

NONLINEAR INTERACTIONS OF ACOUSTIC-GRAVITY WAVES

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Submitted to the Department of Earth and Planetary Sciences on August 6, 1976 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The theory of weak nonlinear interactions of internal gravity and acoustic waves in the atmosphere is presented. The development rests on Hamilton's principle for adiabatic motion of an atmosphere in hydrostatic equilibrium with the Lagrange density approximated to third order in the displacement field. Whitham's method of the averaged Lagrangian is reviewed and used to yield the formalism for propagation of wavetrains in a plane-stratified isothermal atmosphere. In the analysis, a variable serving the role of a slowly varying amplitude is achieved by the device of scaling the local displacement with respect to the background density. The Euler-Lagrange equations implied by the action integral give, for the linear approximation, the standard dispersion and polarization relations for acoustic and gravity wave modes. The incorporation of nonlinear terms yields the equations which describe the coupling and energy transfer of waves within and between these modes. The coupling coefficients, which, as expected, depend on the wave amplitudes, relative phases and relative wave polarizations, also increase with height in the atmosphere. The time evolution of interactions for three-wave processes is solved, and particular cases in which one of the waves has zero energy initially is discussed.

The theory is used to propose an explanation for the presence in the upper atmosphere of 2-5 minute period oscillations which have been observed in the region above, and during periods of, thunderstorm disturbance near the ground. These anomalous oscillations are modelled as being infrasonic acoustic waves which are generated by the nonlinear interaction of gravity waves which, in turn, are known to be produced by storm convective activity.

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1.0 Introduction

This thesis presents a development of the theory of weakly nonlinear resonant interactions of internal acoustic-gravity waves in the atmosphere. The earth's atmosphere sustains a large number of wave phenomena. The theory of acoustic-gravity waves is concerned with that part of the wave spectrum in which the effects of compressibility are of the same order of magnitude as those due to buoyancy caused by the density stratification established by gravity. From linear theory we know that these effects produce anisotropic dispersive wave motions which are distinguishable into two types: a higher frequency acoustic regime, in which the elastic force dominates, and a lower frequency gravity wave regime in which buoyancy dominates and the motion is more anisotropic. When we extend the analysis to incorporate nonlinear effects, the possibility arises that waves, which in linear theory propagate independently of each other, couple and effect energy transfer across the wave spectrum, both within and across mode types. The dynamical description of the process of nonlinear coupling of waves in an idealized stratified isothermal atmosphere is the principal purpose of this thesis.

In the theory of nonlinear resonant interactions between waves a necessary condition for coupling is that the waves satisfy a certain resonant or kinematic condition. For a triad of waves with frequencies ω_1 , ω_2 , and ω_3 and corresponding wave number vectors \underline{k}_1 , \underline{k}_2 and \underline{k}_3 , this condition is the simultaneous relations

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$$\omega_1 = \omega_2 + \omega_3$$

$$\underline{k}_1 = \underline{k}_2 + \underline{k}_3 \tag{1}$$

Any three waves may not satisfy this condition since in general there exists a dispersion relation relating ω and \underline{k} of each wave.

The subject of resonant interactions of acoustic-gravity waves has previously been dealt with by Yeh & Liu (1970), and in two short papers by Jurén & Stenflo (1973) and Dysthe, Jurén & Stenflo (1974). These studies can be characterized by their varying interpretations of the kinematic condition, Eqs. (1), as it relates to the acoustic-gravity dispersion relation. Using the Hines (1960) representation of the dispersion relation

$$\omega^4 - \omega^2 c^2 (K_x^2 + K_y^2 + K_z^2) + g^2 (\gamma - 1) (K_x^2 + K_y^2) + i \gamma g \omega_B^2 K_z = 0 \tag{2}$$

for an isothermal atmosphere, where $K_x = k_x$, $K_y = k_y$, $K_z = k_z - i/2H$ (H is the scale height), Yeh & Liu looked for solutions of the resonant conditions in which each wave vector in Eq. (1) is complex. Because of this requirement on both the real and imaginary parts of the wave vectors they were led to the conclusion that the only interaction allowed by Eq. (1) is one in which one of the waves has zero frequency, which they interpreted as a steady sinusoidal wind shear. With this restriction, they developed the equations to describe the trapping of internal gravity waves by this wind structure, arriving at results similar to those given previously by Phillips (1968) for ocean waves.

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In their earlier paper, Jurén & Stenflo (1973) accepted the requirement of Yeh & Liu in that the wave vectors in the kinematic relation be complex, but, using extensions of Eq. (2) by Pitteway & Hines (1963), in which viscous damping is taken into account, they indicated by the use of approximations of these extended dispersion relations that resonant interaction among high frequency atmospheric acoustic waves is possible without the requirement of zero frequency of one of the waves. In their later paper, Dysthe, Jurén & Stenflo (1974) recognized that the resonant condition, with the dispersion relation Eq. (2), can be satisfied with purely real wave vectors, and they displayed graphical examples of resonant triads. However, the results of their brief and sketchy discussion of the nonlinear dynamics of interaction appears to be limited, because of an analysis in terms of a single field variable, pressure.

There are essentially two approaches employed in the analysis of nonlinear wave problems: methods using differential equations; and, more recently, variational methods. The first approach consists of expanding the governing differential equations in a series expansion in some small ordering parameter. In the older version nonlinear effects give rise to secular terms that increase rapidly with time and space, limiting the value of the analysis. To overcome this difficulty, techniques of averaging or of slowly varying functions have been developed. These methods are described by Kinsman (1965) and Phillips (1969), for water wave interactions;

Sagdeev & Galeev (1969), and Davidson (1972) discuss plasma wave applications. However, a forbidding aspect that accompanies the use of this approach is the vast quantity of algebraic manipulation required in deriving the interaction equations. The arduousness follows from the fact that, in using the primary differential equations, a great many interaction terms arise, most of which are eliminated by application of the kinematic conditions, Eqs. (1), but only after these terms are encountered.

By contrast, the Lagrangian method, which we employ in this work, considerably reduces the labor in computing the coupled equations. The method requires that the governing equations for the system be derivable from a variational principle such as Hamilton's. For wave perturbation about an equilibrium state, trial solutions in the form of wavetrains, with slowly varying amplitudes, frequencies and wave numbers, are substituted into the action integral, which is then subjected to a local space and time averaging to smooth out the fast oscillations of the underlying wave motion. The Euler-Lagrange equations implied by the variational principle with this 'averaged' Lagrangian then yield the governing equations for the slowly varying parameters. For cubic and higher order non-linear terms in the action integral, the kinematic relations, if satisfied, give rise to terms in the action integral which represent coupling of waves. The essential utility of this variational approach is that the Lagrangian is considerably simplified before the variational principle is applied.

The particular device of forming an averaged Lagrangian for

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waves is due to Whitham (1965). Its employment for describing resonant interactions among waves was briefly indicated by Whitham (1967) using a model Lagrangian. However, more systematic extensions for this purpose were developed by Simmons (1969), for application to capillary-gravity waves, and by Dougherty (1970), who provides a useful introduction to the method of the averaged Lagrangian. In this thesis we use a methodology which is essentially a synthesis of the procedures of Simmons and of Dougherty. In order to orient the reader to the method as we shall employ it, a general review is provided in Section 3.0, unencumbered by the complexities which attends its use in the description of acoustic-gravity waves.

The variational method requires an expression for the integrand of the action integral, i.e. a Lagrange density, or lagrangian as we call it, for continuum systems. Tolstoy (1963) gave such an expression for linear acoustic-gravity waves by writing down quadratic energy terms for kinetic, elastic and buoyancy energies in terms of displacement of the fluid particle from equilibrium. He showed that the Euler-Lagrange equations for them lead to the same wave dispersion and polarization properties as those derived from the linearized perturbation solutions of the Euler fluid equations. Tolstoy's lagrangian, however, is somewhat ad hoc and not simply extendable to treatment of nonlinear motion.

In Section 2.0 we develop a Lagrange density for the adiabatic motion of an atmosphere in hydrostatic equilibrium in the

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Lagrangian or particle-coordinate representation, and we show via Hamilton's principle that it is consistent with the usual Euler equation of motion. In Section 2.1 this lagrangian is expanded in terms of the displacement from equilibrium. This is carried out to the third order in order that we might describe the lowest order nonlinear effects.

Using the averaged Lagrangian method, Section 4.0 develops the formalism for the quasilinear description of slowly varying acoustic-gravity wavetrains in an isothermal atmosphere. In order to deal with the exponential nonuniformity of the background density we define a new variable, which serves the role of a slowly varying amplitude, by scaling the displacement field with respect to the local density. The standard acoustic-gravity dispersion and polarization relations are obtained.

In Section 5.0 the nonlinear coupled equations are derived by extending the analysis of the previous section to incorporate the third order terms in the lagrangian. The exponential variation of the background density is shown to have the effect of increasing the coupling with altitude, and also introduces a coupling coefficient term which is absent in uniform media. The resonant or kinematic relations, Eqs. (1), are examined in Section 5.1, in which we tabulate the allowed possibilities for interactions among gravity and acoustic wave modes, and we describe graphical procedures for finding solutions of Eqs. (1). Section 5.2 presents approximations to the coupling coefficients which obtain for large magnitudes

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of wave vectors when it can be assumed that the polarization of gravity waves is transverse. The coupled interaction equations have known analytical solutions only when we specialize to one independent variable, which we take to be time. In 5.3 we employ a method of solution due to Armstrong et al. (1962) to describe the time evolution of the exchange of energy between interacting waves, and we discuss particular cases in which one of the waves starts out with zero energy but is generated by interaction of the other two waves.

The subject of this thesis had its genesis in the attempt to provide an explanation for the presence of coherent quasi-sinusoidal oscillations in the ionospheric regions above, and during periods of, severe storm activity near the ground. The spectrum of these oscillations is of relatively narrow bandwidth with periods in the range of 2 to 5 minutes. The phenomenology of these anomalous ionospheric oscillations was reviewed by Georges (1973). The evidence is consistent with the interpretation usually given that the oscillations are caused by the passage of infrasonic acoustic waves propagating upwards from the disturbed weather system below. In Section 6.0 it is proposed that these acoustic waves are generated by the nonlinear interaction of gravity waves which are known to be emitted by storm convective activity. Using a simple interaction model as a basis for calculation, and an approximate coupling coefficient derived in earlier sections, the feasibility of the source mechanism is assessed by comparison with

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the observed ionospheric perturbation. This source model stands in contrast to the explanations of Jones (1970) and Chimonas & Peltier (1974) who assert that the storm associated oscillations are the result of selective wave transmission characteristics of the atmosphere; objections to these views, which are based on multimode waveguide theories, are discussed in Appendix D. Our model was originally proposed by Moo & Pierce (1972); that analysis was based on concepts advanced by Lighthill (1952) in the theory of aerodynamic generation of sound. However, the present author, along with others, developed grave reservations about the validity of the analytical techniques employed by Lighthill, so that line of development was abandoned.

2.0 Lagrange Density

The equations governing the propagation of acoustic-gravity waves are usually derived from the dynamic equations for the adiabatic motion of an ideal gas atmosphere in a gravitational field. These are the equation of mass conservation:

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underline{v} + (\underline{v} \cdot \nabla) \rho = \frac{D\rho}{Dt} + \rho \nabla \cdot \underline{v} = 0, \quad (1)$$

the assumption of adiabaticity:

$$\rho \left(\frac{\partial p}{\partial t} + \underline{v} \cdot \nabla p \right) = \gamma p \left(\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho \right),$$

or, equivalently,

$$\frac{D}{Dt} (p\rho^{-\gamma}) = 0, \quad (2)$$

and the Euler force equation:

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} \right) = \rho \frac{D\underline{v}}{Dt} = - \nabla p + \rho \underline{g} \quad (3)$$

in which the pressure gradient and gravity are the only operative forces causing acceleration. Here ρ , \underline{v} , and p are the density, velocity, and pressure, respectively, at a point in space, and \underline{g} is the gravitational acceleration. γ is the ratio of specific heats for the gas and is taken to be constant ($\gamma = 1.4$ for air).

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The time derivative D/Dt refers to a material point moving with the fluid and is related to the time derivative at a fix spatial point:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla. \quad (4)$$

Our task now is to develop the Lagrange density from which the equations (1) - (3) may be derived via Hamilton's variational principle. We shall do this in the Lagrange or particle-coordinate representation of a fluid in which one considers the position vector \underline{X} of a fluid element as a function of the independent coordinates of time t and reference position \underline{x} :

$$X_i = X_i(\underline{x}, t); \quad (5)$$

and one specifies the reference configuration of the system by giving the density and pressure as functions of \underline{x} :

$$\rho_0 = \rho_0(\underline{x}) \quad (6)$$

$$p_0 = p_0(\underline{x}) \quad (7)$$

We shall write \dot{X}_i for the generalized velocity in configuration space; this is simply the partial derivative of X_i , Eq. (5), with respect to t , and should be identified with the time derivative D/Dt , Eq. (4), i.e.

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$$\dot{\underline{x}}_i(\underline{x}, t) = DX_i/Dt. \quad (8)$$

Conservation of mass, Eq. (1), requires that

$$\int \rho_0(\underline{x}) d^3x = \int \rho(\underline{X}, t) d^3X \quad (9)$$

over any volume containing a fixed number of particles. This in turn, by the manner in which volume integrals are transformed, requires

$$\rho_0(\underline{x}) = J\rho(\underline{X}, t), \quad (10)$$

where J is the Jacobian of the transformation from \underline{x} to \underline{X} coordinates:

$$J = \frac{\partial(\underline{X})}{\partial(\underline{x})} = \det(\partial X_i/\partial x_j). \quad (11)$$

The adiabatic condition, Eq. (2), may be written

$$p\rho^{-\gamma} = p_0\rho_0^{-\gamma}, \quad (12)$$

which, using Eq. (10), becomes

$$\widehat{p}_0(\underline{x}) = J^\gamma p(\underline{X}, t). \quad (13)$$

Hamilton's variational principle states that the time integral of the Lagrangian is stationary for all variations in the path leaving the initial and final configurations fixed:

$$\delta \int_{t_1}^{t_2} dt \int L d^3x = 0, \quad (14)$$

where the variation satisfies

$$\delta X_i(\underline{x}, t_1) = \delta X_i(\underline{x}, t_2) = 0 \quad (15)$$

and where L is the Lagrange density, or lagrangian. In classical mechanics L has the form

$$L = T - V \quad (16)$$

where T and V are kinetic and potential energy densities. Here we shall write down the form of the energy densities from physical considerations and then show that the Euler equation of motion, Eq. (3), follows from L . Appendix A gives the explicit procedure for the construction of L by the variational principle.

The kinetic energy per unit volume of \underline{x} space, i.e. T , is simply

$$T = \frac{1}{2} \rho_0(\underline{x}) \dot{\underline{x}}^2, \quad (17)$$

whereas V is the sum of two parts: the gravitational energy V_G and the internal thermodynamic energy V_I . The gravitational energy density is in general

$$V_G = \rho_0 \phi(\underline{x}), \quad (18)$$

where ϕ is the potential such that $g = -\nabla\phi$, and for our purposes

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we may put

$$\phi = gZ \quad (19)$$

where Z is the vertical component of the fluid particle coordinate and g is the value of the acceleration of gravity, assumed constant.

The internal potential energy is computed with the aid of the thermodynamic relation

$$dE = T dS - p d(1/\rho), \quad (20)$$

where E and S are the internal energy and entropy per unit mass, and T is the absolute temperature. During an adiabatic process S stays constant while p and ρ vary, so that

$$dE = -p d(1/\rho) = - (p/\rho_0) dJ, \quad (21)$$

where we have used the mass conservation relation, Eq. (10).

The internal potential energy density V_I is the net change in internal energy per unit volume of \underline{x} space, or

$$V_I = \rho_0 \int dE = - \int_1^J p dJ. \quad (22)$$

Using the adiabatic relation for an ideal gas, Eq. (13) thus becomes

$$V_I = p_0 \frac{J^{1-\gamma}}{\gamma-1}. \quad (23)$$

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Finally, our Lagrange density L is given by Eqs. (16), (17), (18) and (23) as

$$L = \frac{1}{2} \rho_0 \dot{\underline{x}}^2 - \rho_0 \phi(\underline{x}) = \rho_0 \frac{J^{1-\gamma}}{\gamma-1}. \quad (24)$$

The Euler-Lagrange equation, giving the equation of motion in Lagrangian form, appropriate to this L is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) + \frac{\partial}{\partial x_j} \left(\frac{\partial L}{\partial X_{ij}} \right) - \frac{\partial L}{\partial x_i} = 0, \quad (25)$$

for $i, j = 1, 2, 3$ and using the summation convention for repeated indices. Here we have used the abbreviation $X_{ij} = \partial x_i / \partial x_j$. More explicitly, using Eq. (24), the equation of motion is

$$\rho_0 \left(\ddot{x}_i + \frac{\partial \phi}{\partial x_i} \right) = \frac{\partial}{\partial x_j} \left(\frac{\partial J}{\partial X_{ij}} \frac{\partial V_I}{\partial J} \right) = \frac{\partial}{\partial x_j} \left(\frac{\partial J}{\partial X_{ij}} P \right), \quad (26)$$

where for the second equality we used Eq. (22). However, one may show that the identities

$$\frac{\partial}{\partial x_j} \left(\frac{\partial J}{\partial X_{ij}} \right) = 0 \quad (27)$$

and

$$\frac{\partial J}{\partial X_{ij}} = J \frac{\partial x_j}{\partial x_i} \quad (28)$$

are true, and so that

§ 2.0

$$\frac{\partial J}{\partial X_{ij}} \frac{\partial}{\partial x_j} = J \frac{\partial}{\partial X_i} . \quad (29)$$

Then, with Eq. (10), we find Eq. (26) reduces to

$$\rho(\ddot{\underline{X}} + \nabla_{\underline{X}}\phi) = -\nabla_{\underline{X}}p, \quad (30)$$

where $\nabla_{\underline{X}}$ is the gradient operator in X space. This is just Euler's equation of motion for a fluid, Eq. (3), if we identify the Eulerian velocity \underline{v} with the Lagrange velocity $\dot{\underline{X}}$ as

$$v_i(\underline{X}, t) = \dot{X}_i(\underline{x}, t). \quad (31)$$

2.1 Perturbation Expansion of L

To put the lagrangian L, Eq. (24), in a form convenient for describing small disturbances in the atmosphere, we let

$$\underline{X} = \underline{x} + \underline{\xi} \quad (32)$$

where $\underline{\xi}$ is a perturbation displacement on the background state defined by \underline{x} . We consider $\underline{\xi}$ as a field function of \underline{x} and t. In our case, in which there is no background flow (wind), the lagrangian becomes

$$L = \frac{1}{2} \rho_0 \left(\frac{\partial \underline{\xi}}{\partial t} \right)^2 - \rho_0 \phi (\underline{x} + \underline{\xi}) - p_0 \frac{J^{1-\gamma}}{\gamma-1}, \quad (33)$$

and the Jacobian is given by

$$\begin{aligned} J &= \frac{\partial(\underline{X})}{\partial(\underline{x})} = \det (\delta_{ij} + \partial \xi_i / \partial x_j) \\ &= 1 + \frac{\partial \xi_k}{\partial x_k} + \frac{1}{2} \left(\frac{\partial \xi_i}{\partial x_i} \frac{\partial \xi_j}{\partial x_j} - \frac{\partial \xi_j}{\partial x_i} \frac{\partial \xi_i}{\partial x_j} \right) + \frac{\partial(\underline{\xi})}{\partial(\underline{x})}, \end{aligned} \quad (34)$$

or

$$J = 1 + \nabla \cdot \underline{\xi} + \frac{1}{2} [(\nabla \cdot \underline{\xi})^2 - \xi_{ij} \xi_{ji}] + \nabla \xi_1 \cdot (\nabla \xi_2 \times \nabla \xi_3). \quad (35)$$

To describe waves and three wave interactions we expand L up to third order in $\underline{\xi}$. Thus we set

$$J^{1-\gamma} = 1 - (\gamma-1) \nabla \cdot \underline{\xi} + F_2 + F_3 + O(\xi^4), \quad (36)$$

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where

$$F_2 = -(\gamma-1) J_2 + [(\gamma-1)/2] J_1^2 \quad (37)$$

$$F_3 = -(\gamma-1) J_3 + \gamma(\gamma-1) J_1 J_2 - (1/6)(\gamma+1)\gamma(\gamma-1) J_1^3 \quad (38)$$

$$J_1 = \nabla \cdot \underline{\xi} \quad (39)$$

$$J_2 = (1/2) [(\nabla \cdot \underline{\xi})^2 - \xi_{ij} \xi_{ji}] \quad (40)$$

$$J_3 = \nabla \xi_1 \cdot (\nabla \xi_2 \times \nabla \xi_3) = \partial(\underline{\xi})/\partial(\underline{x}). \quad (41)$$

The lagrangian is then given by

$$L = L_0 + L_1 + L_2 + L_3 + O(\xi^4) \quad (42)$$

where

$$L_0 = -\rho_0 \phi(\underline{x}) - p_0 (\gamma-1)^{-1} \quad (43)$$

$$L_1 = -\rho_0 \underline{\xi} \cdot \nabla \phi + p_0 \nabla \cdot \underline{\xi} \quad (44)$$

$$L_2 = \frac{1}{2} \rho_0 \dot{\xi}^2 - \frac{1}{2} p_0 [(\gamma-1) (\nabla \cdot \underline{\xi})^2 + \xi_{ij} \xi_{ji}] \quad (45)$$

$$L_3 = p_0 \nabla \xi_1 \cdot (\nabla \xi_2 \times \nabla \xi_3) + \frac{\gamma p_0}{2} (\nabla \cdot \underline{\xi}) [\xi_{ij} \xi_{ji} - \frac{2-\gamma}{3} (\nabla \cdot \underline{\xi})^2] \quad (46)$$

The zeroth order term L_0 gives the energy of the background state, while L_1 leads under variation to the equation of motion which should vanish identically if the background satisfies the equations of hydrodynamics, viz.

§ 2.1

$$\rho_0 \nabla \phi + \nabla p_0, \quad (47)$$

which, in our case is the hydrostatic equation

$$\partial p_0 / \partial z = -\rho_0 g. \quad (48)$$

Since ϕ is nearly linear, the ϕ terms in L_2 and L_3 have been neglected; they are important perhaps only for motions on a galactic scale.

