NONLINEAR EFFECTS IN THE INTERACTION

BETWEEN WAVES AND TOPOGRAPHY

by

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ABSTRACT

This thesis consists of three separate problems unified by two basic features. First, in all the problems a nonlinear effect either in the form of a wave-wave interaction or a wave-zonal flow interaction plays a key role. Second, all the problems involve the instability of a topographically forced wave for which resonance plays a key role. The method of multiple time scales is used to analyze each problem which exploits the assumptions of weak nonlinearity in each case as well as the assumption that the flow is nearly neutral or near resonance or both depending on the problem under consideration.

In Chapter 2 the weakly nonlinear effects on the instability of a basic state consisting of a topographically forced wave in an inviscid, barotropic beta-plane model are studied. The results obtained differ substantially from those obtained when the basic state is a free Rossby wave. Here the basic-state wave is fixed in phase with respect to the mountain, while the amplitude of the topographic wave and perturbation evolve. The nonlinear feedback between the topographic wave and perturbation gives rise to an oscillation for a topographically subresonant zonal flow and an explosive nonlinear instability for a topographically superresonant zonal flow. In the subresonant case, the effect of the perturbation on the forced wave is a dissipative one, when averaged over the course of the nonlinear oscillation. The standing topographic wave interacts with the traveling instability on the topographic wave through the convergence of Reynolds' stresses which is suggestive of the way in which standing and traveling eddies interact in the atmosphere.

In Chapter 3, under the assumption of quasi-resonant flow, the topographic instability of a normal mode of the Charney problem is examined and found to draw its energy from both the available potential energy of the mean flow and from the zonal momentum via form drag. No apriori truncation is necessary. The instability occurs for either superresonant or subresonant flow depending on the zonal and meridional scale of the topography. In addition it is limited to narrow bands in meridional wavenumber with relatively little dependence on zonal wavenumber. It is suggested that this may explain the more well defined meridional structure of blocking.

In Chapter 4, the evolution equations are obtained for the instability of a small but finite-amplitude topographically forced nearly neutral Charney mode embedded in a mean shear flow. The topography is found to have a stabilizing effect on the baroclinic instability, but a new topographically induced stationary instability arises even for baroclinically subcritical flow. It is most easily excited in superresonant flow near resonance for narrow bands in meridional wavenumber. This topographic instability of the Charney mode draws its energy from the available potential energy of the mean flow and survives even in the presence of $O(1)$ Ekman friction if the Ekman friction is not too strong. Thus, it is suggested that the topographic instability of the Charney mode may be just as dynamically important as the baroclinic instability of the Charney mode. In addition, it is suggested that the normal modes of the Charney model, due to their topographic instability, are just as dynamically important as the baroclinically unstable modes of the Charney model in the presence of topography.

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Chapter 1

Introduction

Historically speaking, most stability problems with atmospheric applications have treated zonally uniform basic states which, in general, were allowed to vary in both the vertical and meridional directions. The assumption of zonality in these unforced and usually inviscid stability problems was considered justified as a rough approximation to the quasi-steady Hadley circulation and by its success in explaining various aspects of the observed global circulation. In particular, such stability problems have been successful in explaining the existence of the midlatitude cyclones by the structure of the most unstable modes arising from the assumed basic state.

Of course as these wave disturbances grow the assumption of linearity breaks down which forces one to consider the effects of nonlinearity on the evolution of the disturbance. The first systematic study of this sort was given in a paper by Pedlosky (1970) in which a small but finite-amplitude disturbance was shown to be able to modify the linear instability such that the disturbance grew exponentially only in its nascent stage. In particular he demonstrated that as the baroclinic disturbance grows, it decreases the vertical shear, on which it depends for its growth. This slows the growth of the disturbance and eventually causes it to decay. Then as the disturbance decays the zonal available potential energy is replenished until the disturbance again becomes unstable. In this manner an oscillation ensues due to the nonlinear feedback between the finite-amplitude disturbance and the vertical shear of the zonal flow.

However, having established that the baroclinic disturbance can grow to finite amplitude, one is faced with a further question. What are the stability properties of the resulting wave disturbances? This question forces the consideration of more general basic states.

As a prototype model of wave instability in the atmosphere Lorenz (1972) considered the instability of a stationary one-dimensional Rossby wave. He found that the Rossby wave is indeed unstable and suggested that such instability explained a portion of the unpredictability of the atmosphere. Gill (1974) considered a basic state Rossby wave which varied both cross-stream and downstream and showed that, for small basic wave amplitude, the instability is confined to a small portion of wavenumber space, which corresponds to the locus of wavenumbers satisfying the kinematic resonance conditions (or near resonance conditions) for three waves. For small amplitudes, instability is thus limited in wavenumber space, while instability, for larger basic wave amplitudes, studied by Lorenz (1972), is much less limited in wavenumber space. Gill refers to the instability as resonant instability for small basic wave amplitude and Rayleigh instability for large basic wave amplitude. The weakly nonlinear behavior of a resonant triad of waves has been examined by Longuet-Higgins and Gill (1967). In that work they described the nonlinear oscillation between the three resonant waves which ensues after the initial stage of the linear instability. Recently I (i.e. Deininger, 1982) have considered the weakly nonlinear evolution of the Rayleigh type wave instability. The result obtained was analogous to that obtained by Pedlosky (1970) but differed in a fundamental way due to the wavy basic state I was dealing with as opposed to the vertically sheared zonal flow in Pedlosky's model. Namely, there was a feedback between the growing perturbation and Rossby wave amplitude analogous to the feedback between the growing baroclinic wave and zonal shear in Pedlosky's case but, in addition, there was a feedback between the disturbance and Rossby wave phase. This phase feedback has no analog in the case considered by Pedlosky (1970) and was suggested as a possible mechanism for the nonconstant speed of progression of midlatitude high and low pressure systems.

Lin (1980a,b) has studied the stability of the more realistic basic state consisting of a one-dimensional free neutral baroclinc wave embedded in a vertically sheared zonal flow. He demonstrates as one might expect that for strong wave amplitude as compared to zonal shear the instability is fundamentally wavelike, that is, the instability excites a spectrum of waves which is generally not centered at the scale of the most unstable mode of the vertically sheared zonal flow alone. For relatively weak basic state wave amplitude the instability behaves like the instability of the vertically sheared zonal flow but slightly modified by the presence of the wave. However, had he considered a basic state wave with meridional structure as well, he would have found that a locus of other wavenumbers might also be excited corresponding to those neutral modes which can form a resonant triad with most unstable baroclinic mode. In fact Loesch (1974) considered the weakly nonlinear evolution of a weakly unstable baroclinic wave which formed a resonant triad with two neutral baroclinic waves. He demonstrated that as the unstable baroclinic wave draws available potential energy out of the mean flow it can be leaked to the neutral baroclinic modes via the resonant triad wave-wave interaction.

Until now only free wave instabilities have been discussed. Another mechanism for distorting the zonal uniformity of the Hadley circulation involves the diversion of the zonal flow by topography. The linear stability of such a topographically forced wave has received relatively little treatment until recently. Charney and Flierl (1981) analyzed the linear instability of a barotropic wave forced by the diversion of a uniform zonal flow by sinusoidal topography. In general the instability of the topographically forced wave is due both to the shear of the forced wave and the way in which the disturbance interacts with the topography. Thus, the instability of the forced wave is shown to be unstable to Rayleigh or resonant triad instabilities as is the free Rossby wave. In Chapter 2, I examine the weakly nonlinear evolution of a topographically forced wave and the traveling instability which develops on it in an effort to understand in a simple internally consistent manner, the interaction of their atmospheric relatives. The basic technique employed in this regard is the method of multiple time scales and is the same as that used in the weakly nonlinear analysis of the free Rossby wave instability problem (i.e., Deininger, 1982). Briefly stated, we proceed as follows: The perturbation nonlinearity is required to be weak. That is to say, while the interaction between the basic state wave and perturbation is O(1), the interaction of the perturbation with itself and with corrections to the basic state wave is required to be weak. Having made this assumption we choose the small amplitude parameter such that the weak nonlinearity balances the slow growth of the slightly unstable perturbation. The slow growth is attained by hinging the nonlinear stability analysis about the point of neutral stability. However, the results of the nonlinear forced wave problem in Chapter 2 differ markedly from those of the free wave problem. Therefore a comparison of the two problems will be particularly interesting.

In the linear stability analysis of Charney and Flierl (1981), they also show that there is a limit for which the instability is not of the Rayleigh or resonant triad type. In this limit, if the zonal flow is superresonant, the instability becomes stationary and is referred to as a form-drag instability or topographic instability. It was first examined in a paper by Charney and Devore (1979) in which they sought to explain the quasi-steady existence of anomalously high pressure in certain geographical regions (blocking). They suggested that the stationary topographic instability is responsible for the transition from the normal relatively zonal state of the atmosphere to the blocking situation as the zonal flow became superresonant. The mechanism for the instability can be most simply stated as follows: An increase in the topographically forced wave amplitude increases the form drag. This increase in form drag then decreases the zonal flow on which the forced wave depends. If the flow is superresonant the decrease in zonal flow increases the amplitude of the forced wave which in turn further decreases the zonal flow via the form drag. Clearly this cycle leads to instability. The form drag removes momentum from the zonal flow which increases the forced wave amplitude.

The study of Charney and Devore (1979) was later extended to a two-layer baroclinic model by Charney and Strauss (1980) with no zonal flow at the ground. Again topographic instability is found but in this case the perturbation grows at the expense of the zonal available potential energy. The topography and therefore the form drag merely act as the catalyst for this energy transfer. Had there been nonzero flow at the ground there would also have been a conversion of zonal momentum by the form drag.

In both the studies of Charney and Devore (1979) and Charney and Strauss (1980) an *ad hoc* severely truncated spectral representation of the motion is employed. This has led subsequent authors to consider ways of isolating the topographic instability in a more systematic manner. Hart (1979) did so in a barotropic model by restricting the topographic features he considered to those of nearly infinite meridional extent. In this manner he neglected the asymptotically small Reynolds stresses which were neglected in an *ad hoc* manner in the truncated models. The limit taken in the forced wave instability study of Charney and Flierl (1981) for which the wave instability reduced to topographic instability, corresponds to the assumptions made by Hart (1979) for the validity of his asymptotic theory. However, Hart's analysis remains unsatisfactory due to the unrealistic assumption of topography with nearly infinite meridional extent -- to begin with the very assumption which allowed the simplified analysis.

An alternative approach has been taken by Pedlosky (1981), Plumb (1981a,b) and Jacobs (1979a,b) within the context of various models based on the analysis of the forced Duffing equation (Stoker, 1950). The asymptotic technique employed by these authors exploits the importance of the linear resonance in the topographic instability and the quasi-linear nature of the topographic instability. The latter point, based on the previous studies of topographic instability, says that the important interaction for the sake of the topographic instability is between the wave field topography and the zonal flow. As a result wave-wave interactions can be neglected, although this should be done in a systematic manner. To accomplish this, the flow was restricted to near linear resonance and weak nonlinearity.

Both Pedlosky (1981) and Jacobs (1979a) worked within the barotropic model. Pedlosky pointed out that the topographic instability may also occur for subresonant zonal flow if the zonal wavelength is long enough. He then extended this study to a two-layer model with non zonal flow at the ground as was the case in the work of Charney and Strauss (1980). In an independent study, Plumb (1981a) employed a similar approach in the context of the two-layer model, however he did not consider a real topographic lower boundary but rather he imposed a moving forcing on the lower boundary which eliminated the feedback between the zonal flow and wave field at the ground.

The topic of Chapter 3 is a study of the topographic instability in the continuously stratified Charney (1947) model, employing most closely, the asymptotic technique used by Pedlosky (1981) in the barotropic and two-layer baroclinic models. This study is undertaken in order to understand, in a deductive way, the nature of the topographic instability in a simple continuously stratified model. Specifically the topographic instability of a normal mode of the Charney model is examined. The normal modes which are found on the long wave or weak vertical shear side of the unstable Charney mode neutral curve, become stationary and therefore resonant only for a given eastward zonal velocity at the ground. The exact value depends on the mode in question. Thus, it is to be expected that the disturbance will draw its energy from both the available potential energy of the vertically sheared zonal flow with the topography acting as a catalyst and from the zonal momentum via the form drag. As in the barotropic model this topographic instability might be thought of as a special case of a more general wave instability of the Rayleigh type, in which the topography acts as a catalyst for the instability of a normal mode of the Charney problem. In this limit the normal modes would not be unstable in the absence of topography. Plumb (1981b) and Jacobs (1979b) have also considered the question of topographic instability in a continuously stratified model. The difference between those studies and mine will also be discussed in Chapter 3.

In Chapter 4 the main topic of study is the stability of a small but finite-amplitude topographically forced Charney mode which, of course, is embedded in a uniform vertically sheared flow. Note we distinguish between the neutral modes of the Charney problem and the normal modes of the Charney problem, the latter of which can only become neutral, or nearly neutral, on a line in parameter space. We will be restricted to the nearly neutral portion of parameter space. Thus, Chapter 4 deals with a mode which can be weakly unstable even in the absence of topography in contrast to the subject of Chapter 3. This fact brings to mind two questions we shall attempt to answer in Chapter 4. (1) How does the presence of the sinusoidal topography affect the baroclinic instability of the zonal flow and (2) does any new instability arise in parameter regimes for which the zonal flow is baroclinically stable due to the presence of topography?

One might view this study as the analog of Lin's (1980a,b) study of a neutral baroclinic wave embedded in a vertical shear flow in the limit of small but finite wave amplitude except, in the present work, the wave is forced by topography and not limited to strict neutrality because of its small amplitude. The method of analysis is very similar to that used by Pedlosky (1979) to study the finite amplitude dynamics of a growing Charney mode. In particular the analysis pivots about the point of neutral stability for the Charney mode. The method of multiple time scales is used to obtain the evolution of a weakly growing Charney mode under the assumption of weak nonlinearity <u>and</u>, in the present case, weak sinusoidal topography.

For the purpose of summary I will reiterate the subjects to be discussed in Chapters 2, 3 and 4. In

Chapter 2 the finite-amplitude dynamics of the topographically forced wave instability in a barotropic model are discussed. The linear disturbance draws its energy from the shear of the forced wave. A special case of this linear instability is the topographic instability. In Chapter 3 the topographic instability of a normal mode of the Charney model is discussed. This normal mode would not be unstable in the absence of topography. The subject of Chapter 4 is the instability of a topography forced Charney mode which can be unstable in the absence of topography. As will be shown, all these topics involve nonlinear effects in the interaction between waves and topography.

Chapter 2 Topographically Forced Wave Instability at Finite Amplitude

1. Introduction

The interaction between standing and transient eddies in the atmosphere is of fundamental importance to meteorologists. In an observational study of this interaction, Holopainen (1978) has shown the vertically averaged horizontal fluxes of relative vorticity by small- and large-scale transient eddies to be important in maintaining the vertically averaged, annual mean, standing eddies' vorticity balance. In this chapter, as a first step in understanding the interaction between the standing and transient eddies in the atmosphere, a barotropic model will be studied in which the nonlinear evolutions of a stationary topographically forced wave and the travelling instability which develops on it are considered. The topographic wave is forced by the diversion of a uniform zonal current by sinusoidal topography. The tacit assumptions here are that the barotropic topographically forced wave will represent the vertically averaged standing wave in the atmosphere and the traveling, linear, barotropic disturbance which develops on the topographic wave will represent the large-scale vertically averaged transient atmospheric disturbances. Obviously, the cyclone-scale transient eddies cannot be properly represented in a barotropic model. Although these representations of atmospheric topographic stationary and large-scale transient eddies are crude, the interactions between them can be studied in a simply internally consistent manner in order to provide some insight into the interaction of their atmospheric counterparts.

To study the problem of the interaction between a forced standing eddy and traveling disturbances which develop on it as instabilities, the finite-amplitude characteristics of linearly unstable perturbations to a topographically forced wave will be analyzed. The linear stability problem has been discussed by Charney and Flierl (1981). In order to study the nonlinear effects on the evolution of the perturbation and the topographic wave, the non-linearity is required to be weak. Then, as in Deininger (1982, hereafter referred to as DE), the method of multiple time scales can be used to close the problem for a slowly growing perturbation. In performing this analysis, it will be found that the nonlinear evolution of this topographically forced wave stability problem is markedly different from that of the free Rossby wave stability problem described in DE. This contrast will make the comparison of the nonlinear aspects of the topographically forced wave stability problem and the free Rossby wave stability problem particularly interesting.

Clearly, the atmosphere is baroclinic; however, based on some preliminary calculations, the behavior to be described in this paper using the barotropic model to be discussed in the next section, does have a counterpart in baroclinic models. Therefore, the results of this chapter should be regarded as providing a foundation for the understanding of stationary and transient wave interactions in more realistic baroclinic models.

2. The model

The nondimensional vorticity equation governing the barotropic motion of a quasigeostrophic, inviscid, homogeneous fluid on an infinite beta-plane, bounded in the vertical direction by an upper flat horizontal plate and by a lower corrugated plate to act as topography, is

$$\frac{\partial}{\partial t} \nabla^2 \Psi + \beta \frac{\partial}{\partial x} \Psi + J(\Psi, \nabla^2 + \eta) = 0 \tag{2.1}$$

where $\eta = h_B/\varepsilon D$ is $O(1)$ and represents the topographic variation. h_B is the height of the topography, $\varepsilon = U/f_0 L$ is the Rossby number, and D is the mean depth of the fluid. All other variables have their conventional meanings and are the same as those in DE. The topography is assumed to be sinusoidal, i.e.,

$$\eta = \eta_0 \sin\theta, \tag{2.2a}$$

where

$$\theta = kx + ly. \tag{2.2b}$$

3. Formulation of the nonlinear problem

An exact solution of (2.1) for sinusoidal topography (2.2) whose stability Charney and Flierl (1981) studied, is the topographically forced wave solution

$$\Psi = -Uy + F \sin\theta, \qquad (3.1)$$

where

$$F = \frac{\eta_0 U}{K^2(U - c)}, \qquad (3.2)$$

and $c = \dfrac{\beta}{K^2}$.

This solution exhibits a singularity when

$$U = c, \qquad (3.3)$$

in which case the Doppler-shifted wave field becomes stationary relative to the topography. In the subsequent analysis it will be assumed $U-c=O(1)$, so the forced wave is topographically nonresonant.

If ψ is the disturbance streamfunction which is to be superimposed on topographic wave solution, i.e.,

$$\Psi = -Uy + F \sin\theta + \psi, \qquad (3.4)$$

substitution of (3.4) into (2.1) using (3.2) yields the problem for the perturbation streamfunction, which can be written

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) \nabla^2\psi + \beta \frac{\partial}{\partial x} \psi + \frac{\eta_0}{K^2(U - c)}$$

$$\times \cos\theta\left(k \frac{\partial}{\partial y} - l \frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi = - J(\psi, \nabla^2\psi). \qquad (3.5)$$

If ψ is infinitesimally small, linearization of (3.5) is justified and yields

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi + \frac{\eta_o}{K^2(U - c)}$$

$$\times \cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi = 0. \qquad (3.6)$$

Eq. (3.6) is a more general form of the linear stability problem solved by Charney and Flierl (1981). The solution to (3.6) is of the form

$$\psi = e^{i\lambda t}\sum_n P_n e^{i\theta_n} + *, \qquad (3.7)$$

with

$$(k_n, l_n) = (k_0, l_0) + n(k,l).$$

Substitution of (3.7) into (3.6) results in a recursion relation for the P_n quite similar in form to Gill's (1974) recursion relation for the stability of a free Rossby wave. It is

$$\eta_o d a_{n+1} Q_{n+1} + (\lambda + \delta_n)Q_n + \eta_o d a_{n-1} Q_{n-1} = 0, \qquad (3.8)$$

where

$$d = \frac{b}{K^2(U - c)}, \qquad (3.9a)$$

$$b = \frac{1}{2}(kl_0 - lk_0), \qquad (3.9b)$$

$$a_n = U - c_n, \qquad (3.9c)$$

$$c_n = \beta/K_n^2, \qquad (3.9d)$$

$$K_n^2 = k_n^2 + l_n^2, \qquad (3.9e)$$

$$\delta_n = k_n a_n, \tag{3.9f}$$

$$Q_n = K_n^2 P_n. \tag{3.9g}$$

Solution of (3.8) will require the truncation of the infinite set of homogeneous equations for P_n generated by (3.8). As in the free Rossby wave analysis of DE, the qualitative nature of the weakly nonlinear dynamics can be reasonably represented by the two mode approximation, i.e.,

$$\psi = e^{i\lambda t}\,(P_0 e^{i\theta_0} + P_1 e^{i\theta_1}) + *. \tag{3.10}$$

The analysis to follow can be carried out for <u>any</u> finite trnucation and yield the same qualitative results, so the truncation of (3.10) is not as restrictive as it first may seem. The results of a three mode truncation of (3.7) are given in Appendix A.

After truncating the sum (3.7) to (3.10) the infinite set of equations reduces to

$$(\lambda + \delta_1)Q_1 + \eta_0 d a_0 Q_0 = 0,$$

$$\eta_0 d a_1 Q_1 + (\lambda + \delta_0)Q_0 = 0. \tag{3.11}$$

For the nontrivial solution of (3.11) it is required that

$$\lambda = -\left(\frac{\delta_1 + \delta_0}{2}\right) \pm \frac{1}{2}\,[(\delta_1 - \delta_0)^2 + 4\eta_0^2 d^2 a_1 a_0]^{1/2}\ . \tag{3.12}$$

Then for instability to occur,

$$a_1 a_0 < 0 \tag{3.13a}$$

is required and the mountain height must exceed a critical value which is

$$\eta_c^2 = -\frac{(\delta_1 - \delta_0)^2}{4d^2 a_1 a_0} . \tag{3.13b}$$

Since the perturbation was chosen such that $K_0^2 < K^2 < K_1^2$, it follows that $\beta/K_1^2 < \beta/K^2 < \beta/K_0^2$. Then, from (3.13a) along with (3.9c,d) it can be seen that both subresonant and superresonant flow is possible within the parameter range corresponding to linear instability.

If the topography is just slightly higher than η_c by a small amount $\Delta(\Delta << \eta_c)$ such that

$$\eta_o = \eta_c + \Delta, \tag{3.14}$$

the growth rate of the perturbation is proportional to $|\Delta|^{1/2}$, i.e.,

$$\lambda_i^2 = -2d^2\eta_c^2 a_1 a_0 \Delta . \tag{3.15}$$

As in the free Rossby wave problem, (3.15) suggests the long time scale

$$T = |\Delta|^{1/2} t$$

as the one over which the nonlinear evolution will occur when the perturbation is allowed to be finite but small. Now the time operator in (3.5) is replaced by

$$\frac{\partial}{\partial t} + |\Delta|^{1/2} \frac{\partial}{\partial T}$$

so (3.5) can be rewritten as

$$\left(\frac{\partial}{\partial t} + |\Delta|^{1/2} \frac{\partial}{\partial T} + U \frac{\partial}{\partial x}\right) \nabla^2 \psi + \beta \frac{\partial}{\partial x} \psi$$

$$+ \frac{\eta_c + \Delta}{K^2(U - c)} \cos\theta\left(k \frac{\partial}{\partial y} - 1 \frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi$$

$$= - J(\psi, \nabla^2\psi). \tag{3.16}$$

where (3.14) was also used. It now remains to choose the appropriate expansion for ψ so the weak effects of nonlinearity and instability can be balanced. As in DE, the appropriate choice is

$$\Psi = |\Delta|^{1/2}\psi^{(1)} + |\Delta|\psi^{(2)} + |\Delta|^{3/2}\psi^{(3)} + \cdots. \tag{3.17}$$

Substitution of (3.17) into (3.16) results in a series of problems for the successive powers of $|\Delta|^{1/2}$. The $O(|\Delta|^{1/2})$ problem leads to the specification of the neutral perturbation already calculated. The neutral solution is

$$\psi^{(1)} = \sum_{n=0}^{1} L_n P_0 e^{i\hat{\theta}_n} + *, \tag{3.18}$$

where

$$\hat{\theta}_n = \theta_n - \left(\frac{\delta_1 + \delta_0}{2}\right)t, \tag{3.19a}$$

$$L_1 = \frac{(\delta_1 - \delta_0)K_0^2}{2d\eta_c a_1 K_1^2}, \tag{3.19b}$$

$$L_0 = 1. \tag{3.19c}$$

So far the analysis is essentially the same as in DE except the time stretching necessary in the free Rossby wave analysis has not been done here. Doing so would later prove unnecessary. This is due to the fact that the basic-state wave is forced, causing the basic state wave structure, produced by the self-interaction of the perturbation at the next order, to be nonresonant. As a result, the self-interaction of the perturbation produces a forced solution instead of producing a phase change of the basic state wave which in DE required the use of a stretched time coordinate. This foreshadows the major difference between the free Rossby and topographically forced wave analyses. This difference and the effects of nonlinearity will begin to be apparent at the next order.

4. The nonlinear analysis

The nonlinearity at $O(|\Delta|)$ is due to the self-interaction of the perturbation field. Using (3.18) to calculate the inhomogeneous terms at $O(|\Delta|)$, and retaining only those terms consistent with the original truncation, results in the equation

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \nabla^2 \psi^{(2)} + \beta \frac{\partial}{\partial x} \psi^{(2)} + \frac{\eta_c}{K^2(U - c)}$$

$$\times \cos\theta\left(k \frac{\partial}{\partial y} - l \frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi$$

$$= K_0{}^2 \frac{\partial P_o}{\partial T} e^{i\hat{\theta}_0} + K_1{}^2 L_1 \frac{\partial P_o}{\partial T} e^{i\hat{\theta}_1}$$

$$+ 2b(K_1{}^2 - K_0{}^2)L_1 P_0{}^2 e^{i\vartheta_1} -$$

$$- 2b(K_1{}^2 - K_0{}^2)L_1 |P_0|^2 e^{i\theta} + *, \tag{4.1}$$

where

$$\theta_1 = \bar{k}_1 x + \bar{l}_1 y - (\delta_1 - \delta_0)t,$$

$$(\bar{k}_n, \bar{l}_n) = (2k_0 + nk,\ 2l_0 + nl).$$

The first three inhomogeneous terms in (4.1) are the same as in the free Rossby wave analysis and therefore give rise to the particular solutions

$$\psi_P^{(2)} = \sum_{n=0}^{1} P_n^{(2)} e^{i\hat{\theta}} + *, \tag{4.2}$$

where

$$P_1^{(2)} = \frac{2iL_1}{\delta_1 - \delta_0} \frac{\partial P_0}{\partial T},$$

$$P_0^{(2)} = 0$$

and

$$\psi_f^{(2)} = \sum_{n=0}^{1} F_n^{(2)} e^{i\hat{\theta}_n} + *, \tag{4.3}$$

where

$$F_n^{(2)} = if_n(P_0)^2 .$$

the f_n are given in Appendix B. The remaining inhomogeneous term is analogous to the one which in the free Rossby wave analysis was resonant and therefore produced a phase correction of the Rossby wave. However, here it is nonresonant for $U \neq c$ and produces a forced solution in the basic state structure which represents an amplitude correction to the topographically forced basic state wave. This solution is

$$\psi_B^{(2)} = F_B e^{i\theta} + *, \tag{4.4a}$$

$$F_B = i f_B |P_0|^2, \tag{4.4b}$$

and

$$f_B = - \frac{(K_1^2 - K_0^2)(\delta_1 - \delta_0)K_0^2}{2k\eta_c a_1 K_1^2} \tag{4.4c}$$

This difference is due to the basic state being forced as opposed to being free. The solution to $O(|\Delta|)$ in the basic state structure using (3.1), (3.2) and (4.4), is

$$\Psi_B = - Uy + [\frac{(\eta_c + \Delta)}{K^2(U - c)} U + |\Delta| \frac{(K_1^2 - K_0^2)(\delta_1 - \delta_0)K_0^2}{k\eta_c a_1 K_1^2} |P|^2] \sin\theta, \tag{4.5}$$

and has an interesting property. Note the $O(|\Delta|)$ correction to the basic state wave is always a positive contribution to the effective basic wave amplitude which is defined as the coefficient of $\sin\theta$ in (4.5). This is so since the conditions $a_1 > 0$ and $a_0 < 0$ must be true for instability to occur. For subresonant flow ($U < c$) the absolute value of the effective basic state wave amplitude decreases as the perturbation grows, but for superresonant flow ($U > c$) the effective basic-state wave amplitude increases. Thus, for superresonant flow there seems to be a further destabilization which should be apparent when the behavior of P_0 is determined at the next order.

The $O(|\Delta|^{3/2})$ problem can be written

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \nabla^2\psi^{(3)} + \beta \frac{\partial}{\partial x} \psi^{(3)} + \frac{\eta_c}{K^2(U - c)}$$

$$\times \cos\theta\left(k \frac{\partial}{\partial y} - 1 \frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi^{(3)}$$

$$= - \frac{\partial}{\partial T} \nabla^2\psi^{(2)} - \frac{\Delta}{|\Delta|} \frac{1}{K^2(U - c)}$$

$$\times \cos\theta\left(k \frac{\partial}{\partial y} - 1 \frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi^{(1)}$$

$$- J(\psi^{(1)}, \nabla^2\psi^{(2)}) - J(\psi^{(2)}, \nabla^2\psi^{(1)}). \qquad (4.6)$$

Using (3.18), (4.2), (4.3) and (4.4), the inhomogeneities of (4.6) can be evaluated. At this order only terms proportional to $e^{i\theta_0}$ and $e^{i\theta_1}$ need be retained, as the other terms only produce forced solutions which are not needed to determine the behavior of P_0 on the long time scale T. After evaluating the relevant inhomogeneous terms, (4.6) becomes

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \nabla^2\psi^{(3)} + \beta \frac{\partial}{\partial x} \psi^{(3)} + \frac{\eta_c}{K^2(U - c)}$$

$$\times \cos\theta\left(k \frac{\partial}{\partial y} - l \frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi^{(3)}$$

$$= \left[K_1{}^2 \frac{\partial P_1{}^{(2)}}{\partial T} + \frac{\Delta}{|\Delta|} ida_0K_0{}^2P_0 - 2ib(K_0{}^2 - K^2)f_BP_0|P_0|^2\right.$$

$$\left. + 2ib(K_0{}^2 - \overline{K_1{}^2})f_1P_0|P_0|^2\right] e^{i\hat{\theta}_1}$$

$$+ \left[\frac{\Delta}{|\Delta|} ida_1K_1{}^2L_1P_0 - 2ib(K_1{}^2 - K^2)L_1f_1P_0|P_0|^2\right] e^{i\hat{\theta}_0}$$

$$+ *. \tag{4.7}$$

Removal of the secularities from (4.7) results in the evolution equation for P_0 which is

$$\frac{d^2P_0}{dT^2} - \frac{\Delta}{|\Delta|} \sigma_i{}^2P_0 + NP_0|P_0|^2 = 0, \tag{4.8}$$

where the growth rate

$$\sigma_i{}^2 = \frac{\lambda_i{}^2}{\Delta} = -2d^2\eta_c a_1 a_0,$$

agrees with the linear result (3.15). The nonlinear coefficient

$$N = N_B + N_H,$$

is comprised of a part due to the feedback with the basic state, i.e.,

$$N_B = \frac{b^2K_0{}^2}{kK^2K_1{}^2} \frac{(K_1{}^2 - K_0{}^2)(\delta_1 - \delta_0)}{a_1(U - c)}$$

$$\times \quad [(\frac{K_0{}^2 - K^2}{K_0{}^2})a_1 + (\frac{K_1{}^2 - K^2}{K_1{}^2})a_0] \ , \qquad (4.9)$$

and a part due to the feedback with higher harmonics, i.e.,

$$N_H = \frac{bf_1(\delta_1 - \delta_0)}{L_1} \ \frac{(K_0{}^2 - K_1{}^2)\ (\overline{K_1}{}^2 - K^2)}{(K_1{}^2 - K^2)} \ . \qquad (4.10)$$

The latter feedback is essentially the same as discussed in the free Rossby wave analysis. However, the former feedback mechanism is unique to this chapter. Upon examination of N_B it can be seen for subresonant flow ($U < c$) that

$$N_B > 0,$$

which says the nonlinear feedback is stabilizing and therefore gives rise to an oscillatory exchange of energy between the effective basic state wave amplitude and perturbation. When this oscillation is averaged over one cycle, the effect of the perturbation reinforces the topographic wave. This dissipative effect on the topographic wave is in qualitative agreement with the observational and numerical model results of Youngblut and Sasamori (1980). But for superresonant flow $N_B < 0$, so the nonlinear feedback is destabilizing and results in an explosive instability in which P_0 blows up in a finite time. This was to be expected from the solution at $O(|\Delta|)$, which showed the effective amplitude of the basic state to be increased as the perturbation grew, causing it to be more unstable. This explosive instability is somewhat reminiscent of the form-drag instability discussed by Charney and Devore (1979) in that it occurs for superresonant flow. However, the explosive instability here is actually a distinct phenomenon. It arises by the interaction of the topographically forced wave with its perturbation via the convergence of Reynolds' stresses. Thus, this instability is independent of form drag (note N_B is independent of η_c), whereas the form-drag instability is due to the interaction of the topographically forced wave with a zonal flow via the form drag. However, whether or not the explosive instability occurs actually depends on the sign of the full nonlinear coefficient N, not just N_B. Thus the sign and magnitude of N_H relative to N_B is also important. In fact, if N_H is positive and large as compared to N_B, the explosive instability could occur only near resonance where the analysis is not valid. Therefore, a specific numerical calculation of N is desirable. Unfortunately, I would not expect the calculation of N using the two-mode truncation to be representative of the N calculated using less restrictive truncations. Therefore, this calculation would not be meaningful until the convergence properties of N can be established, which is beyond the scope of the present work.

As previously mentioned, the flow can be barotropically resonant only for unrealistically large values of zonal velocity for planetary-scale waves in this model, making the explosive instability unlikely. However, in a baroclinic system, resonances corresponding to free baroclinic modes may be more easily realized from the point of view of the zonal velocity required, which may allow the explosive instability to be more easily excited provided the topographically forced wave and linear instability are vertically trapped. Also, in a baroclinic model the baroclinic nature of the topogrpahically forced basic state (e.g., Holopainen, 1970) as well as short wavelength transient eddies and some large-scale transient eddies (e.g., Bottger and Fraedrich, 1980) can be accounted for. Therefore, extending this analyses to a baroclinic model is worth further consideration. The neglect of baroclinic and frictional effects in this study precludes any comparison of the present results with the atmosphere.

A few remarks concerning truncation which apply to this analysis as well as that given in DE are in order. In the course of this analysis it has been convenient to truncate the perturbation (3.7) to (3.18); however, it is possible to carry out this analysis for any finite truncation. For less restrictive truncations the growth rate, ${\sigma_i}^2$, and the nonlinear coefficient N of (4.8) are, of course, different from those given here and depend on parameters such as the phase speed and the critical amplitude, which are determined in the course of the linear analysis. It is known that successive estimates of the eigenvalue do converge to a value (although somewhat slowly in some parameter regimes) as the truncation is made less severe (Tung, 1976). Thus, we expect $\sigma_i{}^2$ will converge to a finite value. It is not clear, however, that the nonlinear ocefficient N will converge to a finite value. What would a diverging nonlinear coefficient imply? It would suggest that the nonlinearity is felt immediately and would then imply that the weakly nonlinear analysis as well as the linear stability analysis is meaningless in the region of parameter space in which N diverged. This is because the time scale of the nonlinearity is much faster than that of the linear instability when N diverges, which implies that the separation of the linear instability and nonlinear time scales used to close the problem is no longer valid. In this case we might expect turbulent behavior to be present immediately as opposed to the case of finite N where turbulent behavior may develop more slowly as the wavenumber spectrum fills out and stronger nonlinearity develops. Presumably, this occurs over times scales longer than the time $|\Delta|^{-1/2}$ after which the theory in this study becomes no longer valid. Hence, the question of truncation is a serious one and whould be borne in mind in interpreting any study which employs truncation, such as the present work, or numerical simulations where truncation is implied.

5. Concluding remarks

In this chapter the nonlinear evolution equations were obtained which govern the interaction between a topographically forced wave and its weakly unstable perturbation. This analysis, valid away from topographic resonance, showed that the topographically forced wave remained in a fixed position relative to the mountain while its amplitude was altered as the perturbation evolved. An explosive instability occurs for superresonant flow for which the analysis is not valid after a short time and a smooth oscillatory exchange of energy takes place between the perturbation and topographic wave for subresonant flow. When this oscillation is averaged over one cycle, the effect of the perturbation on the topographically forced wave is a dissipative one. This behavior is markedly different from that of the free-wave analysis by Deininger (1982) in which the phase of the basic-state Rossby wave was altered as the oscillatory exchange took place between the Rossby wave amplitude and perturbation.

The mechanism behind the nonlinear feedback in this study is the convergence of vorticity flux, i.e., essentially the tilted trough mechanism. This mechanism has been shown by Holopainen (1978) to be important in the maintenance of the standing eddies' vorticity balance in the atmosphere. The preceeding analysis, then, is suggestive of a type of interaction which may occur between the topographically forced standing eddies and transient eddies in the atmosphere.

Appendix A

Results for the Three-Term Truncation

When (3.7) is truncated to

$$\psi^{(1)} = L_1 P_0 e^{i\hat{\theta}_1} + P_0 e^{i\hat{\theta}_0} + L_{-1}\, P_0 e^{i\hat{\theta}_{-1}}, \qquad \text{(A1)}$$

where

$$\hat{\Theta}_n = \Theta_n + \hat{\lambda}_r t, \qquad \text{(A2a)}$$

$$L_n = \frac{\eta_c d a_0 K_0{}^2}{(\hat{\lambda}_r + \delta_n) K_n{}^2}, \text{ for } n = \pm 1 \qquad \text{(A2b)}$$

and

$$\hat{\lambda}_r = \lambda_r + \delta_0. \qquad \text{(A2c)}$$

the coefficients of (4.8) become

$$\sigma_i{}^2 = \frac{2}{H\eta_c} \left(\frac{a_1}{\lambda_r + \delta_1} + \frac{a_{-1}}{\lambda_r + \delta_{-1}} \right), \qquad \text{(A3a)}$$

$$N = -\frac{8b^2(K_0{}^2 - K^2)}{HkK^2(U - c)} \left[\frac{K_0{}^2 - K_1{}^2}{K_1{}^2(\hat{\lambda}_r + \delta_1)} + \frac{K_{-1}{}^2 - K_0{}^2}{K_{-1}{}^2(\hat{\lambda}_r + \delta_{-1})}\right]$$

$$\times \left[\frac{a_1}{\lambda_r + \delta_1} + \frac{a_{-1}}{\lambda_r + \delta_{-1}}\right]$$

$$+ \frac{8\eta_c{}^2 b^2 d^2 (K_0{}^2 - K^2)^2 (K_{-1}{}^2 - K_1{}^2)}{HK_1{}^4 K_{-1}{}^4 (\hat{\lambda}_r + \delta_1)^2 (\hat{\lambda}_r + \delta_{-1})^2 kK^2 (4U - c)}$$

$$\times \left[(K_1{}^2 - K^2)(4K^2 - K_{-1}{}^2) + (K_{-1}{}^2 - K^2)(4K^2 - K_1{}^2)\right]$$

$$+\frac{1}{H}\left\{\frac{2b}{d\eta_c}\,\frac{(\overline{K_1}^2 - K^2)(\overline{K_1}^2 - K_0^2)f_1}{K_1^2(K_0^2 - K^2)(\hat{\lambda}_r + \delta_1)}\right.$$

$$+\frac{4b(K_1^2 - K_{-1}^2)\,(K^2 - \overline{K_0}^2)f_0}{K_1^2K_{-1}^2(\hat{\lambda}_r + \delta_1)(\hat{\lambda}_r + \delta_{-1})}$$

$$\left.-\frac{2b\,(\overline{K_{-1}}^2 - K^2)(K_{-1}^2 - K_0^2)f_{-1}}{d\eta_c\,K_{-1}^2\,(K_0^2 - K^2)(\hat{\lambda}_r + \delta_{-1})}\right\} \tag{A3b}$$

where

$$H = \frac{a_1}{(\lambda_r + \delta_1)^3} + \frac{a_{-1}}{(\lambda_r + \delta_{-1})^3}\,, \tag{A4a}$$

$$f_1 = \frac{2}{\overline{K_1}^2(\delta_1 + 2\hat{\lambda}_r)}\,[-\eta_c d(\overline{K_0}^2 - K^2)f_0$$

$$+ b(K_1^2 - K_0^2)]\,, \tag{A4b}$$

$$f_{-1} = \frac{-2}{\overline{K_{-1}}^2(\delta_{-1} + 2\lambda_r)}\,[\eta_c d(\overline{K_0}^2 - K^2)f_0$$

$$+ b(K_{-1}^2 - K_0^2)]\,, \tag{A4c}$$

$$f_0 = \frac{1}{\overline{K_0}^2 D}\,[4b(K_1^2 - K_{-1}^2)L_1L_{-1}(\bar{\delta}_1 + 2\hat{\lambda}_r)(\bar{\delta}_{-1} + 2\hat{\lambda}_r)$$

$$- 4bd\eta_c\bar{a}_1\,(K_1^2 - K_0^2)(\bar{\delta}_{-1} + 2\hat{\lambda}_r)L_1$$

$$- 4bd\eta_c\bar{a}_{-1}\,(K_0^2 - \overline{K_{-1}}^2)(\bar{\delta}_1 + 2\hat{\lambda}_r)L_{-1}] \tag{A4d}$$

$$D = -4d^2\eta_c{}^2\bar{a}_0[\bar{a}_1\ (\bar{\delta}_{-1} + 2\hat{\lambda}_r) + \bar{a}_{-1}(\bar{\delta}_1 + 2\hat{\lambda}_r)]$$

$$+ (\bar{\delta}_1 + 2\hat{\lambda}_r)(\bar{\delta}_0 + 2\hat{\lambda}_r)(\bar{\delta}_{-1} + 2\hat{\lambda}_r), \qquad \text{(A4e)}$$

and η_c and $\hat{\lambda}_r$ can be determined from the two equations

$$\eta_c{}^2 = \frac{3\hat{\lambda}_r{}^2 + 2(\delta_1 + \delta_{-1})\hat{\lambda}_r + \delta_1\delta_{-1}}{d^2a_0(a_1 + a_{-1})}, \qquad \text{(A5a)}$$

$$\hat{\lambda}_r{}^3 + (\delta_1 + \delta_{-1})\hat{\lambda}_r{}^2 + [\delta_1\delta_{-1} - \eta_c{}^2d^2a_0(a_1 + a_{-1})]\hat{\lambda}_r{}^2$$

$$- \eta_c{}^2d^2a_0(a_1\delta_{-1} + a_{-1}\delta_1) = 0. \qquad \text{(A5b)}$$

The singularity in N when $U = c/4$ corresponds to a resonance with the second harmonic of the basic-state wave and the analysis is not valid there. For the special case in which $l = k_0 = 0$, one must go to the $O(|\Delta|^2)$ to obtain closure as done in Deininger and Loesch (1982).

