OPTIMAL STRATEGIES IN WAITING FOR COMMON BUS LINES

by

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Ecole Nationale des Ponts et Chaussées
(1980)

SUBMITTED TO THE DEPARTMENT OF
CIVIL ENGINEERING IN PARTIAL
FULFILLMENT OF THE
REQUIREMENTS FOR THE
DEGREE OF

MASTER OF SCIENCE IN
TRANSPORTATION

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1981

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ABSTRACT

In many cities, bus routes sometimes share common sections. When several bus routes share a common section between two points A and B, the passenger willing to go from A to B is faced by the problem of choosing which bus line(s) to take. Because some routes might have a very long travel time between A and B, he may disregard them. We assume that the passenger will choose an *optimal strategy*: he/she will select a subset among the set of all routes passing through A and B so as to minimize his total expected travel time (expected waiting time + expected in-vehicle travel time). Once this subset is selected, the passenger will take the first vehicle serving one of the routes in this subset, arriving at A.

The purpose of this thesis is to study the problem of the passenger route-choice decision in a probabilistic framework: on each bus line, the headway has a certain known distribution and the passenger arrives at a random time at the bus stop, so that the time for the first bus of each line to arrive at A is also a random variable. Because we show that only the expected value of in-vehicle travel time between A and B for each line matters, in-vehicle travel times are considered deterministic.

First, we review the problem of choosing a subset and show that the shortest travel time route is always included in the optimal strategy both for the case of the passenger who has just arrived and for the case of the passenger who has already waited for a while. We review the Chriqui and Robillard heuristic solution for the optimal strategy, find conditions under which it fails and provide counterexamples. The failure of this heuristic in the general case prompts the performance of an extensive investigation of the case of three bus lines. We demonstrate that many properties which we might expect to be true are indeed valid in the cases of negative exponential and deterministic headways. But they appear to be false in the most general case; there is at least one class of distributions for which these properties fail to hold. There seem to be very few general statements that can be made regarding optimal strategies unless the waiting time distributions are further constrained. The case of four bus lines also show how much more complicated the problem becomes as the number of routes increases.

The problem of the "clever" passenger, who has already waited for a certain time, and takes this time into account in making his route-choice decision is described subsequently. We discover that it is possible to find some distributions for which the remaining waiting time actually increases with the time the passenger has already waited.

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ACKNOWLEDGEMENTS

I would like to express my sincere appreciation particularly to Professor Amedeo Odoni for his significant contribution of his time and expertise, continual enthusiasm, critical comments, and helpful suggestions, which inspired many elements within this thesis.

I also extend my thanks and gratitude to Professor Nigel Wilson for his interest in this study.

Finally, I also acknowledge the considerable efforts and the patience of Susan Feuerbacher for her excellent typing work.

May 8, 1981

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Chapter 1

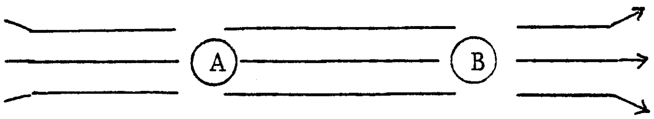
The Problem and Some Observations

1.1 Introduction

In large cities operating an extensive mass transit network, passengers are often faced with the problem of "common bus lines". Some routes share common route segments and passengers must select the buses and bus lines they will ride on.

Claude Chriqui and Pierre Robillard [1] have formulated this problem as an optimization problem, within a probabilistic context, which gives the optimal subset of routes "to be selected", based on "perceived" travel time.

As an example, suppose that several routes share a common section between points A and B:



A passenger wishing to travel from A to B has the choice of using just one of the bus lines connecting A and B or all bus lines, or more generally a subset of these bus lines. The purpose of this thesis is to study this problem within a probabilistic framework, taking into account the frequencies and the speeds of each competing line, in a fashion similar to the work of Chriqui and Robillard.

The central idea is that a passenger wishing to travel from A to B will not necessarily use just any one of the routes connecting A and B, but may disregard some of them which might have a very long travel time between A and B.

We assume that the passenger will make his decision regarding the choice of routes to use based on total travel time. His problem is to select a subset among the set of all routes passing through A and B so as to minimize his total travel time, made up of his expected waiting time and his expected in-vehicle travel time.

Once this subset is selected, the passenger will take the first vehicle serving one of the routes in this subset to arrive at A.

Three basic assumptions are used to solve this problem:

i) The routes are statistically independent.

ii) The passenger arrives at the bus stop in A at a random time and independently of the route schedules.

iii) However, the passenger knows the frequency (i.e. the distribution of the headway) and the in-vehicle travel time between A and B for all routes, so that he can make "intelligent" decisions regarding the choice of routes to use.

In this thesis, a heuristic solution of the problem, proposed by Chriqui and Robillard will be discussed and contradicted. Then, in the search for a general solution to the problem, the different strategies available will be compared, first in particular cases of waiting times distributions and then in the general case.

An extension of the problem is then developed, involving the case of the "clever" passenger. The "clever" passenger, looking at the time he/she has already waited at the bus-stop in A, will make his/her decision depending on this time and may change his/her choice as this time increases. Thus the optimal strategy of the clever passenger may change with the passage of time. The Chriqui and Robillard problem is, in fact, a special case of this new one: it is the case where the passenger has just arrived at the bus stop in A (the time he/she has already waited is zero).

1.2 Notations

Routes are numbered according to the in-vehicle travel time between A and B, in the order of increasing in-vehicle travel time. If there are n routes passing through A and B, route number 1 is the one with the shortest in-vehicle time, route number n is the one with the longest.

Strategies are represented by choice-vectors of length n (number of routes) filled with 0's and 1's. A 1 in the i^{th} position means that route i is included and used in the strategy, a 0 means that route i is not selected.

Example:

(1,1, . . .,1): all routes are chosen

(1,0, . . .,0): the passenger will only take a route number 1 bus.

Functions and Parameters:

t_r: in-vehicle travel time on route r (assumed deterministic); $t_1 < t_2 < \ldots < t_n$

$f_r(t)$: probability density function of the headway on route r at A.

$F_r(t)$: cumulative distribution of this headway.

$p_r(t)$: probability density function of the waiting time for the first bus of route r to arrive at A.

$P_r(t)$: cumulative distribution of this waiting time.

$\overline{P}_r(t) = 1 - P_r(t)$.

$X = (. \; . \; . \; . \; . \;)$: a choice-vector (strategy)

$p_x(t)$: probability density function of the waiting time for the first bus from the subset of routes represented by X, to arrive at A.

$P_x(t)$: cumulative distribution of this waiting time.

$\overline{P}_x(t) = 1 - P_x(t)$.

w_r: waiting time for the first bus of route r to arrive at A (random variable).

w_x: waiting time for the first bus from the subset of routes represented by x, to arrive at A, (random variable).

$H_r(x)$: probability that among the subset of routes represented by x, a bus of route r will arrive first.

t_x: in-vehicle time with the strategy represented by x.

E( ): expectation of the random variable used in argument, (discrete random variable).

T_x: Expected total time with the strategy represented by x.

The optimal stategy is represented by the choice vector x minimizing T_x over all possible strategies.

$\rightarrow$: arrow used in logical relationships. (P) $\rightarrow$ (Q) means "P implies Q", i.e. proposition Q is right whenever proposition P is right. The $\rightarrow$ symbol is transitive: it is legitimate to write (P) $\rightarrow$ (Q) $\rightarrow$ (R) . . .

$\leftrightarrow$: logical equivalence. (P) $\leftrightarrow$ (Q) means that we have both (P) $\rightarrow$ (Q) and (Q) $\rightarrow$ (P)

1.3 Derivation of the Expected Total Travel Time

Since the passenger is supposed to arrive at the bus stop at A, at a random time, we have a random incidence process and we can apply a formula derived by R.C. Larson and A.R. Odoni [2]. The probability density function for the waiting time for the first bus of route r to arrive is given as a function of the headway of route r by:

$$p_r(t) = \frac{1 - F_r(t)}{E(\text{headway})}$$

where $F_r(t)$ is the cumulative distribution of the headway. The expected value of the headway is given by $\int_0^\infty f_r(u)du$ which after integrating by parts equals $\int_0^\infty (1 - F_r(u))du$. Therefore, the formula for $p_r(t)$ is

$$p_r(t) = \frac{1 - F_r(t)}{\int_0^\infty (1 - F_r(u))\,du} \qquad (1.1)$$

This is the expression given by Chriqui and Robillard in their paper. According to this equation, since $F_r(t)$ is obviously an increasing function of t, $p_r(t)$ must be a decreasing function of t.

Furthermore, any decreasing function defined on $\mathbb{R}^+$ and, integrating to 1 can be expressed in the form (1.1) with $F_r(t)$ an increasing function such that $F_r(o) = 0$ and $\lim_{t \to +\infty} F_r(t) = 1$, and therefore can be the probability density function of a waiting time. Since p_r is decreasing, P_r is concave-shaped and $\overline{P}_r$ is convex-shaped. ($P_r(t) = \int_0^t p_r(u)du$: p_r is the derivative of P_r; $\overline{P}_r(t) = 1 - P_r(t)$).

The derivation of the probability density functions of the waiting times for all routes constitutes an initial step of the derivation of the expected total travel time for a strategy defined by a choice-vector X. Chriqui and Robillard carried out this derivation in their paper and it is summarized here.

The expected waiting time for a bus of route r is

$$E(w_r] = \int_0^\infty \bar{P}_r(t)\,dt \qquad (1.2)$$

This expression is obtained by integrating $\int_0^\infty tp_r(t)dt$ by parts.

The waiting time for a bus from the subset of routes defined by X is

$$w_X = \min_{r \in X} w_r \qquad (1.3)$$

The probability that w_x is greater than t is therefore the probability that all w_r's are greater than t which is

$$\bar{P}_X(t) = \prod_{r\in X} \bar{P}_r(t) \qquad (1.4)$$

Since the routes are statistically independent (i.e. w_r and $w_{r'}$ are independent random variables for $r \neq r'$).

Thus, the expected waiting time for a bus included in X is

$$E(w_X) = \int_0^\infty \bar{P}_X(t)\,dt = \int_0^\infty \prod_{r\in X} \bar{P}_r(t)\,dt \qquad (1.5)$$

The probability that among the subset of routes represented by X, a bus route r will arrive first is given by

$$H_r(X) = \text{prob}(w_r = w_X) = \int_0^\infty p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\,dt \qquad (1.6)$$

Then, the expected in-vehicle time for a bus from the subset of routes defined by X is

$$E(t_X) = \sum_{r\in X} t_r H_r(X) = \int_0^\infty \sum_{r\in X} t_r p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\,dt \qquad (1.7)$$

Hence, the expected total travel time for the passenger choosing strategy X is given by

$$T_X = E(w_X) + E(t_X) = \int_0^\infty \Big[\prod_{r\in X} \bar{P}_r(t)\,dt + \sum_{r\in X} t_r p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\Big]dt \qquad (1.8)$$

1.4 A Simple Property

An interesting property, useful and intuitively understandable follows immediately from this expression of the expected total travel time. The passenger always includes route number 1, the route with the shortest in-vehicle travel time between A and B, in his/her selected routes; route number 1 must necessarily be on element of the subset of routes associated with the optimal strategy.

Furthermore, as we will see in the "clever" passenger study, this property remains true, no matter how long the passenger has already waited.

Let X be a strategy not including route number 1 and $X' = X +$ (1, 0....0) i.e. the strategy including the same routes as X plus route number 1.

We can prove that $w_{X'} \leq w_X$ (1.9)

and $E(t_{X'}) < E(t_X)$ (1.10)

Proof:

- In-vehicle time

From equation (1.6), we can write

$$H_r(X) = \int_0^\infty p_r(t) \prod_{\substack{i \in X \\ i \neq r}} \bar{P}_i(t)\,dt \qquad (1.11)$$

$$H_r(X') = \int_0^\infty p_r(t)\, \bar{P}_1(t) \prod_{\substack{i \in X \\ i \neq r}} \bar{P}_i(t)\,dt \qquad \text{for } r \neq 1 \qquad (1.12)$$

$$H_1(X') = \int_0^\infty p_1(t) \prod_{i \in X} \bar{P}_i(t)\,dt = \int_0^\infty p_1(t)\, \bar{P}_X(t)\,dt \qquad (1.13)$$

Using (1.7), we deduce

$$E(t_X)= \sum_{r\in X} t_r H_r(X) = \int_0^\infty \sum_{r\in X} t_r p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\,dt \tag{1.14}$$

$$E(t_{X'}) = t_1 H_1(X') + \sum_{r\in X} t_r H_r(X') = \int_0^\infty t_1 p_1(t)\bar{P}_X(t)\,dt + \int_0^\infty t_r p_r(t)\bar{P}_1(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\,dt \tag{1.15}$$

We have

$$\int_0^\infty p_1(t)\bar{P}_X(t)\,dt = \int_0^\infty p_X(t)\bar{P}_1(t)\,dt \qquad \text{(integrating by parts)}$$

$$= \int_0^\infty \sum_{r\in X} p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\bar{P}_1(t)\,dt \tag{1.16}$$

because $p_X(t)$ is the derivative of $-\bar{P}_X(t)$ and $\bar{P}_X(t) = \prod_{r\in X} \bar{P}_r(t)$ according to (1.14).

Then, since $t_1 < t_r$ for all $r \neq 1$, the following inequality holds:

$$\int_0^\infty t_1 p_1(t)\bar{P}_X(t)\,dt = \int_0^\infty \sum_{r\in X} t_1 p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t) P_1(t)\,dt < \int_0^\infty \sum_{r\in X} t_r p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\bar{P}_1(t)\,dt \tag{1.17}$$

Hence, we get

$$t_1 H_1(X') + \sum_{r\in X} t_r H_r(X') < \int_0^\infty \sum_{r\in X} t_r p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t) P_1(t)\,dt + \int_0^\infty \sum_{r\in X} t_r p_r(t)\bar{P}_1(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\,dt$$

$$= \int_0^\infty \sum_{r\in X} t_r p_r(t)\left(P_1(t) + \bar{P}_1(t)\right) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\,dt$$

$$= \int_0^\infty \sum_{r\in X} t_r p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\,dt \tag{1.18}$$

Because of expressions (1.14) and (1.15), this means that inequality (1.10) holds.

$$E(t_{X'}) < E(t_X) \tag{1.10}$$

- Waiting time

Using (1.3), we have

$$\bar{w}_X = \min_{r \in X} \bar{w}_r \tag{1.19}$$

and

$$\begin{aligned} \bar{w}_{X'} &= \min_{r \in X'} \bar{w}_r \\ &= \min_{\substack{r \in X \\ \text{and } r=1}} \bar{w}_r \\ &= \min\left[\min_{r \in X} \bar{w}_r, \bar{w}_1\right] \\ &= \min\left[\bar{w}_X, \bar{w}_1\right] \end{aligned} \tag{1.20}$$

This shows that inequality (1.9) holds too.

$$\bar{w}_{X'} \leqslant \bar{w}_X \tag{1.9}$$

- Total time

From those two inequalities (1.9) and (1.10), we get

$$T_{X'} = E(\bar{w}_{X'}) + E(t_{X'}) < T_X = E(\bar{w}_X) + E(t_X) \tag{1.21}$$

Strategy X', including route number 1 is better for the passenger than strategy X. This shows that for any strategy not including the quickest route, it is always possible to find a strategy which includes this route and which produces a shorter expected total travel time. Therefore, the optimal strategy includes the quickest route (route number 1) necessarily. (End of proof.)

Because of this property, all further work will deal only with (1, . . .) strategies.

1.5 Non-deterministic In-vehicle Travel Times

In-vehicle travel times have been assumed to be deterministic. If they are probabilistic, it can be shown that in fact we would only be interested in their expected values and that furthermore the expressions already derived are still valid after we replace the in-vehicle travel times by their expected values.

For that reason, the assumption of deterministic in-vehicle travel times is not at all restrictive, but is used only to simplify the terminology.

Proof:

t_r, in-vehicle time on route r, is here a random variable.

E(tr) is its expected value.

$H_r(X)$, the probability that a bus of line r arrives first among the subset of routes represented by X does not depend on the in-vehicle times. Therefore,

$$E(t_X) = \sum_{r \in X} H_r(X)\, E(t_r) \tag{1.22}$$

We take the expectation of the total travel times and the average over the routes available in the strategy.

Then, the expected in-vehicle time is, using equation (1.6),

$$E(t_X)=\int_0^\infty \sum_{r\in X} E(t_r)\, p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\, dt \tag{1.23}$$

and the expected total travel time is, adding the expected waiting time given by (1.5)

$$T_X=\int_0^\infty \Big[\prod_{r\in X} \bar{P}_r(t)+\sum_{r\in X} E(t_r)\, p_r(t) \prod_{\substack{i\in X\\ i\neq r}} \bar{P}_i(t)\Big]\, dt \tag{1.24}$$

The $E(t_r)$'s simply replace the t_r's in the previous formulae.

Chapter 2

C. Chriqui's and P. Robillard's Heuristic Solution

2.1 Algorithm

C. Chriqui and P. Robillard [1] proposed the following heuristic algorithm to solve the problem of the minimization of the expected total travel time T_x.

1. Let $\bar{x} = (1,0,\ldots.0)$

   $x = (1,1,0,\ldots.,0)$

2. Compare T_x with $T_{\bar{x}}$.

   If $T_x > T_{\bar{x}}$ (strategy $\bar{x}$ better than strategy x), the heuristic solution is $\bar{x}$.

   Otherwise, let $\bar{x} = (1,1,0\ldots0)$

   $x = (1,1,1,0..0)$ and compare $T_{\bar{x}}$ with T_x

3. Continue this way until $T_x > T_{\bar{x}}$, in which case, the heuristic solution is $\bar{x}$; or $\bar{x} = (1,1,1,\ldots,1)$ is attained and it is the heuristic solution.

The motivation for the heuristic is the following: it is logical for a passenger to let a bus of a given route go by and wait for a bus with longer in-vehicle time.

However, this statement is misleading. Let r_1, r_2 be two routes such that $t_{r_1} < t_{r_2}$. If the passenger wishes to use a third route, r_o, with a shorter in-vehicle time ($t_{r_o} < t_{r_1} < t_{r_2}$), it is possible to have a strategy which includes routes r_o and r_2 but does not include route r_1, be *better* than another strategy which includes all three routes. The reason essentially is that in letting a bus of route r_1 go by, it may be possible to catch a bus of route r_o, which has a shorter in-vehicle time.

If, however, the first bus to arrive in A, after the disregarded one, is a bus of route r_2, it may be worth taking it, at that time, rather than waiting for the following buses of routes r_o and r_1 that may take a long time to reach A.

For example, in the case of three routes, strategy (1,0,1) can be better than strategy (1,1,1). The following case, with negative exponentially distributed waiting times is an illustration of this. w_1, w_2, w_3, are negative exponentially distributed.

$$p_1(t) = ae^{-at} \qquad a > 0 \tag{2.1}$$

$$p_2(t) = be^{-bt} \qquad b > 0 \tag{2.2}$$

$$p_3(t) = ce^{-ct} \qquad c > 0 \tag{2.3}$$

a,b,c are the respective arrival rates for each route.

This leads to

$$T_{(1,0,1)} = \frac{1 + t_1 a + t_3 c}{a + c}^{*} = \frac{1 + t_1 a + t_3 c - t_2(a+c)}{a + c} + t_2$$

$$T_{(1,1,1)} = \frac{1 + t_1 a + t_2 b + t_3 c}{a + b + c}^{*} = \frac{1 + t_1 a + t_3 c - t_2(a+c)}{a + b + c} + t_2 .$$

Subtracting,

$$T_{(1,0,1)} - T_{(1,1,1)} = \left[1 + (t_1 - t_2)a + (t_3 - t_2)c\right]\left[\frac{1}{a+c} - \frac{1}{a+b+c}\right] \tag{2.4}$$

---

*Note: if the waiting times are negative exponentially distributed for all routes in the subset represented by x, we have:

$$T_x = \frac{1 + \sum_{r \in x} t_r . a_r}{\sum_{r \in x} a_r} \quad \text{when } p_r(t) = a_r e^{-a_r t} \quad [1]$$

This difference will be negative, and the strategy (1,0,1) better than the strategy (1,1,1) for

$$c = \frac{a}{2} \; ; \quad t_1 = \frac{1}{a} \; ; \quad t_2 = \frac{4}{a} \; ; \quad t_3 = \frac{6}{a} \tag{2.5}$$

Furthermore, it can be easily shown that (1,0,0) is the optimal strategy if the conditions given by (2.5) hold.

But, cases where, surprisingly, (1,0,1) is in fact the optimal strategy, will be produced too.

C. Chriqui and P. Robillard experimented with this heuristic solution for different examples in which their algorithm worked out well and gave the optimal solution. They report being unable to find any counterexamples. They found it easy to prove that whenever the arrival times are identically distributed or the travel times are equal, or when the arrival times are negative exponentially distributed, the heuristic solution is the optimal one. They conjectured that the heuristic always produces the optimal solution.

In fact, this heuristic does not always produce the optimal solution and counterexamples can be found.

2.2 Search for a Counterexample of the Algorithm

In the 3-route case, the algorithm says that if the (1,0,0) strategy is better than the (1,1,0) strategy, (1,0,0) should be the optimal strategy and thus (1,0,0) should be better than (1,0,1).

