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by

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Abstract

This paper is concerned with the motion of protons in a frequency-modulated synchrotron. The equations for the phase and radial oscillations are obtained and the conditions under which these oscillations are damped are discussed. The effects of changes in the r-f frequency and voltage are also analysed. It is shown that the condition for stability can be met in a practical design.

SOME THEORETICAL CONSIDERATIONS OF THE PROTON SYNCHROTRON

1. Introduction

The problem of the synchrotron has received attention from a number of writers in the recent past (see Ref. 1). In the majority of cases attention has been confined to the purely relativistic case characteristic of electron synchrotrons with high injection energies, but Dr. J. S. Gooden and his colleagues in Professor Oliphant's laboratory at Birmingham have considered the non-relativistic case, applicable to proton synchrotrons with low injection energies, in considerable detail and have written a number of, as yet unpublished, memoranda on their work (see Ref. 2).

In this paper we have developed the theory of the proton synchrotron by a method slightly more general than that used by Dr. Gooden. Except for one or two minor results the conclusions reached are identical, the most significant being that the phase oscillations will not be damped at energies where the relativistic increase of mass with velocity is unimportant (say kinetic energies less than 100 Mev) unless the amplitude of the r-f accelerating voltage increases with time.

The main purpose of this paper was to provide design data for a 10-Bev proton synchrotron. A number of alternative designs for such a machine are possible, some using a steel, some an air-core magnet, some using betatron and some direct electrostatic injection into the synchrotron phase. For reasons that will be outlined in a companion paper, the initial studies were made on a steel-cored magnet with direct electrostatic injection, but all of the results derived here will be applicable to the air-cored magnet and many of these will be applicable to the betatron injection machine. In all cases, however, it is assumed, that the doughnut in which the protons are accelerated, is a volume of revolution about a vertical axis and that there is a single accelerating gap.

In order to simplify the treatment, a number of restrictions will be imposed, the most important of which will be listed.

a. The magnetic field in the median plane has cylindrical symmetry and is of the form in cylindrical co-ordinates of

$$\underset{\sim}{H} = \left\{0,\ 0,\ H_0(t)\left(r/r_0\right)^{-n}\right\}$$

where $H_0(t)$ is a function of time alone. In a practical case dH_0/dt will be approximately constant over the acceleration period.

b. The effect of scattering due to the presence of gas atoms and molecules in the annular acceleration region will be neglected.

c. Attention will be confined to the case where motion is in the median plane of the synchrotron. The possibility of coupling between the vertical and radial oscillations is then neglected.

d. Space-charge effects are neglected; this is justified by the very low current densities in a high-energy proton synchrotron.

e. Radiation losses are regarded as negligible, an assumption that is quite valid for all practicable proton synchrotrons.

Under these assumptions it has been shown by previous writers[1] that the equations of motion for a charged particle in a synchrotron take the form discussed in the next section.

2. The General Equations of Motion

The following symbols will be used in this and subsequent sections. All quantities are given in Gaussian units unless otherwise stated.

r, ϕ, z	=	cylindrical coordinates of the position of the accelerated particle
$\underset{\sim}{V}$	=	vector velocity of the particle
e	=	charge of a proton
μ	=	relativistic mass of a proton
μ_0	=	relativistic mass of the "stable" proton
m_0	=	rest mass of a proton
V	=	the amplitude of the r-f voltage
f	=	the frequency of the r-f voltage
ω	=	the angular frequency of the r-f voltage
$\underset{\sim}{H}$	=	the magnetic field in the doughnut
$\underset{\sim}{\mathcal{E}}$	=	the electric field vector in the doughnut
$\underset{\sim}{A}$	=	the vector potential in the doughnut
Φ	=	total flux through a circle of radius r
t	=	the time duration since injection
ψ	=	the phase of the radio frequency when the particle reaches the centre of the accelerating gap
r_0	=	the radius of the "stable" orbit
n	=	the fall-off index
x	=	$r-r_0$, the radial deviation from the "stable" orbit
ψ_0	=	the phase of the "stable" particle
ψ_1	=	$\psi - \psi_0$, the departure from the "stable" phase
E	=	the <u>kinetic energy</u> of the proton
E_0	=	the <u>rest energy</u> of the proton
T	=	the duration of the acceleration period
k	=	$(\mu_0^2/m_0^2)/\left(\mu_0^2/m_0^2 - (1-n)\right)$

If radiation losses may be neglected, the vector equation of motion in Gaussian units may be written as

$$\frac{d}{dt}(\mu \underset{\sim}{V}) = e\underset{\sim}{\mathcal{E}} + \frac{e}{c} \underset{\sim}{V} \times \underset{\sim}{H} - \frac{e}{c} \frac{\partial \underset{\sim}{A}}{\partial t} \qquad (1)$$

where $\underset{\sim}{\mathcal{E}}$ is the electric field produced by the accelerating gap, $\underset{\sim}{H}$ is the magnetic field

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1. In particular, by David S. Saxon and Julian Schwinger (see Ref. 1).