Appendix B

Forced Solution Coefficients

The f_n are

$$f_1 = \frac{2b(2\lambda_r + \bar{\delta}_0)(K_1{}^2 - K_0{}^2)L_1}{\overline{K_1}{}^2D}, \qquad \text{(B1a)}$$

$$f_0 = \frac{-4\eta_c bd\bar{a}_1(K_1{}^2 - K_0{}^2)L_1}{\overline{K_0}{}^2D}, \qquad \text{(B1b)}$$

where

$$\lambda_r = -\left(\frac{\delta_1 + \delta_0}{2}\right) , \tag{B2a}$$

and

$$D = (2\lambda_r + \bar{\delta}_0)\ (2\lambda_r + \bar{\delta}_1) - 4\eta_c{}^2 d^2 \bar{a}_1 \bar{a}_0 . \tag{B2b}$$

The barred quantities are

$$\left.\begin{aligned}
\overline{K_n}^2 &= \overline{k_n}^2 + \overline{l_n}^2 \\
\bar{c}_n &= \beta/\overline{K_n}^2 \\
\bar{a}_n &= U - \bar{c}_n \\
\bar{\delta}_n &= \bar{k}_n(U - \bar{c}_n)
\end{aligned}\right\} . \tag{B3}$$

Chapter 3 Topographic instability of the normal modes in the Charney problem

1. Introduction

The first theories of topographic instability (Charney and Devore, 1979; Charney and Strauss, 1980) were based on an *ad hoc* simplification of barotropic and two-layer baroclinic models. Specifically, the *ad hoc* simplification was the severely truncated spectral representation of the already simplified equations of motion. The primary effect of the particular truncation employed was to neglect the wave-wave interactions but ingeniously retain the wave-zonal flow interaction. This wave-zonal flow interaction was fundamental to the topographic instability. However, the above cited studies remain unsatisfying due to the nondeductive nature of the simplifications required to isolate the topographic instability. Hart (1979) was successful in eliminating the necessity of spectral truncation in a barotropic model by considering topography with nearly infinite horizontal extent. The drawback of this study is the unrealistic restriction placed on the topography in order to isolate the instability. Independently Jacobs (1979a), Plumb (1981a) and Pedlosky (1981) studied the topographic instability in barotropic and two-layer baroclinic models using another asymptotically valid technique which neglected the wave-wave interactions. This deductive method exploits the role of linear resonance in the topographic instability and its quasi-linear nature discovered in the earlier studies. Specifically, the flow is assumed to be near linear resonance and weakly nonlinear. Thus the lowest order solution consists of a free, stationary, linear wave. Its amplitude is determined by the wave-mean flow nonlinearity which arises in the presence of topography as the flow is slightly detuned from resonance. As mentioned in Chapter 1, Plumb's (1981b) analysis is somewhat different since he imposes a propagating forcing. Of course a wave resonant with this forcing must be moving with it. Furthermore, as noted by Pedlosky (1981) there is no feedback affecting the topographic forcing.

The purpose of this chapter is to remove one of the simplifications of the above cited models paralleling, most closely, the technique employed by Pedlosky (1981) to isolate the topographic instability in the layered models. Namely, the limiting assumption of a barotropic or two-layer baroclinic model will be removed. Here, topographic instability will be examined in a continuously stratified baroclinic model. The particular model to be used in this chapter and in Chapter 4 is the nonlinear version of the Charney (1947) model with sinusoidal topography, Ekman friction and Newtonian cooling. It is chosen for its relative simplicity which is due to the assumptions of constant vertical shear and constant Brunt-Väisälä frequency. The Charney model admits as its vertically trapped solutions both normal modes and unstable modes. The Charney mode is the most unstable mode. However, the unstable modes can become neutral on a curve in parameter space in which case they are referred to as neutral modes. The neutral modes should not be confused with the normal modes. The normal modes which owe their existence to the presence of vertical shear remain baroclinically neutral throughout parameter space as opposed to the unstable modes. The unstable modes near neutrality is the subject of Chapter 4. In this chapter we examine the topographically induced instability of the normal modes. Thus in a manner analogous to Pedlosky (1981) the O(1) solution corresponds to a free, stationary normal mode. Its amplitude is determined by the topographically induced wave-mean flow interaction with a zonal flow slightly detuned from resonance. Weak Ekman friction and Newtonian cooling are also allowed to affect the dynamics.

The principal differences between the layered baroclinic model and the continuous stratified baroclinic model are due to the way in which Ekman friction and topography are included in each. In the layered baroclinic model, Ekman friction enters the potential vorticity budget in each layer. Topography enters the potential vorticity budget of the entire lower layer. Thus potential vorticity is not conserved and multiple equilibria can result as a balance between Ekman friction and topographic forcing. However, in the continuous model in the absence of internal dissipation potential vorticity is conserved even in the presence of topography and Ekman friction. Therefore internal dissipation in the form of Newtonian cooling is included which destroys the conservation of potential vorticity in the interior. This along with the inclusion of nonlinearity makes possible the existence of multiple equilibria. Ekman friction alone is not sufficient to produce multiple equilibria in a continuous model.

During the course of this work two other studies (Jacobs, 1979b; Plumb, 1981b) of topographic instability in a continuous model became known to me. Both authors consider a more general shear flow which makes their results less analytically explicit. In fact, due to this additional complexity, Jacobs (1979b) found it necessary to limit his consideration to the steady problem. The relative simplicity of the present model allows the derivation of the fully transient amplitude equations and an explicit expression for the growth rate of the topographic instability. Plumb's work also differed from the present work in the type of topography specified. He considered a propagating topography for the purpose of emulating previous studies of stratospheric warming. However this does not allow a feedback between the zonal flow and wave field at the ground. As will be seen, this feedback is a feature of the present study.

2. The Model

The model to be used in this and the next chapter, consists of a quasigeostrophic continuously stratified fluid on a β-plane bounded horizontally by two vertical walls separated by a distance L, and bounded vertically by the ground with its sinusoidal topographic undulations and infinity. The basic density field is assumed to have a constant scale height H and constant Brunt-Väisälä frequency N associated with it. The basic state velocity field consists of a latitudinally uniform zonal flow with constant vertical shear. The flow at the ground is not zero. Thus the full nondimensional stream function ψ can be written

$$\psi\ (x,y,z,t) = -\ \overline{U}y - zy + \varepsilon\phi(x,y,z,t) \qquad (2.1)$$

where x,y,z,t are the zonal, meridional, vertical and time coordinates, respectively. In (2.1) $\overline{U}$ is independent of y and z. The first two terms of (2.1) represent the basic state zonal flow while the third term represents the disturbance field. The variables have been nondimensionalized as follows: horizontal lengths by L, vertical distance by

$$D = \frac{fL}{N} \qquad (2.2a)$$

and horizontal velocities by

$$U_o = D\,\frac{\partial U*}{\partial z} \qquad (2.2b)$$

where $\partial U*/\partial z$ is the constant shear of the basic zonal flow in dimensional form. The parameter ε measures the relative magnitude of the basic state zonal velocity field to the disturbance field. Using the above scaling, the nondimensional version of the potential vorticity equation for the disturbance under the influence of Newtonian cooling takes the form

$$\left[\frac{\partial}{\partial t} + (\bar{U} + z)\frac{\partial}{\partial x}\right]\Pi + q\frac{\partial \phi}{\partial x} + \varepsilon J(\phi,\Pi)$$

$$= -\eta\left(\frac{\partial^2 \phi}{\partial z^2} - h^{-1}\frac{\partial \phi}{\partial z}\right), \tag{2.3}$$

where

$$\Pi = \nabla^2\phi + \frac{\partial^2 \phi}{\partial z^2} - h^{-1}\frac{\partial \phi}{\partial z}, \tag{2.4}$$

is the potential vorticity of the disturbance,

$$q = \beta + h^{-1} \tag{2.5}$$

and the operators J and ∇^2 are defined by

$$J(a,b) \equiv \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\frac{\partial b}{\partial y},$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The nondimensional parameters β, h and η are

$$\beta = \beta_* L^2/U_o \tag{2.6a}$$

$$h = H/D, \tag{2.6b}$$

$$n = L/(U_o \tau_N), \tag{2.6c}$$

where τ_N is the Newtonian cooling time and $\beta*$ is the northward gradient of the Coriolis parameter f.

At the bottom boundary just above the Ekman layer the vertical velocity must match the Ekman pumping velocity plus the vertical velocity due to topographyu. After using this condition the nondimensional lower boundary condition for ϕ at z=0 can be written

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial x} + \varepsilon J\left(\phi,\frac{\partial\phi}{\partial z}\right) = -\frac{1}{\varepsilon}\bar{U}\eta_x$$

$$- J(\phi,\eta) - r\nabla^2\phi - n\frac{\partial\phi}{\partial z} \tag{2.7}$$

where

$$\eta = \frac{h_B}{D}\left(\frac{fL}{U_o}\right), \tag{2.8a}$$

$$r = \frac{1}{2}\frac{\delta_E}{D}\left(\frac{fL}{U_o}\right), \tag{2.8b}$$

In (2.8) $\delta_E = (2\nu'/f)^{1/2}$ is the Ekman layer thickness and $h_B(x,y)$ is the elevation of the lower boundary above z=0.

At the channel sidewalls the meridional velocity must vanish, i.e.

$$\partial\phi/\partial x = 0 \text{ at } y = 0,1, \tag{2.9a}$$

and as a result

$$\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\frac{\partial^2\phi}{\partial y\partial t}\,dx' = 0 \tag{2.9b}$$

must also be true.

The model as stated above is essentially the same model as formulated by Pedlosky (1979) except for the inclusion of topographic effects. It also could be thought of as the nonlinear version of the Charney (1947) model for baroclinic instability with the addition of Newtonian cooling, Ekman pumping and topography.

3. Brief review of the Charney model

The analysis in this chapter and the next chapter exploits the simplifications that occur when the Charney mode or the normal modes of the Charney model are nearly resonant. This linear resonance occurs when these modes become stationary. Thus a brief review of the stationary neutral Charney mode and stationary normal modes of the Charney problem is necessary.

In doing so we consider the linear, undamped version of the basic equations (2.3,2.7,2.9a) for the disturbance ϕ_0 in the absence of topography. They are

$$\left[\frac{\partial}{\partial t} + (\overline{U} + z)\frac{\partial}{\partial t}\right]\left[\nabla^2\phi_o + \frac{\partial^2\phi}{\partial z^2} - h^{-1}\frac{\partial\phi_o}{\partial x}\right] + q\,\frac{\partial\phi_o}{\partial x} = 0, \tag{3.1a}$$

$$\left(\frac{\partial}{\partial t} + \overline{U}\,\frac{\partial}{\partial x}\right)\frac{\partial\phi_o}{\partial z} - \frac{\partial\phi_o}{\partial x} = 0, \tag{3.1b}$$

and

$$\phi_{ox} = 0 \text{ on } y = 0,1. \tag{3.1c}$$

A propagating wave solution of (3.1) can be sought in the form

$$\phi_0 = A\, F_0(z)\, e^{ik(x-ct)} \sin ly + *, \tag{3.2}$$

where l is an integral multiple of π. Substitution of (3.2) into (3.1) results in the following vertical structure equations

$$(z + \overline{U} - c)\left(\frac{d^2F_o}{dz} - h^{-1}\frac{dF_o}{dz} - K^2F_o\right) + qF_o = 0, \tag{3.3a}$$

$$(\overline{U} - c)\frac{dF_o}{dz} - F_o = 0 \quad \text{on } z = 0. \tag{3.3b}$$

The equation for F_o can be reduced to a standard confluent hypergeometric equation with the transformation

$$F_o = (z + \overline{U} - c)e^{\frac{1}{2}(h^{-1} - \mu)z}\, F(\xi), \tag{3.4a}$$

$$\xi = \mu(z + \overline{U} - c), \tag{3.4b}$$

where

$$\mu = (h^{-2} + 4K^2)^{1/2}, \tag{3.4c}$$

$$K^2 = k^2 + l^2 \tag{3.4d}$$

The standard equation and associated lower boundary condition takes the form

$$\xi\, F_{\xi\xi} + (2-\xi)F_\xi - (1-r_o)F = 0, \tag{3.5a}$$

$$\mu(\overline{U}-c)^2\left[F_\xi + \left(\frac{h^{-1} - \mu}{2\mu}\right)F\right] = 0 \text{ on } \xi = \xi_o \equiv \mu(\overline{U}-c) \tag{3.5b}$$

where

$$r_o = \frac{q}{\mu} = \frac{\beta + h^{-1}}{(h^{-2} + 4K^2)} 1/2 , \tag{3.6}$$

Both the neutral Charney mode and normal modes can be discussed relatively simply by considering only integral values of r_o. For integral r_o, the solution to (3.5a) for which F_o decays as $z \to \infty$ can be written as (Abramowitz and Stegun, 1964)

$$F(\xi) = M(1-r_o, 2, \xi), \tag{3.7a}$$

where

$$M(1-r_o,2,\xi) \equiv 1 + \frac{(1-r_o)\xi}{2} + \frac{(1-r_o)(2-r_o)\xi^2}{2 \times 3 \times 2!}$$

$$+\ldots+ \frac{(1-r_o)_n \xi^n}{(2)_n n!} +\ldots = \sum_{m=0}^{\infty} \frac{(1-r_o)_m}{(2)_m} \frac{\xi^m}{m!} , \tag{3.7b}$$

and if a is any number

$$(a)_n = (a)(a+1)(a+2) \ldots (a+n-1); \; (a)_o = 1.$$

For a given integral r_o there are two ways by which the lower boundary condition (3.5b) may be satisfied. On $\xi = \xi_o \equiv \mu(\overline{U}-c)$ either

$$\overline{U} = c, \tag{3.8a}$$

or

$$F_\xi + \frac{(h^{-1} - \mu)}{2\mu} F = 0 \tag{3.8b}$$

In order to distinguish between the two roots (3.8a,b) the convention in terminology adopted here is as follows: the wave corresponding to the double root (3.8a) is referred to as the neutral mode and the wave(s) corresponding to the single root(s) of (3.8b) is(are) referred to as the normal mode(s). The neutral mode becomes unstable for nonintegral r_o. The normal modes also exist for nonintegral r_o and are vertically trapped waves propagating zonally through the shear flow. As will be seen later, the distinction between the modes is important since the two different types of modes require different treatments. Next we discuss the structures of the different types of modes when they are stationary.

When $r_o=1$, the solution (3.7) is simply F=1 for which it is only possible to satisfy (3.8a) not (3.8b). This single root for $r_o=1$ is known as the neutral Charney mode and is stationary when $\overline{U}=0$. In this case the vertical structure is

$$F_o(z) = ze^{-\beta_o z/2}, \qquad (3.9)$$

where $q_o = \mu$ or equivalently

$$\beta_o = -h^{-1} + (h^{-2} + 4K^2)^{1/2} \qquad (3.10)$$

Miles (1964) has shown that near the neutral curve for the Charney mode, i.e.

$$\beta = \beta_o - \delta \text{ with } |\delta| \ll 1, \qquad (3.11)$$

the growth rate is $O(\delta^{1/2})$ for $\delta>0$ and $O(\delta^{3/2})$ for $\delta<0$. The interest here is in the stronger instability for which $\delta>0$. Pedlosky (1979) points out that a small amount of Ekman damping is enough to squelch the weak instability. Notice that F_0 is zero at $z=0$. The analysis of Chapter 4 makes use of these facts concerning the Charney mode as in Pedlosky (1979). But in Chapter 4 the purpose is to study the effect of topography on the baroclinic instability and its equilibration.

If $r_0=2$, then (3.7) implies

$$F(\xi) = 1 - \frac{1}{2}\xi, \tag{3.12}$$

and (3.6) becomes

$$2\mu = \beta + h^{-1} \tag{3.13}$$

In this case there is both a neutral mode which becomes unstable for nonintegral r_0 and a normal mode. Using (3.12) and (3.13) in (3.5b) it can be seen that the normal mode becomes stationary when

$$\overline{U} = \frac{2\beta}{\mu(\mu-h^{-1})} \tag{3.14}$$

The vertical structure function for this simplest stationary normal mode is

$$F_O(z) = (z+\overline{U})[1 - \frac{\mu}{2}(z+\overline{U})]\, e^{\frac{1}{2}(h^{-1} - \mu)z}, \tag{3.15}$$

where μ and $\overline{U}$ are given by (3.13) and (3.14), respectively. This normal mode has an analytic continuation for all $r_o>1$. In addition, for higher values of r_0 more normal modes exist. In fact for any $r_0>1$ there are a-1 normal modes where a is the next integer larger than r_0. However, to keep the calculations relatively simple the example in section 5 is restricted to the $r_0=2$ normal mode. I do not expect the restriction to integral r_0 to alter the fundamental physics of the problem. In fact, as we will see in section 4, the derivation of the evolution equations which govern the destabilization of the normal modes by topography, is independent of r_0. Of course, the numerical results would certainly depend on the structure of the normal modes and therefore on r_0. This dependence should perhaps be investigated in the future.

2.4 Derivation of the evolution equations for the normal modes of the Charney problem in the presence of topography

The method of analysis used to describe the behavior of the normal modes of this continuously stratified baroclinic model is analogous to that used in the barotropic and two layer baroclinic model by Pedlosky (1981). Here as in Pedlosky's work, the analysis pivots about the resonance of a normal mode described in the last section. This occurs for some value of the zonal current, say $\bar{u}_r$, which depends on the mode considered and the various parameters in the problem such as β, k, h, and r_0. However, it is not necessary at this point to specify which normal mode is to be dealt with. It suffices to say that the concern here is with one of the stationary normal modes of the Charney problem which exist for either integral or nonintegral r_0.

In the absence of nonlinearity and damping, a normal mode resonant with topography grows secularly in time. In order to prevent this occurrence, weak nonlinearity and damping are included and the flow is slightly detuned from resonance, i.e. with $\varepsilon \ll 1$ we choose

$$r = \varepsilon^2 r', \tag{4.1a}$$

$$n = \varepsilon^2 n', \tag{4.1b}$$

$$\overline{U} = \overline{u}_r + \varepsilon^2 \Delta'. \tag{4.1c}$$

In addition the sinusoidal topography is required to be weak. The appropriate choice for topography is

$$\eta = \varepsilon^3 b = \varepsilon^3 M e^{ikx} \sin ly + *, \tag{4.1d}$$

where * represents the complex conjugate. The choice of $O(\varepsilon^3)$ topography in this case differs from the choice made in the baroclinic case discussed by Pedlosky (1981) but is the same choice made in his barotropic analysis. The reason for this reverts to the properties of the normal modes in each case. Here, as in Pedlosky's barotropic case, the preexisting zonal flow directly forces the O(1) stationary wave in which case only $O(\varepsilon^3)$ topography is required for significant $O(\varepsilon^3)$ wave-topographic interaction. In his baroclinic case he chooses a prescribed zonal flow which does not directly force the stationary wave. In this case an $O(\varepsilon^3)$ wave-topographic interaction depends on the transfer of $O(\varepsilon)$ mean zonal momentum from the upper to lower layer. This, in the presence of $O(\varepsilon)$ topography, forces an $O(\varepsilon^2)$ wave which interacts with the topography at $O(\varepsilon^3)$.

As in Pedlosky (1981), given the above choices, the otherwise stationary normal mode evolves on a time scale

$$T = \varepsilon^2 t. \tag{4.2}$$

If we substitute (4.2) along with (4.1) into the governing equations (2.3), (2.7) and (2.9) they become

$$(\bar{u}_r + z)\frac{\partial \Pi}{\partial x} + q\frac{\partial \phi}{\partial x} = -\varepsilon^2\left(\frac{\partial}{\partial t} + \Delta'\frac{\partial}{\partial x}\right)\Pi - \varepsilon J(\phi, \Pi) \tag{4.3a}$$

$$- \varepsilon^2 n'\left(\frac{\partial^2 \phi}{\partial z^2} - h^{-1}\frac{\partial \phi}{\partial z}\right),$$

$$\bar{u}_r \frac{\partial^2 \phi}{\partial t} - \frac{\partial \phi}{\partial x} = -\varepsilon^2\left(\frac{\partial}{\partial t} + \Delta'\frac{\partial}{\partial x}\right)\frac{\partial \phi}{\partial z} - \varepsilon J\left(\phi, \frac{\partial \phi}{\partial z}\right)$$

$$- \varepsilon^2 r'\nabla^2\phi - \varepsilon^2 n'\frac{\partial \phi}{\partial x} - \varepsilon^2 \bar{u}_r b_x - \varepsilon^3 J(\phi, b) \tag{4.3b}$$

on $z = 0$ and

$$\phi_x = 0, \tag{4.3c}$$

$$\lim_{x\to\infty} \frac{1}{2x}\int_{-x}^{x} \varepsilon^2 \frac{\partial^2 \phi}{\partial y \partial T}\, dx' = 0, \tag{4.3d}$$

on $y = 0,1$.

Substitution of the asymptotic series

$$\phi = \phi_0 + \varepsilon\,\phi_1 + \varepsilon^2\phi_2 + \ldots, \tag{4.4}$$

into (4.3) results in a sequence of linear problems, which upon removal of the secularities at each order, will determine the behavior of the stationary normal mode and the topographically induced zonal flow. The O(1) problem for ϕ_0 is

$$(z+\bar{u}_r)\frac{\partial}{\partial x}\left(\nabla^2\phi_o + \frac{\partial^2\phi_o}{\partial z^2} - h^{-1}\frac{\partial\phi_o}{\partial x}\right) + q\frac{\partial\phi_o}{\partial x} = 0\ , \tag{4.5a}$$

$$\bar{u}_r\frac{\partial^2\phi_o}{\partial z\partial x} - \frac{\partial\phi_o}{\partial x} = 0 \quad \text{on } z = 0, \tag{4.5b}$$

and leads to the specification of what is to O(1), a free stationary normal mode of the Charney problem. As discussed in section 3, it can be written

$$\phi_0 = AF_0(z)\ e^{ikx} \sin ly + * \tag{4.6}$$

where F_0 satisfies

$$(z + \bar{u}_r)\left(\frac{d^2F_o}{dz^2} - h^{-1}\frac{dF_o}{dz} - K^2F_o\right) + qF_o = 0, \tag{4.7a}$$

$$\bar{u}_r\frac{dF_o}{dz} - F_o = 0 \quad \text{on } z = 0, \tag{4.7b}$$

and the value of the zonal current required for resonance $\bar{u}_r$ depends on which mode is chosen. It is not necessary for the purpose of the derivation in this section to make specific the normal mode under consideration, but it is necessary to limit the initial spectrum to just one mode thereby neglecting resonant triad interactions.

Nevertheless, the restriction in the initial spectrum to a single mode is <u>self consistent</u>. This is a consequence of the weak nonlinearity.

The $O(\varepsilon)$ problem can be written

$$(z + \bar{u}_r)\frac{\partial \Pi_1}{\partial x} + q\,\frac{\partial \phi_1}{\partial x} = -\,J(\phi_o,\, \Pi_o)\ , \qquad (4.8a)$$

$$\bar{u}_r\,\frac{\partial^2 \phi_1}{\partial z \partial x} - \frac{\partial \phi_1}{\partial x} = -\,J\Big(\phi_o,\, \frac{\partial \phi_o}{\partial z}\Big)\ , \qquad \text{on } z=0. \qquad (4.8b)$$

Upon integration of (4.5a,b) once in x the relations

$$\Pi_o = -\,\frac{q}{z + \bar{u}_r}\,\phi_o\ , \qquad (4.9a)$$

$$\frac{\partial \phi_o}{\partial z} = \frac{1}{\bar{u}_r}\,\phi_o \quad \text{on} \quad z = 0, \qquad (4.9b)$$

are obtained. Using (4.9a,b) in the Jacobian nonlinearity of (4.8a,b) it is easily seen that the right hand side of (4.8a,b) vanishes since

$$J(\phi_0,\ \phi_0) = 0.$$

As a result (4.8a,b) reduce to

$$(z+u_r)\,\frac{\partial}{\partial x}\Big(\nabla^2 \phi_1 + \frac{\partial^2 \phi_1}{\partial z^2} - h^{-1}\,\frac{\partial \phi_1}{\partial z}\Big) + q\,\frac{\partial \phi_1}{\partial x} = 0, \qquad (4.10a)$$

$$\bar{u}_r\,\frac{\partial^2 \phi_1}{\partial z \partial x} - \frac{\partial \phi_1}{\partial x} = 0 \text{ on } z = 0, \qquad (4.10b)$$

which are identical in form to the O(1) problem (4.5a,b). Therefore a solution with the same structure as (4.6) might be added at this order. However, this solution can be eliminated by renormalizing ϕ_0. Alternatively, if ϕ_0 is fixed as specified in (4.6) we need not include the similar solution at this order.

Based on the experience of Pedlosky (1981) and Plumb (1981a,b) a new nontrivial solution satisfying (4.10a,b) must be included which corresponds to a wave induced correction to the zonal current. This solution which can be written

$$\phi_1 = \Phi_1 \, (y,z,T), \tag{4.11}$$

has the potential vorticity

$$\Pi_1 = \frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z} \tag{4.12}$$

Its evolution and structure will be determined by the removal of zonal secularities at $O(\varepsilon^3)$. Such a solution might have been included at O(1). However, removal of the zonal secularities at $O(\varepsilon^2)$ would show that Φ_0 is not forced by the wave field and is damped by Newtonian cooling and Ekman friction. Thus, if the O(1) zonal correction were zero initially, it would remain so forever which of course obviates its introduciton.

Having specified the free stationary normal mode and the zonal current, we proceed to higher order where the effects of topographic forcing, damping and wave-mean flow nonlinearity appear and determine the evolution of the flow.

The problem at $O(\varepsilon^2)$ is

$$(z+\bar{u}_r)\frac{\partial \Pi_2}{\partial x} + q\frac{\partial \phi_2}{\partial x} = -\left(\frac{\partial}{\partial T} + \Delta'\frac{\partial}{\partial x}\right)\Pi_o - J(\phi_o, \Pi_1)$$

$$- J(\Phi_1, \Pi_o) - n'\left(\frac{\partial^2 \phi_o}{\partial z^2} - h^{-1}\frac{\partial \phi_o}{\partial z}\right), \qquad (4.13a)$$

$$\bar{u}_r\frac{\partial^2 \phi_2}{\partial x \partial z} - \frac{\partial \phi_2}{\partial x} = -\left(\frac{\partial}{\partial T} + \Delta'\frac{\partial}{\partial x}\right)\frac{\partial \phi_o}{\partial z} - J\left(\phi_o, \frac{\partial \Phi_1}{\partial z}\right)$$

$$- J\left(\Phi_1, \frac{\partial \phi_o}{\partial z}\right) - r'\nabla^2\phi_o - n'\frac{\partial \phi_o}{\partial z} \quad \text{on } z = 0. \qquad (4.13b)$$

Using (4.1d), (4.6) and (4.9) the inhomogeneous terms of (4.13) can be evaluated and rewritten as

$$(z+\bar{u}_r)\frac{\partial \Pi_2}{\partial x} + q\frac{\partial \phi_2}{\partial x} = [q\frac{F_o}{z+\bar{u}_r}\left(\frac{\partial}{\partial T} + ik\Delta' + n'\right)A(\sin ly)$$

$$- n'K^2AF_o(\sin ly) - ikF_oA\Pi_{1y}(\sin ly)$$

$$- ikqA\frac{Fo}{z+\bar{u}_r}\Phi_{1y}(\sin ly)]e^{ikx} + *, \qquad (4.14a)$$

$$\bar{u}_r\frac{\partial^2 \phi_2}{\partial x \partial z} - \frac{\partial \phi_2}{\partial x} = [-\frac{F_o}{\bar{u}_r}\left(\frac{\partial}{\partial T} + ik\Delta' + n'\right)A(\sin ly)$$

$$+ r'K^2AF_o(\sin ly) - ikAF_o\Phi_{1yz}(\sin ly)]e^{ikx} + *,$$

$$\text{on } z = 0. \qquad (4.14b)$$

The removal of the secular terms from (4.14) results in the amplitude equation for A. Removal of the secularities can be accomplished by first operating on (4.14a) with the operator which corresponds to multiplying (4.14a) by the adjoint solution of the homogeneous side of (4.14) and integrating over the domain, i.e. the operator is

$$\int_0^\infty dz e^{-z/h} \frac{F_o}{z+\bar{u}_r} \int_0^1 dy 2(\sin \; ly) \overline{e^{-ikx} (\;\;)},$$

where $(\overline{\;\;\;\;})$ denotes a zonal average. This will ensure the validity of the expansion (4.4) by removing those inhomogeneous terms which have a projection on the homogeneous solutions of (4.14). Application of the operator to (4.14a), integrating by parts a number of times and using (4.7a,b), results in the equation

$$- F_o(0) \int_0^1 dy 2(\sin \; ly) \left. \overline{e^{-ikx} \frac{\partial}{\partial x} [\frac{\partial \phi_2}{\partial z} - \frac{1}{\bar{u}_r} \phi_2]} \right|_{z = 0}$$

$$= (\frac{\partial A}{\partial T} + ik\Delta' A + n' A) \; q \int_0^\infty e^{-z/h} \frac{F_o{}^2}{(z+\bar{u}_r)^2} dz$$

$$- n' K^2 A \int_0^\infty e^{-z/h} \frac{F_o{}^2}{(z+\bar{u}_r)} dz - ikA \int_0^1 dy \; 2 \; (\sin^2 ly)$$

$$\times \int_0^\infty dz e^{-z/h} \frac{F_o{}^2}{z+\bar{u}_r} \Pi_{1y} - ikqA \int_0^1 dy 2(\sin^2 ly)$$

$$\times \int_0^\infty dz e^{-z/h} \frac{F_o{}^2}{(z+\bar{u}_r)^2} \Phi_{1y}. \qquad (4.15)$$

Substitution of the boundary condition (4.14b) into (4.15) with the definitions

$$P_{11} \equiv \int_0^1 2 \sin^2 ly \; \Pi_{1y} \; dy, \qquad (4.16a)$$

$$u_{11} \equiv -\int_0^1 2\sin^2 ly\, \Phi_{1y}\, dy = \int_0^1 2\sin^2 ly\, u_1\, dy \qquad (4.16b)$$

$$u_1 = -\frac{\partial \Phi_1}{\partial y} \qquad (4.16c)$$

yields

$$\frac{F_o(0)}{\bar{u}_r^2}\left(\frac{\partial A}{\partial T} + ik\Delta' A + n' A\right) - r' K^2 \frac{F_o^2(0)}{\bar{u}_r} A + ikF_o(0)M$$

$$- ik\frac{F_o^2(0)}{\bar{u}_r} A\left(u_{11z} - \frac{1}{\bar{u}_r} u_{11}\right)_{z=0} = \left(\frac{\partial A}{\partial T} + ik\Delta' A + n' A\right)$$

$$\times\, q\int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}\, dz - n' K^2 A \int_0^\infty e^{-z/h} \frac{F_o^2}{z+\bar{u}_r}\, dz$$

$$- ik\int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}\left[(z+\bar{u}_r)p_{11} - qu_{11}\right] dz, \qquad (4.17)$$

or equivalently

$$\left(\frac{\partial A}{\partial T} + ik\Delta' A + n' A\right)\left[q\int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}\, dz - \frac{F_o^2(0)}{\bar{u}_r^2}\right]$$

$$+ K^2 A\left[r' \frac{F_o^2(0)}{\bar{u}_r} - n' \int_0^\infty e^{-z/h} \frac{F_o^2}{z+\bar{u}_r}\, dz\right]$$

$$- ikA\int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}\left[(z+\bar{u}_r)p_{11} - qu_{11}\right] dz$$

$$+ ik\frac{F_o^2(0)}{\bar{u}_r} A\left(u_{11z} - \frac{1}{\bar{u}_r} u_{11}\right)_{z=0} = ikF_o(0)M. \qquad (4.18)$$

This is the equation governing the evolution of the wave amplitude A which is forced by sinusoidal topography of amplitude M, damped by Ekman friction and Newtonian cooling and which interacts with the correction to the zonal flow.

However, the correction to the zonal flow is as yet undetermined; thus, we seek an equation for the zonal flow correction driven by the wave field. Such an equation can be obtained from the $O(\varepsilon^3)$ zonally averaged problem which is written

$$\frac{\partial \Pi_1}{\partial T} + n'\left(\frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z}\right) + \overline{J(\phi_o, \Pi_2)}$$

$$+ \overline{J(\phi_2, \Pi_o)} = 0 \ , \qquad (4.19a)$$

$$\frac{\partial^2 \Phi_1}{\partial T \partial z} + r' \frac{\partial^2 \Phi_1}{\partial y^2} + n' \frac{\partial \Phi_1}{\partial z} + \overline{J(\phi_o, \frac{\partial \phi_2}{\partial z})}$$

$$+ \overline{J(\phi_2, \frac{\partial \phi_o}{\partial z})} + \overline{J(\phi_o, b)} = 0. \qquad (4.19b)$$

In (4.19a,b) the first terms represent the time change of x-independent potential vorticity and temperature respectively. Terms proportional n' and r' damp the $O(\varepsilon)$ thermal and momentum fields respectively. The Jacobian terms of (4.19a,b) can be rewritten as

$$J(\phi_o, \Pi_2) + J(\phi_2, \Pi_o) = \frac{\partial}{\partial y} (\overline{v_o \Pi_2} + \overline{v_2 \Pi_o}), \qquad (4.20a,b)$$

$$J(\phi_o, \phi_{2z}) + J(\phi_2, \phi_{oz}) = \frac{\partial}{\partial y} (\overline{v_o \theta_2} + \overline{v_2 \theta_o}), \qquad (4.20a,b)$$

where

$$v_i = \phi_{ix},$$

$\theta_i = \phi_{iz}$.

Thus, it is clear the nonlinear terms represent the convergence of the meridional fluxes of potential vorticity produced by the wave field in (4.19a) and the convergence of the meridional flux of heat at the ground produced by the wave field in (4.19b). The final term of (4.19b) represents the form drag caused by the phase shift in the wave field relative to the topography. Note Ekman friction and form drag do not act directly on the interior of the flow described by (4.19a), in contrast to the layered cases discussed by Pedlosky (1981). Rather they only enter in the boundary condition (4.19b).

It is possible to evaluate the nonlinear terms of the zonal flow equations (4.19a,b) or, equivalently according to (4.20a,b), the convergence of potential vorticity and heat fluxes without explicitly calculating ϕ_2 and Π_2. To do this we first use (4.5a,b) to rewrite the potential vorticity and heat fluxes as

$$\frac{\partial}{\partial y}(\overline{v_o\Pi_2} + \overline{v_2\Pi_o}) = -\frac{\partial}{\partial y}\overline{\phi_o\left(\frac{\partial \Pi_2}{\partial x} + \frac{q}{z+\bar{u}_r}\frac{\partial \phi_2}{\partial x}\right)}, \qquad (4.21a)$$

$$\frac{\partial}{\partial y}(\overline{v_o\theta_2} + \overline{v_2\theta_o}) = -\frac{\partial}{\partial y}\phi_o\left(\overline{\frac{\partial^2 \phi_2}{\partial x \partial z}} - \frac{1}{\bar{u}_r}\overline{\frac{\partial \phi_2}{\partial x}}\right). \qquad (4.21b)$$

Then by substituting the $O(\varepsilon^2)$ equations (4.14a,b) into (4.21a,b) in order to eliminate the $O(\varepsilon^2)$ variables in terms of the O(1) wave field, the convergence of the fluxes can be written

$$\frac{\partial}{\partial y}(\overline{v_o \Pi_2} + \overline{v_2 \Pi_o}) = [-q \frac{F_o^2(z)}{(z+\bar{u}_r)^2}(\frac{\partial}{\partial T} + 2n') + 2n'K^2 \frac{F_o^2(z)}{z+\bar{u}_r}] \, |A|^2 \frac{\partial}{\partial y}(\sin^2 ly), \tag{4.22a}$$

$$\frac{\partial}{\partial y}(\overline{v_o \theta_2} + \overline{v_2 \theta_o})_{z=0} = [\frac{F_o^2(0)}{\bar{u}_r^2}(\frac{\partial}{\partial T} + 2n') - 2\frac{F_o^2(0)}{\bar{u}_r} r'K^2] \, |A|^2 \frac{\partial}{\partial y}(\sin^2 ly) - ikF_o(0) \times (AM^* - A^*M)\frac{\partial}{\partial y}(\sin^2 ly) \tag{4.22b}$$

Finally using (4.20a,b) and (4.22a,b) to evaluate the nonlinear terms in the equations for the zonal current (4.19a,b), in terms of the O(1) wave field, yields

$$\frac{\partial}{\partial T}(\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1}\frac{\partial \Phi_1}{\partial z}) + n'(\frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1}\frac{\partial \Phi_1}{\partial z}) + [-q\frac{F_o^2(z)}{(z+\bar{u}_r)^2}(\frac{\partial}{\partial T} + 2n') + 2n'\frac{K^2 F_o^2(z)}{z+\bar{u}_r}]|A|^2 \times \frac{\partial}{\partial y}(\sin^2 ly) = 0 \tag{4.23a}$$

$$\frac{\partial^2 \Phi_1}{\partial T \partial z} + r'\frac{\partial^2 \Phi_1}{\partial y^2} + n'\frac{\partial \Phi_1}{\partial z} + [\frac{F_o^2(0)}{\bar{u}_r^2}(\frac{\partial}{\partial T} + 2n') - \frac{2F_o^2(0)}{\bar{u}_r} r'K^2] \, |A|^2 \frac{\partial}{\partial y}(\sin^2 ly) = 0 . \tag{4.23b}$$

In obtaining (4.23b) note the form drag term present in (4.19b) has been cancelled by a term of the same form but opposite sign arising from the evaluation of the nonlinear term in (4.19b). As a result, both the convergence of the meridional flux of potential vorticity in (4.23a) and the sum of the form drag and the convergence of the meridional flux of heat at the ground in (4.23b) produced by the wave field are written solely in terms of wave transience and damping. There is no explicit topographic contribution to the wave forcing of the zonal flow which consists of the convergence of the meridional potential vorticity flux and the sum of the form drag and convergence of meridional heat flux. Therefore, as in Pedlosky (1979, 1981) the wave forcing of the zonal flow vanishes, as it must according to the noninteraction theorems, for steady waves in the absence of friction. However, just because there is no explicit contribution from topography in the wave forcing of the zonal flow does not mean the wave forcing is completely independent of the topography. In fact, from the energy form of the wave field evolution equation (4.17), i.e.

$$
\begin{aligned}
&\left[q \int_0^{\infty} e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}\, dz \left(\frac{\partial}{\partial T} + 2n'\right) - 2n'K^2 \right. \\
&\left. \times \int_0^{\infty} e^{-z/h} \frac{F_o^2}{z+\bar{u}_r}\, dz\right] |A|^2 = \left[\frac{F_o^2(0)}{\bar{u}_r^2}\left(\frac{\partial}{\partial T} + 2n'\right)\right. \\
&\left. - r'K^2 \frac{F_o^2(0)}{\bar{u}_r}\right] |A|^2 - ikF_o(0)(AM^* - A^*M), \qquad (4.24)
\end{aligned}
$$

it is easily seen that the wave transience forcing of the zonal flow in the absence of damping is due to the topographic form drag. Thus, the form drag drives the wave transience which then drives the zonal flow correction according to (4.23a,b).

