

**WAVE PROPAGATION IN Laterally Varying Media
AND Iterative Inversion for Velocity**

by

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Abstract

In this thesis, wave propagation in laterally varying media is studied using the Gaussian beam method. The method consists tracing rays and then solving the wave equation in the neighborhood of each ray using the parabolic approximation. These local solutions can be expressed in terms of Gaussian beams. The approximate global solution for a given source is then constructed by a superposition of Gaussian beams along particular rays. Forward modelling as well as inverse modelling is investigated. In the first chapter, an overview and brief review of the literature is given. In the second chapter, the Gaussian beam method is applied to 2-D velocity structures. The validity of the method is tested in two ways. First is the application of the reciprocal theorem for Green's functions in an arbitrary heterogeneous medium. The discrepancy between synthetic seismograms for reciprocal cases is considered as a measure of error. Second, the Gaussian beam method is applied to cases for which solutions are known by other approximate methods. This includes the soft basin problem. The method is then used for two applications. First, it is used to study volcanic earthquakes at Mt. St. Helens. The observed large differences in amplitude and arrival time between a station inside the crater and those at the flank can be explained by the combined effects of an anomalous velocity structure and shallow focal depths. The method is then applied to scattering of teleseismic P-waves by a lithosphere with randomly fluctuating velocities. The third chapter discusses several topics in Gaussian beam synthesis. This includes the choice of the beam parameter, ϵ , and its interpretation in terms of a complex source point. Asymptotic and exact evaluations of the Gaussian beam integral for a line source are then investigated. Next, the problem connected with an initial point in a gradient is discussed, which in a vertically varying media reduces to the difference between an initial plane wave and an initial Snell wave. A comparison is done with the reflectivity method for the PP phase in the Imperial Valley structure. Then, a comparison is then done between the slant stack of a wavefield and the Gaussian beam decomposition using

large beam widths, showing good agreement. Beams reflected at an interface are then investigated using a first order correction to the plane wave reflection coefficient typically used in the Gaussian beam method. Finally, the finite frequency scattering effects from an indented interface are computed using the Gaussian beam method and compared with those computed using the Aki-Larner method. In the final chapter, inversion for material parameters using waveform data is investigated. An iterative linearized approach is used in which a linear sensitivity operator must be derived. This can be done economically by using reciprocity of the Green's function. In order to avoid a large matrix inversion, several descent algorithms are described. Data errors and a priori model information are incorporated using covariance operators. A fast and reasonably accurate forward modelling scheme is required, and here we make use of the Gaussian beam method in a slowly varying heterogeneous medium. Different linearizations are possible including a linearization in terms of the field, the Born approximation, and a linearization in terms of the log field, the Rytov approximation. The relative merits are discussed. Finally several numerical examples are performed using a field linearization in order to test the method. Transmitted body waves through a model with sources beneath a heterogeneity and surface stations are used. The results using the waveform data identify the trial structures. A comparison is done with the travel time inversion results.

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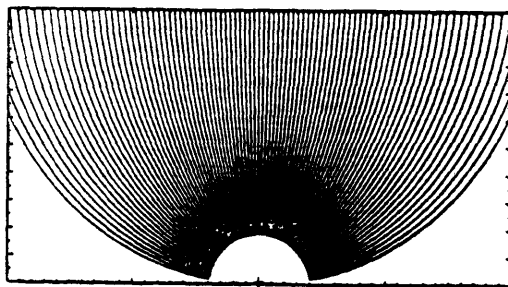
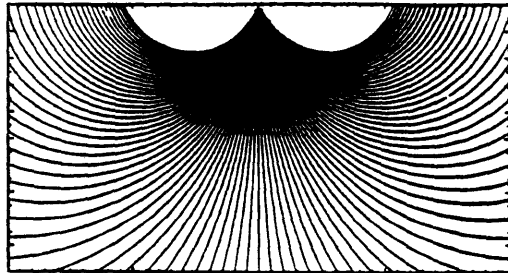
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Chapter 1

Introduction



In recent years, a variety of techniques have been developed to compute seismograms in laterally varying media. These include the finite difference and finite element methods which are general techniques but expensive. Thus they are limited to small grids and lower frequencies (see for example Boore [1972], Smith [1975], Kelly et.al. [1976]). Perturbation techniques are also useful primarily at lower frequencies (see Kennett [1972a, 1972b], Larner [1970], Aki and Richards [1980]). The boundary integral method is another technique recently investigated in the seismological literature (see Sanchez-Sesma and Esquivel [1979], Sanchez-Sesma et.al. [1982]). The remaining methods to be described are primarily high frequency asymptotic methods.

The ray method is one of the most commonly used techniques for propagation in laterally varying media. It is based on a series solution to the wave equation (elastodynamic equation) in inverse powers of frequency (Kline and Kay [1965], Cerveny et.al. [1977]). In practice, usually only the first term is retained. Examples of seismograms computed using the ray method are found in May and Hron [1978], Cerveny [1979], McMechan and Mooney [1980].

Simple estimates of the validity of the ray method are based on several criterion (Kravtsov and Orlov [1980]). First, the parameters of the medium as well as the amplitude and phase gradient of the wavefront should be slowly varying over the cross section of the Fresnel volume. The boundary

of the Fresnel volume is defined by $\omega\tau_g - \omega\tau_v = \pi$, where τ_g is the phase time for the geometrical ray and τ_v is the phase time for a secondary virtual ray. The Fresnel volume is the physical region around a geometric ray for finite frequencies. Next, the phase difference corresponding to two rays arriving at the same point should be greater than π or a caustic results. Finally, the ray method has the range limitation

$$L \ll \frac{a^2}{\lambda}$$

where L is the range, λ is the wavelength, and a is the inhomogeneity scale. This limitation is a result of the increasing size of the Fresnel radius with distance.

A disadvantage of the ray method even at high frequencies is that the method is invalid at caustics where it incorrectly predicts infinite amplitudes. These singularities can be smoothed over by properly incorporating the effects of adjacent rays into the solution. One way to correct the ray method is to construct a local asymptotic solution in the vicinity of the caustic (see Ludwig [1966]). This method may be used to construct a uniform solution, but may be inconvenient for complicated structures.

Certain diffraction effects can be included in the ray method using an extension known as the geometrical theory of diffraction (Keller [1962]). Keller defined new diffracted ray types which include edge rays and vertex

rays. A simple review of this theory is given by Keller [1978]. Extensions to seismology are given for example by Klem-Musatov and Aizenberg [1984]. In this approach, diffraction coefficients must be determined for different cases.

A simplification of the ray method applied to vector wave equations, such as the elastodynamic equation, was the introduction of the ray centered coordinates by Popov and Psencik [1976,1978]. This is a special orthogonal curvilinear coordinate system which moves along a particular ray. In this coordinate system, the first order transport equations for the P and S displacement components decouple. This results in three scalar transport equations rather than vector equations. Writing the wave equation in this local coordinate system along a given ray is also a fundamental step in the Gaussian beam method.

The Maslov method is a semi-classical method which can be applied to propagation problems to extend the ray method (see Kravtsov [1968], Chapman and Drummond [1983], and Ziolkowski and Deschamps [1984]). The method is based on the fact that a ray is a construct dependent not just on location, $\vec{x}$, but also on the direction represented by the slowness vector, $\vec{p}$. In the space $(\vec{x}, \vec{p})$ there are no caustics. This can be seen by noting that $(\vec{x}(t), \vec{p}(t))$ is a state vector for the ray system. Thus if two rays have the same state vector for a given time, then they will be the same for all time since this vector contains all the information needed to

construct the ray. Caustics occur upon projection from this extended space to various three dimensional configuration spaces. The caustics will have different locations depending on the configuration space. In the (x,y,z) space, the method reduces to the standard ray method. In the (p_x,y,z) space in a vertically varying medium, the method would be similar to the WKBJ method discussed by Chapman [1978] where the field is the result of an integral over horizontal slowness, p_x , which is then evaluated by an equal phase method. The idea of the Maslov method is to choose the domain of computation which avoids caustics near different receivers. A problem with this method is that the appropriate domain of computation is not known prior to starting the computations. The final solution is the result of piecing together the asymptotic solutions which are found to be valid in different ranges.

Another technique for the propagation in laterally varying media is based on the parabolic approximation to the wave equation. It was first derived by Leontovich and Fock in 1946 for radio wave propagation (see Fock [1965], ch 11). The parabolic approximation assumes small angle forward scattering about some preferred direction of propagation. It results in a one way operator, since back travelling energy is neglected. An advantage of the derived parabolic equation is that a boundary value problem is replaced by a much simpler initial value problem. The method is valid assuming that the fluctuations of the material parameters are small, and slowly varying with respect to the wavelength.

In relation to the ray method, the parabolic method does not require energy to remain bounded within rigid ray tubes. It accounts for the slow lateral spread of ray energy along the wavefront as the wave propagates. The parabolic method thus includes the effects of diffraction due to lateral diffusion along the wavefront (see review by Malyuzhinets [1959]). This explains the similarity in form between the parabolic equation and the diffusion equation. The parabolic equation is also closely related to the Fresnel approximation in optics (see Goodman [1968], Tappert [1976]).

Geophysical applications of the parabolic method include long range underwater acoustics where the global direction of propagation is assumed to be horizontal (Tappert [1976], McDaniel [1976]), and reflection seismology where the global direction is assumed to be vertical for upward travelling reflected waves (Claerbout [1976]). Applications to elastic waves are discussed by Landers and Claerbout [1972], McCoy [1977], Hudson [1980a, 1980b], and Wales and McCoy [1983].

Derivations of one way parabolic operators by the use of splitting matrices can be found in Coronas [1975] and McDaniel [1976]. The importance in this approach is the potential estimation of coupling between forward and back travelling energy. The splitting method has been extended to the elastic case by Coronas et.al. [1982].

Another method for including non-Fermat information is by using the Kirchhoff integral in which the field is gotten by an integration along some

intermediate wave surface (see Baker and Copson [1950]). For elastic formulations see Pao and Varatharajulu [1976] and Aki and Richards [1980]. For geophysical applications see Trorey [1970,1977], Hilterman [1970,1975], Scott and Helmberger [1983], Hadden and Buchen [1981], and Frazer and Sinton [1984]. Evaluation along many intermediate surfaces leads in the limit to a path integral formulation (Dashen [1979], Flatte et.al. [1979]).

In this thesis, the Gaussian beam method is investigated. It is an extension of both the ray method and the parabolic method in which a global direction must be assumed. The Gaussian beam method has been described in the literature by Popov [1981,1982], and Cerveny et.al. [1982]. These studies were based on the scalar wave equation. The elastic case was first derived by Kirpicnnichova [1972] and more recently by Cerveny [1983] for the 2-D case and Cerveny and Psencik [1983] for the 3-D case.

The Gaussian beam method involves three basic steps. First, a system of rays must be traced from the source to a vicinity of the receiver. Since high frequency energy flows along rays, the system of rays can be thought as the framework on which the wavefield is to be built. This step requires the numerical solution of the kinematic ray tracing system. Next, the wave equation is solved in the 'ray centered' coordinates for each ray using the parabolic approximation. This approximation is a local one related to a particular ray trajectory. These local solutions can be expressed in terms

of Gaussian beams. The final solution is the result of superposing local solutions along individual rays to give an approximate global solution for a given source condition. This step relies on the linearity of the underlying wave equation.

There are several advantages in using the Gaussian beam method. In relation to the ray method, the GB method is finite at caustics. In addition, no prior location of the caustic locations is required as in the Maslov method. The GB method relies on local parabolic approximations and therefore no global direction of propagation is assumed as in the standard parabolic method. Finally, the GB method is fast, comparable to the standard ray method, and possibly faster since no two point ray tracing is required.

This thesis is organized as follows. In chapter 2 the Gaussian beam method is applied to 2-D velocity structures. The validity of the method is tested in two ways. First is the application of the reciprocal theorem for Green's functions in an arbitrary heterogeneous medium. The discrepancy between synthetic seismograms for reciprocal cases is considered as a measure of error. Second, the Gaussian beam method is applied to cases for which solutions are known by other approximate methods. This includes the soft basin problem. The method is then used for two applications. First, it is used to study volcanic earthquakes at Mt. St. Helens. The observed large differences in amplitude and arrival time

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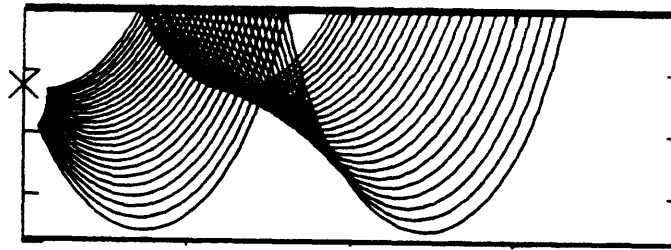
Chapter 2

The 2-D Gaussian beam synthetic method:

Testing and application

by

Robert Nowack and Keiiti Aki



Abstract

The Gaussian beam method of Cerveny et. al. [1982] is an asymptotic method for the computation of wavefields in inhomogeneous media. The method consists of tracing rays and then solving the wave equation in "ray-centered coordinates." The parabolic approximation is applied to find the asymptotic local solution in the neighborhood of each ray. The approximate global solution for a given source is then constructed by a superposition of Gaussian beams along nearby rays. The Gaussian beam method is tested in a 2-D inhomogeneous medium using two approaches. One is the application of the reciprocal theorem for Green's functions in an arbitrarily heterogeneous medium. The discrepancy between synthetic seismograms for reciprocal cases is considered as a measure of the error. The other approach is to apply Gaussian beam synthesis to cases for which solutions are known by other approximate methods. This includes the soft basin problem which has been studied by finite difference, finite element, discrete wavenumber, and glorified optics. We found that the results of these tests were in general satisfactory. We have used the Gaussian beam method for two applications. First, the method is used to study volcanic earthquakes at Mt. St. Helens. The observed large differences in amplitude and arrival time between a station inside the crater and stations on the flanks can be explained by the combined effects of an anomalous velocity structure and a shallow focal depth. The method is also applied to scattering of teleseismic P-waves by a lithosphere with randomly

fluctuating velocities.