Our reduced Lagrange density for describing wave motions up to third order is then

$$L' = L_2 + L_3 \quad (49)$$

Linear wave motion is described by L_2 , being quadratic in ξ , while L_3 gives the lowest order nonlinear effects. Although the equations for L_2 and L_3 do not involve gravity explicitly, they do so implicitly through the equation p_0 and ρ_0 must satisfy, Eq. (16). This is just a reflection of the fact that the role of gravity is to establish the background stratification and the waves are perturbations in this prestressed medium (Biot, 1963).

It is useful to introduce a smallness parameter ϵ in the expansion process, with $\underline{x} = \underline{x} + \epsilon \underline{\xi}$; it follows then that

$$L = L_0 + \epsilon L_1 + \epsilon^2 L_2 + \epsilon^3 L_3 + O(\epsilon^4), \quad (50)$$

and that our wave Lagrange density (cancelling an irrelevant

§ 2.1

ϵ^2 factor) is

$$L' = L_2 + \epsilon L_3, \quad (51)$$

indicating that nonlinear effects are indeed weak.

3.0 Averaged Lagrangian Method

We shall treat the action integral for waves,

$$\int (L_2 + \epsilon L_3 + \dots) d^3x dt, \quad (1)$$

(in which we vary $\underline{\xi}$ assuming the background is given), by looking for oscillating solutions, in space and time, as slowly varying wavetrains, and constructing an "averaged" lagrangian. A wavetrain is a system of almost sinusoidal propagating waves with a recognizable dominant local frequency ω , wave number \underline{k} , and amplitude a . These may vary with position \underline{x} and time t , but only slowly, in the sense that appreciable changes are apparent only over many periods and wavelengths. The problem of describing the slow changes may be approached via the differential equation for $\underline{\xi}$, but such a local description has the disadvantage that the dominant behavior is the fast oscillation. In the method of the averaged lagrangian, introduced into fluid mechanics by Whitham (1965), a trial function with the same form as of a uniform solution is used to average out the rapidly varying parts of the lagrangian. This section provides a description of the method in the manner we shall use it in the remainder of this work. Mathematical justification of the method is given by Whitham (1970).

The motivation for the analytical procedure may be given

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by considering, for simplicity, a system with one dependent variable $u(\underline{x}, t)$ in the variational principle

$$\delta \int L(u, u_t, u_{\underline{x}}) d^3x dt = 0, \quad (2)$$

with Euler-Lagrange equation

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial u_t} \right) + \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\partial L}{\partial u_{\underline{x}}} \right) - \frac{\partial L}{\partial u} = 0. \quad (3)$$

If we are concerned with dispersive wave problems in a strictly linear system this equation has as special solutions

$$u = U(\phi), \quad \phi = \underline{k} \cdot \underline{x} - \omega t, \quad (4)$$

where $U(\phi)$ turns out to be a periodic function of ϕ , and the wave number \underline{k} and frequency ω are constants. Eq. (3) is a second-order equation in u , so that there will be two constants of integration: the amplitude a , and an arbitrary shift in the phase. Thus, in linear problems we have

$$U = a \cos (\underline{k} \cdot \underline{x} - \omega t + \theta), \quad (5)$$

where θ is the constant phase shift. The parameters ω , \underline{k} , a , θ will not be independent, but must satisfy a "dispersion relation" $G(\omega, \underline{k}, a, \theta) = 0$, which for linear problems is independent of a and θ , i.e. of the form $\lambda(\omega, \underline{k}) = 0$.

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For nearly periodic wavetrains, a and θ in Eq. (5) will not be constant, but slowly varying. The slow rates of change can be ordered by assuming

$$\left. \begin{aligned} \partial a / \partial t &\sim O(\mu \omega), & \partial \theta / \partial t &\sim O(\mu \omega), \\ \nabla a &\sim O(\mu k), & \nabla \theta &\sim O(\mu k), \end{aligned} \right\} \quad (6)$$

where we have introduced a slowness scale factor μ , $0 < \mu \ll 1$. With this ordering, the parameters of the solution U , Eq. (5), are assumed to vary on two scales: the amplitude a changes on the slow scale, while the phase function (the argument of the cosine) changes on both fast and slow scales. Our aim is to average out the fast change and derive equations for the slow parameters. To do this we shall explicitly employ "two timing" solutions (Cole, 1968; Nayfeh, 1973).

In the following discussion of the method, for clarity of exposition and reduction of notational complexity, we shall first develop the dynamics of a single wavetrain for linearized waves (i.e., L_3 and higher order terms equal to zero in Eq. (1)), then proceed to consider a sum of wavetrains to explain the nonlinear interactions among waves, involving L_2 and L_3 . Also, we confine the analysis in this section to 'uniform' wavetrains, in the sense that modulation of waves due to strong stratification of the background is not discussed. A particular modification, by scaling with respect to the background, allows the

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the method to be used with such a stratification; this is introduced in the next section for application to acoustic-gravity waves.

With the considerations given above, we assume as a trial function that, to lowest order in ϵ , Eq. (1) has solutions of the form of a wavetrain

$$\underline{\xi} = \hat{\underline{a}}(\underline{x}_2, t_2) \left[e^{i\phi(\underline{x}_1, t_1; \underline{x}_2, t_2)} + e^{-i\phi(\underline{x}_1, t_1; \underline{x}_2, t_2)} \right], \quad (7)$$

where $\hat{\underline{a}}$ is a real amplitude vector varying on the slow scale \underline{x}_2 and t_2 and ϕ is the phase function varying on fast and slow scales (\underline{x}_1, t_1) and (\underline{x}_2, t_2) respectively. We explicitly take ϕ to have the form

$$\phi = \phi(\underline{x}_1, t_1) + \theta(\underline{x}_2, t_2), \quad (8)$$

a sum of fast and slow phases. It is then convenient to introduce the complex amplitude vectors

$$\underline{a} = \hat{\underline{a}} e^{i\theta}, \quad \underline{a}^* = \hat{\underline{a}} e^{-i\theta} \quad (9)$$

so that Eq. (7) may be written

$$\underline{\xi} = \underline{a}(\underline{x}_2, t_2) e^{i\phi(\underline{x}_1, t_1)} + \text{c.c.} \quad (10)$$

Where, as here, a term is followed by its complex conjugate we shall write ' +c.c. ' for brevity.

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For time and space derivatives of ξ we use the derivative expansions

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} + \mu \frac{\partial}{\partial t_2} ; \quad t_1 = t, \quad t_2 = \mu t, \quad (11)$$

$$\frac{\partial}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}_1} + \mu \frac{\partial}{\partial \underline{x}_2} ; \quad \underline{x}_1 = \underline{x}, \quad \underline{x}_2 = \mu \underline{x} .$$

We define the local frequency and wave number as

$$\omega = - \frac{\partial \phi}{\partial t} = - \frac{\partial \phi}{\partial t_1} ; \quad \underline{k} = \frac{\partial \phi}{\partial \underline{x}} = \frac{\partial \phi}{\partial \underline{x}_1} . \quad (12)$$

Corrections of $O(\mu)$ to these definitions are provided by adding $-\partial\theta/\partial t$ and $\partial\theta/\partial \underline{x}$ respectively. It is more convenient, however, to treat this by use of the complex amplitude vector \underline{a} . (Throughout this work we adopt the convention that $\omega > 0$.)

The time and space derivatives of $\underline{\xi}$ are now given by

$$\begin{aligned} \frac{\partial \xi_i}{\partial t} &= (-i\omega a_i + \dot{a}_i) e^{i\phi} + \text{c.c.}, \\ \frac{\partial \xi_i}{\partial x_j} &= (ik_j a_i + a_{ij}) e^{i\phi} + \text{c.c.}, \end{aligned} \quad (13)$$

where $i = 1, 2, \dots, n$; $j = 1, 2, 3$;

and

$$\dot{a}_i \equiv \frac{\partial a_i}{\partial t} = \mu \frac{\partial a_i}{\partial t_2} ; \quad a_{ij} \equiv \frac{\partial a_i}{\partial x_j} = \mu \frac{\partial a_i}{\partial x_{2j}} . \quad (14)$$

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The conversion of the action integral, Eq. (1), to an integral over an averaged lagrangian, in which only slow variables remain, is achieved by substituting Eqs. (10) & (13) into Eq. (1) and retaining terms in which the fast phases cancel. Thus, for L_2 , we continually use the fact that if $\psi = f e^{i\phi} + \text{c.c.}$ and $\chi = g e^{i\phi} + \text{c.c.}$, where f and g are slow complex variables, then the average

$$\langle \psi \chi \rangle = fg^* + f^*g. \quad (15)$$

Appendix B shows that considering L_2 as a quadratic form in ξ_i , $\partial \xi_i / \partial t$ and $\partial \xi_i / \partial x_j$, and carrying out the averaging, leads to

$$\langle L_2 \rangle = \langle L_2(0) \rangle + \langle L_2(\mu) \rangle + O(\mu^2) \quad (16)$$

where

$$\langle L_2(0) \rangle = A_{ij} a_i a_j^*, \quad (17)$$

$$\begin{aligned} \langle L_2(\mu) \rangle = & - \frac{i\mu}{2} \left[\frac{\partial A_{ij}}{\partial \omega} \left(a_i \frac{\partial a_j^*}{\partial t} - a_j^* \frac{\partial a_i}{\partial t} \right) \right. \\ & \left. - \frac{\partial A_{ij}}{\partial k_\beta} \left(a_i \frac{\partial a_j^*}{\partial x_\beta} - a_j^* \frac{\partial a_i}{\partial x_\beta} \right) \right]; \quad (18) \end{aligned}$$

the summation convention on repeated indices is implied. Also, we drop the subscript 2 on t and x since no confusion arises as all variables, dependent and independent, are slow variables

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after averaging.

$\langle L_2(0) \rangle$ is a Hermitian form in the a_i 's, while $O(\mu)$ terms of $\langle L_2(\mu) \rangle$ take variation of the amplitudes into account. The elements A_{ij} of the Hermitian matrix are functions of ω , k and the background parameters. In a strictly linear system the action integral is Eq. (1) with $\langle L \rangle = \langle L_2(0) \rangle$, in which case the Euler-Lagrange equations are obtained by variation with respect to \underline{a} and \underline{a}^* :

$$\frac{\partial \langle L \rangle}{\partial a_i} = A_{ij} a_j^* = 0, \quad (i = 1, 2, \dots, n) \quad (19)$$

or,

$$\frac{\partial \langle L \rangle}{\partial \underline{a}} = A \cdot \underline{a}^* = 0,$$

and similar relations for $\delta \underline{a}^*$. These equations contain information about the dispersion properties and polarization of waves in the system. Eq. (19) leads in the usual way to the eigenvalue dispersion relation

$$\det \{A(\omega, \underline{k})\} = \lambda_1(\omega, \underline{k}) \lambda_2(\omega, \underline{k}) \dots \lambda_n(\omega, \underline{k}) = 0 \quad (20)$$

If a particular (ω, \underline{k}) represents a wave in the system, one of the eigenvalues, λ_s say, must be zero, and the wave belongs to the s -th mode or branch of propagation. To this eigenvalue there corresponds an eigenvector defined by Eq. (19), and the relative

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ratios of the components of this vector give the so called polarization relations of the wave. If $\lambda(\omega, \underline{k})$ and \underline{a} are a solution of Eq. (19) then it follows from Eq. (17) that in the linear system

$$\langle L_2 \rangle = \lambda \underline{a} \cdot \underline{a}^*. \quad (21)$$

For a wavetrain in the quasilinear (or weakly nonlinear) system represented by Eqs. (16)-(18) we expect that the amplitude vector is always nearly a solution of the linear problem. This is indicated by the following argument: if \underline{a}_0 is an eigenvector of the matrix A, with eigenvalue λ , the admission of $O(\mu)$ terms in $\langle L_2 \rangle$ slightly changes the solution so that we may write the new solution as a perturbation series

$$\underline{a} = \underline{a}_0 + \mu \underline{a}_1 + \mu^2 \underline{a}_2 + \dots, \quad (22)$$

where \underline{a}_1 can be taken to be orthogonal to \underline{a}_0 (i.e., $\underline{a}_0 \cdot \underline{a}_1^* = 0 = \underline{a}_0^* \cdot \underline{a}_1$). Putting this into the expression for $\langle L_2 \rangle$, Eqs. (16)-(18), shows that \underline{a}_1 appears only in terms of $O(\mu^2)$. It follows then that $\langle L_2 \rangle$ can be written in the form

$$\begin{aligned} \langle L_2 \rangle = & \lambda \underline{a} \cdot \underline{a}^* - \frac{i\mu}{2} \left[\frac{\partial \lambda}{\partial \omega} \left(\underline{a} \cdot \frac{\partial \underline{a}^*}{\partial t} - \underline{a}^* \cdot \frac{\partial \underline{a}}{\partial t} \right) \right. \\ & \left. - \frac{\partial \lambda}{\partial k_\beta} \left(\underline{a} \cdot \frac{\partial \underline{a}^*}{\partial x_\beta} - \underline{a}^* \cdot \frac{\partial \underline{a}}{\partial x_\beta} \right) \right] + O(\mu^2). \end{aligned} \quad (23)$$

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This equation is similar to one given by Dysthe (1974) for one dimensional wave systems.

With Eq. (23) in the action integral, variation with respect to \underline{a}^* produces the equation

$$\lambda \underline{a} - \frac{i\mu}{2} \left[2 \left(\frac{\partial \lambda}{\partial \omega} \frac{\partial \underline{a}}{\partial t} - \frac{\partial \lambda}{\partial k_\beta} \frac{\partial \underline{a}}{\partial x_\beta} \right) + \frac{1}{2} \left(\frac{\partial}{\partial t} \left(\frac{\partial \lambda}{\partial \omega} \right) - \frac{\partial}{\partial x_\beta} \left(\frac{\partial \lambda}{\partial k_\beta} \right) \right) \underline{a} \right] = 0, \quad (24)$$

or, since $\lambda = 0$,

$$\frac{\partial \lambda}{\partial \omega} \left(\frac{\partial}{\partial t} + \underline{v}_G \cdot \nabla \right) \underline{a} = - \frac{1}{2} \left(\frac{\partial}{\partial t} \left(\frac{\partial \lambda}{\partial \omega} \right) + \nabla \cdot \left(\frac{\partial \lambda}{\partial \omega} \underline{v}_G \right) \right) \underline{a}, \quad (25)$$

where

$$\underline{v}_G = - \left(\frac{\partial \lambda}{\partial \underline{k}} \right) / \left(\frac{\partial \lambda}{\partial \omega} \right) = \left. \frac{\partial \omega}{\partial \underline{k}} \right|_{\lambda=0} \quad (26)$$

is the group velocity. In the literature of asymptotic (WKB) wave theory Eq. (25) is called the transport equation. It determines the evolution of the amplitude of a wavetrain, from its value at a specified initial point in time and space, as it moves in a slowly varying background state, the gradients of which are implied by the right-hand side of Eq. (25). Thus in quasilinear systems the wavetrain amplitude changes are due to background parameter changes. Lewis (1965) and Hayes (1970) give expositions of modern wave theory relating to Eq. (25) and ray theory.

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Instead of the complex amplitudes \underline{a} and \underline{a}^* it is sometimes convenient to reintroduce the real amplitude \hat{a} and slow phase θ , defined by Eq. (9), and define the wave action density

$$N = \frac{\partial \lambda}{\partial \omega} \underline{a} \cdot \underline{a}^* = \frac{\partial \lambda}{\partial \omega} |\hat{a}|^2. \quad (27)$$

Then $\langle L_2 \rangle$, Eq. (23), has the compact representation

$$\langle L_2 \rangle = \lambda |\hat{a}|^2 - \mu N \left[\frac{\partial}{\partial t} + \underline{v}_G \cdot \nabla \right] \theta. \quad (28)$$

The Euler-Lagrange equations for this lagrangian with respect to variations in θ and \hat{a} are, respectively,

$$\frac{\partial N}{\partial t} + \nabla \cdot \underline{v}_G N = 0, \quad (29)$$

$$\frac{\partial \theta}{\partial t} + \underline{v}_G \cdot \nabla \theta = 0. \quad (30)$$

The former equation expresses the well known law of conservation of wave action in slowly varying media (Bretherton & Garrett, 1968); the latter equation shows that the phase shift following the group velocity is zero.

With the results of the mechanics of a single wavetrain as background, we now turn to multiple wave systems and their non-linear interaction. To treat the presence of several waves we write the field variable $\underline{\xi}$ as a sum

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$$\underline{\xi} = \sum_{\ell} \underline{a}^{\ell} e^{i\phi_{\ell}} + \text{c.c.} \quad (31)$$

Here ℓ labels the wavetrains, each of which has its own slowly varying complex amplitude and fast phase as in the single wave case, Eq. (10). We also define the local frequency and wave number by

$$\omega_{\ell} = - \frac{\partial \phi_{\ell}}{\partial t}, \quad \underline{k}^{\ell} = \frac{\partial \phi_{\ell}}{\partial \underline{x}}. \quad (32)$$

Eq. (31) and its derivatives are inserted into the action integral, Eq. (1), and the aim, as before, is to approximate the integrand by averaging over the fast scale and so render the variational principle as

$$\delta \int (\langle L_2 \rangle + \epsilon \langle L_3 \rangle) d^3x dt = 0, \quad (33)$$

where $\langle L \rangle = \langle L_2 \rangle + \epsilon \langle L_3 \rangle$ is a function of the set of complex amplitudes $\{\underline{a}^{\ell}, a^{\ell*}\}$, or, equivalently, of the set of real amplitudes and phases $\{\hat{a}^{\ell}, \theta_{\ell}\}$ defined by

$$\underline{a}^{\ell} = \hat{a}^{\ell} e^{i\theta_{\ell}}. \quad (34)$$

The Euler-Lagrange equations in these slow variables describe the dynamics of the wavetrains.

The procedure for averaging L_2 is just the same as for a

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wavetrain. The single wavetrain solutions in Eq. (31) may be superposed since in the averaging procedure terms that come from products of fields belonging to two different wavetrains disappear. (Exceptions to this due to degeneracy of the local dispersion relation will not be discussed, because they do not arise in this study.) Thus, by inserting Eq. (31) into L_2 , and averaging, we obtain

$$\langle L_2 \rangle = \sum_{\ell} A_{ij}(\omega_{\ell}, \underline{k}^{\ell}) a_i^{\ell} a_j^{\ell*} + O(\mu), \quad (35)$$

and if \underline{a}^{ℓ} is an eigenvector of A , with corresponding eigenvalue λ_{ℓ} for each wave labelled ℓ , we can then write this as

$$\begin{aligned} \langle L_2 \rangle = & \sum_{\ell} \lambda_{\ell} \underline{a}^{\ell} \cdot \underline{a}^{\ell*} - \frac{i\mu}{2} \sum_{\ell} \frac{\partial \lambda_{\ell}}{\partial \omega_{\ell}} \left(\underline{a}^{\ell} \cdot \frac{\partial \underline{a}^{\ell*}}{\partial t} - \underline{a}^{\ell*} \cdot \frac{\partial \underline{a}^{\ell}}{\partial t} \right) \\ & - \frac{\partial \lambda_{\ell}}{\partial k_{\beta}} \left(\underline{a}^{\ell} \cdot \frac{\partial \underline{a}^{\ell*}}{\partial x_{\beta}} - \underline{a}^{\ell*} \cdot \frac{\partial \underline{a}^{\ell}}{\partial x_{\beta}} \right) + O(\mu^2), \end{aligned} \quad (36)$$

or, alternatively, as

$$\langle L_2 \rangle = \sum_{\ell} \lambda_{\ell} |\hat{a}^{\ell}|^2 - \mu \sum_{\ell} N_{\ell} \left[\frac{\partial}{\partial t} + \underline{v}_G^{\ell} \cdot \nabla \right] \theta_{\ell} + O(\mu^2), \quad (37)$$

where

$$N_{\ell} = \frac{\partial \lambda_{\ell}}{\partial \omega_{\ell}} |\hat{a}^{\ell}|^2, \quad \underline{v}_G^{\ell} = - \left(\frac{\partial \lambda_{\ell}}{\partial \underline{k}} \right) / \left(\frac{\partial \lambda_{\ell}}{\partial \omega_{\ell}} \right) = \frac{\partial \omega_{\ell}}{\partial \underline{k}} \Big|_{\lambda_{\ell}=0} \quad (38)$$

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are the ℓ -th wave action density and group velocity, respectively.

If L_3 and other higher order terms in the lagrangian were absent then the Euler-Lagrange equations for Eqs. (37) or (38) would show that the wavetrains behave independent of each other, and that the amplitudes change only as the ambient changes. The inclusion of the nonlinear terms changes this picture. The waves can still propagate nearly independently but are subject to small interactions. This occurs if three or more waves satisfy a resonance condition between their frequencies, and between their corresponding wave number vectors, so that there are terms in Eq. (1), besides that from L_2 , which do not oscillate rapidly, contributing in the averaging process to a modification of the Euler-Lagrange equations.

L_3 is a given cubic expression in $\underline{\xi}$ and its derivatives, so that substituting Eq. (31) into it yields a large summation over all possible triplets of waves. Each combination of three waves (which we here label ℓ , m and n) provides eight terms containing the factors

$$e^{i\eta} \quad \text{where } \eta = \pm \phi_\ell \pm \phi_m \pm \phi_n, \quad (39)$$

for all choices of sign. In general we may expect that $\eta \neq 0$ for any choice of signs, in which case these terms oscillate on the fast scale and their contribution to the integral

$$\epsilon \int L_3 d^3x dt \quad (40)$$

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average out to zero. The dynamics of these waves is as in linear theory since the interaction term has no effect, because in making variations of \underline{a}^ℓ , etc., in applying Hamilton's principle, variations in action due to them are negligible.