In general, at every height z the meridional flux of potential vorticity has contributions due to the convergence of both Reynolds stress and meridional heat flux (e.g. Pedlosky, 1979), i.e.

$$e^{-z/h}\left(\overline{v_o \Pi_2} + \overline{v_2 \Pi_o}\right) = -e^{-z/h} \frac{\partial}{\partial y}\left(\overline{u_o v_2} + \overline{u_2 v_o}\right)$$

$$+ \frac{\partial}{\partial z} e^{-z/h}\left(\overline{v_o \theta_2} + \overline{v_2 \theta_o}\right) \qquad (4.25)$$

Thus both terms contribute to the wave forcing in (4.22a) and in the interior zonal flow equation (4.23a) which is the convergence of this potential vorticity flux. However, the density weighted vertical integral of the convergence of the Reynolds stress vanishes. To show this we use the meridional integral of (4.22a) in (4.25) and then integrate over the vertical domain to obtain

$$\left[- q\int_o^{\infty} e^{-z/h} \frac{F_o^2}{(z-u_r)^2}\, dz \left(\frac{\partial}{\partial T} + 2n'\right) + 2n'K^2\right.$$

$$\left.\times \int_0^{\infty} e^{-z/h} \frac{F_o^2}{z+\overline{u}_r}\, dz\right] |A|^2 \,(\sin^2 ly)$$

$$= - \int_0^{\infty} e^{-z/h} \left(\overline{u_o v_2} + \overline{u_2 v_o}\right) dz - \left(\overline{v_o \theta_2} + \overline{v_2 \theta_o}\right)_{z\,=\,0} \qquad (4.26)$$

Then using the meridional integral of the evaluation of the convergence of the heat flux at the ground in (4.22b) to eliminate the heat flux in (4.26), we obtain

$$[-q\int_0^\infty \frac{e^{-z/h} F_o^2(z)}{(z+\bar{u}_r)^2} dz \left(\frac{\partial}{\partial T} + 2n'\right) + 2n'K^2\int_0^\infty \frac{e^{-z/h} F_o^2(z)}{z+\bar{u}_r}$$

$$\times dz] |A|^2 (\sin^2 ly) = --\int_0^\infty e^{-z/h} \frac{\partial}{\partial y} (\overline{u_o v_2} + \overline{u_2 v_o}) dz$$

$$- [\frac{F_o^2(0)}{\bar{u}_r^2} \left(\frac{\partial}{\partial T} + 2n'\right) - \frac{2F_o^2(0)}{\bar{u}_r} r'K^2] |A|^2 (\sin^2 ly)$$

$$+ ikF_o(0)(AM^* - A^*M) \sin^2 ly. \qquad (4.27)$$

Now, if the modular wave field equation (4.24) is used in (4.27), the desired result obtains, i.e.

$$\int_o^\infty e^{-z/h} \frac{\partial}{\partial y} (\overline{u_o v_2} + \overline{u_2 v_o}) dz = 0. \qquad (4.28)$$

Therefore the vertical integral of the convergence of meridional momentum flux is zero. As a result the convergence of the momentum flux at each height z merely redistributes the x independent potential vorticity $\bar{\Pi}_1$ in the vertical but does not alter

$$\int_0^\infty e^{-z/h} (\overline{v_0 \Pi_2} + \overline{v_2 \Pi_0}) dz$$

and

$$\int_0^\infty e^{-z/h} \Pi_1 dz.$$

Eq. (4.28) also has a consequence on the vertically integrated x-independent momentum balance. In general, the vertically integrated x-independent momentum can be changed only by friction such as Ekman friction, the vertically integrated Coriolis torque due to the wave induced meridional circulation and the vertically integrated convergence of the Reynolds stress. By (4.28) the last of these is zero leaving only friction and the Coriolis torque to alter the vertically integrated zonal momentum. The Coriolis torque can be related to the form drag using the zonally averaged heat and mass conservation equations for the wave induced meridional circulation. However, we can more directly obtain an equation for the changes in the vertically integrated x-independent zonal momentum in terms of friction and form drag, by first multiplying $e^{-z/h}$ times (4.23a) or equivalently times

$$\frac{\partial}{\partial T}\left(\frac{\partial^2\Phi_1}{\partial y^2} + \frac{\partial^2\Phi_1}{\partial z^2} - h^{-1}\frac{\partial\Phi_1}{\partial z}\right) + n'\left(\frac{\partial^2\Phi_1}{\partial z^2} - h^{-1}\frac{\partial\Phi_1}{\partial z}\right)$$

$$= -\frac{\partial}{\partial y}\left(\overline{u_o v_2} + \overline{u_2 v_o}\right) + e^{z/h}\frac{\partial}{\partial z}e^{-z/h}\left(\overline{v_o\theta_2} + \overline{v_2\theta_o}\right),$$

and integrating in z using (4.28) to obtain

$$\frac{\partial}{\partial T}\left(\frac{\partial^2\Phi_1}{\partial y^2} + \frac{\partial^2\Phi_1}{\partial z^2} - h^{-1}\frac{\partial\Phi_1}{\partial z}\right) = -\frac{\partial}{\partial y}\left(\overline{u_o v_2} + \overline{u_2 v_o}\right)$$
$$+ \frac{1}{e^{-z/h}}\frac{\partial}{\partial z}e^{-z/h}\left(\overline{v_o\theta_2} + \overline{v_2\theta_o}\right), \tag{4.29}$$

Then using the evaluation of the heat flux at the ground (4.22b) and the bottom boundary zonal flow equation (4.23b) in (4.29) and integrating once in y using the sidewall boundary condition (4.3d), the desired relation can be written

$$\int_0^\infty dz e^{-z/h} \frac{\partial u_1}{\partial T} = - r' \; u_1 \; (z=0)$$

$$+ ikF_o(0)(AM^* - A^*M)(\sin^2 ly), \qquad (4.30)$$

where

$$u_1 = -\partial \Phi_1/\partial y.$$

Thus the vertically integrated momentum at a given latitude is damped by Ekman friction at the ground and forced by the topographic form drag at the ground. The ;sign of the topographic effect depends upon whether or not the wave is leading or lagging behind the topography.

Note the form drag here is proportional to AM*-A*M as it is in the barotropic case discussed by Pedlosky (1981). However, this is in contrast to his barocliic case where the form drag is proportional to $(dA/dT)M^* - (dA^*/dT)M$. Again this is a reflection of the different properties previously alluded to of the O(1) stationary waves in each case. In Pedlosky's two-layer baroclinic case the O(1) stationary wave is nonzero in the upper layer only and therefore is not directly forced by the topography. As a result, the development of form drag relies on the following essentially transient (in the absence of friction) process: the O(1) wave field forces an $O(\varepsilon)$ zonal flow in the lower layer proportional to $d|A|^2/dT$. This interacts with the $O(\varepsilon)$ topography to produce an $O(\varepsilon^2)$ wave field in the lower layer proportional to dA/dT. The $O(\varepsilon^2)$ wave then interacts with the topography to produce an $O(\varepsilon^3)$ form drag proportional to $(dA/dT)M^* - (dA^*/dT)M$. This is in contrast to the present case and Pedlosky's barotropic analysis where the normal mode of amplitude A is directly forced by the topography and hence, it directly interacts with the $O(\varepsilon^3)$ topography to produce an $O(\varepsilon^3)$ form drag proportional to AM^*-A^*M.

In closing this section we restate the evolution equations and compare them to those derived by Plumb (1981b). The closed set of nonlinear equations describing the topographically induced wave-mean flow interaction consists of (4.18), (4.23a,b). With $u_1 = -\partial\Phi_1/\partial y$ they can be rewritten

$$\left(\frac{\partial A}{\partial T} + ik\Delta' A + n' A\right)\left[q\int_0^\infty e^{-z/h}\frac{F_o^2}{(z+\bar{u}_r)^2}\,dz - \frac{F_o^2(0)}{\bar{u}_r^2}\right]$$

$$+ K^2 A\left[r'\frac{F_o^2(0)}{\bar{u}_r} - n'\int_0^\infty e^{-z/h}\frac{F_o^2}{(z+\bar{u}_r)^2}\,dz\right] - ikA$$

$$\times\int_0^\infty e^{-z/h}\frac{F_o^2}{(z+\bar{u}_r)^2}\left[(z+\bar{u}_r)p_{11} - qu_{11}\right]dz$$

$$+ ik\frac{F_o^2(0)}{\bar{u}_r}A\left(\frac{\partial u_{11}}{\partial z} - \frac{1}{\bar{u}_r}u_{11}\right)_{z=0} = ikF_oM, \qquad (4.31a)$$

$$\frac{\partial}{\partial T}\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial u_1}{\partial z^2} - h^{-1}\frac{\partial u_1}{\partial z}\right) + n'\left(\frac{\partial^2 u_1}{\partial z^2} - h^{-1}\frac{\partial u_1}{\partial z}\right)$$

$$= \left[-q\frac{F_o^2(z)}{(z+\bar{u}_r)^2}\left(\frac{\partial}{\partial T} + 2n'\right)|A|^2 + 2n'K^2\frac{F_o2(z)}{z+\bar{u}_r}|A|^2\right.$$

$$\times \frac{\partial^2}{\partial y^2} (\sin)^2(ly), \tag{4.31b}$$

$$\frac{\partial^2 u_1}{\partial T \partial z} + r' \frac{\partial^2 u_1}{\partial y^2} + n' \frac{\partial u_1}{\partial z} = [\frac{F_o^2(0)}{\bar{u}_r^2} (\frac{\partial}{\partial T} + 2n')|A|^2$$

$$- 2 \frac{F_o^2(0)}{\bar{u}_r} r'K^2|A|^2] \frac{\partial^2}{\partial y^2} (\sin^2 ly), \tag{4.31c}$$

where

$$p_{11} = - \int_0^1 2 (\sin^2 ly) (\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} - h^{-1} \frac{\partial u_1}{\partial z}) dy, \tag{4.32a}$$

$$u_{11} = \int_0^1 2 (\sin^2 ly) u_1 dy. \tag{4.32b}$$

The zonal velocity u_1 must satisfy the boundary condition

$$\partial u_1/\partial T = 0, \tag{4.33}$$

on $y = 0,1$.

The evolution equations (4.31a,b,c,d), even in the absence of Newtonian cooling and Ekman friction differ fundamentally from those derived by Plumb (1981b). First, in contrast to Plumb's model the present model does not have a top which here eliminates the possibility of any spurious reflection. However, since the normal modes of the Charney problem are vertically trapped, the constraint of a top would not alter the basic physics of the wave mean flow interaction. More importantly, Plumb, lead by his desire to parallel other imposed forcing treatments of stratospheric warmings, is not considering real topographic forcing which in effect eliminates the feedback between the zonal flow and wave field _at the ground_. This is evidenced by the absence of the entire right hand side of (4.31c) and the last term on the left side of (4.31a) in his comparable equations (2.26) and (2.39). This is not important for the existence of the initial topographic instability. Such a feedback is only necessary when describing the later stages of real topographic instability. However, the presence of the feedback does alter the growth rate obtained.

5. The topographic instability and its energetics

In this section we shall demonstrate the existence of the topographic instability. In order to simplify the analysis, only the instability in the absence of damping will be demonstrated. Without damping the evolution equations (4.31) become

$$\left(\frac{\partial A}{\partial T} + ik\Delta' A\right)\left[q \int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}\, dz - \frac{F_o^2(0)}{\bar{u}_r^2}\right]$$

$$- ikA \int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2} \left[(z+\bar{u}_r)p_{11} - qu_{11}\right] dz + ikA$$

$$\times \frac{F_o^2(0)}{\bar{u}_r} \left(\frac{\partial u_{11}}{\partial z} - \frac{1}{\bar{u}_r} u_{11}\right)_{z=0} = ikF_o(0)M, \tag{5.1a}$$

$$\frac{\partial}{\partial T}\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} - h^{-1} \frac{\partial u_1}{\partial z}\right) = - q \frac{Fo^2(z)}{(z+\bar{u}_r)^2} \frac{d|A|^2}{dT}$$

$$\times \frac{\partial^2}{\partial y^2} (\sin^2 ly), \tag{5.1b}$$

$$\frac{\partial^2 u_1}{\partial T \partial z} = \frac{F_o^2(0)}{\bar{u}_r^2} \frac{d|A|^2}{dT} \frac{\partial^2}{\partial y^2} (\sin^2 ly), \tag{5.1c}$$

on $z = 0$, and

$$\partial u_1/\partial T = 0, \tag{5.1d}$$

on $y = 0,1$.

The steady solution of (5.1) whose stability we will examine is

$$A_s = \frac{F_o(0)M}{\Delta'\left(qI_1 - \frac{F_o^2(0)}{\bar{u}_r^2}\right)} \tag{5.21}$$

$$u_{1s} \equiv 0, \tag{5.2b}$$

where

$$I_1 = \int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}\, dz \, .$$

For simplicity M and therefore A_s is taken to be real. The steady solution corresponds to a topographically forced wave. Since the wave is steady and inviscid, by (4.27) we know there is no form drag associated with it. Note exact resonance ($\Delta' = 0$) can't be considered because the steady wave amplitude would be infinite in the absence of damping.

The instability of the topographically forced wave (5.2) can be studied by linearizing (5.1) about the steady solution. Therefore if we write

$$A = A_s + A'$$

$$u_1 = u'_1,$$

the linearized equations are

$$\left(\frac{\partial A'}{\partial T} + ik\Delta' A'\right)\left[qI_1 - \frac{F_o^2(0)}{\bar{u}_r^2}\right] - ikA_s \int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}$$

$$\times \left[(z+\bar{u}_r)p'_{11} - qu'_{11}\right] dz + ikA_s \frac{F_o^2}{\bar{u}_r} \left(\frac{\partial u'_{11}}{\partial z}\right.$$

$$\left.- \frac{1}{\bar{u}_r} u'_{11}\right)_{z=0} = 0, \tag{5.3a}$$

$$\frac{\partial}{\partial T}\left(\frac{\partial^2 u'_1}{\partial y^2} + \frac{\partial^2 u'_1}{\partial z^2} - h^{-1} \frac{\partial u'_1}{\partial z}\right) = -\,2q \frac{F_o^2(z)}{(z+\bar{u}_r)^2} A_s$$

$$\times \frac{dA_r'}{dT} \frac{\partial^2}{\partial y^2} (\sin^2 ly), \tag{5.3b}$$

$$\frac{\partial u'_1}{\partial T \partial z} = 2 \frac{F_o^2(0)}{\bar{u}_r^2} A_s \frac{dA_r'}{dT} \frac{\partial^2}{\partial y^2} (\sin^2 ly) \quad \text{on } z=0 \;, \tag{5.3c}$$

where A'_r is the real part of A'. A'_i will be the imaginary part. It is particularly easy to calculate p_{11} by multiplying (5.3b) by $-2 \sin^2 ly$ and integrating over the meridional domain using the definition (4.32a). The result of this is

$$\frac{\partial p_{11}}{\partial T} = -\,2ql^2 \frac{Fo^2(z)}{(z+\bar{u}_r)^2} A_s \frac{dAr'}{dT} \;,$$

which when integrated in time yields

$$p_{11} = -\,2ql^2 \frac{F_o^2(z)}{(z+\bar{u}_r)^2} A_s A_r' \;, \tag{5.4}$$

plus a constant which for the purpose of the stability problem can be set equal to zero.

In order to obtain u_1 we are forced to solve (5.3b-d) directly. Following Pedlosky (1979), the solution to u_1 is sought in the form

$$u'_1 = \sum_j u_j\,(z,T) \quad \sin jy, \tag{5.5}$$

where $j = J\pi$, $J = 1,2,3,\ldots.$ Note (5.5) satisfies the sidewall boundary condition (5.1d). Substituting (5.5) into the zonal flow equation (5.3b,c), multiplying each by sinjy, integrating over the meridional domain and time yields

$$\frac{\partial^2 u_j}{\partial z^2} - h^{-1}\frac{\partial u_j}{\partial z} - j^2 u_j = -\,8ql^2\,\frac{j(1-(-1)^J)}{j^2-4l^2}$$

$$\times\,\frac{F_o{}^2(z)}{(z+\bar{u}_r)^2}\,A_s A_r{}', \tag{5.6a}$$

$$\frac{\partial u_j}{\partial z} = 8l^2 j\,\frac{(1-(-1)^J)}{j^2-4l^2}\,\frac{F_o{}^2(0)}{\bar{u}_r{}^2}\,A_s A'_r \quad \text{on } z=0. \tag{5.6b}$$

Again, in obtaining (5.6a,b) since the concern is the stability problem and not the initial value problem, the constants arising from the initial value problem are taken to be zero. The solution to (5.6a) is of the form

$$u_j = -8ql^2\,\frac{j(1-(-1)^J)}{j^2-4l^2}\,A_s A_r{}'\,(B_j(z) + C_j e^{-\alpha j z}), \tag{5.7}$$

where

$$\alpha_j = \frac{1}{2h} + \left(\frac{1}{4h^2} + j^2\right)^{1/2} , \tag{5.8}$$

and $B_j(z)$ is the forced solution of the equation

$$\frac{\partial^2 B_j}{\partial z^2} - h^{-1} \frac{\partial B_j}{\partial z} - j^2 B_j = \frac{F_o^2(z)}{(z+\bar{u}_r)^2} . \tag{5.9}$$

The constant C_j can be obtained by demanding that the solution (5.7) satisfy the lower boundary condition (5.6b). The result of this is

$$C_j = \frac{1}{\alpha_j} \left(\frac{F_o^2(0)}{q\bar{u}_r^2} + \frac{\partial B_j(0)}{\partial z}\right) , \tag{5.10}$$

Then by (5.5), (5.7) and (5.10) the correction to the zonal current can be written

$$u'_1 = \sum_j - 8ql^2 \frac{j(1-(-1)^J}{j^2 - 4l^2} A_s A_r' \, [B_j(z)$$

$$+ \frac{1}{\alpha_j} \left(\frac{F_o^2(0)}{q\bar{u}_r^2} + \frac{\partial B_j(0)}{\partial z}\right) e^{-\alpha_j z}] \sin jy, \tag{5.11}$$

which in combination with (4.16b) yields

$$u'_{11} = \sum_j - 32ql^4 \frac{(1-(-1)^J}{(j^2 - 4l^2)} A_s A_r' \, [Bj(z)$$

$$+ \frac{1}{\alpha_j} \left(\frac{F_o^2(0)}{q\bar{u}_r^2} + \frac{\partial Bj(0)}{\partial z}\right) e^{-\alpha_j z}] . \tag{5.12}$$

Finally, using (5.4) and (5.12) in the linearized wave field equation (5.3a) to write the wave-zonal flow interaction in terms of wave amplitude results in the following equation for the wave amplitude

$$\partial A'/\partial T + ik\Delta'A' - ik\, N_0\, A_s^2\, A_r' = 0 \tag{5.13}$$

where

$$N_0 = -\left(qI_1 - \frac{F_0^2(0)}{\bar{u}_r^2}\right)^{-1} \Big\{ql^2\int_0^\infty e^{-z/h} \frac{F_0^4}{(z+\bar{u}_r)^2}\, dz$$

$$+ 32q^2l^4 \sum_j \frac{(1-(-1)^J)^2}{(j^2 - 4l^2)^2} \int_0^\infty e^{-z/h} \frac{F_0^2}{(z+\bar{u}_r)^2} [Bj(z)$$

$$+ \frac{1}{\alpha_j} \left(\frac{F_0^2(0)}{q\bar{u}_r^2} + \frac{\partial Bj}{\partial z}\right) e^{-\alpha_j z}]\, dz - 32ql^4 \frac{F_0^2(0)}{\bar{u}_r}$$

$$\times \Big[\frac{F_0^2(0)}{q\bar{u}_r^2} \sum_j \frac{(1-(-1)^J)}{(J^2 - 4l^2)^2} + \sum_j \frac{(1-(-1)^J)^2}{(J^2 - 4l^2)^2} [Bj(0)$$

$$+ \frac{1}{\alpha_j} \left(\frac{F_0^2(0)}{q\bar{u}_r^2} + \frac{\partial Bj(0)}{\partial z}\right)]\Big]\Big\}. \tag{5.14}$$

Separating (5.13) into real and imaginary parts and solving for the imaginary part yields

$$\frac{d^2A_i}{dT^2} = k^2\, (\Delta' N_0 A_s^2 - \Delta'^2)\, A_i. \tag{5.15}$$

If a solution is sought in the form

$$A_i = A_i(0)e^{\sigma T}, \tag{5.16}$$

the growth rate σ is given by

$$\sigma^2/k^2 = (\Delta' N_0 A_s^2 - \Delta'^2). \tag{5.17}$$

From (5.17) we see the topographically forced wave is unstable whenever

$$|\Delta' N_0| > 0, \tag{5.18a}$$

and

$$|N_0 A_s^2| > \Delta'. \tag{5.18b}$$

Thus in this continuously stratified case as in Pedlosky's (1981) and Plumb's (1981a,b) analyses, instability exists for either super- or sub-resonant flow, depending on the sign of N_0. According to (5.18), if $N_0>0$ the instability occurs on the superresonant side of resonance whereas for $N_0<0$ the opposite is true. It is interesting to note that N_0 is analogous to the quantity $\varepsilon^2(3k^2-l^2)/8$ in Pedlosky's (1981) equation (3.7). In his case the side of resonance on which the topographic instability occurs depends on the relative meridional and zonal scales of the topography. In the present case it also depends on the vertical structure of the Charney mode excited. However, the exact parameter dependence of N_0 on zonal and meridional wave numbers and vertical structure is extremely complicated. As a result, an explicit calculation of N_0 is done only for the normal Charney mode with the simplest vertical structure (i.e. $r_0=2$) under the Boussinesq-like assumption $\beta \gg h^{-1}$. In Figure (3.5.2a) the regions for which N_0 is positive and

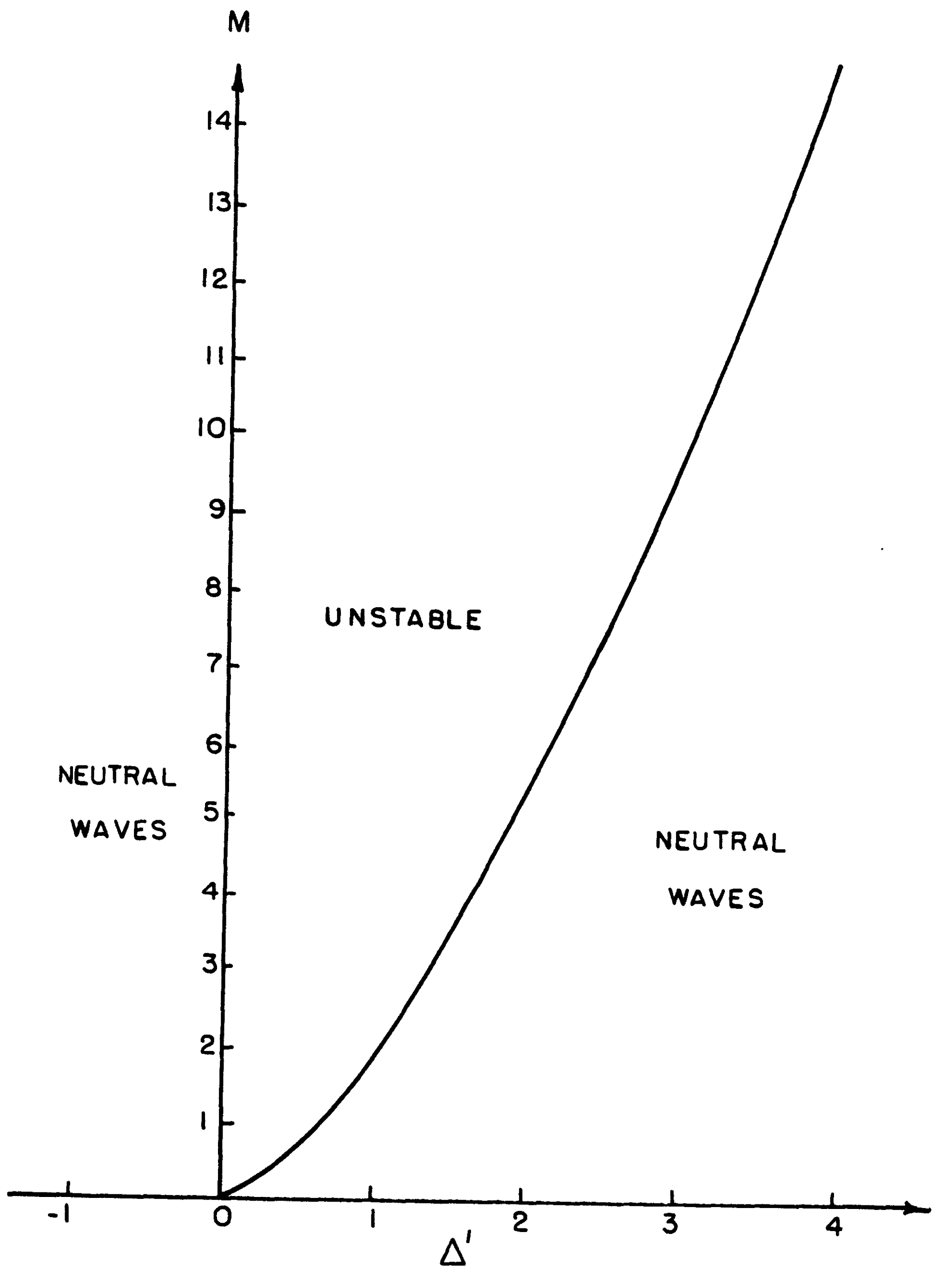

Fig. (3.5.1) Stability regime in M, Δ' space for the topographic instability of the $r_o = 2$ normal mode of the Charney problem assuming $\beta \gg h^{-1}$ and taking $k = l = 1$.

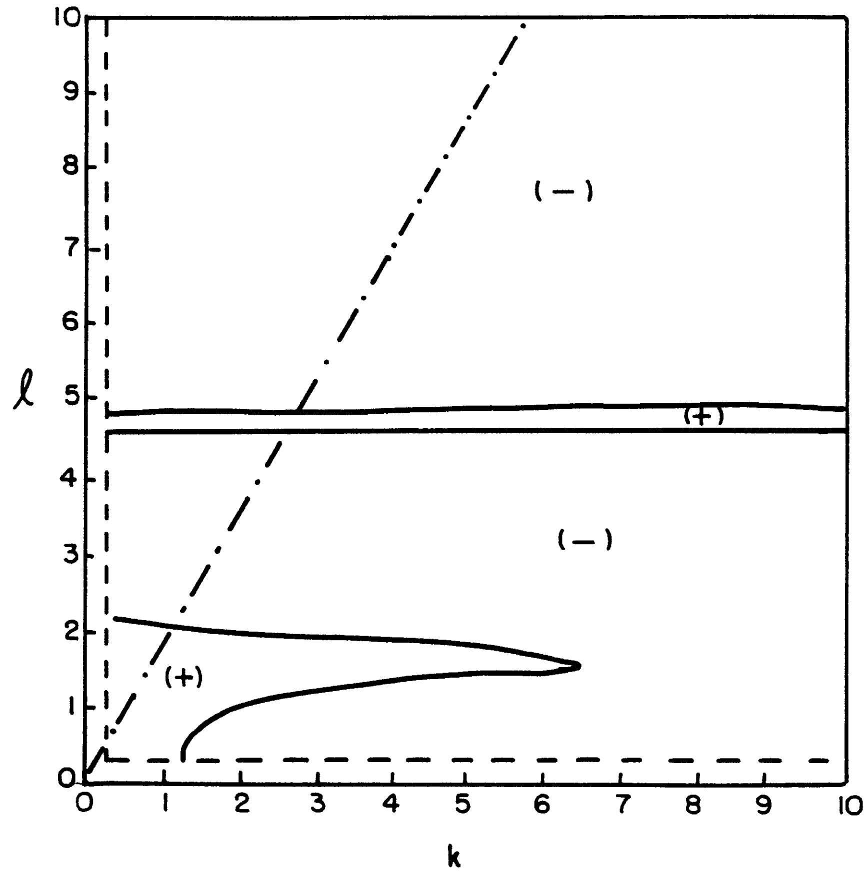

Fig. (3.5.2a) Regions of wavenumber space for which $N_0 > 0$ and $N_0 < 0$. In the region below the dot-dashed line, the quantity $\epsilon^2(3k^2-\ell^2)/8$ is greater than zero. This is the counterpart in Pedlosky's (1981) barotropic analysis to the more complicated N_0 of the present analysis.

negative are shown. N_0 is positive in a band centered at $l = 4.75$ and in a wedge centered at $l \simeq 1.5$ for $k<6.5$. The region above the dot-dashed line is the region of wavenumber space for which the nonlinear coefficient in the barotropic topotropic instability analysis of Pedlosky (1981) is greater than zero, i.e.

$$\frac{\varepsilon^2(3k^2-l^2)}{8} > 0$$

Thus it is clear the regions for which the topographic instability occurs for subresonant or superresonant flow differ markedly from the barotropic model to the baroclinic model considered here. This is due to the fact that now N_0 contains information on the vertical structure as well as information on the horizontal structure of the mode. In Figure (3.5.1) the stability regimes of the normal mode $k=l=1.0$ (for which $N_0>0$) are plotted against the topography height M and the detuned zonal flow Δ'. The principal feature of this diagram is its demonstration that less topographic height is required for instability as resonance is approached. This is reminiscent of the earlier studies of topographic instability which showed that topographic instability is a quasi-resonant phenomenon. However, as M is increased, the topographic instability can be excited further from resonance. In Figure (3.5.2b) the stability regimes in wavenumber space are shown. Due to the strength of the topography, the regions of instability correspond quite closely to where N_0 plotted in Figure (3.5.2a) is

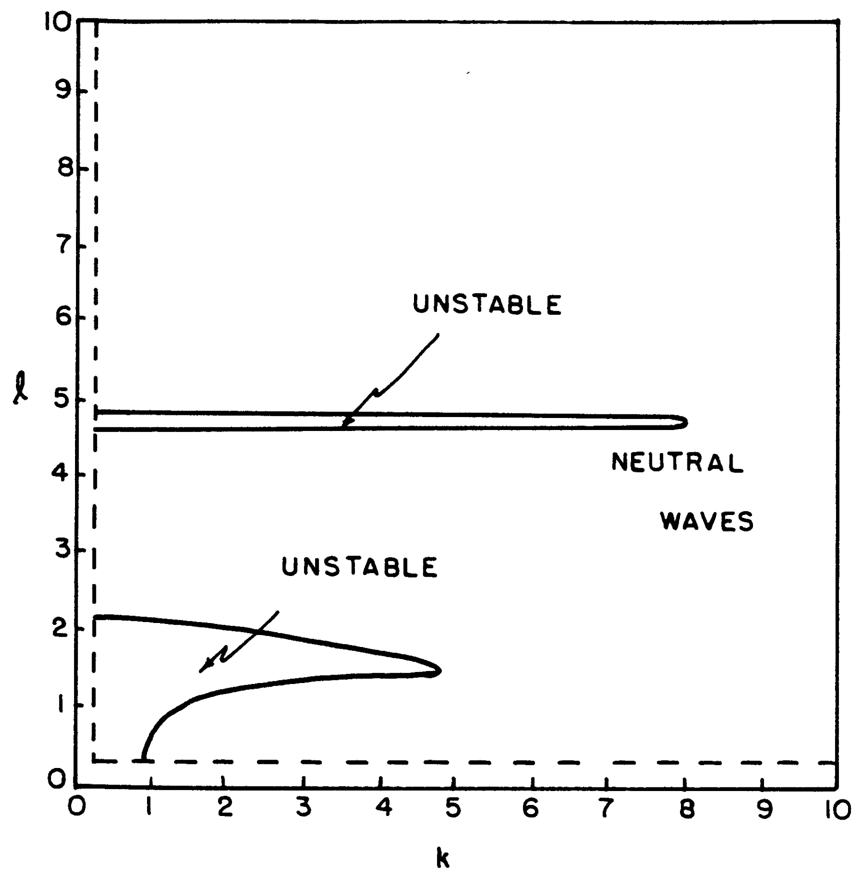

Fig. (3.5.2b) Stability regime in wavenumber space for the topographic instability with $\Delta'=1.0$ and $m=4.0$.

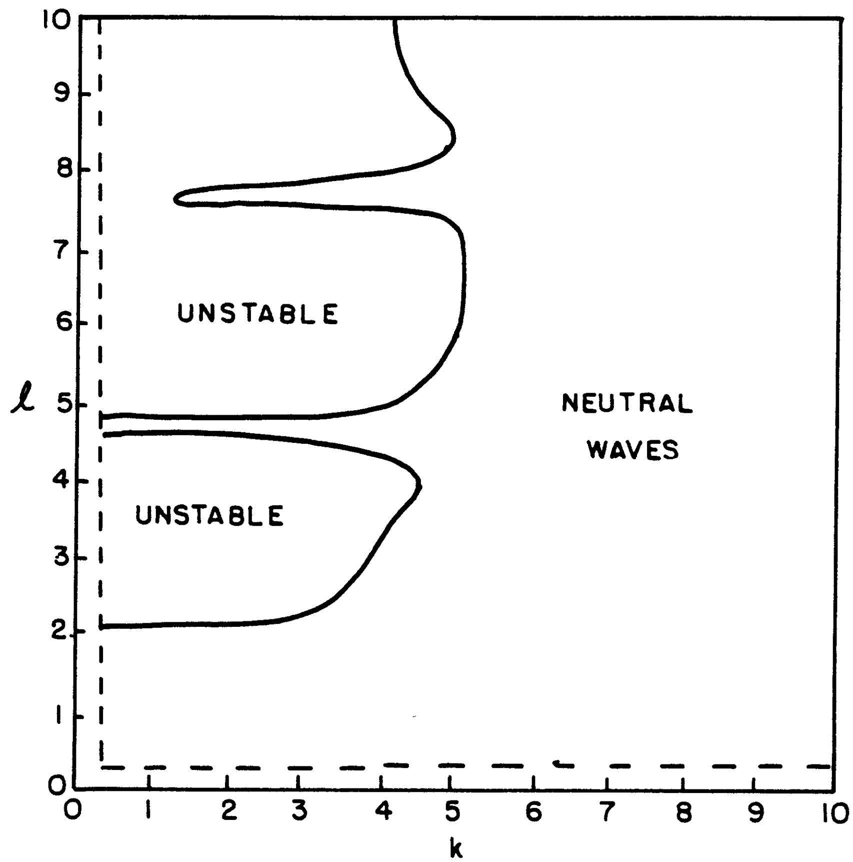

Fig. (3.5.2c) Same as (b) except $\Delta' = -1.0$ and $M = 5.0$.

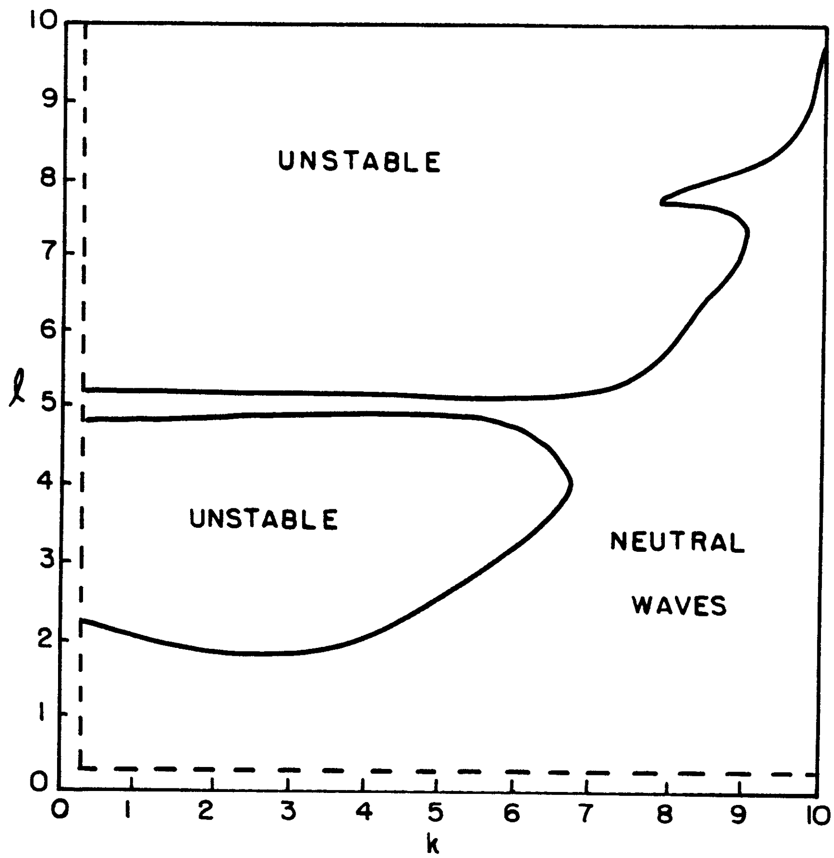

Fig. (3.5.2d) Same as (c) except $\Delta' = -1.0$ and $M = 7.5$.

greater than zero. These are, as we have already mentioned, the only regions in which the instability can occur for superresonant flow when $N_0>0$. However, note that not all waves with larger zonal wavenumber which could be unstable when $N_0>0$, are unstable. Thus, the instability is limited to the larger scales for which $N_0>0$. In Figures (3.5.2c,d) the stability diagrams in wavenumber space are plotted for the subresonant flow $\Delta' = -1.0$ and increasing topographic amplitude M = 5.0 and M = 7.5, respectively. As the size of topography is increased, the regions of instability, which now occupy regions in Figure (3.5.2a) for which $N_0<0$, grow toward larger values of k. This can be seen by comparing Figures (3.5.2c,d). These regions of instability avoid, as they must, regimes where $N_0>0$ which can be seen by comparing Figures (3.5.2c,d) to Figure (3.5.2a).

Above the existence of the topographic instability was demonstrated for the normal modes of the Charney problem. In order to better understand this study of topographic instability we now investigate its energetics. The equation for the wave field energetics of the inviscid topographic instability is

$$\varepsilon^4 \frac{\partial}{\partial T} \int_0^{\infty} e^{-z/h} \frac{1}{2} \overline{[\phi_{ox}^2 + \phi_{oy}^2 + \phi_{oz}^2]} dz$$

$$= \varepsilon^6 \int_0^{\infty} e^{-z/h} [\overline{\phi_{ox}\ \phi_{2y}} + \overline{\phi_{2x}\ \phi_{oy}}]\ \overline{u}_{1y}\ dz$$

$$+ \varepsilon^4 \int_0^\infty e^{-z/h} [\overline{\phi_{ox}\,\phi_{2z}} + \overline{\phi_{2x}\,\phi_{oz}}] \frac{\partial}{\partial z} (\overline{u}_r + z)\, dz$$

$$+ \varepsilon^4\, \overline{u}_r\, [\overline{\phi_o b_x}] + O(\varepsilon^6) \tag{5.19}$$

where [ ] represents the meridional average. The left side of (5.19) is the time change of total wave energy, kinetic plus available. The terms on the right side of (5.19) are in their order of appearance the rate of conversion of mean flow kinetic energy to perturbation kinetic energy by the Reynolds stress, the rate of conversion of mean available potential energy to perturbation available potential energy by the eddy heat flux, and the rate of conversion of mean flow kinetic energy to perturbation energy by the form drag. Note the Reynolds stress conversion of energy from the horizontally sheared part of the zonal flow is $O(\varepsilon^6)$ and therefore negligible compared to the $O(\varepsilon^4)$ terms. This is due to the absence of horizontal shear in the $O(1)$ zonal flow. The $O(\varepsilon^4)$ energy balance, i.e.

$$\frac{\partial}{\partial T} \int_0^\infty e^{-z/h}\, \frac{1}{2} [\overline{\phi_{ox}^2 + \phi_{oy}^2 + \phi_{oz}^2}]dz = \int_0^\infty e^{-z/h}$$

$$\times [\overline{\phi_{ox}\,\phi_{2z}} + \overline{\phi_{2x}\,\phi_{oz}}] \frac{\partial}{\partial z} (\overline{u}_r + z)\, dz + \overline{u}_r\, [\overline{\phi_o b_x}], \tag{5.20}$$

is somewhat different than that of the layered baroclinic case considered by Pedlosky (1981). Here both the heat flux conversion of energy from the vertically sheared part of the zonal flow and the form drag conversion of energy from the zonal flow contribute to the topographic instability. The later conversion was $O(\varepsilon^6)$ in Pedlosky's (1981) baroclinic case since the O(1) zonal flow was zero in the lower layer. Thus, only the heat flux conversion contributed to the instability in his case. In barotropic models of topographic instability such as those discussed in Charney and Devore (1979) and Pedlosky (1981) only the form drag conversion contributes to the topographic instability. Thus, both the form drag conversion, singly responsible for topographic instability in previous barotropic models, and the heat flux conversion, singly responsible for the topographic instability in previous baroclinic models, are responsible for the instability in the present baroclinic model.

It is a simple matter to calculate the contributions to the time change of the total wave energy in the wave energy equation (5.20). The left side of (5.20) can be calculated using the solution (4.6) for Φ_0, integrating by parts employing the equations for the vertical eigenfunction (4.7a,b). The result for the time change of the total wave energy is

$$\frac{\partial}{\partial T} \int_0^\infty e^{-z/h} \frac{1}{2} \overline{[\phi_{ox}^2 + \phi_{oy}^2 + \phi_{oz}^2]} dz$$

$$= \frac{1}{2} \left(q \int_0^\infty e^{-z/h} \frac{F_o 2}{z+\overline{u}_r} dz - \frac{F_o^2(0)}{\overline{u}_r}\right) \frac{d|A|^2}{dT} \, . \tag{5.21}$$

The two terms on the right side of (5.20) must add up to the result (5.21). To calculate the heat flux conversion, we first integrate by parts in z and use the y-averaged version of the relation between the vorticity flux, Reynolds stress and heat flux (4.25), i.e.

$$\frac{\partial}{\partial z} e^{-z/h} [\overline{v_o \theta_2} + \overline{v_2 \theta_o}] = e^{-z/h} [\overline{v_o \Pi_2} + \overline{v_2 \Pi_o}]. \tag{5.22}$$

Then using the inviscid form of (4.22a,b) to write the vorticity flux and the heat flux at the ground in terms of the wave amplitude yields the heat flux conversion

$$\int_0^\infty e^{-z/h} [\overline{\phi_{ox} \phi_{2z}} + \overline{\phi_{2x} \phi_{oz}}] \frac{\partial}{\partial z} (\bar{u}_r + z) dz$$

$$= \frac{1}{2} (q \int_0^\infty e^{-z/h} \frac{F_o^2}{z+\bar{u}_r} dz - \frac{F_o^2(0)}{\bar{u}_r}) \frac{d|A|^2}{dT}$$

$$+ \frac{1}{2} ik\bar{u}_r F_o(0) (AM^* - A^*M). \tag{5.23}$$

Using the O(1) solution (4.6) and (4.1d) to calculate the form drag term yields

$$\bar{u}_r[\overline{\phi_o b_x}] = - \frac{1}{2} ik\bar{u}_r F_o(0) (AM^* - A^*M). \tag{5.24}$$

Thus, the sum of the heat flux conversion (5.23) and form drag conversion (5.24) is equal to the time change of total wave field energy and is proportional to $\partial|A|^2/\partial T$. We note from the inviscid form of (4.24), i.e.

$$[q \int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2}\, dz - \frac{F_o^2(0)}{\bar{u}_r^2}] \frac{d|A|^2}{dT}$$

$$= -ikF_o(0)(AM^*-A^*M) \tag{5.25}$$

the wave transience is entirely due to the presence of topography. Thus, the topography has acted as a catalyst causing a normal mode of Charney problem, which is stable to the usual baroclinic instabiltiy, to be unstable.