Introduction

In this paper, we will discuss applications of the Gaussian beam method. The Gaussian beam method is a variation of both the asymptotic ray method [Cerveny et. al., 1977] and the parabolic approximation method [Tappert, 1976; Claerbout, 1976]. The Gaussian beam method has been recently described in the literature by Babich and Popov [1981], Popov [1981, 1982] and Cerveny et. al. [1982]. These studies were based on the scalar wave equation. The elastic case was first derived by Kirpicnichova [1971] and more recently by Cerveny and Psencik [1983a] for the 2-D case and Cerveny and Psencik [1983b] for the 3-D case.

There are three basic steps involved in the Gaussian beam method. First, a system of rays must be traced from the source. Since high-frequency energy flows along rays, the system of rays can be thought of as the framework upon which the wavefield is to be built. This step requires the numerical solution of the kinematic ray tracing system. Second, the wave equation (elastodynamic equation) is solved in "ray centered" curvilinear coordinates for each ray using the parabolic approximation. This is now a local parabolic approximation related to a particular ray trajectory. These local solutions can be expressed in terms of Gaussian beams. The final solution is the result of superposing local solutions along individual rays to give an approximate global solution for a given source condition. This step is justified by the linearity of the underlying wave

equation.

There are several advantages to using the Gaussian beam method. In relation to the asymptotic ray method, the Gaussian beam method is always finite at caustics. In addition, no prior knowledge of caustic locations is required as in the Maslov method [Kravtsov, 1968; Chapman and Drummond, 1983]. Since the Gaussian beam method relies on local parabolic approximations, no global direction of propagation must be assumed as in the standard parabolic approximation. Finally, the Gaussian beam method is comparable in cost to ray methods, and possibly even faster since no two-point ray tracing is done.

The organization of this paper is as follows. First, a more detailed description of the Gaussian beam method is given along with computational procedures. Then, various examples are given in order to illustrate different uses of the method as well as to test its validity. Two procedures are used to test the method. First, the reciprocal theorem, valid for an arbitrary heterogeneous medium, is applied. The discrepancy between reciprocal seismograms is considered as a measure of the error in the Gaussian beam solution. Second, the Gaussian beam synthesis is applied to several cases, including the 2-D soft basin problem, for which solutions are known by other approximate methods. The Gaussian beam method is then used for two applications. First, the method is used to study volcanic earthquakes recorded under Mt. St. Helens. Then, the

method is applied to scattering of teleseismic P waves by a lithosphere with randomly fluctuating velocities.

Review of the Gaussian beam method

In this section, a more detailed description of the Gaussian beam method is given. The discussion is limited to the 2-D case, for which the examples are given, and follows the results of Cerveny et. al. [1982], Cerveny and Psencik [1983a], and Cerveny [1983].

Before solving for the rays and the beam solutions, the velocity model must be parameterized. The velocities are input as discrete points, and then interpolated using cubic splines. This results in continuous velocities as well as first and second derivatives. For velocities given on a mesh in 2-D or 3-D, this involves a tensor product of 1-D splines [De Boor, 1980]. Velocity discontinuities are included by introducing interfaces between layers with smoothly varying velocities. Curved interfaces are input as a discrete set of points and then spline interpolated.

Spline interpolation of the velocity may introduce some oscillation in the interpolated function depending on the input grid values [Azbel et. al., 1980]. Care must be taken to check the interpolated function and to have the grid velocities change slowly. Several methods can be used to avoid the oscillations in the interpolating function. The first is to use smoothed splines as described by Pretlova [1976]. A second approach is to use splines under tension [see Chine, 1981]. For the remainder of this paper, careful use is made of simple bicubic spline interpolation by comparing the desired and interpolated functions.

The first step of the Gaussian beam method is to solve the kinematic ray tracing system. The rays are extremal of Fermat's integral and are determined by solving the ray equations:

$$\frac{dx_i}{ds} = vp_i, \quad \frac{dp_i}{ds} = -v^{-2} \frac{\partial v}{\partial x_i} \quad i = 1, 2 \quad (1)$$

with initial conditions:

$$\vec{x}(s_0) = \vec{x}_0, \quad \vec{p}(s_0) = \vec{p}_0$$

where $\vec{x}$ is the location along the ray and $\vec{p}$ is the slowness vector tangent to the ray. In an elastic medium, the velocity, v , is either α or β . This system can be solved by standard numerical techniques such as the Runge-Kutta method. At an interface, Snell's law is applied locally. These ray equations are appropriate for an isotropic nondispersive media. In an isotropic dispersive media, wave packets will travel at the group velocity, but their trajectories will be determined by the local phase velocity.

The second step of the Gaussian beam method involves solving the wave equation locally in ray-centered coordinates using the parabolic approximation. The ray-centered coordinate system is an orthogonal curvilinear system that follows along a particular ray and was introduced to seismology by Popov and Psencik [1976, 1978]. In a 2-D medium, the ray-centered coordinates can be specified by the unit vector $\vec{t}$ tangent to the ray and the unit vector $\vec{n}$ normal to the ray (see Figure 1). An element of infinitesimal length in the ray-centered coordinates can be written

$$|d\vec{r}|^2 = h_1^2 ds^2 + h_2^2 dn^2 \quad (2)$$

where the scale factors, h_1 and h_2 , are given by

$$h_1 = 1 + v^{-1} v_{,n} n \quad h_2 = 1$$

where v and $v_{,n}$ are evaluated on the central ray with $n=0$. The ray-centered coordinates comprise an orthogonal system. Since the local flow of high-frequency energy is along rays, the ray-centered coordinate system is an appropriate system for approximating the wave equation by local one-way operators.

The elastodynamic equation is then written in orthogonal curvilinear coordinates, as in Aki and Richards [1980, eqn. 2.48], using scale factors for the ray-centered system. The basic step in deriving a one-way operator from the elastodynamic equation is the substitution

$$u_j(s, n, \omega, t) = U_j(s, n, \omega) e^{-i\omega(t - \tau(s))} \quad (3)$$

where, $\tau(s) = \int v(s)^{-1} ds$ is the phase delay along the central ray, $v(s) = \alpha$ or $v(s) = \beta$, and U_j is a slowly varying envelope. The envelope is then expanded as $U_j = \sum_{i=0}^{\infty} U_j^i \omega^{-i/2}$. Note that this is in half powers of ω^{-1} as opposed to the typical ray series expansion in integral powers. In 2-D, U_s and U_n are the in-plane components along the ray and normal to it, and U_b is the out of plane component normal to the ray. After substitution of equation (3) into the elastodynamic equation, the resulting equations are approximated for large ω . The following results are obtained by retaining only terms of

order ω^7 , $\gamma \geq 1$, and letting $\nu = \omega^{1/2} n$ assuming that $\nu = O(1)$. To this order of approximation, the P and S components decouple. For an incident P wave [Cerveny and Psencik, 1983]:

$$\begin{aligned} U_s^0(s, \nu) &= \frac{1}{\sqrt{\alpha(s)\rho(s)}} W^a(s, \nu) \\ U_n^0 &= 0 \\ U_n^1 &= -i\alpha U_{s,\nu}^0 \\ U_b &= 0 \end{aligned} \quad (4)$$

where W^a satisfies the following parabolic equation

$$2i\alpha^{-1} W_{,s}^a + W_{,\nu\nu}^a - \alpha^{-3} \alpha_{,nn} \nu^2 W^a = 0 \quad (5)$$

For an incident SV wave

$$\begin{aligned} U_n^0(s, \nu) &= \frac{1}{\sqrt{\beta(s)\rho(s)}} W^b(s, \nu) \\ U_s^0 &= 0 \\ U_s^1 &= i\beta U_{n,\nu}^0 \\ U_b &= 0 \end{aligned} \quad (6)$$

where W^b satisfies the parabolic equation

$$2i\beta^{-1} W_{,s}^b + W_{,\nu\nu}^b - \beta^{-3} \beta_{,nn} \nu^2 W^b = 0 \quad (7)$$

and for an incident SH wave

$$U_b^0(s, \nu) = \frac{1}{\sqrt{\beta(s)\rho(s)}} W^b(s, \nu) \quad (8)$$

$$U_s = U_n = 0$$

with W^b satisfying the same parabolic equation as for the SV case. All

velocities and densities are evaluated along the central ray. These results were derived from asymptotic analysis of the elastodynamic equation. The same basic parabolic equation for W can also be obtained from the eikonal equation as shown in appendix 1.

The last term in the parabolic equation (5) or (7) comes from the approximation of order ω

$$h_1 \omega^2 \left[\frac{1}{v(s,n)^2} - \frac{1}{h_1^2 v(s)^2} \right] W \approx -\omega v(s)^{-3} v_{,nn} v^2 W$$

where $v(s)$ and $v_{,nn}$ are evaluated along the central ray, $v(s,n)$ is the velocity in the vicinity of the central ray, and h_1 is the scale factor for the ray-centered coordinates. Without this approximation for the last term, the parabolic equation (5) or (7) is similar in form to those given by Aki and Richards [1980, eqn. 13.154] or Claerbout [1976, 10-3-9]. The parabolic equation (5) or (7) is valid for small angle forward scattering about the central ray direction assuming large scale inhomogeneities with respect to the wavelength. In addition, the validity of the above approximation requires that the velocities be smoothly varying within a certain beam-width of the central ray. Since, as will be seen, the beam solutions to (5) or (7) spread with distance, this will result in a range limitation for the Gaussian beam method.

A particular solution to the parabolic equation (5) or (7) can be written as

$$W(s) = \frac{1}{\sqrt{q(s)}} e^{\frac{i}{2} \frac{p(s)}{q(s)} \nu^2} \quad (9)$$

where $q(s)$ and $p(s)$ satisfy

$$\frac{d}{ds} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 0 & \nu \\ -\nu & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \quad (10)$$

with $\nu(s)$ and $\nu_{,nn}$ evaluated along the central ray. These are the same equations as the dynamic ray tracing system for ray theory, except here $p(s)$ and $q(s)$ have to be complex. $p(s)$ and $q(s)$ must also satisfy certain conditions, among them the condition $\text{Im}(p/q) > 0$. In this case the energy will be confined to a vicinity of the central ray, and the solution will be a Gaussian beam (see Figure 1). For ray theory p and q are real, and $q(s)$ has the interpretation of spreading and $\nu(s) \frac{p(s)}{q(s)}$ has the interpretation of wavefront curvature. The equations in (10) are in transmission-line form, where the ratio p/q is related to the characteristic admittance for the system.

The complete solution to the parabolic equation for a particular central ray is made up of an infinite system of linearly independent beam solutions or modes [Cerveny et.al., 1982] of the form

$$W_k(s, \nu) = \frac{1}{\sqrt{q(s)}} \left(\frac{q'}{q} \right)^{k/2} H_k(\nu \sqrt{\text{Im}(p/q)}) e^{\frac{i}{2} \frac{p}{q} \nu^2} \quad k=0,1,2 \quad (11)$$

where H_k are hermite polynomials. Here, only the zeroth order beam solution, (9), is used for each central ray and no mode coupling between higher order beam solutions is assumed.

The localized solution to the 2-D elastodynamic equation along a particular central ray, specified by a ray index parameter, δ , can be written in the following form [Cerveny and Psencik, 1983a]

$$\vec{u}_\delta(s, n, \omega) = \frac{1}{\sqrt{v(s)\rho(s)q(s)}} \vec{r} e^{-i\omega(t - \int_0^s \frac{ds}{v(s)} - \frac{1}{2} \frac{p(s)}{q(s)} n^2)} \quad (12)$$

where for P waves, $v(s) = \alpha(s)$, $\vec{r} = [\vec{t} + \vec{n}\alpha(s) \frac{p(s)}{q(s)} n]$;

for SV waves, $v(s) = \beta(s)$ $\vec{r} = [\vec{n} - \vec{t}\beta(s) \frac{p(s)}{q(s)} n]$;

and for SH waves, $v(s) = \beta(s)$ $\vec{r} = \vec{\delta}$ (*out of plane component*);

where p and q solve (10) and are in general complex, and $\vec{r}$ gives the orientation of the displacement vector nearby the central ray. Note that the beam solutions are valid only where the ray-centered coordinates are well defined and regular in some region nearby the central ray.

The beam solutions must be modified in the presence of any interfaces. Following Popov [1982], it is assumed that to first approximation the beam solutions, (11), for a given ray remain uncoupled at a smoothly curved interface. Then, the particular beam solution, (9), is only modified by a possibly complex reflection or transmission coefficient and the change in $p(s)$ and $q(s)$ due to the interface. Formulas for how real-valued $p(s)$ and $q(s)$ transform at an interface are given by Cerveny and Psencik [1979], and for curvature by Cerveny and Hron [1980]. Formulas for how

complex-valued $p(s)$ and $q(s)$ transform at an interface are given by Cerveny [1983c], where the equations are written so that $q(s)$ is a relative spreading across each interface. This avoids writing an additional factor in the beam amplitude.