The only terms contributing to Eq. (40) from the substitution of Eq. (31), and averaging, are then those for which $\eta = 0$, or, equivalently,

$$(\pm \underline{k}^\ell \pm \underline{k}^m \pm \underline{k}^n) \cdot \underline{x} - (\pm \omega_\ell \pm \omega_m \pm \omega_n) t = 0 \quad (41)$$

for particular choice of signs. Because of our sign convention for ω , i.e. $\omega > 0$, two of the signs must be opposite to a third, so following from Eq. (41) let us choose as the standard relation among waves

$$\left. \begin{aligned} \omega_\ell &= \omega_m + \omega_n \\ \underline{k}^\ell &= \underline{k}^m + \underline{k}^n \end{aligned} \right\} \cdot \quad (42)$$

This relation, variously called the resonant, kinematic, or synchronism, condition for nonlinear wave interaction might hold permanently or over a substantial time and space range. As Eq. (40) is a correction of order ϵ to the action integral it is sufficient to test the kinematic condition using the local linear dispersion relation of the waves.

The contribution to the integrand of Eq. (40) coming from

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interactions of resonant triplets of waves is thus of the form

$$\langle L_3 \rangle = \sum_{\ell, m, n} (V_{\ell mn} : \underline{a}^\ell \underline{a}^{m*} \underline{a}^{n*} + \text{c.c.}), \quad (43)$$

where the summation is over all wave triplets satisfying Eq. (42), and the three vertical dots indicate the triadic inner product. $V_{\ell mn}$ is more explicitly given when L_3 is specified. If L_3 involves derivatives of $\underline{\xi}$, $V_{\ell mn}$ would also contain the ω 's and \underline{k} 's of the waves. For simplicity, we assume in this section that $V_{\ell mn}$ is real. With the representation $\underline{a}^\ell = \hat{a}^\ell e^{i\theta_\ell}$, etc., $\langle L_3 \rangle$ can then be put in the form

$$\langle L_3 \rangle = 2 \sum_{\ell, m, n} V_{\ell mn} : \hat{a}^\ell \hat{a}^m \hat{a}^n \cos \theta \quad (44)$$

where $\theta = \theta_\ell - \theta_m - \theta_n$ is the relative phase.

The Euler-Lagrange equations for $\langle L \rangle = \langle L_2 \rangle + \epsilon \langle L_3 \rangle$ with respect to variations in θ_ℓ and \hat{a}^ℓ are, respectively,

$$\left(\frac{\partial}{\partial t} + \nabla \cdot \mathbf{v}_G^\ell \right) N_\ell = - 2 \frac{\epsilon}{\mu} \sum_{m, n} V_{\ell mn} : \hat{a}^\ell \hat{a}^m \hat{a}^n \sin \theta, \quad (45)$$

$$N_\ell \left(\frac{\partial}{\partial t} + \mathbf{v}_G^\ell \cdot \nabla \right) \theta_\ell = \frac{\epsilon}{\mu} \sum_{m, n} V_{\ell mn} : \hat{a}^m \hat{a}^n \hat{a}^n \cos \theta, \quad (46)$$

to lowest order in ϵ and μ ; this follows from Eqs. (37) & (44) and because $\lambda_\ell = 0$. The summation is over all waves (m, n) in

resonance with wave ℓ , satisfying the kinematic relations like Eq. (42). If there are just three waves in resonance, then similar variations with respect to θ_m and θ_n give the action transfer relations

$$\begin{aligned} \frac{\partial N_\ell}{\partial t} + \nabla \cdot (\underline{v}_G^\ell N_\ell) &= - \left[\frac{\partial N_m}{\partial t} + \nabla \cdot (\underline{v}_G^m N_m) \right] = - \left[\frac{\partial N_n}{\partial t} + \nabla \cdot (\underline{v}_G^n N_n) \right] \\ &= - \frac{2\epsilon}{\mu} V_{\ell mn} : \hat{a}^\ell \hat{a}^m \hat{a}^n \sin \theta. \end{aligned} \quad (47)$$

The first two equations (called the Manley & Rowe (1956) relations in nonlinear electronics) give information about the relative rate of transfer of action between the three waves. These equations have the obvious interpretation that wavetrains m and n each lose, and wave ℓ gains, action at equal rates. The right-hand side of Eq. (47) gives the "cross-section" of the process, the direction and absolute value of the transfer rate. This is calculated from L_3 , and, besides being proportional to the product of the amplitudes and the relative phase θ , depends on the relative geometric polarization of the waves which follows from the detailed form of L_3 .

Examination of Eqs. (45)-(47) shows that if the variation of action of each wave is due to nonlinear resonance between wave triplets, then the scale of nonlinearity, represented by ϵ , and the scale of slowness of amplitude variations, represented

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by μ , must be of the same order of magnitude. Thus, from the beginning, we could have put $\epsilon = \mu$, which we shall hereafter do. (However, if the local dispersion relation does not allow three-wave kinematic relation solutions, such as Eq. (42), then $\langle L_3 \rangle = 0$, and for nonlinear action transfer we would have to move to a four-wave process with $\langle L \rangle = \langle L_2 \rangle + \epsilon^2 \langle L_4 \rangle$, and a resulting identification of $\mu = \epsilon^2$ for significant nonlinear coupling between waves.)

The approach which we follow in treating nonlinear interactions with the averaged lagrangian method is due to Simmons (1969) and Dougherty (1970). Whitham (1967) had briefly indicated a treatment of resonant wave interactions, using a model equation. However, Simmons explicitly introduced a slowly varying phase term and kept the local frequency and wave number constant, as we do. Dougherty adopted a complex amplitude formalism which reduces the algebraic manipulation required in averaging the lagrangian. Curiously, he and Dysthe (1974), who elaborated on Dougherty's discussion on nonlinear wave interactions, carried along an additional slow phase term for each wave with which to effect the variation with respect to phase in the averaged lagrangian. We absorb the slow phase into the phase of the complex amplitude of each wavetrain, and, after averaging the lagrangian, our slow variables are \underline{a}^r and \underline{a}^{r*} (or, equivalently, N_r and θ_r) with which we carry out the variation in the averaged variational principle.

4.0 Quasilinear Waves

Here, we describe the theory of the propagation of plane small-amplitude wavetrains in an isothermal atmosphere, employing L_2 and the averaged lagrangian method introduced previously. Although the actual atmosphere varies considerably in temperature with height, an isothermal atmosphere is analytically tractable and still allows an understanding of the dispersive properties of the waves and of their nonlinear coupling, which we shall take up in Section 5.0.

Since our medium is inhomogeneous, with the background pressure and density decreasing exponentially with height, we first have to introduce a scaling of our dependent variable, the displacement $\underline{\xi}$, so that as a wave progresses vertically its "amplitude" will remain small and constant, or at most slowly varying, as in a uniform medium. A clue to a way to do this is provided by considering the second order kinetic energy density

$$T = \frac{1}{2} \rho_0 \dot{\underline{\xi}}^2,$$

which is bounded for a wavetrain even though $\underline{\xi}$ is an exponentially growing function of altitude as ρ_0 decreases. In L_2 , Eq. (2.45), we thus assume as our solution of the governing equations a modified plane wavetrain of the generic form

$$\underline{\xi} = \frac{1}{\sqrt{\rho_0}} (\underline{a} e^{i\phi} + \underline{a}^* e^{-i\phi}) \quad (1)$$

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where $\underline{a} = \underline{a}(\epsilon t, \epsilon \underline{x})$ is our complex amplitude varying on a slow scale, and $\phi = \phi(\underline{x}, t)$ is a function of the fast scale whose space and time derivatives give the local wavenumber and frequency, i.e.,

$$\phi = \underline{k} \cdot \underline{x} - \omega t, \quad \underline{k} = \nabla \phi, \quad \omega = -\partial \phi / \partial t. \quad (2)$$

Any slow variation of the total phase of the wave is described by the phase of \underline{a} . It should perhaps be stated at this point that in scaling with respect to ρ_0 in Eq. (1) we are implicitly making the reasonable assumption that the vertical displacement ξ_z is small compared to the scale height H , i.e. $\xi_z \ll H$, where $\rho_0 \sim e^{-z/H}$.

With perhaps an appropriate multiplicative constant, we could call \underline{a} the 'potential' amplitude in analogy with the potential temperature of an adiabatically transported fluid particle in the atmosphere, in meteorological description.

With Eq. (1) the kinetic energy density becomes

$$T_2 = \frac{1}{2} [(\epsilon \dot{\underline{a}} - i\omega \underline{a}) e^{i\phi} + \text{c.c.}]^2, \quad (3)$$

and averaging out the fast oscillation we have

$$\langle T_2 \rangle = \omega^2 \underline{a} \cdot \underline{a}^* + i\epsilon \omega (\dot{\underline{a}} \cdot \underline{a}^* - \dot{\underline{a}}^* \cdot \underline{a}) + O(\epsilon^2). \quad (4)$$

In this section we retain $O(\epsilon)$ terms because we are interested

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in wavetrain solutions: quasilinear waves which behave locally as linear waves, described by zero order terms, but whose amplitude and phase change slowly, adding small corrections to a strictly linear process.

Before proceeding to the equations for the isothermal atmosphere, it is interesting to note that the potential energy density in Eq. (2.45) has two physically identifiable parts

$$V_2 = V_e + V_v, \quad (5)$$

where, the fluid being compressible, there is the elastic energy

$$V_e = \frac{1}{2} \lambda (\nabla \cdot \underline{\xi})^2 \quad (6)$$

with $\nabla \cdot \underline{\xi}$ as the incremental volume change or strain, and, in the notation of Tolstoy (1963),

$$\lambda = \rho_0 c^2 \quad (7)$$

is the bulk modulus, with $c^2 = \gamma p_0 / \rho_0$ being the velocity of sound.

$$V_v = - \frac{p_0}{2} [(\nabla \cdot \underline{\xi})^2 - \xi_{ij} \xi_{ji}] \quad (8)$$

is the work done by the stress $-p_0$ in producing the second order volume increment, as may be seen in the expansion of J , Eq. (2.35).

For the isothermal atmosphere (constant scale height) the

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density variation with height may be written

$$\rho_0 = \rho_{00} e^{-z/H} = \rho_{00} e^{-2\nu z}, \quad (9)$$

where ρ_{00} is the density at $z=0$ and

$$H = \frac{c^2}{\gamma g} = \frac{RT}{g} \equiv \frac{1}{2\nu} \quad (10)$$

is the scale height. For the representation of $\underline{\xi}$ as given by Eqs. (1) & (2) and ρ_0 given by the above, one has for the vertical derivative

$$\frac{\partial \xi_i}{\partial x_z} = \frac{1}{\sqrt{\rho_0}} [(\epsilon a_{i,z} + (ik_z + \nu) a_i) e^{i\phi} + \text{c.c.}]. \quad (11)$$

In order to have formally similar expressions for vertical and horizontal derivatives it is convenient to define an auxiliary complex wave number vector:

$$\underline{K} = (k_x, k_y, k_z - i\nu) = (K_x, K_y, K_z) \quad (12)$$

where k_x, k_y, k_z are real. Then, in general (any i, j),

$$\frac{\partial \xi_i}{\partial x_j} = \frac{1}{\sqrt{\rho_0}} [(\epsilon a_{ij} + iK_j a_i) e^{i\phi} + (\epsilon a_{ij}^* - iK_j^* a_i^*) e^{-i\phi}] \quad (13)$$

When one carries out the averaging, the terms in V_2 then become

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$$\langle \xi_{ii} \xi_{jj} \rangle = \frac{2}{\rho_0} [K_i K_j^* a_i a_j^* + i\epsilon (K_i a_i a_{jj}^* - K_i^* a_i^* a_{jj})] + O(\epsilon^2), \quad (14)$$

$$\langle \xi_{ij} \xi_{ji} \rangle = \frac{2}{\rho_0} [K_i^* K_j a_i a_j^* + i\epsilon (K_j a_i a_{ji}^* - K_j^* a_i^* a_{ji})] + O(\epsilon^2), \quad (15)$$

and so the average of L_2 is, in ascending order of powers of ϵ ,

$$\langle L_2 \rangle = \langle L_2(0) \rangle + \langle L_2(\epsilon) \rangle + O(\epsilon^2) \quad (16)$$

where

$$\langle L_2(0) \rangle = [\omega^2 \delta_{ij} - \frac{P_0}{\rho_0} ((\gamma-1) K_i K_j^* + K_i^* K_j)] a_i a_j^*, \quad (17)$$

$$\begin{aligned} \langle L_2(\epsilon) \rangle = i\epsilon [& \omega (\dot{a}_i a_j^* - \dot{a}_i^* a_j) \delta_{ij} - (\gamma-1) \frac{P_0}{\rho_0} (K_i a_i a_{jj}^* - K_i^* a_i^* a_{jj}) \\ & - \frac{P_0}{\rho_0} (K_j a_i a_{ji}^* - K_j^* a_i^* a_{ji})]. \end{aligned} \quad (18)$$

The lowest order part $\langle L_2(0) \rangle$, an Hermitian quadratic form in \underline{a} , describes the dominant and locally linear behavior, while $\langle L_2(\epsilon) \rangle$ gives the $O(\epsilon)$ contribution to $\langle L \rangle$ arising from the slow variation of amplitude in time and space.

Defining the Hermitian matrix element

$$A_{ij} = \omega^2 \delta_{ij} - \frac{P_0}{\rho_0} ((\gamma-1) K_i K_j^* + K_i^* K_j), \quad (19)$$

then

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$$\frac{\partial A_{ij}}{\partial \omega} = 2\omega \delta_{ij}, \quad (20)$$

$$\frac{\partial A_{ij}}{\partial k_\beta} = -\frac{\rho_0}{\rho_0} [\delta_{j\beta} ((\gamma-1) K_i + K_i^*) + \delta_{i\beta} ((\gamma-1) K_j^* + K_j)], \quad (21)$$

and Eqs. (17) & (18) can be put in the compact form

$$\langle L_2(0) \rangle = A_{ij} a_i a_j^* \quad (22)$$

$$\langle L_2(\epsilon) \rangle = \frac{i\epsilon}{2} \left[\frac{\partial A_{ij}}{\partial \omega} (\dot{a}_i a_j^* - \dot{a}_i^* a_j) \delta_{ij} + \frac{\partial A_{ij}}{\partial k_\beta} (a_i a_{j\beta}^* - a_i^* a_{j\beta}) \right]. \quad (23)$$

Straightforward application of Hamilton's principle to the action integral with $\langle L_2 \rangle$ given by Eqs. (16), (22) & (23) would give the equations of motion for the complex amplitudes \underline{a} and \underline{a}^* . However, since $\langle L_2(\epsilon) \rangle$ is $O(\epsilon)$, it is more convenient to solve for \underline{a} with $\langle L \rangle$ approximated by the dominant term $\langle L_2(0) \rangle$ and to iterate such solutions into $\langle L \rangle = \langle L_2(0) \rangle + \langle L_2(\epsilon) \rangle$, and solve for the variation of \underline{a} in space and time. The same consideration applies in weakly nonlinear dynamics in which L_3 , etc., are $O(\epsilon)$, etc., following from the perturbation expansion of L . It is possible to justify this procedure, a posteriori, by an orthogonal expansion of the amplitude, as indicated in the previous section. Actually, the argument is tautologous since, from the beginning, we assumed that solutions have the quasilinear form

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of Eq. (1); this implies that \underline{a} and \underline{a}^* are "nearly" always solutions of the linear part, $\langle L_2(0) \rangle$, of $\langle L \rangle$, although their magnitude $\hat{a} = (\underline{a} \cdot \underline{a}^*)^{1/2}$ can change appreciably; i.e., we take \underline{a} to be parallel to the linear normal mode solution following from $\langle L_2(0) \rangle$.

In the linear approximation the Euler-Lagrange equations are obtained by variation with respect to \underline{a} and \underline{a}^* , yielding

$$\frac{\partial \langle L_2(0) \rangle}{\partial \underline{a}} = 0, \quad \frac{\partial \langle L_2(0) \rangle}{\partial \underline{a}^*} = 0. \quad (24)$$

These are independent linear homogeneous equations in \underline{a}^* and \underline{a} , respectively, and may be considered as eigenvector equations of the system. The eigenvalues of either of these equations yield the dispersion relations for acoustic-gravity waves, while the corresponding eigenvectors determine the polarization of the waves. Note that if Eqs. (24) are satisfied then $\langle L_2(0) \rangle = 0$; this is an instance of a general result (discussed, among others, by Whitham (1965)) of the equality of averaged kinetic and potential wave energy.

To exhibit the dispersion relation in detail, we write the second of Eqs. (24) out explicitly, using Eqs. (19) & (22), as

$$\frac{\partial \langle L_2(0) \rangle}{\partial a_j^*} = A_{ij} a_i = \left[\omega^2 \delta_{ij} - \frac{p_0}{\rho_0} \left((\gamma-1) K_i K_j^* + K_i^* K_j \right) \right] a_i = 0. \quad (25)$$

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This has the matrix representation $\underline{a} \cdot A = 0$, with

$$A = \begin{bmatrix} \omega^2 - c^2 k_x^2 & -c^2 k_x k_y & -(\gamma k_z - i(2-\gamma)v) k_x c^2 / \gamma \\ -c^2 k_x k_y & \omega^2 - c^2 k_y^2 & -(\gamma k_z - i(2-\gamma)v) k_y c^2 / \gamma \\ -(\gamma k_z + i(2-\gamma)v) k_x c^2 / \gamma & -(\gamma k_z + i(2-\gamma)v) k_y c^2 / \gamma & \omega^2 - c^2 (k_z^2 + v^2) \end{bmatrix}, \quad (26)$$

or, equivalently,

$$\begin{bmatrix} -\omega^2 / c^2 + k_x^2 & k_x k_y & k_x (k_z + i\beta v) \\ k_x k_y & -\omega^2 / c^2 + k_y^2 & k_y (k_z + i\beta v) \\ k_x (k_z - i\beta v) & k_y (k_z - i\beta v) & -\omega^2 / c^2 + k_z^2 + v^2 \end{bmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = 0, \quad (27)$$

where $\beta = (2/\gamma) - 1$. The existence of a solution requires the vanishing of the determinant of the matrix. The determinant of A is readily computed to be

$$\det = \omega^2 \left(\omega^4 - \omega^2 ((k_x^2 + k_y^2 + k_z^2) c^2 + \omega_A^2) + (k_x^2 + k_y^2) c^2 \omega_B^2 \right), \quad (28)$$

$$\omega_A^2 = c^2 v^2 = c^2 / 4H^2; \quad \omega_B^2 = c^2 v^2 (1 - \beta^2) = (\gamma - 1) g^2 / c^2. \quad (29)$$

ω_A is the acoustic cutoff frequency, and ω_B is the Brunt-Väisälä frequency. These are critical frequencies, determined by the atmospheric background parameters, which delineate the acoustic

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mode, with $\omega > \omega_A$, from the gravity wave mode, with $\omega < \omega_B$. Since for air $\gamma = 1.4$, then $\omega_A > \omega_B$, and $\omega_A^2 = 1.225 \omega_B^2$. The vanishing of the determinant gives the well known dispersion relation for acoustic-gravity waves (Eckart, 1960; Hines, 1960; Pierce, 1963; Tolstoy, 1963):

$$D = \omega^4 - \omega^2(k^2 c^2 + \omega_A^2) + k_H^2 c^2 \omega_B^2 = 0, \quad (30)$$

where we have put

$$k^2 = k_x^2 + k_y^2 + k_z^2, \quad k_H^2 = k_x^2 + k_y^2. \quad (31)$$

Since the matrix equation $\underline{a} \cdot A(\omega, \underline{k}) = 0$ implies

$$\det \{A(\omega, \underline{k})\} = \lambda_1(\omega, \underline{k}) \lambda_2(\omega, \underline{k}) \lambda_3(\omega, \underline{k}) = 0 \quad (32)$$

where λ_i ($i=1,2,3$) are the eigenvalues of A , then if the pair (ω, \underline{k}) is to correspond to an allowed wave, or normal mode, one of the eigenvalues, say λ_m , must be zero, and \underline{a} should be the corresponding eigenvector. It follows that a wave satisfying the dispersion relation

$$\lambda_m(\omega, \underline{k}) = 0 \quad (33)$$

induces from Eq. (22) the representation

$$\langle L_2(0) \rangle = \lambda_m \underline{a} \cdot \underline{a}^*, \quad (34)$$

where here \underline{a} must be understood to have a direction parallel to that of the eigenvector of A corresponding to λ_m . The eigenvalues of the system are readily found from Eq. (28) to be

$$\lambda_0 = \omega^2 \quad (35)$$

$$\lambda_a = \omega^2 - \frac{1}{2}(k^2 c^2 + \omega_A^2) - \left[\frac{1}{4}(k^2 c^2 + \omega_A^2)^2 - k_H^2 c^2 \omega_B^2 \right]^{1/2}, \quad (36)$$

$$\lambda_g = \omega^2 - \frac{1}{2}(k^2 c^2 + \omega_A^2) + \left[\frac{1}{4}(k^2 c^2 + \omega_A^2)^2 - k_H^2 c^2 \omega_B^2 \right]^{1/2}. \quad (37)$$

The first (since it is zero only when $\omega^2=0$) relates to the mechanics of steady flow, so is of no particular interest for us, while λ_a and λ_g correspond to the acoustic and internal gravity wave modes, respectively.