6. Multiple equilibria

In the interest of completeness, we now briefly discuss the evolution of the initial linear instability which will lead to the consideration of the special requirements for multiple equilibria in this continuously stratified model. The nonlinear complex amplitude equation corresponding to the fully nonlinear set of inviscid evolution equations (5.1a-d) is simply

$$\frac{dA}{dT} + ik\Delta'A - \frac{1}{2}ikN_oA\,(|A|^2 - |A(0)|^2) = ik\Delta'A_s, \tag{6.1}$$

where A_s is given by (5.2a) and A(0) is the initial value of the amplitude. This describes the inviscid nonlinear oscillations which occur after the initial instability. We note however, that these oscillations never lose their memory of the initial condition. This is due to the neglect of viscosity thus far. Therefore, we now consider the damped equations (4.31a-c) of this continuously stratified baroclinic model. This is done to show the existence of the multiple equilibria to which the instability may tend hence losing sight of the initial value. Such multiple equilibria have been shown to exist in previous barotropic models, layered baroclinic models, and in the continuously stratified model considered by Jacobs (1980). All the multiple equilibria previously calculated were the result of a balance between the form drag and the damping. In barotropic models Ekman friction is sufficient to produce multiple equilibria. However, this is not necessarily the case in baroclinic models. In a layered baroclinic model Ekman friction only enters into the energy budget of the layer on which it acts. As a result, a lower Ekman layer in a two layer model only determines the zonal flow in terms of the wave amplitude in the lower layer. The upper layer remains infinitely degenerate. In a continuously stratified model the situation is even more severe where the Ekman friction only determines the zonal flow at the ground. The interior is left infinitely degenerate. Therefore in order to completely determine the steady problem, an internal dissipation is necesary.

In the present model, the internal dissipation is the Newtonian cooling. We now consider the steady form of (4.31a,b,c), i.e.

$$ik\Delta'(qI_1 - \frac{F_o^2(0)}{\bar{u}_r^2})\bar{A} + [n'(qI_1 - k^2 I_2 - \frac{F_o^2(0)}{\bar{u}_r^2})$$

$$+ r'k^2 \frac{F_o^2(0)}{\bar{u}_r}]\bar{A} - ik\bar{A}\int_0^\infty e^{-z/h} \frac{F_o^2}{(z+u_r)^2} [(z+\bar{u}_r)\bar{p}_{11}$$

$$- q\bar{u}_{11}]dz + ik \frac{F_o^2(0)}{\bar{u}_r}\bar{A} (\frac{\partial \bar{u}_{11}}{\partial z} - \frac{1}{\bar{u}_r}\bar{u}_{11})_{z=0} = ikF_o(0)M, \tag{6.2a}$$

$$\frac{\partial^2 \bar{u}_1}{\partial z^2} - h^{-1}\frac{\partial \bar{u}_1}{\partial z} = \frac{4l^2 F_o^2(z)}{(z+\bar{u}_r)^2} [(z+\bar{u}_r)k^2 - q] |\bar{A}|^2 \cos 2ly \tag{6.2b}$$

$$r' \frac{\partial^2 \bar{u}_1}{\partial z^2} - n' \frac{\partial \bar{u}_1}{\partial z} = \frac{4l^2 F_o^2(0)}{\bar{u}_r^2} (n' - r'\bar{u}_r K^2)|A|^2 \cos 2ly$$

on $z=0$, (6.2c)

where

$$I_1 = \int_0^\infty e^{-z/h} \frac{F_o^2}{(z+\bar{u}_r)^2} dz \tag{6.3a}$$

$$I_2 = \int_0^\infty e^{-z/h} \frac{F_o^2}{z+\bar{u}_r} dz \tag{6.3b}$$

The steady problem (6.2a,b,c) is similar to the one considered by Pedlosky (1979) where Newtonian cooling was also introduced to break the conservation of potential vorticity in the interior of the fluid. Here, as in that study, the steady limit of (4.31b) is singular in the y interval. This singularity gives rise to narrow transient boundary layers at the sidewalls whose widths narrow according to $T^{1/2}$ as the solution approaches the steady state. Therefore, as a steady state is approached, the steady zonal flow equation (6.2b) is valid over a larger portion of the meridional domain. As a result (6.2b) can be used to calculate the frictionally induced steady zonal flow.

Solving for the zonal flow and substituting the result into the steady wave equation (6.2a) results in the steady complex amplitude equation for $\bar{A}$. Separating that into real and imaginary parts yields

$$-k\Delta'\bar{A}_i + \left(qI_1 - \frac{F_o^2(0)}{\bar{u}_r^2}\right)^{-1} \left[n' \left(qI_1 - \frac{F_o^2(0)}{\bar{u}_r^2} - K^2 I_2\right) + r'K^2 \frac{F_o^2(0)}{\bar{u}_r}\right]\bar{A}_r - kP\bar{A}_i\,(\bar{A}_r^2 + \bar{A}_i^2) = 0, \tag{6.4a}$$

$$k\Delta'\bar{A}_i + \left(qI_1 - \frac{F_o^2(0)}{\bar{u}_r^2}\right)^{-1} \left[n' \left(qI_1 - \frac{F_o^2(0)}{\bar{u}_r^2} - K^2 I_2\right) + r'K^2 \frac{F_o^2(0)}{\bar{u}_r}\right]\bar{A}_i + kP\bar{A}_r\,(\bar{A}_r^2 + \bar{A}_i^2) = k\Delta' A_s, \tag{6.4b}$$

where A_s is given by (2.5.2a) and P is

$$P = \left(qI_1 - \frac{F_o^2(0)}{\bar{u}_r^2}\right)^{-1} \left\{- 2 \int_0^\infty e^{-z/h} \frac{F_o^2}{(z+u_r)^2} [4l^2 (z+\bar{u}_r) \times (\hat{u}(z) + De^{-z/h})dz - \frac{F_o^2}{z+\bar{u}_r} ((z+\bar{u}_r)K^2-q)]\right.$$

$$+ 2q \int_0^\infty e^{-z/h} \frac{F_o^2}{(z+u_r)^2} (\hat{u}(z) + De^{-z/h})dz$$

$$+ 2 \frac{F_o^2(0)}{\overline{u}_r^2} [\hat{u}_z(0) - h^{-1} D - \frac{1}{\overline{u}_r} (\hat{u}(0) + D)]\}, \qquad (6.5)$$

where $\hat{u}$ is the forced solution of

$$\hat{u}_{zz} - h^{-1} \hat{u}_z = \frac{F_o^2(z)}{(z+\overline{u}_r)^2} [(z+\overline{u}_r)K^2 - q]$$

and

$$D = \frac{n'\hat{u}_z(0) - 4l^2 r'\hat{u}(0)}{4l^2 r' + h^{-1} n'} - \frac{F_o^2(0)}{\overline{u}_r^2} (\frac{n' - r'\overline{u}_r K^2}{4l^2 r' + h^{-1} n'})$$

It is very difficult to solve (5.33a,b) in general. However, in the limit of small friction it is relatively simple to show the existence of multiple equilibria. If r', '→0 keeping the ratio r'/ ' fixed a solution to (6.4a,b) can be found in the form

$$\overline{A}_r = \overline{A}_r(0) + n' \overline{A}_r(1), \qquad (6.6a)$$

$$\overline{A}_i = \qquad + n' \overline{A}_i(1) \qquad (6.6b)$$

Note from (6.4a) $\overline{A}_i{}^{(0)}$ vanishes. $\overline{A}_r{}^{(0)}$ satisfies

$$P\overline{A}_r{}^{(0)3} + \Delta' A_r{}^{(0)} - \Delta' A_s = 0 \qquad (6.7)$$

The cubic (6.7) has multiple real roots or equivalently, multiple equilibria exist when

$$\Delta'/P < - 27/4 \; A^2{}_s \; . \qquad (6.8)$$

The essence of this result is that multiple equilibria exist only for a sufficiently detuned zonal flow. This was also found to be the case in the earlier studies of topographic instability. The result (6.8) is particularly analogous to that obtained in the barotropic analysis of Pedlosky (1981) where it is also demonstrated that multiple equilibria exist for sufficiently detuned flow. In addition, he finds an upper bound on the detuning parameter for multiple equilibria to exist which is inversely proportional to the Ekman friction. Therefore as the Ekman friction is decreased the range of the detuning parameter for which multiple equilibria occur becomes larger and in fact becomes infinite as the Ekman friction approaches zero. In the present study, the condition (6.8) was obtained assuming Ekman friction approached zero. As is the case in Pedlosky (1981), it is not possible to obtain an upper limit on Δ' in the limit of very small friction. In order to obtain this upper limit, the Ekman friction must be allowed to be larger. In contrast to the case studied by Pedlosky, this calculation is technically difficult and therefore is not carried out here.

Previous studies (e.g,. Jacobs (1979)) have demonstrated that we can expect one of the equilibria to be unstable. However, to describe the evolution of the solution away from the unstable equilibrium requires the solution of the fully transient and damped evolution equations (4.31a-c) which is beyond the scope of the present work.

7. Conclusions

In this chapter, the fully transient evolution equations were derived governing the topographic instability of a general normal mode of the Charney problem. These evolution equations include the wave-zonal flow interaction at the ground not found by Plumb (1981b) due to the nature of the forcing used in his problem. The growth rate of the topographic instability was obtained in analytic but unclosed form for the normal mode of the Charney model with the simplest vertical structure under a Boussinesq-like assumption. Thus, this calculation is not quite as explicit as that in the two-layer model (Pedlosky, 1981), but it is easier to follow than the analogous calculation in the study of topographic instability in a continuously stratified model by Plumb (1981b). It was found that while the wave-zonal flow interaction at the ground (not present in Plumb's (1981b) analysis) is not required for the existence of the topographic instability, this interaction does modify the growth rate of the disturbance. The topographic instability is excited most easily near resonance as in earlier studies and exists for both subresonant and superresonant flow depending on the sign of the nonlinear interaction coefficient N_0. Multiple equilibria also exist in this model for a sufficiently detuned flow as in the earlier studies. The regions of subresonant and superresonant instability differ markedly from those in the barotropic model. In the present model the superresonant instability is very limited in meridional wavenumber. This strong dependence of the instability on meridional wavenumber is due to the strong meridional dependence of the nonlinear wave-mean flow interaction. The fact that meridional dependence is not a feature of the barotropic analysis of Pedlosky (1981) suggests that this is due to the more realistic vertical structure in the present model. However, the role of the channel walls in this regard should be investigated in the future. The subresonant instability is not so limited in meridional wavenumber but for both subresonant and superresonant flow the larger zonal scales are most easily excited by the topographic instability. Overall this suggests that the topographic instability is quite sensitive to the vertical structure of the mean flow and normal mode under study. The strong dependence on meridional wavenumber suggests that given the zonal flow in the atmosphere, one might be able to locate a stationary instability with a fairly well defined meridional structure. However, we must be cautious about the applicability of the instability found at very large space scales in this model. At these scales, the quasi-Boussinesq approximation breaks down and moreover, the quasi-geostrophic theory begins to break down. Thus more realistic models of the large-scale atmospere should be considered in the future.

Chapter 4. The Charney mode under the influence of topography.

1. Introduction

In this chapter we examine the behavior, in the presence of topography, of the Charney mode which is either weakly stable or unstable and embedded in a uniform shear flow. The two questions to be answered are: (1) what effect does topography have on the unstable Charney mode and (2) does the presence of topography lead to instability in parameter regimes which in teh absence of topography would be stable? The studies in the next four sections follow the work of Pedlosky (1979) quite closely with the important exception that weak topography is allowed as well as a weak uniform zonal flow in some cases. In each section either the size of topography or the size of the weak zonal flow are altered in order to examine the above stated questions in different parameter regimes. Pedlosky (1979) considered the finite-amplitude dynamics of a weakly unstable Charney mode. To examine this oproblem he employs the method of multiple time scales and pivots the analysis about the point of neutral stability to obtain the necessary time scale separation. The expansion parameter is then chosen such that the amplitude of the weak instability can be determined by the wave-mean flow nonlinearity. We shall see that this wave-mean flow nonlinearity will be important in modifying the instability.

In Section 2 the effect of sinusoidal topography on an infinitesimal weakly-growing Charney mode, of the same structure as the topography, will be studied. Pedlosky (1981) studied this question in a two-layer baroclinic model and found the effect to be stabilizing. There is a difference between his analysis and the one in section 2 which is a consequence of structure of the neutral baroclinic wave in each case. In the two-layer model the neutral wave amplitude does not vanish in the lower layer. Thus it is necessary to require that it be in phase with the topography so as not to produce a form drag. Form drag is only produced by the transient phase shifted part of the growing wave. In the continuous model the neutral Charney mode vanishes at the ground making such a requirement unnecessary but the form drag is still produced only by the transient phase-shifted part of the wave. The effect of topography on the instability of the Charney mode in section 2 is a result of the topographically induced wave-zonal flow interaction at the ground.

Section 3 is quite similar to section 2 but a very weak uniform zonal flow is allowed. The diversion of this zonal flow by the topography forces a nearly neutral Charney mode. The presence of this forced wave futher alters the stability properties of the previous section by further modification of the wave-mean flow interaction at all depths of the fluid.

In section 4 the size of the uniform zonal flow is increased again. As a result the topography must be assumed to be weaker in order that the Charney mode is not too strongly forced. The consequence of this is to eliminate the portion of the wave-zonal feedback at the ground which was solely responsible for modifying the instability in section 2 and partially responsible in section 3. However, the uniform zonal velocity is now so large that the advective time scale associated with it is of the order of the long time scale for the wave-mean flow interaction. Therefore, the weakly unstable Charney mode is now propagating at the speed of the zonal flow in the absence of topography. Plumb (1979) considered an analogous problem in the two layer model with propagating topography. His analysis is restricted to weak baraclinic stability. However, he was able to show even in baroclinically stable parameter regimes that there was an instability if the topography were strong enough. He found taht as the topographic height increased, one of the neutral modes became stationary and therefore coherent with the forced wave which allowed a strong wave-zonal flow interaction leading to the instability. In addition, it was shown the energy for the instability is drawn from the available potential energy of the mean flow. In section 4 we will examine this effect for baroclinically subcritical flow and also the effect of increasing topography on teh propagating baroclinic instability (i.e., also for baroclinically supercritical flow). Section 5 contains an analogous problem to that considered in section 4 but with strong Ekman friction and small Newtonian cooling.

2. Effect of topography on a stationary weakly growing Charney mode.

In the previous chapter the nearly stationary normal modes of the Charney problem were shown to be unstable due to the wave-mean flow interaction for which the topography acted as a catalyst. In this chapter the role of topography in altering the baroclinic instability of the stationary, nearly-neutral Charney mode is examined. We will show that this alteration is due to the topographically induced modification of the wave mean flow interaction which, in the absence of topography, arises from the finite-amplitude equilibration of the stationary, weakly unstable Charney mode. Pedlosky (1979) studied the equilibration of the weakly growing Charney mode in the absence of topography. The present analysis is identical to that in his section 3 with the important exception that weak $O(\varepsilon)$ sinusoidal topography is included here.

The analysis pivots about the stationary neutral Charney mode for which, according to (3.3.10),

$$\beta_o + h^{-1} = (h^{-2} + 4K^2)^{1/2} , \tag{2.1a}$$

and

$$\overline{U} = 0. \tag{2.1b}$$

As described in seciton (3.3), if β is perturbed from the value defined by (2.1a) by an amount of $O(\varepsilon^2)$, i.e.,

$$\beta = \beta_o - \varepsilon^2\delta , \tag{2.2}$$

the growth rate is $O(\varepsilon)$ for $\delta > 0$. This fact motivates the choice of the long time scale

$$T = \varepsilon t. \tag{2.3}$$

If (2.1a,b), (2.2) and (2.3) are used in the potential vorticity equation (3.2.3) and the Newtonian cooling is assumed to be very small ($\varkappa << O(\varepsilon^2)$), (3.2.3) becomes

$$z \frac{\partial \Pi}{\partial x} + q_o \frac{\partial \phi}{\partial x} = - \varepsilon \frac{\partial \Pi}{\partial T} - \varepsilon^2 \delta \frac{\partial \phi}{\partial x} - \varepsilon J(\phi, \Pi), \tag{2.4}$$

where

$$q_o = \beta_o + h^{-1} \tag{2.5}$$

Upon inspection of (2.4) it becomes apparent that there is a singularity near the ground. This is due to the fact that near the ground ($z \sim O(\varepsilon)$) the advective time scale matches the $O(\varepsilon)$ slow time scale of the temporal evolution, whereas for $z \sim O(1)$ the advective derivative due to the vertical shear of the basic state flow is $O(1)$. In order to deal with this critical layer singularity, the inner independent variable

$$\zeta = \frac{z}{\varepsilon} \tag{2.6}$$

is defined. The inner stream function $\hat{\phi}(x,y,\zeta,T)$ then satisfies the potential vorticity equation

$$\left(\frac{\partial}{\partial T} + \zeta \frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \hat{\phi}}{\partial \zeta^2} - \varepsilon h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta} + \varepsilon^2 \nabla^2 \hat{\phi}\right) + \varepsilon(q_o - \varepsilon^2 \delta)\frac{\partial \hat{\phi}}{\partial x}$$

$$= -J\left(\hat{\phi}, \frac{\partial^2 \hat{\phi}}{\partial \zeta^2} - \varepsilon\, h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta} + \varepsilon^2 \nabla^2 \hat{\phi}\right), \qquad (2.7)$$

subject to a boundary condition at the ground. If we choose the topography according to

$$\eta = \varepsilon b = \varepsilon M e^{ikx} (\sin ly) + *, \qquad (2.8)$$

and if the Ekman spinup time is assumed to be of the order of the long time scale, i.e.,

$$r = \varepsilon r' \qquad (2.9)$$

then the lower boundary condition for $\hat{\phi}$ on $\zeta=0$ can be written

$$\frac{\partial^2 \hat{\phi}}{\partial T \partial \zeta} - \frac{\partial \hat{\phi}}{\partial x} + J\left(\hat{\phi}, \frac{\partial \hat{\phi}}{\partial \zeta}\right) = -\varepsilon r' \nabla^2 \hat{\phi} - \varepsilon J(\hat{\phi}, b). \qquad (2.10)$$

Therefore, the potential vorticity equation (2.4) defines the outer problem for ϕ whereas the potential vorticity equation (2.7) and lower boundary condition (2.10) define the inner problem for $\hat{\phi}$. By expanding both ϕ and $\hat{\phi}$ in powers of ε both the outer and inner problems can be solved independently yielding the equations for the wave amplitude and the associated zonal flow correction. Then the problem can be closed by the asymptotic matching of the inner solution to the outer solution which passes information about the effect of topography and other lower boundary information in $\hat{\phi}$ to the fluid away from the boundary described by ϕ.

We begin by considering the outer problem. The appropriate outer expansion is

$$\phi = \phi o + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots . \tag{2.11}$$

Using (2.11) in the outer potential vorticity equation (2.4) results in a sequence of problems for consecutive powers of ε. The O(1) problem is

$$z \frac{\partial \Pi_o}{\partial x} + q_o \frac{\partial \phi_o}{\partial x} = 0, \tag{2.12}$$

and leads to the specification of a single stationary neutral Charney mode which can be written

$$\phi_o = AF_o(z)e^{ikx} (\sin ly) + * , \tag{2.13a}$$

where

$$F_o(z) = ze^{-\beta_o z/2} , \tag{2.13b}$$

and β_o is given by (2.1a). It is interesting to note that the vertical eigenfunction given in (2.13b) vanishes at z=0. Therefore the stationary neutral Charney mode does not interact directly with the topography to produce form drag. In addition note that ϕ_o near the ground is only $O(\varepsilon)$, i.e.,

$$\phi_o \sim \varepsilon\zeta Ae^{ikx} (\sin ly) + *. \tag{2.14}$$

As a result the inner solution which will match the neutral Charney mode will be $O(\varepsilon)$. It will be helpful to keep this in mind when we get to the inner sequence of problems.

The $O(\varepsilon)$ outer problem can be written

$$z \frac{\partial \Pi_1}{\partial x} + q_0 \frac{\partial \phi_1}{\partial x} = q_0 \frac{dA}{dT} e^{-\beta_0 z/2} e^{ikx}(\sin ly) + *, \tag{2.15}$$

where the $O(1)$ solution has already been used to evaluate the inhomogeneous terms. The full solution to (2.15) is

$$\phi_1 = \frac{1}{ik} \frac{dA}{dT} e^{-\beta_0 z/2} e^{ikx}(\sin ly) + * + \Phi_1(y,z,T), \tag{2.16}$$

where Φ_1 is the $O(\varepsilon)$ correction of the zonal flow. It will be apparent at the next order that the introduction of Φ_1 is necessary due to the forcing provided by the self interaction of the wave field. The solution (2.16) has only x-independent potential vorticity, i.e.,

$$\Pi_1 = \frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z} \tag{2.17}$$

The $O(\varepsilon)$ solution (2.16) remains $O(\varepsilon$ near the ground, thus requiring the matchable inner solution to be $O(\varepsilon)$. Since the transient wave part of ϕ_1 does not vanish at the ground we can expect form drag to be produced by its interaction with topography which in turn alters the zonal flow. In addition, if from the inner problem at $O(\varepsilon)$ it is determined that Φ_1 does not vanish at the ground, the zonal flow correction at the ground $\Phi_1(y,0,T)$, will interact with the topography thereby altering the wave amplitude. Thus we anticipate the presence of a wave-zonal flow feedback loop due to the presence of topography which is in addition to the topography independent wave-zonal flow feedback loop discussed by Pedlosky (1979) in his section 3. However, to clearly see this requires the consideration of the inner sequence of problems. We postpone this until after consideration of the final outer problem at $O(\varepsilon^2)$.

The $O(\varepsilon^2)$ outer equation is

$$z \frac{\partial \Pi_2}{\partial x} + q_0 \frac{\partial \phi_2}{\partial x} = - \frac{\partial \Pi_1}{\partial T} + \delta \frac{\partial \phi_0}{\partial x} - J(\phi_0, \Pi_1) - J(\phi_1 \Pi_0). \tag{2.18}$$

The zonally average version of (2.18) can be written

$$\frac{\partial}{\partial T} \left(\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z} \right) = - \overline{J(\phi_0, \Pi_1)} - \overline{J(\phi_1, \Pi_0)} \tag{2.19}$$

The nonlinear terms in (2.19) are easily evaluated using the solutions at previous orders (2.13) and (2.16). After doing so (2.19) becomes

$$\frac{\partial}{\partial T} \left(\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z} \right) = q_0 e^{-\beta_0 z} \times \frac{d|A|^2}{dT} \frac{\partial}{\partial y} (\sin^2 ly), \tag{2.20}$$

which describes the evolution of the zonal flow in the body of the fluid in terms of the forcing provided by the transient wave field. A lower boundary condition on Φ_1 is still required.

We now seek the equation for the wave amplitude. Using the O(1) and O(ε) solutions to evaluate the wave-like inhomogeneous terms of (2.18) yields

$$z \frac{\partial \Pi_2}{\partial x} + q_o \frac{\partial \phi_2}{\partial x} = ik\delta F_o A e^{ikx} (\sin ly) + *$$

$$- ikF_o \frac{\partial \Pi_1}{\partial y} Ae^{ikx} (\sin ly) + * - ik \frac{\partial \Phi_1}{\partial y}$$

$$\times q_o \frac{F_o}{z} Ae^{ikx} (\sin ly) + *. \tag{2.21}$$

Removal of the secularities from (2.21) results in the desired amplitude equation. This can be accomplished by multiplying (2.21) by

$$e^{-z/h} \left(\frac{F_o}{z}\right) e^{-ikx} \, 2 \, (\sin ly)$$

and integrating over the domain. The resulting solvability condition is

$$\int_0^1 dy \; 2 \, (\sin ly) \; \overline{e^{-ikx} \; \phi_2(x,y,0,T)} = \frac{\delta}{q_o^2} A$$

$$- A\int_0^\infty dz e^{-q_o z} \left(zp_{11} - \frac{\partial u_{11}}{\partial z}\right) + Au_{11} \, (0,T), \tag{2.22}$$

where

$$p_{11} = \int_0^1 2(\sin^2 ly) \frac{\partial \Pi_1}{\partial y} \, dy, \tag{2.23a}$$

and

$$u_{11} = - \int_0^1 2(\sin^2 ly) \frac{\partial \Phi_1}{\partial y} \, dy. \tag{2.23b}$$

In order to obtain the amplitude equation, $\phi_2(x,y,0,T)$ must first be found. Thus, the sequence of inner problems is considered next for the purpose of determining $\phi_2(x,y,0,T)$ and obtaining a lower boundary condition on the zonal flow correction Φ_1.

The appropriate inner expansion takes the form

$$\hat{\phi} = \hat{\phi}_o + \varepsilon\hat{\phi}_1 + \varepsilon^2 \hat{\phi}_2 + \ldots \quad . \tag{2.24}$$

However, recall from the outer problem that the outer stream function near the ground is $O(\varepsilon)$ or higher order. Therefore no $O(1)$ inner solution is matchable to the outer solution which requires

$$\hat{\phi}_o = 0. \tag{2.25}$$

The first nonzero inner solution appears at $O(\varepsilon)$. In view of (2.25) the $O(\varepsilon)$ inner equations can be written

$$\left(\frac{\partial}{\partial T} + \zeta\frac{\partial}{\partial x}\right) \hat{\phi}_{1\zeta\zeta} = 0. \tag{2.26a}$$

$$\frac{\partial\hat{\phi}_1}{\partial T\partial\zeta} - \frac{\partial\hat{\phi}_1}{\partial x} = 0 \qquad \text{on } \zeta = 0. \tag{2.26b}$$

The solution to (2.26a,b) which can be matched to the outer solution is

$$\hat{\phi}_1 = (D_1\zeta + C_1)\, e^{ikx}(\sin ly) + * + f_1\,(y,t) \tag{2.27}$$

Matching the inner solution (2.27) to the outer solution up to $O(\varepsilon)$ it is found

$$C_1 = \frac{1}{ik}\frac{dA}{dT} , \tag{2.28a}$$

$$D_1 = A(T), \tag{2.28b}$$

$$f_1 = \Phi_1(y,0\ T). \tag{2.28c}$$

As a result of (2.28a,b,c) the $O(\varepsilon)$ inner solution takes the final form

$$\hat{\phi}_1 = (A\zeta + \frac{1}{ik}\frac{dA}{dT})\ e^{ikx}\ (\sin ly) + * + \Phi_1(y,0,T). \tag{2.29}$$

From the inner solution (2.29) it is now clear there is both nonvanishing wave and zonal flow at the ground. The consequence of this in the presence of bottom topography will be apparent in the $O(\varepsilon^2)$ inner problem.

The $O(\varepsilon^2)$ inner equations are

$$(\frac{\partial}{\partial T} + \zeta\frac{\partial}{\partial x})(\frac{\partial^2\hat{\phi}}{\partial\zeta^2} + \beta_o A e^{ikx}\ (\sin ly) + *) = 0 \tag{2.30a}$$

$$\frac{\partial^2\hat{\phi}_2}{\partial T\partial\zeta} - \frac{\partial\hat{\phi}_2}{\partial x} = - r'\nabla^2\hat{\phi}_1 - J(\hat{\phi}_1,b) - J(\hat{\phi}_1, \frac{\partial\hat{\phi}_1}{\partial\zeta}) \tag{2.30b}$$

where the inhomogeneous terms in the vorticity equation have already been evaluated using the $O(\varepsilon)$ solution (2.29) and the definition of q_o (2.5). The solution of the vorticity equation (2.30a) is

$$\hat{\phi}_2 = - \frac{\beta_o}{2}\zeta^2 A e^{ikx}\ (\sin ly) + * + (C_2+D_2\zeta)e^{ikx}(\sin\ ly)$$

$$+ * + \zeta\chi_2\ (y,T). \tag{2.31}$$

If (2.31) is substituted into the lower boundary condition (2.30b) and a zonal average is taken, the lower boundary condition on the zonal flow is obtained, i.e.,

$$\frac{\partial \chi_2}{\partial T} + r' \frac{\partial^2 \Phi_1}{\partial y^2}(y,0,T) = -\frac{\partial}{\partial y}\overline{(\hat{v}_1, \theta_1)}. \tag{2.32}$$

In (2.32) χ_2 is as yet undetermined,

$$\hat{v}_1 = \frac{\partial \hat{\phi}_1}{\partial x}, \quad \hat{\theta}_1 = \frac{\partial \hat{\phi}_1}{\partial \zeta}$$

and the Jacobians have been rewritten to make clear the contribution to the wave forcing from the form drag and convergence of heat flux at the ground. The form drag is due to the interaction of the nonvanishing $O(\varepsilon)$ wave field at the ground with the topography. This forms an essential portion of the additional wave-zonal feedback loop due to the presence of topography. The heat flux contribution to (2.32) would exist even in the absence of topography. Using (2.29) to evaluate the right side of (2.32) yields

$$\frac{\partial \chi_2}{\partial T} + r' \frac{\partial^2 \Phi_1}{\partial y^2}(y,0,T) = -\frac{d}{dT}\left[|A|^2 + AM^* + A^*M\right] \times \frac{\partial}{\partial y}(\sin^2 ly), \tag{2.33}$$

from which it becomes clear that the wave forcing of the zonal flow at the ground is due to the wave transience.

When the $O(\varepsilon)$ and $O(\varepsilon^2)$ solutions (2.29) and (2.31) respectively, are substituted into the $O(\varepsilon^2)$ lower boundary condition (2.30b), the wave-like portion yields

$$C_2 = \frac{1}{ik}\frac{\partial D_2}{\partial T} + \frac{r'K^2}{k^2}\frac{dA}{dT} - (M + A)\frac{\partial \Phi_1}{\partial y}(y,0,T). \tag{2.34}$$

Note from the next to last term in (2.34) that the zonal flow at the ground interacts with the topography. This is the other necessary ingredient for the topographically induced portion of the wave-zonal flow feedback loop.

In (2.34) D_2 is as yet undetermined as is χ_2 in (2.33) and $\phi_2(x,y,0,T)$ in (2.22). To determine these quantities the outer solution is Taylor expanded about z=0, i.e.,

$$\begin{aligned}\phi &= A[\varepsilon\zeta - \frac{\beta_o}{2}\varepsilon^2\zeta^2 + \ldots]\, e^{ikx}\,(\sin ly) + * \\ &+ \varepsilon\frac{1}{ik}\frac{dA}{dT}(1 - \varepsilon\frac{\beta_o}{2}\zeta + \ldots)e^{ikx}\,(\sin ly) + \varepsilon\Phi_1(y,0,T) \\ &+ \varepsilon^2\zeta\frac{\partial\Phi_1}{\partial z}(y,0,T) + \varepsilon^2\phi_2\,(x,y,0,T) + O(\varepsilon^3),\end{aligned} \tag{2.35}$$

and matched to the inner solution, i.e.,

$$\begin{aligned}\hat{\phi} &= \varepsilon(A\zeta + \frac{1}{ik}\frac{dA}{dT})e^{ikx}\,(\sin ly) + * + \varepsilon\Phi_1\,(y,0,T) \\ &+ \varepsilon^2\Big\{-\frac{\beta_o}{2}\zeta^2 A + D_2\zeta + \frac{1}{ik}\frac{\partial D_2}{\partial T} + \frac{r'K^2}{k^2}\frac{dA}{dT} \\ &- (A + M)\frac{\partial\Phi_1}{\partial y}(y,0,T)\Big\}\, e^{ikx}\,(\sin ly) + * \\ &+ \varepsilon^2\,\zeta\chi_2 + O(\varepsilon^3)\end{aligned} \tag{2.36}$$

The results are

$$\chi_2 = \frac{\partial \Phi_1}{\partial z}(y,0,T) \tag{2.37a}$$

$$D_2 = -\frac{\beta}{2ik}\frac{dA}{dT} \tag{2.37b}$$

$$\phi_2(x,y,0,T) = \left[\frac{\beta o}{2k^2}\frac{d^2A}{dT^2} + \frac{r'k^2}{k^2}\frac{dA}{dT} - (A+M)\frac{\partial \Phi_1}{\partial y}(y,0,T)\right]$$

$$\times\ e^{ikx}(\sin ly) + *. \tag{2.37c}$$

Substitution of (2.37c) into the solvability condition (2.22) and substitution of (2.37a) into the lower boundary condition for the zonal flow (2.33) yields, along with the equation for the zonal flow (2.20) already obtained, the final closed set of nonlinear equations. They are

$$\frac{d^2A}{dT^2} + \frac{2r'K^2}{\beta_o}\frac{dA}{dT} - \frac{2k^2\delta A}{\beta_o q^2{}_o} + \frac{2k^2A}{\beta_o}\int_0^\infty e^{-q_o z}\,(zp_{11}$$

$$-\frac{\partial u_{11}}{\partial z})dz + \frac{2k^2}{\beta_o}Mu_{11}(0,T) = 0, \tag{2.38a}$$

$$\frac{\partial}{\partial T}\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} - h^{-1}\frac{\partial u_1}{\partial z}\right) = -q_o e^{-\beta_o z}$$

$$\times\ \frac{d|A|^2}{dT}\frac{\partial^2}{\partial y^2}(\sin^2 ly), \tag{2.38b}$$

$$\frac{\partial^2 u_1}{\partial T \partial z}(y,0,T) + r' \frac{\partial^2 u_1}{\partial y^2}(y,0,T) = \frac{d}{dT}(AM^* + A^*M + |A|^2) \frac{\partial^2}{\partial y^2}(\sin^2 ly), \tag{2.38c}$$

where

$$u_1 = -\frac{\partial \Phi_1}{\partial y}, \tag{2.39}$$

and

$$\frac{\partial u_1}{\partial T} = 0, \tag{2.40}$$

on $y = 0,1$.

The set of equations (2.38a,b,c) governs the equilibration of the weakly unstable stationary Charney mode in the presence of topography and the associated wave mean flow interaction. They are identical to those derived by Pedlosky (1979) (his equations (3.47), (4.3) and (4.4a)) except for the presence of the form drag term in (2.38c) and the presence of the last term in (2.38a) which represents the interaction of the topography with the zonal flow correction at the ground. These terms are responsible for the topographically induced portion of the wave mean flow interaction.

We now investigate their effect on the wave mean flow interaction. This requires the solution of (2.38a,b,c) which entails the elimination of p_{11} and u_{11} from the amplitude equation (2.38a). It is a simple matter to solve for p_{11} in terms of wave amplitude keeping in mind its definition

$$p_{11} = \int_0^1 2(\sin^2 ly) \frac{\partial \Pi_1}{\partial y} dy = - \int_0^1 2(\sin^2 ly)$$

$$\times \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} - h^{-1} \frac{\partial u_1}{\partial z}\right) dy$$

Multiplying (2.38b) by $2(\sin^2 ly)$. integrating over the meridional domain and once in time yields

$$p_{11} = - q_o l^2 \left(|A|^2 - |A(0)|^2\right) e^{-\beta_o z} , \qquad (2.41)$$

in order to obtain

$$u_{11} = \int_0^1 2(\sin^2 ly)\, u_1\, dy,$$

first the solution of u_1 must be found. If a solution to u_1 is sought in the form

$$u_1 = \sum_j u_{1j}\, (z,T) \sin jy, \qquad (2.42)$$

where

$$j = J\pi, \qquad J = 1,\ 2,\ 3,\ \ldots\ .$$

Using (2.38b,c) to solve for $u_{1j}(z,T)$ yields

$$u_{ij} = \frac{-8 q_o l^2\, j}{(\beta_o q_o - j^2)(j^2 - 4l^2)} \left[\left(|A|^2 - |A(0)|^2\right)\right.$$

$$\times \left. \left(e^{-\beta_o z} - \frac{j^2}{\alpha_j q_o} e^{-\alpha_j z} + Bje^{-\alpha_j z}\right]\right., \qquad (2.43)$$

where

$$\alpha_j = -\frac{1}{2h} + \left(j^2 + \frac{1}{4h^2}\right)^{1/2}, \tag{2.44}$$

and

$$\frac{dBj}{dT} + \frac{r'j^2}{\alpha_j} Bj = -r' \frac{j^2}{\alpha_j}\left(1 - \frac{j^2}{q_o\alpha_j}\right)(|A|^2 - |A(0)|^2)$$

$$+ \frac{\beta_o q_o - j^2}{q_o \alpha_j} \frac{\partial}{\partial T} (AM^* + A^*M). \tag{2.45}$$

From (2.43), u_{11} can be calculated. If the result is substituted in (2.38a), the amplitude equation becomes

$$\frac{d^2A}{dT^2} + \frac{2r'k^2}{\beta_o} \frac{dA}{dT} - \frac{2k^2\delta}{\beta_o q_o^2} A + A[N_o(|A|^2 - |A(0)|^2)$$

$$+ \sum_j N_j B_j] + M[Q_o(|A|^2 - |A(0)|^2) + \sum_j Q_j B_j] = 0, \tag{2.46}$$

where

$$N_o = \frac{2k^2}{\beta_o} \sum_j \left[\frac{16 q_o l^4 (1-(-1)^J)^2}{(\beta_o q_o - j^2)(j^2 - 4l^2)^2}\left(\frac{\beta_o}{\beta_o + q_o} - \frac{j^2}{q_o(\alpha_j + q_o)}\right)\right.$$

$$\left. - \frac{q_o l^2}{(q_o + \beta_o)^2}\right], \tag{2.47a}$$

$$N_j = \frac{2k^2}{\beta_o} \frac{16 q_o l^4 (1-(-1)^J)^2 \alpha_j}{(\beta_o q_o - j^2)(j^2 - 4l^2)^2 (q_o + \alpha_j)}, \tag{2.47b}$$

$$Q_o = \frac{2k^2}{\beta_o} \sum_j \frac{16 q_o l^4 (1-(-1)^J)^2}{(\beta_o q_o - j^2)(j^2 - 4l^2)^2} \left(1 - \frac{j^2}{q_o \alpha_j}\right), \qquad (2.47c)$$

$$Q_j = \frac{2k^2}{\beta_o} \frac{16 q_o l^4 (1-(-1)^J)^2}{(\beta_o q_o - j^2)(j^2 - 4l^2)^2} . \qquad (2.47d)$$

The solution of the coupled amplitude equations (2.45) and (2.46) determines the evolution of the wave amplitude given the inital conditions. In general they are very difficult to solve. However, in the absence of friction the effect of the topography on the linear baroclinic instability is easy to see. Consider the inviscid linearized version of (2.45) and (2.46), i.e.,

$$\frac{d^2A}{dT^2} - \frac{2k^2\delta}{\beta_o q_o^2} A + M \sum_j Q_j B_j = 0, \qquad (2.48a)$$

$$B_j = \frac{\beta_o q_o - j^2}{q_o \alpha_j} (AM^* + A^*M). \qquad (2.48b)$$

If, for simplicity, A and M are assumed to be real, the elimination of B_j from (2.48a) yields

$$\frac{d^2A}{dT^2} - \sigma^2 A = 0, \qquad (2.49)$$

where

$$\sigma^2 = -2 \left(M^2 \sum_j \frac{\beta_o q_o - j^2}{q_o \alpha_j} Q_j - \frac{k\delta}{\beta_o q_o^2}\right. \qquad (2.50)$$

From the growth rate (2.50) it is easily seen that the topography modifies the baroclinic instability towards stability provided

$$\sum_j \frac{\beta_o q_o - j^2}{q_o \alpha_j} Q_j = 64M^2 \frac{k^2 l^4}{\beta_o} \sum_j \frac{(1-(-1)^J)^2}{\alpha_j (j^2-4l^2)^2} > 0. \tag{2.51}$$

Indeed this is the case. In fact each term of the sum is greater than zero and the sum converges. In Fig. (4.2.1) the stability diagram plotted in topographic amplitude M and supercriticality δ space when the Boussinesq-like approximation $\beta_o >> h^{-1}$ is made. It clearly shows that as the topographic amplitude increases the baroclinic instability is stabilized and that as the supercriticality is increased the topographic amplitude required for stabilization is greater.

The physical mechanism for this topographic stabilization is as follows: As the baroclinically unstable wave grows, a wave, phase shifted from the initial baroclinic disturbance, develops, which interacts with the topography to produce the form drag which according to (2.38c) modifies the zonal shear at the ground in such a way as to stabilize the baroclinic wave.

Closer examination of the growth rate rewritten in the form

$$\sigma^2 = - \frac{2k^2}{\beta_o} [32\ M^2 l^4 \sum_j \frac{(1-(-1)^J)^2}{\alpha_j (j^2 - 4l^2)^2} - \frac{\delta}{q_o{}^2}] , \tag{2.52}$$

reveals that the topographic stabilization is most effective for waves with short meridional scale (i.e., $l^4 >> 1$). Furthermore, for those scales for which

$$M^2 > \frac{\delta/q_o^2}{32l^4 \sum_j \frac{(1-(-1)^J)^2}{\alpha_j(j^2 - 4l^2)^2}}, \qquad (2.53)$$

the baroclinic instability is rendered stable. Thus the main effect of topography on baroclinic instability is to limit the baroclinic instability to the larger meridional scales. This is apparent in Fig. (4.2.2) in which the stability regimes are plotted in k,l space under the Boussinesq-like assumption $\beta_o >> h^{-1}$. This stabilizing effect of topography was also noted by Pedlosky (1981) in the context of a two layer model. He also pointed out that the presence of topography caused the inviscid nonlinear oscillation to be asymmetric about zero amplitude. Inspection of the present amplitude equations (2.45) and (2.46) in their inviscid limit reveals this to be the case here also.