While trying to prove this result mathematically in the general case, it slowly became evident that this statement may not always be true. Finally, it was possible to show that this result is not necessarily true even in the case of uniformly distributed waiting times.

Let A = (1,0,0)

B = (1,1,0)

C = (1,0,1)

(1,0,0) better than (1,1,0) is written $T_B - T_A > 0$

(1,0,0) better than (1,0,1) is written $T_C - T_A > 0$

The logical relationship to be studied is

$$T_B - T_A > 0 \rightarrow T_C - T_A > 0 \qquad \text{(A)}$$

We have according to the general formula (1.8)

$$T_A = \int_0^\infty [\bar{P}_1(t) + t_1 p_1(t)]\, dt = \int_0^\infty \bar{P}_1(t)\, dt + t_1 \qquad (2.6)$$

$$T_B = \int_0^\infty [\bar{P}_1(t)\bar{P}_2(t) + t_1 p_1(t)\bar{P}_2(t) + t_2 p_2(t)\bar{P}_1(t)]\, dt \qquad (2.7)$$

$$T_C = \int_0^\infty [\bar{P}_1(t)\bar{P}_3(t) + t_1 p_1(t)\bar{P}_3(t) + t_3 p_3(t)\bar{P}_1(t)]\, dt \qquad (2.8)$$

Subtracting (2.6) from (2.7) and from (2.8),

$$T_B - T_A = \int_0^\infty [-\bar{P}_1(t)P_2(t) - t_1 p_1(t) P_2(t) + t_2 p_2(t)\bar{P}_1(t)]\, dt \qquad (2.9)$$

$$T_C - T_A = \int_0^\infty [-\bar{P}_1(t)P_3(t) - t_1 p_1(t) P_3(t) + t_3 p_3(t)\bar{P}_1(t)]\, dt \qquad (2.10)$$

Then

$$T_B - T_A = \int_0^\infty \left[-\bar{P}_1(t) P_2(t) + (t_2 - t_1) p_1(t) P_2(t) - t_2 p_1(t) P_2(t) + t_2 p_2(t) \bar{P}_1(t) \right] dt \tag{2.11}$$

$$T_C - T_A = \int_0^\infty \left[-\bar{P}_1(t) P_3(t) + (t_2 - t_1) p_1(t) P_3(t) + (t_3 - t_2) p_3(t) \bar{P}_1(t) - t_2 p_1(t) P_3(t) + t_2 p_3(t) \bar{P}_1(t) \right\} dt \tag{2.12}$$

Certain parts of both expressions can be integrated directly.

$$T_B - T_A = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] P_2(t) dt + t_2 \left[\bar{P}_1(t) \bar{P}_2(t) \right]_0^\infty \tag{2.13}$$

$$T_C - T_A = \int_0^\infty \left[\left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] P_3(t) + (t_3 - t_2) p_3(t) \bar{P}_1(t) \right] dt + t_2 \left[\bar{P}_1(t) P_3(t) \right]_0^\infty \tag{2.14}$$

This reduces to

$$T_B - T_A = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] P_2(t) \, dt \tag{2.15}$$

$$T_C - T_A = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] P_3(t) dt + (t_3 - t_2) \int_0^\infty p_3(t) \bar{P}_1(t) dt \tag{2.16}$$

It is interesting to have the expression of $T_B - T_A$ and $T_C - T_A$ in this form. With $t_3 - t_2$ being a positive real number which can be arbitrarily small, relationship (A) can only be true if the following relationship holds too.

$$\int_0^\infty \varphi(t)\cdot P_2(t)\,dt > 0 \quad \Rightarrow \quad \int_0^\infty \varphi(t)\cdot P_3(t)\,dt > 0 \qquad \text{(B)}$$

for $$\varphi(t) = k\,p_1(t) - \bar{P}_1(t) \qquad (2.17)$$

where $k = t_2 - t_1$ is given positive real number and p_1 a given *decreasing* probability density function. (See page 11)

When the problem is expressed this way, we get a strong indication that relationships (A) and (B) may be false, since φ, P_2, and P_3 are completely independent. Relationship (B) states that a certain property which is true with a particular function P_2 should be true with any function P_3.

The heuristic will not give an optimal result if we can find $(k=t_2-t_1;P_1)$, P_2 and P_3 such that

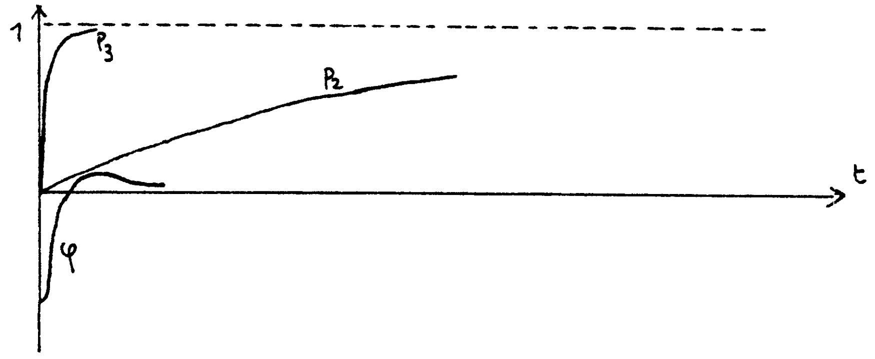
$$\int_0^\infty \varphi(t)\cdot P_2(t)\,dt > 0 \quad \text{and} \quad \int_0^\infty \varphi(t)\cdot P_3(t)\,dt < 0 \qquad (2.18)$$

and then we simply have to choose

$$t_3 - t_2 < \frac{-\int_0^\infty \varphi(t)\,P_3(t)\,dt}{\int_0^\infty \varphi(t)\,P_2(t)\,dt} \qquad (2.19)$$

to get $T_B - T_A > 0$ and $T_C - T_A < 0$ which will constitute the counterexample.

A situation for which $\int_0^\infty \varphi(t).P_2(t)\,dt > 0$ and $\int_0^\infty \varphi(t).P_3(t)dt < 0$ can be obtained graphically:



$P_2(t) = \int_0^t p_2(u)du$ and $P_3(t) = \int_0^t p_3(u)du$ are continuous, increasing, concave (p_2 and p_3 are decreasing) functions of t with limit 1 as t approaches infinity.

$$\varphi(t) = k p_1(t) - \bar{P}_1(t) = k p_1(t) - 1 + P_1(t) = k p_1(t) - 1 + \int_0^t p_1(u)\,du$$

is a function with limit 0 as t approaches infinity, which can have positive and/or negative values depending on the values of k (a positive real number) and the function p_1.

If p_1 is a negative exponential probability density function, φ will have a constant sign and cannot have a shape as that represented on the graph. We have seen that indeed the algorithm works in the negative exponential case.

$$p_1(t) = ae^{-at} \quad \text{gives} \quad \begin{array}{l} \varphi(t) = (ka-1)\,e^{-at} \\ \varphi(t) > 0 \text{ if } ka > 1 \\ \varphi(t) < 0 \text{ if } ka < 1 \end{array} \qquad (2.20)$$

But, if p_1 is a uniform distribution, it is possible to obtain for a shape such as the one shown on the graph.

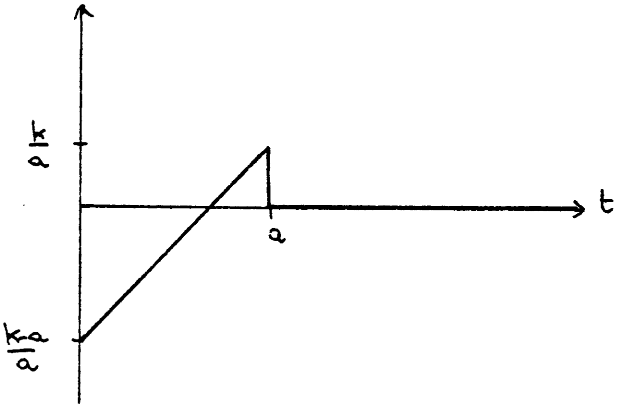
$$\begin{array}{l} p\ (t) = \frac{1}{a} \text{ if } t < a \\ p\ (t) = 0 \text{ if } t > a \end{array} \qquad (2.22)$$

Then,

$$\begin{array}{lll} \varphi(t) = \frac{t}{a} + \frac{k-a}{a} & \text{if} & t < a \\ \varphi(t) = 0 & \text{if} & t > a \end{array} \qquad (2.23)$$

and with $k < a$, we have a function $p(t)$ with negative values for t small, and then positive values.

$$\varphi(0) = \frac{k-a}{a} < 0 \qquad \varphi(a^-) = \frac{k}{a} > 0 \tag{2.24}$$



Thus, it seems very likely that we can get a counterexample $\left(\int_0^\infty \varphi(t) P_2(t)\,dt > 0 \right.$ whereas $\left. \int_0^\infty \varphi(t) P_3(t)\,dt < 0 \right)$ with waiting times uniformly distributed.

2.3 Counterexample With Uniformly Distributed Waiting Times

Let w_1, w_2, w_3 be uniformly distributed the following way

$$\begin{aligned} p_1(t) &= \frac{1}{a} & t &< a \\ p_1(t) &= 0 & t &> a \end{aligned} \tag{2.25}$$

which gives

$$\begin{aligned} P_1(t) &= \frac{t}{a} & t &< a \\ P_1(t) &= 1 & t &> a \end{aligned} \tag{2.26}$$

and

$$\varphi(t) = k p_1(t) - \bar{P}_1(t, \quad \begin{cases} \frac{t}{a} + \frac{k-a}{a} & t < a \\ 0 & t > a \end{cases} \tag{2.27}$$

where $k = t_2 - t_1$.

$$\begin{aligned} p_2(t) &= \frac{1}{b} & t &< b \\ p_2(t) &= 0 & t &> b \end{aligned} \tag{2.28}$$

which gives
$$P_2(t) = \frac{t}{b} \qquad t<b$$
$$P_2(t) = 1 \qquad t>b \tag{2.29}$$

$$p_3(t) = \frac{1}{c} \qquad t<c$$
$$p_3(t) = 0 \qquad t>c \tag{2.30}$$

which gives
$$P_3(t) = \frac{t}{c} \qquad t<c$$
$$P_3(t) = 1 \qquad t>c \tag{2.31}$$

This corresponds to a case in which the headways are deterministic and of value a, b, and c respectively. Deterministic headways and uniform waiting times are equivalent concepts.

Counterexample of the heuristic:

$$I = \int_0^\infty \varphi(t) \cdot P_2(t)\, dt > 0 \tag{2.32}$$

$$J = \int_0^\infty \varphi(t) \cdot P_3(t)\, dt < 0 \tag{2.33}$$

We can substitute the expressions of α, P_2 and P_3 given by (2.23), (2.26), (2.28) in I and J.

If $a<b$
$$I = \int_0^b \left(\frac{t}{a} + \frac{k-a}{a}\right) \frac{t}{b}\, dt = \frac{3ka - a^2}{6b} \tag{2.34}$$

If $a>b$
$$J = \int_0^b \left(\frac{t}{a} + \frac{k-a}{a}\right) \frac{t}{b}\, dt + \int_b^a \left(\frac{t}{a} + \frac{k-a}{a}\right) dt$$

$$= \frac{-b^2 + 3ab - 3kb + 6ka - 3a^2}{6a}$$

$$= F\left(\frac{b}{a}\right) \cdot \frac{a}{6} \tag{2.35}$$

where F is a function defined by

$$F(u) = -u^2 + 3(1-\lambda)u + 6\lambda - 3 \tag{2.36}$$

with $\lambda = \frac{k}{a}$ (2.37)

Similarly,

if a < c $\quad J = \frac{3ka - a^2}{6c}$ (2.38)

if a > c $\quad J = F(\frac{c}{a}) \cdot \frac{a}{6}$ (2.39)

Now we can compare the signs of I and J.

If a < b and a < c, I and J have the same sign. No counterexample is possible.

In all other cases, the function F(u) defined by (2.36) will play an important role, therefore, it is useful to study its sign, for u > 0. Considering F(u) as a second-order polynomial function, we obtain that

if $\quad 0 < \lambda < \frac{1}{3} \quad F(u) < 0$ (2.40)

if $\quad \frac{1}{3} \leq \lambda < \frac{1}{2} \quad F(u) \geq 0$ if $u_1 \leq u \leq u_2$ (2.41)

$F(u) < 0$ if $0 < u < u_1$ or $u > u_2$ (2.42)

if $\quad \lambda \geq \frac{1}{2} \quad F(u) \geq 0$ if $0 < u \leq u_2$ (2.43)

$F(u) < 0$ if $u > u_2$ (2.44)

where u_1 and u_2 are given by

$$u_1 = \frac{3(1-\lambda) - \sqrt{3(3\lambda-1)(\lambda+1)}}{2} \qquad (2.45)$$

$$u_2 = \frac{3(1-\lambda) + \sqrt{3(3\lambda-1)(\lambda+1)}}{2} \qquad (2.46)$$

u_1 and u_2 very as functions of λ in the following way:

λ	1/3		1/2		1		$+\infty$
u_1	1	$\searrow$	0	$\searrow$	$-\sqrt{3}$	$\searrow$	$-\infty$
u_2	1	$\nearrow$	3/2	$\nearrow$	$\sqrt{3}$	$\nearrow$	2

Therefore using this study and the expression for I and J given by (2.34), (2.35), (2.38), and (2.39), we obtain the following results for each case.

- If $c < a < b$

  $I > 0$ and $J < 0$ for $1/3 < \lambda < 1/2$ and $0 < \frac{c}{a} < u_1$ (2.47)

- If $b < a < c$

  if $J < 0$, necessarily $I < 0$ too. No counterexample.

- If a > b, a > c

$$I > 0 \text{ and } J < 0 \text{ for } \quad \frac{1}{3} < \lambda < \frac{1}{2} \text{ and } u_1 < \frac{a}{b} < 1 \text{ and } 0 < \frac{c}{a} < u_1 \tag{2.48}$$

Therefore, using (2.37) and (2.45) we will get I > 0 and J < 0 under the following conditions

$$\frac{a}{3} < t_2 - t_1 < \frac{a}{2} \tag{2.49}$$

$$0 < \frac{c}{a} < u < \frac{b}{a} \quad \text{with} \quad u = \frac{3(1-\lambda) - \sqrt{3(3\lambda-1)(\lambda+1)}}{2}$$

$$\text{where} \quad \lambda = \frac{t_2 - t_1}{a} \tag{2.50}$$

Now taking into account the term depending on $t_3 - t_2$, we have according to equations (2.15), (2.16), and (2.17)

$$T_B - T_A = I > 0 \quad \text{and}$$

$$T_C - T_A = J + (t_3 - t_2)\int_0^\infty p_3(t)\,\bar{P}_1(t)\,dt = J + (t_3 - t_2)\int_0^\infty p_3(t)(1 - P_1(t))\,dt < 0$$

when *the above conditions are satisfied* and by using the expresions of p_3, P_1 and J given by (2.30), (2.26), and (2.39)/(2.36) we have moreover,

$$t_3 - t_2 < \frac{c^2 - 3ac + 3(t_2 - t_1)c - 6(t_2 - t_1)a + 3a^2}{6a - 3c} \tag{2.51}$$

The three conditions (2.49), (2.50), and (2.51) are sufficient and necessary to obtain a counterexample of the heuristic in the case of three routes with deterministic headways.

This result can be checked with an example verifying these conditions. These conditions are not "easy" to meet: t has to be relatively close to t_3, while, on the other hand, b has to be relatively large with respect to c.

The counterexample is obtained with routes 1,2,3 such that the frequency 1/b (reverse of the headway) is quite low on route 2 whereas the frequency 1/c on a route 3 is much higher while the in-vehicle time between A and B is not much more penalizing (not much greater) than the one for route 2.

Let $t_1 = 8$ $a = 5$

$t_2 = 10$ $b = 10$

$t_3 = 10.2$ $c = 1$

(values verifying conditions (2.49), (2.50), (2.51))

we obtain $T_A = 10.5$

$T_B = 10.58$

$T_C = 10.45$

In that case, strategy (1,0,0) is better than strategy (1,1,0) *but* strategy (1,0,1) is better than strategy (1,0,0); this constitutes therefore a violation of the heuristic algorithm.

However, from a practical point of view, the three strategies are not much different with respect to expected total travel time.

Interestingly, it can be checked that in this particular case, the optimal solution is (1,1,1) with an expected total travel time T = 10.43

(still not very different).

Chapter 3

Comprehensive Study of the 3-Route Case, Extension to the 4-Route Case

3.1 Introduction to the 3-Route Case

Since the proposed algorithm does not work, it is interesting to try to discover possible properties of the expected total travel times of the various strategies offered to the passenger and, more exactly, properties of the differences between these expected total travel times, in order to compare the strategies.

Such properties, in the form of *logical relationships* between differences in expected total travel times could lead to discovery of a new approach to finding the optimal solution, a new way of solving the problem.

A study of the general case (n routes) requires extremely complicated mathematical expressions. The 3-route case is the first non-trivial case and its study could give fairly good ideas on the way the general case works. For the 3-route case, general formulae for $T_x - T_y$ will be derived and used in the cases of negative exponential (for route no. 1) and uniform distributions for waiting times. The results obtained in these particular cases are then studies in the case of general distributions.

3.2 General Formulae

This part is devoted to the derivation of T_x, expected total travel time of strategy x and to $T_x - T_y$ difference between strategies x and y, for all strategies x and y.

Let A = (1,0,0)

B = (1,1,0)

C = (1,0,1)

D = (1,1,1)

These are all possible strategies that include route number one.

T_A, T_B, T_C, $T_B - T_A$, and $T_C - T_A$ have already been expressed or calculated previously in equations (2.6), (2.7), (2.8), (2.15), and (2.16).

$$T_A = \int_0^\infty [\bar{P}_1(t) + t_1 p_1(t)]\,dt = \int_0^\infty \bar{P}_1(t)\,dt + t_1 \tag{2.6}$$

$$T_B = \int_0^\infty [\bar{P}_1(t)\bar{P}_2(t) + t_1 p_1(t)\bar{P}_2(t) + t_2 p_2(t)\bar{P}_1(t)]\,dt \tag{2.7}$$

$$T_C = \int_0^\infty [\bar{P}_1(t)\bar{P}_3(t) + t_1 p_1(t)\bar{P}_3(t) + t_3 p_3(t)\bar{P}_1(t)]\,dt \tag{2.8}$$

$$T_B - T_A = \int_0^\infty [(t_2 - t_1)p_1(t) - \bar{P}_1(t)]\,p_2(t)\,dt \tag{2.15}$$

$$T_C - T_A = \int_0^\infty [(t_2 - t_1)p_1(t) - \bar{P}_1(t)]\,p_3(t)\,dt + (t_3 - t_2)\int_0^\infty p_3(t)\,\bar{P}_1(t)\,dt \tag{2.16}$$

subtracting (2.16) from (2.15), we get

$$T_C - T_B = \int_0^\infty [(t_2 - t_1)p_1(t) - \bar{P}_1(t)](p_3(t) - p_2(t))\,dt + (t_3 - t_2)\int_0^\infty p_3(t)\,\bar{P}_1(t)\,dt \tag{3.1}$$

$$= \int_0^\infty [(t_2 - t_1)p_1(t) - \bar{P}_1(t)](\bar{P}_2(t) - \bar{P}_3(t))\,dt + (t_3 - t_2)\int_0^\infty p_3(t)\,\bar{P}_1(t)\,dt \tag{3.2}$$

T_D is obtained from the general formula (1.8).

$$T_D = \int_0^\infty \left[\bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t) + t_1 p_1(t)\bar{P}_2(t)\bar{P}_3(t) + t_2 p_2(t)\bar{P}_1(t)\bar{P}_3(t) + t_3 p_3(t)\bar{P}_1(t)\bar{P}_2(t) \right] dt \tag{3.3}$$

Then, we can derive $T_D - T_C$ similarly to $T_B - T_A$ and $T_C - T_A$ in chapter 2. Subtracting (2.8) from (3.2) we get,

$$T_D - T_C = \int_0^\infty \left[-\bar{P}_1(t)\bar{P}_3(t)P_2(t) - t_1 p_1(t)\bar{P}_3(t)P_2(t) - t_3 p_3(t)\bar{P}_1(t)P_2(t) + t_2 p_2(t)\bar{P}_1(t)\bar{P}_3(t) \right] dt \tag{3.4}$$

Then

$$\begin{aligned} T_D - T_C = \int_0^\infty \Big[& -\bar{P}_1(t)\bar{P}_3(t)P_2(t) + (t_2 - t_1) p_1(t)\bar{P}_3(t)P_2(t) - (t_3 - t_2) p_3(t)\bar{P}_1(t)P_2(t) \\ & + t_2\left[-p_1(t)\bar{P}_3(t)P_2(t) - p_3(t)\bar{P}_1(t)P_2(t) + p_2(t)\bar{P}_1(t)\bar{P}_3(t) \right] \Big] dt \end{aligned} \tag{3.5}$$

Certain parts of both expressions can be integrated directly.

$$T_D - T_C = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] P_2(t)\bar{P}_3(t)\,dt - (t_3 - t_2)\int_0^\infty p_3(t)\bar{P}_1(t)P_2(t)\,dt + t_2\left[\bar{P}_1(t)\bar{P}_3(t)P_2(t) \right]_0^\infty \tag{3.6}$$

This reduces to

$$T_D - T_C = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] P_2(t)\bar{P}_3(t)\,dt - (t_3 - t_2)\int_0^\infty p_3(t)\bar{P}_1(t)P_2(t)\,dt \tag{3.7}$$

Adding (3.6) and (3.1) gives the expression of $T_D - T_B$

$$T_D - T_B = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] \left(P_2(t) \bar{P}_3(t) + \bar{P}_2(t) - \bar{P}_3(t) \right) dt + (t_3 - t_2) \int_0^\infty p_3(t)\, \bar{P}_1(t) \left(1 - P_2(t) \right) dt \quad (3.8)$$

$$\text{i.e. } T_D - T_B = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] \bar{P}_2(t) P_3(t)\, dt + (t_3 - t_2) \int_0^\infty p_3(t) \bar{P}_1(t) \bar{P}_2(t)\, dt \quad (3.9)$$

Adding (3.9) and (2.15) gives the expression of $T_D - T_A$

$$T_D - T_A = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] \left(\bar{P}_2(t) P_3(t) + P_2(t) \right) dt + (t_3 - t_2) \int_0^\infty p_3(t)\, \bar{P}_1(t) \bar{P}_2(t)\, dt \quad (3.10)$$

$$\text{i.e. } T_D - T_A = \int_0^\infty \left[(t_2 - t_1) p_1(t) - \bar{P}_1(t) \right] \left(1 - \bar{P}_2(t) \bar{P}_3(t) \right) dt + (t_3 - t_2) \int_0^\infty p_3(t) \bar{P}_1(t) \bar{P}_2(t)\, dt \quad (3.11)$$

Expressions (2.15), (2.16), (3.1) or (3.2), (3.4), (3.9) and (3.11) show the dependency on $t_2 - t_1$ and $t_3 - t_2$ of $T_x - T_y$ for any two strategies X and Y, among A, B, C, D. Not surprisingly, only the relative values of the in-vehicle times t_1, t_2, t_3, (their differences) matter.