Using the normal mode solutions for $\langle L_2(0) \rangle$, just discussed, as a local description for a wavetrain we can now write $\langle L_2 \rangle$, from Eqs. (16), (22), (23) & (34), as

$$\begin{aligned} \langle L_2 \rangle = & \lambda \underline{a} \cdot \underline{a}^* + \frac{i\varepsilon}{2} \left[\frac{\partial \lambda}{\partial \omega} (\dot{\underline{a}} \cdot \underline{a}^* - \dot{\underline{a}}^* \cdot \underline{a}) + \frac{\partial \lambda}{\partial k_\beta} \left(\underline{a} \cdot \frac{\partial \underline{a}^*}{\partial x_\beta} - \underline{a}^* \cdot \frac{\partial \underline{a}}{\partial x_\beta} \right) \right] \\ & + O(\varepsilon^2), \end{aligned} \quad (38)$$

where $(\lambda, \underline{a}, \underline{a}^*)$ belong to the same eigenmode. Let us introduce the group velocity

$$\underline{v}_G \equiv - \left(\frac{\partial \lambda}{\partial \underline{k}} \right) / \left(\frac{\partial \lambda}{\partial \omega} \right) = \left(\frac{\partial \omega}{\partial \underline{k}} \right)_{\lambda=0} \quad (39)$$

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for acoustic-gravity waves, which, from Eqs. (30) or (36) & (37), becomes, in component form,

$$\underline{v}_G = \frac{Rc^2}{\omega} \left[k_x, k_y, \left(\frac{\omega^2}{\omega^2 - \omega_B^2} k_z \right) \right], \quad (40)$$

with

$$R = (\omega^2 - \omega_B^2) / (2\omega^2 - \omega_A^2 - k^2 c^2).$$

Then, since $\partial\lambda/\partial\omega = 2\omega$, Eq. (38) may be expressed as

$$\begin{aligned} \langle L_2 \rangle = \lambda \underline{a} \cdot \underline{a}^* + i\epsilon\omega \left[(\dot{\underline{a}} \cdot \underline{a}^* - \dot{\underline{a}}^* \cdot \underline{a}) + v_{G\beta} \left(\underline{a}^* \cdot \frac{\partial \underline{a}}{\partial x_\beta} - \underline{a} \cdot \frac{\partial \underline{a}^*}{\partial x_\beta} \right) \right] \\ + O(\epsilon^2). \end{aligned} \quad (41)$$

With Eq. (41) in the action integral, variation with respect to \underline{a}^* gives the transport equation

$$\left(\frac{\partial}{\partial t} + \underline{v}_G \cdot \nabla \right) \underline{a} = - \frac{1}{2\omega} (\nabla \cdot \underline{v}_G) \underline{a}, \quad (42)$$

where we have used the fact that $\lambda = 0$. This equation, together with its conjugate obtained from the \underline{a} variation, determines the amplitude of a wavetrain along the direction of the group velocity (rays) as it moves in an atmosphere with varying local background parameters, the gradients of which are represented by the right-hand side of Eq. (42). This is explicitly shown by introducing the real amplitude \hat{a} and phase θ by

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$$\underline{a} = \hat{a} e^{i\theta}, \quad \underline{a}^* = \hat{a} e^{-i\theta}. \quad (43)$$

Eqs. (42) and its conjugate are then equivalent to the equations

$$\left(\frac{\partial}{\partial t} + \underline{v}_G \cdot \nabla \right) \hat{a} = - \frac{1}{2\omega} (\nabla \cdot \underline{v}_G) \hat{a}, \quad (44)$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_G \cdot \nabla \right) \theta = 0. \quad (45)$$

The second equation indicates the phase is invariant along rays. This can be interpreted as showing that there are no frequency and wavenumber shifts along the group path.

A simple expression for changes in the amplitude magnitude $|\underline{a}| = |\hat{a}|$ is shown by a change to action and phase angle variables. Defining the wave action by

$$N = \frac{\partial \lambda}{\partial \omega} \underline{a} \cdot \underline{a}^* = \frac{\partial \lambda}{\partial \omega} |\hat{a}|^2 = 2\omega |\hat{a}|^2, \quad (46)$$

then, with Eqs. (43), $\langle L_2 \rangle$ from Eqs. (38) & (41) can be expressed as

$$\langle L_2 \rangle = \lambda |\hat{a}|^2 - \epsilon N \left(\frac{\partial \theta}{\partial t} + \underline{v}_G \cdot \nabla \theta \right) + O(\epsilon^2). \quad (47)$$

With this in the action integral, variation with respect to θ gives

$$\frac{\partial N}{\partial t} + \nabla \cdot (\underline{v}_G N) = 0, \quad (48)$$

the conservation equation for wave action, which gives the relation governing the magnitude $|\hat{a}|$. A posteriori, using the definition for N of Eqs. (46), this equation is seen to be just the sum of Eq. (42) and its conjugate. The variation of Eq. (47) with respect to N gives the phase relation previously found, Eq. (45).

As we have just seen, changes in amplitude of a wavetrain, in the quasilinear approximation, are caused by changes in the background parameters as revealed in the gradients of the local group velocity. We shall not pursue this topic further as it is properly in the realm of asymptotic wave theory, with which we are not primarily concerned. We shall see in the following section that amplitude changes of wavetrains are possible, even in the absence of background parameter gradients, through the nonlinear interaction of three or more waves, if they satisfy a resonance condition.

5.0 Nonlinear Interactions

The major development of this thesis, a dynamical description of nonlinear interactions of acoustic-gravity waves, is presented in this section, described to lowest order in nonlinearity. The equations of motion are now governed by the action principle with the lagrangian

$$L = L_2 + \epsilon L_3, \quad (1)$$

where L_2 is as in the previous section, or Eq. (2.45), and, from Eq. (2.46),

$$L_3 = p_0 \left[\frac{\gamma}{2} (\xi_{kk} \xi_{ij} \xi_{ji} - \alpha (\nabla \cdot \underline{\xi})^3) + \frac{\partial(\underline{\xi})}{\partial(\underline{x})} \right], \quad \alpha = \frac{2-\gamma}{3}. \quad (2)$$

In Eq. (1) we have introduced the smallness parameter ϵ which follows from scaling the displacement $\underline{\xi}$ in the perturbation expansion of L in Section 2.1. Thus only weak nonlinearity is implied.

We assume that the displacement field is given by a sum of elementary terms

$$\underline{\xi} = \underline{\xi}^{\ell} + \underline{\xi}^m + \underline{\xi}^n + \dots = \sum_r \underline{\xi}^r, \quad (3)$$

and each term is considered as a slowly varying wavetrain

$$\underline{\xi}^r = \frac{1}{\sqrt{\rho_0}} (\underline{a}^r e^{i\phi_r} + \underline{a}^{r*} e^{-i\phi_r}), \quad r = \ell, m, n, \dots \quad (4)$$

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with its own slowly varying complex amplitude given by

$$\underline{a}^r(\epsilon t, \epsilon \underline{x}) = \hat{a}^r e^{i\theta_r}, \quad (5)$$

where \hat{a}^r is the real amplitude and θ_r is the phase. As in the previous section $\phi_r = \phi_r(\underline{x}, t)$ is the fast phase whose space and time derivatives define the local wavenumber and frequency:

$$\phi_r = \underline{k}^r \cdot \underline{x} - \omega_r t, \quad \underline{k}^r = \nabla \phi_r, \quad \omega_r = -\partial \phi_r / \partial t. \quad (6)$$

It should be noted that we have identified the slowness scale ϵ in Eqs. (4) & (5) with the smallness scale of Eq. (1) as we consider here only the lowest order nonlinear term L_3 .

From Eqs. (3) & (4) we thus have

$$\xi_i = \frac{1}{\sqrt{\rho_0}} \sum_r a_i^r e^{i\phi_r} + \text{c.c.}, \quad r = l, m, n, \dots \quad (7)$$

This is inserted into the expressions for L_2 and L_3 in Eq. (1), and the action principle gives the dynamical equations for the waves. The action integral is now approximated as an integration of an averaged lagrangian in which rapid fluctuations have been removed and only slow variables remain, as in the treatment of a single wavetrain given before.

The form of $\langle L_2 \rangle$ is just the superposition of single wavetrain averaged lagrangians, the structure of each of which was discussed in the last section. This is because after putting

Eq. (7) into L_2 the only terms in which the sum of fast phases ϕ_r cancel, in carrying out the averaging, are those belonging to the same wave. Thus, following from Eqs. (4.16-4.23),

$$\langle L_2 \rangle = \langle L_2(0) \rangle + \langle L_2(\epsilon) \rangle + O(\epsilon^2), \quad (8)$$

$$\langle L_2(0) \rangle = \sum_r A_{ij}(\omega_r, \underline{k}^r) a_i^r a_j^{r*}, \quad (9)$$

$$\langle L_2(\epsilon) \rangle = \frac{i\epsilon}{2} \sum_r \left[\frac{\partial A_{ij}}{\partial \omega_r} (a_i^r a_j^{r*} - a_i^{r*} a_j^r) \delta_{ij} + \frac{\partial A_{ij}}{\partial k_\beta^r} (a_i^r a_{j\beta}^{r*} - a_i^{r*} a_{j\beta}^r) \right], \quad (11)$$

and where the Hermitian matrix element

$$A_{ij}(\omega_r, \underline{k}^r) = \omega_r^2 \delta_{ij} - \frac{p_0}{\rho_0} ((\gamma-1)K_i^r K_j^{r*} + K_i^{r*} K_j^r). \quad (11)$$

Substituting the derivatives of Eq. (7) into the cubic expression for L_3 , Eq. (2) results in a summation over all possible triplets of waves. Each triplet, which we shall label (ℓ, m, n) , gives eight terms, each term containing the factor

$$e^{i\eta} \quad \text{where} \quad \eta = \pm \phi_\ell \pm \phi_m \pm \phi_n \quad (12)$$

with a particular choice of signs. To effect the averaging, i.e. $\eta = 0$, two of the signs must be opposite to a third. Thus, as a standard choice, we shall put

$$\pm (\phi_\ell - \phi_m - \phi_n) = 0. \quad (13)$$

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This implies, via Eq. (6), the kinematic relations

$$\left. \begin{aligned} \omega_l &= \omega_m + \omega_n \\ \underline{k}^l &= \underline{k}^m + \underline{k}^n \end{aligned} \right\} \quad (14)$$

be satisfied by any triplet of waves contributing to $\langle L_3 \rangle$.

With the resonant conditions of the form of Eqs. (14) the terms in L_3 , Eq. (2), which remain after averaging are as follows:

$$\begin{aligned} \frac{\gamma p_0}{2} \langle \xi_{kk} \xi_{ij} \xi_{ji} \rangle &= - \frac{ic^2}{2\sqrt{\rho_0}} [(\underline{K}^l \cdot \underline{a}^l) (\underline{K}^{m*} \cdot \underline{a}^{n*}) (\underline{K}^{n*} \cdot \underline{a}^{m*}) \\ &+ (\underline{K}^{m*} \cdot \underline{a}^{m*}) (\underline{K}^l \cdot \underline{a}^{n*}) (\underline{K}^{n*} \cdot \underline{a}^l) + (\underline{K}^{n*} \cdot \underline{a}^{n*}) (\underline{K}^l \cdot \underline{a}^{m*}) (\underline{K}^{m*} \cdot \underline{a}^l)] \\ &+ \text{c.c.} \end{aligned} \quad (15)$$

$$\frac{\gamma p_0}{2} \alpha \langle (\nabla \cdot \xi)^3 \rangle = - \frac{3ic^2}{2\sqrt{\rho_0}} \alpha (\underline{K}^l \cdot \underline{a}^l) (\underline{K}^{m*} \cdot \underline{a}^{m*}) (\underline{K}^{n*} \cdot \underline{a}^{n*}) + \text{c.c.} \quad (16)$$

$$\begin{aligned} p_0 \left\langle \frac{\partial(\xi)}{\partial(\underline{x})} \right\rangle &\equiv p_0 \langle \nabla \xi_x \cdot \nabla \xi_y \times \nabla \xi_z \rangle \equiv p_0 \left\langle \frac{\partial \xi}{\partial x} \cdot \frac{\partial \xi}{\partial y} \times \frac{\partial \xi}{\partial z} \right\rangle \\ &= - \frac{ic^2}{\gamma \sqrt{\rho_0}} (\underline{K}^l \cdot \underline{K}^{m*} \times \underline{K}^{n*}) (\underline{a}^l \cdot \underline{a}^{m*} \times \underline{a}^{n*}) + \text{c.c.} \end{aligned} \quad (17)$$

We can thus write $\langle L_3 \rangle$ as the sum of all triads of

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acoustic-gravity waves satisfying the kinematic relations of the form Eq. (14) as

$$\langle L_3 \rangle = \sum_{\ell, m, n} v^{\ell mn} \vdots \underline{a}^{\ell} \underline{a}^{m*} \underline{a}^{n*} + \text{c.c.} \quad (18)$$

where the three dots signify the inner product of the complex amplitudes and the triadic

$$\begin{aligned} v^{\ell mn} &= - \frac{ic^2}{2\sqrt{\rho_0}} \left[(\underline{K}^{\ell} \underline{K}^{n*} \underline{K}^{m*} + \underline{K}^{n*} \underline{K}^{m*} \underline{K}^{\ell} + \underline{K}^{m*} \underline{K}^{\ell} \underline{K}^{n*}) + 3\alpha (\underline{K}^{\ell} \underline{K}^{m*} \underline{K}^{n*}) \right. \\ &\quad + \frac{2}{\gamma} (\underline{K}^{\ell} \underline{K}^{m*} \underline{K}^{n*} - \underline{K}^{\ell} \underline{K}^{n*} \underline{K}^{m*} + \underline{K}^{n*} \underline{K}^{\ell} \underline{K}^{m*} \\ &\quad \left. - \underline{K}^{m*} \underline{K}^{\ell} \underline{K}^{n*} + \underline{K}^{m*} \underline{K}^{n*} \underline{K}^{\ell} - \underline{K}^{n*} \underline{K}^{m*} \underline{K}^{\ell}) \right]. \\ &= - \frac{ic^2}{2\sqrt{\rho_0}} \left[\left(1 - \frac{2}{\gamma} \right) (\underline{K}^{\ell} \underline{K}^{n*} \underline{K}^{m*} + \underline{K}^{n*} \underline{K}^{m*} \underline{K}^{\ell} + \underline{K}^{m*} \underline{K}^{\ell} \underline{K}^{n*}) \right. \\ &\quad \left. + \left(3\alpha + \frac{2}{\gamma} \right) \underline{K}^{\ell} \underline{K}^{m*} \underline{K}^{n*} + \frac{2}{\gamma} (\underline{K}^{n*} \underline{K}^{\ell} \underline{K}^{m*} + \underline{K}^{m*} \underline{K}^{n*} \underline{K}^{\ell}) \right]. \quad (19) \end{aligned}$$

The dynamics of the wavetrains is given by the action principle with the averaged lagrangian

$$\langle L \rangle = \langle L_2(0) \rangle + \langle L_2(\varepsilon) \rangle + \varepsilon \langle L_3 \rangle, \quad (20)$$

a function of the complex amplitudes $\underline{a}^r, \underline{a}^{r*}$. We proceed in a

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parallel fashion as in the analysis of single wavetrains; i.e. since the second and third terms of Eq. (20) are $O(\epsilon)$ we substitute in the expressions for $\langle L \rangle$ the eigenvector solutions for $\langle L_2(0) \rangle$, and this approximation should be valid to $O(\epsilon^2)$. Thus, from the results of the previous section, we can write Eqs. (8-11) as

$$\begin{aligned} \langle L_2 \rangle = & \sum_r \left[\lambda_r \underline{a}^r \cdot \underline{a}^{r*} + i\epsilon \omega_r \left\{ (\dot{\underline{a}}^r \cdot \underline{a}^{r*} - \underline{a}^{r*} \cdot \dot{\underline{a}}^r) \right. \right. \\ & \left. \left. + v_{G\beta}^r \left(\underline{a}^{r*} \cdot \frac{\partial \underline{a}^r}{\partial x_\beta} - \underline{a}^r \cdot \frac{\partial \underline{a}^{r*}}{\partial x_\beta} \right) \right\} + O(\epsilon^2) \right] \end{aligned} \quad (21)$$

and $\langle L_3 \rangle$ is as above, Eqs. (18) & (19), with \underline{a}^l , etc., understood as also solutions of $\langle L_2(0) \rangle$ which are in resonance. The eigenvalues associated with \underline{a}^r are given by

$$\begin{aligned} (\lambda_r)_{a,g} &= \omega_r^2 - \frac{1}{2}(k^{r2}c^2 + \omega_A^2) \mp \left[\frac{1}{4}(k^{r2}c^2 + \omega_A^2)^2 - k_H^{r2}c^2\omega_B^2 \right]^{1/2} \\ &= 0 \end{aligned} \quad (22)$$

and the sign depends on whether wave r is on the acoustic or gravity wave branch of the dispersion relation. Also, the group velocities

$$\underline{v}_G^r = \left(\frac{\partial \omega_r}{\partial \underline{k}^r} \right)_{\lambda_r=0} \quad r = l, m, n, \dots \quad (23)$$

The dynamical equations for the wavetrains may be obtained

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by substituting the terms for $\langle L \rangle$ into the action integral and carrying out the variations with respect to \underline{a}^r and \underline{a}^{r*} ($r = \ell, m, n, \dots$), a straightforward procedure. For example, for three waves (ℓ, m, n) in an isothermal atmosphere and obeying the resonant conditions Eq. (14), variation of $\langle L \rangle$ with respect to $\underline{a}^{\ell*}$, \underline{a}^{m*} and \underline{a}^{n*} gives the coupled equations

$$\begin{aligned} 2\omega_{\ell} \underline{a}^{\ell*} \cdot \left(\frac{\partial}{\partial t} + \underline{v}_G^{\ell} \cdot \nabla \right) \underline{a}^{\ell} &= iV^{\ell mn*} : \underline{a}^{\ell*} \underline{a}^m \underline{a}^n, \\ 2\omega_m \underline{a}^{m*} \cdot \left(\frac{\partial}{\partial t} + \underline{v}_G^m \cdot \nabla \right) \underline{a}^m &= iV^{\ell mn} : \underline{a}^{\ell} \underline{a}^{m*} \underline{a}^{n*}, \\ 2\omega_n \underline{a}^{n*} \cdot \left(\frac{\partial}{\partial t} + \underline{v}_G^n \cdot \nabla \right) \underline{a}^n &= iV^{\ell mn} : \underline{a}^{\ell} \underline{a}^{m*} \underline{a}^{n*}. \end{aligned} \quad (24)$$

However, to explicate the physical meaning of the process it is convenient now to change variables to real amplitudes and phases, via Eqs. (5), and introduce the wave action

$$N_r = 2\omega_r |\hat{a}^r|^2, \quad r = \ell, m, n, \dots \quad (25)$$

Eq. (21) then becomes

$$\langle L_2 \rangle = \sum_r \left[\lambda_r |\hat{a}_r|^2 - \epsilon N_r \left(\frac{\partial}{\partial t} + \underline{v}_G^r \cdot \nabla \right) \theta_r \right] + O(\epsilon^2), \quad (26)$$

and

$$\langle L_3 \rangle = \sum_{\ell, m, n} \hat{a}^{\ell} \hat{a}^m \hat{a}^n : \left[(V^{\ell mn} + V^{\ell mn*}) \cos \theta_{\ell mn} + i(V^{\ell mn} - V^{\ell mn*}) \sin \theta_{\ell mn} \right], \quad (27)$$

where

$$\theta_{\ell mn} = \theta_{\ell} - \theta_m - \theta_n \quad (28)$$

is the relative phase of each of the resonant triads in the summation. Recalling that the complex wave vector $\underline{k}^r = (k_x^r, k_y^r, k_z^r - i/2H)$ then, using Eq. (19), the triadic factors in $\langle L_3 \rangle$ become

$$(V^{\ell mn} + V^{\ell mn*}) = - \frac{c^2}{\sqrt{\rho_0}} \frac{(5-\gamma)}{2H^3} \hat{u}_z \hat{u}_z \hat{u}_z, \quad (29)$$

where \hat{u}_z is the vertical unit vector, and

$$i(V^{\ell mn} - V^{\ell mn*}) = \frac{4c^2}{\sqrt{\rho_0}} [\underline{k}^{\ell} \underline{k}^m \underline{k}^n + \underline{k}^n \underline{k}^m \underline{k}^{\ell} + \underline{k}^m \underline{k}^{\ell} \underline{k}^n + (2-\gamma) \underline{k}^{\ell} \underline{k}^m \underline{k}^n]. \quad (30)$$

Those terms in the latter equation which correspond to the sum of terms in the third parenthesis of the first of Eq. (19) have been eliminated since they are equivalent to $(\underline{k}^{\ell} \cdot \underline{k}^m \times \underline{k}^n)$, which, because of the kinematic relation $\underline{k}^{\ell} = \underline{k}^m + \underline{k}^n$, is zero.

Defining unit vectors \underline{u}^r parallel to \hat{a}^r , $r = \ell, m, n, \dots$, and using the fact that $|\hat{a}^r| = (N_r/2\omega_r)^{1/2}$, the total lagrangian for waves up to third order in \underline{a}^r can then be put into a standard form:

$$\langle L \rangle = \sum_r -N_r \left(\frac{\partial}{\partial t} + \underline{v}_G^r \cdot \nabla \right) \theta_r + \sum_{\ell, m, n} (N_{\ell} N_m N_n)^{1/2} (F_{\ell mn} \cos \theta_{\ell mn} + G_{\ell mn} \sin \theta_{\ell mn}), \quad (31)$$

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where

$$F_{\ell mn} \equiv (8\omega_\ell \omega_m \omega_n)^{-1/2} \underline{u}^\ell \underline{u}^m \underline{u}^n: (V^{\ell mn} + V^{\ell mn*}), \quad (32)$$

$$G_{\ell mn} \equiv i(8\omega_\ell \omega_m \omega_n)^{-1/2} \underline{u}^\ell \underline{u}^m \underline{u}^n: (V^{\ell mn} - V^{\ell mn*}); \quad (33)$$

we have cancelled an ϵ factor, and used the fact in Eq. (26) that $\langle L_2(0) \rangle = \Sigma \lambda_r |a^r|^2 = 0$ in the local eigenvector representation.

The Euler-Lagrange equations for $\langle L \rangle$, with respect to variations in θ_ℓ and N_ℓ (or, equivalently, $|a^\ell|$) are, respectively,

$$\left(\frac{\partial}{\partial t} + \nabla \cdot \underline{v}_G^\ell \right) N_\ell = \sum_{m,n} (N_\ell N_m N_n)^{1/2} (F_{\ell mn} \sin \theta_{\ell mn} - G_{\ell mn} \cos \theta_{\ell mn}), \quad (34)$$

$$2N_\ell \left(\frac{\partial}{\partial t} + \underline{v}_G^\ell \cdot \nabla \right) \theta_\ell = \sum_{m,n} (N_\ell N_m N_n)^{1/2} (F_{\ell mn} \cos \theta_{\ell mn} + G_{\ell mn} \sin \theta_{\ell mn}), \quad (35)$$

where each summation is over all waves (m,n) in resonance with wave ℓ . If there are none such then the dynamics of wave ℓ is as in linear theory. Equations similar to Eqs. (34) and (35) obtain for the other wavetrains m, n , etc.