of the magnet, $\underset{\sim}{A}$ is the vector potential, and $\underset{\sim}{V}$ is the vector velocity. In the case where motion is confined to the median plane, it can be shown that Eq. (1) yields two scalar equations

$$\frac{d}{dt}(\mu\dot{r}) = \mu r\dot{\varphi}^2 - \frac{e}{c}\, r\dot{\varphi}\, H_o(t)(r/r_o)^{-n} \tag{2}$$

$$\frac{d}{dt}(\mu r^2\dot{\varphi}) = \frac{eV}{2\pi}\sin\psi + \frac{e}{2\pi c}\frac{d\Phi}{dt} \tag{3}$$

where ψ is the r-f phase when the particle reaches the accelerating gap, and where $\Phi(r,t)$ is the total flux through a circle of radius r and is related to $\Phi_o(t)$, the total flux through the stable orbit $r = r_o$, by the equation

$$\Phi(r,t) = \Phi_o(t) + \frac{2\pi}{2-n} H_o(t) r_o^{\,2}\left[(r/r_o)^{2-n} - 1\right] . \tag{4}$$

These three equations are sufficient to determine the motion of a proton in terms of its initial conditions. Before solving the general equations, we shall consider briefly the restrictions on the r-f frequency and voltage that must be imposed if a particle can move around the stable orbit $r = r_o$ at a fixed phase to the radio frequency.

3. The "Stable" Orbit

The phase of the radio frequency when the particle reaches the accelerating gap is related to the proton angular velocity $\dot{\varphi}$ by the equation

$$\dot{\psi} = \dot{\varphi} - \omega \tag{5}$$

where ω is the r-f angular frequency.

If the phase ψ is to be constant and equal to ψ_o while r is constant and equal to r_o, the following equations must be satisfied

$$\left.\begin{aligned} 0 &= \frac{d}{dt}(\mu\dot{r}) = r_o\dot{\varphi}\left(\mu\dot{\varphi} - \frac{e}{c}H_o(t)\right) \\ 0 &= \dot{\psi} = \dot{\varphi} - \omega \\ \frac{d}{dt}(\mu r_o^{\,2}\dot{\varphi}) &= \frac{d}{dt}\frac{e r_o^{\,2}}{c}H_o(t) = \frac{eV}{2\pi}\sin\psi_o + \frac{e}{2\pi c}\frac{d\Phi_o}{dt} \end{aligned}\right\} \tag{6}$$

Hence the r-f angular frequency must obey the law

$$\mu\,\omega_o = \frac{e}{c}H_o(t) \tag{7}$$

and the r-f voltage must obey the law

$$V\sin\psi_o = 2\pi\frac{d}{dt}\frac{r_o^{\,2}H_o(t)}{c} - \frac{1}{c}\frac{d\Phi_o}{dt} \tag{8}$$

These equations determine the motion of the so-called "stable" particle.

In the synchrotron phase $d\Phi_0/dt$ will be small compared with $2\pi r_0^2 H_0(t)$ so that $V \sin \psi_0$, the voltage gain per turn, is given approximately by

$$V \sin \psi_0 \approx \frac{2\pi r_0^2}{c} \frac{d}{dt} H_0(t).$$

This approximation will not be made in working out the general case, but it is useful here in giving an approximate value for the voltage across the accelerating gap.

To see quantitatively what these restrictions mean, it is best to express everything in terms of the kinetic energy E and the rest energy E_0 of the proton. If we do this, the following simple relations are obtained:

(1) Product of orbital radius and magnetic field

$$\begin{aligned} H_0 r_0 &= \frac{E_0}{e} \sqrt{(2 + E/E_0)E/E_0} \\ &\sim \frac{E}{e} \text{ for } E/E_0 >> 1 . \end{aligned} \tag{9}$$

The maximum magnetic field obtainable in practice with a steel magnet is of the order of 15,000 gauss and is independent of the size of the orbit. This means that the radius of the orbit increases linearly with the kinetic energy for energies above 5 Bev (E_0 for a proton is 948×10^6 ev).

(2) RF Angular frequency

$$\left.\begin{aligned} \omega_0 &= (c/r_0) \sqrt{E/E_0(2 + E/E_0)}\Big/(1 + E/E_0) \\ &= (c/r_0)(er_0H_0/E_0)\Big/\sqrt{1 + (er_0H_0/E_0)^2} \\ &\sim c/r_0 = \omega_{00} \quad \text{for } E/E_0 >> 1 \end{aligned}\right\} \tag{10}$$

where ω_{00} is the limiting angular frequency equal to the natural angular frequency of the orbit.

