .

It would be tempting to write down a third equation relating the sum of the component fields to the input wave so that the absolute value of the components could be determined. Such a relation is not forthcoming however, since the field configurations associated with the three propagation constants differ. Calculations could be made of the power propagated with each component wave, the sum of the powers being equated to the input power. However, for our purposes the relative magnitudes of the three component axial electric fields are sufficient. For convenience we write the following normalized equation:

$$E_{z1} + E_{z2} + E_{z3} = 1 \quad (46)$$

and solve for E_{z1} , E_{z2} , E_{z3} at the input.

$$\left. \begin{aligned} E_{z1} &= \frac{(\gamma_1 - j \frac{\omega}{v_0})^2 (\gamma_2 - \gamma_3)}{D} \\ E_{z2} &= \frac{(\gamma_2 - j \frac{\omega}{v_0})^2 (\gamma_3 - \gamma_1)}{D} \\ E_{z3} &= \frac{(\gamma_3 - j \frac{\omega}{v_0})^2 (\gamma_1 - \gamma_2)}{D} \end{aligned} \right\} \quad (47)$$

where

$$D = (\gamma_1 - j \frac{\omega}{v_0})^2 (\gamma_2 - \gamma_3) + (\gamma_2 - j \frac{\omega}{v_0})^2 (\gamma_3 - \gamma_1) + (\gamma_3 - j \frac{\omega}{v_0})^2 (\gamma_1 - \gamma_2).$$

Within the range of amplification, the voltage gain along a tube of length ℓ is evidently

$$\text{Voltage Gain} = \left| E_{z2} \right| e^{-\alpha_2 \ell} \quad (48)$$

since the longitudinal electric field at the input is taken as a unity.

The normalized amplitudes and phases of the E_{z1} , E_{z2} and E_{z3} at the input are shown in Figs. 9 and 10. Over the amplifying range, E_{z2} and E_{z3} are conjugate quantities. At the high velocity end, the amplitudes of E_{z2} and E_{z3} exceed unity. It should be remembered that here the amplification constant is rather small. At the low velocity end, the amplitude of E_{z2} is rather small. However, this is compensated by the high a-c current over this portion of the curve.

Beyond the amplification range, the three waves have phases of 0 or 180°. We see that for very low or very high electron velocities, where the interaction between the beam and the fields is essentially zero (no amplification), the waves with phase constants associated with that of the electron beam are but weakly excited. On the other hand, the wave whose phase constant lies close to that of the unperturbed electron-free wave has a normalized amplitude approaching unity. Thus, when the electrons are traveling too slowly to interact, E_{z1} with a phase constant very near β_0 is the predominant wave; when they move much too fast for energy exchange, E_{z3} is the dominant wave. This is to be expected from physical reasoning since at high or low electron velocities the beam should have but little perturbing effect on the propagation along the helix.

In any event, if conditions are right for amplification, it is fair to conclude that an appreciable fraction of the input field is converted into the amplified mode associated with γ_2 .

2.19 Signal Level and Limiting Efficiency. So long as we limit our discussion to a low level of r-f power all through the tube, the assumption of a constant electron velocity v_0 is valid, since the energy transferred from the electron beam to the r-f field will be a negligible fraction of the total kinetic energy of the electrons. We observe from the above theory that within the amplification range, the electron beam moves at a higher velocity than the interacting wave. Each individual electron must consequently undergo many periods of acceleration and deceleration along the tube. It must also lose more energy during the deceleration than the energy gained during the acceleration. On the time average all the electrons must lose energy gradually and probably at the same average rate. If the last statement is correct, we can apply the above theory, with modification, to a tube with a high signal level. We can consider the theory valid for a very short section of the tube. As the wave and electrons move along the tube the d-c electron velocity is reduced because of the net decrease in kinetic energy of the electrons. The phase velocity of the wave is slowed down accordingly with