To gain insight into the properties of the nonlinear interactions we shall consider the case of only three waves (ℓ, m, n) in resonance, with kinematic relations Eq. (14), in a strictly

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isothermal atmosphere with uniform group velocity. The interaction equations following from variations with respect to the action and phase variables for ℓ , m and n are then

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \underline{v}_G^\ell \cdot \nabla \right) N_\ell &= - \left(\frac{\partial}{\partial t} + \underline{v}_G^m \cdot \nabla \right) N_m = - \left(\frac{\partial}{\partial t} + \underline{v}_G^n \cdot \nabla \right) N_n \\ &= (N_\ell N_m N_n)^{1/2} (F \sin \theta - G \cos \theta), \end{aligned} \quad (36)$$

$$\begin{aligned} N_\ell \left(\frac{\partial}{\partial t} + \underline{v}_G^\ell \cdot \nabla \right) \theta_\ell &= N_m \left(\frac{\partial}{\partial t} + \underline{v}_G^m \cdot \nabla \right) \theta_m = N_n \left(\frac{\partial}{\partial t} + \underline{v}_G^n \cdot \nabla \right) \theta_n \\ &= \frac{1}{2} (N_\ell N_m N_n)^{1/2} (F \cos \theta + G \sin \theta), \end{aligned} \quad (37)$$

where for convenience we have dropped the subscript ℓmn on the coupling coefficients F and G and the relative phase θ . Eqs. (36) give the variation of the amplitude envelopes of the wave-trains, which propagate with the group velocity, due to the lowest order nonlinear effects. The first two equations indicates that the action of wave ℓ increases at the rate at which waves m and n lose action, and the absolute rate, given by the last term, is a function of the product of the three amplitudes, the relative phase θ and the coupling coefficients F and G . Similarly, Eqs. (36) gives the phase change of each wave or, alternately, the frequency and wavenumber shifts, due to the modulation given by the right-hand term divided by each wave's action density N_r , $r = \ell, m, n$.

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If we define wave energy and wave momentum as $E_r = N_r \omega_r$ and $\underline{p}^r = N_r \underline{k}^r$, respectively, then conservation of wave energy and momentum follow from Eqs. (35). Writing down the equations for N_ℓ , N_m , N_n , multiplying by ω_ℓ , ω_m , ω_n , respectively, and adding the resulting relations gives, by virtue of the kinematic relation $\omega_\ell = \omega_m + \omega_n$,

$$\sum_r \left(\frac{\partial}{\partial t} + \nabla \cdot \underline{v}_G^r \right) E_r = 0 \quad r = \ell, m, n \quad (37)$$

Similarly, for wave momentum we obtain

$$\sum_r \left(\frac{\partial}{\partial t} + \nabla \cdot \underline{v}_G^r \right) \underline{p}^r = \underline{0} \quad (39)$$

The form of Eqs. (34-39) allows the interpretation of the nonlinear wave-wave interactions as a scattering process in which the "cross-section" is given by the last term of Eqs. (36).

The corpus of diagram techniques and interaction rules developed in quantum theory is useful in this regard, especially for higher order nonlinear processes (Hasselmann, 1966).

No general solution to the initial value problem posed by the nonlinear interaction equations, like Eqs. (36) and (37), is known (Benney, 1971). At the present time, and for the past decade or so, these equations (or their equivalent form in complex amplitudes, Eqs. 24) are under intense investigation, mainly by direct numerical computation, using model equations

in one space dimension. For a review of this work see Scott, et al, (1973).

The nonuniformity of the ambient density ρ_0 has two discernable roles in the interaction of acoustic gravity waves; these peculiar effects are usually absent in wave coupling in uniform media. First, and more important, the presence of the $1/\sqrt{\rho_0}$ factor in the complex triadic coefficient $V^{\ell mn}$, Eq. (19), shows that from Eqs. (24), (29), (30), (32) & (33), the coupling between waves increases with height in the atmosphere, since $\rho_0 \sim e^{-z/H}$. Second, the existence of the coefficient $F_{\ell mn}$, Eq. (32), in contrast to that of $G_{\ell mn}$, is the direct result of the imaginary part of the auxiliary complex wave number vectors $\underline{K}^r = (k_x^r, k_y^r, k_z^r - i/2H)$, $r = \ell, m, n$, in the expression for $V^{\ell mn}$ and subsequently in Eqs. (29) & (32), as may be seen by retracing their derivation from L_3 . It then follows from the definition of \underline{K}^r , and comparing Eqs. (29) & (30), that in the ranges in which $|k_z^r| \gg 1/2H$, or $|k^r| \gg 1/2H$, that $F_{\ell mn}$ should be negligible compared to $G_{\ell mn}$.

5.1 Kinematic Conditions

In deriving the interaction equations, and the coupling coefficients for them, we implicitly assumed that the resonance kinematic relations can in fact be satisfied by the linear dispersion relation for acoustic-gravity waves. We now examine this requirement.

From the previous analysis we have seen that the necessary kinematic condition for the three-wave interaction to take place is the simultaneous solution of the equations

$$\left. \begin{aligned} \omega_\ell &= \omega_m + \omega_n \\ \underline{k}^\ell &= \underline{k}^m + \underline{k}^n \\ \lambda(\omega_r, \underline{k}^r) &= 0 \end{aligned} \right\} \quad (40)$$

where $r = \ell, m, n$, and $(\omega_\ell, \underline{k}^\ell)$, $(\omega_m, \underline{k}^m)$ and $(\omega_n, \underline{k}^n)$ represent the three waves. From $\lambda_r = 0$ in Eq. (22) the particular dispersion relation to be satisfied is

$$\omega_r^2 = \frac{1}{2} (k^{r2} c^2 + \omega_A^2) \pm \left[\frac{1}{4} (k^{r2} c^2 + \omega_A^2)^2 - k_H^{r2} c^2 \omega_B^2 \right]^{1/2}, \quad (41)$$

for waves in the acoustic (+ sign) and gravity (- sign) wave modes.

Although in principle calculating roots of Eqs. (40) is an elementary algebraic problem, in practice machine computation

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is required since manageable expressions for the roots are hard to find because of the complexity of the algebraic system. However, the main question does not concern the numerical roots of the system, but rather the existence of such roots. For that purpose, a heuristic or graphical indication of solutions should suffice.

For a candidate set of three waves (ℓ, m, n) for Eq. (40), each wave being either a member of the acoustic (a) or the gravity (g) mode, there are six possibilities. These are listed in Table 1. Class 5 (g,a,a) and Class 6 (g,g,a) are inadmissible as the frequency condition $\omega_\ell = \omega_m + \omega_n$ cannot be satisfied. This follows from the separation of the dispersion relation Eq. (41) into the frequency regimes $\omega_a \geq \omega_A > \omega_B \geq \omega_g$, with frequencies ω_a in the acoustic mode and frequencies ω_g in the gravity mode. Thus, since all frequencies are non-negative, with either ω_m or ω_n in the acoustic mode ω_ℓ cannot be a gravity wave. In the table, the remaining wave Classes 1-4 are allowed. Since for gravity waves $\omega_g \leq \omega_B$ we have the limitation for Class 1 interactions (a,g,g) that at most $\omega_a = 2\omega_B$. Also, we know (see Appendix C) that in the limit of high frequencies or large wave numbers waves in the acoustic mode are nondispersive and isotropic, as in ordinary sound, so that, trivially, in that range the kinematic relation for Class 3 (a,a,a) can always be satisfied.

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It is useful to represent the solution of the resonant relations graphically. These graphs demonstrate the existence of solutions of Eqs. (40), and their utility is enhanced in that they can be visualized and the superposition carried out mentally. Usually, the dispersion relation is drawn with ω as the vertical coordinate and the components of \underline{k} as orthogonal coordinates. Choosing a wave $(\omega_1, \underline{k}_1)$, say, a second graph of the dispersion relation is then overlaid with its origin at $(\omega_1, \underline{k}_1)$ of the first graph and the respective axes of the two graphs parallel. It should be clear that any points of intersection of the two graphs so superimposed satisfy the kinematic relations, Eqs. (40), because an intersection represents waves $(\omega_2, \underline{k}_2)$ and $(\omega_3, \underline{k}_3)$; the three points $(\omega_1, \underline{k}_1)$, $(\omega_2, \underline{k}_2)$ and $(\omega_3, \underline{k}_3)$ are vertices of a parallelogram on the first graph, whose origin is the fourth vertex. Fig. 1 shows examples of resonant triads for the coupling of acoustic and gravity waves, Class 1 (a,g,g) and Class 2 (a,a,g) in the table, constructed by the above procedure, for colinear horizontal propagation. If we try to find resonant triads for Class 4 (g,g,g) we will find it not possible since the gravity mode dispersion relation for horizontal propagation is convex downwards. Resonant triads for Class 3 (a,a,a) would be found on the linear part of the acoustic mode curve, off the graph as displayed in Fig. 1.

The geometrical procedure just described can only be used

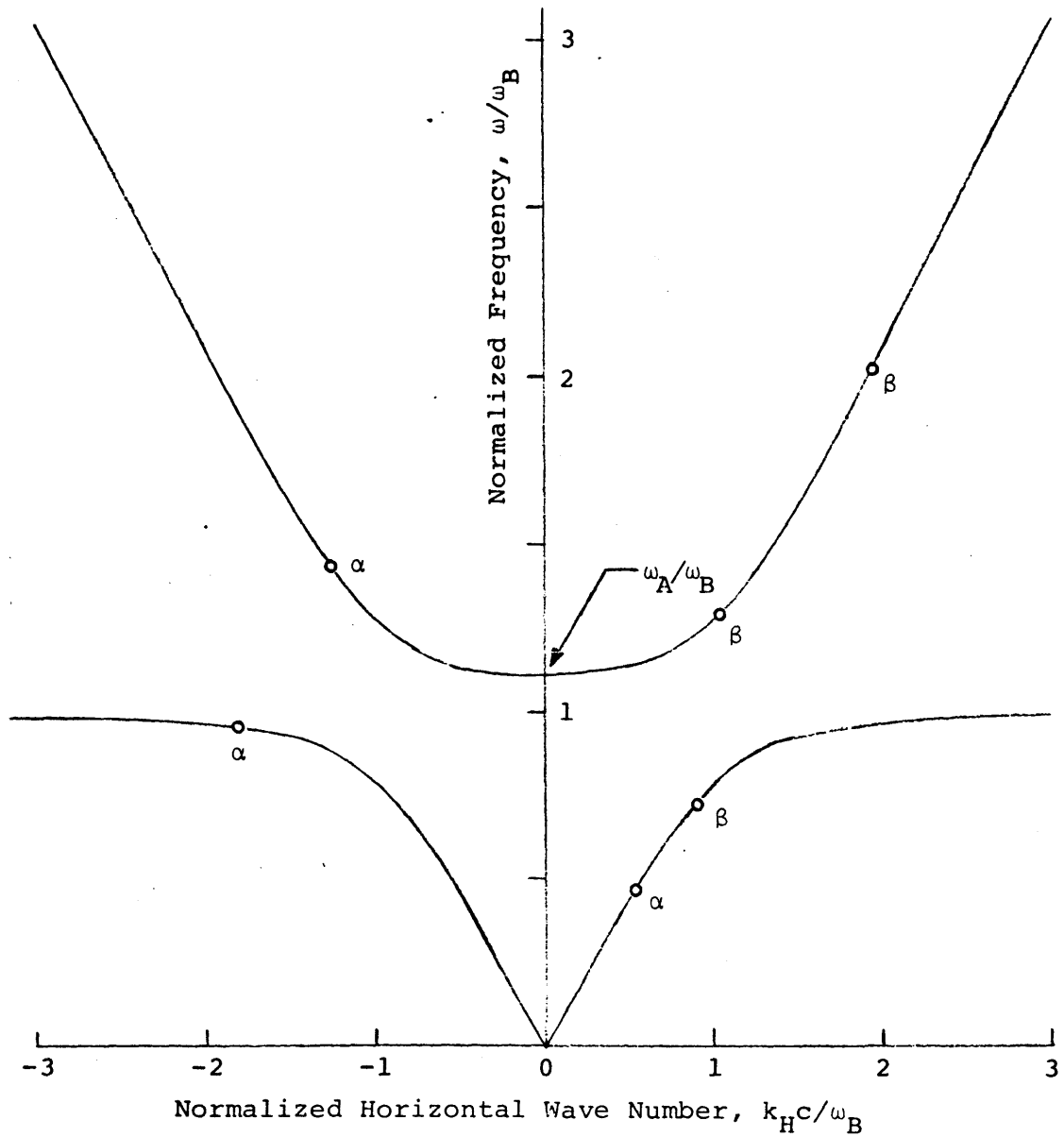


Fig. 1. Two examples of nonlinear interaction of acoustic-gravity waves in collinear horizontal propagation. α : (a,g,g) interaction. β : (a,a,g) interaction.

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for one and two-dimensional propagation systems, and was so employed by Ball (1964), Thorpe (1966) and Simmons (1969) for interactions of water waves. For propagation in a vertical plane the acoustic-gravity wave dispersion relation with ω, k_x, k_z as coordinates has the geometrical structure formed by ellipses and hyperbolas generated as contours for constant ω values in the acoustic and gravity wave modes, respectively (Appendix C). Thus, with the above procedure, a triad of waves in a vertical plane obeying the kinematic relations would be represented by a parallelogram whose vertices are at the origin and on the (ω, k_x, k_z) surface.

For interactions of waves propagating generally in three dimensions a graphical representation of resonant triads has to resort to the use of the $\underline{k}(\omega)$ propagation surfaces defined by the dispersion relation. As described in Appendix C, these are nested surfaces, parametric in ω and symmetrical about the k_z axis, which are ellipsoids in the acoustic mode, and hyperboloids of one sheet in the gravity wave mode. A point on any surface is a value of \underline{k} allowed by the dispersion relation. The recipe for construction of a resonant triad is similar to that above: at \underline{k}_1 on the ω_1 surface place the origin of coordinates of the ω_2 surface with k_x, k_y, k_z axes of both surfaces parallel; the locus of intersection with the $\omega_3 = \omega_1 + \omega_2$ surface (whose coordinate origin is the same as that of the ω_1 surface), if any, will

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be those values of \underline{k}_2 such that $\underline{k}_3 = \underline{k}_1 + \underline{k}_2$. Each resonant triad is represented on a set of dispersion surfaces by a parallelogram in \underline{k} -space with vertices at 0 , $\underline{k}_1(\omega_1)$, $\underline{k}_2(\omega_2)$ and $\underline{k}_3(\omega_3)$. An example of a triad satisfying the kinematic relations for each class of allowed interactions is displayed in Fig. 2, for propagation in a vertical plane and constructed in the manner described.

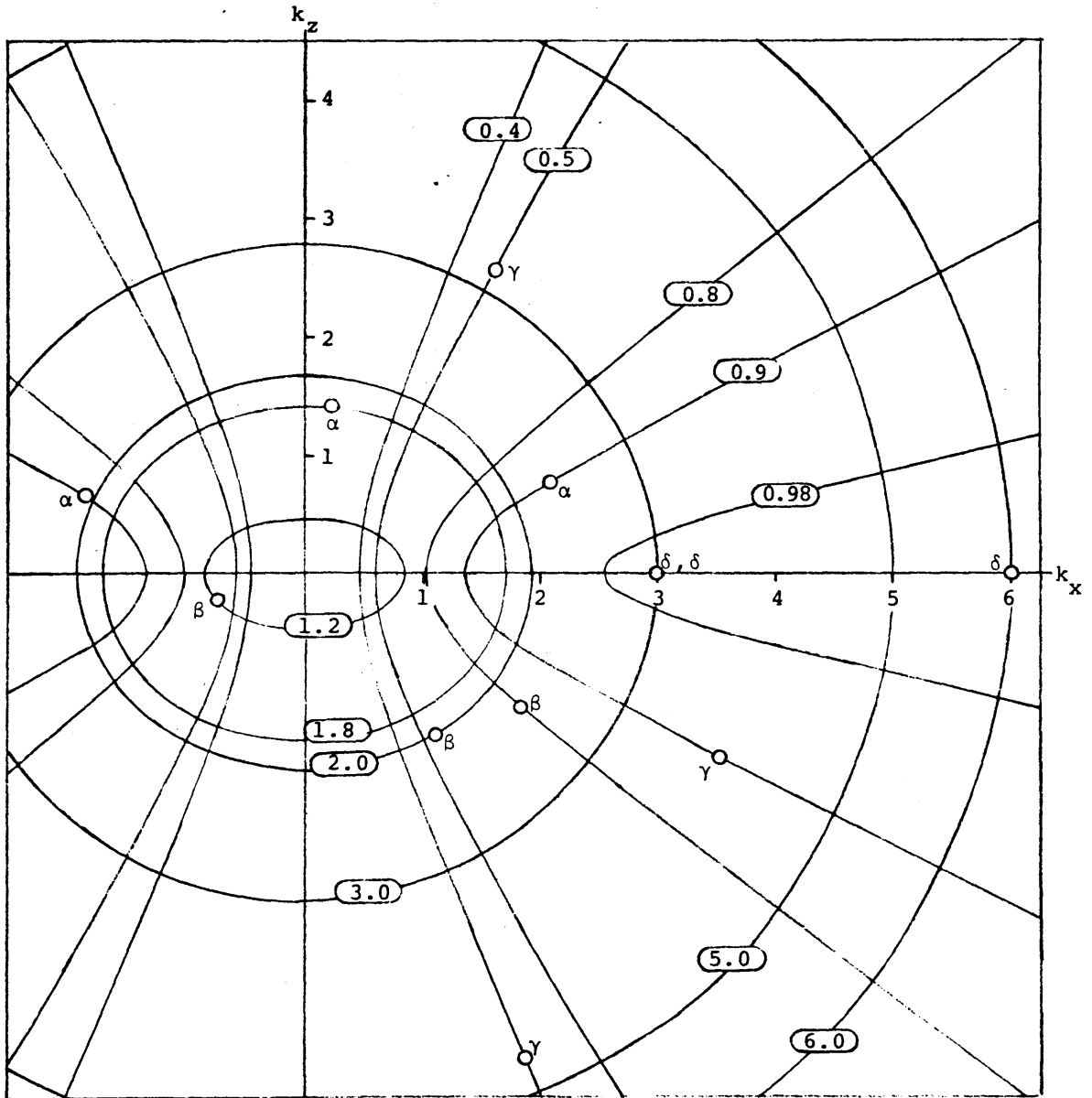


Fig. 2. Examples on acoustic-gravity propagation curves (constant ω contours in \underline{k} space) of resonant triads of waves propagating in a vertical plane. Units are normalized wave number kc/ω_B , and normalized frequency ω/ω_B . α : (a,g,g), β : (a,a,g), γ : (g,g,g), δ : (a,a,a) interactions.

| | l | m | n |
|---|-----|-----|-----|
| 1 | a | g | g |
| 2 | a | a | g |
| 3 | a | a | a |
| 4 | g | g | g |
| 5 | g | a | a |
| 6 | g | g | a |

Table 1. Candidate triads for acoustic and gravity wave interaction. Classes 5 & 6 are inadmissible, since the relation $\omega_l = \omega_m + \omega_n$ cannot be satisfied.

| l | m | n | $G_{lmn} / \frac{4c^2}{\sqrt{\rho_0}} (8\omega_l \omega_m \omega_n)^{-1/2}$ |
|-----|-----|-----|---|
| g | g | g | 0 |
| a | g | g | $(\underline{k}^l \cdot \underline{u}^l) (\underline{k}^n \cdot \underline{u}^m) (\underline{k}^m \cdot \underline{u}^n)$ |
| a | a | g | $(\underline{k}^l \cdot \underline{u}^l) (\underline{k}^n \cdot \underline{u}^m) (\underline{k}^m \cdot \underline{u}^n) + (\underline{k}^n \cdot \underline{u}^l) (\underline{k}^m \cdot \underline{u}^m) (\underline{k}^l \cdot \underline{u}^n)$ |
| a | a | a | Eq. (42) |

Table 2. Simplified expressions for G_{lmn} obtained from the transverse polarization approximation for gravity waves.

5.2 Coupling Coefficient Approximations

Some simplification of the expressions for the coupling coefficients is possible in the range of wavelengths of the interacting waves in which $|\underline{k}^r| \gg 1/2H$, i.e., for wavelengths of the order of the scale height, or less. It is known that in these ranges the polarization of acoustic-gravity can be approximated as being longitudinal, $\hat{a}^r \times \underline{k}^r = 0$, in the acoustic mode, and transverse, $\hat{a}^r \cdot \underline{k}^r = 0$, in the gravity wave mode (Appendix C). In this wavelength range also, it was indicated at the end of Section 5.0 that $F_{\ell mn}$ should be small compared to $G_{\ell mn}$.

From Eqs. (33) & (30) $G_{\ell mn}$ can be written out as follows

$$G_{\ell mn} = \frac{4c^2}{\sqrt{\rho_0}} (8\omega_\ell \omega_m \omega_n)^{-1/2} \left[(\underline{k}^\ell \cdot \underline{u}^\ell) (\underline{k}^n \cdot \underline{u}^n) + (\underline{k}^n \cdot \underline{u}^\ell) (\underline{k}^m \cdot \underline{u}^m) (\underline{k}^\ell \cdot \underline{u}^n) \right. \\ \left. + (\underline{k}^m \cdot \underline{u}^\ell) (\underline{k}^\ell \cdot \underline{u}^m) (\underline{k}^n \cdot \underline{u}^n) + (2-\gamma) (\underline{k}^\ell \cdot \underline{u}^\ell) (\underline{k}^m \cdot \underline{u}^m) (\underline{k}^n \cdot \underline{u}^n) \right]. \quad (42)$$

In considering the four classes of interactions of waves (ℓ, m, n) as being in the acoustic or gravity wave modes, discussed in the previous section, we see that if $\underline{a}^r \cdot \underline{k}^r = 0$ ($r=\ell, m, n$) then inspection of Eq. (42) shows that those interactions in which waves in the gravity mode participate lead to the simplified form of $G_{\ell mn}$ given in Table 2.

It is noteworthy that no coupling of three gravity waves (g, g, g)

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is possible via the $G_{\ell mn}$ coefficient in the large $|\underline{k}^r|$ range (which corresponds to the so called Boussinesq approximation for internal gravity water waves), so that interaction must be achieved by means of the coefficient $F_{\ell mn}$.