(3) Voltage gain per turn

$$V \sin \psi_0 \approx \frac{2\pi r_0^2}{c} \frac{d H_0}{dt} \approx \frac{2\pi r_0}{cT} \frac{E_0}{e} \sqrt{(E_T/E_0)(2 + E_T/E_0)} \tag{11}$$

$$\sim (2\pi/\omega_{00} T) \frac{E_T}{e} \text{ for } E_T/E_0 >> 1$$

where it has been assumed that the average value of dH_0/dt is equal to H_{0T}/T, where H_{0T}, E_T are the final values of the magnetic field and kinetic energy, respectively, and T is the duration of the acceleration period. This equation tells us that for large final energies and fixed final magnetic field, the voltage gain per turn, and hence the amplitude of the r-f voltage across the accelerating gap, must increase directly as the square of the final energy and inversely as the duration of the acceleration period.

(4) Distance traveled in synchrotron

$$D = cT\,(E_o/E_T)\left(\sqrt{1 + (E_T/E_o)^2} - 1\right) \qquad (12)$$

where D is the total distance traveled and the other symbols have the same significance as before. It is assumed here that the injection energy is small compared with E_T and E_o. The magnitude of this term is important when calculating the number of particles that will be lost by collision with the nuclei of the gas atoms in the annulus.

(5) The energy as a function of the magnetic field

The kinetic energy at any time is related to the magnetic field at that time by the equation

$$E = E_o\sqrt{1 + (er_oH_o/E_o)^2} - 1\,. \qquad (13)$$

To see what all these results mean quantitatively we will give the design data for two machines, one intended to yield 1-Bev particles, the other to yield 10-Bev particles. In both cases it will be assumed that the fixed value of the magnetic field is 15,000 gauss and that the duration of the acceleration period is 1 sec.

To illustrate the above results, we give a table of design data for 1-Bev and 10-Bev proton synchrotrons.

Table I. Design Properties of 1-Bev and 10-Bev Proton Synchrotrons

Property	1 Bev	10 Bev
Radius	3.76 metres	24.10 metres
Voltage Gain per revolution	131 ev	5500 ev
Distance traveled	1.3×10^5 km	3×10^5 km
Limiting frequency f_∞	12.7 Mc/sec	1.98 Mc/sec
Final frequency	11.1 Mc/sec	1.96 Mc/sec
Input frequency for 300 kev injection	320 kc/sec	50.5 kc/sec
Initial magnetic field for 300 kev injection	210 gauss	32.7 gauss
Frequency range for 300 kev injection	34.7:1	38.8:1
Input frequency for 4 Mev injection	1.17 Mc/sec	185 kc/sec
Frequency range for 4 Mev injection	9.5:1	10.65:1
Initial magnetic field for 4 Mev injection	769 gauss	114 gauss

4. First-order Oscillations about the Stable Orbit.

In a practical synchrotron the width of the doughnut in which the particles are accelerated is small compared with the radius of the stable orbit. This enables us to use a first-order approximation and assume that the amplitude of the radial oscillations about the stable orbit is small. If the radial oscillations are large, or increase with time, the particles will be lost to the walls. Besides executing radial oscillations about the stable orbit, the particles will also execute oscillations in phase about the stable phase. The amplitude of these oscillations will not in general be small, and care must therefore be taken when making approximations.

In this section it will be assumed that the radio frequency and the voltage across the accelerating gap have the values given by Eqs. (7) and (8). The effect of varying these quantities will be considered later.

To find the first-order oscillation, we put

$$\begin{aligned} \psi &= \psi_o + \psi_1 \\ r &= r_o + x \\ \dot{\varphi} &= \omega_o + \dot{\varphi}_1 \end{aligned} \tag{14}$$

in Eqs. (2) and (3) when x and $\dot{\varphi}_1$ are first-order small quantities.

As a first step we must analyze the effect on μ, the mass of the particle, of a first-order change of radius. Now

$$\mu = \frac{m_o}{\sqrt{1 - v^2/c^2}} = \frac{m_o}{\left(1 - \frac{r_o^2 \omega_o^2}{c^2} - 2\frac{(r_o x \omega_o^2 + r_o^2 \omega_o \dot{\varphi}_1)}{c^2} - \frac{\dot{x}^2}{c^2}\right)^{1/2}} \tag{15}$$

neglecting powers of x and $\dot{\varphi}$ above the first.

Hence

$$\mu = \frac{m_o}{\left(1 - \frac{r^2 \omega_o^2}{c^2}\right)^{1/2}} \left\{ \frac{1 - \frac{2r_o^2 \omega_o^2}{c^2}\left(\frac{x}{r_o} + \frac{\varphi_1}{\omega_o}\right)}{\left(1 - \frac{\omega_o^2 r_o^2}{c^2}\right)} \right\}^{-1/2}$$

and if

$$\left(\frac{x}{r_o} + \frac{\dot{\varphi}_1}{\omega_o}\right) \frac{r_o^2 \omega_o^2}{c^2 - \omega_o^2 r_o^2} << 1 \tag{16}$$

$$\mu \approx \mu_o \left\{ 1 + \frac{\mu_o^2}{m_o^2} \frac{r_o^2 \omega_o^2}{c^2} \left(\frac{x}{r_o} + \frac{\dot{\varphi}_1}{\omega_o}\right)\right\} \tag{17}$$