$T_B - T_A$ increases with $t_2 - t_1$.

$T_C - T_A$, $T_D - T_B$ and $T_D - T_A$ increase with $t_2 - t_1$ and $t_3 - t_2$.

$T_D - T_C$ increases with $t_2 - t_1$ and decreases with $t_3 - t_2$.

$T_C - T_B$ increases with $t_3 - t_2$, but there is no general rule with respect to $t_3 - t_2$.

These variations as a function of $t_2 - t_1$ and $t_3 - t_2$ coincide with our intuition.

Furthermore, the way these differences $T_x - T_y$, for any two strategies x and y, is expressed is advantageous. Using general functions $\Phi(f)$ and $\psi(f)$ of which the argument is a generic function of time f(t), and defined by

$$\Phi(f) = \int_0^\infty \left[(t_2 - t_1)\, p_1(t) - \bar{P}_1(t) \right] f(t)\, dt \qquad (3.12)$$

and $$\psi(f) = (t_3 - t_2) \int_0^\infty p_3(t)\, \bar{P}_1(t)\, f(t)\, dt \qquad (3.13)$$

all formulae can be re-written in the shorter forms.

$$T_B - T_A = \Phi(P_2) \qquad (3.14)$$

$$T_C - T_A = \Phi(P_3) + \psi(1) \qquad (3.15)$$

$$T_D - T_C = \Phi(P_2 \bar{P}_3) - \psi(P_2) \qquad (3.16)$$

$$T_D - T_B = \Phi(\bar{P}_2 P_3) + \psi(\bar{P}_2) \qquad (3.17)$$

$$T_D - T_A = \Phi(1 - \bar{P}_2 \bar{P}_3) + \psi(\bar{P}_2) \qquad (3.18)$$

$$T_C - T_B = \Phi(P_3 - P_2) + \psi(1) = \Phi(\bar{P}_2 - \bar{P}_3) + \psi(1) \qquad (3.19)$$

Another advantage of this notation is that Φ is linear and that ψ is proportional to $t_3 - t_2$ which is a real positive number than can be made arbitrarily small, and that ψ is linear and increasing (i.e. $\psi(g) \geq \psi(f)$ when $g(t) \geq f(t)$ for any t) and positive if f(t) is positive for any t, as well. Therefore, if for any four strategies X, Y, Z, U, among A, B, C, D, the following logical relationship, Strategy Y better than strategy X → Strategy U better than strategy Z i.e. $T_x - T_y > 0 \rightarrow T_2 - T_u > 0$ (see notation for usage of symbol →) with $T_x - T_y = \Phi(f_{xy}) + \psi(g_{xy})$ and $T_2 - T_u = \Phi(f_{zu}) + \psi(g_{zu})$, is true, the logical relationship (consisting of the same expressions, dropping the terms in ψ), $\Phi(f_{xy}) > 0 \rightarrow \Phi(f_{zu}) > 0$ must be true too.

Conversely, if the relationship $\Phi(f_{xy}) > 0 \rightarrow \Phi(f_{zu}) > 0$ is false, the relationship $T_x - T_y > 0 \rightarrow T_z - T_u > 0$ is equally false. This property will be used extensively below with the uniform distribution and the general cases, with the propositions involving only terms in Φ being tested first.

3.2 Case of Three Routes With a Negative Exponentially Distributed Waiting Time w_1

In this section, we examine the case of three routes, for which only the waiting time for a bus of route number one (w_1) is restricted to be negative-exponentially distributed. This is explained by the following reasoning: the interesting results about logical relationships concerning differences of expected total travel times, that we can obtain with all three waiting times negative exponentially distributed are still valid when we drop the assumptions of w_1 and w_2 being negative exponential.

w_1 is negative-exponentially distributed:

$$p_1(t) = ae^{-at} \tag{3.20}$$

which gives

$$(t_2-t_1)p_1(t) - \bar{P}_1(t) = [(t_2-t_1)a - 1]e^{-at} \tag{3.21}$$

Thus the expression $(t_2 - t_1)\ p_1(t) - \overline{P}_1(t)$ has a constant sign as time varies, positive if $t_2 - t_1 > 1/a$, negative if $t_2 - t_1 < 1/a$. Therefore, the general function $\Phi(f)$ defined by (3.12) will be positive (resp. negative) when its argument function $f(t)$ is always positive (resp. negative), if $t_2 - t_1 > 1/a$, and will be negative (resp. positive) when its argument is always positive(resp. negative), if $t_2 - t_1 < 1/a$. Also, $\Phi(f)$ will be an increasing function (i.e. $\Phi(f) \geq \Phi(g)$ when $f(t) \geq g(t)$ for any time t) if $t_2 - t_1 > 1/a$ and a decreasing function (i.e. $\Phi(f) < \Phi(g)$ when $f(t) > g(t)$ for any time t). This is part of the explanation why the assumption of w_1 alone being negative-exponentially distributed is so powerful.

Because of these properties and because $\psi(f)$ is always positive and using equations (3.14) to (3.18), we always have

$$T_B-T_A,\ T_C-T_A,\ T_D-T_B \quad\text{and}\quad T_D-T_A > 0 \quad\text{when}\quad t_2-t_1 > \frac{1}{a} \tag{3.22}$$

and

$$T_B-T_A \quad\text{and}\quad T_D-T_C < 0 \quad\text{when}\quad t_2-t_1 < \frac{1}{a} \tag{3.23}$$

Furthermore, because of the complete identity between the signs of $(t_2 - t_1) - 1/a$ and $T_B - T_A$, the following relationship holds

$$T_D-T_C>0 \Rightarrow T_B-T_A>0 \Rightarrow T_D-T_A,\ T_C-T_A,\ T_D-T_B>0 \tag{A}$$

Moreover, we also have

$$T_D-T_A>0 \Rightarrow T_C-T_A>0 \tag{B}$$

and

$$T_D-T_A>0 \Rightarrow T_D-T_B>0 \tag{C}$$

After (3.22), if $t_2 - t_1 > 1/a$, we have $T_D-T_A > 0$ and $T_C-T_A > 0$ anyway. If $t_2 - t_1 < 1/a$, we have $T_D-T_C < 0$ according to (3.23). Hence

$$T_C-T_A = (T_D-T_A)-(T_D-T_C) > T_D-T_A \tag{3.24}$$

which implies relationship (B).

Similarly, after (3.22) if $t_2 - t_1 > 1/a$, we have $T_D-T_A > 0$ and $T_D-T_B > 0$ anyway. If $t_2 - t_1 < 1/a$, we have $T_B-T_A < 0$ according to (3.23). Hence,

$$T_D-T_B = (T_D-T_A)-(T_B-T_A) > T_D-T_A \tag{3.25}$$

which implies relationship (C).

Combining relationships (A), (B), (C) and using the transitivity of symbol →, we obtain relationship (E).

$$T_D-T_C > 0 \Rightarrow T_B-T_A > 0 \Rightarrow T_D-T_A > 0 \Rightarrow T_C-T_A > 0, T_D-T_B > 0 \tag{E}$$

which means

$$\begin{aligned} (1,0,1) \text{ better than } (1,1,1) &\rightarrow (1,0,0) \text{ better than } (1,1,0) \\ &\rightarrow (1,0,0) \text{ better than } (1,1,1) \\ &\rightarrow (1,0,0) \text{ better than } (1,0,1) \\ &\quad\ \text{and } (1,1,0) \text{ better than } (1,1,1) \end{aligned} \tag{F}$$

which is remarkable.

Interestingly, we can see from this logical relationship that (1,0,1) cannot be the optimal strategy in this case, because if (1,0,1) is better than (1,1,1), this proves that, in fact, (1,0,0) will be the optimal strategy.

The second important result is that, in the case studied in this section, which is more general than the all negative-exponential case, Chriqui's and Robillard's heuristic algorithm is still valid.

A last remark is that relationship (E) is composed entirely of logical equivalencies ($\leftrightarrow$) if we have

$$T_B - T_C > 0 \tag{3.26}$$

Then
$$T_B - T_A = (T_C - T_A) + (T_B - T_C) > T_C - T_A \tag{3.27}$$

which implies

$$T_C - T_A > 0 \Rightarrow T_B - T_A > 0 \tag{G}$$

and combining with (E)

$$T_C - T_A > 0 \Rightarrow T_B - T_A > 0 \Rightarrow T_D - T_A > 0 \Rightarrow T_C - T_A, T_D - T_B > 0 \tag{H}$$

Similarly, because of (3.26),

$$T_D - T_C = (T_D - T_B) + (T_B - T_C) > T_D - T_B \tag{3.28}$$

which implies

$$T_D - T_B > 0 \Rightarrow T_D - T_C > 0 \qquad \text{(I)}$$

and combining with (E).

$$T_D - T_B > 0 \Rightarrow T_D - T_C > 0 \Rightarrow T_B - T_A > 0 \Rightarrow T_D - T_A > 0 \Rightarrow T_C - T_A , T_D - T_B > 0 \qquad \text{(J)}$$

Finally, a combination of (H) and (J) can be written

$$T_D - T_C > 0 \Leftrightarrow T_B - T_A > 0 \Leftrightarrow T_D - T_A > 0 \Leftrightarrow T_C - T_A > 0 \Leftrightarrow T_D - T_B > 0 \qquad \text{(K)}$$

This means that if strategy (1,0,1) is better than strategy (1,1,0), then

(1,0,1) better than (1,1,1) ↔ (1,0,0) better than (1,1,0)
↔ (1,0,0) better than (1,1,1) (L)
↔ (1,0,0) better than (1,0,1)
↔ (1,1,0) better than (1,1,1)

The practical consequence of (L) is that as soon as any one of the propositions above is true and if strategy (1,0,1) is better than strategy (1,1,0), (1,0,0) is the optimal strategy.

3.3 Case of Three Routes with Uniformly Distributed Waiting Times

In this section, we examine now the case of three routes for which the respective headways are deterministic with values a,b,c, i.e. the probability density functions and the cumulative distributions of the waiting times w_1, w_2, w_3 are respectively, (as seen in equations (2.25) to (2.31)),

$$\begin{aligned} p_1(t) &= \frac{1}{a} & t<a \\ p_1(t) &= 0 & t>a \end{aligned} \qquad (2.25)$$

$$\begin{aligned} P_1(t) &= \frac{t}{a} & t<a \\ P_1(t) &= 1 & t>a \end{aligned} \qquad (2.26)$$

with $$\varphi(t) = k p_1(t) - \overline{P}_1(t) = \begin{cases} \frac{t}{a} + \frac{k-a}{a} & t<a \\ 0 & t>a \end{cases} \qquad (2.27)$$

where $k = t_2 - t_1$

$$\begin{aligned} p_2(t) &= \frac{1}{b} & t<b \\ p_2(t) &= 0 & t>b \end{aligned} \qquad (2.28)$$

$$\begin{aligned} P_2(t) &= \frac{t}{b} & t<b \\ P_2(t) &= 1 & t>b \end{aligned} \qquad (2.29)$$

$$\begin{aligned} p_3(t) &= \frac{1}{c} & t<c \\ p_3(t) &= 0 & t>c \end{aligned} \qquad (2.30)$$

$$\begin{aligned} P_3(t) &= \frac{t}{c} & t<c \\ P_3(t) &= 1 & t>c \end{aligned} \qquad (2.31)$$

A systematic study has been performed with a programmable calculator to test the signs of the terms in Φ in the expressions (3.14) to (3.19) of the difference of the expected total travel times between any two strategies X and Y, for all types of values of the numbers k, a, b, and c. From this systematic study, results in the form of logical relationships like in the previous section, are conjectured with a very high level of confidence, after these calculations.

For each of the following cases, $\Phi(P_2)$, $\Phi(P_3)$, and $\Phi(P_2P_3)$ i.e.

$$\Phi(P_2) = \int_0^\infty \varphi(t) \cdot P_2(t)\, dt \qquad (3.29)$$

$$\Phi(P_3) = \int_0^\infty \varphi(t).\, P_3(t)\, dt \tag{3.30}$$

$$\Phi(P_2P_3) = \int_0^\infty \varphi(t)\, P_2(t)\, P_3(t)\, dt \tag{3.31}$$

are derived in a similar way as I and J in expressions (2.34) to (2.39), after equations (2.27) to (2.31). Then $\Phi(\overline{P}_2P_3)$, $\Phi(P_2\overline{P}_3)$, $\Phi(1-\overline{P}_2\overline{P}_3)$, and $\Phi(\overline{P}_2-\overline{P}_3)$ are computed in the programs, using the following formulae, true because of the linear property of the general function $\Phi(f)$.

$$\Phi(\overline{P}_2P_3) = \Phi(P_3 - P_2P_3) = \Phi(P_3) - \Phi(P_2P_3) \tag{3.32}$$

$$\Phi(P_2\overline{P}_3) = \Phi(P_2 - P_2P_3) = \Phi(P_2) - \Phi(P_2P_3) \tag{3.33}$$

$$\Phi(1-\overline{P}_2\overline{P}_3) = \Phi(P_2+P_3-P_2P_3) = \Phi(P_2) + \Phi(P_3) - \Phi(P_2P_3) \tag{3.34}$$

$$\Phi(\overline{P}_2-\overline{P}_3) = \Phi(P_3-P_2) = \Phi(P_3) - \Phi(P_2) \tag{3.35}$$

- If $a < b < c$

$$\Phi(P_2) = \frac{3ka-a^2}{6b} \tag{3.36}$$

$$\Phi(P_3) = \frac{3ka-a^2}{6c} \tag{3.37}$$

$$\Phi(P_2P_3) = \frac{4ka^2-a^3}{12bc} \tag{3.38}$$

We conjecture, with near certainty, that we have the following logical relationship.

$$\Phi(\bar{P}_2 P_3)>0 \Rightarrow \Phi(P_2\bar{P}_3)>0 \Rightarrow \Phi(1-\bar{P}_2\bar{P}_3)>0 \Rightarrow \Phi(P_2)>0 \Leftrightarrow \Phi(P_3)>0 \Leftrightarrow \Phi(\bar{P}_3-\bar{P}_2)>0 \qquad \text{(M)}$$

- If a < c < b

The same formulae as for the first case hold here. But it is now conjectured that

$$\Phi(P_2\bar{P}_3)>0 \Rightarrow \Phi(\bar{P}_2 P_3)>0 \Rightarrow \Phi(1-\bar{P}_2\bar{P}_3)>0 \Rightarrow \Phi(P_2)>0 \Leftrightarrow \Phi(P_3)>0 \Leftrightarrow \Phi(\bar{P}_2-\bar{P}_3)>0 \qquad \text{(N)}$$

- If b < a < c

$$\Phi(P_2) = \frac{-b^2+3ab-3kb+6ka-3a^2}{6a} \qquad (3.39)$$

$$\Phi(P_3) = \frac{3ka-a^2}{6c} \qquad (3.40)$$

$$\Phi(\bar{P}_2 P_3) = \frac{-b^3-2kb^2+6ab^2-2a^3+6ka^2}{12ac} \qquad (3.41)$$

We conjecture that

$$\Phi(\bar{P}_2 P_3)>0 \Rightarrow \Phi(P_2\bar{P}_3)>0 \Rightarrow \Phi(\bar{P}_3-\bar{P}_2)>0 \Rightarrow \Phi(1-\bar{P}_2\bar{P}_3)>0 \Rightarrow \Phi(P_2)>0 \Rightarrow \Phi(P_3)>0 \qquad \text{(0)}$$

- If b < c < a

$$\Phi(P_2) = \frac{-b^2+3ab-3kb+6ka-3a^2}{6a} \qquad (3.42)$$

$$\Phi(P_3) = \frac{-c^2+3ac-3kc+6ka-3a^2}{6a} \tag{3.43}$$

$$\Phi(P_2P_3) = \frac{-b^3-2kb^2+6ab^2-2c^3-6kc^2+6ac^2-6a^2c+12kac}{12ac} \tag{3.44}$$

The same logical relationship, (0), is conjectured here.

- If c < a < b

$$\Phi(P_2) = \frac{3ka-a^2}{6b} \tag{3.45}$$

$$\Phi(P_3) = \frac{-c^2+3ac-3kc+6ka-3a^2}{6a} \tag{3.46}$$

$$\Phi(P_2P_3) = \frac{-c^3-2kc^2+2ac^2-2a^3+6ka^2}{12ab} \tag{3.47}$$

We conjecture that

$$\begin{tikzcd}
 & & \Phi(\bar{P}_2-\bar{P}_3)>0 \arrow[dr, Rightarrow] & & \\
\Phi(P_2\bar{P}_3)>0 \arrow[r, Rightarrow] & \Phi(\bar{P}_2P_3)>0 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \Phi(P_3)>0 \arrow[r, Rightarrow] & \Phi(P_2)>0 \\
 & & \Phi(1-\bar{P}_2\bar{P}_3)>0 \arrow[ur, Rightarrow] & &
\end{tikzcd} \tag{P}$$

- If c < b < a

$$\Phi(P_2) = \frac{-b^2+3ab-3kb+6ka-3a^2}{6a} \tag{3.48}$$

$$\Phi(P_3) = \frac{-c^2+3ac-3kc+6ka-3a^2}{6a} \tag{3.49}$$

$$\Phi(P_2P_3) = \frac{-c^3 - 2kc^2 + 2ac^2 - 2b^3 - 6kb^2 + 6ab^2 - 6a^2b + 12kab}{12ab} \tag{3.50}$$

We conjecture, similarly, that

$$\begin{tikzcd}
 & \Phi(\bar{P}_2-\bar{P}_3)>0 \arrow[dr, Rightarrow] & & & \\
\Phi(P_2\bar{P}_3) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \Phi(1-\bar{P}_2\bar{P}_3)>0 \arrow[r, Rightarrow] & \Phi(P_3)>0 \arrow[r, Rightarrow] & \Phi(P_2)>0 \\
 & \Phi(\bar{P}_2P_3)>0 \arrow[ur, Rightarrow] & & &
\end{tikzcd} \tag{Q}$$

Thus, taking the common part of relationships (M), (N), (O), (P), and (Q), the following logical relationship will hold in all cases.

$$\begin{tikzcd}
\Phi(\bar{P}_2P_3)>0 \arrow[dr, Rightarrow] & & \Phi(P_2)>0 \\
 & \Phi(1-\bar{P}_2\bar{P}_3)>0 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
\Phi(P_2\bar{P}_3)>0 \arrow[ur, Rightarrow] & & \Phi(P_3)>0
\end{tikzcd} \tag{R}$$

Nevertheless, because of the terms in ψ which are proportional to $t_3 - t_2$, in the expressions of $T_x - T_y$, (3.14) to (3.19), for any two strategies X and Y, certain relationships included in (R) do not remain valid just for any value of $t_3 - t_2$.

However, some of them do. The following relationships hold.

$$\Phi(P_2\bar{P}_3) - \psi(P_2) > 0 \Rightarrow \Phi(1-\bar{P}_2\bar{P}_3) + \psi(\bar{P}_2) > 0 \Rightarrow \Phi(P_3) + \psi(1) > 0 \tag{S}$$

$$\Phi(P_2\bar{P}_3) - \psi(P_2) > 0 \Rightarrow \Phi(P_2) > 0 \tag{T}$$

$\psi(f)$ is positive whenever $f(t)$ is positive for any t.

