XVII. NONLINEAR CIRCUITS

Prof. E. A. Guillemin
Prof. S. J. Mason
Prof. H. J. Zimmermann
J. Gross
T. E. Stern
R. D. Thornton

A. PIECEWISE-LINEAR NETWORK ANALYSIS

Investigation of the algebra of inequalities, described in the Quarterly Progress Report of January 15, 1955, page 108, was pursued further with a view to discovering its limitations, making it more generally applicable, and increasing the facility with which it can be handled. It appears that one of the basic assets of this algebra is its ability to deal with piecewise-linear functions in which one or more parameters are given in literal form only. To demonstrate this, an example is presented here in which we want to determine the behavior of the circuit of Fig. XVII-1: the parallel combination of a capacitor, an inductor, and a voltage-controlled negative resistance N whose characteristic is shown in Fig. XVII-2. G_c and R_L represent losses associated with the capacitor and inductor. A phase-plane representation in which the two coordinates are the inductor current and the capacitor voltage is used. The algebraic symbolism is employed to obtain expressions for the isoclines. Although the circuit is quite simple, the fact that both energy storage elements are dissipative makes the graphical method of Liénard (1) inapplicable; therefore, some method for the calculation of isoclines is necessary.

An important theorem, which was not presented before, that will be used throughout the following discussion is the implicit equation theorem.

Theorem: Given the implicit equation

$$F(x, y) = [f_1(x, y), f_2(x, y), \ldots, f_n(x, y)]\, \phi^{\pm} = 0$$

in which

1. Each f_p is monotonically increasing (decreasing) and continuous in y for any constant x.
2. For each f_p, there is some point, (x_o, y_o), that yields $f_p(x_o, y_o) = 0$.
3. Each f_p is continuous in x.

Then $F(x, y) = 0$ can be solved explicitly for y in the form,

$$y = G(x) = [g_1(x), g_2(x), \ldots, g_n(x)]\, \phi^{\mp(\pm)}$$

where $y = g_p(x)$ is the explicit solution of the equation, $f_p(x, y) = 0$.

Figure XVII-3 shows the circuit rearranged to include all of the resistive elements in a two terminal-pair unit with current and voltage reference directions as shown.

The admittance function of the piecewise-linear branch is

$$i_N = [(e_2,\ 1 - e_2)\, \phi^+,\ 2e_2 + 2]\, \phi^- = g(e_2) \tag{1}$$

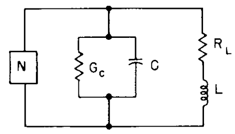

Fig XVII-1. Negative-resistance oscillator.

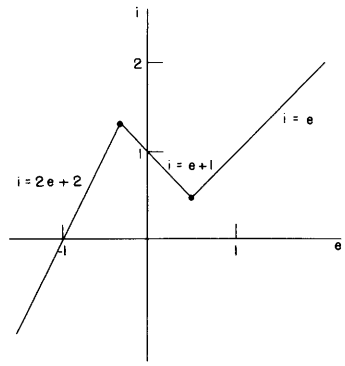

Fig. XVII-2. Negative-resistance characteristic.

and the relations defining the phase-plane trajectories are

$$e_1 = R_L i_1 + e_2 \tag{2}$$

$$i_2 = i_N - i_1 + G_c e_2 = g(e_2) - i_1 + G_c e_2 \tag{3}$$

where

$$e_1 = -L \frac{di_1}{dt} \tag{4}$$

and

$$i_2 = -C \frac{de_2}{dt} \tag{5}$$

Substituting Eq. 4 in Eq. 2, and Eqs. 5 and 1 in Eq. 3, and then dividing Eq. 2 by Eq. 3, we obtain the isocline equation

$$\frac{di_1}{de_2} = \frac{C}{L} \cdot \frac{R_L i_1 + e_2}{[(e_2, 1 - e_2)\,\phi^+,\ 2e_2 + 2]\,\phi^- - i_1 + G_c e_2} = M \tag{6}$$

where the parameter M is the particular trajectory derivative associated with each isocline. Multiplying Eq. 6 by the denominator of the fraction and combining terms, we can put it into the form

$$\left\{\left[\left(1 - \frac{C}{LM} + G_c\right) e_2 - \left(\frac{R_L C}{LM} + 1\right) i_1,\ 1 - \left(1 + \frac{C}{LM} - G_c\right) e_2 - \left(\frac{R_L C}{M} + 1\right) i_1\right] \phi^+,\right.$$
$$\left.\left(2 - \frac{C}{LM} + G_c\right) e_2 + 2 - \left(\frac{R_L C}{LM} + 1\right) i_1\right\} \phi^- = 0 \tag{7}$$

If we let the complete term, $[(R_L C/M) + 1]\, i_1$, represent the variable for which this equation is to be solved, the implicit equation theorem can be applied twice to Eq. 7 to yield the explicit isocline equation

$$i_1 = \frac{LM}{R_L C + LM} \left\{\left[\left(1 - \frac{C}{LM} + G_c\right) e_2,\ 1 - \left(1 + \frac{C}{LM} - G_c\right) e_2\right] \phi^+,\ 2 + \left(2 - \frac{C}{LM} + G_c\right) e_2\right\} \phi^- \tag{8}$$

Substituting various values of M in Eq. 8 enables us to plot the family of isoclines. However, it is interesting to observe, first, some of the significant features of this family. In the following discussion it will be assumed that G_c and R_L are much less than unity; that is, the dissipation is a small effect compared with the power associated with the piecewise-linear branch.