For Gaussian beams, $p(s)$ and $q(s)$ can be written as [see Cerveny et. al., 1982]

$$q(s) = \varepsilon q_1(s) + q_2(s) \quad (13)$$

$$p(s) = \varepsilon p_1(s) + p_2(s)$$

where $p(s)$ and $q(s)$ satisfy equation (10), ε is a complex valued parameter to be determined, and $\begin{bmatrix} q_1 & q_2 \\ p_1 & p_2 \end{bmatrix}$ is the fundamental matrix of linearly independent real solutions of (10) with initial conditions $\begin{bmatrix} 1 & 0 \\ 0 & 1/v_0 \end{bmatrix}$. From these initial conditions

$$K_1(s_0) = v_0 \frac{p_1(s_0)}{q_1(s_0)} = 0 \quad \rightarrow \quad \text{initial plane wave at source}$$

$$K_2(s_0) = v_0 \frac{p_2(s_0)}{q_2(s_0)} = \infty \quad \rightarrow \quad \text{initial line source}$$

where $v_0 = v(s_0)$. Thus, the complex solution is a linear combination of real solutions for a plane wave and line source weighted by ε as in (13). ε is a factor chosen so that a) $q(s) \neq 0$; resulting in no singularities in amplitude along the ray, and b) $\text{Im}(\frac{p(s)}{q(s)}) > 0$; implying that the solution is concentrated near the ray. Following Cerveny et. al. [1982], ε is written

in the form

$$\varepsilon = S_0 - i \frac{\omega}{2v_0} L_M^2 \quad (14)$$

where, geometrically, L_M is the beam-width at a specific location along the beam known as the beam waist, and S_0 is a parameter which shifts the location of the beam waist away from the source location, s_0 , along the ray. Cerveny et. al.[1982] showed that conditions a) and b) above are both satisfied when $L_M \neq 0$. This result depends on the fact that the determinant of the fundamental matrix above is a nonzero constant along the ray, and therefore, $q_1(s)$ and $q_2(s)$ cannot simultaneously be zero for a given s . Thus, the method is constructed to ensure no singularities in amplitude.

The fundamental beam solution can now be written from (12) as

$$\vec{u}_s = \frac{1}{\sqrt{v(s)\rho(s)q(s)}} \vec{r} e^{-i\omega(t - \int_0^s \frac{ds}{v(s)}) + \frac{i\omega}{2v} K(s)n^2 - \frac{n^2}{L(s)}} \quad (15)$$

where

$$q(s) = [S_0 - i \frac{\omega}{2v_0} L_M^2] q_1 + q_2 = \text{complex spreading} \quad (16)$$

$$K(s) = v(s) \text{Re} \left[\frac{p(s)}{q(s)} \right] = \text{phase front curvature of the beam} \quad (17)$$

$$K(s) = \frac{v(s) [(S_0 p_1 + p_2)(S_0 q_1 + q_2) + (\frac{\omega L_M^2}{2v_0})^2 p_1 q_1]}{[(S_0 q_1 + q_2)^2 + (\frac{\omega L_M^2}{2v_0})^2 q_1^2]}$$

and

$$L(s) = \left[\frac{\omega}{2} \text{Im} \left(\frac{p(s)}{q(s)} \right) \right]^{-1/2} = \text{beam halfwidth} \quad (18)$$

$$L(s) = [L_M^2 q_1^2 + \left(\frac{2v_0}{\omega L_M} \right)^2 (S_0 q_1 + q_2)^2]^{1/2}$$

For convenience, the parameter L_0 will sometimes be used to specify the initial beam-width at the beam waist. L_0 is specified by $L_M = \frac{2v_0}{\omega}^{1/2} L_0$, and is independent of frequency since from (18), $L_M = L(s_0 - S_0) = O(\omega^{-1/2})$.

The properties of a Gaussian beam in a homogeneous medium are briefly reviewed in Appendix 2. An example of a Gaussian beam in a homogeneous medium with $L_M = 1$ and $\lambda = \frac{\pi}{4}$ is shown in Figure 2. The beam waist is located at $s = 0$. The solid lines show the spread of the beam with distance and the dashed straight lines are the far field asymptotes. The distance s_c separates the columnated near zone from the diverging far zone of the beam and is proportional to the initial beam width squared over the wavelength. The near zone has planar phase fronts while the far zone diverges as if from a point. By adjusting the initial beam-width, the beam-width observed at a given station range can be changed. Adjusting the initial beam-width so that the station distance is just s_c results in the smallest possible beam-width at the station. This will be called the critical initial beam-width for a given station range. It also places the station just between the columnated planar part of the beam and the far field spreading part. For a homogeneous medium the critical

initial beam-width is $L_{M_{cr}} = \sqrt{\lambda s / \pi}$. For the general inhomogeneous case

$$L_{M_{cr}} = \left(\frac{2\nu_0}{\omega} \right)^{1/2} \left(\left| \frac{S_0 q_1 + q_2}{q_1} \right| \right)^{1/2} \quad (19)$$

[see Cerveny et.al., 1982]

An alternative way to generate a Gaussian beam in a homogeneous medium is to use an initial planar Gaussian amplitude distribution and diffract it using the Fresnel diffraction integral [see Marcuse, 1982]. Since the Gaussian beam is a known solution to the Fresnel integral for a given initial amplitude distribution, the field from an arbitrary initial amplitude distribution can be approximated as a sum of basis Gaussian beams.

The final step of the Gaussian beam method is a superposition of beam solutions to represent a given source using the proper weight functions, $\Phi(\delta)$, where δ is the ray index parameter. For an initial line source, δ represents an initial takeoff angle for a given ray. For an initial plane wave, δ represents the distance along the wavefront for a given ray. Each beam solution solves the wave equation in the vicinity of a ray, and together, the weighted superposition approximately satisfies a source condition for a given source. Thus, the following integral over ray index parameter and frequency can be written

$$\bar{u}(\vec{x}, t) = \frac{1}{\pi} \text{Re} \int_0^\infty \int_{\delta_0}^{\delta_M} \Phi(\delta) F(\omega) \bar{u}_\delta(s, \pi, \omega, t) d\delta d\omega \quad (20)$$

where, $F(\omega)$ is the source spectra, and δ_0, δ_M define the range of the ray

index parameter. δ_0 and δ_M must be chosen to adequately cover the area of interest with rays. Using the slowness approach [Chapman, 1978]

$$\vec{u}(\vec{x}, t) = \int_{\delta_0}^{\delta_M} \Phi(\delta) \vec{g}(\vec{x}, t, \delta) d\delta \quad (21)$$

where $\vec{g}(\vec{x}, t)$ is a wave packet [see Cervený, 1983]. In the examples, $F(\omega)$ is the source spectra for the following damped cosine wavelet [see Cervený et.al., 1977]

$$f(t) = \exp(-(\omega t / \gamma)^2) \cos(\omega t + \varphi) \quad (22)$$

Using this source wavelet, the wave packet $\vec{g}(\vec{x}, t, \delta)$ is approximately Gaussian in both time and space [Cervený, 1983]. Equation (21) is then approximated by the finite sum

$$\vec{u}(\vec{x}, t) = \sum_{i=0}^N \Phi(\delta_i) \vec{g}(\vec{x}, t, \delta_i) \Delta\delta \quad (23)$$

where $\Delta\delta$ must be chosen small enough to smoothly interpolate the beam solutions and to adequately sample the medium.

The weight factors $\Phi(\delta)$ for a given source must then be specified. The weight function for an acoustic line source is given by Popov [1982] and Cervený et. al. [1982], and for a plane wave by Cervený [1982]. In Appendix 3, the weight function for a 2-D point force in an elastic medium is obtained by comparing the Gaussian beam superposition with the exact plane wave decomposition.

There are thus several free parameters when computing the Gaussian beam solution for a given problem: δ_0 , δ_M , and $\Delta\delta$ for the finite sum, the initial beam-width at the beam waist L_M , and the shift of the beam waist away from the source given by the parameter S_0 . The sensitivity of the method on these parameters, particularly L_M , will be investigated in several of the following examples.

Applications

In this section, several applications of the Gaussian beam method are given in order to illustrate the uses of the method and test its range of applicability. The computer program for the Gaussian beam synthesis was modified from the ray tracing program, RAY81, written by I. Psencik [1983]. For all examples, the slowness method is used in which the frequency integral is evaluated first assuming Gaussian packets and a damped cosine source wavelet [Cerveny, 1983].

Example A

In this example, an initial plane wave in a homogeneous medium is decomposed into a number of centered Gaussian beams. Each beam is propagated along the straight line rays shown in Figure 3a. The initial plane wave is located at $z=-50$ km. The station is shown by the triangle at $x=25$ km, $z=0$ km. The velocity is 6 km/sec and the source wavelet parameters are, $f=2$ Hz, $\gamma=4$, $\varphi_0=0$ (see eqn. (22)). The contribution of each beam to the resulting sum is shown for 3 different initial beam-widths in Figures 3b-d. The initial beam-width is represented by L_0 , a frequency independent parameter where $L_M = \left(\frac{2v_0}{\omega}\right)^{1/2} L_0$. L_M has units of km and L_0 has units of $\text{km}^{1/2}$. For this example, $L_M = .98 L_0$. The value $L_0 = 7 \text{ km}^{1/2}$ represents the critical initial beam-width that gives the smallest beam-

width at the station range of 50 km. Larger than critical initial beam-widths result in planar beams at the receiver, while smaller than critical initial beam-widths result in diverging phase fronts at the receiver as if from a point.

The 'ray number' in Figures 3b-d represents the ray index parameter, which for this example measures distance along the initial plane wavefront. The initial ray spacing is 2 km. The rays in the ray diagram (Figure 3a) are assigned a number from 1 to 25. The beam going directly to the station follows along ray number 13. The scaled sum for each initial beam-width is shown to the right in Figures 3b-d.

For $L_0 = 1 \text{ km}^{1/2}$, less than critical, only a limited number of beams around the direct ray contribute to the resulting sum due to phase interference (see Figure 3d). There is also a spurious end effect in the resulting sum caused by lack of cancellation of the end beams. As the initial beam-width is changed to $L_0 = 7 \text{ km}^{1/2}$, near critical, again only a limited number of beams contribute to the resulting sum (see Figure 3c). This is now caused by amplitude decay of beams away from the station. The spurious end effects have been avoided in the resulting sum by using the critical initial beam-width. Finally for $L_0 = 100 \text{ km}^{1/2}$, larger than critical, the beams are more planar at the station distance and no amplitude or phase interference limits the number of rays contributing to the resulting sum (Figure 3b). The sum will then depend on the ray

aperture used.

Figure 3e shows the amplitude of the resulting sum as a function of initial beam-width. The critical initial beam-width is shown. The true amplitude at the receiver is one. For larger than critical initial beam-widths, the amplitude of the Gaussian beam sum falls off since a finite aperture of rays is used.

The following comments can be made concerning decomposing an initial plane wave in a homogeneous medium into Gaussian beams. For larger than critical initial beam-widths, the amplitude is not stable using a finite ray aperture. For near critical initial beam-widths, the result is stable by amplitude decay of beams away from the receiver. The end effects are avoided here by amplitude decay. For smaller than critical initial beam-widths, a stable result occurs by phase interference, but spurious end effects are noticeable in the resulting sum. This is similar to a decomposition of the initial wavefront into point-like sources as viewed at the receiver.

Example B

In this example, an initial half plane is decomposed into Gaussian beams. The half plane is located at a depth of 2.5 km and terminates at $x=15$ km as seen in Figure 4a. Receivers are located on the surface from

12 to 18 km with a spacing of 0.2 km. The velocity is 2.5 km/sec and the source wavelet parameters are, $f=20$ hz, $\gamma=4$, $\varphi_0=\pi/2$.

The initial beam-width, L_M , is related to L_0 by $L_M=0.2L_0$. The critical value of L_0 for a receiver range of 2.5 km is $L_{0cr}=\sqrt{s}=1.58 \text{ km}^{1/2}$. Figure 4b shows the resulting seismograms for $L_0=1.5 \text{ km}^{1/2}$. Since the initial amplitude on the edge of the half plane goes to zero gradually over 0.3 km, the amplitude at the receivers smoothly goes to zero at $x=15$ km with no evident edge diffraction. When L_0 is reduced to $.05 \text{ km}^{1/2}$, the initial amplitude on the edge of the half plane goes to zero much more sharply. A diffracted wave from the edge of the half plane can now be seen (Figure 4c).

Example C

In this example, the decomposition of a line source in a homogeneous medium into Gaussian beams is investigated. Each beam is propagated along the rays shown in the ray diagram in Figure 5a, for a source at $x=25$ km, $z=-50$ km. The station at which the resulting sum is evaluated is shown by the triangle located at $x=25$ km, $z=0$ km. The velocity is 6 km/sec and the source wavelet parameters are, $f=2$ hz, $\gamma=4$, $\varphi_0=0$. The contribution of each beam to the resulting sum at the station is shown for three initial beam-widths in Figures 5b-d. The initial beam-width is represented by L_0 where, $L_M=.98L_0$. The 'ray number' corresponds to the ray index

parameter which for this case is the initial takeoff angle at the source. The rays in the ray diagram (Figure 4a) are assigned a number from 1 to 18. The beam going directly to the station is represented by ray number 10. The scaled sum for each initial beam-width is shown to the right in Figures 5b-d.