where

$$\mu_o = m_o \Big/ \left(1 - \frac{r^2 \omega_o^2}{c^2}\right)^{1/2}.$$

From Eq. (5) we get

$$\dot{\psi} = \dot{\psi}_1 = \dot{\varphi}_1 . \tag{18}$$

Now from (2) we have

$$\frac{d}{dt}\left(\frac{eH_o\dot{x}}{\omega_o c}\right) = \mu_o r_o \omega_o^2 - \frac{e}{c} r_o H_o + \mu_o r_o \omega_o^2 \left(\frac{x}{r_o} + \frac{\dot{\varphi}_1}{\omega_o}\right)\frac{\mu_o^2}{m_o^2}$$

$$+ \mu_o(2r_o\dot{\varphi}_1\omega_o + x\,\omega_o^2) - \frac{eH_o}{c}(1-n)\,\omega_o x - \frac{eH_o}{c} r_o\dot{\varphi}_1$$

using Eq. (7).

Hence

$$\frac{d}{dt}\left(\frac{eH_o\dot{x}}{\omega_o c}\right) - \frac{eH_o\omega_o x}{c}\left(\frac{\mu_o^2}{m_o^2} - (1-n)\right) = \dot{\varphi}_1 \frac{\mu_o^2}{m_o^2}\frac{eH_o r_o}{c} \qquad (19)$$

In the purely relativistic case where $\mu_o^2/m_o^2 >> 1$ this equation becomes

$$\frac{-x}{r_o} = \frac{\dot{\varphi}_1}{\omega_o} \qquad (20)$$

a result, already given by Saxon and Schwinger, that is valid even when Eq. (16) is not satisfied. In the pure non-relativistic case

$$\frac{\mu_o^2}{m_o^2} = 1$$

and then Eq. (17) becomes

$$\frac{d}{dt}\left(\frac{eH_o\dot{x}}{\omega_o c}\right) - \frac{eH_o\omega_o x}{c}\,n = \frac{\dot{\varphi}_1 eH_o r_o}{c}\ . \qquad (21)$$

Substituting from (2) in (3) we have, correct to the first order,

$$\frac{e}{c}\frac{d}{dt}\left\{H_o r_o^2\,(r/r_o)^{2-n} + \frac{r_o}{\omega_o}\frac{d}{dt}\left(\frac{H_o x}{\omega_o}\right)\right\}$$

$$= \frac{eV(t)}{2\pi}\sin\psi + \frac{e}{2\pi c}\frac{d\Phi_o}{dt} + \frac{er_o^2}{c}\frac{d}{dt}r_o^2\,H_o\left(\left(\frac{r}{r_o}\right)^{2-n} - 1\right) \qquad (22)$$

Expanding in powers of x, neglecting second-order quantities and using Eq. (8), we get

$$\frac{er_o}{c}\frac{d}{dt}\frac{1}{\omega_o}\frac{d}{dt}\left(\frac{H_o\dot{x}}{\omega_o}\right) + \frac{er_o}{c}(1-n)\frac{d}{dt}(H_o x) = \frac{eV}{2\pi}\sin\psi - \frac{eV}{2\pi}\sin\psi_o . \qquad (23)$$

Equations (19) and (23) determine the phase and radial oscillations.

To determine the general form of the oscillations let us put $\sin\psi \approx \psi$, neglect the derivatives of all functions that vary slowly with the time and eliminate ψ between (19) and (23). Then

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2}\left\{\omega_o^2(1-n) - \frac{m_o^2\, Vc\omega_o}{\mu_o^2 2\pi r_o^2\, H_o}\right\} + x\left\{\left[\frac{\mu_o^2}{m_o^2} - (1-n)\right]\frac{m_o^2\,\omega_o^3\, cV}{\mu_o^2\, 2\pi r_o^2\, H_o}\right\} = 0 \quad (24)$$

If

$$2B = \omega_o^2(1-n) - \frac{m_o^2\, Vc\,\omega_o}{\mu_o^2\, 2\pi r_o^2\, H_o} \quad (25)$$

and

$$C = \left(\frac{\mu_o^2}{m_o^2} - (1-n)\right)\frac{m_o^2}{\mu_o^2}\frac{\omega_o^3\, cV}{2\pi r_o^2\, H_o} \quad (26)$$

then the frequencies of the oscillation are given by

$$\gamma^2 \approx B\left\{1 \pm \sqrt{1 - C/B^2}\right\}. \quad (27)$$

If $C/B^2 << 1$, the frequencies are given by

$$\gamma^2 = 2B; \quad \gamma^2 = C/2B$$

and the fourth-order equation separates into two second-order equations.