The stabilization of the baroclinic instability by topography is of special interest in light of the result in the previous chapter. Namely, while topography destabilizes the normal modes of the Charney model via topographic instability, it stabilizes the baroclinically unstable Charney mode. Thus, in the presence of topography the

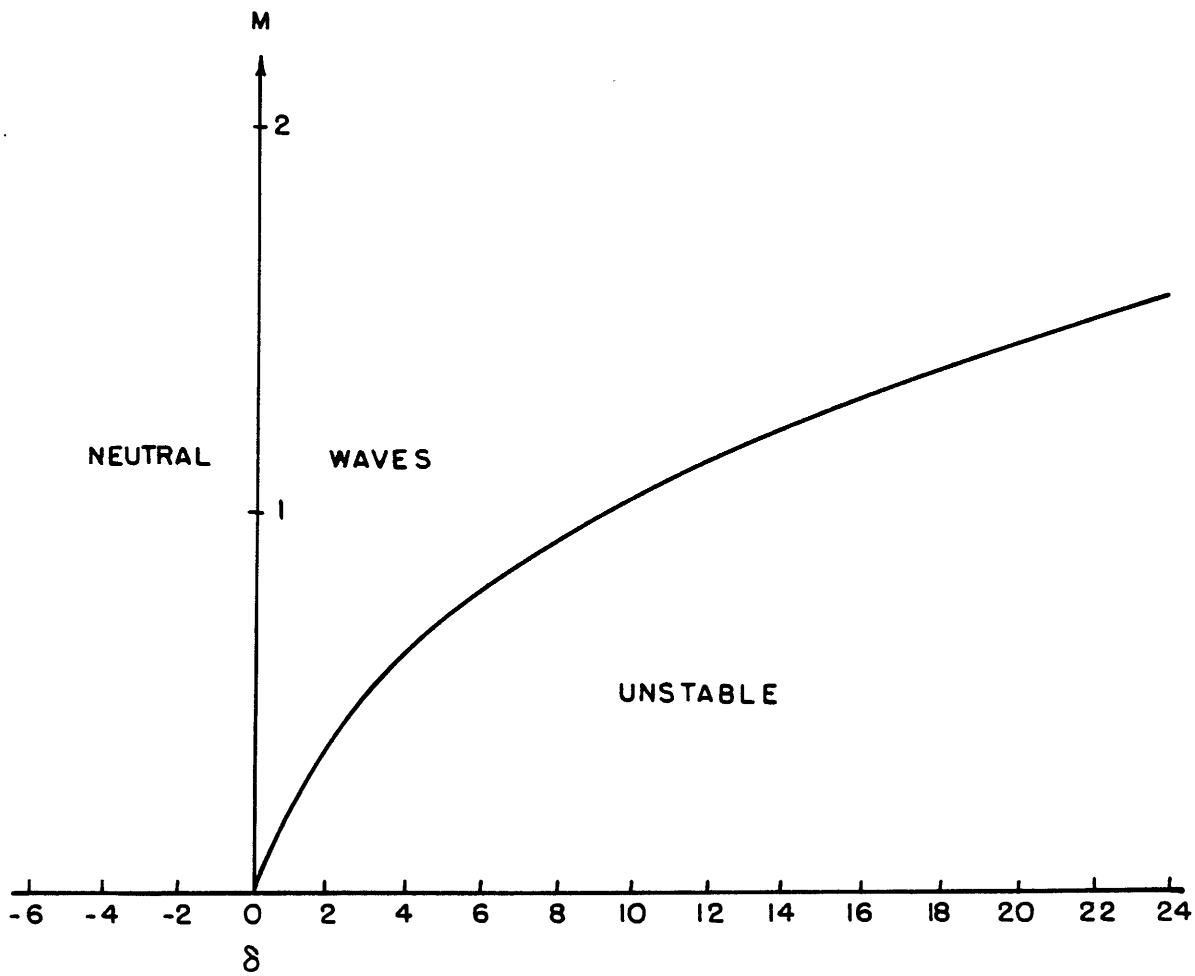

Fig.(4.2.1) Stability regime diagram vs. M and δ for k=l=1.0.

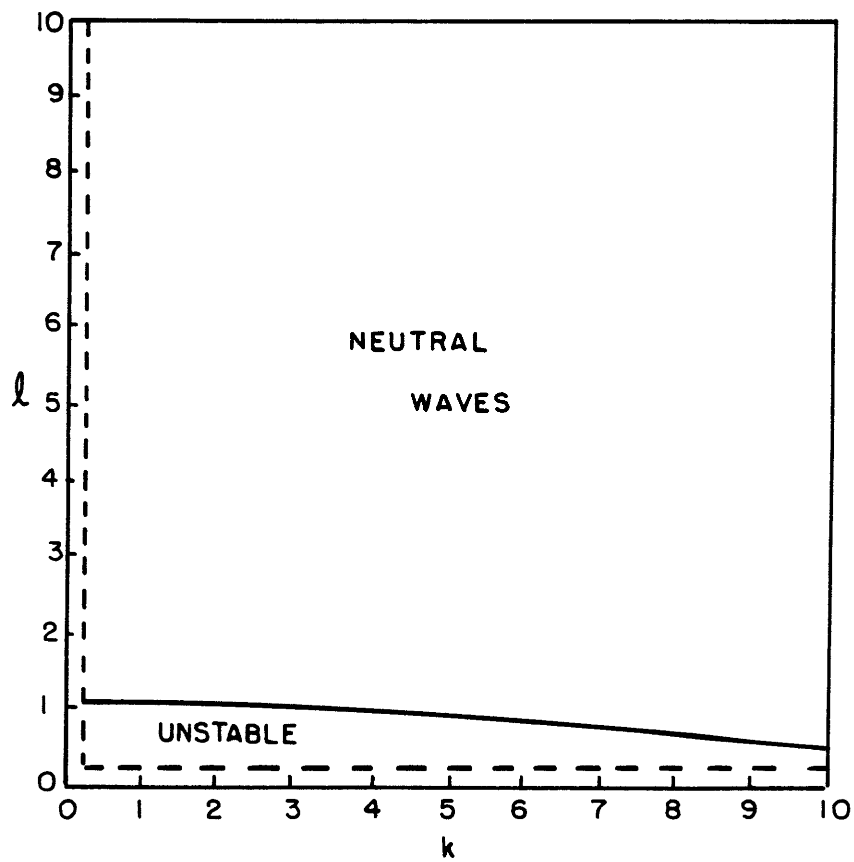

Fig. (4.2.2) Stability regime diagram in k,l space for fixed topography $\mathcal{M}$ =.5 and supercriticality δ=12.0. In the absence of topography the entire range of wavenumbers would be unstable.

normal modes are at least as dynamically important as the baroclinically unstable modes in the Charney model.

3. Instability of a topographically forced Charney mode with very weak zonal flow at the ground.

In this section we will see how the stability problem of section 2 is altered as the zonal flow at the ground is increased enough to force a small but finite-amplitude Charney mode near neutrality. In section 2 we studied the effect of topography on a weakly unstable infinitesimal free Charney mode. Here and in sections 4 and 5 the stability of a topographically forced nearly neutral Charney mode embedded in a shear flow will be examined. The wave is forced by the diversion of the zonal flow by the sinusoidal topography. The difference between this and the next section is due to the choice of the size of the zonal velocity at the ground and the topography. Here the choice of topography is the same as in the last section but now a very weak zonal flow will be allowed at the ground, i.e.,

$$\overline{U} = \varepsilon^2 \Delta' . \tag{3.1}$$

Except for the presence of this weak zonal flow all aspects of this analysis are identical to those of the last seciton. Then if (2.1b) is replaced with (3.1) but the choices (2.2), (2.3), (2.6), (2.8) and (2.9) are made as before the outer and inner problems, respectively, become

$$z \frac{\partial \Pi}{\partial x} + q_o \frac{\partial \phi}{\partial x} = - \varepsilon \frac{\partial \Pi}{\partial T} - \varepsilon^2 \Delta' \frac{\partial \Pi}{\partial x} + \varepsilon^2 \delta \frac{\partial \phi}{\partial x} - \varepsilon J(\phi, \Pi), \tag{3.2a}$$

and

$$\frac{\partial}{\partial T} + \varepsilon \Delta' \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial x})(\frac{\partial^2 \hat{\phi}}{\partial \zeta^2} - \varepsilon\, h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta} + \varepsilon^2 \nabla^2 \hat{\phi})$$

$$+ \varepsilon\, (q_o - \varepsilon^2 \delta) \frac{\partial \hat{\phi}}{\partial x} + J(\hat{\phi}, \frac{\partial^2 \hat{\phi}}{\partial \zeta^2} - \varepsilon h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta} + \varepsilon^2 \nabla^2 \hat{\phi}) = 0 \tag{3.2b}$$

subject to

$$\frac{\partial^2 \hat{\phi}}{\partial T \partial \zeta} - \varepsilon \Delta' \frac{\partial^2 \hat{\phi}}{\partial x \partial \zeta} - \frac{\partial \hat{\phi}}{\partial x} + J(\hat{\phi}, \frac{\partial^2 \hat{\phi}}{\partial \zeta^2}) = - \varepsilon r' \nabla^2 \hat{\phi}$$

$$- \varepsilon^2 \Delta' b_x - \varepsilon J(\hat{\phi}, b), \tag{3.2c}$$

on z=0.

Again outer and inner expansions of the form (2.11) and (2.24) will be used. Before proceeding it is useful to note that (3.2a,b,c) differ from their counterparts in section 2 only by the presence of terms proportional to Δ'. Examination of these terms in (3.2a,b,c) reveals that they all have a zero zonal average. Therefore, the equations (2.38b,c) for the zonal flow obtained in the previous section also obtain here, i.e.,

$$\frac{\partial}{\partial T} (\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} - h^{-1} \frac{\partial u_1}{\partial z}) = - q_o e^{-\beta_o z} \frac{d|A|^2}{dT}$$

$$\times \frac{\partial^2}{\partial y^2} (\sin^2 ly), \tag{3.3a}$$

$$\frac{\partial^2 u_1}{\partial T \partial z} (y,0,T) + r' \frac{\partial^2 u_1}{\partial y^2} (y,0,T) = \frac{d}{dT} (AM^* + A^*M$$

$$+ |A|^2) \frac{\partial^2}{\partial y^2} (\sin^2 ly). \tag{3.3b}$$

Thus it is only possible for the additional terms to alter the wave equation. In fact a further examination of (3.2a,b,c) reveals that only the $O(\varepsilon^2)$ outer and inner wave field problems are different from those in section 2. As a result the outer and inner wave field solutions up to $O(\varepsilon)$ are the same as in section 2 which are

$$\phi = Aze^{-\beta_o z/2} e^{ikx} (\sin ly) + * + \frac{\varepsilon}{ik} \frac{dA}{dT}$$

$$\times \; e^{-\beta_o z/2} e^{ikx} (\sin ly) + * , \tag{3.41}$$

and

$$\hat{\phi} = \varepsilon \left(A\zeta + \frac{1}{ik} \frac{dA}{dT}\right) e^{ikx} (\sin ly) + * . \tag{3.4b}$$

respectively.

Now we turn to the consideration of the $O(\varepsilon^2)$ outer problem. Using (3.4a) to evaluate the inhomogeneous terms the $O(\varepsilon^2)$ outer equation for the wave field takes the form

$$z \frac{\partial}{\partial x} (\nabla^2 \phi_2 + \frac{\partial^2 \phi_2}{\partial z^2} - h^{-1} \frac{\partial \phi_2}{\partial z}) + q_o \frac{\partial \phi_2}{\partial x}$$

$$= \quad ik\Delta' \, q_o \frac{F_o}{z} Ae^{ikx} (\sin ly) + ik\delta F_o Ae^{ikx}(\sin ly)$$

$$- ik \, F_o \frac{\partial}{\partial y} (\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z}) Ae^{ikx}(\sin ly)$$

$$- h^{-1} \frac{\partial \Phi_1}{\partial z}) \, Ae^{ikx} \, (\sin \; ly) - ik \frac{\partial \Phi_1}{\partial y} q_o \frac{F_o}{z} Ae^{ikx}$$

$$\times (\sin ly) + * \,. \tag{3.5}$$

The solvability condition for (3.5) is

$$\int_0^1 2(\sin ly) \, \overline{e^{-ikx} \, \phi_2 \, (x,y,0,T)dy} = \Delta' A + \frac{\delta}{q_o{}^2} A$$

$$- A\int_0^\infty dze^{-q_o z} \, (zp_{11} - \frac{\partial u_{11}}{\partial z}) + Au_{11} \, (0,T). \tag{3.6}$$

In (3.6) it appears as if the term $\Delta' A$ will play a role in the amplitude equation. However, shortly we will see that a contribution from the lefthand side of (3.6) will cancel with $\Delta' A$ on the right side of (3.6).

Now the $O(\varepsilon^2)$ inner problem will be considered whose solution when matched to the outer solution will determine $\phi_2(x,y,0,T)$. If (3.4b) is used to evaluate the inhomogeneous terms, the $O(\varepsilon^2)$ inner wave field equations can be written

$$\frac{\partial^2 \hat{\phi}_2}{\partial \zeta^2} = - \beta_o A e^{ikx} (\sin ly) + *, \qquad (3.7a)$$

$$\frac{\partial \hat{\phi}_2}{\partial \zeta^2} - \frac{\partial \hat{\phi}_2}{\partial x} = -ik\Delta' A e^{ikx} (\sin ly) + \frac{r'K^2}{ik} \frac{dA}{dT}$$

$$\times \quad e^{ikx} (\sin ly) - ik\Delta' M e^{ikx} (\sin ly)$$

$$- ik(A + M) \Phi_{1y} (y,O,T) e^{ikx} (\sin ly) + * . \qquad (3.7b)$$

The wave solution to (3.7a) which can be matched to the outer solution is

$$\hat{\phi}_2 = - \frac{\beta_o}{2} \zeta^2 A e^{ikx} (\sin ly) + (- \frac{\beta_o}{2ik} \frac{dA}{dT} \zeta + C_2)$$

$$\times e^{ikx} (\sin ly) + * . \qquad (3.8)$$

When (3.5) is substituted into the lower boundary condition (3.7b), C_2 can be determined. It is

$$C_2 = [\frac{\beta_o}{2k^2} \frac{d^2A}{dT^2} + \frac{r'k^2}{k^2} \frac{dA}{dT} + (A + M) \frac{\partial \Phi_1}{\partial y}(y,O,T)$$

$$+ \Delta' (A + M)] e^{ikx} (\sin ly) + *. \qquad (3.9)$$

Matching $\hat{\phi}_2$ to ϕ_2 yields

$$\phi_2 (x,y,O,T) = C_2 e^{ikx} (\sin ly) + *. \qquad (3.10)$$

where C_2 is given by (3.9). Using (3.10) in the solvability condition (3.6) we obtain the wave amplitude equation

$$\frac{d^2A}{dT^2} + \frac{2r'k^2}{\beta_o}\frac{dA}{dT} - \frac{2k^2\delta}{\beta_o q_o^2} A + \frac{2k^2}{\beta_o} A \int_0^\infty e^{-q_o z} (zp_{11}$$

$$- \frac{\partial u_{11}}{\partial z}) \, dz + \frac{2k^2}{\beta_o} M \, u_{11} \, (0,T) = - \frac{2k^2\Delta'}{\beta_o} M. \tag{3.11}$$

Equation (3.11) differs from its counterpart in section 2 only by the presence of the inhomogeneous term. No other terms proportional to Δ' arise because the advective time scale associated with the weak zonal flow $\varepsilon^2\Delta'$ is $O(\varepsilon^2)$ which is an order slower than the long time scale.

The coupled set of equations (3.11) and (3.3a,b) allow a steady solution of the form

$$A_s = \frac{q_o^2\Delta' M}{\delta} , \tag{3.12a}$$

$$u_{1s} = 0 . \tag{3.12b}$$

The study of the stability of this steady solution is the point of this section. It corresponds to a topographically forced wave. Notice the wave has a resonance at $\delta = 0$. This is the point at which the neutral propagating waves become stationary relative to the topography. It also happens to correspond to the stability threshold for an infinitesmal baroclinic wave in the absence of topography. Thus we expect the instability of the forced wave to depend on whether the flow is superresonant ($\delta > 0$) or subresonant ($\delta < 0$). In any case, in order to avoid the resonance, the following stability analysis must be restricted to values of δ which are O(1).

If the Ekman friction is assumed to be negligible, the evolution equations, when linearized about the topographically forced wave solution (3.12a,b) take the form

$$\frac{d^2A'}{dT^2} - \frac{2k^2\delta}{\beta_o q_o^2} A' + \frac{2k^2}{\beta_o} A_s \int_0^\infty e^{-q_o z} \left(zp'_{11} - \frac{\partial u'_{11}}{\partial z}\right) dz$$

$$+ \frac{2k^2}{\beta_o} M\, u'_{11}\,(0,T) = 0, \tag{3.13a}$$

$$\frac{\partial}{\partial T}\left(\frac{\partial^2 u'_1}{\partial y^2} + \frac{\partial^2 u'_1}{\partial z^2} - h^{-1}\frac{\partial u'_1}{\partial z}\right) = - q_o e^{-\beta o z}$$

$$\times \left(2A_s \frac{dA_r'}{dT}\right) \frac{\partial^2}{\partial y^2} (\sin^2 ly), \tag{3.13b}$$

$$\frac{\partial^2 u'_1}{\partial T \partial z} (y,0,T) = \left(2A_s \frac{dA_r'}{dT} + 2M \frac{dA_r'}{dT}\right) \frac{\partial^2}{\partial y^2} (\sin^2 ly), \tag{3.13c}$$

$$\frac{\partial u'_1}{\partial T} (0,z,T) = \frac{\partial u'_1}{\partial T} (1,z,T) = 0, \tag{3.13d}$$

where A_r' and and A_i' are the real and imaginary parts of the perturbation wave amplitude A' and M is assumed to be real for simplicity.

The small amplitude solution of (3.11) consists of a baroclinically unstable mode plus a topographically forced wave. Thus, the equation for the perturbation wave amplitude (3.13a) now involves the interaction of the perturbation zonal flow with both the forced wave and the topography which modifies the baroclinic instability. This is in contrast to the previous section where the perturbation zonal flow at the ground only interacts with the topography. Also in section 2, the perturbation zonal flow was only forced by the interaction of the wave distubance with the topography. However, in the present case, the perturbation zonal flow is also forced by the interaction of the wave disturbance with the topographically forced wave. Thus in addition to the topographically induced modifications of the wave-mean flow interaction in the last section there is a further modification due to the presence of the forced wave. Thus, the stability problem is altered fundamentally.

The linearized equations for the perturbation zonal flow (3.13b,c,d) can be solved just as the nonlinear zonal flow equations were solved in the last section. When the resultant perturbation zonal flow is substituted into (3.13a) the resulting perturbation wave equation is

$$\frac{d^2A'}{dT^2} - \frac{2k^2}{\beta_o}\frac{\delta}{q_o^2}A' + \frac{4k^2M^2}{\beta_o}\left[\frac{q_o^4\Delta'^2X_o}{\delta^2} + \frac{q_o^2\Delta'}{\delta}(X_1 + X_2) + X_3\right]A_r' \tag{3.14}$$

The growth rate $\sigma = kc$, corresponding to (3.14) is

$$c^2 = \frac{2}{\beta_o}\left\{\frac{\delta}{q_o^2} - 2M^2\left[\frac{q_o^4\Delta'^2X_o}{\delta^2} + \frac{q_o^2\Delta'}{\delta}(X_1 + X_2) + X_3\right]\right\} \tag{3.15}$$

where

$$X_0 = \frac{\beta_o}{2k^2} N_o \tag{3.16a}$$

$$X_1 = \frac{\beta_o}{2k^2} Q_o \tag{3.16b}$$

$$X_2 = \sum_j \frac{\beta_o}{2k^2} Nj \left(\frac{\beta_o q_o - j^2}{q_o \alpha_j}\right) \tag{3.16c}$$

$$X_3 = \sum_j \frac{\beta_o}{2k^2} Qj \left(\frac{\beta_o q_o - j^2}{q_o \alpha_j}\right) \tag{3.16d}$$

where N_o, N_j, Q_o and Q_j are defomed by (2.47a,b,c,d) respectively. The term responsible for the stabilization of the baroclinic instability in section 2 is X_3 in the square brackets of (3.15) which is always greater than zero. The remaining terms in the square brackets are due to the presence of the forced wave. Their effect on the instability depends on their sign and relative magnitudes. In order to evaluate the nonlinear coefficients X_o, X_1, X_2 and X_3 more easily, the Boussinesq-like approximation

$$\beta_o >> h^{-1} \tag{3.17}$$

will be made. Using (3.17) the dispersion relation becomes

$$c^2 = \frac{1}{K} \left\{\frac{\delta}{4K^2} - 2M^2 \left[\frac{16K^4\Delta'^2}{\delta^2} X_o + \frac{8K^2\Delta'}{\delta} X_1 + X_3\right]\right\} , \tag{3.18}$$

where

$$X_o = \frac{2l^4}{\pi^5}\left\{\sum_{m=1}^{\infty}\frac{(m + \frac{1}{2}(\frac{K}{\pi} - 1))}{(m + \frac{K}{\pi} - \frac{1}{2})^2((m - \frac{1}{2})^2 - \frac{l^2}{\pi^2})^2} - \frac{\pi^5}{16Kl^2}\right\}, \tag{3.19a}$$

$$X_1 = X_2 = \frac{2l^4}{\pi^5}\sum_{m=1}^{\infty}\frac{1}{(m + \frac{K}{\pi} - \frac{1}{2})\,((m - \frac{1}{2})^2 - \frac{l^2}{\pi^2})^2}, \tag{3.19b}$$

$$X_3 = \frac{2l^4}{\pi^5}\sum_{m=1}^{\infty}\frac{1}{(m - \frac{1}{2})\,((m - \frac{1}{2})^2 - \frac{l^2}{\pi^2})^2} \tag{3.19c}$$

From (3.19b,c) it is easy to see X_1, X_2 and X_3 are greater than zero. Numerical calcuations show that X_o is greater than zero except for extremely large scale waves. The term proportional to X_3 is always stabilizing and the terms proportional to X_o is stabilizing for all but the largest scale waves. Thus the relative signs and magnitudes of Δ' and δ determine whether or not the term proportional to X_1 in (3.18) is stabilizing or destabilizing. The dispersion relation and therefore the instability depends on whether the flow is eastward or westward. When the flow is

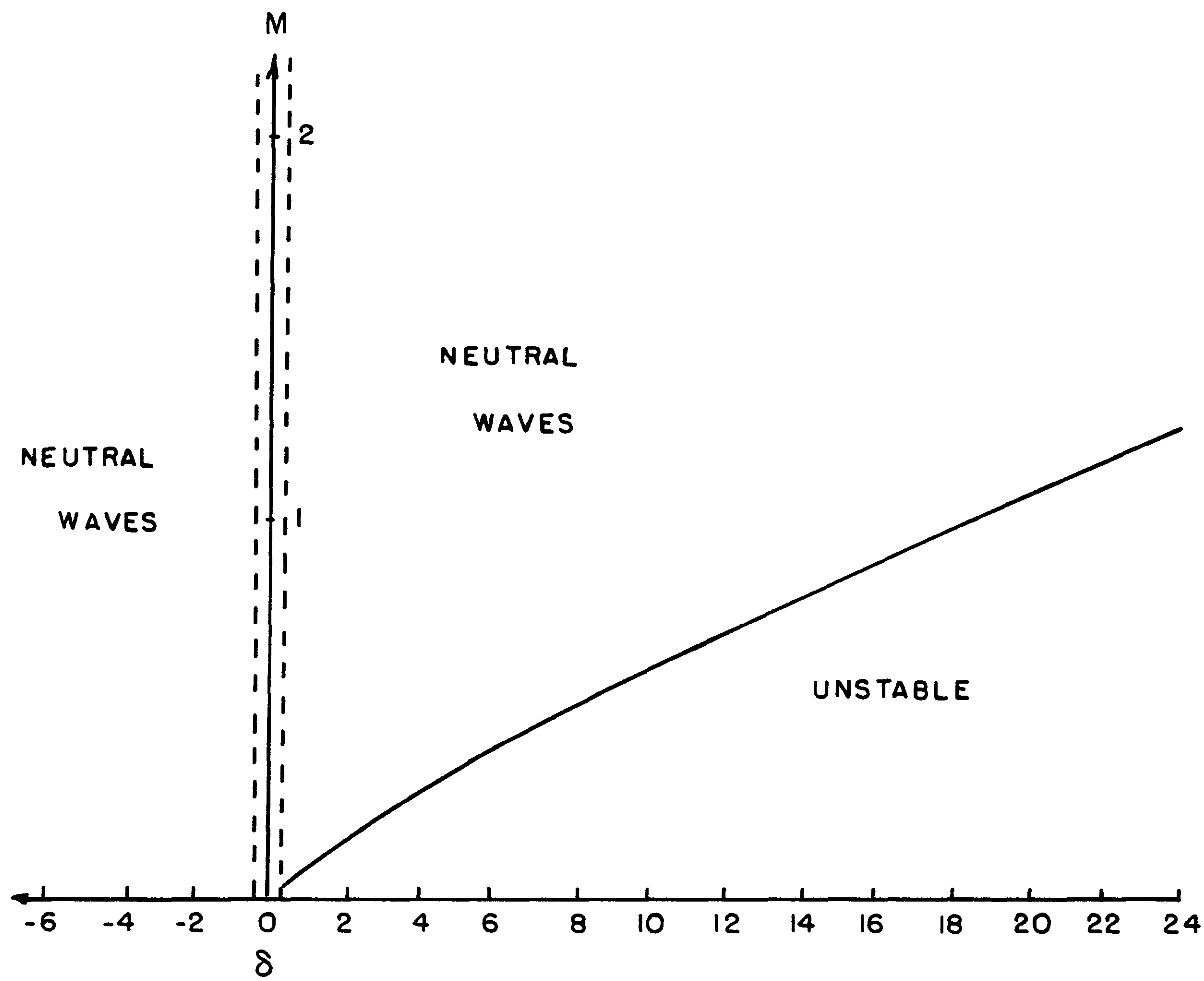

Fig. (4.3.1a) Stability regimes in M, δ space with k=l=1.0 and Δ'=1.0.

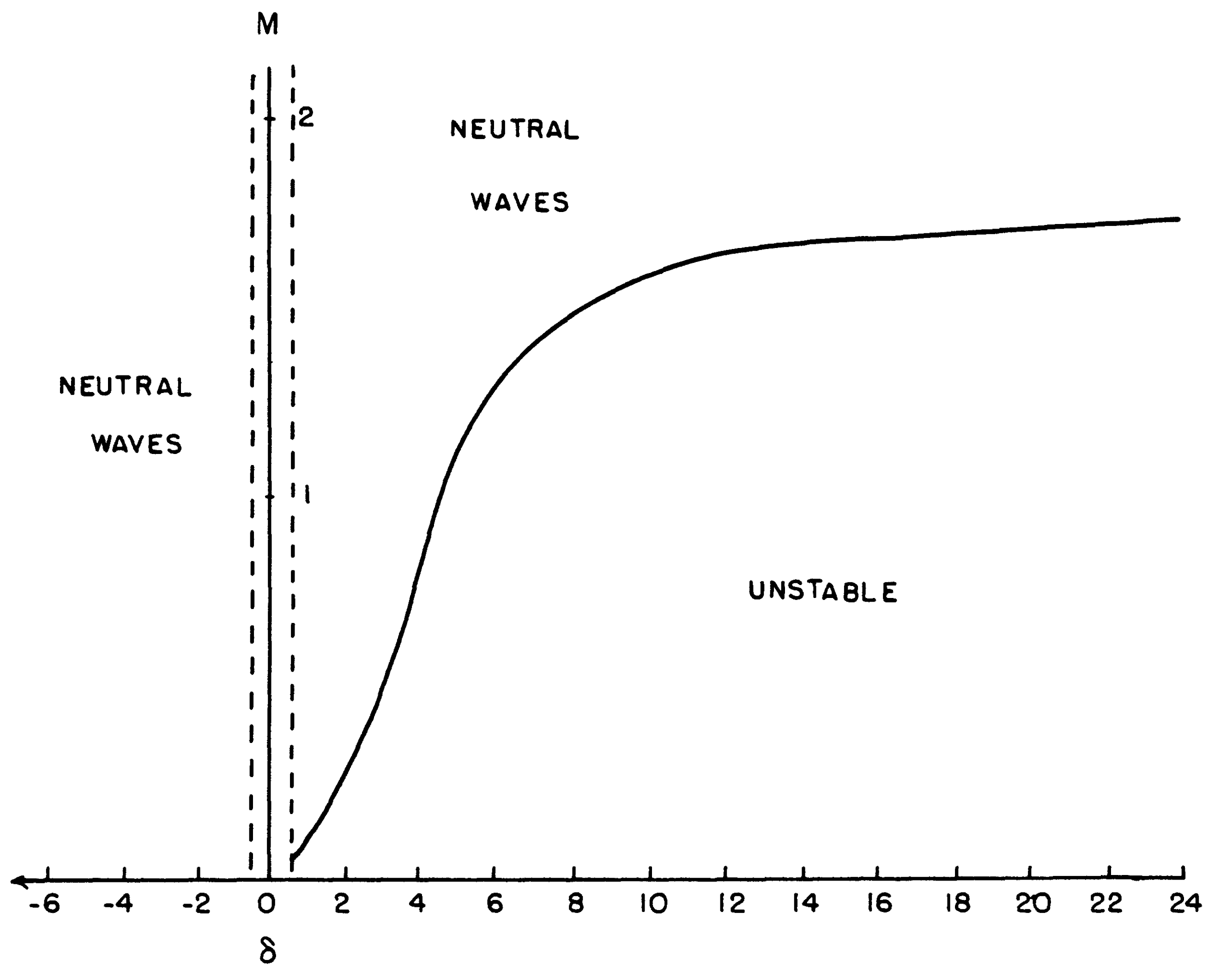

Fig.(4.3.1b) Same as (a) but Δ'=-1.0.

baroclinically supercritical (δ>0) the term proportional to X_1 is stabilizing for eastward flow (Δ'>0) and destabilizing for westward flow (Δ'<0). Just the opposite is true for baroclinically subcritical flow (δ<0). However, it seems clear that overall, we should expect an increase in topography to be stabilizing. The numerical computations displayed in Figs. (4..3.1a,b) and (4.3.2a,b) bear this out. In Figs. (4.3.1a,b) we have plotted the stability regimes in M,δ space for Δ' = 1.0 and Δ' = -1.0 respectively. The eastward flow case shows stronger stabilization than in the westward flow case. The eastward flow case is also more stable than the Δ' = 0 case of Fig. (4.2.1). This is due to the interaction of the forced wave with the perturbation zonal flow. The westward flow case in Fig. (4.3.1b) is less stabilized than the Δ' = 0 case of Fig. (4.2.1). This is a result of the term in (3.18) proportional to X_1 which can be destabilizing for westward flow (Δ'<0). However, on the whole topography is stabilizing except for unlikely parameter regimes.

We will close this section by considering the wave-field energetics. In general the total wave energy can be modified by the conversion of mean flow kinetic energy to perturbation kinetic energy by Reynold's stress, conversion of mean flow kinetic energy to perturbation energy by the form drag and conversion of mean available potential energy to perturbation available potential energy by the eddy heat flux. In the present case, the energy budget can be written

$$\varepsilon^3 \frac{\partial}{\partial T} \int_0^\infty e^{-z/h} \frac{1}{2} [\overline{\phi_{ox}{}^2} + \overline{\phi_{oy}{}^2} + \overline{\phi_{oz}{}^2}]dz$$

$$= - \varepsilon^5 \int_0^\infty e^{-z/h} [\overline{\phi_{ox}\ \phi_{1y}} + \overline{\phi_{1x}\ \phi_{oy}}]\ \Phi_{1yy}\ dz$$

$$+ \varepsilon^3 \int_0^\infty e^{-z/h} [\overline{\phi_{ox}\ \phi_{1z}} + \overline{\phi_{1x}\ \phi_{oz}}]\ \frac{\partial (z)}{\partial z}\ dz$$

$$+ \varepsilon^4 (\Delta' - \Phi_{1y})[\overline{\phi_o b_x}]_{z=0}, \qquad (3.20)$$

where [ ] is a meridional avrage and $(\overline{\ })$ is a zonal average. Eq. (3.20) shows the total energy is modified primarily by the conversion of available zonal potential energy. The other contributions are one or two orders of magnitude smaller. The energetics of the previous section are the same as (3.20) but $\Delta' = 0$. Thus it is clear that the $O(\varepsilon^2)$ zonal flow at the ground has altered the instability problem but leaves the leading order energy budget unaltered.

In the next section the zonal flow at the ground will be $O(\varepsilon)$ but the topography will be $O(\varepsilon^2)$ instead of $O(\varepsilon)$. The leading order energetics will remain unaltered from (3.20) but the term

$$- \Phi_{1y} [\overline{\phi_o b_x}]_{z=0}$$

will fall to $O(\varepsilon^5)$. However, the instability will be modified further.

4. Influence of topography on the baroclinic instability of the Charney mode and its topographic instability.

The topic of the present section is, as was the topic of the last section, the linear instability of a small but finite-amplitude topographically forced Charney mode near baroclinic neutrality. In the last section the uniform zonal flow which forced the Charney mode is so weak that in the absence of topography it has no effect on the problem to the order of interest. In this section, the uniform zonal flow is allowed to be stronger such that the stationary weak baroclinic instability of the last section will become a slowly propagating weak baroclinic instability in this section.

Nevertheless, the present analysis is very similar to that of the last two sections and section 3 of Pedlosky (1979). This will allow the discussion of its development to be considerably more brief. Again the analysis pivots about the stationary neutral Charney mode which, in the absence of forcing, will grow or decay on the long time scale

$$T = \varepsilon t, \tag{4.1}$$

if

$$\beta = \beta_o - \varepsilon^2 \delta \tag{4.2}$$

and

$$r = \varepsilon^2 r' \tag{4.3}$$

As in section 3 this weak instability is then balanced against the forcing and the usually stabilizing nonlinearity, but now the weak advection of the Charney mode by the uniform zonal flow also comes into play. The wave is forced by a stronger weak uniform zonal current flowing over the topography than in the last section. We choose the magnitude of this current such that the advection time scale associated with it matches the e-folding time of the disturbance which is also the time scale of wave-zonal flow interaction. Therefore we write $\overline{U}$ as

$$\overline{U} = \varepsilon\Delta' \tag{4.4}$$

Then the appropriate choice for the topography is

$$\eta = \varepsilon^2 b = \varepsilon^2 \, Me^{ikx} \, (\sin \; ly) + * \, , \tag{4.5}$$

With (4.5) the forcing will be balanced against the growth, friction and nonlinear terms. This approach was also used by Plumb (1979) in a two-layer model Note that the topography is an order of magnitude smaller here than in the last section. As a result the topographically induced portion of the wave mean flow interaction of the ground in the last two sections will be an order weaker. Thus it will be negligible to the order of interest here.

Substituting (4.1-5) into (3.2.3) and (3.2.7) the outer, inner and lower boundary equations can be written

$$z \frac{\partial \Pi}{\partial x} + q_0 \frac{\partial \phi}{\partial x} = - \varepsilon\left(\frac{\partial}{\partial T} + \Delta' \frac{\partial}{\partial x}\right)\Pi + \varepsilon^2\delta \frac{\partial \phi}{\partial x} - \varepsilon J(\phi, \Pi), \tag{4.6a}$$

$$\left(\frac{\partial}{\partial T} + \Delta' \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \hat{\phi}}{\partial \zeta^2} - \varepsilon\, h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta} + \varepsilon^2 \nabla^2 \hat{\phi}\right) + \varepsilon\, (q_o$$

$$- \varepsilon^2 \delta) \frac{\partial \hat{\phi}}{\partial x} = - J\left(\hat{\phi}, \frac{\partial^2 \hat{\phi}}{\partial \zeta^2} - \varepsilon h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta} + \varepsilon^2 \nabla^2 \hat{\phi}\right), \qquad (4.6b)$$

and

$$\frac{\partial^2 \hat{\phi}}{\partial T \partial \zeta} + \Delta' \frac{\partial^2 \hat{\phi}}{\partial x \partial \zeta} - \frac{\partial \hat{\phi}}{\partial x} + J\left(\hat{\phi}, \frac{\partial \hat{\phi}}{\partial \zeta}\right) = \varepsilon r' \nabla^2 \hat{\phi} - \varepsilon^2 \Delta' b_x$$

$$- \varepsilon^2 J(\hat{\phi}, b), \qquad (4.6c)$$

respectively. In (4.6a,6b,6c), $\hat{\phi}$ and δ are the inner stream function and inner vertical coordinate, respectively as in the previous sections. Expanding ϕ and $\hat{\phi}$ as before, i.e.,

$$\phi = \phi_o + \varepsilon \phi_1 + \varepsilon^2\, \phi_2 + \ldots, \qquad (4.7a)$$

$$\phi = \quad + \varepsilon \hat{\phi}_1 + \varepsilon^2\, \hat{\phi}_2 + \ldots, \qquad (4.7b)$$

and substituting into (4.6a,6b,6c) results in a sequence of equations for both the inner and outer problems. The outer problem as defined by (4.6a) is identical to that solved in section 2 except for the presence of the $-\varepsilon\Delta'\Pi_x$ term. Thus the O(1) outer equation is the same as (2.12) whose solution is the stationary neutral Charney mode, i.e.

$$\phi_o = Aze^{-\beta_o z/2}\, e^{ikx}\, (\sin ly) + * , \qquad (4.8)$$

At $O(\varepsilon)$ the effect of the additional term is felt. Using the $O(1)$ solution (4.8) the $O(\varepsilon)$ outer problem can be written as

$$z \frac{\partial \Pi_1}{\partial x} + q_o \frac{\partial \phi_1}{\partial x} = q_o \left(\frac{dA}{dT} + ik\Delta' A\right) e^{-\beta_o z/2} e^{ikx}$$

$$\times \ (\sin ly) + * \ . \tag{4.9}$$

It has the solution

$$\phi_1 = \left(\frac{1}{ik} \frac{dA}{dT} + \Delta' A\right) e^{-\beta_o z/2} e^{ikx} (\sin \ ly) + *$$

$$+ \Phi_1 (y,z,T). \tag{4.10}$$

Again only the x-independent part of (4.10) contributes to the potential vorticity

$$\Pi_1 = \frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z} \ . \tag{4.11}$$

Note that the contribution to the solution (4.10) from the additional term $-\varepsilon\Delta' \Pi_x$ is not phase shifted relative to the $O(1)$ neutral Charney mode solution. As a result its interaction with the $O(1)$ solution in the $O(\varepsilon^2)$ outer problem does not produce additional wave forcing of the zonal flow. Because of this fact and because the $O(\varepsilon)$ potential vorticity remains x-independent, the result of the $O(\varepsilon^2)$ outer problem is the same as in section 2 where (2.20) describes the evolution of the zonal flow in terms of the transient wave forcing and (2.22) is the solvability condition on the wave-like inhomogenieties. The solvabiliy condition (2.22) is

$$\int_0^1 dy2(\sin ly)\,\overline{e^{-ikx}\,\phi_2(x,y,0,T)} = \frac{\delta}{q_o^2}$$

$$- A\int_0^\infty dz e^{-q_o z}\,(zp_{11} - \frac{\partial u_{11}}{\partial z}) + Au_{11}(0,T), \tag{4.12}$$

where p_{11} and u_{11} are defined by (2.23a,b), respectively. Again $\phi_2(x,y,0,T)$ is undetermined by the outer problem which forces us to turn to the consideration of the inner problem for its determination as well as for the purpose of obtaining a lower boundary condition on the zonal flow.

Based on the experience of section 2 the solution to the $O(\varepsilon)$ inner problem can easily be obtained by taking the limit of the $O(1)$ and $O(\varepsilon)$ outer solutions (4.8) and (4.10) near the ground. Thus the $O(\varepsilon)$ inner solution is found to be

$$\hat{\phi}_1 = (A\zeta + \frac{1}{ik}\frac{dA}{dT} + \Delta' A)\, e^{ikx}(\sin ly) + * + \Phi_1(y,0,T) \tag{4.13}$$

Since the $O(\varepsilon^2)$ inner problem differsn more substantially from that of section 2, it will be given more careful consideration. If we use (4.13) as well as (4.5) to evaluate the inhomogeneous terms, the $O(\varepsilon^2)$ inner equations can be written as

$$(\frac{\partial}{\partial T} + \Delta' \frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial x})(\frac{\partial^2 \hat{\phi}_2}{\partial \zeta^2} + \beta_o A e^{ikx}(\sin ly) + *) = 0, \tag{4.14a}$$

$$\frac{\partial^2\hat{\phi}_2}{\partial T\partial\zeta} + \Delta' \frac{\partial^2\hat{\phi}_2}{\partial x\partial\zeta} - \frac{\partial\hat{\phi}_2}{\partial x} = r'K^2\left(\frac{1}{ik}\frac{dA}{dT} + \Delta'A\right) e^{ikx}(\sin ly)$$

$$+ * - ikAe^{ikx}\,\Phi_{1y}(y,0,T)\,(\sin ly) + * - ik\Delta'Me^{ikx}$$

$$\times (\sin ly) + * - r'\frac{\partial^2\Phi_1}{\partial y^2}(y,0,T) - \frac{d|A|^2}{dT}\frac{\partial}{\partial y}(\sin^2 ly) \tag{4.14b}$$

As opposed to the $O(\varepsilon^2)$ lower boundary condition in section 2, (4.14b) contains additional terms proportional to $\Delta'A$ and the forcing $\Delta'M$ but does not contain a form drag term and an interaction term between the topography and the zonal flow correction at the ground which formed the topographically induced portion of the wave mean flow interaction in section 2. As before, the solution (4.14a,b) is of the form

$$\hat{\phi}_2 = -\frac{\beta_0}{2}\zeta^2 Ae^{ikx}(\sin ly) + * + (C_2 + D_2\zeta)\,e^{ikx}$$

$$\times (\sin ly) + * + \zeta\frac{\partial\Phi_1}{\partial z}(y,0,T) \tag{4.15}$$

Matching this solution to the otuer solution yields

$$D_2 = -\frac{\beta_0}{2}\left(\frac{1}{ik}\frac{dA}{dT} + \Delta'A\right) \tag{4.16a}$$

$$\phi_2(x,y,0,T) = C_2\,e^{ikx}(\sin ly) + * \tag{4.16b}$$

Substituting the $O(\varepsilon^2)$ inner solution (4.15) along with

(4.16a) into the lower boundary condition (4.14b) then yields

$$C_2 = \frac{\beta_o}{2k^2}\frac{d^2A}{dT^2} + \frac{i\beta_o\Delta'}{k}\frac{dA}{dT} - \frac{\beta_o\Delta'^2A}{2} + \frac{r'K^2}{k^2}\frac{dA}{dT}$$

$$+ \frac{ir'K^2}{k}\Delta'A + \Delta'M - A\frac{\partial\Phi_1}{\partial y}(y,0,T), \tag{4.17a}$$

and

$$\frac{\partial^2\Phi_1}{\partial T\partial z}(y,0,T) + r'\frac{\partial^2\Phi_1}{\partial y^2}(y,0,T) = -\frac{d|A|^2}{dT}\frac{\partial}{\partial y}(\sin^2 ly) \tag{4.17b}$$

The later result is the required lower boundary condition on the zonal flow. Use of (4.16b) along with (4.17a) in the solvability condition (4.12) results in the wave amplitude equation which along with (2.20) and (4.17b) form the closed set of equations governing this damped and forced wave mean flow interaction. This set of equations is

$$\frac{d^2A}{dT^2} + 2(ik\Delta' + \frac{r'K^2}{\beta_o})\frac{dA}{dT} + (2ir'\frac{kK^2}{\beta_o}\Delta' - k^2\Delta'^2$$

$$- \frac{2\delta k^2}{\beta_o q_o^2})A + \frac{2k^2}{\beta_o}A\int_0^\infty e^{-q_o z}(zp_{11} - \frac{\partial u_{11}}{\partial z}) = -\frac{2k^2}{\beta_o}\Delta'M, \tag{4.18a}$$

$$\frac{\partial}{\partial T}(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} - h^{-1}\frac{\partial u_1}{\partial z}) = -q_o e^{-\beta_o z}\frac{d|A|^2}{dT}\frac{\partial^2}{\partial y^2}(\sin^2 ly), \tag{4.18b}$$

$$\frac{\partial^2 u_1}{\partial T \partial z}(y,0,T) + r' \frac{\partial^2 u_1}{\partial y^2}(y,0,T) = \frac{d|A|^2}{dT^2} \frac{\partial^2}{\partial y^2}(\sin^2 ly). \tag{4.18c}$$

The zonal velocity is also subject to the sidewall boundary condition

$$\frac{\partial u_1}{\partial T} = 0, \tag{4.19}$$

on $y = 0,1$.