The initial beam-width of $L_0=10 \text{ km}^{1/2}$ is slightly larger than the critical value of $L_0=7 \text{ km}^{1/2}$. For a given station distance, L_0 equal to critical separates planar beams from the beams that diverge as if from a point. For $L_0=100 \text{ km}^{1/2}$, larger than critical, only a limited number of beams contribute around the direct ray due to phase interference (see Figure 5b). A noticeable end effect from the finite ray aperture can be seen in the sum. For $L_0=10 \text{ km}^{1/2}$, again only a limited number of rays about the direct ray contribute to the sum (see Figure 5c). This is now due to amplitude decay of beams away from the station. The end effects have been avoided in this case. For $L_0=1 \text{ km}^{1/2}$, smaller than critical, neither amplitude or phase interference limits the number of beams contributing about the direct ray (see Figure 5d). The result then will depend on the ray aperture used.

The amplitude of the Gaussian beam seismogram as a function of initial beam-width is shown in Figure 5e. The amplitude of 1.0 in Figure 5e corresponds to the amplitude of the far field approximation for a line source, where $U(D) \approx -1/4\pi\sqrt{\lambda/r}e^{i\omega r/v}e^{i\pi/4}$. For $r=50 \text{ km}$,

$|U(D)| = .01949$. The amplitude of the Gaussian beam sum for $L_0 = 10 \text{ km}^{1/2}$ is $|U_{GB}| = .01958$. For larger than critical initial beamwidths, the amplitude of the Gaussian beam synthesis is stable within 1% of the far field line source solution. For smaller than critical initial beamwidths, the amplitude of the Gaussian beam sum falls off.

The following conclusions can be made concerning decomposing a line source into Gaussian beams. For larger than critical L_0 , a stable result occurs by phase interference. This is similar to a decomposition of a line source into plane waves. Spurious end effects occur for this case due to the finite ray aperture. Using the critical L_0 , a stable result occurs by amplitude decay and end effects are avoided. Finally for smaller than critical L_0 , the amplitude is not stable.

Example D

In this example, an SH line source above a welded interface is investigated. The velocities are 3.7 km/sec over 4.6 km/sec with the source and receivers located 30 km above the interface. The receivers range from 20 km to 200 km with a spacing of 20 km. The model is shown in Figure 6a.

Figure 6b shows the seismograms computed using the Cagniard method [see Aki and Richards, 1980]. The Cagniard result is convolved with

a source wavelet with parameters $f=3 \text{ Hz}$, $\gamma=4$, $\varphi_0=0$, and the seismograms are plotted in reduced time, $T-X/4.6$. The precritical reflection coefficient goes to zero near 60 km and then changes sign. The critical distance is located at 81.2 km, and the amplitude becomes large at a distance slightly greater than the critical distance. For post-critical distances, total reflection occurs with a phase shift. A small head wave can also be seen emerging past the critical distance. The Gaussian beam result using a larger than critical initial beam-width, $L_0=75 \text{ km}^{1/2}$, is shown in Figure 6c. The Gaussian beam result is similar to the Cagniard result, including the head wave phase. But, there also exists a spurious end effect cutting across the Gaussian beam seismograms from about 6 seconds at 100 km to about 12 seconds at 200 km.

The Gaussian beam result for three initial beam-widths is shown in Figures 7a-c. The value of L_0 equal to 20 is slightly larger than critical for the farthest station. For $L_0=20 \text{ km}^{1/2}$, the small head wave phase is not generated, but the other phases compare with the Cagniard result. For larger L_0 , the head wave begins to form. In addition, an unwanted end effect begins to appear.

Example E

In this example, the effect of a single caustic in a layer over a gradient is reviewed. This is a standard example, but illustrates several features of

the Gaussian beam solution [see Cervený et. al., 1977; Cervený et. al., 1982; Klimes and Cervený, 1983]. The velocity model is constant to 15 km with a velocity of 5.6 km/sec and increases linearly from 15 km down to 40 km with a velocity of 8 km/sec. The ray diagram is shown in Figure 8a using 36 rays and a ray spacing of 1 degree. The wavelet parameters are $f = 4 \text{ Hz}$, $\gamma = 3$, $\varphi_0 = 0.0$.

A caustic intersects the surface at 120 km. Ray theory would predict an infinite amplitude at the caustic and no energy in the caustic shadow. The Gaussian beam seismograms are shown in Figure 8b-c for two initial beam-widths, $L_0 = 8 \text{ km}^{1/2}$ near the average critical value for the set of rays, and $L_0 = 128 \text{ km}^{1/2}$. S_0 is chosen to be 0.0, locating the beam-waist at the source. The Gaussian beam seismograms show several interesting features including, the shift of the peak amplitude to the right of the caustic location and the penetration of energy into the caustic shadow. As a result of phase interference between beams, the $L_0 = 128 \text{ km}^{1/2}$ does not penetrate any further into the caustic shadow than the $L_0 = 8 \text{ km}^{1/2}$ solution. But, spurious phases can be seen in the $L_0 = 128 \text{ km}^{1/2}$ solution which don't quite cancel with the ray set used.

In order to investigate how each beam contributes to the final seismograms, the individual beam solutions are plotted for several station locations and initial beam-widths. The results for a station at 140 km are shown in Figure 9. Figure 9a shows the ray diagram using 18 rays and a

ray spacing of 2 degrees. The station is located by the triangle, and two geometric arrivals are seen to arrive at this station. Figures 9b-c shows the individual beam contributions for $L_0=8 \text{ km}^{1/2}$ and $L_0=128 \text{ km}^{1/2}$. The 'ray number' represents the ray index parameter. The rays are here simply indexed from 1 to 18 from a shallow initial takeoff angle to a steep initial takeoff angle from horizontal. The two geometric arrivals are seen in the scaled sum to the right in Figures 9b-c.

For $L_0=8 \text{ km}^{1/2}$, near the average critical value, the energy is concentrated around the two direct beams to the station with ray numbers 4 and 17. The individual beam wavelets in Figure 9b map out a curve which has stationary points corresponding to the direct beams to the station. For $L_0=128 \text{ km}^{1/2}$, the energy is more evenly distributed among the beams as seen in Figure 9c. The beam solutions away from the stationary points now phase interfere giving a similar resulting sum as for the $L_0=8 \text{ km}^{1/2}$ case.

For very large L_0 , each beam solution is effectively an initial plane wave contribution. For a large L_0 and a one dimensional medium, the individual beam contributions in phase time and ray number are similar to plane wave decompositions in phase time, $\Theta = \tau + px$, and horizontal ray parameter [see Chapman, 1978].

Figure 10 shows the results for a station at $x=120 \text{ km}$, where the caustic intersects the surface. Here the two geometric arrivals merge as

one. Figure 10b shows the individual beam contributions for $L_0 = 8 \text{ km}^{1/2}$. Again the 'ray number' represents the ray index parameter and indexes the rays from 1 to 18. There is now only one stationary point in the curve represented by the locus of individual beam wavelets, located at ray number 10. The result for $L_0 = 128 \text{ km}^{1/2}$ is shown in Figure 10c. The sum shows a spurious end effect caused by the lack of cancellation of the end beams. The $L_0 = 8 \text{ km}^{1/2}$ case avoids this end effect by amplitude decay of beams away from the station.

In Figure 11, the beam contributions are shown for a station at $x=110$ km in the caustic shadow. Here no geometric arrival occurs and the individual beam wavelets shown in Figures 11b-c have no stationary points. The $L_0 = 8 \text{ km}^{1/2}$ case shows a clean resulting sum, whereas the $L_0 = 128 \text{ km}^{1/2}$ case shows spurious end effects which are the same order as the actual signal.

Example F

In this example, the effect of caustics on $q(s)$ is investigated. The caustics result from multiple bounces of rays in a surface waveguide. The velocity model consists of a 0.1 km homogeneous surface layer having a velocity of 1.8 km/sec over a linear velocity gradient from 1.8 km/sec to 5.6 km/sec at 50 km. The ray diagram for a source in the thin surface layer is shown in Figure 12a. The complex function $q^r(s)$, is shown in

Figure 12b as a function of distance along the ray. $q^r(s)$ is a relative spreading across each interface and for this case is related to $q(s)$ by

$$q^r(s) = (-1)^{\alpha_r} q(s) = (-1)^{\alpha_r} [\epsilon q_1(s) + q_2(s)]$$

where α_r is the number of reflections from an interface. $q^r(s)$ is the actual spreading factor which occurs in the amplitude of the beam [see Cervený and Psencik, 1983c]. For simplicity, the parameter ϵ is set to $-i$. The horizontal scale for $q^r(s)$ is ten times larger than the vertical scale.

For ray theory, caustics occur when $q_2(s) = \text{Real}(q(s))$ goes to zero. Several zero crossings for $\text{Real}(q^r(s))$ are shown in Figure 12b, for example, between points b and c. By combining the results for an initial plane wave and an initial line source, the magnitude of the complex function $q^r(s)$ never equals zero. In Figure 12b, the plot of $q^r(s)$ cycles counterclockwise about the origin, but never passes through it.

Using ray theory, a phase advance of $\pi/2$ must be introduced at each caustic. This is done using the KMAH index. Using a complex $q^r(s)$, the phase of $q^r(s)$ automatically changes by π between zero crossings of $\text{real}(q^r(s))$. Thus, as long as the phase of $q^r(s)$ is kept continuous, taking a square root results in the proper number of $\pi/2$ phase shifts.

Example G

In this example the effect of the scale of a velocity inhomogeneity on the Gaussian beam solution is investigated. A single heterogeneity is considered with an incident plane wave impinging from below. The background velocity is 6 km/sec and the wavelet parameters are, $f=1$ hz, $\gamma=4$, $\varphi_0=0$. The velocity heterogeneity is circular and the velocity varies smoothly in a Gaussian manner. The radius of the heterogeneity ranges from 15 to 25 km with a velocity contrast of 10% lower than the background. The Gaussian beam solution is then compared with the finite difference solution of the 45 degree parabolic equation typically used in seismic exploration and forward scattering simulations [see Claerbout, 1976]. The ray diagram for a 25 km radius heterogeneity is shown in Figure 13.

Figure 14 shows the seismograms for an inhomogeneity with a radius of 25 km, a velocity contrast of 10%, a distance from the initial wavefront to the heterogeneity of 50 km, and a distance from the initial wavefront to the stations of 120 km. The finite difference solution of the parabolic equation, Parab45, is shown in Figure 14a. Figures 14b-d show the Gaussian beam solutions for three initial beam-widths. The critical initial beam-width for the station distance in a homogeneous medium is $L_{0cr} = 11 \text{ km}^{1/2}$. From the results of Example A, the decomposition of an initial plane wave in a homogeneous medium requires initial beam-widths comparable or smaller than the critical initial beam-width for the receiver distance. The maximum amplitude for each solution is shown in Figure 14.

The Gaussian beam results for the different initial beam-widths are seen to be stable and compare with the Parab45 result.

Figure 14 also shows the beam-width in km at the inhomogeneity distance for each initial beam-width solution. The smallest beam-width at the heterogeneity distance is 14 km resulting from an initial beam width of $L_0 = 7 \text{ km}^{1/2}$, and is smaller than the heterogeneity radius. From example A, this smallest beam-width defines the zone of contributing beams to the final sum. For smaller than critical initial beam-widths a similar limited region is defined by phase interference about the stationary direct arrival and is related to the first Fresnel zone.

Figure 15 shows the same experiment, except now the initial wavefront is 140 km from the heterogeneity and 210 km from the stations. The Parab45 result is shown in Figure 15a, and the Gaussian beam seismograms for four initial beam-widths are shown in Figures 15b-e. Due to the spread of the beams with distance, the beam-widths at the heterogeneity are larger than the previous case. The smallest beam-width at the heterogeneity distance is 23 km and is just slightly smaller than the heterogeneity radius. The Gaussian beam results are still stable and compare with the Parab45 result. The $L_0 = 4 \text{ km}^{1/2}$ result has a much larger beam-width of 50 km at the heterogeneity distance but the number of beams contributing is now limited by phase interference. The amplitude for this case is fairly stable but a spurious secondary phase can be seen.

In the final case, the inhomogeneity radius is reduced to 15 km with a velocity contrast of 10%, a distance from the initial wavefront to the heterogeneity of 30 km, and a distance from the initial wavefront to the stations of 210 km. The Parab45 result is shown in Figure 16a. The Gaussian beam results are shown in Figures 16b-e for several initial beam-widths. For this case the Gaussian beam amplitudes are less stable. The Gaussian beam solution with the smallest beam-width at the heterogeneity compares best with the Parab45 result. All the Gaussian beam solutions have a larger secondary arrival than the Parab45 solution. This may in part be due to dip filtering in the Parab45 formulation.

These preliminary results indicate that when using the Gaussian beam method in a heterogeneous medium, the critical beam-width be smaller than the heterogeneity scale. This is more restrictive than simply small wavelength with respect to heterogeneity scale, and is related to the approximation made in the final term in equations (5) or (7). This restriction is similar to that given for geometric optics by Kravtsov and Orlov [1980] in which the medium and wave parameters must be slowly varying over the first Fresnel zone.