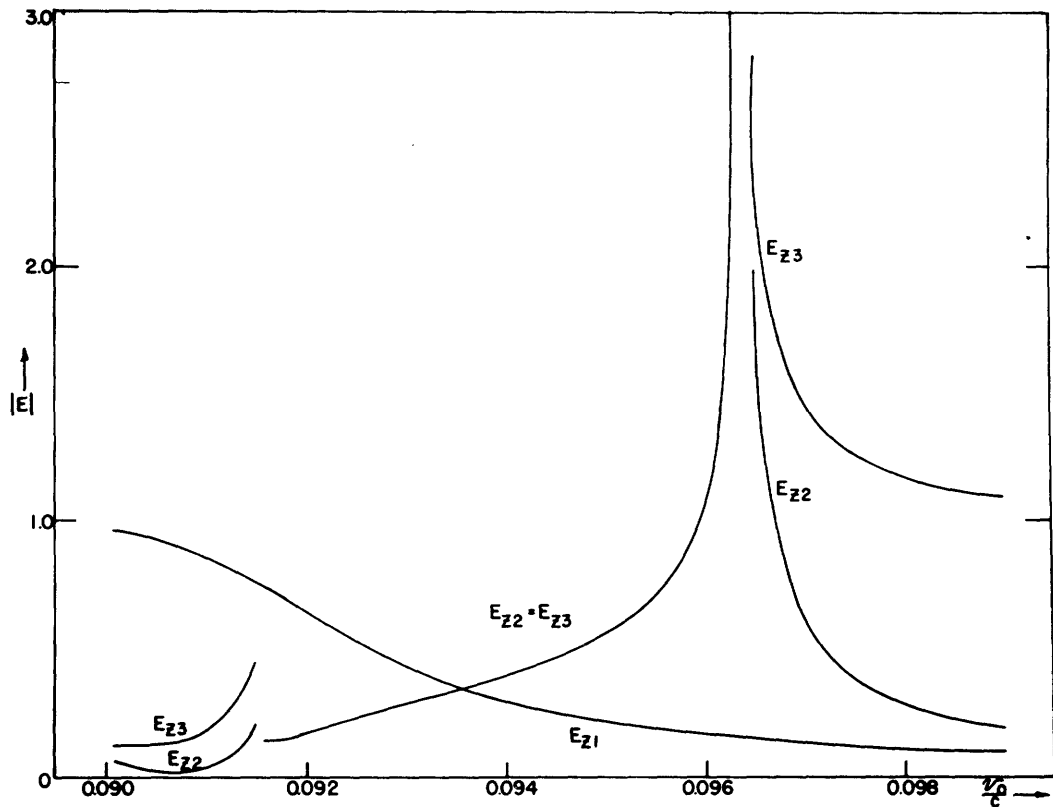


Figure 9. Initial relative magnitudes of the three component axial electric fields as functions of the average electron velocity (v_0). $I = 10^{-3}$ amp.

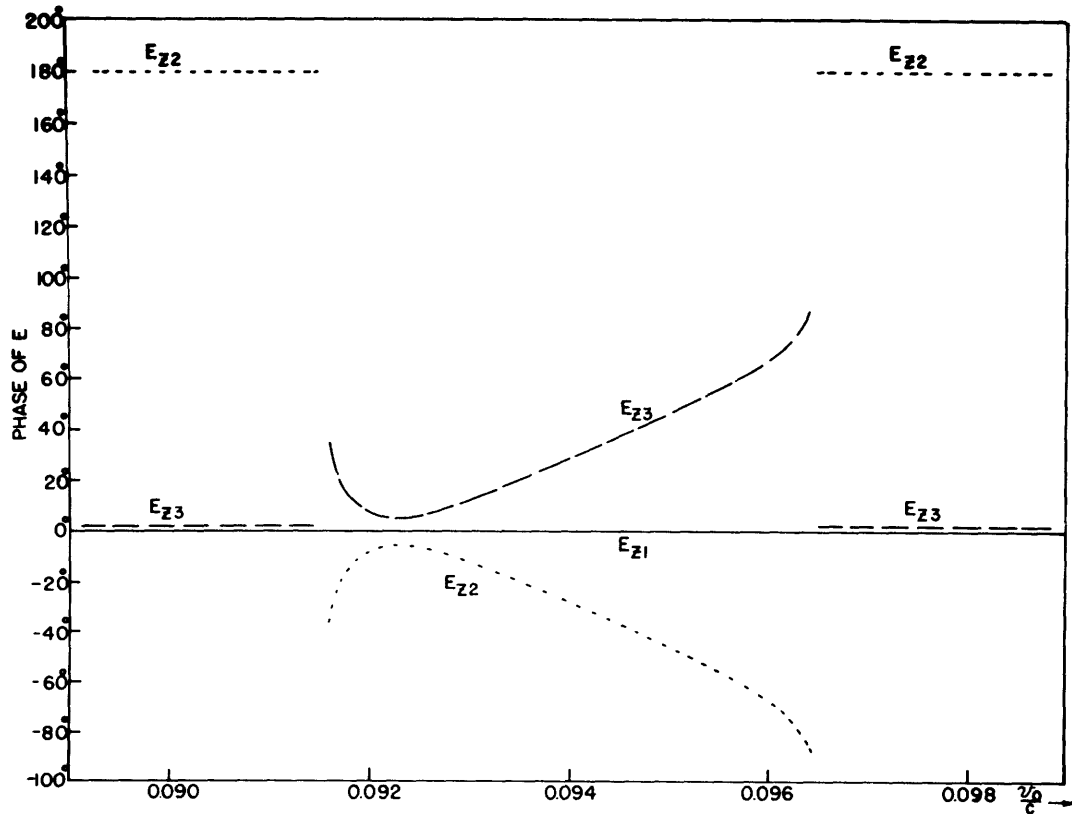


Figure 10. Initial relative phases of the three component axial electric fields as functions of the average electron velocity (v_0). $I = 10^{-3}$ amp.

a corresponding change of the field structure. We can imagine that the point of operation in Fig. 6 shifts gradually to the left. The amplification constant is continuously changing. The decrease of the average electron velocity will be slow at the beginning. Because of the exponential increase of the r-f power level, the average electron velocity must decrease rapidly at the end. If this picture is correct, there are two conclusions which we can draw.

On the assumption that a sufficiently long lossless helix is used, the maximum energy exchange would take place if the electrons enter the helical guide with a velocity v_a corresponding to the upper end of the amplification range, and leave with a velocity v_b corresponding to the lower end of the range. Therefore, the maximum limit of the efficiency of energy conversion ϵ can be given by:

$$\epsilon = \frac{v_a^2 - v_b^2}{v_a^2} \quad (49)$$

For the typical tube discussed here, the upper limit of maximum efficiency possible is of the order of 10 to 25 per cent depending upon the beam current and beam radius. Practically, the efficiency of a practical device would be limited to a value much less than that by the relatively short length of helix employed, terminating conditions, and other factors.

The second conclusion is that the ratio of the a-c electron current to the longitudinal electric field increases as the d-c velocity of the electrons slows down. This is obvious from the conductance curve in Fig. 7. At the output of a helix-type traveling-wave tube, we shall find that the beam is highly bunched, more so than what is expected from an exponential increase of the a-c current. It is interesting to note that it might be possible to extract the r-f power from the beam at the output by some klystron type cavity.

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