From Table 1 we see that $C/B^2 \lessdot \frac{1}{20}$ for the 10-Bev machine

$\lessdot \frac{1}{1000}$ for the 1-Bev machine

and in what follows we shall assume that this separation is justified. In this case the equations for the oscillations become

$$\frac{d^2x}{dt^2} + \omega_o^2(1-n)x = 0 \quad (28)$$

since

$$\omega_o^2(1-n) >> \frac{m_o^2\, Vc\,\omega_o}{\mu_o^2\, 2\pi r_o^2\, H_o}, \quad (29)$$

and

$$\frac{d^2x}{dt^2} + \frac{\omega_o^2\, cV}{2\pi r_o^2\, H_o(1-n)}\left(\frac{\mu_o^2}{m_o^2} - (1-n)\right)\frac{m_o^2}{\mu_o^2}x = 0. \quad (30)$$

The first equation gives the oscillation characteristic of the betatron, the second that characteristic of the synchrotron. The betatron oscillation has a frequency $\omega_0\sqrt{1-n}$ and as it is usual to take $n \approx 0.75$, the betatron oscillation has a frequency approximately half that of the radio frequency, while the synchrotron oscillation frequency is very much lower.

If the particle is injected into the synchrotron diplaced by x_0 from the stable orbit and with initial radial velocity $\dot{x}_0$, it can be shown that the displacement of the particle due to the betatron oscillation at any subsequent time is given by

$$x = \frac{x_0 H_0^{-1/2}}{H_1^{-1/2}} \cos\sqrt{1-n}\int \omega_0 dt + \frac{\dot{x}_0 H_0^{-1/2}}{\sqrt{1-n}\,\omega_1 H_1^{-1/2}} \sin\sqrt{1-n}\int \omega_0 dt \qquad (31)$$

where ω_1 is the angular frequency and H_1 the magnetic field at injection. The total displacement at any time can be found by adding the displacement due to betatron oscillation to the displacement due to synchrotron oscillation. It is with these synchrotron oscillations that we shall henceforth be concerned.

5. The Synchrotron Phase Oscillations

In setting up the equations for the radial oscillations in the previous section, it was necessary to assume $\sin\psi = \psi$, $\sin\psi_0 = \psi_0$ in order to be able to eliminate ψ. However, in general this approximation is unjustified and it is better to eliminate x. When discussing the synchrotron oscillations alone we can use the fact that the synchrotron oscillation frequency is very much less than the radio frequency. In this case Eq. (19) assumes the simpler form

$$-\frac{x}{r_0}\left(\frac{\mu_0^2}{m_0^2} - (1-n)\right) = \frac{\dot{\varphi}_1}{\omega_0}\frac{\mu_0^2}{m_0^2} = \frac{\dot{\psi}}{\omega_0}\frac{\mu_0^2}{m_0^2} . \qquad (32)$$

Substituting in Eq. (21) and neglecting the fourth-order term, we find that the equation of motion for the phase oscillations in the synchrotron is

$$\frac{r_0^2(1-n)}{c}\frac{d}{dt}\left(\frac{H_0\dot{\psi}}{\omega_0}k\right) + \frac{V}{2\pi}\sin\psi = \frac{V}{2\pi}\sin\psi_0 \qquad (33)$$

where

$$k = \frac{\mu_0^2/m_0^2}{(\mu_0^2/m_0^2 - (1-n))} \qquad (34)$$

(1)

= 1 in the pure relativistic case

= 1/n in the non-relativistic case.

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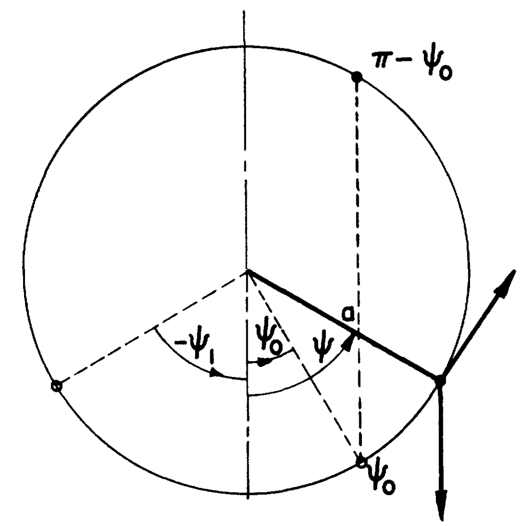
1. It may be noted that k is a very slowly varying function of time and for most purposes may be taken as constant.

It is not possible to solve Eq. (33) exactly when H_o, ω_o, V, and k are all functions of time, but an approximate BKW solution can be obtained to provide all the information needed in a practical design. If we neglect for the moment the fact that H varies with the time, the equation of motion for the phase oscillations assumes the form

$$\ddot{\psi} + \frac{V\, c\, \omega_o}{2\pi r_o^2 (1-n) k H_o} \sin\psi = \frac{V\, c\, \omega_o}{2\pi r_o^2 (1-n) k H_o} \sin\psi_o \qquad (35)$$

which is the equation of angular motion of a particle constrained to move on a vertical circular wire under the action of gravity and of a constant tangential force as in Fig. 1.

Figure 1.