Example H

In this example, SH waves in a 2-D basin structure are investigated. This problem has been studied by a number of investigators using

techniques including; finite difference, finite element, discrete wavenumber, and Glorified Optics(a ray method) [Boore et. al., 1971; Hong and Helmberger, 1977; Bard and Bouchon, 1980; Aki and Richards, 1980]. The basin structure varies in thickness from 1 km at the edge to 6 km in the center over a range of 25 km. The interface has a cosine shape. The material parameters in the basin are $v_1=7 \text{ km/sec}$ and $\rho_1=2.0 \text{ gm/cm}^3$ and the material parameters of the basement are $v_2=3.5 \text{ km/sec}$ and $\rho_2=3.3 \text{ gm/cm}^3$.

Figure 17 shows the ray diagrams for the direct wave and the first 2 bounces. The rays represent an upward traveling SH plane wave normally incident from below. The Gaussian beam synthesis was made up of 12 bounces. The incident plane wave was expressed by equally spaced rays in the basement. No two-point ray tracing was required; the rays were simply allowed to reverberate in the basin. For this case, 150 rays were used with a ray spacing of 1 km.

The damped cosine source wavelet used in the Gaussian beam synthesis (see eqn. 21) was adjusted to simulate an often used Ricker wavelet [see Bard and Bouchon, 1980]. In Figure 10, the solid line is the Ricker wavelet, and the dashed line is the damped cosine wavelet used in the Gaussian beam synthesis with $T=16.5 \text{ sec}$ and $\gamma=3.2$. The value of L_0 was set at $2.0 \text{ km}^{1/2}$.

Figure 19 shows a comparison of four different methods for the 2-D basin structure. The Gaussian beam seismograms have been added to Figure 13.26 in Aki and Richards [1980]. The center of the basin is marked as 0 km, and the edge of the basin as 20 km. For each distance, the Gaussian beam seismogram is at the bottom. The second phase on the seismograms represents energy propagating across the basin. The seismograms calculated by the Gaussian beam method compare well with the other solutions at the earlier portions of the seismograms. Discrepancies among the various methods are evident in the latter portions of the seismograms. This may be due to the fact that both the Gaussian beam method and Glorified Optics are high-frequency methods, while the discrete wavenumber method is applicable over a broader range of frequencies.

Example I

In this example, the Gaussian beam method is applied to the study of volcanic earthquakes under Mt. St. Helens. During the summer of 1981, a 9-station array was deployed on the flanks and within the crater of the Mt. St. Helens volcano. This was a joint effort undertaken by Oregon State University, Massachusetts Institute of Technology, and the U.S. Geological Survey [Fehler and Chouet, 1982]. Two of the more significant observations of the recorded high-frequency volcanic events were: 1) the

arrival times at the crater station were between 0.4 and 0.7 seconds earlier than those at the flank stations, and 2) the amplitudes at the crater station were about 10 times greater than those recorded at the flank stations.

Figure 20 shows the 2-D model of the velocity structure which was tentatively adopted for Mt. St. Helens. This model is based on evidence obtained by various workers on Mt. St. Helens [Malone, 1982] as well as on other similar types of volcanoes such as St. Augustine in Alaska [Kienle et. al., 1979] and Showa-shinzan in Japan [Hayakawa, 1957]. In all these cases, the internal structure of a volcano is depicted as a high-velocity central body with a P velocity of about 4 km/sec covered by lower-velocity surficial layers.

It was first thought that the observed high amplitude and early arrival time could be explained by locating the earthquake foci at the bottom of the hypothesized high-velocity column. Figure 20 shows the ray diagram for a focus located 3.5 km below the summit. The high-velocity column tends to cause energy to diverge from the summit. Vertical component Gaussian beam seismograms for stations located across the volcano from 2.5 km to 7.5 km along the surface are shown in Figure 21 using frequencies of 5, 10, and 20 Hz. The amplitudes at the flank stations appear to be comparable to those at the crater, and the differences in arrival time are within 0.25 seconds.

If the focal depth is moved upward to 1 km below the summit, the seismic signal changes dramatically. The ray diagram for a focus 1 km below the summit is shown in Figure 20. In this case, the ray paths going directly upward are dense and short, while those toward the flank of the volcano are sparse and take a detour. The resultant seismograms for 5, 10, and 20 Hz are shown in Figure 22. These show an amplitude at the crater station an order of magnitude greater than at the flank stations, and about 0.5 sec earlier arrival, in agreement with observation.

In order to check the accuracy of the computed seismograms, reciprocal seismograms were compared in Figure 23. In case A, a 2-D vertical point force is applied at a point 3.5 km beneath the summit, and the receiver is on the flank of the volcano. In case B, the vertical point force is on the flank of the volcano and the receiver is located 3.5 km beneath the summit. The free surface is neglected here. The vertical component seismograms are shown, for both cases, for frequencies of 5, 10, and 20 Hz. There is a satisfactory agreement in waveform for all three frequencies. The amplitudes agree to within 1% for 20 Hz, 6% for 10 Hz, and 12% for 5 Hz, showing better accuracy for higher frequencies, as expected.

Thus, the P-velocity structure depicted in Figure 20 and focal depths less than 1 km from the summit can explain the two significant observations. A similar observation was made for St. Augustine volcano by

Lalla and Kienle [1982], who found a correlation between the amplitude ratio of summit to flank stations and the focal depth. For focal depths shallower than about 2 km, the summit amplitude was greater by an order of magnitude than the flank amplitude, and for deeper events, the opposite was observed.

In any case, the sensitivity of the seismic signal to the focal depth is strong, due to the heterogeneous structure under a volcano. This result also suggests a promising way of accurately locating volcanic earthquakes using variations in amplitude as well as in arrival time.

Example J

In this final example, the Gaussian beam method is used to study the influence of small-scale fluctuations on teleseismic P-waveforms. The models consist of several realizations of a lithosphere with randomly fluctuating velocities. The layer thickness is 120 km with a correlation length of 15 km. The average velocity is 8 km/sec and the root mean square fluctuation is 3%, roughly representing the results obtained for LASA [Aki, 1973; Capon, 1974].

Normally distributed random velocities were used to construct a 2-D velocity grid with a 15-km spacing between mesh points. This was then interpolated using bicubic splines to give continuous velocities as well as

first and second derivatives. The first realization for the randomly fluctuating velocities is shown in Figure 24. Above is shown a contour plot of the random velocities contoured at 0.1 km/sec interval. Below is shown a printer plot of the random velocities where each integer on the plot represents a 0.3 km/sec velocity interval and the integer "4" represents the velocity interval 7.7 to 8.0 km/sec.

Figure 25 shows, the ray diagram for a vertically incident plane wave perturbed by the random velocity layer. The receivers are located just above the random layer at $z=75$ km. The range of receivers is from 125 to 275 km, spaced at 5 km intervals. Figure 26 shows the resulting wavefield for two dominant frequencies, 2 Hz and 1 Hz. The amplitude and phase have both been significantly distorted by the random velocity layer. A multipath effect can also be seen in the range from 130 to 160 km. No significant P coda with a duration beyond about 3 seconds has developed as a result of the random velocity layer.

In order to check the accuracy of the method for this example, seismograms from reciprocal geometries have been compared using a 2-D point force as a source. The first comparison, shown in Figure 27, is for a source and receiver 300 km above and below the random velocity layer. In this geometry, the rays for the reciprocal cases sample the same region of the random layer. There is a good agreement in waveshape between reciprocal cases. Also, the amplitudes agree within 5%. For this case, the

parameter, S_0 , was specified to put the minimum beam width (beam waist) just before the ray entered the random layer, and L_0 was specified as $12. \text{ km}^{1/2}$.

In addition to reciprocity checks, the calculation was repeated using the finite difference solution of the 45 degree equation [Claerbout, 1976]. Figure 28 shows a comparison between the parabolic result, Parab45, and the the Gaussian beam result for the 1 hz case. The results are similar in the significant details.

In Figure 29, a second realization for the random velocity layer is shown. The perturbed rays for an incident plane wave are shown in Figure 30. The wavefields observed on the surface of the random velocity layer for the two dominant frequencies of 2 Hz and 1 Hz are shown in Figure 31. This realization perturbs the wavefield less than the previous realization. The results have been checked using reciprocity and compared against the Parab45 solution and showed good agreement. For this realization, a comparison has been made between the ln-amplitude and the residual phase delay. This is shown in Figure 32. The positive correlation between $\ln(A)$ and the phase delay residual is in agreement with observations of teleseismic P waves at the Montana LASA by Aki [1973].

Conclusions

There are several advantages in using the Gaussian beam method. First, the method is always finite at caustics. In addition, no prior knowledge of caustic locations is required as in other methods. Next, since the Gaussian beam method relies on local parabolic approximations, no global direction of propagation must be assumed as in the standard parabolic approximation. Finally, the Gaussian beam method is comparable in cost to ray methods, and possibly faster since no two-point ray tracing is done.

The Gaussian beam result depends on the choice of the initial beam-width. The expansion of a plane wave by Gaussian beams in a homogeneous medium requires initial beam-widths comparable or smaller than the critical initial beam width for a given station distance. This is similar to an expansion of an initial plane wave into point-like sources as viewed from the station. The approximate expansion of a line source into Gaussian beams in a homogeneous medium requires initial beam-widths comparable or larger than the critical initial beam-width. This is similar to an expansion of a line source into plane waves. The use of initial beam-widths near critical avoids end effects but doesn't simulate certain arrivals, such as head waves in the welded interface example.

The Gaussian beam method in a heterogeneous medium has been tested using two approaches. The first is the application of the reciprocal

theorem where the discrepancy between seismograms calculated for reciprocal cases is considered as a measure of error. The second is the application of the Gaussian beam method to cases, including the 2-D basin problem, which have solutions from other approximate methods. Preliminary results for a single heterogeneity using the finite difference solution to the parabolic equation and the Gaussian beam method gave best comparisons for critical beam-widths smaller than the heterogeneity scale.

The Gaussian beam method was then applied to the study of volcanic earthquakes under Mt. St. Helens. The observed differences in amplitude and arrival time between a station inside the crater and those at the flank can be explained by the combined effects of an anomalous velocity structure and a shallow focal depth. The method was also applied to study the influence of small-scale velocity fluctuations on teleseismic P waveforms. Significant amplitude and phase fluctuations due to a lithosphere with randomly fluctuating velocities were found, in agreement with observations by Aki [1973] at the Montana LASA.

Appendix 1

In this appendix, we give a derivation of the parabolic equation from the eikonal equation [see Gloge and Marcuse, 1969; and Marcuse, 1982]. By considering a variable endpoint in Fermat's integral, the eikonal or Hamilton-Jacobi equation can be obtained

$$|\nabla\tau|^2 = v^{-2} \quad (\text{A1.1})$$

The rays are the characteristics of this equation. In ray-centered coordinates, the eikonal equation can be written

$$\frac{\partial\tau}{\partial s} - \frac{h_1}{v} [1 - v^2 (\frac{\partial\tau}{\partial n})^2]^{1/2} = 0 \quad (\text{A1.2})$$

where h_1 is the scale factor in (2). The Hamiltonian is written as

$$H(n, p_n) = -\frac{h_1}{v} [1 - v^2 p_n^2]^{1/2} \quad (\text{A1.3})$$

and $p_n = \frac{\partial\tau}{\partial n}$ is the slowness in the n direction and is identified as the generalized momentum. The Hamiltonian can be approximated to second order about the central ray [see Cerveny, 1981] as:

$$H(n, p_n) = -\frac{1}{v} + \frac{1}{2} v p_n^2 + \frac{1}{2} v^{-2} v_{,nn} n^2 \quad (\text{A1.4})$$

Following Marcuse [1982], a parabolic equation can be derived by treating all variables as operators. By analogy to quantum mechanics

$$n \rightarrow n, \quad p_n \rightarrow -i\kappa \frac{\partial}{\partial n}, \quad H \rightarrow i\kappa \frac{\partial}{\partial s} \quad (\text{A1.5})$$

where κ is a parameter to be determined. Applying the operator H to a wave function, ψ ,

$$i\kappa \frac{\partial}{\partial s} \psi = H\psi \quad (\text{A1.6})$$

Using equations (A1.4) and (A1.5), this is written

$$\frac{2i}{v\kappa} \psi_{,s} + \psi_{,nn} - \frac{1}{\kappa^2 v^3} v_{,nn} n^2 \psi + \frac{2}{v^2 \kappa^2} \psi = 0 \quad (\text{A1.7})$$

Writing ψ in terms of the envelope, $\psi = We^{-i\omega(t - \int_0^s \frac{ds}{v(s)})}$, and assuming $v = \omega^{1/2} n$ where $v = O(1)$, gives

$$\frac{2i}{\kappa \omega v(s)} W_{,s} + W_{,vv} - \frac{1}{\kappa^2 \omega^2 v(s)^3} v_{,nn} v^2 W + \left(\frac{2}{\kappa^2 \omega v(s)^2} - \frac{2}{\kappa v(s)^2} \right) W = 0 \quad (\text{A1.8})$$

Letting $\kappa = \omega^{-1}$ gives

$$\frac{2i}{v(s)} W_{,s} + W_{,vv} - v(s)^{-3} v_{,nn} v^2 W = 0 \quad (\text{A1.9})$$

This is equivalent to the parabolic equation in (7) derived from asymptotic analysis. Note that in quantum mechanics $\kappa = \frac{h}{2\pi}$ where h is Planck's constant, but here $\kappa = v_0 \frac{\lambda_0}{2\pi}$. v_0 can be avoided here by writing the eikonal equation (A1.2) in terms of $\tau' = v_0 \tau$. Classical ray theory is valid when λ_0 is much smaller than the heterogeneity scale. The natural analogy between mechanics and ray theory thus provides a link between the eikonal equation and the parabolic equation. Further results from this analogy can be found in Marcuse [1982] and Marcuvitz [1980].