The equations (4.18a,b,c) and (4.19) are identical to the equations (3.47), (4.3) and (4.4a,b) obtained by Pedlosky (1979) except for the presence of terms proportional to the weak zonal velocity Δ' and an inhomogeneous term proportional to the topogrpahic amplitude M. We now consider the linear instability of the topographically forced wave in the absence of Ekman friction whose steady amplitude is

$$A_s = \frac{2\Delta' M}{\beta_o(\Delta'^2 + \dfrac{2\delta}{\beta_o q_o^2})} . \tag{4.20}$$

By (4.20) is the topographic amplitude is assumed to be real, then A_s also is. Note that for baroclinically subcritical ($\delta<0$) flow A_s becomes infinite when

$$\Delta'^2 = - \frac{2\delta}{\beta_o q_o^2} .$$

This corresponds to a resonance which occurs when the subcritical propagating neutral mode becomes stationary relative to the topography. In contrast to section 3, since the advective time scale of the zonal flow $\varepsilon\Delta'$ is the same as that of the wave-mean flow interaction, the resonance point no longer corresponds to the stability transition for an infinitesimal baroclinic wave. Nevertheless, based on the experience of the two previous chapters we can expect the resonance to play an important role in the instability. Our inviscid stability analysis must avoid this point. However, the stability analysis in section 5 will include friction which will allow this restriction to be removed.

Linearizing the evolution equations (4.18a,b,c) about the steady wave solution (4.20) yields

$$\frac{d^2A'}{dT^2} + 2ik\Delta' \frac{dA'}{dT} - k^2 \left(\Delta'^2 + \frac{2\delta}{\beta_o q_o^2}\right) A'$$

$$+ \frac{2k^2}{\beta_o} A_s \int_0^\infty e^{-q_o z} \left(zp'_{11} - \frac{\partial u'_{11}}{\partial z}\right) dz = 0, \qquad (4.21a)$$

$$\frac{\partial}{\partial T} \left(\frac{\partial^2 u'_1}{\partial y^2} + \frac{\partial^2 u'_1}{\partial z^2} - h^{-1} \frac{\partial u'_1}{\partial z}\right) = - q_o e^{-\beta_o z} 2A_s$$

$$\times \frac{dA_r'}{dT} \frac{\partial^2}{\partial y^2}(\sin^2 ly), \qquad (4.21b)$$

$$\frac{\partial^2 u'_1}{\partial T \partial z} (y,0,T) = 2A_s \frac{dA_r'}{dT} \frac{\partial^2}{\partial y^2} (\sin^2 ly), \qquad (4.21c)$$

$$\frac{\partial u'_1}{\partial T} = 0 \ , \tag{4.21d}$$

on $y = 0,1$. The definitions of p'_{11} and u'_{11} are

$$p'_{11} = - \int_0^1 2(\sin^2 ly) \left(\frac{\partial^2 u'_1}{\partial y^2} + \frac{\partial^2 u'_1}{\partial z^2} - h^{-1} \frac{\partial u'_1}{\partial z}\right) dy, \tag{4.22a}$$

$$u'_{11} = - \int_0^1 2(\sin^2 ly)\ u'_1\ dy \ . \tag{4.22b}$$

The last term in (4.21a) represents the interaction of the forced wave with the perturbation zonal flow. According to (4.21b,c) the disturbance zonal flow is forced by the interaction of the topographically forced wave and the wave disturbance. Thus there is a feedback loop between the forced wave and the disturbance wave field and zonal flow which is key in the instability of the forced Charney mode.

Again the zonal velocity can be obtained by solving (4.21b,c,d) as in previous sections. The result can then be used in (4.22a,b) to write p'_{11} and u'_{11} in terms of the wave amplitude alone. When p'_{11} and u'_{11} are substituted into (4.21a) the folowing linear equation for the perturbation wave amplitude is obtained:

$$\frac{d^2 A'}{dT^2} + 2ik\Delta' \frac{dA'}{dT} - k^2 \left(\Delta'^2 + \frac{2\delta}{\beta_o q_o^2}\right) A' + 2N_o A_s^2 A_r' = 0, \tag{4.23}$$

where A_r' is the real part of the perturbation amplitude and

$$N_o = k^2 N' = \frac{2k^2}{\beta_o} \{ \sum_j \frac{16 q_o l^4 \; (1-(-1)^J)^2}{(\beta_o q_o - j^2)(j^2 - 4l^2)^2} \; (\frac{\beta_o}{\beta_o + q_o}$$

$$- \frac{j^2}{q_o(\alpha_j + q_o)}) - \frac{q_o l^2}{(q_o + \beta_o)^2} \}$$

If (4.23) is split into the real and imaginary parts the result is

$$\frac{d^2 A_r'}{dT^2} - 2\, k\Delta' \frac{dA_i'}{dT} - k^2 \, (\Delta'^2 + \frac{2\delta}{\beta_o q_o^2}) A_r' + 2 N_o A_s^2 A_r' = 0, \tag{4.25a}$$

$$\frac{d^2 A_r'}{dT^2} + 2\, k\Delta' \frac{dA_r'}{dT} - k^2 \, (\Delta'^2 + \frac{2\delta}{\beta_o q_o^2}) A_i' = 0, \tag{4.25b}$$

where A_i' is the imaginary part of A'. The growth rate σ associated with (4.25a,b) is

$$\frac{\sigma^2}{k^2} = - \, (N' A_s^2 + \Delta'^2 - \frac{2\delta}{\beta_o q_o^2}) \pm \{ (N' A_s^2 + \Delta'^2 - \frac{2\delta}{\beta_o q_o^2})^2$$

$$+ \, (\Delta'^2 + \frac{2\delta}{\beta_o q_o^2}) [2N' A_s^2 - (\Delta'^2 + \frac{2\delta}{\beta_o q_o^2})] \}^{1/2} \tag{4.26}$$

The first thing to note about (4.26) is that where Δ' appears it does as Δ'^2 so remarks for westerly flow at the ground ($\Delta' > 0$) also apply to easterly flow at the ground. In order to simplify the calculations the Boussinesq-like approximation $\beta_o \gg h^{-1}$ will be made. Under this approximation

$$q_o = \beta_o \tag{4.27a}$$

$$\alpha_j = j = J\pi \tag{4.27b}$$

$$\beta_o = 2a_o \pi \tag{4.27c}$$

where

$$a_o = \frac{K}{\pi} \tag{4.27d}$$

As a consequence of (4.27) the nonlinear coefficient (4.24) reduces to

$$N_o = \frac{2k^2L^4}{\pi^2 a_o} \{\sum_{m=1}^{\infty} \frac{m + \frac{1}{2}(a_o - 1)}{(m + a_o - \frac{1}{2})^2 ((m - \frac{1}{2})^2 - L^2)^2}$$

$$- \frac{\pi^2}{16 a_o L^2}\}, \tag{4.28}$$

and the growth rate (2.7.26) reduces to

$$\frac{\sigma^2}{k^2} = -(N'A_s{}^2 + \Delta'^2 - \frac{\delta}{4K^3}) \pm \{(N'A_s{}^2 + \Delta'^2 - \frac{\delta}{4K^3})^2$$

$$+ (\Delta'^2 + \frac{\delta}{4K^3})(2N'A_s{}^2 - \Delta'^2 - \frac{\delta}{4K^3})\}^{1/2} , \tag{4.29}$$

where

$$m = \frac{J}{2} + 1, \quad L = \frac{1}{\pi} \quad \text{and} \quad N' = \frac{N_o}{k^2} .$$

$N_o > 0$ for all but the smallest meridional wavenumber. This implies that the nonlinearity would stabilize a baroclinically growing wave as in Pedlosky (1979). However, the contribution to the growth rate (4.29) from the terms proportional to N' is not necessarily stabilizing. To show this the parameter regime of (4.29) will be considered corresponding to baroclinic stability of an infinitesimal disturbance, i.e.

$$\delta < 0. \tag{4.30}$$

Plumb (1979) discussed this parameter regime in the context of his two-layer model. If μ is defined such that

$$\mu^2 = -\frac{\delta}{4K^3}, \tag{4.31}$$

and the steady wave amplitude in (4.29) is rewritten as

$$A_s^2 = \frac{\Delta'^2 M^2}{K^2(\Delta'^2 - \mu^2)^2}, \tag{4.32}$$

the growth rate takes the form

$$\frac{\sigma^2}{k^2} = -\left[\Delta'^2 + \mu^2 + \frac{N'\Delta'^2 M^2}{K^2(\Delta'^2 - \mu^2)^2}\right] \pm \left\{\left[\Delta'^2 + \mu^2 + \frac{N'\Delta'^2 M^2}{K^2(\Delta'^2 - \mu^2)^2}\right] + (\Delta'^2 + \mu^2)\left[-(\Delta'^2 - \mu^2) + \frac{2N'\Delta'^2 M^2}{k^2(\Delta'^2 - \mu^2)^2}\right]\right\}^{1/2}. \tag{4.33}$$

If the operations within the curly brackets in (4.33) are carried out it is easy to see that σ^2/k^2 must always be real. When $\sigma^2/k^2<0$ the modes are propagating, but when $\sigma^2/k^2>0$ one mode becomes unstable. If the nonlinear effects are assumed to be zero for the moment, (4.33) reduces to

$$\frac{\sigma^2}{k^2} = -(\Delta' \mp \mu)^2 \tag{4.34}$$

whose roots correspond to a slowly propagating mode (+ sign) and to a faster propagating mode (-sign). Now, by increasing the forcing a small amount, we can determine which mode's phase speed is decreased. This is the mode which will become unstable. For small forcing σ^2/k^2 can be approximated by

$$\frac{\sigma^2}{k^2} = -(\Delta' \mp \mu^2) + \frac{N'\Delta'^2M^2}{K^2(\Delta'^2 - \mu^2)^2}\left(\frac{\mp\Delta' - \mu}{\mu}\right) \tag{4.35}$$

When the zonal flow is subresonant, i.e.

$$\Delta'^2 - \mu^2 < 0 \tag{4.36}$$

the phase speeds of both the modes increase due to the nonlinear effect. Thus, we expect both of them to remain stable. Indeed this is the case. From (4.33) it is easy to see that there is no instability possible for subresonant flow. However, if the flow is superresonant, i.e.

$$\Delta'^2 - \mu^2 > 0 \tag{4.37}$$

(4.35) shows that the nonlinearity decreases the phase speed of the slower mode while increasing the speed of the faster mode. Thus the slower mode is expected to be unstable. This decrease in phase speed continues as the forcing increases until

$$2N'\Delta'^2M^2 = K^2\ (\Delta'^2 - \mu^2)^3 \tag{4.38}$$

at which point $\sigma^2/k^2 = 0$ for the slower mode. From this point on the disturbance is in phase with the forced wave which allows a coherent interaction between the perturbation, forced wave and mean flow. Therefore, as the forcing increases further such that

$$2N'\Delta'^2M^2 > K^2\ (\Delta'^2 - \mu^2)^3 \tag{4.39}$$

instability occurs. Thus the nonlinear effect is destabilizing here.

Recall that as

$$\mu^2 = -\frac{\delta}{4K^3}$$

increases the flow becomes more baroclinically subcritical. As μ is increased toward Δ', a resonance is approached for which the slowest baroclinically subcritical neutral mode becomes stationary relative to the topography. Thus, since the neutral modes are slower as resonance is approached, less topographic height should be necessary to excite the stationary instability. As is shown by (4.39) and Figure (4.41), less topography is required to make the flow unstable as resonance is approached. This behavior is similar to that of the topographic instability discussed in Chapter 3 which is also more easily excited near resonance for superresonant flow only. This stationary instability, then, might be thought of as the topographic instability of the Charney mode as opposed to the topographic instability of a normal mode of the Charney model.

We now consider the parameter regime of (4.29) corresponding to the baroclinic instability of an infinitesimal disturbance, i.e.

$$\delta > 0. \tag{4.40}$$

In this case μ is defined as

$$\mu = \frac{\delta}{4K^3} \tag{4.41}$$

and the growth rate is written as

$$\frac{\sigma^2}{k^2} = -\left[\Delta'^2 - \mu^2 + \frac{N'\Delta'^2M^2}{k^2(\Delta'^2 + \mu^2)^2}\right] \pm \left\{\left[\Delta'^2 - \mu^2 + \frac{N'\Delta'^2M^2}{K^2(\Delta'^2 + \mu^2)^2}\right]^2 + (\Delta'^2 + \mu^2)\left[-(\Delta'^2 + \mu^2) + \frac{2N'\Delta'^2M^2}{K^2(\Delta'^2 + \mu^2)^2}\right]\right\}^{1/2} . \tag{4.42}$$

Here, when the topographic forcing is zero, the instability corresponds to an unstable baroclinic wave propagating as the speed of the zonal velocity, i.e. for the unstable modes

$$\frac{\sigma}{K} = \mu \pm i\Delta. \tag{4.43}$$

As the forcing M is increased by a small amount, the growth rate of the unstable modes is decreased. This is easily seen by an examination of te approximation to (4.42) for small M, i.e.

$$\frac{\sigma}{k} = \mu \pm i\Delta' - \frac{N'\Delta'^2M^2}{2\mu K^2(\Delta'^2 + \mu^2)} \tag{4.44}$$

The phase speed is also modified but this effect is of higher order. The numerical calculations shown in Figure (4.41) demonstrate that this tendency continues until the quantity in curly brackets in (4.42) is equal to zero, i.e.

$$\frac{N'\Delta'^2M^2}{K^2(\Delta'^2 + \mu^2)^2} + 4\Delta'^2 \left[\frac{N'\Delta'^2M^2}{K^2(\Delta'^2 + \mu^2)^2} - \mu^2 \right] = 0 \tag{4.45}$$

at which point the growth rate vanishes leaving neutral propagating waves of equal phase speed. As M is increased further, this quantity continues to increase. The result is to cause the phase speeds of the two modes to diverge. The mode whose phase speed is decreased with increasing M becomes stationary when M is increased enough such that

$$\frac{2N'\Delta'^2M^2}{K^2(\Delta'^2 + \mu^2)^2} = \Delta'^2 + \mu^2, \tag{4.46}$$

and becomes unstable when

$$\frac{2N'\Delta'^2M^2}{K^2(\Delta'^2 + \mu^2)^2} > \Delta'^2 + \mu^2 . \tag{4.47}$$

Therefore, the role of topography depends on its size. Smaller topography tends to stabilize the propagating baroclinic wave which grows at the expense of the mean vertical shear. However, as the topography becomes larger, the presence of the topographically forced wave modifies the nonlinear wave-mean flow interaction enough to give rise to an unstable disturbance. This instability, which still draws its energy from the available potential energy of the mean flow, occurs because the neutral propagating wave disturbance becomes stationary relative to the forced wave as M is increased and therefore coherent with the forced wave. This is the same instability mechanism as in the case for which $\delta<0$. In fact, in Figure (4.41) note how the transition line from neutral waves to the stationary instability continues smoothly from negative to positive super-criticalities. Thus, the topographic instability of the Charney mode occurs for both subcritical and supercritical baroclinic flow.

Also of interest is the dependence of the instability on wavenumber space, all else being fixed. In Figures (4.4.2a,b,c), for three different values of the shear supercriticality, the regions of stationary instability, propagating instability and neutral propagating waves have been plotted vs. both k and l. The most striking feature of these figures is the relatively weak dependence of the instability on k and the very strong dependence on l. For negative shear supercriticality ($\delta<0$) the only instability

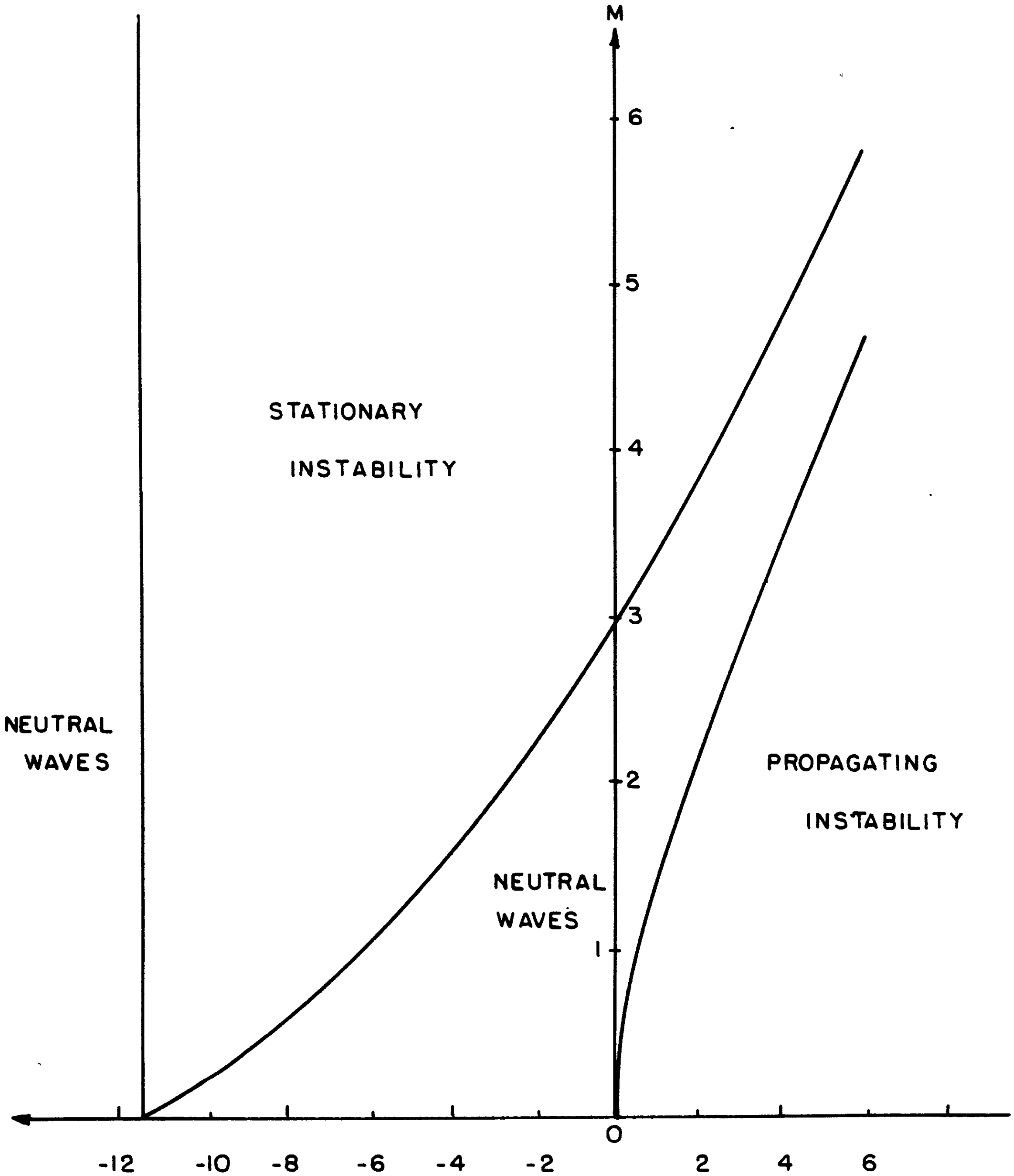

Fig. (4.4.1) The regions of the topographic amplitude $\mathcal{M}$, and shear supercriticality δ , space for which there are neutral waves(N), stationary(S) or propagating(P) instabilities. The calculation was done with $\Delta'=1$, k=1, and l=1.

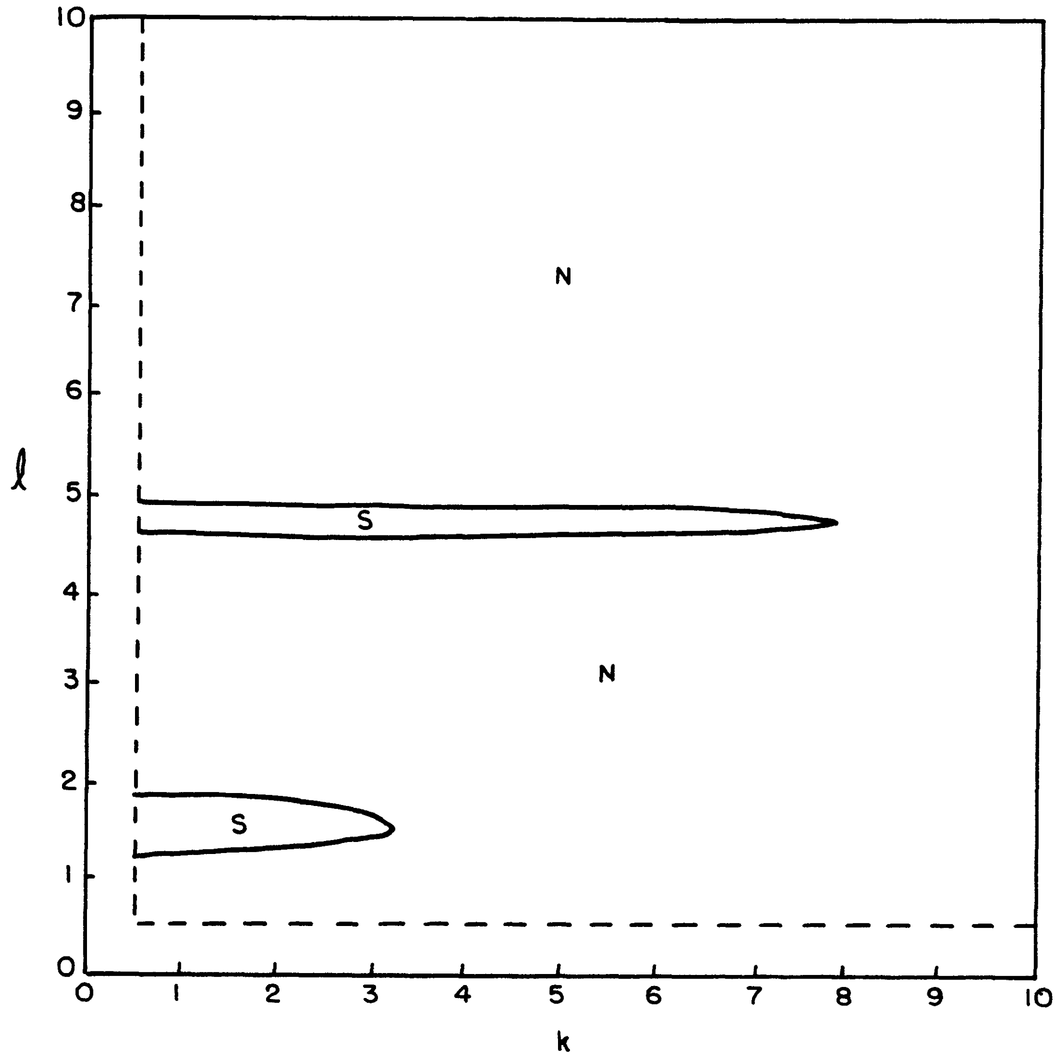

Fig. (4.4.2a) For negative shear supercriticality the regions of zonal and meridional wavenumber space for which there are neutral waves, stationary or propagating instabilities. In the calculation $\Delta'=1$, $\mathcal{M}=1$ and $\delta=-1$.

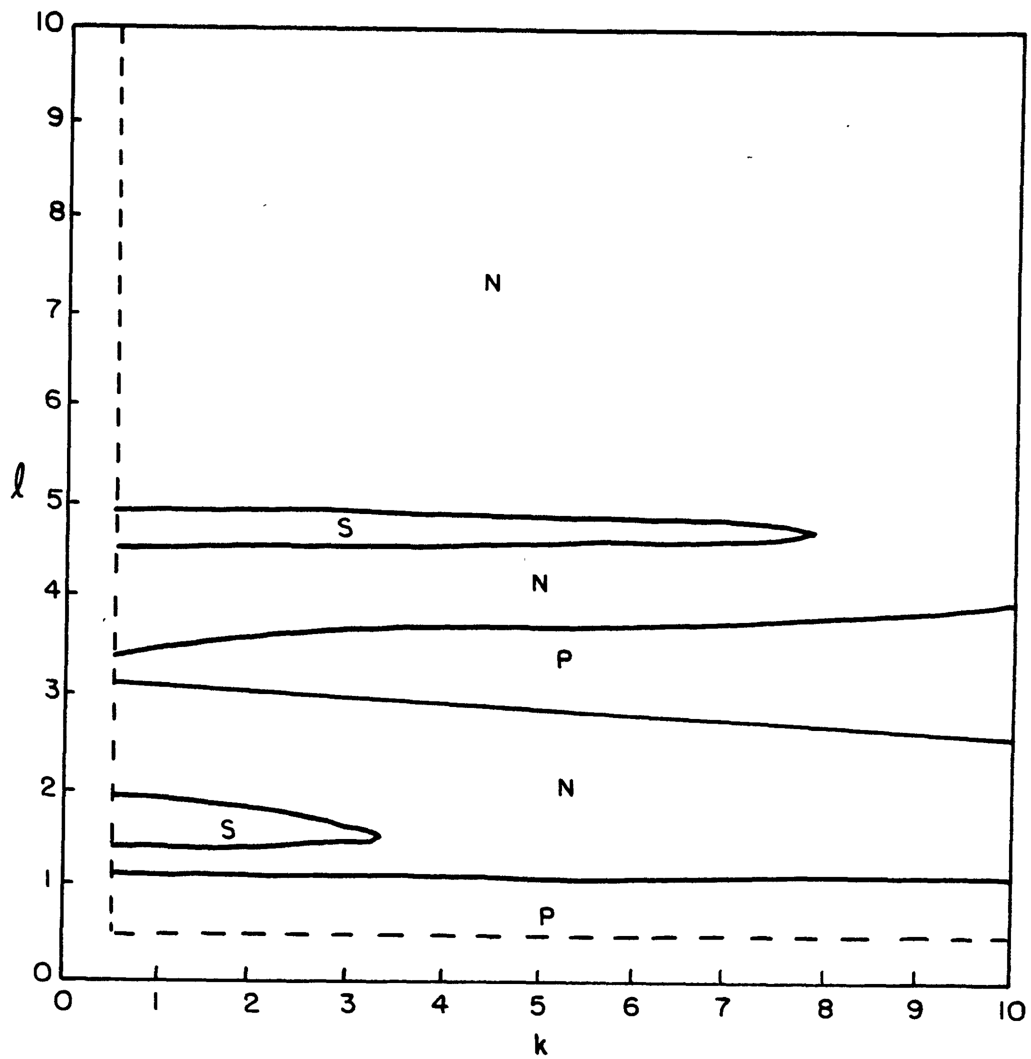

Fig. (4.4.2b) Same as (a) except for positive shear supercriticality (δ =1).

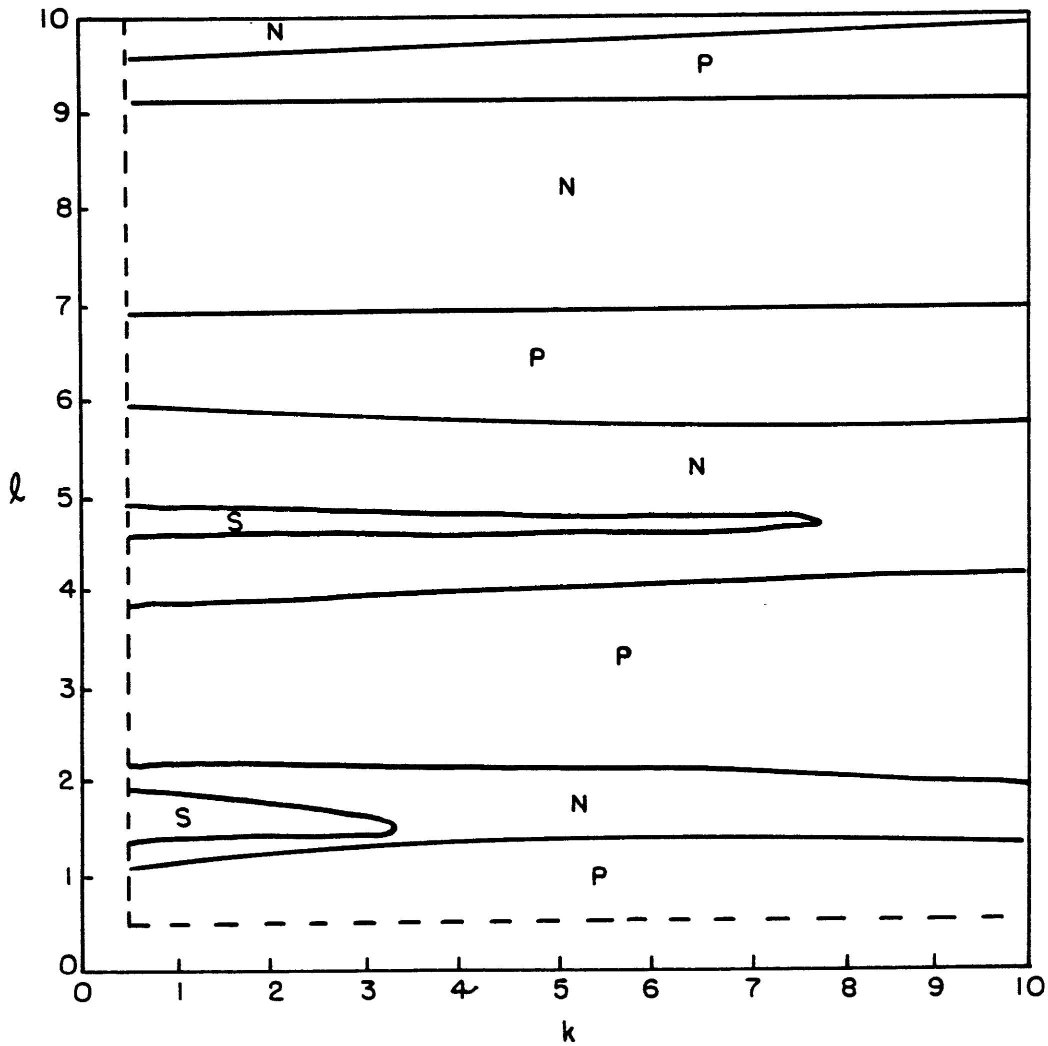

Fig. (4.4.2c) Same as (b) except $\delta = 3$.

possible is of the stationary type and is limited to two very narrow bands centered at $l \simeq 1.75$ for $k \leq 4$ and $l=4.75$ for $k<8$ shown in Figure (4.4.2a). This instability which relies on the nonlinear wave-mean flow interaction remains virtually unchanged for positive shear supercriticalities as shown in Figures (4.4.2b,c). However, when $\delta>0$ wider bands of propagating instability begin to appear which tend to get wider as k gets larger. For $\delta=1$ these bands are centered at $l=.75$ and $l=3.25$. As the flow becomes more baroclinically unstable for larger $\delta=3$ two additional bonds of propagating instability centered at $l=6.5$ and $l=9.5$ develop. Thus, for larger supercriticality, the propagating instability is not limited to the larger meridional scales as it is for smaller supercriticality. To summarize, it can be said that the stationary instability or topographic instability which owes its existence to the topographically induced wave-mean flow interaction is limited to larger meridional scales while the propagating instability which is essentially a baroclinic wave modified by the topographically induced wave-mean flow interaction is limited to larger meridional scales for only weak supercriticalities. Furthermore, the topographic instability tends to occupy the larger zonal scales while the baroclinic instability, under the influence of topography tends to be more effective at shorter zonal scales. Finally, an increase in topographic amplitude tends to stabilize the barolcinic instability of the zonal flow but gives rise to a new topographic instability.

In the next section we will examine the role to strong friction on the instability of the forced wave.

5. The topographically forced Charney mode and the influence of O(1) Ekman friction

The subject of this section is a study of the effect of stronger friction on the instability of a topographically forced Charney mode near neutrality. The basic approach is the same as in section 4 in that the analysis pivots about the neutral Charney mode which is weakly forced by the diversion of weak zonal flow by topography and made weakly unstable by letting

$$\beta = \beta_o - \varepsilon^2 \delta \tag{5.1}$$

The nonlinearity due to the wave mean flow interaction is then balanced against the linear growth or decay and forcing terms in order to obtain an equation for the wave amplitude. However, in the present analysis both the Ekman friction and Newtonian cooling are assumed to be stronger than iin section 4. In fact Ekman friction is assumed to be O(1). This modifies the properties of the instability. The study of these modifications is the purpose of this sction.

Pedlosky (1979), in section 5, studied the nonlinear evolution and eventual equilibration of the weakly unstable Charney mode in the presence of O(1) Ekamn friction but in the absence of forcing. The present problem is essentially identical to that except for the inclusion of topographic forcing. Here as in Pedlosky (1979) the Newtonian cooling parameter is assumed to be $O(\varepsilon^2)$, i.e.

$$n = \varepsilon^2 n' \tag{5.2}$$

As a result, the growth rate associated with the instability is $O(\varepsilon^2)$ as opposed to $O(\varepsilon)$ in section 4. Therefore, the appropriate long time scale is

$$T = \varepsilon^2 t \ . \tag{5.3}$$

This slower long time scale necessitates a change in the definition of the inner independent variable ζ, which now becomes

$$\zeta = z/\varepsilon^2 \ . \tag{5.4}$$

As previously mentioned, the wave forcing is provided by the diversion of a weak zonal flow by sinusoidal topography. Again the uniform zonal velocity is chosen such that the advection time scale associated with it is of the same order as the e-folding time scale. In this case, the choice for the uniform zonal velocity is

$$U = \varepsilon^2 \Delta' \ . \tag{5.5}$$

The topography is $O(\varepsilon)$ as in section 4, i.e.

$$\eta = \varepsilon b = \varepsilon M e^{ikx}(\sin ly) + * . \tag{5.6}$$

Under the above assumptions the outer, inner and lower boundary equations can be written

$$z\frac{\partial \Pi}{\partial x} + q_o\frac{\partial \phi}{\partial x} = -\varepsilon^2\left(\frac{\partial}{\partial T} + \Delta'\frac{\partial}{\partial x}\right)\Pi + \varepsilon^2\delta\frac{\partial \phi}{\partial x}$$

$$-\varepsilon J(\phi,\Pi) - \varepsilon^2 n'\left(\frac{\partial^2 \phi}{\partial z^2} - h^{-1}\frac{\partial \phi}{\partial z}\right), \tag{5.7a}$$

$$\left[\frac{\partial}{\partial T} + (\Delta' + \zeta)\frac{\partial}{\partial x}\right]\left(\frac{\partial^2 \hat{\phi}}{\partial \zeta^2} - \varepsilon^2 h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta} + \varepsilon^4 \nabla^2 \hat{\phi}\right)$$

$$+ \varepsilon^2 (q_o - \varepsilon^2 \delta) \frac{\partial \hat{\phi}}{\partial x} = - \frac{1}{\varepsilon} J\left(\hat{\phi}, \frac{\partial^2 \hat{\phi}}{\partial \zeta} + \varepsilon^2 h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta} + \varepsilon^4 \nabla^2 \hat{\phi}\right)$$

$$- n'\left(\frac{\partial^2 \hat{\phi}}{\partial \zeta^2} - \varepsilon^2 h^{-1} \frac{\partial \hat{\phi}}{\partial \zeta}\right), \tag{5.7b}$$

and

$$\frac{\partial^2 \hat{\phi}}{\partial T \partial \zeta} + \Delta' \frac{\partial^2 \hat{\phi}}{\partial x \partial \zeta} - \frac{\partial \hat{\phi}}{\partial x} + \frac{1}{\varepsilon} J\left(\hat{\phi}, \frac{\partial \hat{\phi}}{\partial \zeta}\right) = - \varepsilon J(\hat{\phi}, b)$$

$$- r \nabla^2 \hat{\phi} - n' \frac{\partial \hat{\phi}}{\partial \zeta} - \varepsilon^2 \Delta' b_x , \tag{5.7c}$$

respectively.

Again, by expanding the outer and inner stream functions ϕ and $\hat{\phi}$ as

$$\phi = \phi_o + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \ldots , \tag{5.8a}$$

$$\hat{\phi} = \hat{\phi}_o + \varepsilon \hat{\phi}_1 + \varepsilon^2 \hat{\phi}_2 + \ldots , \tag{5.8b}$$

we begin consideration of the resulting sequence of outer and inner problems. As is now the usual the O(1) outer problem leads to the specification of the neutral Charney mode which is written as

$$\phi_o = Aze^{-\beta_o z/2} e^{ikx} (\sin ly) + * . \tag{5.9}$$

Keeping (5.9) in mind, the $O(\varepsilon)$ outer equation takes the form

$$z \frac{\partial \Pi_1}{\partial x} + q_o \frac{\partial \phi_1}{\partial x} = 0, \tag{5.10}$$

to which the only new solution is

$$\phi_1 = \Phi_1 (y,z,T). \tag{5.11}$$

It will become apparent that this solution represents the $O(\varepsilon)$ zonal flow correction forced by the self interaction of the wave fields.

Using the solutions at previous orders (5.9) and (5.11) to evaluate the inhomogeneous terms of the $O(\varepsilon^2)$ equation becomes

$$z \frac{\partial \Pi_2}{\partial x} + q_o \frac{\partial \phi_2}{\partial x} = [q_o (\frac{dA}{dT} + ik\Delta' A + n' A)\frac{F_o}{z} + (ik\delta - n' K^2) F_o A - ikF_o \Pi_{1y} A + ikq_o \frac{F_o}{z} u_1 A] e^{ikx} (\sin ly) + * , \tag{5.12}$$

where

$$F_o(z) = ze^{-\beta_o z/2} \tag{5.13}$$

Attempting to remove the secular terms from (5.12) as before results in the solvability condition

$$\int_0^1 2 (\sin ly)\overline{e^{-ikx} \phi_2(x,y,0,T)}\, dy = \frac{1}{ik} \frac{dA}{dT} + \Delta' A + \frac{\delta}{q_o^2} A$$

$$- \frac{n'}{ik} \left(\frac{K^2}{q_o^2} - 1 \right) A - A \int_0^\infty dz e^{-q_o z} \left(z p_{11} - \frac{\partial u_{11}}{\partial z} \right)$$

$$+ A u_{11} (0,T) \; . \tag{5.14}$$

Evaluation of the term on the left hand side of (5.14) will lead to the determination of the amplitude equation for the wave. However, to do so requires the knowledge of $\phi_2(x,y,0,t)$ which can only be determined through the asymptotic matching of ϕ and $\hat{\phi}$ after consideration of the $O(\varepsilon^2)$ inner problem. Thus, we are forced to postpone the final evaluation of (5.14) until then.

We now consider the $O(\varepsilon^3)$ zonally averaged problem. If (5.11) is used, it takes the form

$$\frac{\partial}{\partial T} \left(\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z} \right) + n' \left(\frac{\partial^2 \Phi_1}{\partial z} - h^{-1} \frac{\partial \Phi_1}{\partial z} \right)$$

$$= - J \, (\overline{\phi_o, \, \Pi_2}) - J(\overline{\phi_2, \, \Pi_o}), \tag{5.15}$$

or equivalently

$$\frac{\partial}{\partial T} \left(\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z} \right) + n' \left(\frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1} \frac{\partial \Phi_1}{\partial z} \right)$$

$$= \frac{\partial}{\partial y} \overline{\phi_o \left(\Pi_{2x} + \frac{q_o}{z} \phi_{2x} \right)}, \tag{5.16}$$

where the relation

$$\Pi_o = - \frac{q_o}{z} \phi_o \; , \tag{5.17}$$

was used as was integration by parts in x. When the Jacobian terms of (5.15) are put into the convenient form in (5.16) it is easy to see that (5.12) can be used to eliminate the $O(\varepsilon^2)$ variables in (5.16), which have not been explicitly calculated, in terms of the known O(1) wave field. The result of doing so is

$$\frac{\partial}{\partial T}\left(\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1}\frac{\partial \Phi_1}{\partial z}\right) + n'\left(\frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1}\frac{\partial \Phi_1}{\partial z}\right)$$

$$= q_o e^{-\beta_o z}\left[\frac{d|A|^2}{dT} + 2n'\left(1 - \frac{zK^2}{q_o^2}\right)|A|^2\right]\frac{\partial}{\partial y}(\sin^2 ly). \tag{5.18}$$

This is the desired equation for the zonal flow correction forced by the transient and internally damped wave field. The next step is to solve the sequence of inner problems to obtain $\phi_2(x,y,0,T)$ and a lower boundary condition on Φ_1. Since the only wave-like solution up to $O(\varepsilon^2)$ is given by (5.9) which also becomes $O(\varepsilon^2)$ near the ground it is easy to see that there are no matchable wave-like, inner solutions until $O(\varepsilon^2)$. However, the outer $O(\varepsilon)$ zonal solution Φ_1 (y,z,T) must have a counterpart in the $O(\varepsilon)$ inner problem since Φ_1 need not necessarily vanish at the ground. If the $O(\varepsilon)$ zonal inner solution is written $f_1(y,T)$, the $O(\varepsilon)$ zonally averaged lower boundary condition yields

$$r'\frac{\partial^2 f_1}{\partial y^2} = 0, \tag{5.19}$$

which implies

$$f_1 = 0 \tag{5.20}$$

Then matching f_1 to $\Phi_1(y,0,T)$ yields

$$\Phi_1(y,0,T) = 0 \tag{5.21}$$

which is the desired lower boundary condition on Φ_1. Thus, the inner solution is at most $O(\varepsilon^2)$.