Thus,

$$\Phi(P_2\bar{P}_3)-\psi(P_2)<\Phi(P_2\bar{P}_3) \tag{3.51}$$

and

$$\Phi(1-\bar{P}_2\bar{P}_3)+\psi(\bar{P}_2)>\Phi(1-\bar{P}_2\bar{P}_3) \tag{3.52}$$

Therefore, since

$$\Phi(P_2\bar{P}_3)>0 \Rightarrow \Phi(1-\bar{P}_2\bar{P}_3)>0 \tag{U}$$

is true according to (R), relationship (V) is true too.

$$\Phi(P_2\bar{P}_3)-\psi(P_2)>0 \Rightarrow \Phi(1-\bar{P}_2\bar{P}_3)+\psi(\bar{P}_2)>0 \tag{V}$$

We have $\Phi(P_3)+\psi(1) > \Phi(P_3)$ (3.53)

- if $\Phi(1-\bar{P}_2\bar{P}_3)>0$: $\Phi(P_3)>0$ because of (R), and then $\Phi(1-\bar{P}_2\bar{P}_3) + \psi(\bar{P}_2)>0$ using (3.52), and $\Phi(P_3) + \psi(1)>0$ using (3.53).

- if $(1-\bar{P}_2\bar{P}_3)<0$: $\Phi(P_2\bar{P}_3)<0$ because of (R) and then $\Phi(P_2\bar{P}_3)-\psi(P_2)<0$ using (3.51). Thus, using also the linear properties of Φ and ψ

$$\begin{aligned}\Phi(P_3)+\psi(1) &= \Phi(P_2+P_3-P_2P_3)-\Phi(P_2-P_2P_3)+\psi(\bar{P}_2)+\psi(P_2)\\ &= \Phi(1-\bar{P}_2\bar{P}_3)-\Phi(P_2\bar{P}_3)+\psi(\bar{P}_2)+\psi(P_2)\\ &= [\Phi(1-\bar{P}_2\bar{P}_3)+\psi(\bar{P}_2)]-[\Phi(P_2\bar{P}_3)-\psi(P_2)]\\ &> \Phi(1-\bar{P}_2\bar{P}_3)+\psi(\bar{P}_2)\end{aligned} \tag{3.54}$$

Therefore, the following relationship will hold in both cases $\Phi(1-\bar{P}_2\bar{P}_3)>0$ and $\Phi(1-\bar{P}_2\bar{P}_3)<0$.

$$\Phi(1-\bar{P}_2\bar{P}_3)+\psi(\bar{P}_2)>0 \Rightarrow \Phi(P_3)+\psi(1)>0 \tag{W}$$

Combining (V) and (W) proves that relationship (S) is true.

Similarly, using (3.51) and knowing that

$$\Phi(P_2\bar{P}_3)>0 \Rightarrow \Phi(P_2)>0 \tag{X}$$

is true according to (R), we have the proof that (T) is true too.

All other relationships that we could infer from (R) are violated for high values of $t_3 - t_2$.

Finally, expressing (S) and (T) in terms of T_x-T_y, using equations (3.14) to (3.18), we obtain the set of relationships

$$T_D-T_C>0 \Rightarrow T_D-T_A>0 \Rightarrow T_C-T_A>0 \tag{Y}$$

and $T_D-T_C>0 \Rightarrow T_B-T_A>0$

This means that for uniformly distributed waiting times, i.e. deterministic headways, we have relationships (Z).

(1,0,1) better than (1,1,1) → (1,0,0) better than (1,1,1) →
(1,0,0) better than (1,0,1); and (1,0,1) better than (1,1,1) →
(1,0,0) better than (1,1,0) . (Z)

This shows that strategy (1,0,1) still cannot be the optimal solution.

We can see also that those relationships valid in this case are valid in the negative exponentially w_1 case, but the other relationships true in the negative exponential case, are no longer valid in the uniform distribution case.

The logical relationships discovered here can be verified through the particular example used as the counterexample for the heuristic solution in Chapter 2.

3.4 Case of Three Routes with General Distributions

In this section, the results which were shown valid in the previous special cases are now studied in the general case.

The first interesting result was the strategy (1,0,1) cannot be the optimal one in the cases when w_1 is negative exponentially distributed or in the uniform distributions case. Unfortunately, this result, which can be thought to be true intuitively, can be contradicted in the general case: a counterexample can be found. This counterexample uses the probability density function

$$p(t) = \frac{na^n}{(t+a)^{n+1}} \quad , \quad t \in \mathbb{R}^+ \tag{3.55}$$

This type of distribution will also be used extensively below, to contradict properties predicted by intuition. It is, therefore, interesting before going on with our study to have a look at some specific properties of this distribution which is powerful in contradicting propositions we would think to be true and produces results different from other distributions, particularly the negative exponential and uniform ones.

Two characteristic peculiarities of this type of distribution may be seen:

- the probability density function p(t) converges very slowly to 0, as t goes to infinity.
- if the order of the distribution is n (i.e. $p(t) = \frac{na^n}{(t+a)^{n+1}}$), all moments of order greater than or equal to n are infinite.

Moreover, interestingly with respect to our waiting time problem, if the waiting time is distributed at the order n, this means that the headway is distributed at the order n+1. Using formula (1.1) linking the probability density function p(t) of the waiting time and the cumualtive distribution $F(t) = \int_0^u f(u)du$ of the headway, we get that

$$p(t) = \frac{na^n}{(t+a)^{n+1}} \tag{3.55}$$

if $$F(t) = 1 - \frac{a^{n+1}}{(t+a)^{n+1}} \tag{3.56}$$

i.e. $$f(t) = \frac{(n+1)a^{n+1}}{(t+a)^{n+2}} \tag{3.57}$$

Going back to our study, let us assume that strategy (1,0,1) is the optimal one. This means that

$$T_C - T_A < 0 \tag{3.58}$$

$$T_D - T_C > 0 \tag{3.59}$$

$$T_B - T_C > 0 \tag{3.60}$$

which can be rewritten, according to (3.15), (3.16), (3.19),

$$\Phi(P_3) + \psi(1) < 0 \tag{3.61}$$

$$\Phi(P_2\bar{P}_3) - \psi(P_2) > 0 \tag{3.62}$$

$$\Phi(\bar{P}_3 - \bar{P}_2) - \psi(1) > 0 \tag{3.63}$$

As previously, and following our usual method, the study is first conducted by dropping the terms in ψ, as explained in section 3.2. Thus, we have, using again the shortened notation $k = t_2 - t_1$

$$\int_0^\infty [k p_1(t) - \bar{P}_1(t)] P_3(t)\,dt < 0 \tag{3.64}$$

$$\int_0^\infty [k p_1(t) - \bar{P}_1(t)] P_2(t)\bar{P}_3(t)\,dt > 0 \tag{3.65}$$

$$\int_0^\infty [k p_1(t) - \bar{P}_1(t)][\bar{P}_3(t) - \bar{P}_2(t)]\,dt > 0 \tag{3.66}$$

(3.64), (3.65), (3.66) are respectively equivalent to

$$k < \frac{\int_0^\infty \bar{P}_1(t) P_3(t)\,dt}{\int_0^\infty p_1(t) P_3(t)\,dt} \tag{3.67}$$

$$k > \frac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t)\,dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t)\,dt} \tag{3.68}$$

$$k < \frac{\int_0^\infty \bar{P}_1(t)\,[\bar{P}_3(t) - \bar{P}_2(t)]\,dt}{\int_0^\infty p_1(t)\,[\bar{P}_3(t) - \bar{P}_2(t)]\,dt} \tag{3.69}$$

when $$\int_0^\infty p_1(t)\,[\bar{P}_3(t) - \bar{P}_2(t)]\,dt < 0 \tag{3.70}$$

Combining (3.67) and (3.68), and (3.68) and (3.69), we necessarily have

$$\frac{\int_0^\infty \bar{P}_1(t)\,P_3(t)\,dt}{\int_0^\infty p_1(t)P_3(t)\,dt} > \frac{\int_0^\infty \bar{P}_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt}{\int_0^\infty p_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt} \tag{3.71}$$

and

$$\frac{\int_0^\infty \bar{P}_1(t)\,[\bar{P}_3(t) - \bar{P}_2(t)]\,dt}{\int_0^\infty p_1(t)\,[\bar{P}_3(t) - \bar{P}_2(t)]\,dt} > \frac{\int_0^\infty \bar{P}_1(t)\,P_2(t)\bar{P}_3(t)\,dt}{\int_0^\infty p_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt} \tag{3.72}$$

Reducing the denominators, using the fact that $\int_0^\infty p_r(t)dt = 1$ and $P_r(t) = 1-\bar{P}_r(t)$, and simplifying, we obtain

$$\begin{aligned} &\int_0^\infty \bar{P}_1(t)dt\int_0^\infty p_1(t)\bar{P}_3(t)dt + \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)dt\int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,\bar{P}_2(t)\,\bar{P}_3(t)\,dt \\ &\quad > \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)dt + \int_0^\infty \bar{P}_1(t)\,\bar{P}_2(t)\bar{P}_3(t)dt\int_0^\infty p_1(t)\bar{P}_3(t)dt + \int_0^\infty \bar{P}_1(t)dt\int_0^\infty p_1(t)\,\bar{P}_2(t)\bar{P}_3(t)dt \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} &\int_0^\infty \bar{P}_1(t)\bar{P}_3(t)dt\int_0^\infty p_1(t)\,\bar{P}_2(t)dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)dt\int_0^\infty p_1(t)\,\bar{P}_3(t)dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)dt\int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)dt \\ &< \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt\int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt\int_0^\infty p_1(t)\,\bar{P}_2(t)dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)dt\int_0^\infty p_1(t)\,\bar{P}_3(t)dt \end{aligned} \tag{3.74}$$

We introduce now the type of distribution described above, substitute it in expressions (3.70), (3.73), and (3.74), and check that the inequalities are verified.

$$p_1(t) = \frac{na^n}{(t+a)^{n+1}} \tag{3.75}$$

and thus $$\bar{P}_1(t) = \frac{a^n}{(t+a)^n} \tag{3.76}$$

$$p_2(t) = \frac{pa^p}{(t+a)^{p+1}} \tag{3.77}$$

and thus $$\bar{P}_2(t) = \frac{a^p}{(t+a)^p} \tag{3.78}$$

$$p_3(t) = \frac{qa^q}{(t+a)^{q+1}} \tag{3.79}$$

and thus $$\bar{P}_3(t) = \frac{a^q}{(t+a)^q} \tag{3.80}$$

The various integrals are then

$$\int_0^\infty p_1(t)\,\bar{P}_2(t)\,dt = \int_0^\infty \frac{na^{n+p}}{(t+a)^{n+p+1}}\,dt = \frac{n}{n+p} \tag{3.81}$$

$$\int_0^\infty p_1(t)\,\bar{P}_3(t)\,dt = \int_0^\infty \frac{na^{n+q}}{(t+a)^{n+q+1}}\,dt = \frac{n}{n+q} \tag{3.82}$$

$$\int_0^\infty p_1(t)\,\bar{P}_2(t)\bar{P}_3(t)\,dt = \int_0^\infty \frac{na^{n+p+q}}{(t+a)^{n+p+q+1}}\,dt = \frac{n}{n+p+q} \tag{3.83}$$

$$\int_0^\infty \bar{P}_1(t)\,dt = \int_0^\infty \frac{a^n}{(t+a)^n}\,dt = \frac{a}{n-1} \tag{3.84}$$

$$\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\,dt = \int_0^\infty \frac{a^{n+p}}{(t+a)^{n+p}}\,dt = \frac{a}{n+p-1} \tag{3.85}$$

$$\int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt = \int_0^\infty \frac{a^{n+q}}{(t+a)^{n+q}}\,dt = \frac{a}{n+q-1} \tag{3.86}$$

$$\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt = \int_0^\infty \frac{a^{n+p+q}}{(t+a)^{n+p+q}}\,dt = \frac{a}{n+p+q-1} \tag{3.87}$$

We substitute these expressions in, reduce and simplify (3.70), (3.73), and (3.74), which become respectively

$$p < q \tag{3.88}$$

$$pq\,(p+q) > 0 \tag{3.89}$$

$$pq\,(p-q) < 0 \tag{3.90}$$

Therefore, with the waiting times distributed according to equations (3.75) to (3.80), and the condition p < q, (3.88), (3.89), (3.90) are true, and (3.71) and (3.72) will be verified, so that it will be possible to choose k according to inequalities (3.67), (3.68), (3.69), i.e.

$$\frac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t)\,dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t)\,dt} < k < \min\left[\frac{\int_0^\infty \bar{P}_1(t) P_3(t)\,dt}{\int_0^\infty p_1(t) P_3(t)\,dt}, \frac{\int_0^\infty \bar{P}_1(t)[\bar{P}_3(t)-\bar{P}_2(t)]\,dt}{\int_0^\infty p_1(t)[\bar{P}_3(t)-\bar{P}_2(t)]\,dt}\right] \tag{3.91}$$

to insure inequalities (3.64), (3.65), (3.66), i.e. using Φ,

$$\Phi(P_3)<0 \tag{3.92}$$

$$\Phi(P_2\bar{P}_3)>0 \tag{3.93}$$

$$\Phi(\bar{P}_3-\bar{P}_2)>0 \tag{3.94}$$

Finally (3.61), (3.62), and (3.63) will be true, proving that (1,0,1) is the optimal strategy, if we take $t_3 - t_2$ small enough, i.e. precisely, [to have all three (3.61), (3.62), and (3.63) true]

$$t_3-t_2<\min\left[\frac{-\Phi(P_3)}{\psi(1)},\ \frac{\Phi(P_2\bar{P}_3)}{\psi(P_2)},\ \frac{\Phi(\bar{P}_3-\bar{P}_2)}{\psi(1)}\right] \tag{3.95}$$

Summarizing this important result, we have shown here that the optimal strategy for the passenger can be (1,0,1) if the waiting times w_1, w_2, w_3 are distributed, respectively in the following way:

$$p_1(t)=\frac{na^n}{(t+a)^{n+1}}\ ,\quad p_2(t)=\frac{pa^p}{(t+a)^{p+1}}\ ,\quad p_3(t)=\frac{qa^q}{(t+a)^{q+1}}$$

with the conditions

- $p < q$
- $$\frac{\int_0^\infty \bar{P}_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt}{\int_0^\infty p_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt}<t_2-t_1<\min\left[\frac{\int_0^\infty \bar{P}_1(t)\,P_3(t)\,dt}{\int_0^\infty p_1(t)\,P_3(t)\,dt},\ \frac{\int_0^\infty \bar{P}_1(t)\left[\bar{P}_3(t)-\bar{P}_2(t)\right]dt}{\int_0^\infty p_1(t)\left[\bar{P}_3(t)-\bar{P}_2(t)\right]dt}\right]$$

(which is possible after the first condition); and, using the expressions of Φ and ψ given by (3.12) and (3.13),

- $$t_3 - t_2 < \min\left[\frac{-\int_0^\infty [(t_2-t_1)p_1(t) - \bar{P}_1(t)]P_3(t)\,dt}{\int_0^\infty p_3(t)\bar{P}_1(t)\,dt}, \frac{\int_0^\infty [(t_2-t_1)p_1(t) - \bar{P}_1(t)]P_2(t)\bar{P}_3(t)\,dt}{\int_0^\infty p_3(t)\bar{P}_1(t)P_2(t)\,dt}, \frac{\int_0^\infty [(t_2-t_1)p_1(t) - \bar{P}_1(t)][\bar{P}_3(t) - \bar{P}_2(t)]\,dt}{\int_0^\infty p_3(t)\bar{P}_1(t)\,dt} \right]$$

(which is possible after the two other conditions)

The other results valid in the two previously examined particular cases (sections 3.2 and 3.3) are the following logical relationship (Z).

(1,0,1) better than (1,1,1) → (1,0,0) better than (1,1,0); and
(1,0,1) better than (1,1,1) → (1,0,0) better than (1,1,1) → (Z)
(1,0,0) better than (1,0,1) .

This means four separate relationships to prove in the general case, or to contradict

(1,0,1) better than (1,1,1) → (1,0,0) better than (1,1,0) (AA)
(1,0,1) better than (1,1,1) → (1,0,0) better than (1,1,1) (AB)
(1,0,0) better than (1,1,1) → (1,0,0) better than (1,0,1) (AC)
(1,0,1) better than (1,1,1) → (1,0,0) better than (1,0,1). (AD)

The last relationship, (AD), must be false, because otherwise strategy (1,0,1) could never be the optimal one.

In fact, all four relationships turn out to be false under the following conditions, with certain values of $t_2 - t_1$ and small $t_3 - t_2$.

- $$p_1(t) = \frac{na^n}{(t+a)^{n+1}} \tag{3.96}$$

  $$\text{i.e. } \bar{P}_1(t) = \frac{a^n}{(t+a)^n} \tag{3.97}$$

- $$p_3(t) = \frac{qa^q}{(t+a)^{q+1}} \tag{3.98}$$

  $$\text{i.e. } \bar{P}_3(t) = \frac{a^q}{(t+a)^q} \tag{3.99}$$

- $$P_2(t) \simeq 1 \qquad \left(\bar{P}_2(t) \simeq 0\right), \quad \text{for } t \in \mathbb{R}^+_* \tag{3.100}$$

(P_2, concave, increasing on $\mathbb{R}^+$ from 0 to 1 can be arbitrarily close to 1 at any point except 0.) Mathematically, this is valid. Φ is continuous in the topological space of functions integrating on $\mathbb{R}^+$. $\bar{P}_2$ can be arbitrarily close to the zero-function with respect to the norm of this space defined by $\|f\| = \int_0^\infty |f(t)|\,dt$. Limits can be taken as $\bar{P}_2$ approaches zero.

Relationship (AA) is expressed by

$$T_D - T_C > 0 \Rightarrow T_B - T_A > 0 \tag{AE}$$

i.e. using equations (3.14) and (3.16), and dropping the terms in ψ as usual

$$\Phi(P_2\bar{P}_3) > 0 \Rightarrow \Phi(P_2) > 0 \tag{AF}$$

which is, after (3.12), and noting $k = t_2 - t_1$.

$$\int_0^\infty [k p_1(t) - \bar{P}_1(t)] P_2(t) \bar{P}_3(t)\, dt > 0 \Rightarrow \int_0^\infty [k p_1(t) - \bar{P}_1(t)] P_2(t)\, dt > 0 \qquad \text{(AG)}$$

$$\text{i.e.} \quad k > \frac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t)\, dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t)\, dt} \Rightarrow k > \frac{\int_0^\infty \bar{P}_1(t) P_2(t)\, dt}{\int_0^\infty p_1(t) P_2(t)\, dt} \qquad \text{(AH)}$$

which is only possible for all k when,

$$\frac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t)\, dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t)\, dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t) P_2(t)\, dt}{\int_0^\infty p_1(t) P_2(t)\, dt} \qquad (3.101)$$

If $P_2(t) = 1$, this becomes

$$\int_0^\infty \bar{P}_1(t) \bar{P}_3(t)\, dt \geqslant \int_0^\infty \bar{P}_1(t)\, dt \int_0^\infty p_1(t) \bar{P}_3(t)\, dt \qquad (3.102)$$

With the distributions defined by expressions (3.96) to (3.99), we have

$$\int_0^\infty \bar{P}_1(t) \bar{P}_3(t)\, dt - \int_0^\infty \bar{P}_1(t)\, dt \,.\, \int_0^\infty p_1(t) \bar{P}_3(t)\, dt$$

$$= \frac{a}{n+p-1} - \frac{na}{(n-1)(n+p)} = \frac{-pa}{(n+p-1)(n-1)(n+p)} < 0 \qquad (3.103)$$

which shows that inequality (3.102) is not verified. Thus, relationship (AA) is not true for waiting times distributed according to expressions (3.96) to (3.100), certain values of $t_2 - t_1$ and small $t_3 - t_2$. These values could be determined similarly as previously where $t_2 - t_1$ and $t_3 - t_2$ were found to be given by (3.91) and (3.95), respectively.

Relationship (AB) is expressed by

$$T_D - T_C > 0 \Rightarrow T_D - T_A > 0 \tag{AI}$$

i.e. using equation (3.16) and (3.18), and dropping the terms in ψ

$$\Phi(P_2\bar{P}_3) > 0 \Rightarrow \Phi(1 - \bar{P}_2\bar{P}_3) > 0 \tag{AJ}$$

which proceeding as previously gives

$$k > \frac{\int_0^\infty \bar{P}_1(t)\, P_2(t)\, \bar{P}_3(t)\, dt}{\int_0^\infty p_1(t)\, P_2(t)\, \bar{P}_3(t)\, dt} \Rightarrow k > \frac{\int_0^\infty \bar{P}_1(t)\left[1 - \bar{P}_2(t)\bar{P}_3(t)\right] dt}{\int_0^\infty p_1(t)\left[1 - \bar{P}_2(t)\bar{P}_3(t)\right] dt} \tag{AK}$$

which is only possible for all k when

$$\frac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t)\, dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t)\, dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\left[1 - \bar{P}_2(t)\bar{P}_3(t)\right] dt}{\int_0^\infty p_1(t)\left[1 - \bar{P}_2(t)\bar{P}_3(t)\right] dt} \tag{3.104}$$

If $P_2(t) = 1$ this reduces to inequality (3.102) exactly as in the previous analysis (3.101). Thus in the same way as (AA), relationship (AB) is not true for waiting times distributed according to expressions (3.96) to (3.100), certain values of $t_2 - t_1$ and small $t_3 - t_2$.