This equation yields two points of equilibrium: a stable point at $\psi = \psi_o$ and an unstable point at $\psi = \pi - \psi_o$. If the angular displacement ever exceeds $\pi - \psi_o$, the particle will not oscillate but will describe the circle in the same direction with ever increasing velocity. Now from Eq. (32) we see that $-x \propto \dot{\varphi}$ so that if the particle enters the synchrotron at too large a phase difference from the stable phase, it will spiral in and hit the walls. To prove this, multiply both sides of Eq. (35) by $\dot{\psi}$ and integrate. Then

$$\dot{\psi}^2 = \frac{2V\, c\, \omega_o}{2\pi r_o^2 (1-n) k H_o} \left\{ \cos\psi + \psi \sin\psi_o + A \right\}$$

where A is a constant of the motion. If ψ ever exceeds $\pi - \psi_o$, $\dot{\psi}$ will be increased in ratio

$$\left(\frac{2\pi \sin\psi_o + A + 1}{A + 1} \right)^{1/2}$$

on every cycle and the particle will thus be lost to the walls. On the other hand, if $\dot{\psi} = 0$ when $\psi = \alpha$, where $\alpha < \pi - \psi_o$ we see that

$$\dot{\psi}^2 = \frac{2V\, c\, \omega_o}{2\pi r_o^2 (1-n) k H_o} \left\{ (\psi - \alpha) \sin\psi_o + \cos\psi - \cos\alpha \right\} \qquad (36)$$

and the particle executes oscillation about $\psi = \psi_o$ with a period T given by

$$\left\{\frac{2V\ c\ \omega_o}{2\pi r_o^{\,2}(1-n)kH_o}\right\}^{1/2} T = 2\int_{-\psi_1}^{\alpha}\frac{d\psi}{\left[(\psi-\alpha)\sin\psi_o + \cos\psi - \cos\alpha\right]^{1/2}} \tag{37}$$

where

$$-\psi_1 \sin\psi_o + \cos\psi_1 = \alpha\sin\psi_o + \cos\alpha \tag{38}$$

$$-\psi_o < \psi_1 < \pi .$$

The maximum phase acceptance angle is given by

$$-\psi_2 < \psi < \pi - \psi_o$$

where

$$\cos\psi_2 - \psi_2\sin\psi_o = (\pi - \psi_o)\sin\psi_o - \cos\psi_o \tag{39}$$

$$-\psi_o < \psi_2 < \pi .$$

The maximum phase velocity occurs when $\psi = \psi_o$ and

$$\dot{\psi}^2_{max} = \frac{2V\ c\ \omega_o}{2\pi r_o^{\,2}(1-n)kH_o}\left\{(\psi_o - \alpha)\sin\psi_o + \cos\psi_o - \cos\alpha\right\}. \tag{40}$$

The corresponding value for the maximum radial displacement of the particle from its stable orbit is given by Eq. (32).

Equation (36) cannot be integrated directly in terms of a finite number of known functions except when the stable phase angle ψ_o is zero in which case the equation assumes the familiar form of an elliptic integral. It is still theoretically possible to apply the BKW method to find the effect on the amplitude of these oscillations of slowly varying H_o, but for our present purposes it will be sufficient to consider the case when $\psi - \psi_o$ is small so that we may take $\sin(\psi - \psi_o) = \psi - \psi_o$.

In this case, the general equation of motion for the phase becomes

$$\frac{r_o^{\,2}(1-n)}{c}\ \frac{d}{dt}\left(\frac{H_o k}{\omega_o}\dot{\theta}\right) + \frac{V\cos\psi_o}{2\pi}\theta = 0 \tag{41}$$

where $\theta = \psi - \psi_o$. The BKW solution for this equation is

$$\theta = A_1 \frac{1}{\left(\frac{H_o k V}{2\pi \omega_o}\right)^{1/4}} \left\{ a \cos\left[\sqrt{\frac{c}{2\pi r_o^2(1-n)}} \int \sqrt{\frac{\omega_o V \cos\psi_o}{H_o k}}\, dt\right] + b \sin\left[\sqrt{\frac{c}{2\pi r_o^2(1-n)}} \int \sqrt{\frac{\omega_o V \cos\psi_o}{H_o k}}\, dt\right]\right\} \tag{42}$$

The constants a and b are determined from the initial values of θ and x. Thus if the particle is injected at $x = x_i$ with relative phase $\theta = \theta_i$ then

$$\theta_i \left(\frac{H_o k V}{2\pi \omega_o}\right)_i^{1/4} = a \tag{43a}$$

and as

$$\frac{x_i}{r_o} = -\frac{k\,\psi_i}{\omega_o} = -\frac{k\theta_i}{\omega_o}$$

$$\frac{\omega_c x_i}{r_o k}\left(\frac{H_o k V}{2\pi\omega_o}\right)_i^{1/4} \frac{2\pi r_o^2(1-n)}{c}\,\frac{H_o k}{\omega_o V \cos\psi_o} = b \tag{43b}$$

where all variable quantities are assumed to have their injection values. The conditions that phase damping take place is that

$$\left(\frac{2\pi\,\omega_o}{H_o k V}\right)^{1/4} \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty.$$

In the non-relativistic case $k=1/n$ and $m_o\omega_o = eH_o/c$ so that the required condition is

$$V^{-1/4} \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty .$$

If the voltage follows the law given by Eq. (7), this implies that

$$\left(\frac{dH_o}{dt}\right)^{-1/4} \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty$$

or that $H_o \propto t^s$ where $s > 1$. It will be shown later, however, that there is little need for Eq. (7) to be satisfied and that the important requirement is that

$$V^{-1/4} \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty$$

irrespective of how H_o increases with t provided only that $V^{1/4}/H_o$ does not increase with t.