We now push on to the $O(\varepsilon^2)$ inner problem which is

$$\left[\frac{\partial}{\partial T} + (\zeta + \Delta')\frac{\partial}{\partial x} + n'\right] \frac{\partial^2 \hat{\phi}}{\partial \zeta^2} = 0 , \tag{5.22a}$$

$$\frac{\partial^2 \hat{\phi}_2}{\partial T \partial \zeta} + \Delta' \frac{\partial^2 \hat{\phi}_2}{\partial x \partial \zeta} - \frac{\partial \hat{\phi}_2}{\partial x} + r\nabla^2 \hat{\phi}_2 + n' \frac{\partial \hat{\phi}_2}{\partial \zeta} = - \Delta' b_x . \tag{5.22b}$$

The solution of (5.22a) which will match to the outer solution is

$$\hat{\phi}_2 = (A\zeta + C_2)\, e^{ikx} (\sin ly) + * , \tag{5.23}$$

where C_2 can be determined by requiring that the solution (5.23) satisfy the lower boundary condition (5.22b). It is

$$C_2 = \frac{1}{ik + rK^2} \left(\frac{dA}{dT} + ik\Delta' A + n' A\right) + \frac{ik\Delta'}{ik + rK^2} M. \tag{5.24}$$

Matching the inner solution (5.23) to the outer solution yields

$$\phi_2\,(x,y,0,T) = C_2\, e^{ikx} (\sin ly) + * ,$$

which becomes

$$\phi_2\ (x,y,0,T) = [\frac{1}{ik + rK^2}\ (\frac{dA}{dT} + ik\Delta'A + n'A)$$

$$+ \frac{ik\Delta'}{ik + rK^2}\ M]\ e^{ikx}\ (\sin ly) + * \ , \tag{5.25}$$

when (5.24) is used.

Now that $\phi_2(x,y,0,T)$ is determined, it is a simple matter to obtain the equation for the wave amplitude. Substituting (5.25) into the solvability condition (5.14) and performing the indicated operations yields the following equation for the wave amplitude, i.e.

$$(\frac{dA}{dT} + ik\Delta'A + n'A) + \frac{A}{q_o^2}\ [n'(q_o^2 - K^2) - \frac{k^2}{rK^2}\ \delta]$$

$$+ \frac{ikA}{q_o^2}\ (\delta - \frac{n'}{r}) + Ak^2\ (\frac{1}{rK^2} - \frac{i}{k})\int_0^\infty e^{-q_o z}\ (zp_{11} - \frac{\partial u_{11}}{\partial z})dz$$

$$+ \frac{k^2\Delta'}{rk^2}\ M = 0 \ , \tag{5.26a}$$

which along with the following set of equations for the zonal flow, i.e.

$$\frac{\partial}{\partial T}\ (\frac{\partial^2\Phi_1}{\partial y^2} + \frac{\partial^2\Phi_1}{\partial z^2} - h^{-1}\ \frac{\partial\Phi_1}{\partial z}) + n'(\frac{\partial^2\Phi_1}{\partial z} - h^{-1}\ \frac{\partial\Phi_1}{\partial z^2})$$

$$= q_o e^{-\beta_o z}\left[\frac{d|A|^2}{dT} + 2n'\left(1 - \frac{zK^2}{q_o^2}\right)|A|^2\right]\frac{\partial}{\partial y}(\sin^2 ly), \tag{5.26b}$$

$$\Phi_1(y,0,T) = 0\ , \tag{5.26c}$$

and

$$\frac{\partial^2\Phi_1}{\partial T\partial y} = 0, \tag{5.26d}$$

on y=0,1 form the closed set of equations describing the evolution of the wave-zonal flow interaction. Notice that the wave equation (5.26a) is first order in time as opposed to second order in time which was the case in previous sections. The strong friction present in this case is responsible for this decrease in order. Such a frictional decrease in order was first noted by Pedlosky (1970) in the context of the two layer model.

The zonal flow equations (5.6b,c,d) are identical to the zonal flow equations (5.23), (5.24a,b) of Pedlosky (1979). The equation for the wave amplitude (2.8.26a) is the same as Pedlosky's wave amplitude equation (5.27) except for the presence of a term proportional to the weak zonal velocity Δ' and an inhomogeneous term proportional to the topographic amplitude M. This topographic inhomogeneous term forces a wave whose stability we will now study in the absence of Newtonian cooling. Assuming $n'=0$, the amplitude equations (5.26a-d) take the form

$$\frac{dA}{dT} + ik\,(\Delta' + s)A - ksfA + Ak(f-i)\int_0^\infty e^{-q_o z}$$

$$\times\ (zp_{11} - \frac{\partial u_{11}}{\partial x})\,dz + kf\Delta' M = 0, \qquad (5.27a)$$

$$\frac{\partial}{\partial T}\left(\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial z^2} - h^{-1}\frac{\partial \Phi_1}{\partial z}\right) = q_o e^{-\beta_o z}\ \frac{d|A|^2}{dT}\ \frac{\partial}{\partial y}(\sin^2 ly) \qquad (5.27b)$$

$$\Phi_1(y,0,T) = 0 \qquad (5.27c)$$

$$\frac{\partial^2 \Phi_1}{\partial T \partial^2}\,(0,z,T) = \frac{\partial^2 \Phi_1}{\partial T \partial^2}\,(1,z,T) = 0 \qquad (5.27d)$$

where the inverse friction parameter, f, is defined

$$f = k/rK^2 \qquad (5.28a)$$

and the supercriticality parameter, s, is defined

$$s = \delta/q_o^2 \qquad (5.28b)$$

The steady topographically forced wave solution of (5..27a-d) is

$$A_s = f\Delta'\,\frac{[fs + i(\Delta' + s)]M}{f^2 s^2 + (\Delta' + s)^2} \qquad (5.29)$$

If M is real, the real part of A_s reaches a finite maximum when $\Delta' + s = 0$. Since the maximum is finite it is possible to explore the nature of the instability near this resonance. This is in contrast to the case in section 4 where the stability analysis was valid only away from resonance because the lack of friction lead to an infinite response at resonance.

For simplicity, M will assigned to be real. When the amplitude equations (5.27a-d) are linearized about the topographically forced wave (5.29) and they are solved using the same Fourier expansion technique in y used in previous sections, the linear equation for the disturbance wave amplitude A' takes the form

$$\frac{dA'}{dT} + ik\,(\Delta' + s)A' - ksfA' + 2k(f-i)N_O A_S\,(A_{sr}\,A'_r + A_{si}\,A'_i) = 0, \tag{5.30}$$

where A_{sr} and A_{si} are the real and imaginary parts of A_s, and A'_r and A'_i are the real and imaginary parts of A'. Under the Boussinesq-like approximation

$h^{-1} << \beta_0$,

the nonlinear coefficient, N_0, takes the form

$$N_O = \frac{Kl^4}{\pi^6}\left(\sum_{m=1}^{\infty} \frac{1}{(m + \frac{k}{\pi} - \frac{1}{2})[(m - \frac{1}{2})^2 - \frac{l^2}{\pi^2}]^2}\right) - \frac{l^2}{8K^2}, \tag{5.31}$$

which is nearly always greater than zero and the supercriticality parameter becomes

$$s = \delta/4K^2 . \tag{5.32}$$

Note in (5.30) that as the topographic amplitude is increased the steady wave amplitude A_s is increased which makes the last term of (5.30) larger, hence altering the stability characteristics of (5.30). As in previous sections this term which arises in the linear stability problem (5.30) only when the topography is nonzero, represents the interaction between the topographically forced wave and the perturbation zonal flow which in turn is forced by the interaction of the perturbation wave field and the forced wave.

If A' is assumed to be proportional to $e^{\sigma T}$ in (5.30), after much algebra, the dispersion relation for σ can be written

$$\frac{\sigma}{k} = sf - \frac{N_o M^2 \Delta'^2 f^3}{f^2 s^2 + (\Delta' + s)^2} \pm \left\{ \left[\frac{N_o M^2 \Delta'^2 f^3}{f^2 s^2 + (\Delta' + s)^2} \right]^2 + \frac{2N_o M^2 f^2 \Delta'^2 (\Delta' + s)}{f^2 s^2 + (\Delta' + s)^2} - (\Delta' + s)^2 \right\}^{1/2} . \tag{5.33}$$

In the absence of topography, the forced wave amplitude is zero. Thus, the interaction between the forced wave and disturbance zonal flow is nonexistent when M=0. In this case the dispersion relation (5.33) reduces to

$$\frac{\sigma}{k} = sf \pm i(\Delta' + s) . \tag{5.34}$$

Recall $s = \delta/4K^2$ and $f = k/rK^2$. This is the dispersion relation for an infinitesimally small Charney mode near neutrality under the influence of strong Ekman friction. For s>0 the propagating Charney mode is unstable while for s<0 the propagating Charney mode decays. However, in the presence of Newtonian cooling, absent here, Pedlosky (1979) showed that there is a minimum supercriticality

$$\delta_c = r\,n' \frac{K^2}{k^2}\,(3K^2 + h^{-2}), \tag{5.35}$$

which must be exceeded for instability to occur.

We will now examine how the baroclinic instability of the Charney mode is altered as the topographic amplitude is increased. If the topographic amplitude is small, the dispersion relation will reflect at least a relatively weak interaction between the perturbation mean flow and the forced wave which will help us understand the full dispersion relation (5.33). For $M \ll 1$, (5.33) reduces to

$$\frac{\sigma}{k} = sf - \frac{N_o M^2 \Delta'^2 f^3}{f^2 s^2 + (\Delta' + s)^2} \pm i(\Delta' + s)\left\{1 - \frac{N_o M^2 f^2 \Delta'^2}{(\Delta' + s)[f^2 s^2 + (\Delta' + s)^2]}\right\} + O(M^4). \tag{5.36}$$

We can see immediately from (5.36) that the small increase of topographic amplitude stabilizes the baroclinic instability for s>0 and further stabilizes the baroclinic stability for s<0. The phase speed is decreased for $\Delta'+s>0$ and increased for $\Delta'+s<0$. From Figures (4.5..1a,b,c) which are plots, for $\Delta'=1.0$ and various values of the Ekman parameter r, of the regions in topographic amplitude, supercriticality space for which there is propagating instability (PI), propagating decaying waves (PD), stationary decaying waves (SD) and stationary instability (SI), it is clear that this stabilizing influence of topographic amplitude is increased further. The next transition as M is increased depends on the supercriticality δ and value of r. For $\delta>0$ the regime of propagating instability persists until either (1) the sum of the first two terms in (5.33) becomes negative, i.e.

$$sf - \frac{N_o M^2 \Delta'^2 f^3}{f^2 s^2 + (\Delta' + s)^2} < 0, \qquad (5.37)$$

or (2) the quantity in curly brackets in (5.33) becomes positive, i.e.

$$\left[\frac{N_o M^2 \Delta'^2 f^3}{f^2 s^2 + (\Delta' + s)^2}\right]^2 + \frac{2 N_o M^2 f^2 \Delta'^2 (\Delta' + s)}{f^2 s^2 + (\Delta' + s)^2}$$

$$- (\Delta' + s)^2 > 0. \qquad (5.38)$$

The first possibility corresponds to a transition from a propagating instability regime to a propagating decaying wave regime. This is the situation that occurs in all cases plotted in Figures (4.5.1a,b,c) except for $\delta \geq 3.0$ in Figure (4.5.1a) where r=.2. As the Ekman friction is increased this transition occurs at larger values of M for a given supercriticality. In the case (1) where (5.37) is satisfied, the next transition is from the propagating decaying wave regime to the stationary decaying wave regime and occurs when (5.38) is satisfied. The next transition is to the regime of stationary instability which occurs for

$$\left|\left[\frac{N_o M^2 \Delta'^2 f^3}{f^2 s^2 + (\Delta' + s)^2}\right]^2 + \frac{2 N_o M^2 f^2 \Delta'^2 (\Delta' + s)}{f^2 s^2 + (\Delta' + s)^2}\right.$$

$$\left. - (\Delta' + s)^2 \right| > \left| sf - \frac{N_o M^2 \Delta'^2 f^3}{f^2 s^2 + (\Delta' + s)^2} \right| \qquad (5.39)$$

At this point there is an interesting difference between the transition to the stationary instability here where r is O(1) and the transition to the stationary instability in the inviscid case discussed in the last section. In the last section, before the transition was to occur, the phase speed of one of the neutral modes was slowing down as the topographic amplitude increased. As the topographic amplitude increased further, the neutral mode became stationary relative to the forced wave. Therefore, the neutral mode became coherent with it and could easily draw energy from it leading to the instability. However, here in the presence of friction the mode becomes stationary and yet it still decays until the topography is increased (thereby enhancing the interaction between the perturbation zonal flow and forced wave) enough to overcome the friction. In summary, the inviscid case becomes unstable as soon as the mode becomes coherent with the forced wave but in the viscous case the topography must increase past the point where the waves are stationary to enhance the wave mean flow interaction enough to become unstable.

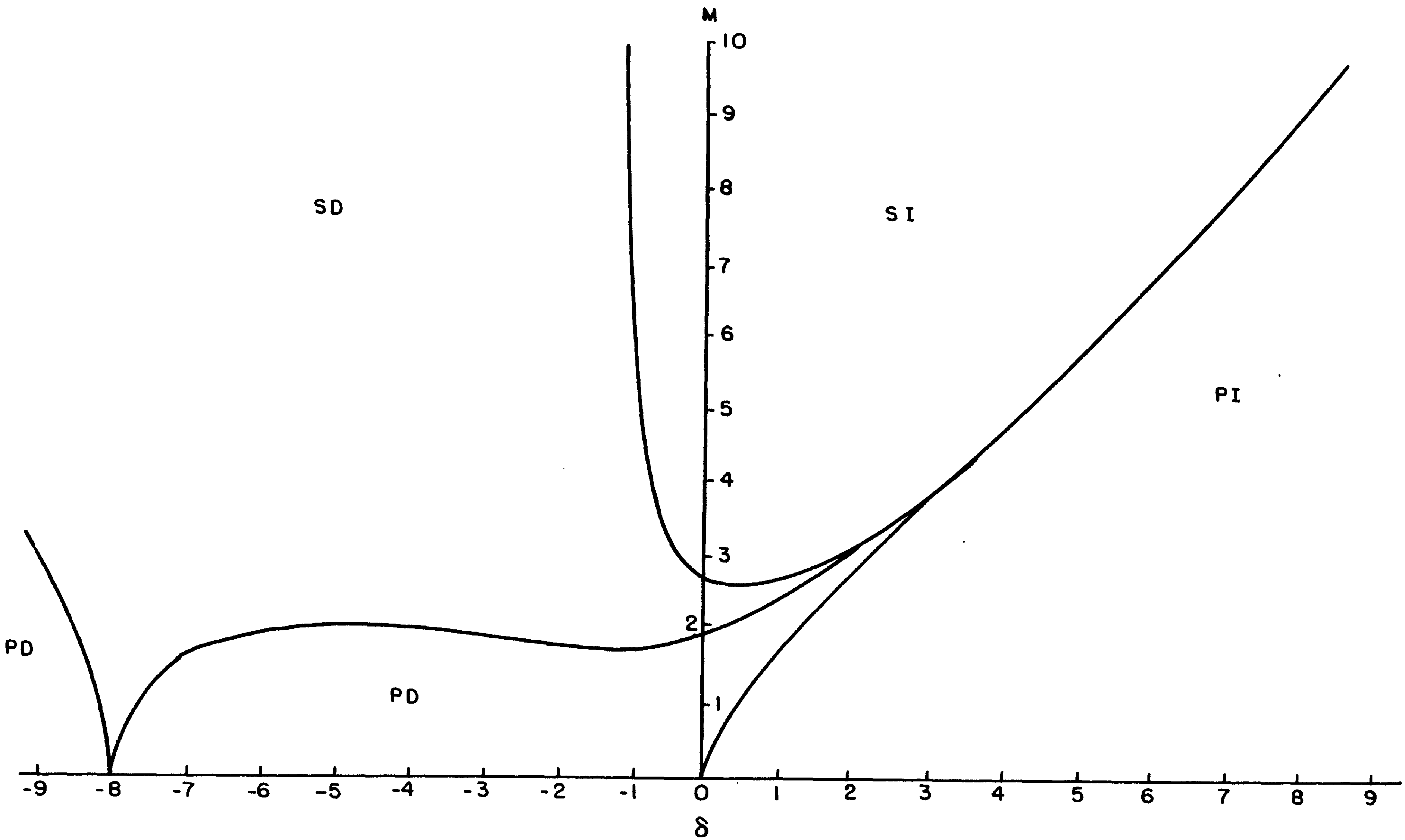

Fig. (4.5.1a) Regions of SI, PI, SD, PD (see text for definitions) plotted vs. topographic amplitude M and supercriticality δ with $\Delta' = 1.0$, $r = .2$, $k = 1.0$, and $l = 1.0$.

When the second possibility (5.38) is satisfied but (5.37) is not, i.e.

$$sf - \frac{N_o M^2 \Delta'^2 f^3}{f^2 s^2 + (\Delta' + s)^2} > 0 \ , \qquad (5.40)$$

the first and only transition is from the propagating instability regime to the stationary instability regime. This occurs in Figure (4.5.1a) where r=.2 for δ>3.0. This also is a uniquely viscous result.

When the flow is subcritical (δ and s<0) there is no regime of propagating instability. The succession of transitions as M increases if from a regime of propagating decaying waves to a regime of stationary decaying waves as (5.38) is satisfied, to a regime of stationary instability when (5.39) is satisfied. That is, the succession of transitions as M is increased if

$$\Delta' + s > 0 \qquad (5.41a)$$

When

$$\Delta' + s < 0 \qquad (5.41b)$$

the only transition is from a regime of propagating decaying waves to one of stationary decaying waves. There is no stationary instability in the region where (5.41b) is satisfied. If

$$\Delta' + s = 0 \qquad (5.41c)$$

the dispersion relation (5.33) reduces to

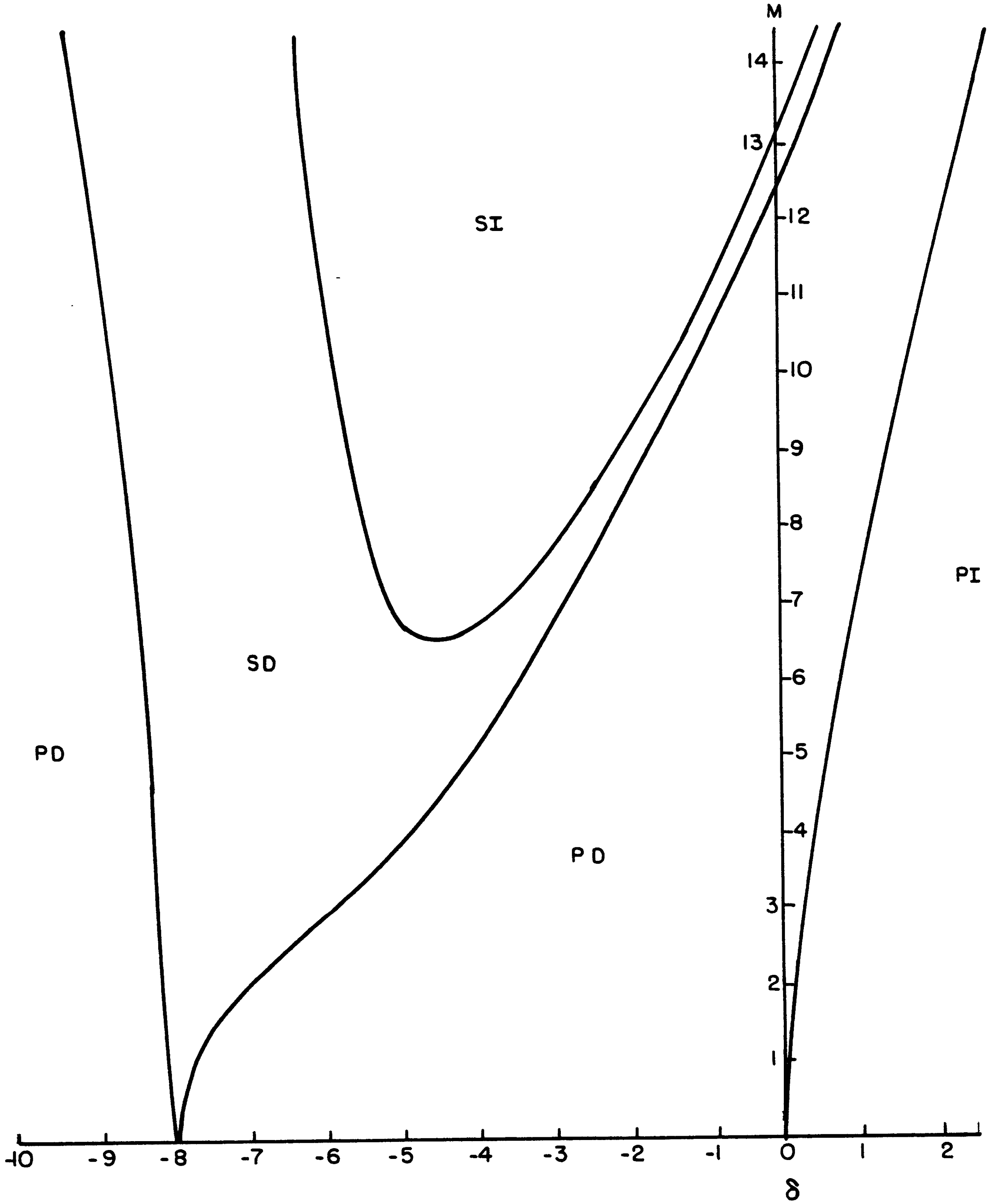

Fig. (4.5.1b) Same as (a) but r=1.0.

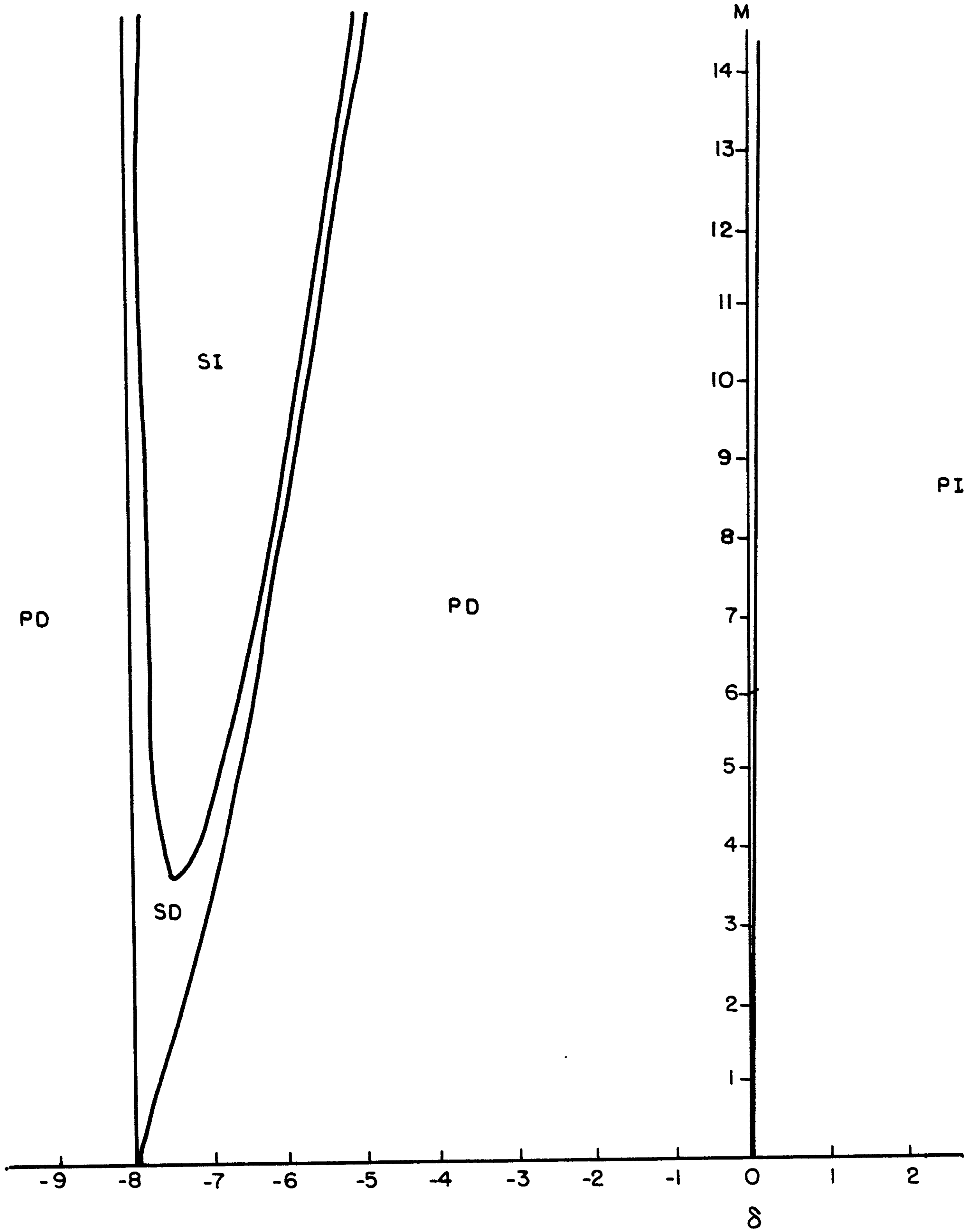

Fig. (4.5.1c) Same as (a) but r=5.0.

$$c = -\Delta' f - N_o M^2 f \pm N_o M^2 f, \tag{5.42}$$

which says there are only stationary decaying modes present for any topographic amplitude M. Thus, the stationary instability is again limited to what might be called a superresonant flow for which

$$\Delta' + s > 0.$$

This is suggestive of its topographic origins.

A further comparison of Figures (4.5.1a,b,c) reveals several interesting features. First note that the stationary instability seems to transcend the supercriticality boundary $\delta=0$ while the propagating instability does not. The latter requires $\delta>0$. As was pointed out in the last section, this is due to the fact that the propagating instability is really just a topographically modified baroclinic instability for which supercriticality is required while the stationary instability relies on the presence of the forced wave to tap the zonal available potential energy. However, in all the Figures (4.5.1a,b,c) the stationary instability under the influence of large friction requires a larger topographic amplitude as resonance is approached. In fact, as previously mentioned there is no stationary instability for

$$\Delta' + \leq 0.$$

This is in direct contrast to the inviscid case discussed in section 4 where the topographic amplitude required for instability decreases as resonance is approached, including the point at which resonance occurs. A second point of interest is that as friction is increased, the transition lines become more vertical, that is to say, highly dependent on δ but not on M. This is particularly evident in Figure (4.5.1c) where some lines may be hard to distinguish from the vertical and the stationary instability is limited to a very narrow region in δ. In addition, the stationary instability is squeezed more closely near the superresonant side of resonance as friction is increased. This we should keep in mind when we examine the regions of instability and decay for propagating waves and stationary waves plotted vs. k,l for r=5.0. The third point of interest is the relatively small area in the diagrams of propagating decaying waves, especially for larger values of friction. This is also a useful fact to keep in mind when we discuss the stability regimes in k and l space.

This strongly damped flow has another feature not present in the inviscid case discussed in section 4. There, the instability is independent of whether the flow at the ground is eastward ($\Delta'>0$) or westward ($\Delta'<0$). Here in the dispersion relation (5.33) Δ' appears not only as Δ'^2 but also as Δ'. Thus, the viscous instability does depend on whether the flow at the ground is eastward of westward. In figure (4.5.2) the stability regimes in M,δ space are plotted for the westward flow at the ground $\Delta'=-1.0$ with r=1.0. The progression of stability transitions as M

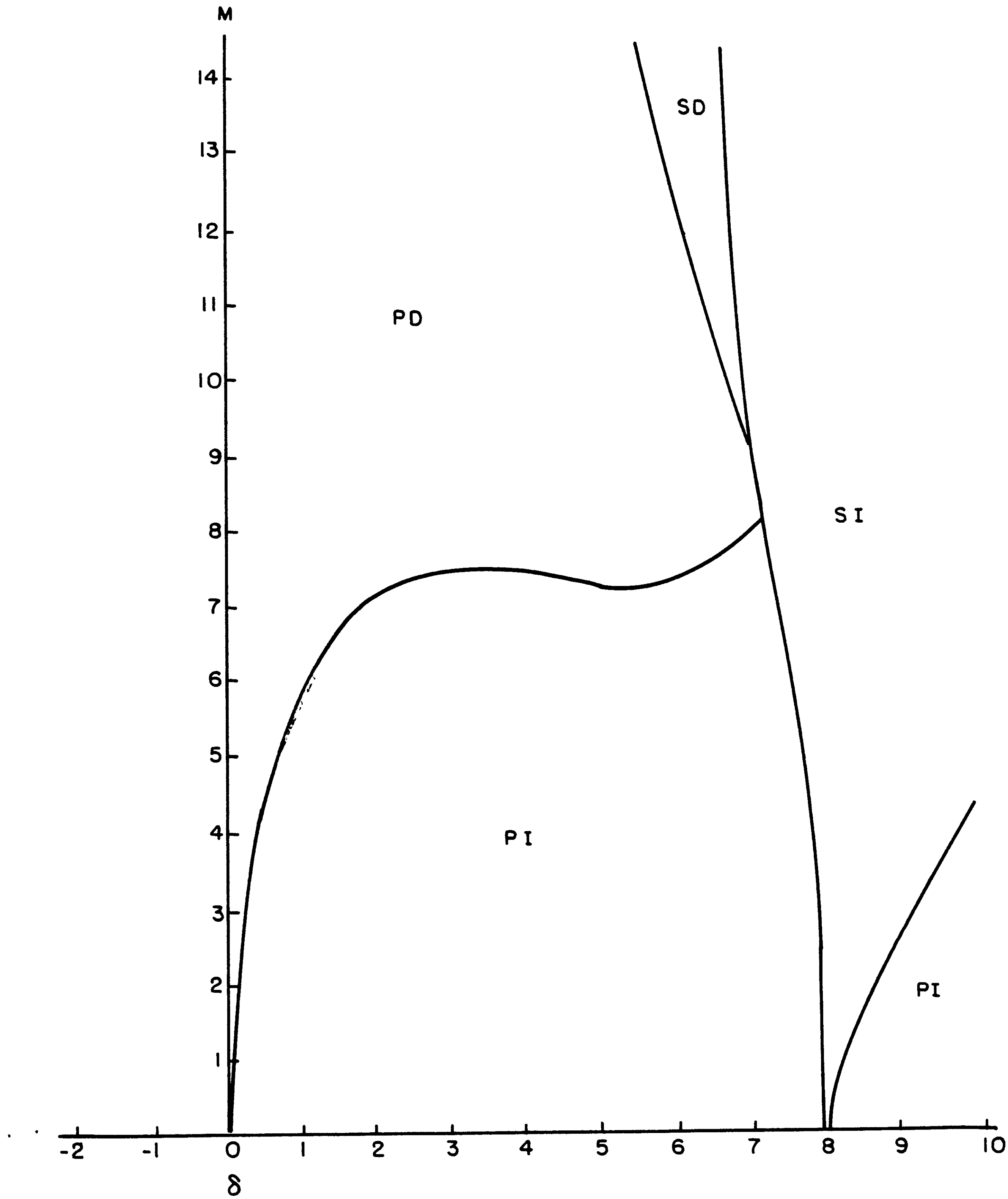

Fig. (4.5.2) Stability regimes plotted vs. M and δ for the westward flow at the ground, Δ'=-1.0 and Ekman friction r=1.0.

increases is essentially the same as that just discussed for $\Delta'=1.0$. In addition, Figure (4.5.2) shows that there are only propagating decaying waves for

$$\delta < 0$$

and that the stationary instability occurs only near $\delta=8.0$. Furthermore, at $\delta=8.0$ the stationary instability occurs for every value of M. This is due to the shifting of the line for which

$$\Delta' + s = 0,$$

from $\delta=-8.0$ to the supercritical value $\delta=8.0$ when $\Delta'=1.0$ is changed to $\Delta'=-1.0$. When $\Delta'+s=0$ the dispersion relationn (5.33) reduces to (5.42) which shows that there is always at least one growing mode for which the growth rate is c=f if $\Delta'=-1.0$ which is in agreement with the numerical results. The instability occuring about $\Delta'=1.0$ is baroclinic and not topographic. Finally we note that since the region of SI is near the resonance defined by $\Delta'+s=0$ the region of SI in k,l space will be near the circle defined by
$k^2+l^2 = -\delta/(4\Delta') = \delta/4$. Thus, when $\Delta'<0$ the stationary instability occurs around the resonance $\Delta'+s=0$ but for $\Delta'>0$ the stationary instability occurs only for superresonant flow, i.e. $\Delta'+s>0$.

We will now examine the regions in k and l space of propagating instability (PI), propagating decaying waves (PD) and stationary instability (SI) for $\Delta'=1.0$. Figures

(4.5.3a,b,c) are the stability regime diagrams for r=.2 and δ=-2.0, 1.0, and 6.0 respectively. Figures (4.5.4a,b,c) are the stability regime diagrams for r=1.0 and δ=-2.0, 0.5 and 1.0, respectively and Figures (4.5.5a,b) are the same type of plot for r=5.0 and δ=.01 and δ=.05, respectively. A figure for r=5.0 and δ<0 is not shown since the entire wavenumber space is occupied by the propagating decay regime. There are regimes of stationary decay near the transition from PD to SI especially for smaller values of friction but they are too small to be drawn. Recall from M vs. δ stability diagrams that regions of PD were small there also. In the present diagrams it is particularly true because of the choise of parameters made.

Let us first compare diagrams for a given value of friction. In Figures (4.5.3a) and (4.5.4a) which are for subcritical flow (δ<0), the only instability present is the stationary type. As in the inviscid case of section 4, it occurs in very narrow bands in l. The rest of the wavenumber space is one of the propagating decaying waves. As the supercriticality is increased to positive values, a band of propagating instability appear first for small k and grow to the left to include larger k values as δ is increased. These wedges develop between the bands of stationary instability which stay essentially the same as δ is increased. The wedges of PI develop more rapidly in k as δ is increased for larger values of friction but the location in l tends to be independent of r. This can be

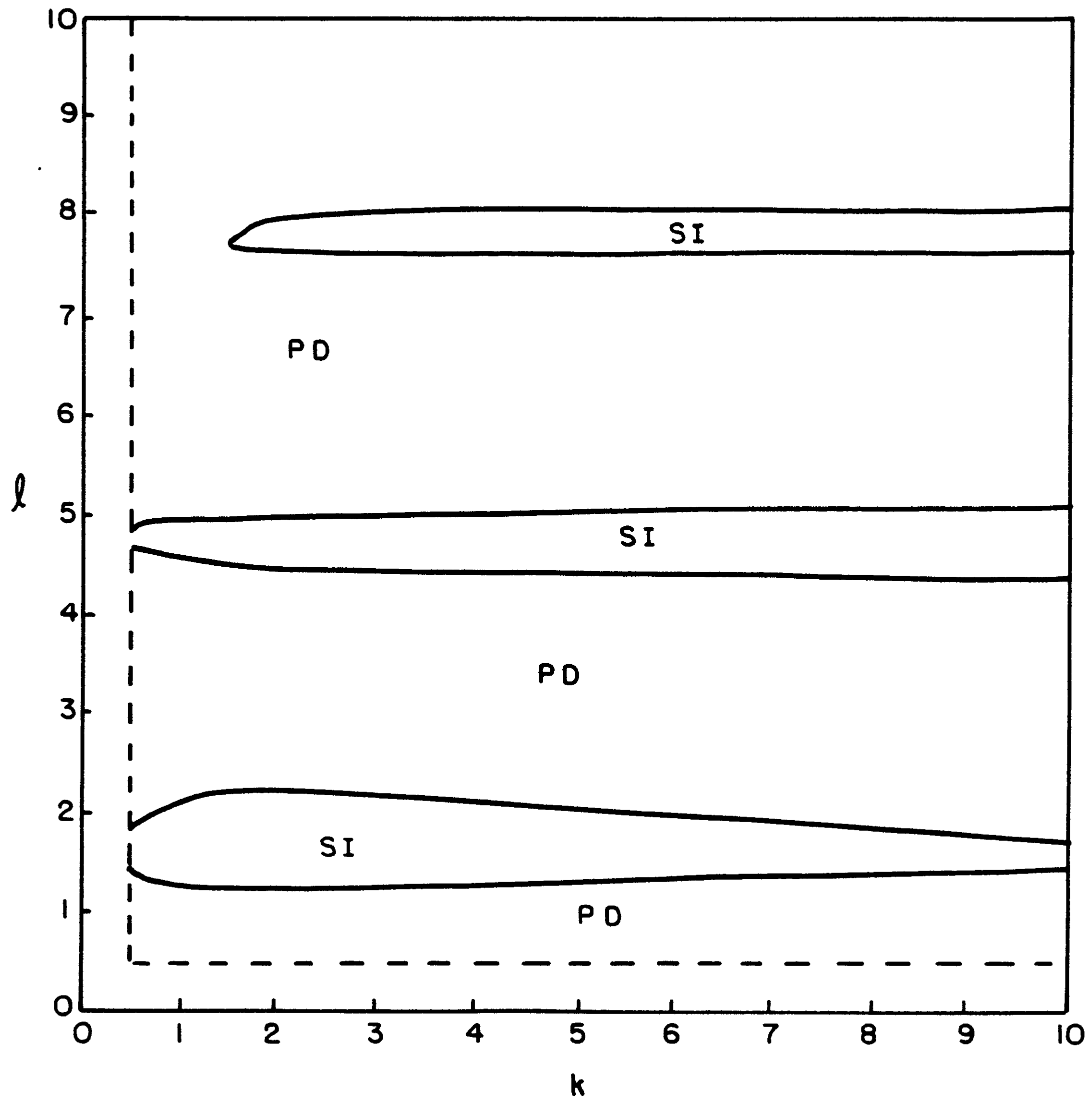

Fig. (4.5.3a) Stability regimes vs. k and l for fixed topographic amplitude M =1.0 and r=.2, δ =.2. See text for the definition of SI, PD, and PI.

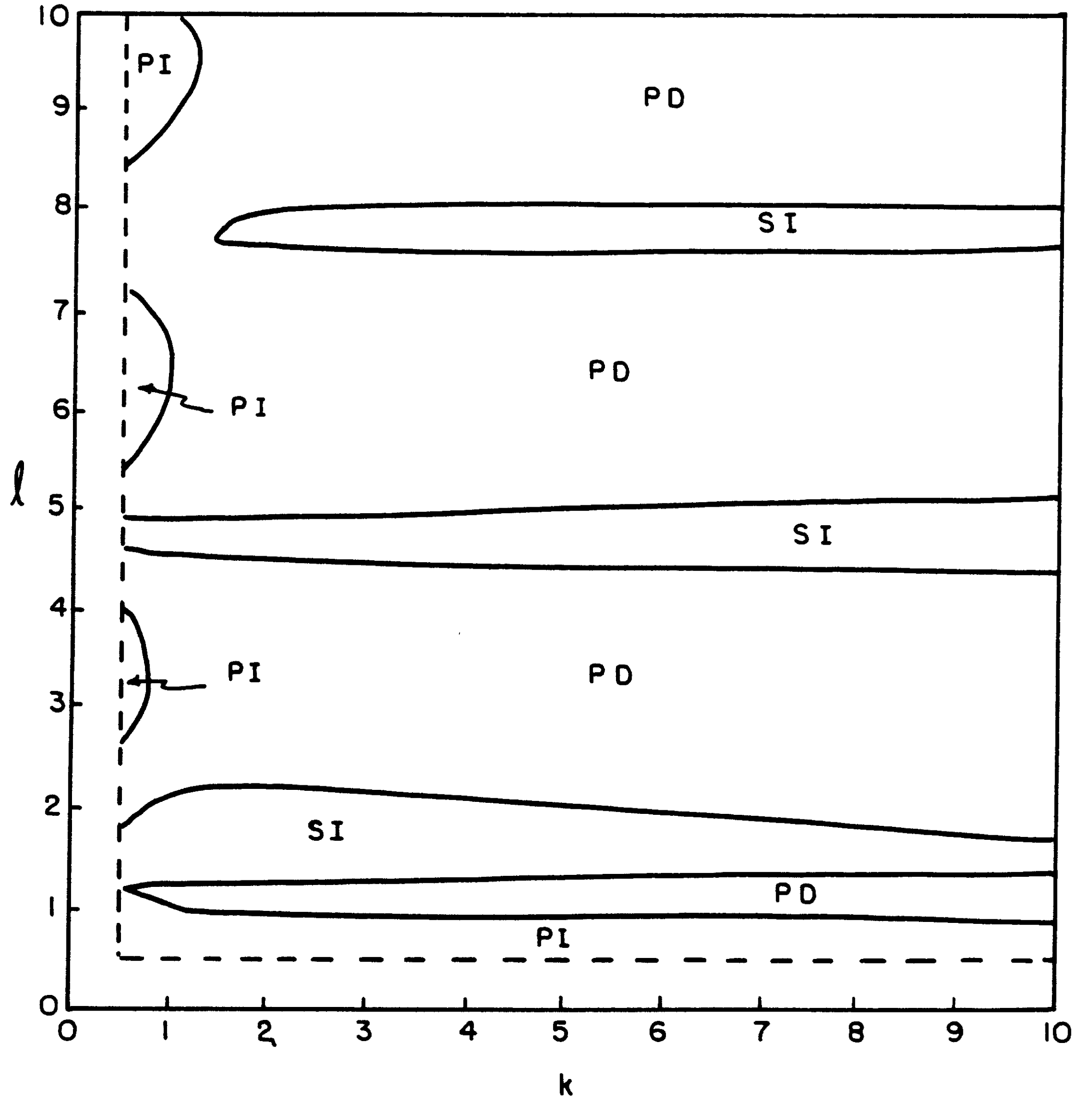

Fig. (4.5.3b) Same as (a) but δ =1.0.