Relationship (AC) is expressed by

$$T_D - T_A > 0 \Rightarrow T_C - T_A > 0 \tag{AL}$$

i.e. using equations (3.18) and (3.15), and dropping the terms in ψ,

$$\Phi(1-\bar{P}_2\bar{P}_3) > 0 \Rightarrow \Phi(P_3) > 0 \tag{AM}$$

which proceeding again as previously gives

$$k > \frac{\int_0^\infty \bar{P}_1(t)\,(1-\bar{P}_2(t)\bar{P}_3(t))\,dt}{\int_0^\infty p_1(t)\,(1-\bar{P}_2(t)\bar{P}_3(t))\,dt} \Rightarrow k > \frac{\int_0^\infty \bar{P}_1(t)P_3(t)\,dt}{\int_0^\infty p_1(t)P_3(t)\,dt} \tag{AN}$$

which is only possible for all k when

$$\frac{\int_0^\infty \bar{P}_1(t)\,(1-\bar{P}_2(t)\bar{P}_3(t))\,dt}{\int_0^\infty p_1(t)\,(1-\bar{P}_2(t)\bar{P}_3(t))\,dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)P_3(t)\,dt}{\int_0^\infty p_1(t)P_3(t)\,dt} \tag{3.105}$$

If $P_2(t) = 1$ this still reduces to equality (3.102). Thus, in the same way as (AA) and (AB), relationship (AC) is not true for waiting times distributed according to expressions (3.96) to (3.100), certain values of $t_2 - t_1$ and small $t_3 - t_2$.

Relationship (AD) is expressed by

$$T_D - T_C > 0 \Rightarrow T_C - T_A > 0 \tag{AO}$$

i.e. using equations (3.16) and (3.15) and dropping the terms in ψ,

$$\Phi(P_2\bar{P}_3) > 0 \Rightarrow \Phi(P_3) > 0 \tag{AP}$$

which still proceeding as previously gives

$$k > \frac{\int_0^\infty \bar{P}_1(t)P_2(t)\bar{P}_3(t)\,dt}{\int_0^\infty p_1(t)P_2(t)\bar{P}_3(t)\,dt} \Rightarrow k > \frac{\int_0^\infty \bar{P}_1(t)P_3(t)\,dt}{\int_0^\infty p_1(t)P_3(t)\,dt} \tag{AQ}$$

which is only possible for all k when

$$\frac{\int_0^\infty \bar{P}_1(t)\, P_2(t)\, \bar{P}_3(t)\, dt}{\int_0^\infty p_1(t)\, P_2(t)\, \bar{P}_3(t)\, dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\, P_3(t)\, dt}{\int_0^\infty p_1(t)\, P_3(t)\, dt} \tag{3.106}$$

If $P_2(t) = 1$ this reduces to equality (3.102) once more. Thus, in the same way as (AA), (AB), and (AC) relationship (AD) is still not true for waiting times distributed according to expressions (3.96) to (3.100), certain values of $t_2 - t_1$ and small $t_3 - t_2$.

3.5 New Logical Relationships

Relationships (AA), (AB), (AC), and (AD) hold true in the special cases of negative exponential or uniform probability density functions for the waiting times. In this section we attempt to develop systematically new "composite" logical relationships that would simply reduce, in the special cases, to consequences of the "single" relationships (AA), (AB), (AC), or (AD).

More precisely, these new logical relationships will be of the type

$$\left.\begin{array}{l} (P_1) \\ \text{and } (P_2) \end{array}\right. \rightarrow (P_3)$$

where, at least one of the relationships $(P_1) \rightarrow (P_3)$, and $(P_2) \rightarrow (P_3)$ is a relationship among (AA), (AB), (AC), and (AD), or their reverse. Most of these new relationships designated by (BA) to (BV) would seem likely to be true in the general case. Of course, those relationships representing just a transitivity relationship are trivially true and will not be included.

Each of the relationships (BA) to (BV) can be studied in a way which is just direct extension of the method used to study relationships (AA), (AB), (AC), and (AD). Therefore, in order to avoid a lengthy presentation here, proofs, similar to those shown for (AA), (AB), (AC), and (AD) are relegated to an appendix. Only the results are presented here.

i)
$$\begin{array}{l}(1,0,1)\text{ better than }(1,1,1)\\ (1,1,0)\text{ better than }(1,1,1)\end{array} \rightarrow (1,0,0)\text{ better than }(1,1,1) \qquad \text{(BA)}$$

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$ $\qquad p_2(t) = p_3(t) = \frac{pa^p}{(t+a)^{p+1}}$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

ii)
$$\begin{array}{l}(1,0,1)\text{ better than }(1,1,1)\\ (1,0,0)\text{ better than }(1,1,1)\end{array} \rightarrow (1,0,0)\text{ better than }(1,0,1) \qquad \text{(BB)}$$

This logical relationship is in fact equivalent to the relationships (AD) and (AC),

$$(1,0,0)\text{ better than }(1,1,1) \rightarrow (1,0,0)\text{ better than }(1,0,1) \qquad \text{(AD)}$$

$$(1,0,0)\text{ better than }(1,1,1) \rightarrow (1,0,0)\text{ better than }(1,0,1) \qquad \text{(AC)}$$

which have been proven to be false in the last section.

iii) (1,0,1) better than (1,1,1)
(1,0,0) better than (1,1,1)
→ (1,0,0) better than (1,1,0) (BC)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$ $\quad p_2(t) = p_3(t) = \frac{pa^p}{(t+a)^{p+1}}$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

iv) (1,0,1) better than (1,1,1)
(1,0,0) better than (1,1,0)
→ (1,0,0) better than (1,1,1) (BD)

No counterexample has been found; and it works with

$$p_1(t) = \frac{na^n}{(t+a)^{n+1}}, \quad p_2(t) = \frac{pa^p}{(t+a)^{p+1}}, \quad p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$$

However, no proof has been found either.

v) (1,1,0) better than (1,1,1)
(1,0,0) better than (1,1,1)
→ (1,0,0) better than (1,0,1) (BE)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$ $\quad p_2(t) = p_3(t) = \frac{pa^p}{(t+a)^{p+1}}$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

vi) (1,0,1) better than (1,1,1)
(1,0,0) better than (1,0,1)
→ (1,0,0) better than (1,1,0) (BF)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$ $\quad p_2(t) = \frac{pa^p}{(t+a)^{p+1}}$ $\quad p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$

with $q > p$, and certain values of $t_2 - t_1$, and small $t_3 - t_2$.

vii) (1,0,1) better than (1,1,1)
(1,0,0) better than (1,1,0) → (1,0,0) better than (1,0,1) (BG)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$, $p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$ and $P_2(t) \simeq 1$ for $t \in \mathbb{R}^+_*$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

viii) (1,0,1) better than (1,1,1)
(1,1,0) better than (1,1,1) → (1,0,0) better than (1,1,0) (BH)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$ $p_2(t) = p_3(t) = \frac{pa^p}{(t+a)^{p+1}}$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

ix) (1,0,1) better than (1,1,1)
(1,1,0) better than (1,1,1) → (1,0,0) better than (1,0,1) (BI)

is not true for same conditions.

x) (1,0,0) worse than (1,0,1)
(1,1,0) worse than (1,1,1) → (1,0,0) worse than (1,1,1) (BJ)

As with relationship (BD), no counterexample has been found; and it also works with the particular type of distributions (3.55). No proof has been found.

xi) (1,0,0) worse than (1,0,1)
(1,0,0) worse than (1,1,0) → (1,0,0) worse than (1,1,1) (BK)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$, $p_2(t) = \frac{pa^p}{(t+a)^{p+1}}$, $p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

xii) (1,0,0) worse than (1,1,0)
(1,0,0) worse than (1,0,1)
→ (1,0,1) worse than (1,1,1) (BL)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$, $p_2(t) = p_3(t) = \frac{pa^p}{(t+a)^{p+1}}$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

xiii) (1,0,0) worse than (1,0,1)
(1,0,0) worse than (1,1,1)
→ (1,0,1) worse than (1,1,1) (BM)

This logical relationship is in fact equivalent to the relationships
(1,0,0) worse than (1,0,1) → (1,0,1) worse than (1,1,1)
(1,0,0) worse than (1,1,1) → (1,0,1) worse than (1,1,1)
i.e. relationships (AD) and (AB),

(1,0,1) better than (1,1,1) → (1,0,0) better than (1,0,1) (AD)
(1,0,1) better than (1,1,1) → (1,0,0) better than (1,1,1) (AB)

which have been proven to be false, in the last section.

xiv) (1,0,0) worse than (1,1,0)
→ (1,0,1) worse than (1,1,1) (BN)
(1,1,0) worse than (1,1,1)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$, $p_2(t) = \frac{pa^p}{(t+a)^{p+1}}$, $p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$

with $q > p$, certain values of $t_2 - t_1$, and small $t_3 - t_2$.

xv) (1,0,0) worse than (1,1,0)
→ (1,0,1) worse than (1,1,1) (BO)
(1,1,0) worse than (1,1,1)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$, $p_2(t) = p_3(t) = \frac{pa^p}{(t+a)^{p+1}}$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

xvi) (1,0,0) worse than (1,0,1)
→ (1,0,1) worse than (1,1,1) (BP)
(1,1,0) worse than (1,1,1)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$, $p_2(t) = \frac{pa^p}{(t+a)^{p+1}}$, $p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$

with $q > p$, certain values of $t_2 - t_1$, and small $t_3 - t_2$.

xvii) (1,0,0) worse than (1,1,1)
→ (1,0,1) worse than (1,1,1) (BQ)
(1,1,0) worse than (1,1,1)

is not true for same conditions.

xviii) (1,0,1) better than (1,1,1)
→ (1,0,0) better than (1,1,1) (BR)
(1,0,1) better than (1,1,0)

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$, $p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$ and $p_2(t) \simeq 1$ for $t \in \mathbb{R}^+_*$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

xix) (1,0,1) better than (1,1,1)
→ (1,0,0) better than (1,1,0) (BS)
(1,0,1) better than (1,1,0)

is not true for same conditions.

xx) (1,0,1) better than (1,1,1)
→ (1,0,0) better than (1,0,1) (BT)
(1,0,1) better than (1,1,0)

is simply false, because we have seen, in the last section, that (1,0,1) could be the optimal strategy.

xxi) (1,0,0) better than (1,1,1)
→ (1,0,0) better than (1,0,1) (BU)
(1,1,0) better than (1,0,1)

is not true for $p_1(t) = p_3(t) = \frac{2a^2}{(t+a)^3}$, $p_2(t) = \frac{3a^3}{(t+a)^4}$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

$$\text{xxii)}\quad \left.\begin{array}{l}(1,0,0)\text{ better than }(1,1,1)\\ (1,0,1)\text{ better than }(1,1,0)\end{array}\right. \rightarrow (1,0,0)\text{ better than }(1,0,1)\quad \text{(BV)}$$

is not true for $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$, $p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$ and $P_2(t) \simeq 1$ for $t \in \mathbb{R}_*^+$

with certain values of $t_2 - t_1$, and small $t_3 - t_2$.

At this point, we can see that most of these composite relationships developed from single relationships that are true in the negative exponentially and uniformly distributed waiting times cases, have a counterexample, and therefore are not true, even if they were thought intuitively to be true. All counterexamples used were constructed with the distribution $p(t) = \frac{na^n}{(t+a)^{n+1}}$, that we have described previously, and have used already extensively. Thus, unfortunately, very few indications are given by the preceding analysis on how it might be possible to develop a new algorithm for the general case.

However, for two of the relationships, (BD) and (BJ), the usual ways of finding and constructing counterexamples have not worked. These relationships that could be true in the general case are

$$\left.\begin{array}{l}(1,0,1)\text{ better than }(1,1,1)\\ (1,0,0)\text{ better than }(1,1,0)\end{array}\right. \rightarrow (1,0,0)\text{ better than }(1,1,1)\quad \text{(BD)}$$

$$\left.\begin{array}{l}(1,0,0)\text{ worse than }(1,0,1)\\ (1,1,0)\text{ worse than }(1,1,1)\end{array}\right. \rightarrow (1,0,0)\text{ worse than }(1,1,1)\quad \text{(BJ)}$$

Both relationships do make sense. With regard to (BD), "(1,0,1) better than (1,1,1)" means that it is better not to use route number 2 and "(1,0,0) better than (1,1,0)" means that it is better not to use route number 3, while "(1,0,0) better than (1,1,1)", on the other side of the → symbol, means that it is better not to use any of routes number 2 and 3. Similarly, with regard to (BJ), "(1,0,0) worse than (1,0,1)" means that it is better to use route number 3 and "(1,1,0) worse than (1,1,1)" means that it is better to use route number 2, while "(1,0,0) worse than (1,1,1)", on the other side of the → symbol, means that it is better to use both routes 2 and 3.

Nevertheless, no proof of relationships (BD) and (BJ) has been found in the general case.

3.6 Study of the 4-Route Case

In the 3-route case, we have seen that $T_x - T_y$, for any two strategies X and Y, could always be expressed using two general functions Φ and ψ, defined by (3.12) and (3.13), in a way showing the dependency on and the variations with the differences between the in-vehicle travel times t_1, t_2, t_3.

We will see now that the 4-route case behaves as a (quite complicated) extension of the 3-route case. Some indications will be given on how fast the complexity of the situation increases as the number of routes increases.

Let $A = (1,0,0,0)$ (3.107)
$B = (1,1,0,0)$ (3.108)
$C = (1,0,1,0)$ (3.109)
$D = (1,1,1,0)$ (3.110)
$E = (1,0,0,1)$ (3.111)
$F = (1,1,0,1)$ (3.112)
$G = (1,0,1,1)$ (3.113)
$H = (1,1,1,1)$ (3.114)

These are all the possible strategies that include route number 1. [Remark: the number of strategies that can be optimal increases very fast with the number of routes n: the number of strategies is 2^{n-1}.] The number of possible comparisons between strategies is then $(2^{n-1}-1)\cdot 2^{n-2}$.

From the 3-line case, we have T_B-T_A, T_C-T_A, T_D-T_C, T_D-T_B, T_D-T_A, and T_C-T_B, given by formulae (3.14) to (3.19). Now, we can calculate the other differences. Derivations are not reproduced here because they are very similar to those of the 3-line case: T_E, T_F, T_G, T_H are obtained in using the general formula (1.81), then the expressions for the differences are derived as previously, factoring out t_2-t_1, t_3-t_2, t_4-t_3 and integrating (and simplifying) the directly integrable parts of the expressions.

Similarly as previously, all T_x-T_y's can be expressed in the form

$$T_X-T_Y = \Phi(f_{XY}) + \psi_1(g_{XY}) + \psi_2(h_{XY}) \tag{3.115}$$

where $\Phi(f)$, $\psi_1(f)$, $\psi_2(f)$, are general functions of which the argument $f(t)$ is a function of time t, and defined by the expressions (3.116), (3.117), (3.118).

$$\Phi(f) = \int_0^\infty [(t_2-t_1)p_1(t)-\bar{P}_1(t)]\, f(t)\, dt \tag{3.116}$$

$$\psi_1(f) = (t_3-t_2)\int_0^\infty f(t)\, dt \tag{3.117}$$

$$\psi_2(f) = (t_4-t_3)\int_0^\infty p_4(t)\, \bar{P}_1(t)\, f(t)\, dt \tag{3.118}$$

All three functions $\Phi(f)$, $\psi_1(f)$, $\psi_2(f)$ are linear in f. $\psi_1(f)$ and $\psi_2(f)$ are respectively proportional to $t_3 - t_2$ and $t_3 - t_4$, and are increasing ($\psi_1(f) > \psi_2(f)$ whenever $f(t) > g(t)$ for any t.) $\psi_1(f)$ and $\psi_2(f)$ are also positive when f(t) is positive for any t. Φ is the same function as previously (see expression (3.12)). ψ_2 is the extension in the 4-route case, of the 3-route case ψ: see expression (3.13).

When $f_{xy}(t) > 0$ (resp. < 0) for any t, $T_x - T_y$ increases (resp. decreases) with $t_2 - t_1$. (CA)

When $g_{xy}(t) > 0$ (resp. < 0) for any t, $T_x - T_y$ increases (resp. decreases) with $t_3 - t_2$. (CB)

When $h_{xy}(t) > 0$ (resp. < 0) for any t, $T_x - T_y$ increases (resp. decreases) with $t_4 - t_3$. (CC)

The type of behavior implied by these properties is consistent with what one might reasonably expect.

General results are given by the following table:

Table 1

$T_x - T_y$

x	y	f_{xy}		g_{xy}		h_{xy}	
B(1100)	A(1000)	P_2	(+)	0		0	
D(1110)	C(1010)	$P_2\overline{P}_3$	(+)	$-p\ \overline{P}\ P$	(-)	0	
F(1101)	E(1001)	$P_2\overline{P}_4$	(+)	$-p_4\overline{P}_1P_2$	(-)	$-P_2$	(-)
H(1111)	G(1011)	$P_2\overline{P}_3\overline{P}_4$	(+)	$-(p_3\overline{P}_4+p_4\overline{P}_3)\overline{P}_1P_2$	(-)	$-P_2\overline{P}_3$	(-)
C(1010)	A(1000)	P_3	(+)	$p_3\overline{P}_1$(or p_1P_3)	(+)	0	
D(1110)	B(1100)	$\overline{P}_2P_3$	(+)	$p_3\overline{P}_1\overline{P}_2$	(+)	0	
G(1011)	E(1001)	$P_3\overline{P}_4$	(+)	$p_1P_3\overline{P}_4$	(+)	$-P_3$	(-)
H(1111)	F(1101)	$\overline{P}_2P_3\overline{P}_4$	(+)	$(p_1\overline{P}_2+ p_2\overline{P}_1)P_3\overline{P}_4$	(+)	$-\overline{P}_2P_3$	(-)
E(1001)	A(1000)	P_4	(+)	$p_4\overline{P}_1$ (or p_4P_1)	(+)	1	(+)
F(1101)	B(1100)	$\overline{p}_2P_4$	(+)	$p_4\overline{P}_1\overline{P}_2$	(+)	$\overline{P}_2$	(+)
G(1011)	C(1010)	$\overline{P}_3P_4$	(+)	$p_1\overline{P}_3P_4$	(+)	$\overline{P}_3$	(+)
H(1111)	D(1110)	$\overline{P}_2\overline{P}_3P_4$	(+)	$(p_1\overline{P}_2+ p_2\overline{P}_1)\overline{P}_3P_4$	(+)	$\overline{P}_2\overline{P}_3$	(+)
D(1110)	A(1000)	$1-\overline{P}_2\overline{P}_3$	(+)	$p_3\overline{P}_1\overline{P}_2$	(+)	0	
H(1111)	E(1001)	$(1-\overline{P}_2\overline{P}_3)\overline{P}_4$	(+)	$p_1(1-\overline{P}_2\overline{P}_3)\overline{P}_4-p_2\overline{P}_1\overline{P}_3\overline{P}_4^{*}$		$1-\overline{P}_2\overline{P}_3$	(+)
F(1101)	A(1000)	$1-\overline{P}_2\overline{P}_4$	(+)	$p_4\overline{P}_1\overline{P}_2$	(+)	$\overline{P}_2$	(+)
H(1111)	C(1010)	$(1-\overline{P}_2\overline{P}_4)\overline{P}_3$	(+)	$p_1(1-\overline{P}_2\overline{P}_4)\overline{P}_3-p_2\overline{P}_1P_3\overline{P}_4^{**}$		$\overline{P}_2\overline{P}_3$	(+)
G(1011)	A(1000)	$1-\overline{P}_3\overline{P}_4$	(+)	$1-\overline{P}_3\overline{P}_4$	(+)	$\overline{P}_3$	(+)
H(1111)	B(1100)	$(1-\overline{P}_3\overline{P}_4)\overline{P}_2$	(+)	$(p_1\overline{P}_2+p_2\overline{P}_1)(1-\overline{P}_3\overline{P}_4)$	(+)	$\overline{P}_2\overline{P}_3$	(+)
H(1111)	A(1000)	$1-\overline{P}_2\overline{P}_3\overline{P}_4$	(+)	$(p_1\overline{P}_2+p_2\overline{P}_1)(1-\overline{P}_3\overline{P}_4)$	(+)	$\overline{P}_2\overline{P}_3$	(+)
C(1010)	B(1100)	$\overline{P}_2-\overline{P}_3$		$p_3\overline{P}_1$ (or p_1P_3)	(+)	0	
G(1011)	F(1101)	$(\overline{P}_2-\overline{P}_3)\overline{P}_4$		$p_1P_3\overline{P}_4+p_4\overline{P}_1P_2$	(+)	$-(\overline{P}_2-\overline{P}_3)$	
E(1001)	B(1100)	$\overline{P}_2-\overline{P}_4$		$p_4\overline{P}_1$ (or p_1P_4)	(+)	1	(+)
G(1011)	D(1110)	$(\overline{P}_2-\overline{P}_4)\overline{P}_3$		$p_1(\overline{P}_2-\overline{P}_4)\overline{P}_3+ p_2\overline{P}_1\overline{P}_3$		$\overline{P}_3$	(+)
E(1001)	C(1010)	$\overline{P}_3-\overline{P}_4$		$p_1(\overline{P}_3-\overline{P}_4)$		1	(+)

Table 1 (continued)

$T_x - T_y$

x	y	f_{xy}	g_{xy}		h_{xy}	
F(1101)	D(1110)	$(\bar{P}_3-\bar{P}_4)\bar{P}_2$	$(p_1\bar{P}_2+p_2\bar{P}_1)(\bar{P}_3-\bar{P}_4)$		$\bar{P}_2$	(+)
G(1011)	B(1100)	$\bar{P}_2-\bar{P}_3\bar{P}_4$	$p_1(1-\bar{P}_3\bar{P}_4)$	(+)	$\bar{P}_3$	(+)
F(1101)	C(1010)	$\bar{P}_3-\bar{P}_2\bar{P}_4$	$p_1(\bar{P}_3-\bar{P}_2\bar{P}_4)-p_2\bar{P}_1\bar{P}_4$		$\bar{P}_2$	(+)
D(1110)	E(1001)	$\bar{P}_4-\bar{P}_2\bar{P}_3$	$p_1(\bar{P}_4-\bar{P}_2\bar{P}_3)-p_2\bar{P}_1\bar{P}_3$		-1	(-)

Notes: * or $-p_4\bar{P}_1(1-\bar{P}_2\bar{P}_3) + p_3\bar{P}_1\bar{P}_2\bar{P}_4$

** or $-p_3\bar{P}_1(1-\bar{P}_2\bar{P}_4) + p_4\bar{P}_1\bar{P}_2\bar{P}_3$

The f_{xy} column gives the argument of Φ in the expression of $T_x - T_y$ as a function of Φ, ψ_1, and ψ_2 for any two strategies X and Y. The g_{xy} and h_{xy} columns give respectively the arguments of ψ_1 and ψ_2.