Appendix 2

In this appendix, a brief review of the properties of a Gaussian beam in a homogeneous medium is given. From the dynamic ray tracing equations, (10), and the initial conditions for a plane wave and line source, one obtains

$$\begin{aligned} q_1(s) &= 1 & p_1(s) &= 0 \\ q_2(s) &= s - s_0 & p_2(s) &= 1/v_0 \end{aligned} \quad (\text{A2.1})$$

For the source at $s_0=0$, with the beam waist at the source, $S_0=0$, and using (16), the complex spreading can be written

$$q(s) = s - i \frac{\omega}{2v} L_M^2 \quad (\text{A2.2})$$

where L_M is the initial beam-width at the beam waist. The beam-width can be written from (17) as

$$L(s) = L_M [1 + (\frac{\lambda s}{\pi L_M^2})^2]^{1/2} \quad (\text{A2.3})$$

The phase front curvature can be written from (18) as

$$K(s) = \frac{1}{s [1 + (\frac{\pi L_M^2}{\lambda s})^2]} \quad (\text{A2.4})$$

At the beam waist, $L(s)=L_M$ and $K(s)=0.0$. Thus at the beam waist, the minimum beam-width is achieved and the wavefront is planar. As $|s| \rightarrow \infty$, the angular spread or the far field diffraction angle can be defined as

$$\varphi = \lim_{|s| \rightarrow \infty} \left[\frac{L(s)}{s} \right] = \frac{\lambda}{\pi L_M}$$

This leads to the uncertainty relation

$$\varphi L_M = \frac{\lambda}{\pi} \quad (\text{A2.5})$$

For a small L_M there is a large far-field spread, whereas for a large L_M , there is a small far-field spread. The distance s_C at which the beam-width starts to significantly differ from the initial beam-width is

$$\varphi \approx \frac{L_M}{s_C} \rightarrow s_C = \frac{\pi L_M^2}{\lambda} \quad (\text{A2.6})$$

This is similar to the Fraunhofer range for an aperture of radius a which has the form $s \gg \frac{a^2}{\lambda}$. For $|s| \gg s_C$, $K(s) \approx \frac{1}{s}$, which is the curvature of the wavefront for a line source. Extrema in the curvature occur at $|s| = s_C$.

The critical value of L_M is the one which gives the smallest beam-width, $L(s)$, at the receiver distance, s . In a homogeneous medium, this places the receiver distance at $s = s_C$. Thus from (A2.6), $L_{\text{opt}} = \sqrt{\lambda s / \pi}$.

Appendix 3

In this appendix, the weight function $\Phi(\delta)$ for a 2-D point force in an elastic medium is obtained by comparing the Gaussian beam synthesis with the exact plane-wave decomposition. For a general point force at the origin directed in the x_3 direction, the displacement field for an isotropic, homogeneous medium is given by

$$U_i = \frac{1}{4\pi\rho} \left[(3\gamma_i\gamma_3 - \delta_{i3}) \left(\frac{r}{i\omega\beta} + \frac{1}{\omega^2} \right) r^{-3} - (\gamma_i\gamma_3 - \delta_{i3}) (\beta^2 r)^{-1} \right] e^{i\omega\frac{r}{\beta}} \quad (\text{A3.1})$$

$$- \frac{1}{4\pi\rho} \left[(3\gamma_i\gamma_3 - \delta_{i3}) \left(\frac{r}{i\omega\alpha} + \frac{1}{\omega^2} \right) r^{-3} - \gamma_i\gamma_3 (\alpha^2 r)^{-1} \right] e^{i\omega\frac{r}{\alpha}}$$

[Aki and Richards, 1980], where a time dependence of $e^{-i\omega t}$ is assumed, and the direction cosines of $\hat{r}$ are given by $\gamma_i = x_i / r$. In order to express the above displacement as an integral superposition of plane waves, the Weyl integral is used to obtain

$$\frac{\partial}{\partial x_3} \frac{\partial}{\partial x_i} \left[\frac{e^{i\omega\frac{r}{c}}}{r} \right] = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_i e^{ik_1 x_1 + ik_2 x_2 + ik_c x_3} dk_1 dk_2 \quad (\text{A3.2})$$

$$= \omega^2 \left[(3\gamma_i\gamma_3 - \delta_{i3}) \left(\frac{r}{i\omega c} + \frac{1}{\omega^2} \right) r^{-3} - \gamma_i\gamma_3 (c^2 r)^{-1} \right] e^{i\omega\frac{r}{c}}$$

where $k_c = +\sqrt{\frac{\omega^2}{c^2} - k_1^2 - k_2^2}$, and c is either α or β . Then, for the 3-D point force

$$U_i = \frac{i}{8\pi^2\rho\omega^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_i e^{i(k_1 x_1 + k_2 x_2 + k_\alpha x_3)} dk_1 dk_2 \quad (\text{A3.3})$$

$$+ \frac{-i}{8\pi^2 \rho \omega^2} \int_{-\infty}^{\infty} \int [k_i - \frac{\delta_{i3}}{k_\beta \beta^2}] e^{i(k_1 x_1 + k_2 x_2 + k_\beta x_3)} dk_1 dk_2$$

Now we will investigate the "P-wave" integral alone, and integrate with respect to x_2 to obtain the line force solution. Thus

$$U_{Li} = \int_{-\infty}^{\infty} U_i dx_2 = \frac{i}{4\pi \rho \omega^2} \int_{-\infty}^{\infty} k_i e^{i(k_1 x_1 + k_\alpha x_3)} dk_1 \quad (A3.4)$$

where $k_\alpha = k_3 = +\sqrt{(\frac{\omega}{\alpha})^2 - k_1^2}$. Let ϑ be the angle with the x_3 axis, then

$k_1 = \frac{\omega}{\alpha} \sin \vartheta$ and $k_3 = \frac{\omega}{\alpha} \cos \vartheta$. Now, writing the integral over ϑ ,

$$U_{L1} = \int_{-\frac{\omega}{\alpha}}^{+\frac{\omega}{\alpha}} U_1 dx_2 = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{i}{4\pi \rho \alpha^2} \cos \vartheta \sin \vartheta e^{i \frac{\omega}{\alpha} (x_1 \sin \vartheta + x_3 \cos \vartheta)} d\vartheta \quad (A3.5)$$

$$U_{L3} = \int_{-\frac{\omega}{\alpha}}^{+\frac{\omega}{\alpha}} U_2 dx_2 = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{i}{4\pi \rho \alpha^2} \cos \vartheta \cos \vartheta e^{i \frac{\omega}{\alpha} (x_1 \sin \vartheta + x_3 \cos \vartheta)} d\vartheta$$

The Gaussian beam solution for a homogeneous medium is written

$$U_{GB}(\vec{x}, \omega) = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \Phi(\delta) u_\delta(s, n, \omega) d\delta \quad (A3.6)$$

$$U_{GB}(\vec{x}, \omega) = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \Phi'(\delta) \sqrt{\frac{\alpha_0 \rho_0 q_0}{\alpha \rho q}} e^{i\omega \frac{r}{\alpha} + i \frac{\omega}{2} \frac{p}{q} n^2} d\delta$$

where it has been assumed that $\Phi(\delta)$ has the form, $\Phi(\delta) = \Phi'(\delta) \sqrt{\alpha_0 \rho_0 q_0}$.

For a homogeneous medium,

$$\begin{aligned}
q &= r + \varepsilon, \quad q_0 = \varepsilon \\
p &= v_0^{-1}, \quad p_0 = v_0^{-1} \\
p/q &= \frac{1}{v_0[r + \varepsilon]}
\end{aligned}$$

where $\varepsilon = S_0 - i \frac{\omega}{2v_0} L_M^2$. To compare with the plane wave solution, let $L_M \rightarrow \infty$

and let $\delta = \vartheta$. Then

$$U_{GB}(\vec{x}, \omega) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Phi'(\vartheta) e^{i \frac{\omega}{\alpha} (x_1 \sin \vartheta + x_3 \cos \vartheta)} d\vartheta \quad (\text{A3.7})$$

Comparison with (A3.5) gives

$$\Phi'(\vartheta) = \frac{i \cos \vartheta}{4\pi \rho_0 \alpha_0^2}$$

and

$$\Phi(\vartheta) = \frac{i \cos \vartheta \sqrt{\varepsilon}}{4\pi \rho_0^{1/2} \alpha_0^{3/2}} \quad (\text{A3.8})$$

where $\Phi(\vartheta)$ is the approximate weight for the Gaussian beam representation of P-wave radiation from a 2-D point force, and ϑ is the angle between the direction of the point force and the beam direction.

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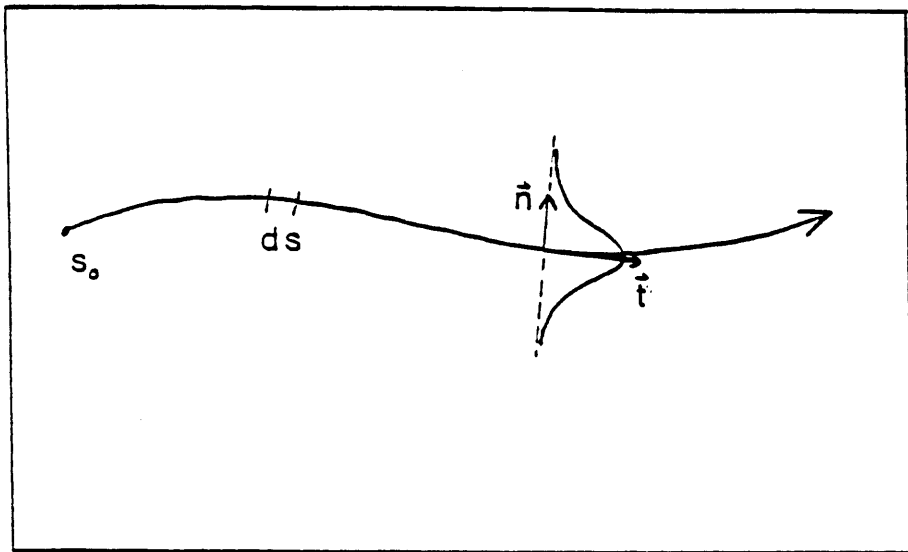


Figure 1

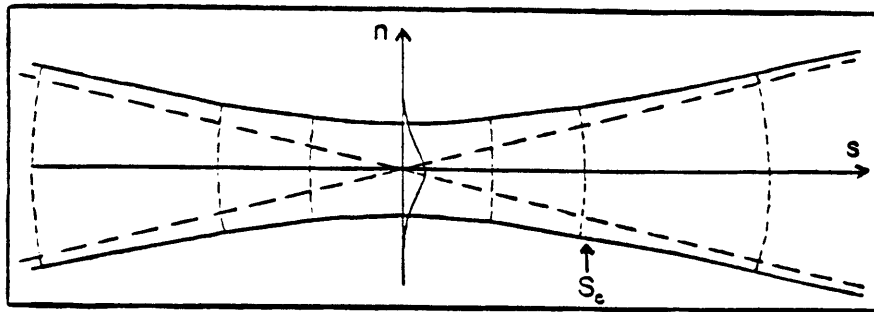


Figure 2

GB SUM FOR PLANE WAVE

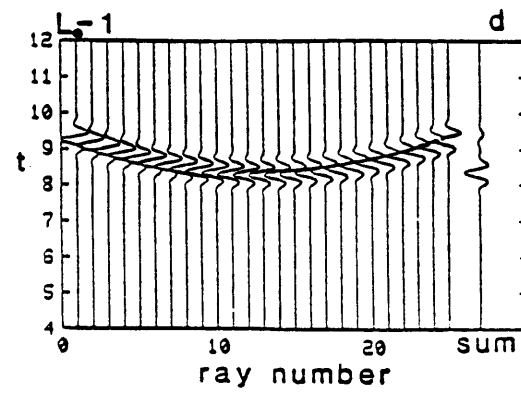
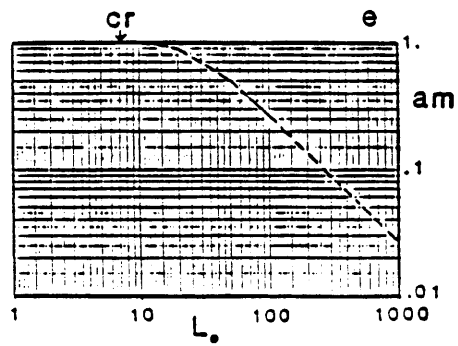
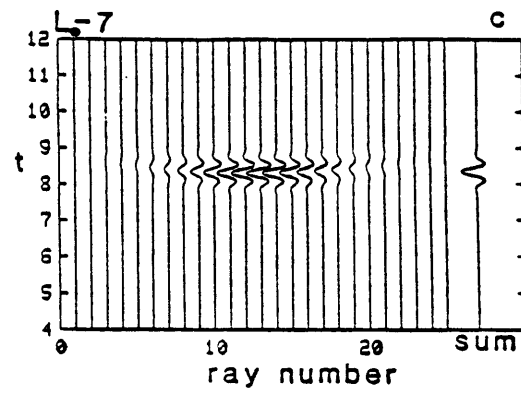
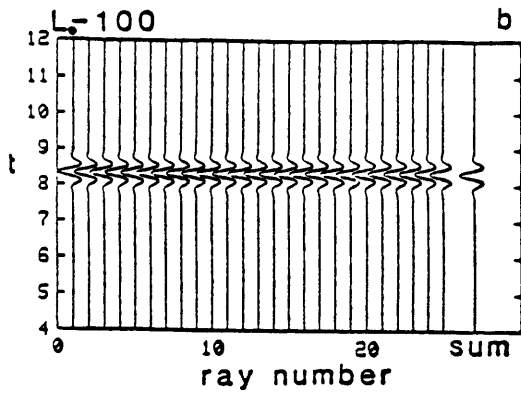
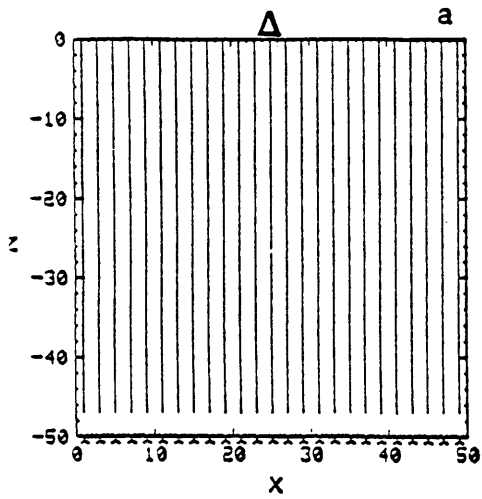