The (a,g,g) interaction is of particular interest to us as the basis for the model of the generation of acoustic waves by interacting gravity waves, an application employed in Section 6.0. Using the approximation from Table 2 we can directly write $\langle L \rangle$ from Eqs. (26), (27) & (30) as

$$\langle L \rangle = -\sum_r N_r \left(\frac{\partial}{\partial t} + \underline{v}_G^r \cdot \nabla \right) \theta_r + \frac{4c^2}{\sqrt{\rho_0}} (\hat{a}^\ell \cdot \underline{k}^\ell) (\hat{a}^m \cdot \underline{k}^n) (\hat{a}^n \cdot \underline{k}^m) \sin \theta_{\ell mn} \quad (43)$$

so that variation with respect to θ_ℓ and \hat{a}^ℓ yields, respectively,

$$\omega_\ell \left(\frac{\partial}{\partial t} + \underline{v}_G \cdot \nabla \right) \hat{a}^\ell = -\frac{c^2}{\sqrt{\rho_0}} \underline{k}^\ell (\hat{a}^m \cdot \underline{k}^n) (\hat{a}^n \cdot \underline{k}^m) \cos \theta_{\ell mn} \quad (44)$$

$$\omega_\ell \hat{a}^\ell \left(\frac{\partial}{\partial t} + \underline{v}_G \cdot \nabla \right) \theta_\ell = \frac{c^2}{\sqrt{\rho_0}} \underline{k}^\ell (\hat{a}^m \cdot \underline{k}^n) (\hat{a}^n \cdot \underline{k}^m) \sin \theta_{\ell mn}. \quad (45)$$

If \underline{k} lies in the x-z plane at angle χ to the z-axis, we can use the rotation matrix operator

$$R = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (46)$$

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to generalize the propagation to the spherical angles χ, ψ , so that we have

$$\underline{k}^r = R\underline{k}^{r'}; \quad \underline{k}^{r'} = k^r (\sin \chi_r, 0, \cos \chi_r), \quad (47)$$

and with the gravity-wave transverse polarization,

$$\hat{a}^r = R\hat{a}^{r'}; \quad \hat{a}^{r'} = a^r (-\cos \chi_r, 0, \sin \chi_r). \quad (48)$$

The spherical angular dependence factor of the coupling, from Eq. (45), is then

$$\begin{aligned} & (\hat{a}^m \cdot \underline{k}^n) (\hat{a}^n \cdot \underline{k}^m) \\ &= a^m a^n k^m k^n (-\cos \psi \cos \chi_m \sin \chi_n + \sin \chi_m \cos \chi_n) (-\cos \psi \sin \chi_m \cos \chi_n \\ &+ \cos \chi_m \sin \chi_n). \end{aligned} \quad (49)$$

This indicates that maximum azimuthal coupling is obtained when $\psi = \pi$ radians, and in that case, the product of angular terms in Eq. (49) becomes $\sin^2(\chi_m + \chi_n)$ which is maximum when $\chi_m + \chi_n = \pi/2$. Thus (a,g,g) coupling is maximized when $\underline{k}^m \cdot \underline{k}^n = 0$ in a vertical plane; this result could be directly deduced by inspection of Eqs. (44) & (45).

5.3 Time Evolution Solutions

Although no general solution to the coupled wave equations is known, some insight into the mechanics of interaction is gained by specializing to one independent variable, which we take to be time. Then we can find solutions for the three-wave process by adapting a method of solution due to Armstrong et al (1962, see Sagdeev & Galeev, 1969). We do this for the coupling coefficient $G_{\ell mn}$.

From Eqs. (36) & (37) the equations to be solved are, in simplified notation,

$$\dot{N}_{\ell} = -(N_{\ell} N_m N_n)^{1/2} G \cos \theta, \quad (50)$$

$$\dot{N}_m = \dot{N}_n = -\dot{N}_{\ell}, \quad (51)$$

$$\dot{\theta} = \frac{1}{2} (N_{\ell} N_m N_n)^{1/2} G \sin \theta \left(\frac{1}{N_{\ell}} - \frac{1}{N_m} - \frac{1}{N_n} \right), \quad (52)$$

where it will be recalled $\theta \equiv \theta_{\ell mn} = \theta_{\ell} - \theta_m - \theta_n$. On integrating Eqs. (51) we find the following constants of the motion:

$$\begin{aligned} C_1 &\equiv N_{\ell}(t) + N_m(t) = N_{\ell}(0) + N_m(0) \\ C_2 &\equiv N_{\ell}(t) + N_n(t) = N_{\ell}(0) + N_n(0) \\ C_3 &\equiv N_m(t) - N_n(t) = N_m(0) - N_n(0). \end{aligned} \quad (53)$$

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Substituting $(N_\ell N_m N_n)^{1/2}$ from Eqs. (50) & (51) into Eq. (52) gives

$$\frac{\partial \theta}{\partial t} = - \tan \theta \frac{\partial}{\partial t} \ln(N_\ell N_m N_n)^{1/2}, \quad (54)$$

which, on integrating, we find

$$(N_\ell N_m N_n)^{1/2} \sin \theta \equiv \Gamma = \text{const.} \quad (55)$$

Note that Γ is zero when any of the N 's is zero, and has the sign of $\sin \theta$ since the N 's are nonnegative.

From Eq. (55) we have

$$\cos \theta = [(N_\ell N_m N_n - \Gamma^2)/N_\ell N_m N_n]^{1/2}, \quad (56)$$

and we can use this to write Eq. (50) in the form

$$\dot{N}_\ell = -G(N_\ell N_m N_n - \Gamma^2)^{1/2}. \quad (57)$$

With the use of the constants of motion, Eqs. (53), Eqs. (50) & (51) can now be written in uncoupled form

$$\dot{N}_\ell = -G[N_\ell (C_1 - N_\ell) (C_2 - N_\ell) - \Gamma^2]^{1/2} \quad (58)$$

$$\dot{N}_m = G[N_m (C_1 - N_m) (N_m - C_3) - \Gamma^2]^{1/2} \quad (59)$$

$$\dot{N}_n = G[N_n (C_2 - N_n) (C_3 + N_n) - \Gamma^2]^{1/2} \quad (60)$$

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To solve these equations we need find a solution only for one of them, since by use of Eqs. (53) we then have the solutions for the other equations. We choose Eq. (58).

If the three real roots of

$$N_{\ell} (C_1 - N_{\ell}) (C_2 - N_{\ell}) - \Gamma^2 = 0 \quad (61)$$

are labeled and ordered as

$$N_{\ell}^{\gamma} \geq N_{\ell}^{\beta} \geq N_{\ell}^{\alpha} \geq 0, \quad (62)$$

then Eq. (58) can be put in the form

$$\dot{N}_{\ell} = -G [(N_{\ell} - N_{\ell}^{\alpha}) (N_{\ell} - N_{\ell}^{\beta}) (N_{\ell} - N_{\ell}^{\gamma})]^{1/2}, \quad (63)$$

or,

$$G(t-t_0) = - \int_{N_{\ell}(t_0)}^{N_{\ell}(t)} \frac{dN_{\ell}}{[(N_{\ell} - N_{\ell}^{\alpha}) (N_{\ell} - N_{\ell}^{\beta}) (N_{\ell} - N_{\ell}^{\gamma})]^{1/2}}. \quad (64)$$

This can be transformed to an elliptic integral by the change of variables

$$y(t) = \left[\frac{N_{\ell}(t) - N_{\ell}^{\alpha}}{N_{\ell}^{\beta} - N_{\ell}^{\alpha}} \right]^{1/2} \quad (65)$$

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$$p = \left[\frac{N_{\ell}^{\beta} - N_{\ell}^{\alpha}}{N_{\ell}^{\gamma} - N_{\ell}^{\alpha}} \right]^{1/2}, \quad 0 < p < 1. \quad (66)$$

Substituting these variables into Eq. (64) yields

$$\frac{G}{2} (t-t_0) (N_{\ell}^{\gamma} - N_{\ell}^{\alpha})^{1/2} = \int_{y(t_0)}^{y(t)} \frac{dy}{[(1-y^2)(1-p^2y^2)]^{1/2}}. \quad (67)$$

If we define the initial time in such a way that the lower limit $y(t_0)$ vanishes, then the integral becomes an elliptic integral of the first kind with parameter p . Therefore,

$$y(t) = \text{sn} \left[\frac{G}{2} (N_{\ell}^{\gamma} - N_{\ell}^{\alpha})^{1/2} (t-t_0) \right] \quad (68)$$

and, by the definition of y , Eq. (65), and Eqs. (53), we get the solutions

$$\begin{aligned} N_{\ell}(t) &= N_{\ell}^{\alpha} + (N_{\ell}^{\beta} - N_{\ell}^{\alpha}) \text{sn}^2 \left[\frac{G}{2} (N_{\ell}^{\gamma} - N_{\ell}^{\alpha})^{1/2} (t-t_0) \right] \\ N_m(t) &= C_1 - N_{\ell}(t) \\ N_n(t) &= C_2 - N_{\ell}(t) \end{aligned} \quad (69)$$

These equations describe the time evolution of the three interacting waves and the periodic exchange of energy between them.

Let us consider two special cases, each with one of the waves

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having zero energy initially.

Case A: Interaction, or mixing, of two finite amplitude waves, ω_m and ω_n , to generate a third ω_ℓ . For example, we could have two gravity waves producing an acoustic wave in an (a,g,g) interaction. We assume that $N_m = N_n$, and $N_\ell = 0$ initially. Then, from Eqs. (53) & (55), we see that

$$N_m(0) = N_n(0) = C_1 = C_2 \equiv C; \quad C_3 = 0; \quad \Gamma = 0, \quad (70)$$

and, directly from Eq. (58):

$$\frac{dN_\ell}{dt} = GN_\ell^{1/2} (N_\ell - C). \quad (71)$$

This can be integrated to give

$$N_\ell^{1/2} = (-C)^{1/2} \tan \left[\frac{G}{2} (-C)^{1/2} t \right]. \quad (72)$$

Since $C \geq 0$, we have from Eqs. (53) the solutions

$$N_\ell(t) = C \tanh^2 \left[\frac{G}{2} C^{1/2} t \right] \quad (73)$$

$$N_m(t) = N_n(t) = C \operatorname{sech}^2 \left[\frac{G}{2} C^{1/2} t \right] \quad (74)$$

Thus, wave ω_ℓ grows while waves ω_m and ω_n decay at the same rate. The period for total transfer is asymptotically infinite

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with initial growth rate $\sqrt{C} G/2$ for the amplitude.

Case B: A finite amplitude wave at ω_m interacting with a low amplitude wave at ω_n to produce a wave at ω_ℓ (parametric amplification; Armstrong et al, 1962). As an example ω_m might be a gravity wave and ω_n either a gravity or acoustic to produce an acoustic wave ω_ℓ in an (a,g,g) or (a,a,g) interaction, respectively. We assume, at $t=0$,

$$N_m(0) \gg N_n(0), \quad N_\ell(0) = 0. \quad (75)$$

Then we have from Eqs. (53) & (55):

$$C_1 = N_m(0), \quad C_2 = N_n(0), \quad \Gamma = 0; \quad (76)$$

and, from Eqs. (61) & (63), the three roots of Eq. (63) are

$$N_\ell^\alpha = 0, \quad N_\ell^\beta = C_2, \quad N_\ell^\gamma = C_1. \quad (77)$$

Now, from Eq. (66), $m^2 = N_\ell^\beta / N_\ell^\gamma = C_2 / C_1 \ll 1$, so in Eq. (67) we can neglect $m^2 y^2$ in comparison with unity. Eq. (67) reduces to

$$-\frac{G}{2} (t-t_0) \sqrt{N_m(0)} = \int_0^{y(t)} \frac{dy}{(1-y^2)^{1/2}} \equiv \pm \sin^{-1} y(t) \quad (78)$$

Thus the solutions are

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$$\begin{aligned}
 N_{\ell}(t) &= N_n(0) \sin^2 \left[\frac{G}{2} \sqrt{N_m(0)} (t-t_0) \right] \\
 N_m(t) &= N_m(0) - N_n(0) \sin^2 \left[\frac{G}{2} \sqrt{N_m(0)} (t-t_0) \right] \\
 N_n(t) &= N_n(0) \cos^2 \left[\frac{G}{2} \sqrt{N_m(0)} (t-t_0) \right]
 \end{aligned} \tag{79}$$

Since in this case $N_m(0) \gg N_n(0)$, we see $N_m(t)$ remains approximately at its initial value and acts as a pump to "parametrically" convert energy between ω_n and ω_{ℓ} . From Eqs. (79) this transfer time is $\pi/G \sqrt{N_m(0)}$ and the initial growth rate for amplitude is $\sqrt{N_m(0)} G/2$.

6.0 Infrasonic Generation by Thunderstorms

In the past decade, experimental evidence, based on high-frequency (HF) radio Doppler sounding of the ionosphere, shows disturbances of the upper atmosphere which are associated with severe weather activity near the ground (Georges, 1968; Baker & Davies, 1969; Detert, 1969; Davies & Jones, 1971; Georges, 1973). These studies reveal that sometimes quasi-sinusoidal oscillations, with periods in the range of 2 min. to 5 min. and frequently for many hours duration, occur when thunderstorms are the region within a radius of 250 km of the ionospheric reflection point. The simultaneous deployment of multiple HF radio frequencies and of several spaced stations shows that these oscillations are coherent over the range of lower ionospheric reflection heights. The experimental evidence is consistent with the interpretation frequently given that the observed oscillations are the manifestation of the vertical passage of long wave-length infrasonic waves through the atmosphere. Since the principal periods (2-5 min.) are somewhat smaller than the typical Brunt-Väisälä periods in the atmosphere, such waves would correspond to the acoustic mode of the acoustic-gravity wave spectrum.

A very striking feature of the HF Doppler data is the relatively narrow bandwidth of the ionospheric oscillations, sometimes with Q greater than 10 for an epoch of several hours. This is suggestive of some kind of resonant phenomenon, although there

are apparently no distinct oscillations in the 2 - 5 min. period range associated with air motion in the troposphere. This leads us to propose in this section a model for the generation of these oscillations. The model is based on the theory of nonlinear interactions of acoustic-gravity waves developed in the previous sections, and on the well-known phenomenon that high convective activity in the troposphere causes turbulent fluctuations and internal gravity waves are generated by the storm cells. Fig. 3 reproduces the power spectrum of atmospheric pressure given in the classic paper by E. Gossard (1960). The longest solid line illustrates average conditions, but the dashed and short solid lines near it represent specific occasions of great excitation. Each of the latter two curves were identified by Gossard as the spectrum of internal gravity wavetrains coming from known regions of convective activity; the observed wave frequencies were near the local Brunt frequency at the time, as computed from radiosonde temperature data. Brunt's period in the troposphere varies typically in the range of 4 to 10 minutes.

The physical model which we propose as providing an explanation of the thunderstorm associated oscillations is as follows: The ionospheric oscillations are infrasonic acoustic waves radiated from lower atmospheric regions of interaction of gravity waves which themselves are emitted by thunderstorm convection cells in the disturbed weather system. With this model, in some volume above the ground area between the storm cells gravity waves

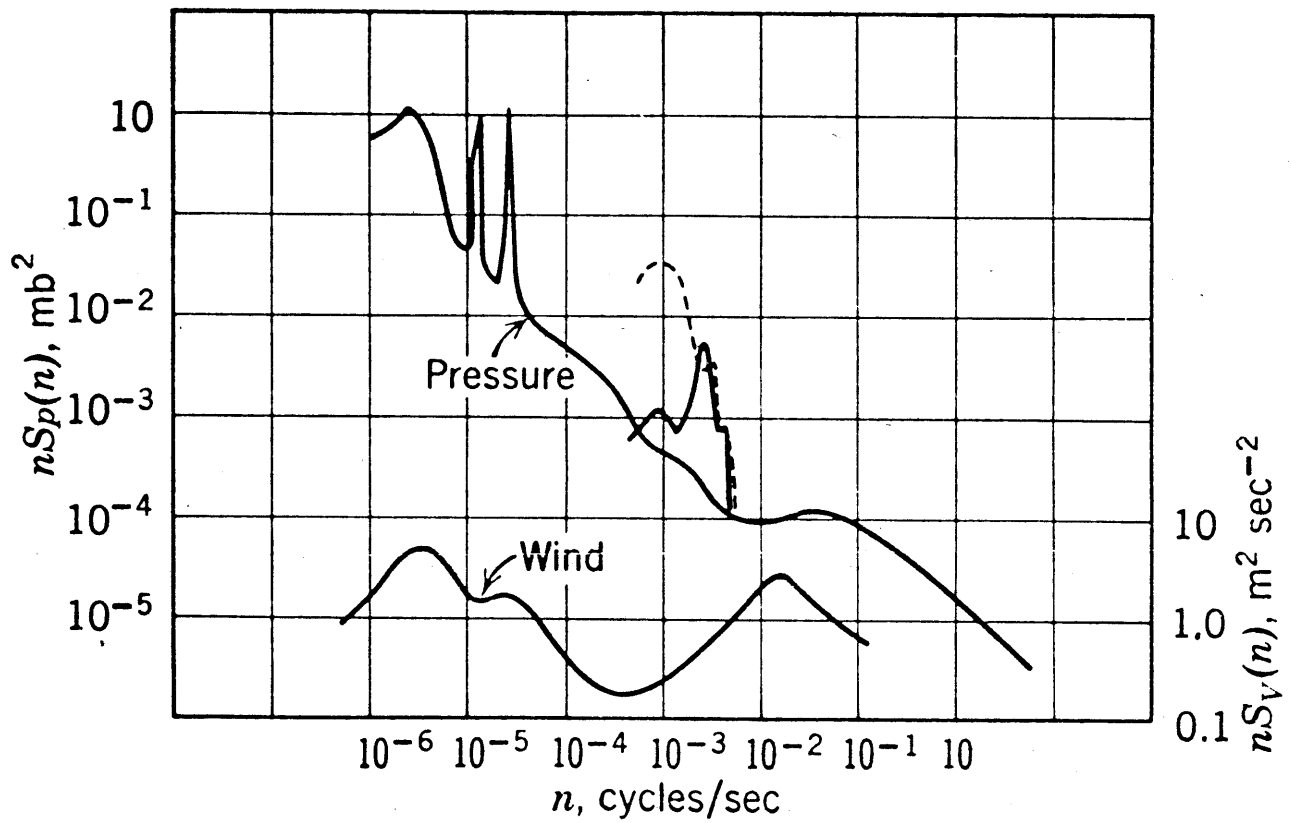


Fig. 3 Pressure power spectrum of the atmosphere, with examples of spectra of gravity waves generated by convective activity. Reprinted from Gossard (1960).

nonlinearly interact to generate acoustic waves in (a,g,g) interactions. Under favorable coupling conditions two interacting gravity waves would generate an acoustic wave whose frequency is the sum of the two primary waves. If, as in Gossard's examples, the gravity waves emitted by thunderstorm cells are confined by the convective process to be near Brunt's frequency ω_B then the generated acoustic wave would also be restricted to be about $2\omega_B$, which is equivalent to the range 2 - 5 min. periods for typical ω_B values in the lower atmosphere. This restriction, however, is not required for the model proposed here.

The proposed mechanism can be considered as a source model for the production of the 2 - 5 min. period waves, via interactions of storm-generated gravity waves, with these acoustic waves travelling upwards, in near vertical direction and experiencing minor refractive effects, to the upper atmosphere to cause the observed ionospheric oscillations. This explanation of the phenomenon is in contrast to those offered by Jones (1971) and Chimonas & Peltier (1974) who describe the oscillations aloft as being the result of selective wave transmission characteristics of the atmosphere, and which are excited by unspecified sources associated with severe storms. The explanations of these authors are based on adaptations of two different aspects of the waveguide multimode theory of atmospheric wave propagation, which has been developed to explain the transmission of wave energy to large horizontal ranges from a source. The use of this theory, which accounts only

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for trapped (i.e., multiply reflected) waves, to explain the presence of waves in the upper atmospheric region above a source, and whose direction and frequencies indicate that they are essentially not trapped, is open to some objections which are discussed in Appendix D. At any rate, the waveguide theories are not really in contradiction with the source mechanism proposed here since those theories require a source to excite the modes, although presumably their mechanisms would allow the possibility that the oscillations aloft could be generated by a broadband source.