A (+) (resp. a (−)) in the f_{xy} column means that $f_{xy}(t)$ is always positive (resp. negative) and therefore that $T_x - T_y$ increases (resp. decreases) as $t_2 - t_1$ increases. Similarly, a (+), (resp. a (−)) in the g_{xy} column means that $g_{xy}^{(t)}$ is always positive (resp. negative) and that $T_x - T_y$ increases (resp. decreases) as $t_3 - t_2$ increases; and a (+) (resp. a (−)) in the h_{xy} column means that $h_{xy}(t)$ is always positive (resp. negative) and that $T_x - T_y$ increases (resp. decreases) as $t_4 - t_3$ increases.

For example, the last row gives that

$$T_D - T_E = \Phi(\bar{P}_4 - \bar{P}_2\bar{P}_3) + \psi_1\left(-p_1(\bar{P}_4 - \bar{P}_2\bar{P}_3) - p_2\bar{P}_1\bar{P}_3\right) + \psi_2(-1) \tag{3.119}$$

i.e. using (3.116), (3.117), (3.118)

$$\begin{aligned} T_D - T_E = & \int_0^\infty \left[(t_2 - t_1)p_1(t) - \bar{P}_1(t)\right]\left[\bar{P}_4(t) - \bar{P}_2(t)\bar{P}_3(t)\right]dt \\ & + (t_3 - t_2)\int_0^\infty \left[-p_1(t)\left(\bar{P}_4(t) - \bar{P}_2(t) \times \bar{P}_3(t)\right) - p_2(t)\bar{P}_1(t)\bar{P}_3(t)\right]dt \\ & - (t_4 - t_3)\int_0^\infty p_4(t)\bar{P}_1(t)\,dt \end{aligned} \tag{3.120}$$

$T_D - T_E$ decreases as $t_4 - t_3$ increases, but there is no rule about the variations with respect to $t_2 - t_1$ and $t_3 - t_2$.

Although the 4-route case is obviously an extension of the 3-route case, it is difficult to detect from the 3-route case results and from this table, any interesting pattern linking the 3-route case and the 4-route case, which could be extended to the general case of n routes and could provide some indications on the behavior of the most general case.

Chapter 4

Clever Passenger

4.1 The "Clever" Passenger

The idea of the "clever" passenger is suggested by C. Chriqui and P. Robillard in their paper.

The "clever" passenger is the passenger who looks at the time he/she already waited at the bus stop and continuously updates the strategy he/she will choose depending on this time.

The choice problem of the "clever" passenger is an extension of the problem studied in the previous chapters. The optimal strategy for the previous problem will not necessarily be always optimal for the "clever" passenger; it will be only the strategy he/she chooses at his/her time of arrival at the bus stop.

4.2 Expected Total Travel Time

Obviously, formula (1.8) derived previously is no longer a valid expression for the expected total travel time. It just expresses the expected total travel time at the moment the passenger arrives at the bus stop.

Let X be the strategy of using just any route, X' a strategy corresponding to a given subset of the set of all routes, and t_o the time the passenger has already waited at the bus stop. Now, since we assume that the passenger has already waited t_o and takes this time into account, we have to make all expectations conditional on the fact, the waiting time w_x is greater than t_o

$$w_x > t_o \tag{4.1}$$

The expected value of the waiting time given this condition is

$$E(w_X/w_X>t_0)=\int_0^\infty t\cdot \text{prob}\,(t<w_X<t+dt/w_X>t_0) \tag{4.2}$$

$$\text{prob}\,(t<w_X<t+dt/w_X>t_0)=\frac{\text{prob}(t<w_X<t+dt)\cdot\text{prob}\,(w_X>t_0/w_X=t)}{\text{prob}\,(w_X>t_0)}$$

$$=\begin{cases}\dfrac{p_X(t)\,dt}{\bar{P}_X(t_0)} & \text{if} \quad t>t_0\\ 0 & \text{if} \quad t<t_0\end{cases} \tag{4.3}$$

Then substituting (4.3) into (4.2), we obtain

$$E(w_X/w_X>t_0)=\frac{\int_{t_0}^\infty t\,p_X(t)\,dt}{\bar{P}_X(t_0)}$$

$$=\frac{[-t\bar{P}_X(t)]_{t_0}^\infty+\int_{t_0}^\infty \bar{P}_X(t)\,dt}{\bar{P}_X(t_0)} \qquad \text{(integrating by parts)}$$

$$=t_0+\frac{\int_{t_0}^\infty \bar{P}_X(t)\,dt}{\bar{P}_X(t_0)}$$

$$=t_0+\frac{\int_0^\infty \bar{P}_X(t+t_0)\,dt}{\bar{P}_X(t_0)} \tag{4.4}$$

Let X" be the strategy of using any route except those used in strategy X. We have similarly:

$$E(w_{X'}/w_X>t_0)=\int_0^\infty t\cdot\text{prob}\,(t<w_{X'}<t+dt/w_X>t_0) \tag{4.5}$$

$$\text{prob}(t<w_{X'}<t+dt/w_X>t_0)=\frac{\text{prob}(t<w_{X'}<t+dt)\cdot\text{prob}(w_X>t_0/w_{X'}=t)}{\text{prob}(w_X>t_0)}$$

$$=\frac{p_{X'}(t)\cdot\text{prob}\,(w_X>t_0/w_{X'}=t)\,dt}{\bar{P}_X(t_0)} \tag{4.6}$$

$$\mathrm{prob}(w_X > t_0 / w_{X'} = t) = \mathrm{prob}(w_{X'} > t_0 / w_{X'} = t) \cdot \mathrm{prob}(w_{X''} > t_0 / w_{X'} = t) \tag{4.7}$$

$$\mathrm{prob}(w_{X'} > t_0 / w_{X'} = t) = \begin{cases} 1 & \text{if } t > t_0 \\ 0 & \text{if } t < t_0 \end{cases} \tag{4.8}$$

$$\mathrm{prob}(w_{X''} > t_0) = \bar{P}_{X''}(t_0) \tag{4.9}$$

Combining (4.6), (4.7), (4.8), and (4.9) and substituting in (4.5), we obtain:

$$E(w_{X'} / w_X > t_0) = \frac{\int_{t_0}^{\infty} t \cdot p_{X'}(t)\,dt \quad \bar{P}_{X''}(t_0)}{\bar{P}_X(t_0)}$$

$$= \frac{\int_{t_0}^{\infty} t \cdot p_{X'}(t)\,dt}{\bar{P}_{X'}(t_0)}$$

$$= t_0 + \frac{\int_{t_0}^{\infty} \bar{P}_{X'}(t)\,dt}{\bar{P}_{X'}(t_0)}$$

(integrating by parts as previously)

$$= t_0 + \frac{\int_{0}^{\infty} \bar{P}_{X'}(t + t_0)\,dt}{\bar{P}_{X'}(t_0)} \tag{4.10}$$

Let r be a route included in strategy X and X_r be the strategy of taking any route but route number r. The probability that a bus of route r arrives first is:

$$H_r(X) = \mathrm{prob}(w_r = w_X / w_X > t_0)$$

$$= \int_0^{\infty} \mathrm{prob}(t < w_r < t + dt / w_X > t_0) \cdot \mathrm{prob}(w_{X_r} > t / w_X > t_0) \tag{4.11}$$

$$\mathrm{prob}(t < w_r < t + dt / w_X > t_0) = \begin{cases} \dfrac{p_r(t)\,dt}{\bar{P}_r(t_0)} & \text{if } t > t_0 \\ 0 & \text{if } t < t_0 \end{cases} \tag{4.12}$$

(using the previous result of prob($t<w_{x'}<t+dt/w_x>t_o$) with x' being the strategy of using route number r only).

$$\text{prob}\,(w_{x_r}>t/w_x>t_o)=\frac{\text{prob}\,(w_{x_r}>t).\ \text{prob}\,(w_x>t_o/w_{x_r}>t)}{\text{prob}\,(w_x>t_o)}$$

$$=\frac{\bar{P}_{x_r}(t).\ \text{prob}(w_x>t_o/w_{x_r}>t)}{\bar{P}_x(t_o)} \qquad (4.13)$$

$$\text{prob}(w_x>t_o/w_{x_r}>t)=\text{prob}\,(w_{x_r}>t_o/w_{x_r}>t)\cdot\text{prob}\,(w_r>t_o)$$

$$=\text{prob}\,(w_{x_r}>t_o/w_{x_r}>t)\ .\ \bar{P}_r(t_o) \qquad (4.14)$$

$$\text{prob}\,(w_{x_r}>t_o/w_{x_r}>t)=\begin{cases}1 & \text{if } t>t_o\\ \dfrac{\text{prob}\,(w_{x_r}>t/w_{x_r}>t_o)\cdot\text{prob}(w_{x_r}>t_o)}{\text{prob}\,(w_{x_r}>t)}=\dfrac{\bar{P}_{x_r}(t_o)}{\bar{P}_{x_r}(t)} & \text{if } t<t_o.\end{cases} \qquad (4.15)$$

Combining (4.13), (4.14), and (4.15), we get

$$\text{prob}(w_{x_r}>t/w_x>t_o)=\begin{cases}\dfrac{\bar{P}_{x_r}(t)\cdot\bar{P}_r(t_o)}{\bar{P}_x(t_o)}=\dfrac{\bar{P}_{x_r}(t)}{\bar{P}_{x_r}(t_o)} & \text{if } t>t_o\\[2ex] \dfrac{\bar{P}_{x_r}(t)\cdot\dfrac{\bar{P}_{x_r}(t_o)}{\bar{P}_{x_r}(t)}\cdot\bar{P}_r(t_o)}{\bar{P}_x(t_o)}=1 & \text{if } t<t_o\end{cases} \qquad (4.16)$$

Substituting (4.12) and (4.16) in (4.11), we obtain

$$H_r(X)=\int_{t_o}^{\infty}\frac{p_r(t)\,\bar{P}_{x_r}(t)}{\bar{P}_r(t_o)\,\bar{P}_{x_r}(t_o)}\,dt$$

$$=\frac{\int_{t_o}^{\infty}p_r(t)\,\bar{P}_{x_r}(t)\,dt}{\bar{P}_r(t_o)\cdot\bar{P}_{x_r}(t_o)}$$

$$=\frac{\int_{0}^{\infty}p_r(t+t_o)\,\bar{P}_{x_r}(t+t_o)\,dt}{\bar{P}_x(t_o)} \qquad (4.17)$$

Let now r be a route included in strategy X' and X'_r be the strategy of taking any route included in strategy X', but route number r. We have similarly:

$$H_r(X') = \int_0^\infty \text{prob}\,(t < w_r < t+dt \,/\, w_X > t_0) \cdot \text{prob}\,(w_{X'_r} > t \,/\, w_X > t_0) \tag{4.18}$$

$$\text{prob}\,(w_{X'_r} > t \,/\, w_X > t_0) = \frac{\text{prob}\,(w_{X'_r} > t) \cdot \text{prob}\,(w_X > t_0 \,/\, w_{X'_r} > t)}{\text{prob}\,(w_X > t_0)}$$

$$= \frac{\bar{P}_{X'_r}(t) \;\; \text{prob}\,(w_X > t_0 \,/\, w_{X'_r} > t)}{\bar{P}_X(t_0)} \tag{4.19}$$

$$\text{prob}\,(w_X > t_0 \,/\, w_{X'_r} > t) = \text{prob}\,(w_{X'_r} > t_0 \,/\, w_{X'_r} > t) \cdot \text{prob}\,(w_r > t_0) \cdot \text{prob}\,(w_{X''} > t_0)$$

$$= \text{prob}\,(w_{X'_r} > t_0 \,/\, w_{X'_r} > t) \cdot \bar{P}_r(t_0) \cdot \bar{P}_{X''}(t_0). \tag{4.20}$$

Similarly to (4.15):

$$\text{prob}\,(w_{X'_r} > t_0 \,/\, w_{X'_r} > t) = \begin{cases} 1 & \text{if } t > t_0 \\ \dfrac{\bar{P}_{X'_r}(t_0)}{\bar{P}_{X'_r}(t)} & \text{if } t < t_0 \end{cases} \tag{4.21}$$

Combining (4.19), (4.20), and (4.21), we get

$$\text{prob}\,(w_{X'_r} > t \,/\, w_X > t_0) = \begin{cases} \dfrac{\bar{P}_{X'_r}(t) \cdot \bar{P}_r(t_0) \cdot \bar{P}_{X''}(t_0)}{\bar{P}_X(t_0)} = \dfrac{\bar{P}_{X'_r}(t)}{\bar{P}_{X'_r}(t_0)} & \text{if } t > t_0 \\[2ex] \dfrac{\bar{P}_{X'_r}(t) \, \dfrac{\bar{P}_{X'_r}(t_0)}{\bar{P}_{X'_r}(t)} \, \bar{P}_r(t_0) \, \bar{P}_{X''}(t_0)}{\bar{P}_X(t_0)} = 1 & \text{if } t < t_0 \end{cases} \tag{4.22}$$

Substituting (4.12) and (4.22) in (4.18), we obtain

$$H_r(X') = \int_{t_0}^{\infty} \frac{p_r(t)\cdot \bar{P}_{X'r}(t)}{\bar{P}_r(t_0)\,\bar{P}_{X'r}(t_0)}\,dt$$

$$= \frac{\int_{t_0}^{\infty} p_r(t)\cdot \bar{P}_{X'r}(t)\,dt}{\bar{P}_{X'}(t_0)}$$

$$= \frac{\int_0^{\infty} p_r(t+t_0)\,\bar{P}_{X'r}(t+t_0)\,dt}{\bar{P}_X(t_0)} \qquad (4.23)$$

Finally, the expected total travel time with strategy X is, using equations (4.4) and (4.17)

$$T_X = t_0 + \frac{\int_0^{\infty}\left(\bar{P}_X(t+t_0) + \sum_{r\in X} t_r\, p_r(t+t_0)\,\bar{P}_{X_r}(t+t_0)\right)dt}{\bar{P}_X(t_0)}$$

$$= t_0 + \frac{\int_0^{\infty}\left(\prod_{r\in X}\bar{P}_r(t+t_0) + \sum_{r\in X} t_r\, p_r(t+t_0)\prod_{\substack{i\in X\\ i\neq r}}\bar{P}_i(t+t_0)\right)dt}{\prod_{r\in X}\bar{P}_r(t_0)} \qquad (4.24)$$

In the same way, the expected total travel time with strategy X' is, using equations (4.10) and (4.23)

$$T_{X'} = t_0 + \frac{\int_0^{\infty}\left(\bar{P}_{X'}(t+t_0) + \sum_{r\in X'} t_r\, p_r(t+t_0)\,\bar{P}_{X'_r}(t+t_0)\right)dt}{\bar{P}_{X'}(t_0)}$$

$$= t_0 + \frac{\int_0^{\infty}\left(\prod_{r\in X'}\bar{P}_r(t+t_0) + \sum_{r\in X'} t_r\, p_r(t+t_0)\prod_{\substack{i\in X'\\ i\neq r}}\bar{P}_i(t+t_0)\right)dt}{\prod_{r\in X'}\bar{P}_r(t_0)} \qquad (4.25)$$

We note that the same formulae apply for X' as for X, but, although this makes sense, it was necessary to prove it, since in both cases, the condition is $w_x > t_o$.

4.3 A Simple Property

As stated in section 1.4, it can be shown that, the passenger always includes route number 1, the route with the shortest in-vehicle travel time between A and B, in his/her selected routes, no matter how long he/she has already waited.

Let X be a strategy not including route number 1 and x' = x + (1,0 ...0) i.e. the strategy including the same routes as x plus route number 1. [There is no relation with the notations x, x' in the last section.] Let w_o denote now the waiting time for the first bus of any route to arrive.

From equations (4.17)/(4.23), we can write

$$H_r(X) = \frac{\int_0^\infty p_r(t+t_o) \prod_{\substack{i\in X \\ i\neq r}} \bar{P}_i(t+t_o)\,dt}{\prod_{r\in X} \bar{P}_r(t_o)} = \frac{\int_0^\infty p_r(t+t_o) \prod_{\substack{i\in X \\ i\neq r}} \bar{P}_i(t+t_o)\,dt}{\bar{P}_X(t_o)} \tag{4.26}$$

$$H_r(X') = \frac{\int_0^\infty p_r(t+t_o)\bar{P}_1(t+t_o) \prod_{\substack{i\in X \\ i\neq r}} \bar{P}_i(t+t_o)\,dt}{\bar{P}_1(t_o)\cdot \prod_{r\in X} \bar{P}_r(t_o)} = \frac{\int_0^\infty p_r(t+t_o)\bar{P}_1(t+t_o) \prod_{\substack{i\in X \\ i\neq r}} \bar{P}_i(t+t_o)\,dt}{\bar{P}_1(t_o)\cdot \bar{P}_X(t_o)} \tag{4.27}$$

$$H_1(X') = \frac{\int_0^\infty p_1(t+t_o) \prod_{i\in X} \bar{P}_i(t+t_o)\,dt}{\bar{P}_1(t_o)\prod_{r\in X} \bar{P}_r(t_o)} = \frac{\int_0^\infty p_1(t+t_o)\bar{P}_X(t+t_o)\,dt}{\bar{P}_1(t_o)\bar{P}_X(t_o)} \tag{4.28}$$

Then, we have

$$E(t_X / w_o > t_o) = \sum_{r\in X} t_r H_r(X) = \frac{\int_0^\infty \sum_{r\in X} t_r p_r(t+t_o) \prod_{\substack{i\in X \\ i\neq r}} \bar{P}_i(t+t_o)\,dt}{\bar{P}_X(t_o)} \tag{4.29}$$

$$E(t_{X'}/w_o > t_o) = t_1 H_1(X') + \sum_{r\in X} t_r H_r(X')$$

$$= \frac{\int_0^\infty t_1 p_1(t+t_o)\bar{P}_X(t+t_o)\,dt}{\bar{P}_1(t_o)\bar{P}_X(t_o)} + \frac{\int_0^\infty \sum_{r\in X} t_r p_r(t+t_o)\bar{P}_1(t+t_o) \prod_{\substack{i\in X \\ i\neq r}} \bar{P}_i(t+t_o)\,dt}{\bar{P}_1(t_o)\bar{P}_X(t_o)} \tag{4.30}$$

We have

$$\int_0^\infty p_1(t+t_0)\,\bar{P}_x(t+t_0)\,dt = -P_1(t_0)\,\bar{P}_x(t_0) + \int_0^\infty p_x(t+t_0)\,P_1(t+t_0)\,dt$$

$$= -P_1(t_0)\,\bar{P}_x(t_0) + \int_0^\infty \sum_{r\in x} p_r(t+t_0)\prod_{\substack{i\in x\\ i\neq r}} \bar{P}_i(t+t_0)\,P_1(t+t_0)\,dt$$

(integrating by parts) (4.31)

because $p_x(t)$ is the derivative of $-\bar{P}_x(t)$ and $\bar{P}_x(t) = \prod_{r\varepsilon x} \bar{P}_r(t)$.