To find how the radial synchrotron oscillations behave with time, we can either set up the equations of motion for x and solve them, or alternatively, we can use Eq. (32) to give

$$x = \frac{r_o k}{\omega_o}\left(\frac{2\pi\,\omega_o}{H_o k V}\right)^{\frac{1}{4}}\left\{ a\sqrt{\frac{\omega_o V c}{2\pi r_o^2(1-n)H_o k}}\ \sin\left(\sqrt{\frac{c}{2\pi r_o^2(1-n)}}\int\sqrt{\frac{\omega_o V\cos\psi_o}{H_o k}}\,dt\right)\right.$$

$$\left. - b\sqrt{\frac{\omega_o V c}{2\pi r_o^2(1-n)H_o k}}\ \cos\left(\sqrt{\frac{c}{2\pi r_o^2(1-n)}}\int\sqrt{\frac{\omega_o V\cos\psi_o}{H_o k}}\,dt\right)\right\} \qquad (44)$$

where a and b have the values given by (43a) and (43b).

The amplitude of the radial oscillations X, thus varies with time as

$$\left(\frac{kH_o V}{\omega_o}\right)^{\frac{1}{4}}\frac{1}{H_o} \qquad (45)$$

and $X \longrightarrow 0$ as $t \longrightarrow \infty$, in the non-relativistic case, as $V^{\frac{1}{4}}/H_o$ a damping even more rapid than that undergone by the radial betatron oscillations for all practical rates of increase of the accelerating voltage.

6. The Effect of Varying the Voltage across the Accelerator Gap by other than the Correct Law.

Up to now it has been assumed that the r-f voltage developed across the accelerator gap is given by Eq. (8) when ψ_o, the stable phase, is a constant. If V follows an arbitrary law, Eq. (8) will still be satisfied but now ψ_o will itself be a function of the time given by

$$\sin\psi_o = \frac{1}{V}\left\{\frac{2\pi r_o^2}{c}\frac{dH_o}{dt} - \frac{1}{c}\frac{d\Phi_o}{dt}\right\}. \qquad (46)$$

The equation of motion of the synchrotron phase is now given by putting

$$\dot{\varphi}_1 = \dot{\theta} + \dot{\psi}_o = \dot{\psi} \qquad (47)$$

into Eqs. (19) and (23). If we do this and neglect derivatives above the second order when considering the synchrotron oscillations, we obtain

$$\frac{r_o^2(1-n)}{c}\frac{d}{dt}\left(\frac{H_o\dot{\psi}k}{\omega_o}\right) + \frac{V}{2\pi}\sin\psi = \frac{V}{2\pi}\sin\psi_o\,. \qquad (48)$$

If ψ is small so that $\sin\psi \approx \psi$, $\sin\psi_o \approx \psi_o$, we obtain

$$\frac{r_o^2(1-n)}{c}\frac{d}{dt}\left(\frac{H_o k}{\omega_o}\dot{\psi}\right) + \frac{V}{2\pi}\theta = 0$$

or

$$\frac{r_o^{\,2}(1-n)}{c}\,\frac{d}{dt}\left(\frac{H_o k}{\omega_o}\,\dot{\theta}\right) + \frac{V}{2\pi}\,\theta = -\frac{r_o^{\,2}(1-n)}{c}\,\frac{d}{dt}\left(\frac{H_o k\,\dot{\psi}_o}{\omega_o}\right). \tag{49}$$

The right-hand side contributes a particular integral to the general solution for θ, which can be expanded as a power series in negative powers of t if V(t) increases more rapidly with time than dH_o/dt (in this case $\dot{\psi}_o$ decreases monotonically to zero). This rapidly damps out and can be neglected so that the solution is just the complementary function which satisfies

$$\frac{r_o^{\,2}(1-n)}{c}\,\frac{d}{dt}\left(\frac{H_o k\dot{\theta}}{\omega_o}\right) + \frac{V}{2\pi}\,\theta = 0 \tag{50}$$

which is identical with Eq. (41) with the BKW solution of Eq. (42). We see from this that when $H_o \propto t$ the phase oscillations will be damped, albeit slowly, if V increases with the time instead of remaining constant. This provides a mechanism for keeping the phase oscillations under control in the period following injection. However, V must not increase too rapidly with time or the radial oscillationswill become unstable. From Eq. (45) we see that X, the amplitude of the radial oscillations, is given by

$$X \propto \frac{(kV\mu)^{1/4}}{H_o} \tag{51}$$

If $H_o \propto t$, V must never increase more rapidly than t^4, in the non-relativistic case, or t^3 in the relativistic case. Such an increase would, of course, be quite impractical so that the limitation is not a real one.