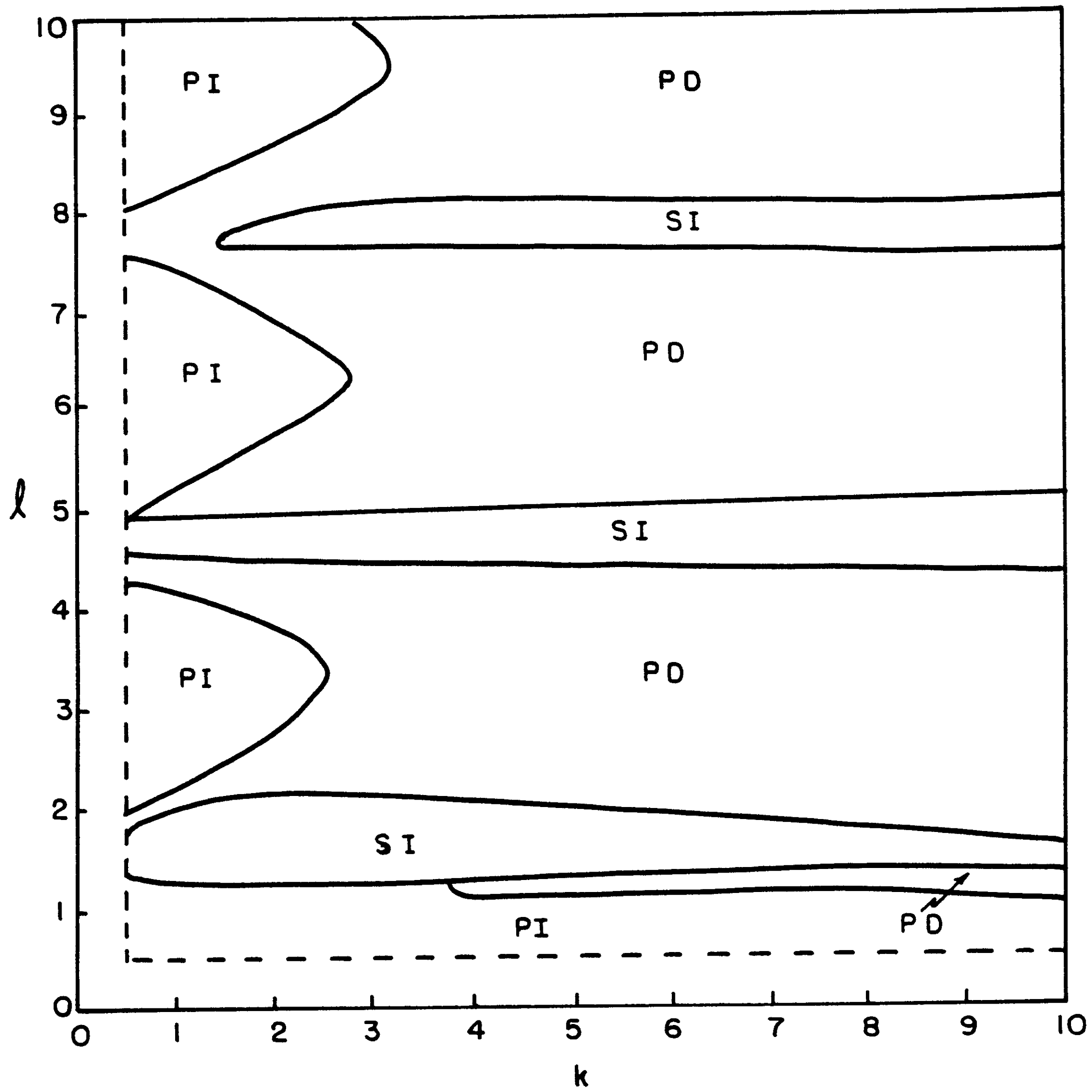

Fig. (4.5.3c) Same as (a) but δ =6.0.

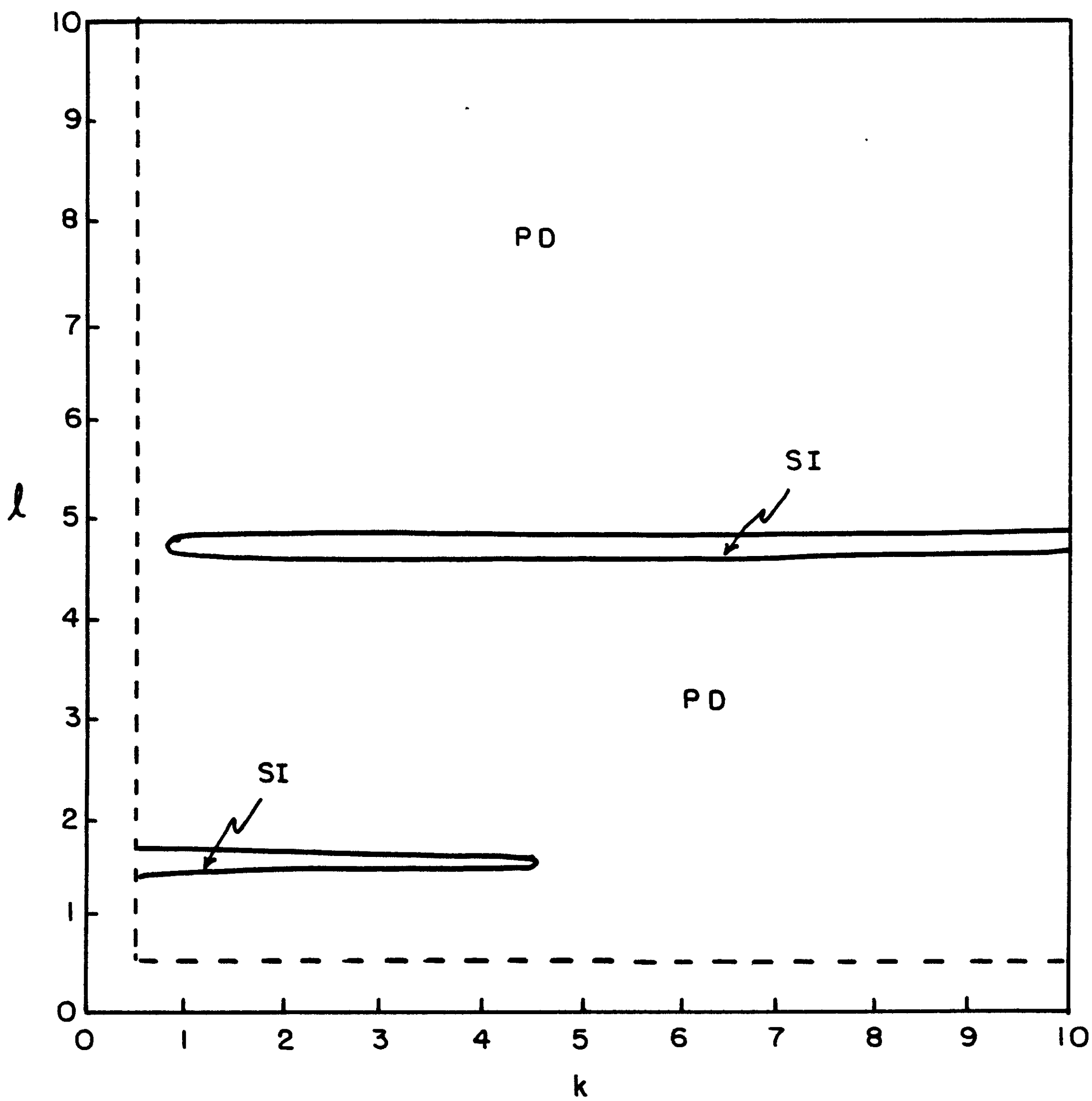

Fig. (4.5.4a) Stability regimes vs. k and l for $\mathcal{M}$=1.0, r=1.0 and δ =-2.0.

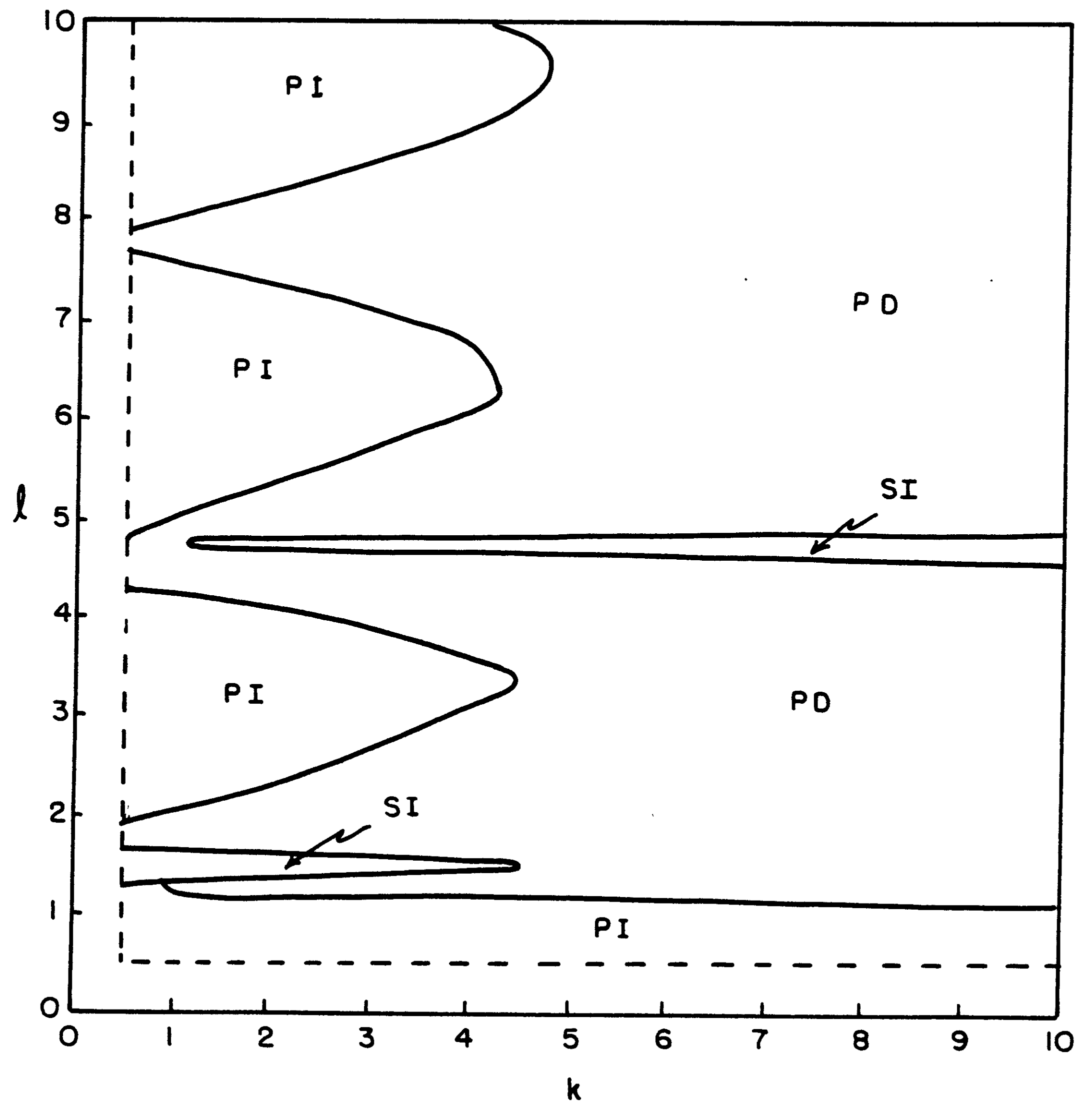

Fig.(4.5.4b) Same as (a) but δ =.5.

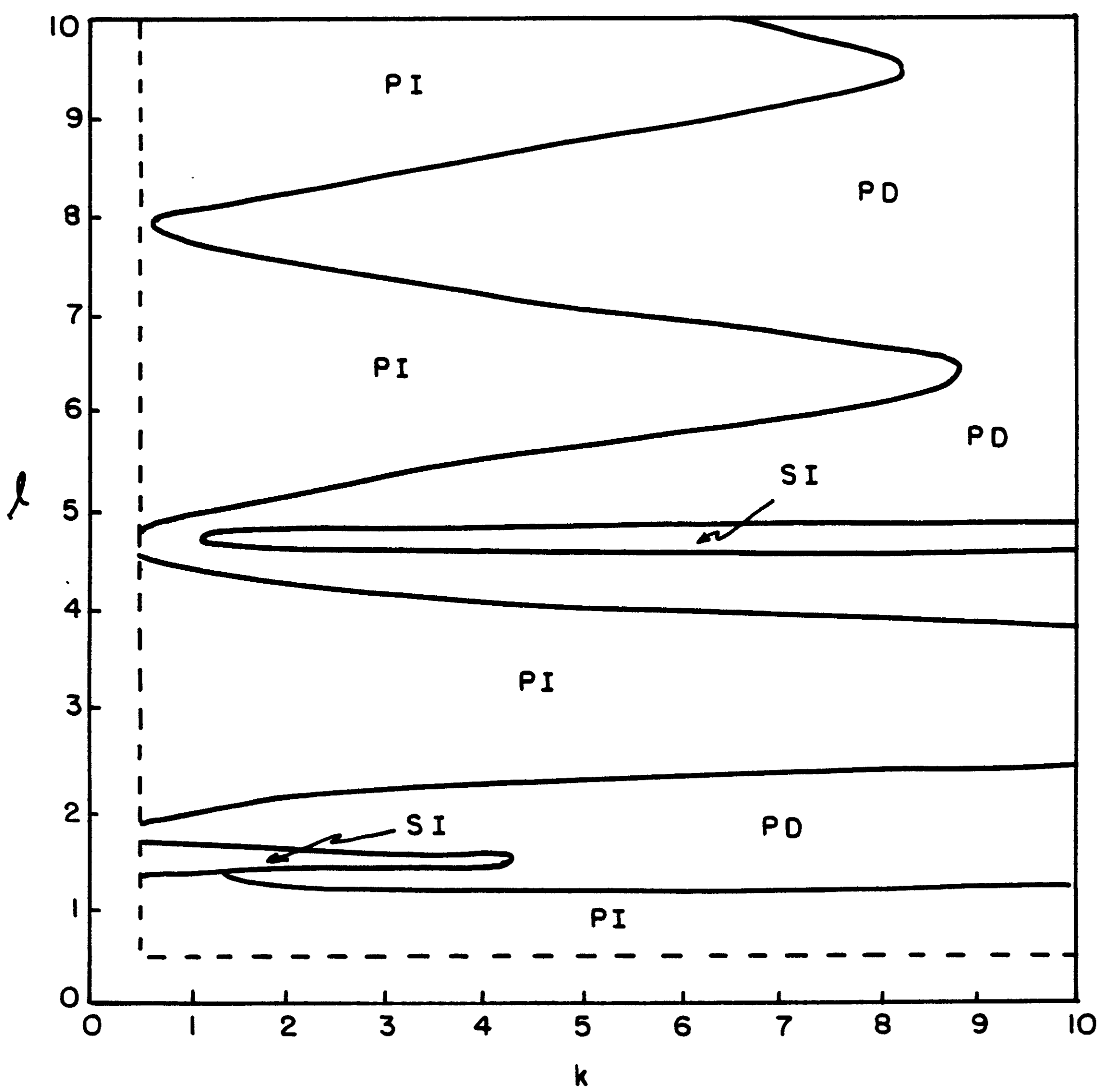

Fig. (4.5.4c) Same as (b) but δ =1.0.

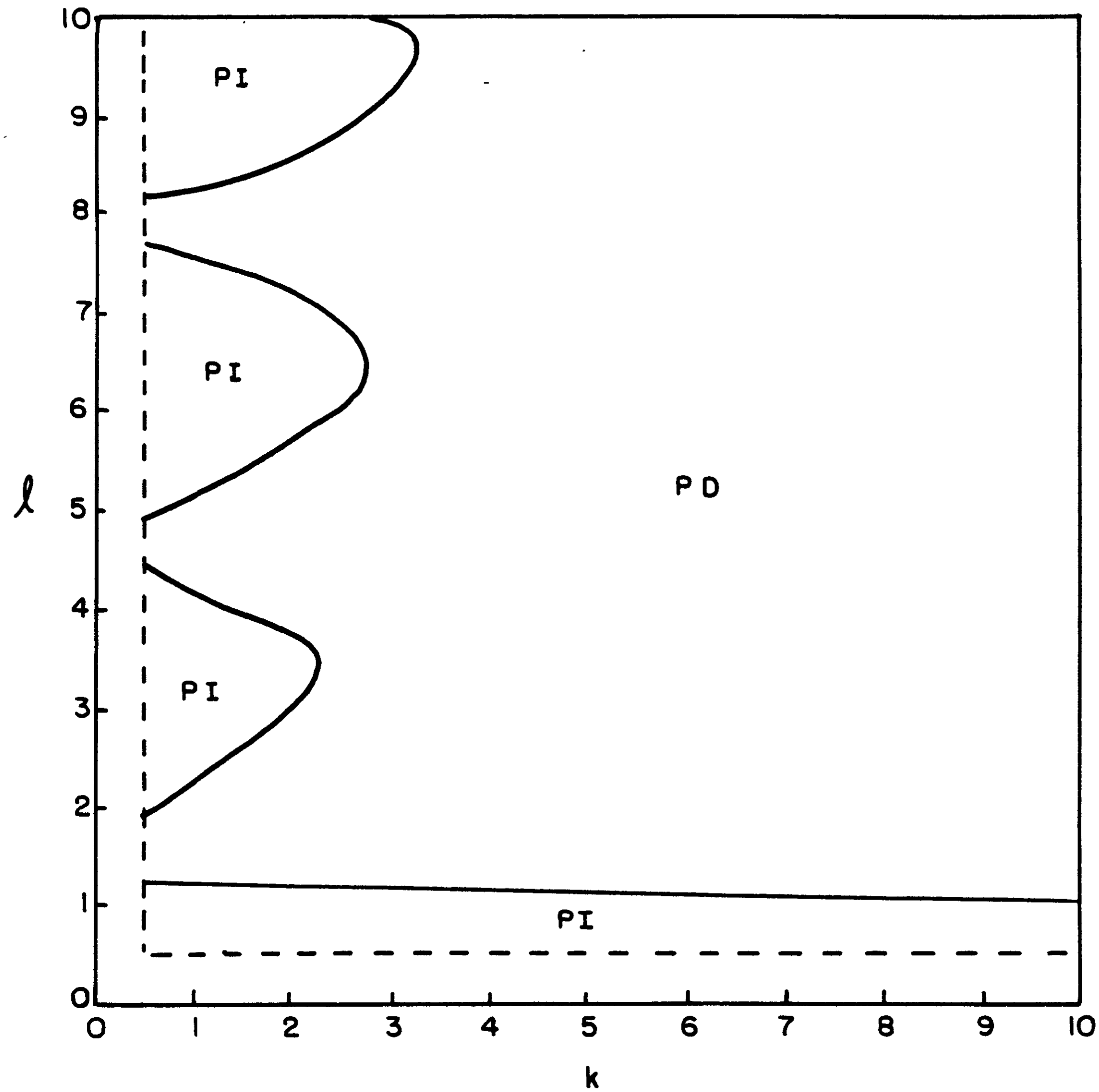

Fig. (4.5.5a) Stability regimes vs. k and l for μ =1.0, r=5.0 and δ =.01.

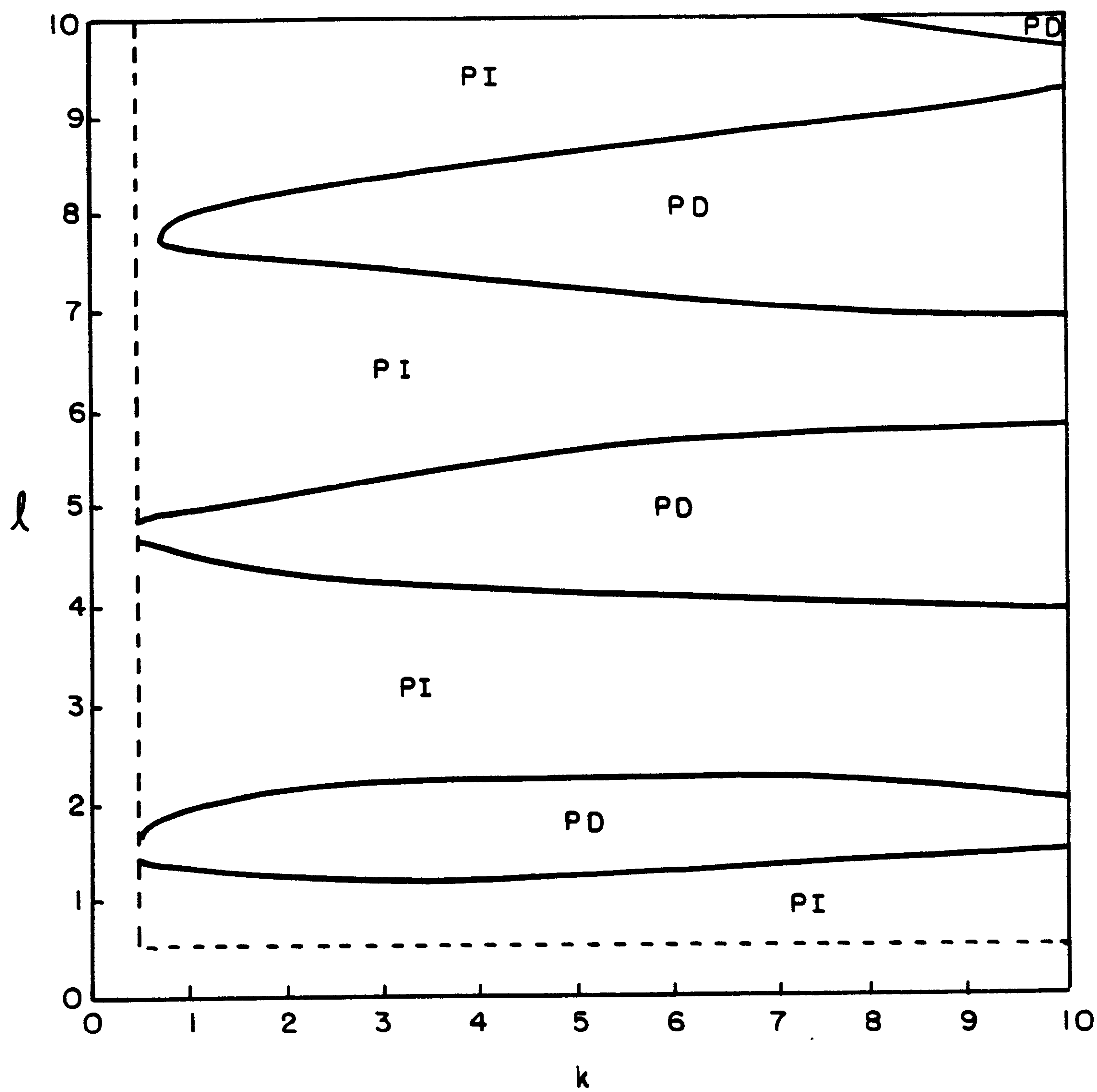

Fig. (4.5.5b) Same as (a) but δ =.05.

easily seen by comparing the figures keeping in mind the values of r and δ. As r gets larger the regions of SI become smaller and in fact disappear for r=5.0. This is also to be expected from the M vs. δ stability diagrams wich showed a decreasing region of SI as r increased. Nevertheless, when regions of SI do appear they appear in approximately the same bands in l independent of the friction. As already mentioned, the center of the wedges of PI in l also tend to be independent of l. Thus, even when there are no bands of SI such as in Figures (4.5.5a,b) for r=5.0, the wedges of PI still develop around those bands in l which contain the stationary instability for lesser values of r.

This discussion can be summarized by making three points. First, the bands of stationary instability transcend the supercriticality boundary $\delta=0$ while the wedges of propagating instability are limited to $\delta>0$. This is due to the different nature of the two types of instability. The propagating instability is essentially a topographically modified baroclinic instability which, of course, requires supercritical flow ($\delta>0$) while the stationary instability which also feeds on the zonal available potential energy requires sufficient topographic amplitude which increases the interaction between the forced wave and perturbation zonal flow thereby instigating the instability. The second point to be made is that the O(1) Ekman friction stabilizes the stationary instability and the propagating instability.

The final point is that the introduction of large viscosity has made the instability depend on the direction of the zonal flow at the ground. For eastward flow at the ground the stationary topographic instability is confined to superresonant flow which exists closer to resonance as the friction is increased. If the flow is westward at the ground, the stationary instability is near resonance but not confined to one side. However, when the flow is westward, the nature of the stationary instability is baroclinic instead of topographic.

6. Conclusions

In this chapter we have considered two questions concerning a Charney mode embedded in a shear flow which is either weakly baroclinically subcritical or supercritical using a weakly nonlinear theory. First, what is the effect of topography on a baroclinically growing Charney mode and second does the presence of topography lead to instability where there would otherwise be none?

In section 2 we found that topography strongly stabilizes the infinitesimal baroclinic instability except for those waves with a very large meridional scale. The stabilizing mechanism is due to the feedback between the zonal flow and wave at the ground. As the wave grows a phase shift develops relative to the topography which produces a form drag. The form drag modifies the mean vertical shear near the ground towards stability, hence stabilizing the disturbance.

In section 2 there is no uniform zonal flow at the ground by which a topographic wave might be forced. The analysis of section 3 allows a very weak zonal flow which is enough to force a Charney mode via the diversion of the zonal flow by the topography. Thus the stability of a small but finite-amplitude topographically forced Charney mode was considered. The results differ somewhat from these of section 2 but again topography is found to stabilize the baroclinic instability except for the very large meridional scales. The additional wave-zonal flow feedback due to the presence of the topographically forced Charney mode is responsible for the differences between section 2 and 3. This feedback depends on the direction of the uniform zonal flow. If the uniform zonal flow at the ground is eastward, this additional feedback makes topography more stabilizing than in section 2 but if the flow at the ground is westward the topography is less stabilizing than in section 2. Thus the essence of sections 2 and 3 is that topography stabilizes all but the largest meridional scales of the Charney mode when the flow at the ground is very weak.

In both sections 4 and 5 the uniform zonal flow at the ground is allowed to be relatively larger; in fact the advection time scale associated with it is the same as the e-folding time scale of the disturbance. The results differ substantially from those of the earlier sections. In section 4 the propagating baroclinic instability is stabilized by the presence of topography. This is especially true for waves of small meridional scale. The stabilization is less effective for waves with small zonal scale. In addition, a topographically induced stationary instability is found which is most easily excited near resonance. The resonance corresponds to the point at which a subcritical neutral mode becomes stationary relative to the topography. The instability whose source is the zonal available potential energy occurs as the disturbance becomes stationary relative to the forced wave which allows a strong coherent interaction between the perturbation, forced wave and mean flow. The topographic instability is limited to superresonant flow and occurs for flow which is either baroclinically subcritical or supercritical. In wavenumber space the topographic instability is limited to very narrow bands in meridional wavenumber centered at moderate and low values of meridional wavenumber which extend from large zonal scales to moderate zonal scales. The the topographic instability found by Plumb (1979) in a two-layer baroclinic model, marked by a strong wave-mean flow interaction, does have a counterpart in the continuous baroclinic Charney model.

In section 5 the effect of O(1) Ekman friction on the topographically forced Charney mode is studied and consequently the effect of Ekman friction on the topographic instability found in section 4 is examined. The results of the analysis differ according to the presence of an eastward or westward zonal flow at the ground. For eastward flow the baroclinic instability is again stabilized by topography. The stabilization is least effective for large zonal or large meridional scales. The stationary instability is topographic in nature and is most easily excited in near resonant flow. In addition, the flow must be superresonant in order for this instability to develop. If the friction is large to begin with and it is increased the topographic instability is harder to excite and confined closer to resonance. This is in contrast to the propagating baroclinic instability which fills more of wavenumber space as friction is increased. This is due to the nature of the Charney mode under the influence of O(1) Ekman friction which changes the vertical phase shift so as to draw energy from the mean flow. As Ekman friction is increased this phase shift becomes more destabilizing. In wavenumber space the topographic instability is confined to very narrow bands in meridional wavenumber but not limited in zonal wavenumber.

The story is quite different when the flow is westward at the ground. In this case there is no instability for baroclinically subcritical flow. The propagating baroclinic instability is again stabilized. The stationary instability occurs for both sub- and superresonant flow and is not topographic in nature but rather baroclinic. The topography only serves to make it a stationary instability as opposed to a propagating one.

Overall the inviscid work of this chapter shows that the topography stabilizes what would otherwise be a baroclinically growing instability except for small zonal or large meridional scales where the stabilization is less effective. Furthermore, there is a new topographic instability of the Charney mode for superresonant flow most easily excited near resonance and similar to that found for the normal modes of the Charney problem. It is limited to narrow bands in meridional wavenumber space and most easily found for larger zonal scales. Thus, on this basis, it is clear the topographic instability and baroclinic instability of the Charney mode are separate phenomena and therefore may be observable. However with O(1) Ekman friction the topographic instability becomes more scarce as compared to the baroclinic instability. This makes less clear, the observability of the topographic instability of the Charney mode in the atmosphere.

Chapter 5

Overall Conclusions

In this thesis I have considered three separate problems concerning the instability of a topographically forced wave. The topographically forced wave considered in each problem is of a different type.

In Chapter 2 the amplitude of the stationary, barotropic, forced, plane wave is O(1). As a result, a traveling instability which develops upon this wave can produce a spectrum of waves which interact with the forced wave and amongst themselves as the instability grows to finite amplitude. The assumptions of weak nonlinearity and weak growth have been made to make the problem analytically tractable. This study provides a self-consistent model for the interaction of the barotropic components of the planetary scale stationary and transient waves in the atmosphere. The study suggests in the atmosphere the possibility of an explosive instability when the zonal flow is superresonant and an oscillation between the planetary scale stationary and transient eddies due to the convergence of Reynolds stresses for subresonant flow. However, there are a number of assumptions made which make difficult the direct comparison of the model with the atmosphere. First, on the long time scale required by the theory, frictional effects become important in the atmosphere. As in other weakly nonlinear stability studies the introduction of small friction may lead to chaotic behavior in time associated with multiple quasi-steady states (e.g. Pedlosky, 1971) and O(1) friction may lead to finite-amplitude steady state equilibration (Pedlosky, 1970). Second, the neglect of baroclinicity is an important limiting factor in the application of the model to the atmosphere. A baroclinic model would be capable of modeling the baroclinic nature of some planetary scale transient eddies (e.g. Bottger and Fraedrich, 1980) and the planetary scale stationary eddies (e.g. Holopainen, 1970) in the atmosphere. Perhaps more importantly, a baroclinic model can model the existence of the cyclone scale eddies and the placement of the cyclone scale eddies relative to the planetary scale stationary eddies (e.g. Neihaus, 1980; Fredriksen, 1979). It is reasonable to expect the relative location of the cyclones to the stationary wave to be important in modeling the interaction between the stationary and transient waves in the atmosphere. Based on some preliminary calculations, it seems a two-layer baroclinic analysis might be carried through in much the same way as the barotropic one in Chapter 2. The details of this calculation are left for future study.

Yet another important restriction has been placed on the problem, that of weak nonlinearity. It seems reasonable to expect that the results of removing this restriction would depend on the degree of truncation employed. The strong nonlinear behavior of the model within the context of the severe truncations employed in Chapter 2 could be examined by numerically solving an ad hoc truncated version of the nonlinear stability problem. In this way the amplitude of each wave in the perturbation is no longer linearly related to each other and it might be determined if the explosive nonlinear instability remains when the weakly nonlinear assumption is removed but the truncation remains severe. In addition, if the severity of the truncation were lessened in this ad hoc truncated model, the cascading of energy in wavenumber space also becomes possible. This effect might also kill the explosive nonlinear instability.

The resolution of the survivability of the explosive nonlinear instability is important since such a strong instability would likely dominate other instabilities in the atmosphere. To determine whether or not the nonlinear instability occurs when the flow is superresonant in the barotropic model, requires the calculation of the feedback between the instability and its harmonics. It is possible that such a feedback is large and stabilizing, hence eliminating the nonlinear instability except near resonance for which the analysis is invalid. However, the calculation depends on the truncation and hence is postponed until the convergence of the entire nonlinear coefficient can be examined.

We might also ask if the instability can still play a role in a baroclinic model. I would expect this to be the case at least in a two-layer baroclinic model. However, it is clear that although the initial tendency of the explosive instability may be meaningful, its long term behavior is not. Thus, it seems likely that an important physical process, as yet undetermined, has been neglected. Above, I have suggested that the assumption of weak nonlinearity or truncation may be responsible. However, there are two other possibilities which might be examined within the context of a weakly nonlinear model. The first possibility is the inclusion of a small amplitude propagating wave with the spatial structure of the topography. This may produce the feedback necessary to stabilize the nonlinear instability. Another possibility (perhaps in combination with the previous one) is to confine the fluid to a channel. This would produce additional nonlinear interaction which would force a correction of the zonal flow. The interaction of this zonal flow with the topographically forced wave could be stabilizing.

In the remainder of this thesis I investigated the influence of topography on the various types of modes possible in a continuously stratified, constant shear model, specifically the Charney model. The first type of modes studied are the normal modes which are baroclinically neutral to the zonal flow. A given normal mode becomes stationary relative to the topography for a value of the westerly zonal current which depends on which normal mode is chosen. The stationary normal mode then can resonate with the topography which leads to a significant wave mean flow interaction. The simplifications which occur when the normal mode is nearly stationary are exploited to study the wave-mean flow interaction. The second type of mode studied is the Charney mode which is unstable to the zonal flow. However, it becomes neutral on a line in parameter space and simultaneously stationary if the zonal flow at the ground is zero. In this case, the neutral mode can interact with the topography to produce a significant wave-mean flow interaction. The simplifications which occur when the flow is assumed to be weakly unstable and nearly stationary are exploited to study this wave-mean flow interaction.

In Chapter 3 I have extended the study of topographic instability to a continuously stratified baroclinic model to examine the topographic instability of a normal mode of the Charney problem. The flow is assumed to be weakly nonlinear and near resonance, which allows the method of multiple scales to be used to obtain the evolution equations without the use of truncation. As in earlier studies of topographic instability, the instability is more easily excited near resonance but draws its energy from both the available potential energy of the mean flow and from the zonal momentum via form drag. In a manner similar to the barotropic study of topographic instability by Pedlosky (1981), the instability can occur for either subresonant or superresonant flow and would develop with a high centered over the topography which would move upstream as the quasi-steady state was approached. However, the regions of wavenumber space for which the topographic instability of the normal mode of the Charney problem is superresonant or subresonant depend strongly on the meridional scale and only very weakly on the zonal scale. Therefore, in contrast to the simpler earlier models, the instability is limited to somewhat narrow bands in meridional wavenumber and is more easily excited for planetary zonal scales. This suggests that the stationary topographic instability may be observable in the atmosphere at planetary zonal scales with a fairly well defined meridional structure but less well defined zonal structure. The instability would have the additional feature of the development of highs initially centered over the peaks of the topography which would move upstream as the quasi-steady state is approached. Then, if as Charney and Devore (1979) first suggested, topographic instability causes the transition in the atmosphere from the zonal state to the blocking state, blocking would propagate upstream initially, involve planetary zonal scales and the meridional structure of blocking would have better definition than its zonal structure. Indeed this seems to be the case. Blocks tend to propagate upstream initially and have planetary zonal scales and a meridional structure which consists of a split jet. However, I remain somewhat baffled as to why the instability is so selective meridionally. It is possible the meridional walls of the channel are artifically playing a role in the meridional selectivity. This seems somewhat unlikely due to the absence of such selectivity in the barotropic channel model study of topographic instability by Pedlosky (1981). Nevertheless the question of the role of the channel geometry requires further investigation. To this end a similar study might be done with spherical geometry. This would also be helpful in another respect. The topographic instability is more easily excited at planetary scales for which spherical effects can be important. Also at planetary scales quasigeostrophic theory begins to break down. Therefore, the inclusion of at least the additional stratification term in the thermodynamic equation which can be important at planetary scales as discussed by White (1977) would be another worthwhile extension of the present theory.

There are two other questions concerning the applicability of the analysis to the atmosphere. In the analysis the flow was assumed to be near global resonance with a single component of the topography. Such resonance is likely at the planetary scales in the atmosphere as long as the wave can't propagate meridionally to any significant extent. However, a resonant response to small scale localized topography is less likely especially in the presence of friction which would damp the response as it moved away from the topography. Therefore, the analysis is only applicable to the large atmospheric scales if the wave is confined by a meridional waveguide. The response of the atmosphere to localized topography involving a "local resonance" remains an open question.

Another assumption made in the course of the analysis was that of weak nonlinearity. As far as the topographic instability is concerned, this is not a serious restriction. However, the assumption limits the interaction of the topographically unstable wave with other waves. Those waves which form a resonant triad with the topographically unstable wave will interact most easily with it. A study of this interaction would be an interesting future study after which the removal of the weakly nonlinear assumption would make sense.

In Chapter 4, the instability of a small but finite-amplitude stationary topographically forced Charney mode embedded in a mean shear flow is examined. The analysis hinges on the neutral stability curve and weak nonlinearity is assumed. With these assumptions the evolution equations for the zonal flow are obtained using the method of multiple time scales. In studying this problem, we seek to find out whether the presence of topography stabilizes or destabilizes the baroclinic instability and if the presence of topography leads to any new instability.

In the second section, the problem treated is somewhat exceptional since the zonal flow at the ground is too weak to force a wave. The effect of topography on the infinitesimal growing baroclinic wave is studied and found to be strongly stabilizing except for large meridional scales. This stabilization is due to the form drag which develops as the wave grows and which alters the mean shear near the ground thus stabilizing the instability.

In the subsequent sections, a uniform zonal flow is present which by its diversion by the topography produces a small but finite-amplitude topographally forced Charney mode near neutrality. Its instability in the presence of the uniform vertically sheared flow is studied. The instability feeds on the available potential energy of the shear flow. Again the effect of topography is stabilizing on the baroclinic instability. This makes less likely the possibility that baroclinic instability leading to a growing Charney mode is the source of the planetary scale transient waves as was suggested by Hartman (1979). Furthermore, it seems likely that this would also be the effect of topography on the Green modes which has also been proposed as the source of the planetary scale transient waves. However, the calculation concerning the effect of topography on the Green modes still remains to be carried out.

A new stationary instability develops even in parameter ranges which are baroclinically stable characterized by a strong wave-mean flow interaction. This strong wave-mean flow interaction is due to the disturbance becoming stationary relative to the forced wave, which allows a strong interaction with it. It is most easily excited near resonance and the flow must be superresonant for it to occur. Thus the instability is reminiscent of the topographic instability of the normal mode. As a result, we refer to it as the topographic instability of the Charney mode. This instability is also limited to narrow bands in meridional wavenumber and is more easily excited at larger zonal scales as is the topographic instability of the normal modes of the Charney problem. Therefore, the topographic instability of the Charney mode may be responsible for the existence of the baroclinic component of the planetary scale stationary eddies in the atmosphere. This might be tested by examining if the planetary scale stationary eddies have a better defined meridional structure than zonal structure. Knowledge of the position of the finite-amplitude stationary wave relative to the topography would also be useful information for an atmospheric test of the theory. Unfortunately, this involves the technically difficult search for the finite-amplitude quasi-steady states of the model. Due to its difficulty, this is left for future consideration.

The presence of O(1) Ekman friction reduces the growth rate of the topographic instability and if Ekman friction becomes strong enough the topographic instability disappears. This casts some doubt on its realizability in the atmosphere. In addition, there are several features of the analysis which make the applicability of the analysis to the atmosphere questionable. The analysis pivots about the neutral point of the Charney mode. Also in the absence of a zonal flow at the ground this neutral Charney mode is stationary which allows it to resonate with the topography.

Therefore the interaction between the topography and the wave is strong. The effect of topography on the barolcinic instability and the existence of topographic instability of the Charney mode away from the neutral curve is unclear. Compounding this problem is that for more realistic vertical profiles of the zonal wind there may be no neutral points (e.g. Geisler and Garcia, 1977) and even if there are neutral points, the waves may not be stationary. Therefore this problem requires further investigation before the actual effect of planetary scale topography on the transient and stationary planetary scale eddies in the atmosphere can be ascertained. Nevertheless, if in the atmosphere there are no neutral modes, it seems physically reasonable to expect the topography to most readily influence those baroclinic waves which travel the slowest relative to the topography.

Based on the various problems considered in this thesis it is clear topography has a profound influence on atmospheric flow. The conclusions are as follows: First, in the atmosphere the interaction between the planetary scale topographically forced stationary eddies and transient eddies which may develop as an instability on the stationary waves is fundamentally different than the interaction between a free stationary wave and the traveling instability which develops on it. In fact the interaction between the topographically forced wave and its instability may lead to an explosive growth for superresonant flow.

Second, planetary scale baroclinically stable modes in the atmosphere become unstable in the presence of topography for selective meridional scales. The selectivity in meridional scale may explain why blocking has a better defined meridional structure than zonal structure.

Finally, propagating planetary scale waves which are baroclinically unstable are stabilized by the presence of topography. Thus, it seems somewhat unlikely that the source of the planetary scale transient eddies is the baroclinic instability of the zonal flow. However, the presence of topography leads to a planetary scale stationary instability which feeds on available potential energy of the flow. This instability may provide a source for the planetary scale transient eddies in the atmosphere.

On the whole the effect of the planetary scale topography on the atmosphere is to act as a catalyst for the removal of energy from the zonal available potential energy and/or the zonal momentum.

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