Figure 3

GB REPRESENTATION OF HALFPLANE

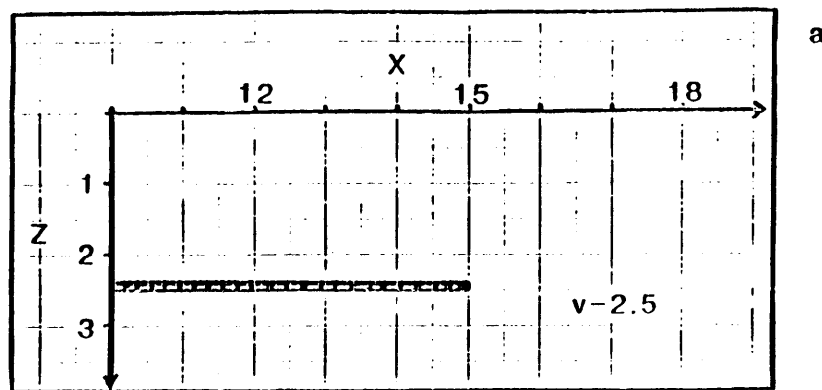
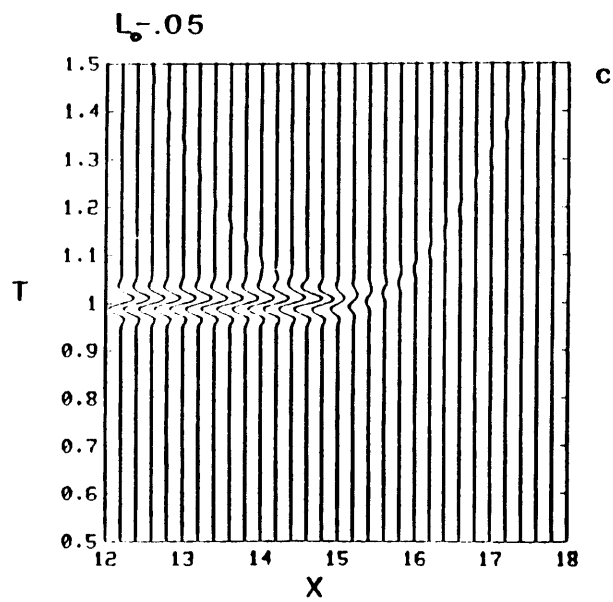
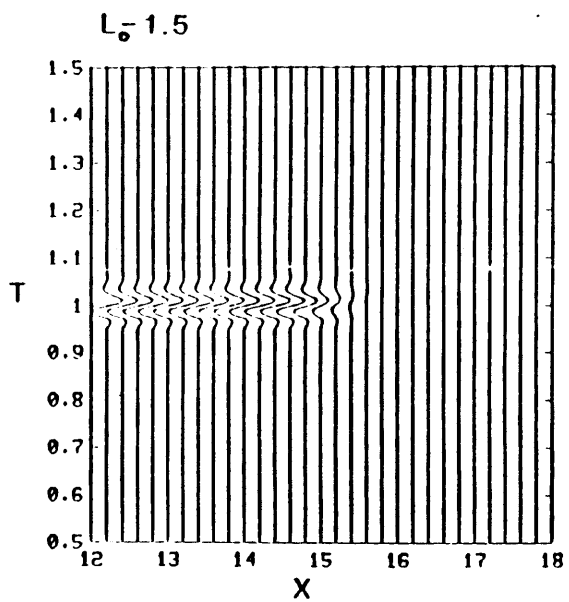


Figure 4



GB SUM FOR LINE SOURCE

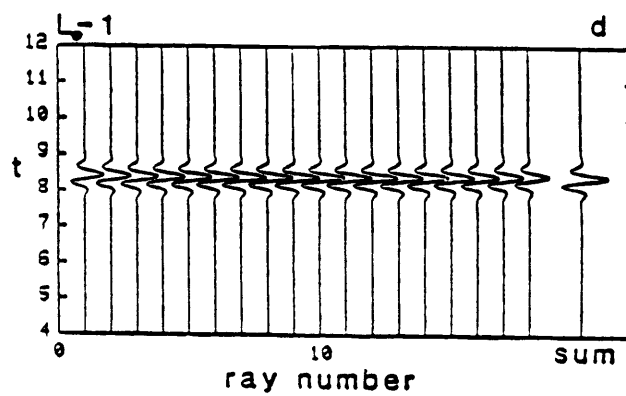
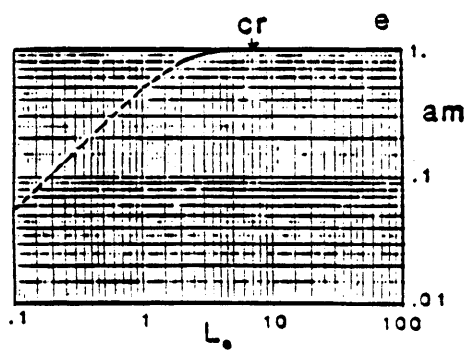
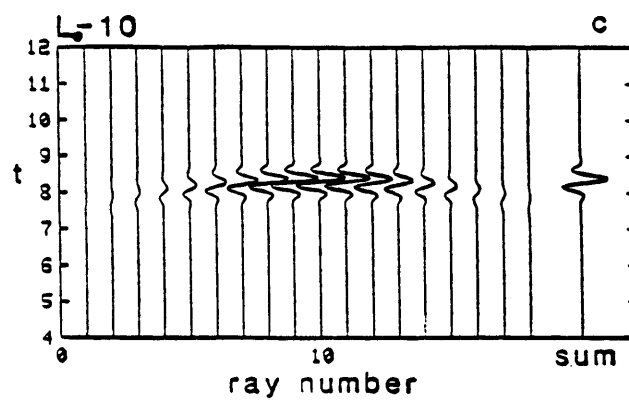
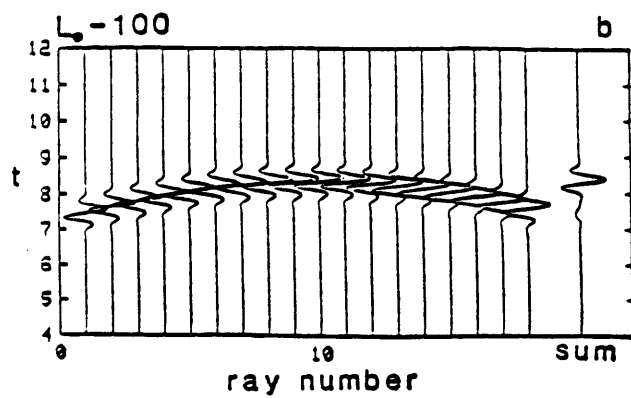
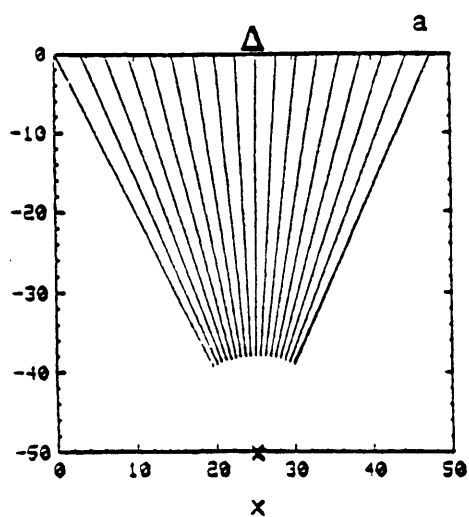


Figure 5

SH LINE
SOURCE

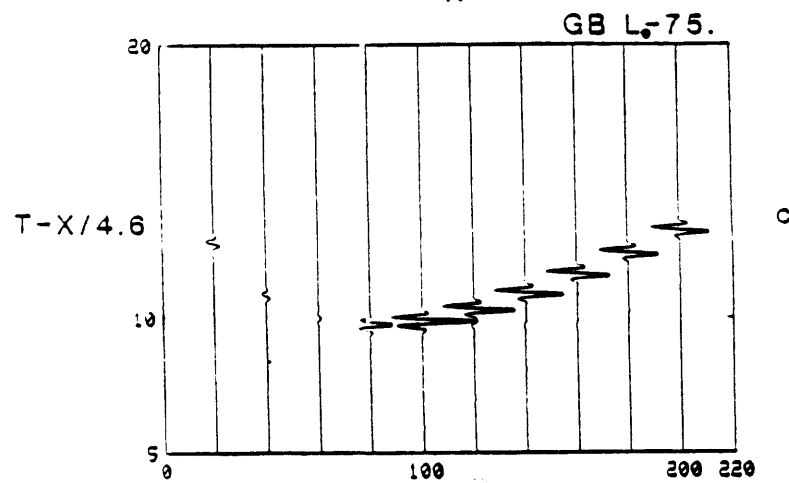
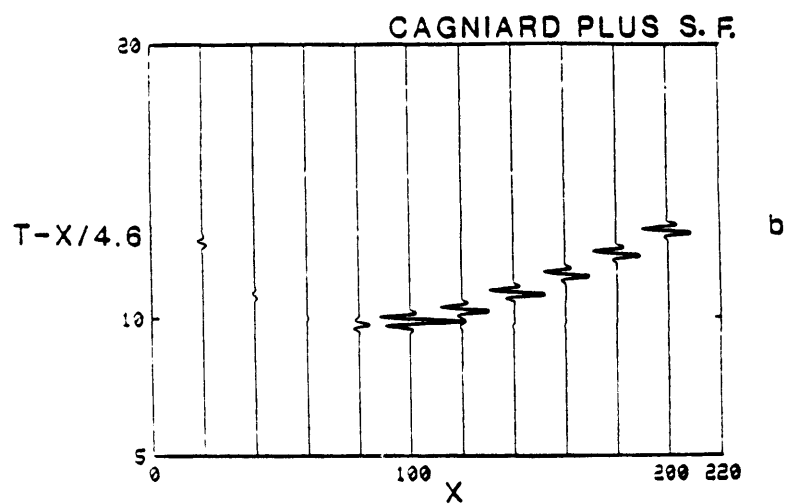
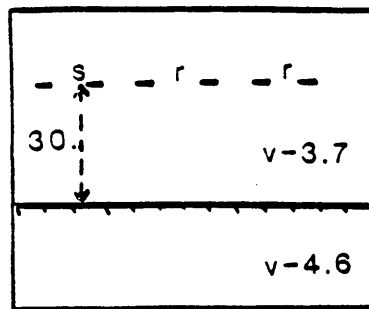