7. The Effect of Letting the Radio Frequency Vary from the Given Law.

Let us suppose that the radio frequency is allowed to vary by a small amount from the given law, so that

$$\omega = \omega_o + \omega_1 \tag{52}$$

is now the angular frequency where ω_1 is small.

To find the effect in the general case we must put

$$\dot{\varphi}_1 = \dot{\psi} + \omega_1 \tag{53}$$

in Eqs. (10) and (23). If we do this and neglect all derivatives above the second when deriving the equation for the synchrotron oscillations, we obtain

$$-\frac{x}{r_o}\left(\frac{\mu_o^{\,2}}{m_o^{\,2}} - (1-n)\right) = \frac{\dot{\psi} + \omega_1}{\omega_o}\,\frac{\mu_o^{\,2}}{m_o^{\,2}} \tag{54}$$

and

$$\frac{r_o^{\,2}(1-n)}{c}\,\frac{d}{dt}\left(\frac{H_o\dot{\psi}k}{\omega_o}\right) + \frac{V}{2\pi}\,\sin\psi = \frac{V}{2\pi}\,\sin\psi_o - \frac{r_o^{\,2}(1-n)}{c}\,\frac{d}{dt}\left(\frac{H_o\omega_1 k}{\omega_o}\right) \tag{55}$$

where $V \sin \psi_0$ is given, as before, by Eq. (18).

The change in radio frequency thus adds a small additional forcing term to the phase oscillations and also changes the radius of the particle orbit. The latter effect is far the more important.

If ω_1 is approximately constant, the forcing term in Eq. (55) can be neglected and the frequency error produces a definite shift in orbit radius. If, however, ω_1 oscillates rapidly, it is better to put $\dot{\theta} = \dot{\psi} + \omega_1$, $\theta = \psi + \int \omega_1 dt$ to give

$$-\frac{x}{r_0}\left(\frac{\mu_0^2}{m_0^2} - (1-n)\right) = \frac{\dot{\theta}}{\omega_0}\frac{\mu_0^2}{m_0^2} \qquad (56)$$

$$\frac{r_0^2(1-n)}{c}\frac{d}{dt}\left(\frac{H_0\dot{\theta}k}{\omega_0}\right) + \frac{V\theta}{2\pi} = V\left(\theta_0 - \int \omega_1 dt\right) \qquad (57)$$

if $\sin \theta \approx \theta$, $\sin \theta_0 \approx \theta_0$.

If $\int \omega_1 dt$ stays small, the only effect is to produce a variation in the stable phase position with accompanying small change in the maximum phase velocity.

A most important special case occurs, in the non-relativistic region, when the frequency error is linearly proportional to the time, say

$$\omega_1 = \alpha t \qquad (58)$$

In this case, the equation determining the phase oscillations becomes

$$\frac{r_0^2(1-n)H_0}{c\omega_0 n}\dot{\psi} + \frac{V \sin \psi}{2\pi} = \frac{V}{2\pi}\sin \psi_0 - \frac{r_0^2(1-n)H_0\,\alpha}{c\omega_0 n} \qquad (59)$$

as H_0k/ω_0 is a constant equal to m_0c/en in the non-relativistic range.

If V is a constant, as will be approximately true, this equation gives

$$\frac{r_0^2(1-n)H_0}{c\omega_0 n}\ddot{\psi} + \frac{V \sin \psi}{2\pi} = \frac{V}{2\pi}\sin \psi_1 \qquad (60)$$

where

$$\psi_1 > \psi_0 \quad \text{if} \quad \alpha < 0$$

$$\psi_1 < \psi_0 \quad \text{if} \quad \alpha > 0 \,.$$

The radial displacement is given by

$$-\frac{xn}{r_0} = \frac{\dot{\psi}}{\omega_0} + \frac{\alpha t}{\omega_0} \qquad (61)$$

The effects of this frequency error are thus twofold. There is an instantaneous jump in the stable phase angle while the orbit about which the radial oscillations are taking place is moved steadily outwards if $\alpha < 0$ and steadily inwards for $\alpha > 0$. This result is utilized by Dennison in his injection scheme for the Rochester race-track electron synchrotron and it is analyzed in detail in a companion paper.

The jump in stable phase angle is unimportant if ψ_0 and ψ_1 are both small. If $\psi_0 - \psi_1$ is large, however, both the amplitude of the phase oscillations and the limiting angles within which oscillation is possible will alter, with possible loss of particles. To minimise this possibility it is advisable to have a fairly small value for the stable phase angle ψ_0, and the r-f peak voltage in a practical f-m synchrotron should be at least twice the acceleration voltage per turn, i.e. $\psi_0 < \pi/3$.

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