Then, since $t_1 < t_r$ for all $r\neq 1$, the following inequality holds.

$$\int_0^\infty t_1\,p_1(t+t_0)\,\bar{P}_x(t+t_0)\,dt = -t_1\,P_1(t_0)\,\bar{P}_x(t_0) + \int_0^\infty \sum_{r\in x} t_1\,p_r(t+t_0)\prod_{\substack{i\in x\\ i\neq r}} \bar{P}_i(t+t_0)\,P_1(t+t_0)\,dt$$

$$< -t_1\,P_1(t_0)\,\bar{P}_x(t_0) + \int_0^\infty \sum_{r\in x} t_r\,p_r(t+t_0)\prod_{\substack{i\in x\\ i\neq r}} \bar{P}_i(t+t_0)\,P_1(t+t_0)\,dt$$

$$< \int_0^\infty \sum_{r\in x} t_r\,p_r(t+t_0)\prod_{\substack{i\in x\\ i\neq r}} \bar{P}_i(t+t_0)\,P_1(t+t_0)\,dt \qquad (4.32)$$

Because of expressions (4.29) and (4.30), this means that we have

$$E(t_{x'}/w_0>t_0) < E(t_x/w_0>t_0) \qquad (4.33)$$

We also have

$$w_{x'} \leq w_x \qquad (4.34)$$

because the derivation of (1.20)/(1.9) in Chapter I is still valid, since it is only concerned with the random variables $w_{X'}$, and w_X and not with any kind of their expected values.

Therefore, from the two inequalities (4.33) and (4.34), we obtain

$$T_{X'} = E(w_{X'}/w_0 > t_0) + E(t_{X'}/w_0 > t_0) < T_X = E(w_X/w_0 > t_0) + E(t_X/w_0 > t_0) \qquad (4.35)$$

Strategy X', including route number 1 is better for the passenger than strategy X. Thus, the best strategy always includes route number 1, the quickest route.

4.4 2-Route Case

In this section, we wish to study the "clever" passenger's optimal strategy, in the simplest case with respect to the number of routes, namely the case of two routes.

Let A = (1,0) (4.36)

B = (1,1) (4.37)

We have according to general formulae (4.24)/(4.25)

$$T_A = t_0 + \frac{\int_0^\infty [\bar{P}_1(t+t_0) + t_1 p_1(t+t_0)]\,dt}{\bar{P}_1(t_0)} = t_0 + \frac{\int_0^\infty \bar{P}_1(t+t_0)\,dt}{\bar{P}_1(t_0)} + t_1 \qquad (4.38)$$

$$T_B = t_0 + \frac{\int_0^\infty [\bar{P}_1(t+t_0)\bar{P}_2(t+t_0) + t_1 p_1(t+t_0)\bar{P}_2(t+t_0) + t_2 p_2(t+t_0)\bar{P}_1(t+t_0)]\,dt}{\bar{P}_1(t_0)\bar{P}_2(t_0)} \qquad (4.39)$$

Substracting (4.38) from (4.39),

$$T_B-T_A = \frac{\int_0^\infty \left[\bar{P}_1(t+t_0)(\bar{P}_2(t+t_0)-\bar{P}_2(t_0)) + t_1\, p_1(t+t_0)(\bar{P}_2(t+t_0)-\bar{P}_2(t_0)) + t_2\, p_2(t+t_0)\bar{P}_1(t+t_0)\right] dt}{\bar{P}_1(t_0)\bar{P}_2(t_0)}$$

$$= \frac{\int_0^\infty \left[(t_2-t_1)p_1(t+t_0)-\bar{P}_1(t+t_0)\right]\left[\bar{P}_2(t_0)-\bar{P}_2(t+t_0)\right] dt}{\bar{P}_1(t_0)\bar{P}_2(t_0)}$$

$$+ \frac{t_2\int_0^\infty \left[p_1(t-t_0)\bar{P}_2(t+t_0)+p_2(t+t_0)\bar{P}_1(t+t_0)-\bar{P}_2(t_0)p_1(t+t_0)\right] dt}{\bar{P}_1(t_0)\bar{P}_2(t_0)} \qquad (4.40)$$

The second term can be integrated directly:

$$T_B-T_A = \frac{\int_0^\infty \left[(t_2-t_1)p_1(t+t_0)-\bar{P}_1(t+t_0)\right]\left[\bar{P}_2(t_0)-\bar{P}_2(t+t_0)\right] dt}{\bar{P}_1(t_0)\bar{P}_2(t_0)} + t_2\,\frac{\bar{P}_1(t_0)\bar{P}_2(t_0)-\bar{P}_2(t_0)\bar{P}_1(t_0)}{\bar{P}_1(t_0)\bar{P}_2(t_0)}$$

$$= \frac{\int_0^\infty \left[(t_2-t_1)p_1(t+t_0)-\bar{P}_1(t+t_0)\right]\left[\bar{P}_2(t_0)-\bar{P}_2(t+t_0)\right] dt}{\bar{P}_1(t_0)\bar{P}_2(t_0)} \qquad (4.41)$$

Intuitively, it seems reasonable that the passenger's optimal strategy might consist of accepting both routes at the beginning of his/her waiting for the bus, but only taking the route 1 bus after the waiting time has reached or exceeded a certain fixed threshold value τ, i.e.

$$\text{optimal strategy:} \begin{cases} (1,1) & \text{for } t_o \leq \tau \\ (1,0) & \text{for } t_o \geq \tau \end{cases} \qquad \text{(DA)}$$

with possibly $\tau = 0$ or $+\infty$,

$$\text{but never} \begin{cases} (1,0) & \text{for } t_o \leq \tau \\ (1,1) & \text{for } t_o \geq \tau \end{cases}$$

To explore whether this is in fact true, we need to look at how T_A and T_B very as a function of t_o.

A simiplified expression (4.38) that facilitates its study can be written for T_A:

$$T_A = t_0 + \frac{\int_0^\infty \bar{P}_1(t+t_0)\,dt}{\bar{P}_1(t_0)} + t_1 \qquad (4.38)$$

The remaining waiting time (the passenger has already waited to) is

$$T_A - t_0 - t_1 = \frac{\int_0^\infty \bar{P}_1(t+t_0)\,dt}{\bar{P}_1(t_0)} \qquad (4.39)$$

In the case of negative exponentially distributed w_1, ($p_1(t) = ae^{-at}$) we can see that

$$T_A = t_0 + \frac{1}{a} + t_1 \qquad (4.40)$$

$$\text{i.e.} \quad T_A - t_0 - t_1 = \frac{1}{a} \qquad (4.41)$$

The remaining waiting time is constant in this case.

In the case of uniformly distributed w_1, $(p_1(t) = \begin{cases} 1/a & \text{for } t<a \\ 0 & \text{for } t>a; \end{cases}$

$\bar{P}_1(t) = \begin{cases} 1-t/a & \text{for } t<a \\ 0 & \text{for } t>a) \end{cases}$

we obtain

$$\int_0^\infty \bar{P}_1(t+t_0)\,dt = \left[t - \frac{t^2}{2a}\right]_{t_0}^{a} = \frac{t_0^2}{2a} - t_0 + \frac{a}{2} \qquad (4.42)$$

and then, simplifying,

$$T_A = t_0 + \frac{a-t_0}{2} + t_1 \qquad (4.43)$$

i.e. $$T_A - t_0 - t_1 = \frac{a-t_0}{2} \qquad (4.44)$$

Hence, in that case,

$$T_A - t_0 - t_1 = 0 \quad \text{for} \quad t_0 = a \qquad (4.45)$$

The remaining waiting time is zero, as we reach the maximum value of the waiting time, a.

This result, intuitively, can be generalized for all waiting times with a finite maximum value a, i.e.

$$p_1(t) = 0 \qquad \text{for} \quad t > a^+ \qquad (4.46)$$

and thus $$\bar{P}_1(t) = 0 \qquad \text{for} \quad t \geqslant a \qquad (4.47)$$

so that, after (4.39),

$$T_A - t_0 - t_1 = \frac{\int_0^\infty \bar{P}_1(t+t_0)\,dt}{\bar{P}_1(t_0)} = \frac{\int_{t_0}^\infty \bar{P}_1(t)\,dt}{\bar{P}_1(t_0)} = \frac{\int_{t_0}^a \bar{P}_1(t)\,dt}{\bar{P}_1(t_0)} \qquad (4.48)$$

The functions of t_o, $\int_{to}^{a} \bar{P}_1(t)dt$ and $\bar{P}_1(t_o)$, can be expanded according to a Taylor series, when t_o is close to a:

$$\int_{t_0}^{a} \bar{P}_1(t)\,dt = p_1(a)\,\frac{(a-t_0)^2}{2} - p_1'(a)\,\frac{(a-t_0)^3}{6} + \cdots \tag{4.49}$$

$$\bar{P}_1(t_0) = p_1(a)(a-t_0) + p_1'(a)\,\frac{(a-t_0)^2}{2} + \cdots \tag{4.50}$$

and then:

$$T_A - t_0 - t_1 = \frac{\int_{t_0}^{a} \bar{P}_1(t)\,dt}{\bar{P}_1(t_0)} = \frac{a-t_0}{2} - \frac{5}{12}\,\frac{p_1'(a)}{p_1(a)}\,(a-t_0)^2 + \cdots \tag{4.51}$$

$$(p_1(a) \neq 0)$$

then $T_A - t_0 - t_1 \simeq \dfrac{a-t_0}{2}$ (4.52)

for to close a, and particularly,

$$T_A - t_0 - t_1 = 0 \qquad \text{for } t_0 = a. \tag{4.45}$$

Moreover, since $T_A - t_0 - t_1 = \int_0^\infty \bar{P}_1(t)dt > 0$ for $t_0 = 0$, the quantity $T_A - t_1 - t_0$, the remaining waiting time, decreases globally as t_0 increases towards a, as we might intuitively expect.

Let us go back, now, to the case of a general distribution, to see if this property of global decrease still always holds. We can differentiate the remaining waiting time with respect to t_0, from (4.39), to obtain:

$$\frac{d(T_A - t_1 - t_0)}{dt_0} = -1 + p_1(t_0)\,\frac{\int_0^\infty \bar{P}_1(t+t_0)\,dt}{\bar{P}_1(t_0)^2} \tag{4.53}$$

From this point, we can study the variations of $T_A-t_1-t_o$ as t_o increases. But, we can also in letting $\frac{(T_A-t_1-t_o)}{dt_o}$ take different particular forms, and find out, in solving the differential equation in $\overline{P}_1$ expressed by (4.53), which distribution will imply each of these particular forms.

Thus, for example, if we set $\frac{d(T_A-t_1-t_o)}{dt_o} = 0$ (4.54)

$$\text{i.e.} \quad T_A-t_1-t_o = \text{constant} \tag{4.55}$$

we have, after (4.39)

$$\frac{\int_{t_o}^{\infty} \overline{P}_1(t)\,dt}{\overline{P}_1(t_o)} = \text{constant} \tag{4.56}$$

$$\frac{\overline{P}_1(t_o)}{\int_{t_o}^{\infty} \overline{P}_1(t)\,dt} = \text{constant } k \tag{4.57}$$

$$\operatorname{Log} \overline{P}_1(t_o) = -kt_o + \text{constant} \tag{4.58}$$

$$\overline{P}_1(t_o) = \text{constant} \cdot e^{-kt_o} \tag{4.59}$$

Since, we must have $\overline{P}_1(0) = 1$ and $\lim_{t_o \to +\infty} \overline{P}_1(t_o) = 0$

k must be positive and $\overline{P}_1(t_o) = e^{-kt_o}$. (4.60)

We, therefore, obtain the negative exponential distribution. We, thus, have shown that the negative exponential distribution is the only one for which the remaining waiting time is constant.

As another example, we can now set

$$\frac{d(T_A-t_1-t_0)}{dt_0}=k \qquad \text{with } k\neq 0 \tag{4.61}$$

i.e. after (4.53),

$$\frac{P_1(t_0)\int_0^\infty \bar{P}_1(t+t_0)\,dt}{\bar{P}_1(t_0)^2}=k+1 \tag{4.62}$$

If we denote $\int_0^\infty \bar{P}_1(t+t_0)dt$ by y, this expression can be re-written as

$$\frac{y''y}{y'^2}=k+1 \tag{4.63}$$

i.e.

$$\frac{y'^2-y''y}{y'^2}=-k \tag{4.64}$$

i.e.

$$\frac{y}{y'}=-k(t_0+a) \tag{4.65}$$

where a is an integration constant

i.e.

$$\frac{y'}{y}=-\frac{1}{k(t_0+a)} \tag{4.66}$$

i.e.

$$\operatorname{Log} y=-\frac{1}{k}\operatorname{Log}(t_0+a)+\text{constant} \tag{4.67}$$

i.e.

$$y=\text{constant}\,(t_0+a)^{-\frac{1}{k}} \tag{4.68}$$

i.e. $$\bar{P}_1(t_0) = -y' = \text{constant} \cdot \frac{1}{(t_0+a)^{\frac{1}{k}+1}} \qquad (4.60)$$

Since we must have $P_1(0) = 1$ and $\int_0^\infty \bar{P}_1(t)dt$ finite, we must have $\frac{1}{k} + 1 > 1$ i.e. $k > 0$ and

$$\bar{P}_1(t_0) = \frac{a^{\frac{1}{k}+1}}{(t_0+a)^{\frac{1}{k}+1}} \qquad (4.61)$$

i.e. with $$n = \frac{1}{k} + 1 \qquad (4.62)$$

$$\bar{P}_1(t_0) = \frac{a^n}{(t_0+a)^n} \qquad (4.63)$$

Thus, we obtain the type of distribution that we have described earlier and used extensively in chapter 3, to contradict different properties. We have shown here that this type of distribution is the only one for which the remaining waiting time is linear (constant derivative) and moreover, that in this case, the slope is necessarily positive ($k>0$), meaning that the remaining waiting time will increase as t_o increases and will even go to infinity as t_o goes to infinity.

Therefore, again, this type of distribution produces a counter-intuitive behavior; the more the passenger has waited already, the more likely he is still going to wait. This distribution exhibits as we have seen in chapter 3, some peculiarities that the more common distributions do not have:

-slow convergence to 0 as time increases

- moments are infinite after a certain order.

However, from a practical point of view, it might be possible to meet some very special cases, where the more the passenger has already waited, the more he is going to wait: this would be true, for example in the case of a strike or of an accident; the more the passenger waits for the bus, the greater the probability that there is a strike or an accident, and the less likely that the bus is going to arrive at the bus stop.

Finally, it must be interesting to look at T_B and T_B-T_A, with this special type of distribution:

let $$p_1(t) = \frac{na^n}{(t+a)^{n+1}} \qquad p_2(t) = \frac{pa^p}{(t+a)^{p+1}} \tag{4.69}$$

i.e. $$\bar{P}_1(t) = \frac{a^n}{(t+a)^n} \qquad \bar{P}_2(t) = \frac{a^p}{(t+a)^p} \tag{4.70}$$

Then using formula (4.38), we obtain,

$$T_A = t_0 + \frac{t_0+a}{n-1} + t_1 \tag{4.71}$$

We retrieve here the fact that with this type of distribution we have

$k = \frac{1}{n-1}$ i.e. $n = \frac{1}{k} + 1$, where k is the slope of the remaining waiting time $T_A-t_o-t_1$ with respect to t_o: expression (4.62).

Using formula (4.41), we obtain after simplification,

$$T_B-T_A = \frac{p}{n+p}(t_2-t_1) - \frac{p}{(n-1)(n+p-1)}(t_0+a) \tag{4.72}$$

Therefore, we can see that T_B-T_A goes to $-\infty$ as t_o goes to $+\infty$, thus T_B-T_A will be negative for high values of t_o, meaning that (1,1) will be the best strategy for high values of t_o. On the other hand, choosing a adequately, we can have T_B-T_A positive for low values of t_o, meaning that (1,0) will be the best strategy for such low values of t_o.

In this way, we will get exactly the reverse of property (DA)--which was thought to be intuitively impossible. Again, the type of distribution given by (4.69) and (4.70) leads to results contrary to what one might reasonably or intuitively expect.

As a matter of information, from (4.71) and (4.72) we can easily deduce T_B:

$$T_B = \frac{n}{n+p} t_1 + \frac{p}{n+p} t_2 + \frac{1}{n+p-1} (t_0 + \alpha) \tag{4.73}$$

Chapter 5

Conclusions

In many cities, bus routes sometimes share common sections. When several bus routes share a common section between two points A and B, the passenger willing to go from A to B is faced by the problem of choosing which bus line(s) to take.

Because some routes might have a very long travel time between A and B, we may disregard them. We assume that the passenger will choose an *optimal strategy*: he/she will select a subset among the set of all routes passing through A and B so as to minimize his total expected travel time (expected waiting time and expected in-vehicle travel time). Once this subset is selected, the passenger will take the first vehicle serving one of the routes in this subset, arriving at A.

The purpose of this thesis was to study the problem of the passenger route-choice decision in a probabilistic framework: on each bus line, the headway has a certain known distribution and the passenger arrives at a random time at the bus stop, so that the time for the first bus of each line to arrive at A is also a random variable. Because we showed that only the expected value of in-vehicle travel time between A and B for each bus line matters, in-vehicle travel times were considered deterministic.

We reviewed the problem of choosing a subset of routes in Chapter 1 and showed that the shortest travel time route is always included in the optimal strategy both for the case of the passenger who has just arrived and for the case of the passenger who has already waited for a while. (We showed the latter in Chapter 4.)

In Chapter 3, we reviewed the Chriqui and Robillard heuristic solution for the optimal strategy, found conditions under which it fails, and provided counterexamples. The failure of this heuristic in the general case prompted the study performed in Chapter 3.

Chapter 3 consisted of an extensive investigation of the case of three bus lines. We demonstrated that many properties which we might expect to be true are indeed valid in the cases of negative exponential and deterministic headways. But, as we have shown these properties are in fact false in the most general case; there is at least one class of distributions for which these properties fail to hold. There seem to be very few general statements that can be made regarding optimal strategies unless the waiting time distributions are further constrained. The case of four bus lines also showed how much more complicated the problem becomes as the number of routes increases.

The problem of the "clever" passenger, who has already waited for a certain time, and takes this time into account in making his route-choice decision, was described in Chapter 4. We discovered there that it is possible to find some distributions for which the remaining waiting time actually increases with the time the passenger has already waited.

Thus, the study of the problem of common bus lines revealed that this problem is much more difficult to solve than anticipated, and we have not been able to find a generally valid algorithm for it.

The cases, for which the counterintuitive results of chapters 3 and 4 were obtained, involved the same family of probability density function. This family of distributions, which is among several that could exhibit the same type of behavior, was used for reasons of algebraic convenience.

It belongs to a set of distributions that have, unlike usual waiting time distributions, peculiar properties; in particular, all moments of the distribution above a certain order are infinite.

In this respect, an area for possible extension of this research is to study further the problem of the optimal strategy of the passenger, "clever" or not, restricting the investigation to waiting time distributions which do not exhibit the peculiar properties mentioned above.

Since, we have found that the Chriqui and Robillard heuristic approach cannot even by applied in the case of deterministic headways, further research would be needed to develop a new one. But, because many intuitive statements are not true in all cases, this research should take place in the *restricted* framework of waiting times or headways distributions that fit more closely the actual behavior of buses along a bus route.

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Appendix A

Logical relationships (BA) to (BV) presented in Chapter III, can be studied in a way which is just a direct extension of the method used to study relationships (AA), (AB), (AC), and (AD) in Chapter 3.

Moreover, across all relationships (BA) to (BV), proofs are very similar to each other, both with respect to the approach taken and to the type of results obtained.