Let us show the feasibility of the proposed mechanism by estimating the velocity of air motion associated with the acoustic wave generated by interacting gravity waves and compare it with the velocities deduced from the radio Doppler data. To do this we consider the simple model of two gravity waves (ω_m, ω_n) with wave vectors $(\underline{k}^m, \underline{k}^n)$ in the same vertical plane but azimuthally 180° apart (Fig. 4). We assume equal angles from the vertical, $\chi_m = \chi_n \equiv \chi$; equal wave amplitude magnitudes, $|\hat{a}^m| = |\hat{a}^n|$; and $|\underline{k}^m| = |\underline{k}^n| \equiv k$. The gravity waves generate a vertical acoustic wave with

$$\underline{k}^\ell = \underline{k}^m + \underline{k}^n = 2k\hat{u}_z \cos \chi, \tag{1}$$

$$\omega_\ell = \omega_m + \omega_n \equiv 2\omega_o.$$

To estimate the generated wave magnitude a^ℓ we could use the approximate coupling coefficient from Section 5.3 and the initial growth rate relation from Eq. (5.73) for the time evolution solution

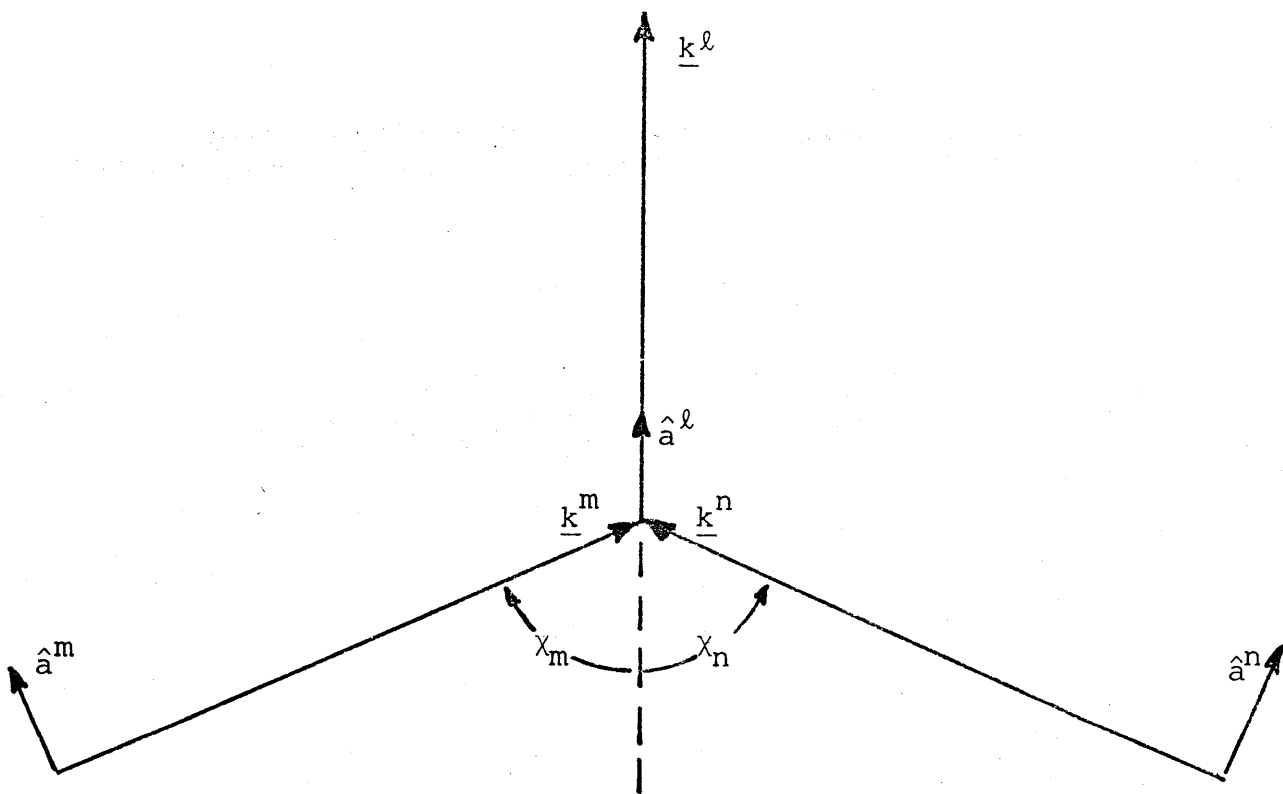


Fig. 4. Sketch of interaction model for calculation of acoustic wave generation by interacting gravity waves.

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for Case A in Section 5.3. We arrive at the same result more conveniently by using Eq. (5.44) and choosing $|\hat{a}^m| = |\hat{a}^n| = \text{const.}$, $\theta_{lmn} = \pm\pi$, and $|a^\ell| = 0$ at $t = 0$. The equation for a^ℓ is then

$$\omega_\ell \frac{\partial \hat{a}^\ell}{\partial t} = \frac{c^2}{\sqrt{\rho_0}} \underline{k}^\ell (\hat{a}^m \cdot \underline{k}^n) (\hat{a}^n \cdot \underline{k}^m). \quad (2)$$

For the model described above, Eq. (5.49) gives the angular factor

$$(\hat{a}^m \cdot \underline{k}^n) (\hat{a}^n \cdot \underline{k}^m) = a_m^2 k^2 \sin^2 2\chi = 4a_z^2 k^2 \cos^2 \chi, \quad (3)$$

where a_z is the vertical component of \hat{a}^m or \hat{a}^n .

In the large $|\underline{k}|$ (Boussinesq) approximation the gravity wave frequency is given by

$$\omega_0 = \omega_B k_H/k = \omega_B \sin \chi. \quad (4)$$

Substituting Eqs. (1), (3) and (4) into Eq. (2) and considering $|a_m| = |a_n| = \text{const.}$ yields

$$a^\ell = \frac{4c^2}{\omega_B \sqrt{\rho_0}} \frac{\cos^3 \chi}{\sin \chi} k^3 a_z^2 \Delta t \quad (5)$$

where Δt is the interaction time interval. This formula should give acceptable values of a^ℓ for Δt equal to a few wave periods of the generated wave ω_ℓ .

We need now an expression relating gravity wave pressure fluctuations, which are recorded on atmospheric wave instruments, and our primary wave amplitudes a^m and a^n . This is afforded by standard first-order perturbation of the Eulerian equations, Eqs.

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(2.1-2.3), with $\rho_0 = \rho_0 + \rho_1$, $\underline{v} = 0 + \underline{v}_1$, $p = p_0 + p_1$. This yields for a plane stratified atmosphere the equation:

$$\frac{\partial p_1}{\partial t} + \frac{\partial p_0}{\partial z} \hat{u}_z \cdot \underline{v} + \gamma p_0 \nabla \cdot \underline{v} = 0 \quad (6)$$

(see, e.g., Eckart (1960), p. 18), and, with $c^2 = \gamma p_0 / \rho_0$, $\partial p_0 / \partial z = -\rho_0 g$, we have

$$\frac{\partial p_1}{\partial t} = \rho_0 [g \hat{u}_z \cdot \underline{v} - c^2 \nabla \cdot \underline{v}]. \quad (7)$$

This is simply

$$p_1 = \rho_0 [g \xi_z - c^2 \nabla \cdot \underline{\xi}] \quad (8)$$

differentiated with respect to time, where $\underline{\xi}$ is the displacement and ξ_z is its vertical component. From Eq. (4.1) we assume a plane wave of the form

$$\underline{\xi} = \frac{2a}{\sqrt{\rho_0}} \cos \phi, \quad \phi = \underline{k} \cdot \underline{x} - \omega t, \quad (9)$$

in an isothermal atmosphere with $\rho_0 \sim e^{-z/H}$, $H = c^2 / \gamma g$. Then substituting Eq. (9) into Eq. (8) gives

$$p_1 = \sqrt{\rho_0} [g(2-\gamma)a_z \cos \phi + 2c^2 \underline{a} \cdot \underline{k} \sin \phi] \quad (10)$$

We employ the transverse polarization approximation for gravity waves, $\underline{a} \cdot \underline{k} = 0$, so that the expression we seek for relating the magnitudes of p_1 and \underline{a} is

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$$p_1 = \sqrt{\rho_0} g(2-\gamma) a_z. \quad (11)$$

In this formula ρ_0 should have the value at the height (say ground level) at which we use measures of wave pressure p_1 .

Substituting Eq. (11) into Eq. (5), and using the definition $\omega_B^2 = (\gamma-1)g^2/c^2$, yields

$$a^{\ell} = \frac{4(\gamma-1)}{(2-\gamma)^2} \frac{p_1^2}{\sqrt{\rho_0} \rho_{00}} \left(\frac{k}{\omega_B} \right)^3 \frac{\cos^3 \chi}{\sin \chi} \Delta t. \quad (12)$$

In the upper atmosphere the mechanical motion of the medium due to the passage of acoustic wave ω_{ℓ} is, from Eq. (9),

$$\frac{\partial \xi}{\partial t} = 2\omega_{\ell} \frac{a^{\ell}}{\sqrt{\rho_{iono}}} \sin \omega_{\ell} t, \quad (13)$$

where ρ_{iono} is the density aloft, so that the velocity magnitude $v = |\dot{\xi}|$ to be compared with radio Doppler data in our simple interaction model is, from Eqs. (12) and (13),

$$v = \frac{8(\gamma-1)}{(2-\gamma)^2} \frac{p_1^2 \omega_{\ell}}{\sqrt{\rho_0} \rho_{00} \sqrt{\rho_{iono}}} \left(\frac{k}{\omega_B} \right)^3 \frac{\cos^3 \chi}{\sin \chi} \Delta t \quad (14)$$

In this formula ρ_{00} is the ground level density where p_1 is measured and ρ_0 is the density at the height of interaction.

In order to provide a basis for discussion of whether the velocities indicated by this model are adequate for our purpose, a typical set of wave and atmospheric parameters in the atmosphere would be:

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$$\begin{aligned}
 \tau_B &= 2\pi/\omega_B = 6 \text{ min} = 3.6 \times 10^2 \text{ sec} \\
 \lambda &= 2\pi/k = 2 \times 10^4 \text{ m} \\
 p_1 &= 1 \text{ mbar} = 10 \text{ kg/m}^2 \\
 \rho_{00} &= 1.2 \text{ kg/m}^3 \\
 \rho_{iono} &= 10^{-10} \text{ kg/m}^2 \text{ at } z = 200 \text{ km.}
 \end{aligned}
 \tag{15}$$

Let us calculate v for the case where the gravity wave frequency ω is near ω_B , with $\omega = 0.9\omega_B$, and for an interaction time Δt equal to one period of generated wave $\omega_\ell = 1.8\omega_B$. This gives the value for wave period $\tau_\ell = 2\pi/\omega_\ell = 200 \text{ sec}$, and $\chi = \sin^{-1} 0.9$. We shall also consider the interaction as taking place at ground level, so $\rho_0 = \rho_{00}$ in Eq. (14). For definiteness, and in the absence of empirical knowledge, the wavelength λ is taken to be about twice the scale height in the lower atmosphere.

With the above values we compute the velocity at 200 km height due to the passage of the acoustic wave generated by this model to be

$$v = 2.3 \times 10^2 \text{ m/sec.} \tag{16}$$

This value is an order of magnitude greater than that deduced from the Doppler sounder data, which indicate frequency shifts of approximately 1 Hz peak to peak during 2 - 5 min. period ionospheric oscillations. Ionospheric Doppler sounding systems employ radio frequencies in the range 2 - 6 MHz. If we assume that most of the change in phase path P for a radio wave travelling vertically up to the ionosphere and back to the ground takes place near the radio

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reflection height h we can use the backscatter Doppler relationship

$$-\frac{c_0}{f} \Delta f = \frac{dP}{dt} \approx 2 \frac{dh}{dt} = 2v, \quad (17)$$

where v is the velocity of the reflector, c_0 is the speed of light, and f is the radio frequency. Using typical data values $\Delta f = 1\text{Hz}$ and $f = 4\text{MHz}$ the estimated velocity in the ionosphere which is associated with the 2 - 5 min. period oscillations is

$$v = 38 \text{ m/sec.} \quad (18)$$

The apparent capability of the proposed interaction source model to more than adequately provide for the production of the acoustic perturbation in the ionosphere, as indicated by comparison of the calculated and empirically deduced velocity values, Eqs. (16) and (18), should be assessed in the light of the idealizations of the model, the sensitivity of Eq. (14) to the parameters used in the illustrative calculation, and the dearth of our knowledge about gravity waves and storm activity. The formula is based on the interaction of three infinite plane waves in an isothermal atmosphere, with the horizontal components of the gravity wave vectors antiparallel and optimum phase relationship between the waves. Departures from these idealizations would reduce the coupling, and the calculated values of a^{λ} and v . If the gravity wave vectors are not in the same vertical plane then an azimuthal angular dependence is indicated by the $\cos \psi$ factor in the expression Eq. (5.49). Similarly, Eq. (5.44) shows that if the relative

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phase $\theta_{\ell mn} \neq \pm\pi$ then the coupling is proportional to $\cos \theta_{\ell mn}$. (In that case, slow detuning of the waves, i.e., changes in the phase of each wave, might occur according to Eq. (5.45) since then $\sin \theta_{\ell mn} \neq 0$.)

The effect on our model of the finite extent of the active source interaction region (and so the influence of sphericity of the waves) is difficult to judge. This depends on the size or area of the active region between storm cells which is related to the mechanism of their generation and the spatial disposition or "pulse length" of the wavetrains, about which we have no information. However, we can estimate if there is any "spreading loss" above the finite source region if we make guesses about its size and employ concepts from ordinary sound theory. The latter indicate spherical divergence of sound sets in only for distances from a source (such as a vibrating piston) greater than $2r^2/\lambda$ where r is the radius of the source area and λ is the wavelength (Kinsler & Frey, 1950, p. 186). Davies & Jones (1971) display maps of weather radar data at the time of the appearance of 2 - 5 min. ionospheric oscillations. From these maps storm cells with tops higher than 40,000 ft. (which these authors, and others, have correlated with the presence of these oscillations) appear to be about 200 km apart. Using 100 km as a measure of the radius of the source area and a wavelength of 60 km (for a 200 sec period wave and 300 m/sec sound speed) we compute the critical altitude above the source area to be about 330 km, which is larger than the heights of radio observation. If we use the acoustic wavelength

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in the thermosphere, this critical height would be reduced by only a factor of three. We infer then that if the interacting region is of the order of the area between storm cells then spherical spreading of waves is probably not significant above the storm regions.

We turn now to the formula Eq. (14), and the parameters, Eqs. (15) and after, which were used in the calculation for v , Eq. (16). Clearly, Eq. (14) is very sensitive to both p_1 and k . However, in arriving at $v = 230$ m/sec we have used relatively conservative numbers. Increasing the wave pressure p_1 to the probably more realistic value of about 10 mbar for highly disturbed weather regimes would raise v by two orders of magnitude, while increasing the gravity wavelength to 60 km (equal to the acoustic wavelength near ground level) reduces v by an order of magnitude. Furthermore, v would be increased by employing: (i) interaction heights higher than ground (via ρ_0), (ii) smaller angle χ (relating the frequencies via Eq. (4)), and (iii) interaction time Δt longer than a single period of the generated wave. Thus it can be accepted that realistic ranges of values of parameters in Eq. (14) can give values of v which are consistent with the observed ionospheric data.

In conclusion, it appears that the model for the generation of ionospheric oscillations proposed in this section has been established as feasible. However, definite proof or further elaboration of the model shall have to await more knowledge of the primary gravity waves and the mechanism of their production by storm convective processes.

7.0 Concluding Remarks

The principal purpose of this investigation has been to develop the theory of the weakly nonlinear propagation and interaction of internal acoustic-gravity waves in the atmosphere, and to propose a model, using this theory, for the generation of ionospheric oscillations associated with tropospheric thunderstorm activity. Within the general context of the theory of atmospheric waves, the major contributions which the present thesis makes to the literature can be summarized as follows:

1. The analysis rests on Hamilton's variational principle for the adiabatic motion of an atmosphere in hydrostatic equilibrium which was shown to be equivalent to the usual nonlinear Euler equations. We start from the original fluid equations as the basis for the development of the wave Lagrange density, in terms of displacement from equilibrium, so we can, in principle, describe wave motion to all orders of nonlinearity. Thus our treatment is more fundamental than that of Tolstoy (1963) since his Lagrange density describes only linear waves and is not easily extendable to consideration of nonlinear wave motion.

2. In deriving the propagation equations for atmospheric wave trains we employ Whitham's averaged Lagrangian method. However, since that theory was originally developed to describe wave trains with slowly varying amplitudes in uniform media, and since the atmosphere is exponentially stratified, we have had to introduce a new variable which serves the role of such a slowly varying

amplitude. This was done with the device of scaling the displacement field by the local density. The use of this scaling technique represents a new departure in the employment of the averaged Lagrangian method in stratified media.

3. The coupled mode equations for the nonlinear resonant interactions of acoustic and gravity waves is presented for the first time in the literature of atmospheric waves. Previous attempts to deal with this problem by the investigators mentioned in the introduction have faltered by their misinterpretation of the nature of the resonant kinematic relations implied by the acoustic-gravity wave dispersion relation, and perhaps by the analytical complexity which attends the usual approach to nonlinear problems using second order perturbation of the standard Euler differential equations. By the use of a variational method we were able to bypass a major part of these complications. It should perhaps also be mentioned in this context that the use of displacement as our field variable, instead of velocity, pressure and density, also reduces the algebraic manipulation in deriving the nonlinear equations.

4. We have proposed a dynamical source model for the generation of the 2 - 5 min period oscillations in the ionosphere which have been observed to be associated with severe storm activity. The model interprets the oscillations as the manifestation of acoustic waves travelling upwards from regions of nonlinear resonant interactions of gravity waves which are

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known to be produced by storm convective activity. Although we calculated estimates of wave energy coupling, using assumed atmospheric and wave parameters, which indicates the feasibility of the model, definite proof of the validity of the model shall have to await better knowledge of the structure of the gravity waves emitted by thunderstorms and, indeed, the mechanism of their production.

APPENDIX A

Hamilton's Principle for Acoustic-Gravity Waves

We here present the procedure for the derivation of the Lagrange density for adiabatic motion in an ideal gas atmosphere from Hamilton's variational principle. We shall employ a notation different from that in Section 2.0 because we extend the description to incorporate a general reference state, and the presence of a background flow (or wind).

Let $\underline{\alpha}$ denote the position of a particle of fluid in its reference state which has a mass density $\rho_R(\underline{\alpha})$ in this reference state. Then, if $\underline{\beta}(\underline{\alpha}, t)$ is the position of the particle during its motion, one has

$$\rho_R d^2 \underline{\beta} / dt^2 = -J \nabla_{\underline{\beta}} p - \rho_R \nabla_{\underline{\beta}} \Phi \quad (1)$$

where $\Phi(\underline{\beta})$ is the potential energy per unit mass due to the earth's gravitational field, $\nabla_{\underline{\beta}}$ is the gradient in the $\underline{\beta}$ coordinate system, and J is the Jacobian

$$J = \det \|\beta_{i,j}\| = \nabla_{\alpha} \beta_1 \cdot (\nabla_{\alpha} \beta_2 \times \nabla_{\alpha} \beta_3) \quad (2)$$

where $\beta_{i,j} = \partial \beta_i / \partial \alpha_j$. We neglect viscosity in the above. The conservation of mass requires that the density in $\underline{\beta}$ space be given by

$$\rho(\underline{\beta}) = J^{-1} \rho_R(\underline{\alpha}) \quad (3)$$

If we neglect thermal conduction, the entropy per unit mass for a parcel of fluid should stay constant, or, for an ideal gas

$$p\rho^{-\gamma} = p_R\rho_R^{-\gamma}$$

This implies that

$$p(\underline{\beta}) = p_R(\underline{\alpha}) J^{-\gamma} \quad (4)$$

where p_R is the pressure in the reference state. Thus Eq. (1) becomes

$$\rho_R d^2\underline{\beta}/dt^2 + J\nabla\beta(p_R J^{-\gamma}) + \rho_R \nabla_{\underline{\beta}}\phi = 0 \quad (5)$$

which may be regarded as three coupled equations for the Cartesian components of $\underline{\beta}$ as functions of $\underline{\alpha}$ and t .

To obtain Hamilton's principle, we take the dot product of the above equation with a small vector function $\delta\underline{\beta}$, then integrate over time and over some given region of $\alpha_1, \alpha_2, \alpha_3$ space, i.e.

$$\int_{t_1}^{t_2} \iiint \{ \rho_R (d^2\underline{\beta}/dt^2) \cdot \delta\underline{\beta} + J\delta\underline{\beta} \cdot \nabla_{\underline{\beta}}(p_R J^{-\gamma}) + \rho_R \delta\underline{\beta} \cdot \nabla_{\underline{\beta}}\phi \} d^3\alpha dt = 0 \quad (6)$$

To place this in a more familiar form, we introduce the variational notation, whereby $Q(\underline{\beta})$ is any function of $\underline{\beta}$, we define

the variation of Q as

$$\delta Q = Q(\underline{\beta} + \delta \underline{\beta}) - Q(\underline{\beta}) \quad (7)$$

Then, to first order in $\delta \underline{\beta}$, it is clear that

$$\delta \underline{\beta} \cdot \nabla_{\underline{\beta}} \phi = \delta \phi \quad (8)$$

A similar procedure may be carried out for the second term in Eq. (6). In terms of $\underline{\alpha}$, we have

$$J \delta \underline{\beta} \cdot \nabla_{\underline{\beta}} (p_R J^{-Y}) = J \delta \beta_j (\partial \alpha_i / \partial \beta_j) (\partial / \partial \alpha_i) (p_R J^{-Y})$$

where the summation convention is used. We note that

$$(J) (\partial \alpha_i / \partial \beta_j) = \partial J / \partial (\partial \beta_j / \partial \alpha_i)$$

where J is considered as a function of the nine quantities $\partial \beta_j / \partial \alpha_i$. (This is a property of the relations between the determinant, inverse, and cofactor matrix of a matrix.) Thus,

$$\begin{aligned} J \delta \underline{\beta} \cdot \nabla_{\underline{\beta}} (p_R J^{-Y}) &= (\partial / \partial \alpha_i) \{ (\partial J / \partial \beta_{j,i}) (\delta \beta_j) p_R J^{-Y} \} \\ &\quad - p_R J^{-Y} (\partial / \partial \alpha_i) \{ (\partial J / \partial \beta_{j,i}) \delta \beta_j \} \end{aligned}$$

We can also show that

$$(\partial / \partial \alpha_i) (\partial J / \partial \beta_{j,i}) = 0$$

$$\partial(\delta\beta_j)/\partial\alpha_i = \delta\beta_{j,i}$$

$$(\partial J/\partial\beta_{j,i})\delta\beta_{j,i} = \delta J$$

$$J^{-\gamma}\delta J = \delta\{J^{1-\gamma}/(1-\gamma)\}$$

Thus we obtain

$$\begin{aligned} J\delta\underline{\beta} \cdot \nabla_{\underline{\beta}}(p_R J^{-\gamma}) &= (\partial/\partial\alpha_i) [(\partial J/\partial\beta_{j,i})(\delta\beta_j)p_R J^{-\gamma}] \\ &+ \delta\{p_R J^{1-\gamma}/(\gamma-1)\} \end{aligned} \quad (9)$$

In a similar manner

$$\begin{aligned} \rho_R d^2\underline{\beta}/dt^2 \cdot \delta\underline{\beta} &= (d/dt) [\rho_R d\underline{\beta}/dt \cdot \delta\underline{\beta}] \\ &- \delta\{(1/2) \rho_R (d\underline{\beta}/dt)^2\} \end{aligned} \quad (10)$$

If we now insert Eqs. (8-10) into Eq. (6) and require that $\delta\underline{\beta} = 0$ at t_1 and t_2 and on the boundary of the region of integration, we obtain the fact that

$$\delta \int_{t_1}^{t_2} \iiint L d^3\alpha dt = 0 \quad (11)$$

to first order in the $\delta\underline{\beta}$. Here the Lagrange density L is given by

$$L = (1/2) \rho_R (d\underline{\beta}/dt)^2 - \rho_R \phi + p_R J^{1-\gamma}/(1-\gamma) \quad (12)$$

The various terms above may be interpreted as kinetic energy density, gravitational potential energy density, and compressional potential energy density.