Singular Points. To find singular points we set the numerator and denominator of Eq. 6 simultaneously equal to zero. From the numerator,

$$R_L i_1 + e_2 = 0$$

$$i_1 = -\frac{e_2}{R_L}$$

From the denominator, setting $i_1 = -\dfrac{e_2}{R_L}$, we obtain

$$\left\{\left[\left(1 + \frac{1}{R_L} + G_c\right) e_2,\ 1 - \left(1 - \frac{1}{R_L} - G_c\right) e_2\right] \phi^+, \left(2 + \frac{1}{R_L} + G_c\right) e_2 + 2\right\} \phi^- = 0$$

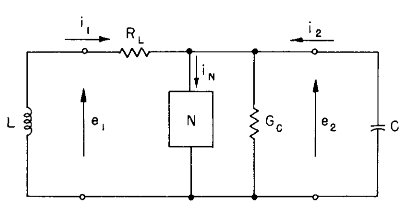

Fig. XVII-3. Rearranged circuit of Fig. XVII-1.

Solving for e_2, we have

$$e_2 = \left[\left(0, \frac{1}{1 - \frac{1}{R_L} - G_c} \right) \phi^-, \frac{-2}{2 + \frac{1}{R_L} + G_c} \right] \phi^+ \approx -R_L$$

so that

$$i_1 = \frac{1}{R_L G_c + 1 - R_L} \approx 1 + R_L$$

This is the only point at which the isoclines can cross.

Significant Isoclines. From Eq. 8,

1. $M = \infty$

$$i_1 = \left\{ [(1 + G_c)\, e_2,\ 1 - (1 - G_c)\, e_2]\, \phi^+,\ 2 + (2 + G_c)\, e_2 \right\} \phi^-$$

2. $M = 0$

$$i_1 = \frac{1}{R_L C} \left\{ [(-C)\, e_2,\ (-C)\, e_2]\, \phi^+,\ (-C)\, e_2 \right\} \phi^- = -\frac{e_2}{R_L}$$

3. $M \to -\frac{R_L C}{L}$

$$i_1 \approx \infty \left\{ \left[\frac{e_2}{R_L},\ 1 + \frac{e_2}{R_L} \right] \phi^+,\ 2 - \frac{e_2}{R_L} \right\} \phi^- = \infty \left(1 + \frac{e_2}{R_L} \right)$$

Breakpoints. Equating the first and second elements of Eq. 8, we obtain

$$e_2 \left(1 - \frac{C}{LM} + G_c \right) = 1 - \left(1 + \frac{C}{LM} - G_c \right) e_2$$

$$e_2 = \frac{1}{2}$$

which is the abscissa of one of the breakpoints of the isoclines. Likewise, equating the second and third elements of Eq. 8, we obtain

$$1 - \left(1 + \frac{C}{LM} - G_c \right) e_2 = 2 + \left(2 - \frac{C}{LM} + G_c \right) e_2$$

$$e_2 = -\frac{1}{3}$$

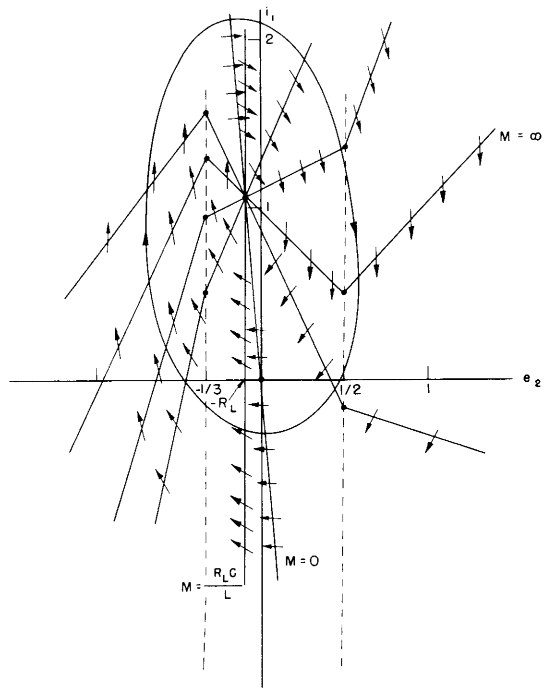

Fig. XVII-4. Phase-plane trajectory.

which is the abscissa of the other breakpoint. Note that these values are independent of M, a fact that could have been predicted by inspection of the original circuit.

It can be observed from Eq. 8 that, if the portions of the isoclines to the left of $e_2 = -1/3$ (determined by the third element in the equation) are extended, they intercept the i_1-axis at $i_1 = 2$. Similarly, the portions to the right of $e_2 = 1/2$ would intercept the i_1-axis at the origin.

This collection of significant features considerably simplifies the sketching of the isoclines. The isoclines are shown in Fig. XVII-4, together with the limit cycle for $(R_L C)/L = 1/2$

T. E. Stern

References

1. J. J. Stoker, Nonlinear Vibrations in Mechanical and Electrical Systems (Interscience Publishers, Inc., New York, 1950), pp. 31-36.