Therefore, only typical proofs are given here, each one corresponding to one of the types of results obtained. The specific relationships to be discussed are (BA), (BB), (BD), (BF), (BG), (BU).

i) Relationship (BA):

$$\left.\begin{array}{l}(1,0,1) \text{ better than } (1,1,1)\\ (1,1,0) \text{ better than } (1,1,1)\end{array}\right. \rightarrow (1,0,0) \text{ better than } (1,1,1), \qquad \text{(BA)}$$

i.e.

$$\begin{array}{l}T_D-T_C>0\\ T_D-T_B>0\end{array} \rightarrow T_D-T_A>0 \qquad \text{(EA)}$$

i.e. using equations (3.16), (3.17) and (3.18), and dropping the terms in ψ as in Chapter 3,

$$\begin{array}{l}\Phi(P_2\bar{P}_3)>0\\ \Phi(\bar{P}_2P_3)>0\end{array} \Rightarrow \Phi(1-\bar{P}_2\bar{P}_3)>0 \qquad \text{(EB)}$$

which is after (3.12) and denoting k= t_2-t_1

$$\begin{array}{l}\int_0^\infty [kp_1(t)-\bar{P}_1(t)]P_2(t)\bar{P}_3(t)\,dt>0\\ \int_0^\infty [kp_1(t)-\bar{P}_1(t)]\bar{P}_2(t)P_3(t)\,dt>0\end{array} \Rightarrow \int_0^\infty [kp_1(t)-\bar{P}_1(t)][1-\bar{P}_2(t)\bar{P}_3(t)]\,dt>0 \qquad \text{(EC)}$$

i.e.
$$k > \frac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t)\,dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t)\,dt} \qquad \Rightarrow\ k > \frac{\int_0^\infty \bar{P}_1(t)\,[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt}{\int_0^\infty p_1(t)\,[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt} \tag{ED}$$
$$k > \frac{\int_0^\infty \bar{P}_1(t) \bar{P}_2(t) P_3(t)\,dt}{\int_0^\infty p_1(t) \bar{P}_2(t) P_3(t)\,dt}$$

which is only possible for all k, when

$$\frac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t)\,dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t)\,dt} \geq \frac{\int_0^\infty \bar{P}_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt}{\int_0^\infty p_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt} \tag{A.1}$$

or

$$\frac{\int_0^\infty \bar{P}_1(t) \bar{P}_2(t) P_3(t)\,dt}{\int_0^\infty p_1(t) \bar{P}_2(t) P_3(t)\,dt} \geq \frac{\int_0^\infty \bar{P}_1(t)\cdot[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt}{\int_0^\infty p_1(t)\,[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt} \tag{A.2}$$

Using the fact that $P_2(t) = 1-\bar{P}_2(t)$ and $\int_0^\infty p_1(t)dt = 1$, (A.1), can be reduced to:

$$\int_0^\infty p_1(t)\bar{P}_3(t)\,dt \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt \cdot \int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt$$

$$\geq \int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty p_1(t)\bar{P}_3(t)\,dt \int_0^\infty \bar{P}_1(t)\,dt \tag{A.3}$$

We can also notice that inequality (A.2) is the same as inequality (A.1), with subscripts 2 and 3 interchanged. Therefore, inequality (A.2) implies an inequality (A.4) which is the same as (A.2) with subscripts 2 and 3 interchanged. Thus, if $\bar{P}_2(t) = \bar{P}_3(t)$, both (A.3) and (A.4) reduce to one inequality (A.5).

$$\int_0^\infty p_1(t)\bar{P}_2(t)dt\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)^2dt+\int_0^\infty \bar{P}_1(t)dt\int_0^\infty p_1(t)\bar{P}_2(t)^2dt+\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)dt$$

$$\geqslant \int_0^\infty p_1(t)\bar{P}_2(t)^2dt\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)dt+\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)^2dt+\int_0^\infty p_1(t)\bar{P}_2(t)dt\int_0^\infty \bar{P}_1(t)dt \tag{A.5}$$

Then if $p_1(t)=\dfrac{na^n}{(t+a)^{n+1}}$ $\qquad p_2(t)=p_3(t)=\dfrac{pa^p}{(t+a)^{p+1}}$ (A.6)

i.e. $\bar{P}_1(t)=\dfrac{a^n}{(t+a)^n}$ $\qquad \bar{P}_2(t)=\bar{P}_3(t)=\dfrac{a^p}{(t+a)^p}$ (A.7)

(6.5) becomes, in a similar way to some derivations in part 3.4,

$$\frac{a}{n-1}\cdot\frac{n}{n+2p}+\frac{a}{n+p-1}+\frac{a}{n+2p-1}\cdot\frac{n}{n+p}\geqslant\frac{a}{n-1}\cdot\frac{n}{n+p}+\frac{a}{n+2p-1}+\frac{a}{n+p-1}\cdot\frac{n}{n+2p} \tag{A.8}$$

i.e. after simplifications,

$$-2p^2 \geq 0 \tag{A.9}$$

which is not true. Therefore, relationship (BA) is not true for waiting times distributed according to (A.6) and (A.7), with certain values of t_2-t_1, and small t_3-t_2.

ii) Relationship (BB):

$$\left.\begin{array}{l}(1,0,1)\text{ better than }(1,1,1)\\(1,0,0)\text{ better than }(1,1,1)\end{array}\right.\rightarrow (1,0,0)\text{ better than }(1,0,1) \tag{BB}$$

i.e.
$$\begin{array}{l} T_D - T_C > 0 \\ T_D - T_A > 0 \end{array} \rightarrow T_C - T_A > 0 \qquad \text{(EE)}$$

i.e. using equations (3.16), (3.18), and (3.15), and dropping the terms in ψ again,

$$\begin{array}{l} \Phi(P_2 \bar{P}_3) > 0 \\ \Phi(1 - \bar{P}_2 \bar{P}_3) > 0 \end{array} \Longrightarrow \Phi(P_3) > 0 \qquad \text{(EF)}$$

which is after (3.12)

$$\begin{array}{l} \int_0^\infty [k p_1(t) - \bar{P}_1(t)] P_2(t) \bar{P}_3(t) \, dt > 0 \\ \int_0^\infty [k p_1(t) - \bar{P}_1(t)][1 - \bar{P}_2(t) \bar{P}_3(t)] \, dt > 0 \end{array} \Longrightarrow \int_0^\infty [k p_1(t) - \bar{P}_1(t)] P_3(t) \, dt > 0 \qquad \text{(EG)}$$

i.e.

$$\begin{array}{l} k > \dfrac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t) \, dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t) \, dt} \\[2ex] k > \dfrac{\int_0^\infty \bar{P}_1(t) [1 - \bar{P}_2(t) \bar{P}_3(t)] \, dt}{\int_0^\infty p_1(t) [1 - \bar{P}_2(t) \bar{P}_3(t)] \, dt} \end{array} \Longrightarrow k > \frac{\int_0^\infty \bar{P}_1(t) P_3(t) \, dt}{\int_0^\infty p_1(t) P_3(t) \, dt} \qquad \text{(EH)}$$

which is only possible when

$$\frac{\int_0^\infty \bar{P}_1(t) P_2(t) \bar{P}_3(t) \, dt}{\int_0^\infty p_1(t) P_2(t) \bar{P}_3(t) \, dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t) P_3(t) \, dt}{\int_0^\infty p_1(t) P_3(t) \, dt} \qquad \text{(A.10)}$$

or

$$\frac{\int_0^\infty \bar{P}_1(t) [1 - \bar{P}_2(t) \bar{P}_3(t)] \, dt}{\int_0^\infty p_1(t) [1 - \bar{P}_2(t) \bar{P}_3(t)] \, dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t) P_3(t) \, dt}{\int_0^\infty p_1(t) P_3(t) \, dt} \qquad \text{(A.11)}$$

Reducing these two inequalities, we find that both of them give:

$$\int_0^\infty \bar{p}_1(t)\,p_2(t)\,dt \int_0^\infty p_1(t)\,p_3(t)\,dt - \int_0^\infty \bar{p}_1(t)\,p_2(t)\,p_3(t)\,dt \cdot \int_0^\infty p_1(t)\,p_3(t)\,dt$$
$$\geqslant \int_0^\infty \bar{p}_1(t)\,p_3(t)\,dt \cdot \int_0^\infty p_1(t)\,p_2(t)\,dt - \int_0^\infty p_1(t)\,p_2(t)\,p_3(t)\,dt \int_0^\infty \bar{p}_1(t)\,p_3(t)\,dt \tag{A.12}$$

This proves that, in fact, (A.10) is true whenever (A.11) is true, and vice versa. But (A.10) is the inequality we obtain when we study

$$\Phi(p_2\bar{p}_3) > 0 \Rightarrow \Phi(p_3) > 0 \tag{EI}$$

and (6.11) is the inequality we obtain when we study

$$\Phi(1-\bar{p}_2\bar{p}_3) > 0 \Rightarrow \Phi(p_3) > 0 \tag{EJ}$$

(EI) and (EJ) are respectively the relationships expressed using Φ, (terms in ψ dropped) associated with relationships (AD) and (AC). Therefore, for small values of t_2-t_1, the three relationships (BB), (AD), and (AC) are equivalent, since they are equivalent to the same inequality (A.12). Thus, relationship (BB) is not true, since (AD) and (AC) are not, according to Chapter 3.

iii) Relationship (BD):

$$\left.\begin{array}{l}(1,0,1) \text{ better than } (1,1,1)\\(1,0,0) \text{ better than } (1,1,0)\end{array}\right. \rightarrow (1,0,0) \text{ better than } (1,1,1) \tag{BD}$$

i.e. $$\begin{array}{l} T_D - T_C > 0 \\ T_B - T_A > 0 \end{array} \rightarrow T_D - T_A > 0 \qquad \text{(EK)}$$

i.e. using equations (3.16), (3.14), (3.18), and dropping the terms in ψ,

$$\begin{array}{l} \Phi(P_2\bar{P}_3) > 0 \\ \Phi(P_2) > 0 \end{array} \Rightarrow \Phi(1-\bar{P}_2\bar{P}_3) > 0 \qquad \text{(EL)}$$

which after (3.12), and processing the inequalities as in the previous cases one gives the following conditions:

$$\frac{\int_0^\infty \bar{P}_1(t)P_2(t)\bar{P}_3(t)\,dt}{\int_0^\infty p_1(t)P_2(t)\bar{P}_3(t)\,dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\left[1-\bar{P}_2(t)\bar{P}_3(t)\right]dt}{\int_0^\infty p_1(t)\left[1-\bar{P}_2(t)\bar{P}_3(t)\right]dt} \qquad \text{(A.13)}$$

or

$$\frac{\int_0^\infty \bar{P}_1(t)P_2(t)\,dt}{\int_0^\infty p_1(t)P_2(t)\,dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\left[1-\bar{P}_2(t)\bar{P}_3(t)\right]dt}{\int_0^\infty p_1(t)\left[1-\bar{P}_2(t)\bar{P}_3(t)\right]dt} \qquad \text{(A.14)},$$

Reducing these inequalities:

$$\begin{aligned} &\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty p_1(t)\bar{P}_2(t)\,dt \int_0^\infty \bar{P}_1(t)\,dt \\ \geqslant\ &\int_0^\infty p_1(t)\bar{P}_2(t)\,dt \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt \int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\,dt \qquad \text{(A.15)} \end{aligned}$$

or

$$\begin{aligned} &\int_0^\infty p_1(t)\bar{P}_3(t)\,dt \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt \int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt \\ \geq\ &\int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty p_1(t)\bar{P}_3(t)\,dt \int_0^\infty \bar{P}_1(t)\,dt \qquad \text{(A.16)} \end{aligned}$$

With $p_1(t) = \frac{na^n}{(t+a)^{n+1}} \qquad p_2(t) = \frac{pa^p}{(t+a)^{p+1}} \qquad p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$ (A.17)

(A.15) reduces to

$$pq(p+q) \geq 0 \tag{A.18}$$

(A.16) reduces to

$$-pq(p+q) \geq 0 \tag{A.19}$$

Therefore, the condition for at least one of the two inequalities to be true is verified. The usual type of distribution, defined by equation (3.55) does not work here as a counterexample.

iv) Relationship (BF):

$$\left.\begin{array}{l} (1,0,1) \text{ better than } (1,1,1) \\ (1,0,0) \text{ better than } (1,0,1) \end{array}\right. \rightarrow (1,0,0) \text{ better than } (1,1,0) \tag{BF}$$

i.e.
$$\left.\begin{array}{l} T_D - T_C > 0 \\ T_C - T_A > 0 \end{array}\right. \rightarrow T_B - T_A > 0 \tag{EM}$$

i.e. using equations (3.16), (3.15), (3.14), and dropping the terms in ψ,

$$\left.\begin{array}{l} \Phi(P_2\bar{P}_3) > 0 \\ \Phi(P_3) > 0 \end{array}\right. \Rightarrow \Phi(P_2) > 0 \tag{EN}$$

which after (3.12), and processing the inequalities as usual gives:

$$\frac{\int_0^\infty \bar{P}_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt}{\int_0^\infty p_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\,P_2(t)\,dt}{\int_0^\infty p_1(t)\,P_2(t)\,dt} \tag{A.20}$$

or

$$\frac{\int_0^\infty \bar{P}_1(t)\,P_3(t)\,dt}{\int_0^\infty p_1(t)\,P_3(t)\,dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\,P_2(t)\,dt}{\int_0^\infty p_1(t)\,P_2(t)\,dt} \tag{A.21}$$

which reduces to

$$\begin{aligned}&\int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\,dt\int_0^\infty p_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt\int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt\\ &\quad + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt\int_0^\infty p_1(t)\bar{P}_2(t)\,dt \geqslant \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt\int_0^\infty p_1(t)\bar{P}_3(t)\,dt\\ &\qquad + \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt\int_0^\infty p_1(t)\bar{P}_2(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\,dt\int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt\end{aligned} \tag{A.22}$$

or

$$\begin{aligned}&\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt\int_0^\infty p_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt\int_0^\infty p_1(t)\bar{P}_2(t)\,dt\\ &\geqslant \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\,dt\int_0^\infty p_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt\int_0^\infty p_1(t)\bar{P}_2(t)\,dt\end{aligned} \tag{A.23}$$

With $p_1(t) = \frac{na^n}{(t+a)^{n+1}}$ $\qquad p_2(t) = \frac{pa^p}{(t+a)^{p+1}}$ $\qquad p_3(t) = \frac{qa^q}{(t+a)^{q+1}}$ (A.24)

(A.22) reduces to

$$-(2n+p+q)(p-1) - 2n(n+q) \geq 0 \tag{A.25}$$

which is false, and (A.23) reduces to

$$pq(p-q) \geq 0 \tag{A.26}$$

which is false too, for p<q. Therefore, relationship (BF) is not true for waiting times distributed according to (A.24), with p<q.

v) Relationship (BG):

$$\left.\begin{array}{l}(1,0,1) \text{ better than } (1,1,1)\\ (1,0,0) \text{ better than } (1,1,0)\end{array}\right\} \rightarrow (1,0,0) \text{ better than } (1,0,1) \tag{BG}$$

i.e.

$$\left.\begin{array}{l}T_D - T_C > 0\\ T_B - T_A > 0\end{array}\right. \rightarrow T_C - T_A > 0 \tag{EO}$$

i.e. using equations (3.16), (3.14), (3.15), and dropping the terms in ψ,

$$\left.\begin{array}{l}\Phi(P_2\bar{P}_3) > 0\\ \Phi(P_2) > 0\end{array}\right. \Rightarrow \Phi(P_3) > 0 \tag{EP}$$

which, processing the inequalities gives

$$\frac{\int_0^\infty \bar{P}_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt}{\int_0^\infty p_1(t)\,P_2(t)\,\bar{P}_3(t)\,dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\,P_3(t)\,dt}{\int_0^\infty p_1(t)\,P_3(t)\,dt} \tag{A.27}$$

$$\frac{\int_0^\infty \bar{P}_1(t)\,P_2(t)\,dt}{\int_0^\infty p_1(t)\,P_2(t)\,dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\,P_3(t)\,dt}{\int_0^\infty p_1(t)\,P_3(t)\,dt} \tag{A.28}$$

This reduces to

$$\begin{aligned}&\int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt \cdot \int_0^\infty p_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt \int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt\\ &\geqslant \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\,dt \int_0^\infty p_1(t)\bar{P}_3(t)\,dt + \int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt \int_0^\infty p_1(t)\bar{P}_2(t)\bar{P}_3(t)\,dt\end{aligned} \tag{A.29}$$

or

$$\int_0^\infty \bar{P}_1(t)\bar{P}_3(t)dt+\int_0^\infty \bar{P}_1(t)\bar{P}_2(t)dt\int_0^\infty p_1(t)\bar{P}_3(t)dt+\int_0^\infty \bar{P}_1(t)\,dt\int_0^\infty p_1(t)\bar{P}_2(t)\,dt$$

$$\geqslant \int_0^\infty \bar{P}_1(t)\bar{P}_2(t)dt+\int_0^\infty \bar{P}_1(t).\int_0^\infty p_1(t)\bar{P}_3(t)dt+\int_0^\infty \bar{P}_1(t)\bar{P}_3(t)dt\cdot\int_0^\infty p_1(t)\bar{P}_2(t)dt \qquad (A.30)$$

If $P_2(t) = 1$, $\bar{P}_2(t) = 0$, both inequalities (A.29) and (A.30) become:

$$\int_0^\infty \bar{P}_1(t)\bar{P}_3(t)\,dt \geqslant \int_0^\infty \bar{P}_1(t)dt.\int_0^\infty p_1(t)\bar{P}_3(t)dt \qquad (A.31)$$

Therefore, just as with relationships (AA), (AB), (AC), (AD), in chapter 3, (A.31) is contradicted, and (BG) is not true for waiting times which are distributed the following way:

$$p_1(t)=\frac{na^n}{(t+a)^{n+1}} \qquad p_3(t)=\frac{qa^q}{(t+a)^{q+1}} \qquad P_2(t)\simeq 1 \qquad (A.32)$$

with certain values $t_2 - t_1$, and small t_3-t_2.

vi) Relationship (BU)

$$\left.\begin{array}{l}(1,0,0) \text{ better than } (1,1,1)\\(1,1,0) \text{ better than } (1,0,1)\end{array}\right\} \rightarrow (1,0,0) \text{ better than } (1,0,1) \qquad (BU)$$

i.e.

$$\left.\begin{array}{l}T_D-T_A > 0\\T_C-T_B > 0\end{array}\right\} \rightarrow T_C-T_A > 0 \qquad (EQ)$$

i.e. using equations (3.18), (3.19), (3.15), and dropping the terms in ψ,

$$\left.\begin{array}{l}\Phi(1-\bar{P}_2\bar{P}_3)>0\\\Phi(\bar{P}_2-\bar{P}_3)>0\end{array}\right\} \Longrightarrow \Phi(P_3)>0 \qquad (ER)$$

which is after (3.12)

$$\left.\begin{array}{l}\int_0^\infty [k p_1(t)-\bar{P}_1(t)][1-\bar{P}_2(t)\bar{P}_3(t)]\,dt > 0 \\ \int_0^\infty [k p_1(t)-\bar{P}_1(t)][\bar{P}_2(t)-\bar{P}_3(t)]\,dt > 0\end{array}\right. \Longrightarrow \int_0^\infty [k p_1(t)-\bar{P}_1(t)]P_3(t)\,dt > 0 \qquad \text{(ES)}$$

i.e.

$$\left.\begin{array}{l}k\int_0^\infty p_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt > \int_0^\infty \bar{P}_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt \\ k\int_0^\infty p_1(t)[\bar{P}_2(t)-\bar{P}_3(t)]\,dt > \int_0^\infty \bar{P}_1(t)[\bar{P}_2(t)-\bar{P}_3(t)]\,dt\end{array}\right. \Longrightarrow k\int_0^\infty p_1(t)P_3(t)\,dt > \int_0^\infty \bar{P}_1(t)P_3(t)\,dt \qquad \text{(ET)}$$

If $\int_0^\infty p_1(t)[\bar{P}_2(t)-\bar{P}_3(t)]dt < 0$ i.e. $\int_0^\infty p_1(t)[\bar{P}_3(t)-\bar{P}_2(t)]dt > 0$ (A.33)

then, we can proceed

$$\left.\begin{array}{l}k > \dfrac{\int_0^\infty \bar{P}_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt}{\int_0^\infty p_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt} \\ k < \dfrac{\int_0^\infty \bar{P}_1(t)[\bar{P}_3(t)-\bar{P}_2(t)]\,dt}{\int_0^\infty p_1(t)[\bar{P}_3(t)-\bar{P}_2(t)]\,dt}\end{array}\right. \Longrightarrow k > \frac{\int_0^\infty \bar{P}_1(t)P_3(t)\,dt}{\int_0^\infty p_1(t)P_3(t)\,dt} \qquad \text{(EU)}$$

This is logically equivalent to

$$\frac{\int_0^\infty \bar{P}_1(t)[\bar{P}_3(t)-\bar{P}_2(t)]\,dt}{\int_0^\infty p_1(t)[\bar{P}_3(t)-\bar{P}_2(t)]\,dt} > \frac{\int_0^\infty \bar{P}_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt}{\int_0^\infty p_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt} \Longrightarrow \frac{\int_0^\infty \bar{P}_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt}{\int_0^\infty p_1(t)[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt} \geq \frac{\int_0^\infty \bar{P}_1(t)P_3(t)\,dt}{\int_0^\infty p_1(t)P_3(t)\,dt} \qquad \text{(EV)}$$

which is logically equivalent to having at least one of the two following inequalities true:

$$\frac{\int_0^\infty \bar{P}_1(t)\,[\bar{P}_3(t)-\bar{P}_2(t)]\,dt}{\int_0^\infty p_1(t)\,[\bar{P}_3(t)-\bar{P}_2(t)]\,dt} \leqslant \frac{\int_0^\infty \bar{P}_1(t)\,[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt}{\int_0^\infty p_1(t)\,[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt} \tag{A.34}$$

or
$$\frac{\int_0^\infty \bar{P}_1(t)\,[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt}{\int_0^\infty p_1(t)\,[1-\bar{P}_2(t)\bar{P}_3(t)]\,dt} \geqslant \frac{\int_0^\infty \bar{P}_1(t)\,P_3(t)\,dt}{\int_0^\infty p_1(t)\,P_3(t)\,dt} \tag{A.35}$$

If
$$p_1(t)=\frac{na^n}{(t+a)^{n+1}} \qquad p_2(t)=\frac{pa^p}{(t+a)^{p+1}} \qquad p_3(t)=\frac{qa^q}{(t+a)^{q+1}} \tag{A.36}$$

(A.35) reduces to

$$-pq(p+q) \geq 0 \tag{A.37}$$

which is false, and (A.36) reduces to

$$(p+q)\,[(n-1)(n+p)(n+q)(n+p+q-1)+n(q-p)\cdot(n+q-1)(n+p+q)] \leqslant 0 \tag{A.38}$$

i.e. for p=q+1,

$$(p+q)\,[(n-1)(n+q+1)(n+q)(n+2q)-n(n+q-1)(n+2q+1)] \leqslant 0 \tag{A.39}$$

which is false with n=q=2, (p=3).

Since in this case, p>q, assumption (A.33) is verified, relationship (BU) will not be true for waiting times distributed the following way

$$p_1(t)=p_3(t)=\frac{2a^2}{(t+a)^3} \qquad p_2(t)=\frac{3a^3}{(t+a)^4} \tag{A.40}$$

with certain values of t_2-t_1, and small t_3-t_2.