To put Eq. (11) in a more convenient form for small disturbances in a moving atmosphere, as in Section 2.1, we let

$$\underline{\beta} = \underline{x} + \underline{\xi}$$

where $\underline{x}(\underline{\alpha}, t)$ defines the ambient state and $\underline{\xi}$ is a perturbation on this state due to the presence of the wave. We consider $\underline{\xi}$ as a function of \underline{x} and t and accordingly change our variables of integration in (11) to \underline{x} and t , with the Jacobian relation

$$d^3 \alpha dt = (1/J_0) d^3 x dt \quad (13)$$

where $J_0 = \|\partial x_i / \partial \alpha_j\|$. One can show from Eqs. (3) and (4) that the ambient density $\rho_0(\underline{x}, t)$ and ambient pressure $p_0(\underline{x}, t)$ are given by

$$\rho_0 = \rho_R / J_0; \quad p_0 = p_R J_0^{-\gamma} \quad (14)$$

Other relevant observations are that

$$J = J_0 J' \quad (15)$$

where

$$J' = \|\delta_{ij} + \partial \xi_i / \partial x_j\| \quad (16)$$

and that

$$d\underline{\beta}/dt = \underline{v} + \frac{\partial \underline{\xi}}{\partial t} \quad (17)$$

where $\underline{v} = d\underline{x}/dt$ is the ambient wind velocity.

With the substitutions enumerated above, the variational principle is transformed into

$$\delta \int_{t_1}^{t_2} \iiint L' d^3x dt = 0 \quad (18)$$

where the Lagrange density L' is given by

$$L' = \frac{1}{2} \rho_0 \left(\underline{v} + \frac{\partial \underline{\xi}}{\partial t} \right)^2 - \rho_0 \Phi(\underline{x} + \underline{\xi}) - p_0 \frac{(J')^{1-\gamma}}{(\gamma-1)} \quad (19)$$

This is the lagrangian employed in this work, but with no wind ($\underline{v} = 0$) and \underline{x} is the background state, i.e. $\underline{a} = \underline{x}$. So, in that case $J' = J$ and we have Eq. (2.33) for the description of waves in a windless atmosphere.

APPENDIX B

Form of $\langle L_2 \rangle$ in Uniform Media

In the following, we derive the form of $\langle L_2 \rangle$ given in Section 3.0. Consider a general quadratic form:

$$L_2 = B_{ij} \xi_i \xi_j + C_{ij} \xi_i \xi_{j,t} + D_{ij}^\beta \xi_i \xi_{j,\beta} + E_{ij}^\beta \xi_{i,t} \xi_{j,\beta} + F_{ij} \xi_{i,t} \xi_{j,t} \\ + G_{ij}^{\alpha\beta} \xi_{i,\alpha} \xi_{j,\beta}$$

where B, C, D, E, F and G are real and may depend weakly on \underline{x} and t . We employ the notation $\xi_{i,\beta} = \partial \xi_i / \partial x_\beta$, $\xi_{i,t} = \dot{\xi}_i = \partial \xi_i / \partial t$. By writing the Euler-Lagrange equations corresponding to L_2 it can be seen that, without loss of generality, B, E, F and G can be taken to be symmetric, and C and D be taken to be antisymmetric, with respect to (i, j) and (α, β) , where relevant.

Substituting

$$\xi_i = a_i e^{i\phi} + a_i^* e^{-i\phi}$$

and its derivatives, using the expansion Eq. (4.14), into L_2 , and averaging, yields

$$\langle L_2 \rangle = 2B_{ij} a_i a_j^* + C_{ij} (2i\omega a_i a_j^* + \mu (a_i \dot{a}_j^* - a_i^* \dot{a}_j)) \\ + D_{ij}^\beta (-2ik_\beta a_i a_j^* + \mu (a_i a_{j,\beta}^* - a_i^* a_{j,\beta})) \\ + E_{ij}^\beta (-2\omega k_\beta a_i a_j^* - i\mu\omega (a_i a_{j,\beta}^* - a_i^* a_{j,\beta}) + i\mu k_\beta (a_i \dot{a}_j^* - a_i^* \dot{a}_j))$$

$$\begin{aligned}
& + F_{ij} (2\omega^2 a_i a_j^* - i\mu 2\omega (a_i \dot{a}_j^* - \dot{a}_i^* a_j)) \\
& + G_{ij}^{\alpha\beta} (2k_\alpha k_\beta a_i a_j^* + i\mu k_\alpha (a_i a_{j,\beta}^* - a_i^* a_{j,\beta}) \\
& \quad + i\mu k_\beta (a_i a_{j,\alpha}^* - a_i^* a_{j,\alpha})) + O(\mu^2)
\end{aligned}$$

This, by inspection, may be written as

$$\langle L_2 \rangle = \langle L_2(0) \rangle + \langle L_2(\mu) \rangle + O(\mu^2)$$

where, $\langle L_2(0) \rangle = A_{ij} a_i a_j^*$,

$$\begin{aligned}
\langle L_2(\mu) \rangle = & - \frac{i\mu}{2} \left[\frac{\partial A_{ij}}{\partial \omega} \left(a_i \frac{\partial a_j^*}{\partial t} - a_j^* \frac{\partial a_i}{\partial t} \right) \right. \\
& \left. - \frac{\partial A_{ij}}{\partial k_\beta} \left(a_i \frac{\partial a_j^*}{\partial x_\beta} - a_j^* \frac{\partial a_i}{\partial x_\beta} \right) \right],
\end{aligned}$$

and

$$A_{ij} = 2(B_{ij} + i\omega C_{ij} - ik_\beta D_{ij}^\beta - \omega k_\beta E_{ij}^\beta + \omega^2 F_{ij} + k_\alpha k_\beta G_{ij}^{\alpha\beta})$$

is an element of the Hermitian matrix A used in Section 3.0.

APPENDIX C

Linear Dispersion and Polarization Properties

We discuss here some of the standard propagation properties of linearized acoustic-gravity waves which follow from the results of Section 4.0, and which are useful to recall in order to interpret the coupling coefficients of the weakly nonlinear resonant interactions of these waves, the theory of which is pursued in Section 5.0. Those coefficients depend on the linear dispersion and polarization properties of the wave modes participating in the interaction.

The structure of the dispersion relation for the acoustic and gravity wave modes is revealed by the shape of the propagation surface $D=0$ if we put Eq. (4.30) in the quadric form

$$\frac{k_x^2 + k_y^2}{a^2} + \frac{k_z^2}{b^2} = 1, \quad (1)$$

where

$$a^2 = \frac{\omega^2}{c^2} \left(\frac{\omega^2 - \omega_A^2}{\omega^2 - \omega_B^2} \right), \quad b^2 = \frac{\omega^2 - \omega_A^2}{c^2}. \quad (2)$$

Thus, the surface, parametric in ω , is an oblate ellipsoid of revolution when $\omega > \omega_A$, the acoustic mode, while in the gravity wave mode, $\omega < \omega_B$, it is a hyperboloid of one sheet. Both surfaces are symmetric around the k_z axis.

The propagation characteristics in each mode can be studied by examining the phase and group velocities. From the dispersion relation the phase velocity v_p is given by

$$v_p^2 = \frac{\omega^2}{k^2} = c^2 \left[\frac{\omega^2 - \omega_B^2 ((k_x^2 + k_y^2)/k^2)}{\omega^2 - \omega_A^2} \right], \quad (3)$$

so that, since $\omega_A > \omega_B$, in the acoustic mode the phase velocity is greater than the sound speed c , while in the gravity mode the phase velocity is less than c , i.e.,

$$v_p|_a \geq c > v_p|_g \quad (4)$$

At very high frequencies v_p approaches c and no longer depends on the relative values of horizontal and vertical wave numbers; the propagation is isotropic and the dispersion surface is spherical, as seen by Eq. (1) as $\omega \rightarrow \infty$. At low frequencies v_p depends on direction. As $\omega \rightarrow 0$,

$$v_p = c \frac{\omega_B}{\omega_A} \frac{(k_x^2 + k_y^2)^{1/2}}{|k|} \quad (5)$$

The gravity wave region is thus a mode of very dispersive anisotropic waves.

The relative behavior of the directions of group velocity v_G and phase velocity v_p is revealed by the dispersion surfaces $D = 0$ for different ω values. In the acoustic mode the surfaces

are nested ellipsoids starting from a point at $\omega = \omega_A$ and increasing in volume and sphericity as $\omega \rightarrow \infty$. Thus for large frequencies v_p and v_G coincide; the propagation is isotropic, as noted before, and as shown by the expressions Eq. (3) and Eq. (4.40). For the gravity mode, the nested hyperboloids start from the interior as the k_z axis at $\omega = 0$ and progress asymptotically to an infinite annulus in the $k_z = 0$ plane as $\omega \rightarrow \omega_B$. Thus, for large $|k|$, v_p and v_G are perpendicular to each other. From the expressions $\underline{v}_p = \underline{k}(\omega/k^2)$ and $\underline{v}_G = -(\partial D/\underline{k})/\partial D/\partial \omega$ the dispersion surfaces for gravity waves show that upwardly propagating waves imply downward movement of group velocity for large $|k|$. For frequencies near $\omega = 0$ and for horizontal propagation $k_z = 0$, v_p and v_G are in the same direction.

The usual and informative approximate dispersion relations for each of the modes can be obtained simply by solving Eq. (4.30) for ω^2 :

$$2\omega^2 = (k^2 c^2 + \omega_A^2) \left[1 \pm \left(1 - 4k_H^2 \omega_B^2 c^2 / (k^2 c^2 + \omega_A^2) \right)^{1/2} \right] \quad (6)$$

The plus sign gives the acoustic mode and the minus sign the gravity wave mode. If in this equation we expand the square root factor and retain the leading terms then we obtain, with the plus sign, the approximate acoustic dispersion relation for frequency ω_a

$$\omega_a^2 = k^2 c^2 + \omega_A^2 = (k^2 + v^2) c^2, \quad (7)$$

and, with the minus sign, the approximate internal gravity wave dispersion relation

$$\omega_g^2 = \frac{k_H^2 \omega_B^2}{k^2 + \omega_A^2 / c^2} = \frac{k_H^2 \omega_B^2}{k^2 + v^2}, \quad (8)$$

Notice that with the approximation Eq. (7) the propagation is isotropic and dispersive. These two approximations show the relative importance of compressibility and gravity in the two modes. This follows from the fact that Eq. (7) is exactly the dispersion relation of an exponentially stratified gas with zero gravity and cutoff frequency ω_A , while Eq. (8) is, in form, the dispersion relation for internal waves in a density stratified incompressible fluid with buoyancy frequency ω_B , with the interpretation of the scale height as $H = 1/2 v$.

The polarization of the waves is most easily discussed by considering, without loss of generality, the propagation wave vector in the x-z plane, so that $k_y = 0$. Then Eq. (4.27) gives

$$\frac{a_x}{a_z} = \frac{k_x (k_z + i\beta v)}{\omega^2 / c^2 - k_x^2}, \quad a_y = 0, \quad (9)$$

which shows that the displacement \underline{a} is entirely in the vertical plane through the wave vector \underline{k} . Using one or other of the

eigenvalues for ω^2 from Eq. (6) gives the exact polarization relation for waves in the acoustic or gravity wave modes. Here we shall be satisfied with approximations. Using Eq. (7) we have for the approximate polarization for acoustic waves

$$\frac{a_x}{a_z} = \frac{k_x (k_z + i\beta v)}{k_z^2 + v^2}, \quad v = \frac{1}{2H} \quad (10)$$

Therefore, if

$$a_z = \cos \omega t, \quad (11)$$

$$a_x = \frac{k_x}{k_z^2 + v^2} [\beta v \sin \omega t + k_z \cos \omega t] \quad (12)$$

and the particle orbit is an ellipse with tilted axes

$$a_x^2 + \frac{k_x^2}{k_z^2 + v^2} a_z^2 - \frac{2k_x k_z}{k_z^2 + v^2} a_x a_z = \left[\frac{k_x \beta v}{k_z^2 + v^2} \right]^2. \quad (13)$$

When $k_z^2 \gg v^2$, or for large frequencies, Eqs. (10) & (13) reduce to

$$\frac{a_x}{a_z} = \frac{k_x}{k_z}, \quad \text{or } \underline{axk} = 0, \quad (14)$$

indicating longitudinal wave motion, as with ordinary sound waves.

For the gravity wave approximation, we may set $c = \infty$, $\beta = -1$, in Eq. (9) to find

$$\frac{a_x}{a_z} = \frac{iv - k_z}{k_x} \quad (15)$$

Again, if $a_z = \cos \omega t$,

$$a_x = \frac{1}{k_x} (v \sin \omega t - k_z \cos \omega t) \quad (16)$$

and the particle orbit becomes

$$a_x^2 + \frac{k_z^2 + v^2}{k_x^2} a_z^2 + 2 \frac{k_z}{k_x} a_x a_z = 0, \quad (17)$$

which, with the approximate dispersion relation Eq. (56), may be written

$$a_x^2 + \left(\frac{\omega_B^2}{\omega^2} - 1 \right) a_z^2 + 2 \frac{k_z}{k_x} a_x a_z = 0; \quad (18)$$

This also is the equation of a tilted ellipse for $\omega < \omega_B$. For large $|k|$, or $k_z^2 \gg v^2$, Eqs. (15-18) becomes

$$\frac{a_x}{a_z} = - \frac{k_z}{k_x}, \quad \text{or } \underline{k} \cdot \underline{a} = 0, \quad (19)$$

i.e., the wave motion is vertical transverse.

The short discussion of polarization and orbits given above

is only an outline. Fuller details are presented by Midgley & Liemohn (1966). The description of polarization properties given above is based on Tolstoy (1963).

APPENDIX D

On the Explanations of Thunderstorm Associated Ionospheric Oscillations

Jones (1970) and Chimonas & Peltier (1974) have attempted to explain the presence of vibrations in the upper atmosphere, with periods of 2 - 5 min, above regions of thunderstorm activity in the troposphere by invoking particular, and differing, characteristics of the theory of wave-guide normal modes of acoustic-gravity wave propagation. These normal modes are established by the nonisothermal (and nonuniform wind) vertical stratification of the atmosphere. This theory accounts only for trapped or multiply reflected waves, and the usefulness of the theory resides essentially in the fact that for an observer or receiver at a point sufficiently removed in horizontal range from a source of excitation in the wave guide only a few of the waves, out of the theoretically infinite number, are of significance in explaining the transfer of energy from source to receiver. Problems, both of utility and interpretation, arise in wave guide mode theory when source and receiver are near each other in distance.

Fig. 5 shows a sketch of acoustic-gravity wave-guide modes with the Press-Harkrider labeling scheme. The particular features in this diagram of horizontal phase velocity versus frequency which are instrumental in the explanations of Jones and Chimonas & Peltier are noted as 'J' and 'C-P', respectively. Jones' explanation of the ionospheric oscillations involves the behavior of the

fundamental mode S_0 and the first gravity mode GR_0 on the phase diagram where they sharply come together to 'kiss' each other. Usually, away from these zones a normal mode of propagation is concentrated in horizontal ducts in which waves are trapped a few tens of kilometers in the vertical, travelling principally in either lower atmospheric or upper atmospheric ducts, with rapid attenuation outside each duct. As the wave frequencies in these kissing regions are passed, however, a normal mode may move from one duct to the other. At this transitional frequency the mode thus extends from low to high atmosphere. This condition allows for the excitation of high level response to low level sources, and Jones explains the ionospheric 2 - 5 min oscillations as the selective transmission filtering of broadband sources associated with thunderstorm activity in the lower atmosphere. The relatively narrow bandwidth of the oscillations aloft he interprets as a manifestation of the narrow-band response filter characteristics of the kissing modes. Although in the usual published phase diagrams this transition region occurs at a period of about 5 min Jones suggests that the 2 - 5 min period range of the ionospheric oscillations is induced by Doppler shifting of the kissing modal point by upper atmosphere winds.

Although we cannot deny the reality of the exchange of wave energy between upper and lower atmosphere ducts via the transition zone modes, Jones' model cannot explain the presence of oscillations in the upper atmospheric region directly above the thunder-

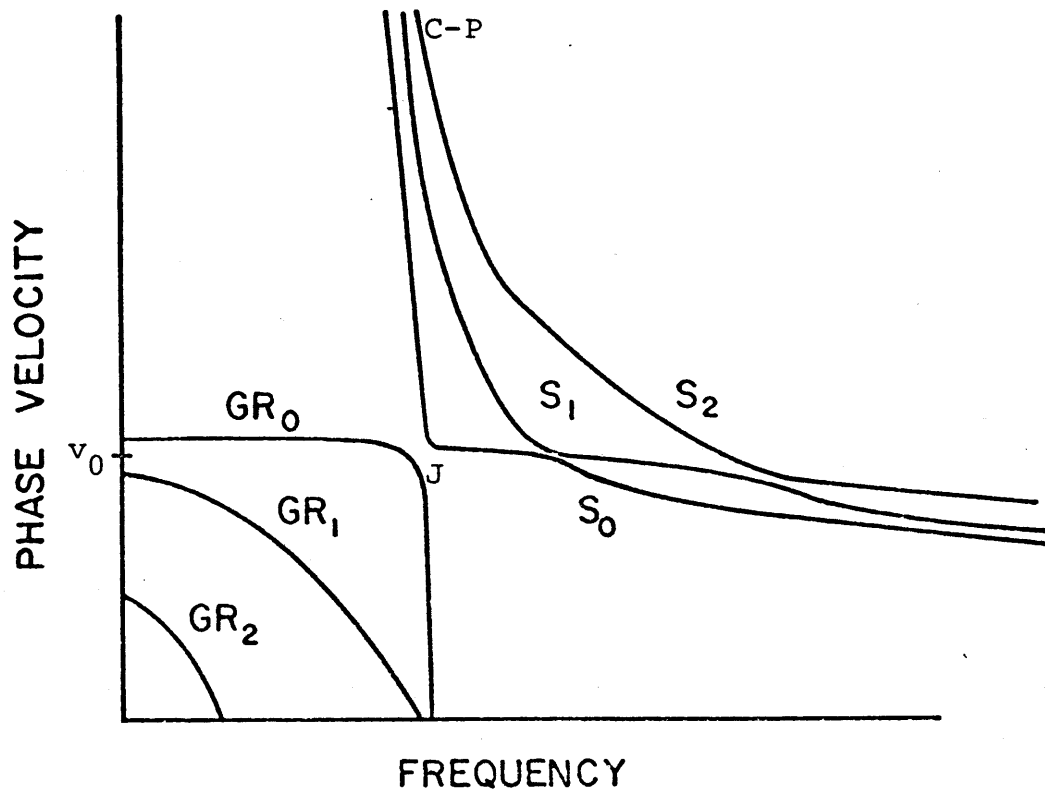


Fig. 5. Sketch of acoustic-gravity wave-guide modes.

storm activity near the ground below. In his analysis Jones dwells on the frequency response function at the modal kissing region and does not consider the implications of the model for the spatial transference of energy between the modes. The latter is most easily discussed in terms of resonance transfer of coupled modes of propagation: Let $v_o = \omega_o/k_o$ be the value of the horizontal phase velocity in the center of the transition gap in phase space, and ω_o and k_o the corresponding frequency and wave number. Let us also suppose that in an initial state the upper duct has zero energy while the lower is excited. Then, from standard coupled mode theory (or from consideration of analogous coupled oscillators) the spatial response of each duct can be written in the form:

$$\xi_l = A \cos \left(\frac{1}{2}x \Delta k \right) \cos (k_o x - \omega_o t)$$

$$\xi_u = -A \sin \left(\frac{1}{2}x \Delta k \right) \sin (k_o x - \omega_o t)$$

where ξ_l signifies the motion in the lower duct and ξ_u the upper. Thus as horizontal range x increases energy will be transferred from the lower to the upper duct. When $x\Delta k = \pi$, the transfer is complete. Jones displays a value for Δk of about 0.001 km^{-1} , which gives a horizontal range required for total resonance transfer to be about 3000 km. Thus, we conclude that the ionospheric oscillations above the thunderstorm region cannot be explained by Jones' model.

For their explanation of the ionospheric 2 - 5 min period oscillations, Chimonas & Peltier noticed that for very large horizontal phase velocities the modal phase diagram shows that the higher acoustic modes, S_1 , S_2 , etc. on Fig. 5 lie in the period range 2 - 5 min. This observation led these authors to suggest that the ionospheric phenomenon is the response of the atmosphere to these modes. However, above a value of phase velocity of about 800 m/s these acoustic modes lose their true modal existence, because beyond that value waves become untrapped as they propagate vertically. To overcome this limitation they propose the maintenance of the modes by invoking the concept of 'leaky modes' in which partial reflections of wave energy take place at the base of the thermosphere and these weak reflections maintain the phase relationship between upgoing and downgoing waves required to establish a mode. The unreflected wave energy travels upward to cause the ionospheric oscillations. When stripped of its genesis in the context of wave-guide modes, this model is simply then a theory in which plane acoustic waves are launched vertically from some source and are then partially reflected so that they have the right phase relation on arrival at the source to resonantly enhance particular wave frequencies.

The explanation of Chimonas & Peltier is open to two main theoretical objections. The first is the requirement for partial reflections. From the known vertical thermal structure of the atmosphere, wave frequencies corresponding to 2 - 5 min periods

are above the acoustic cutoff frequency ω_A , Eq. (4.29), at all heights from ground level to the high thermosphere. Thus, the existence of partial acoustic reflections in this period range is doubtful. The second objection is that in order to launch and maintain the resonance a large coherent source area is required so that spherical loss of energy does not degenerate the resonant enhancement. This area must be at least of the order of the size of the distance to the base of the thermosphere, about 100 km. Such a large source size would seem to demand consideration of the source mechanism of the wave, but this is not discussed by the authors. (Our proposal in terms of nonlinear interactions of gravity waves, given in Section 6.0, provides for such a source.) In summary, then, it appears that the proposal by these authors to explain the phenomena in the ionosphere associated with thunderstorms is less than convincing.

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