Figure 6

SH LINE
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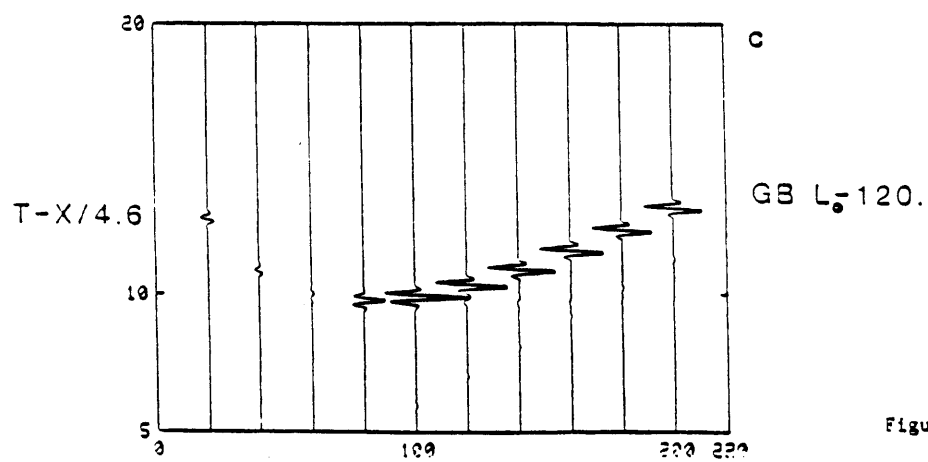
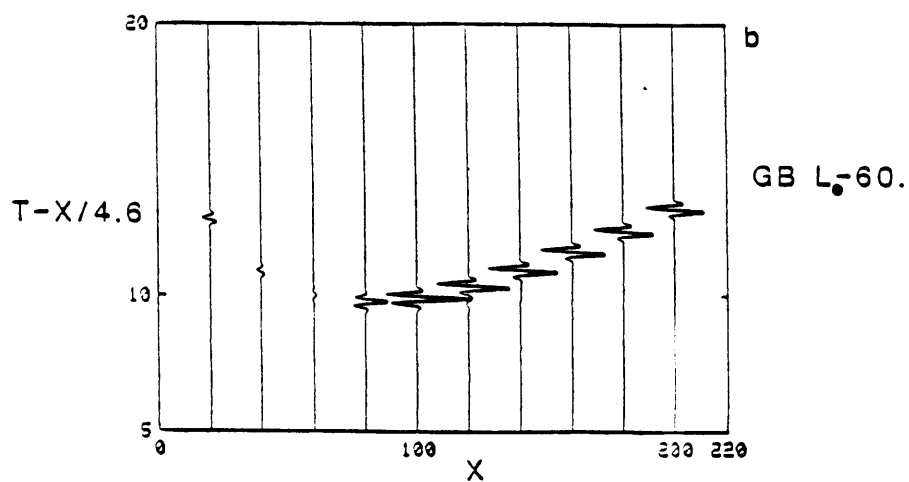
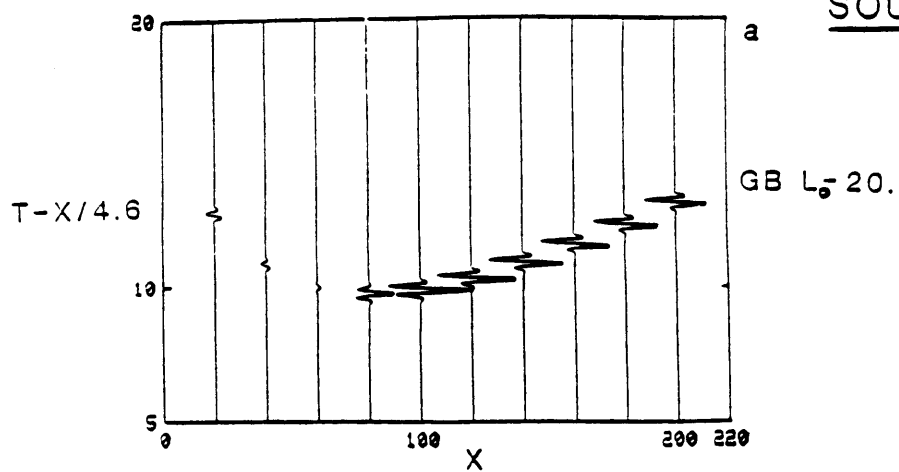


Figure 7

LAYER OVER GRADIENT

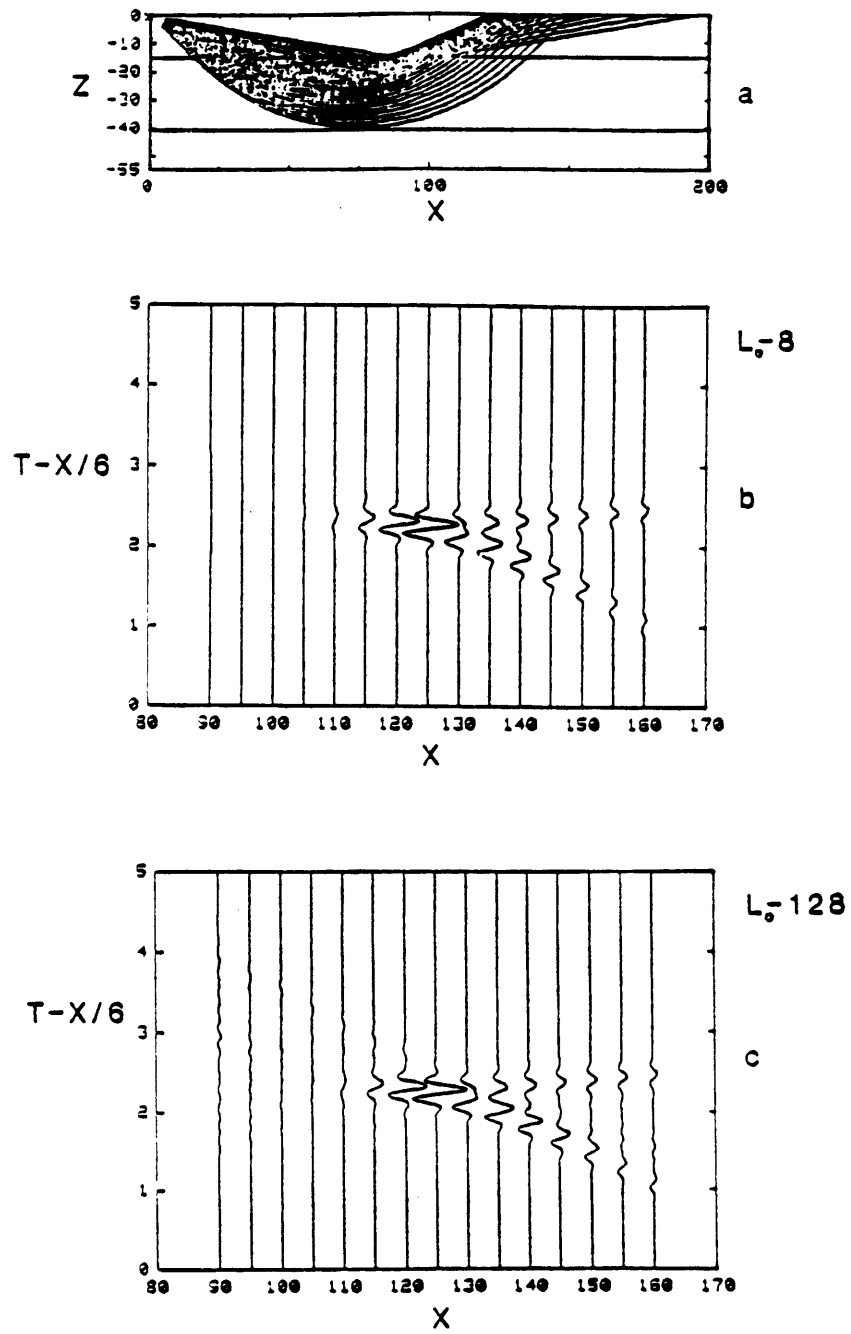


Figure 8

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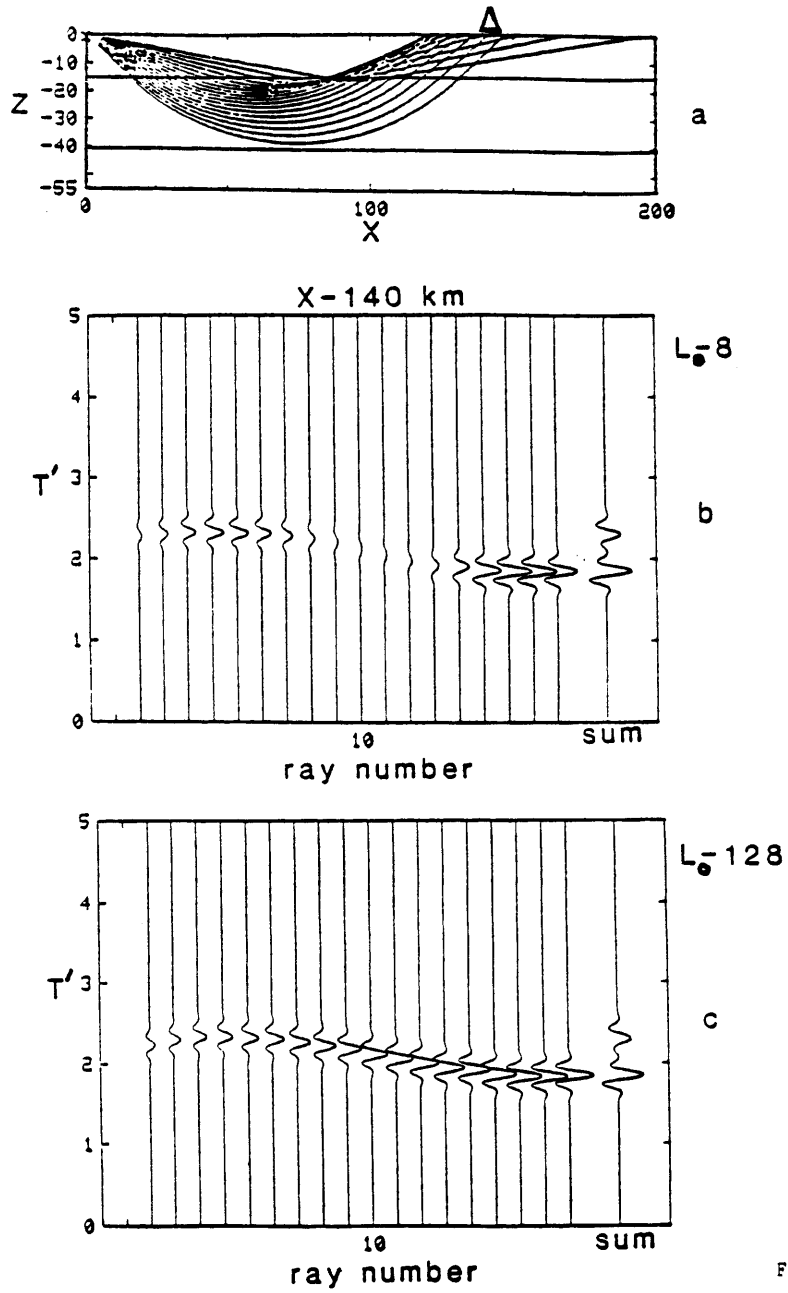


Figure 9

LAYER OVER GRADIENT

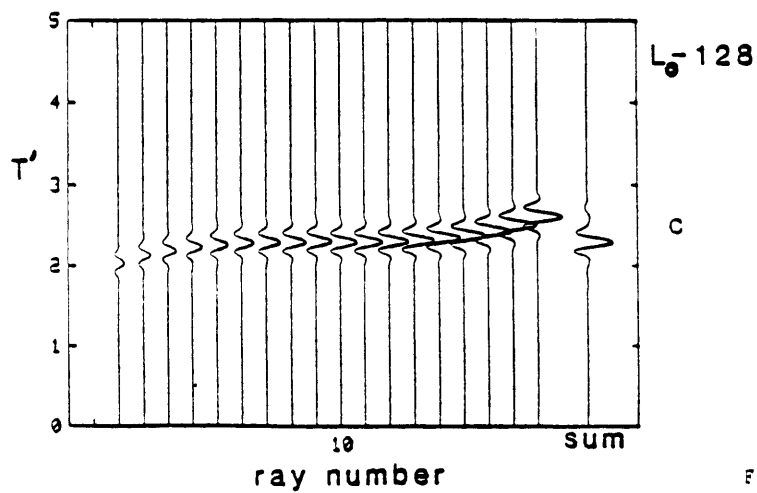
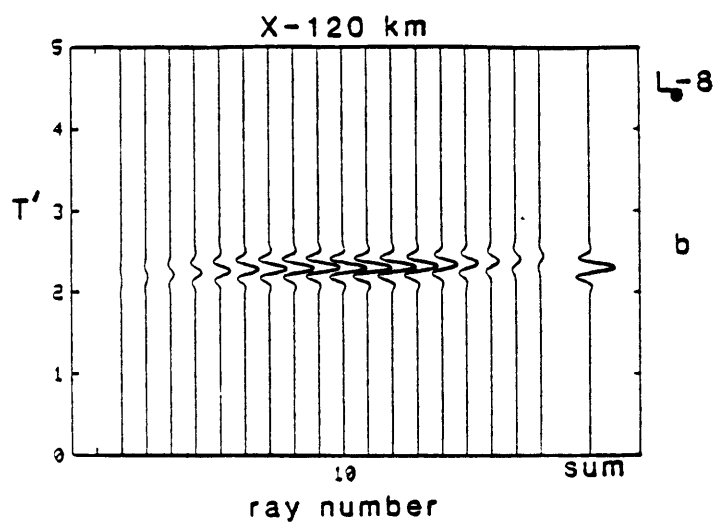
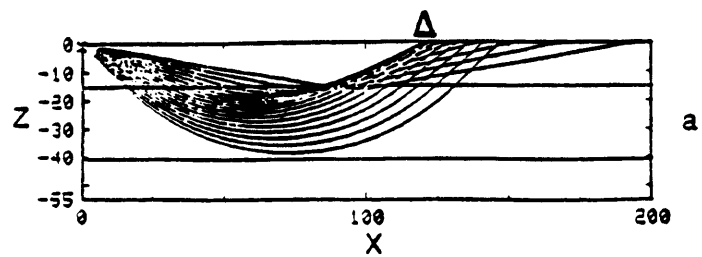


Figure 10

LAYER OVER GRADIENT

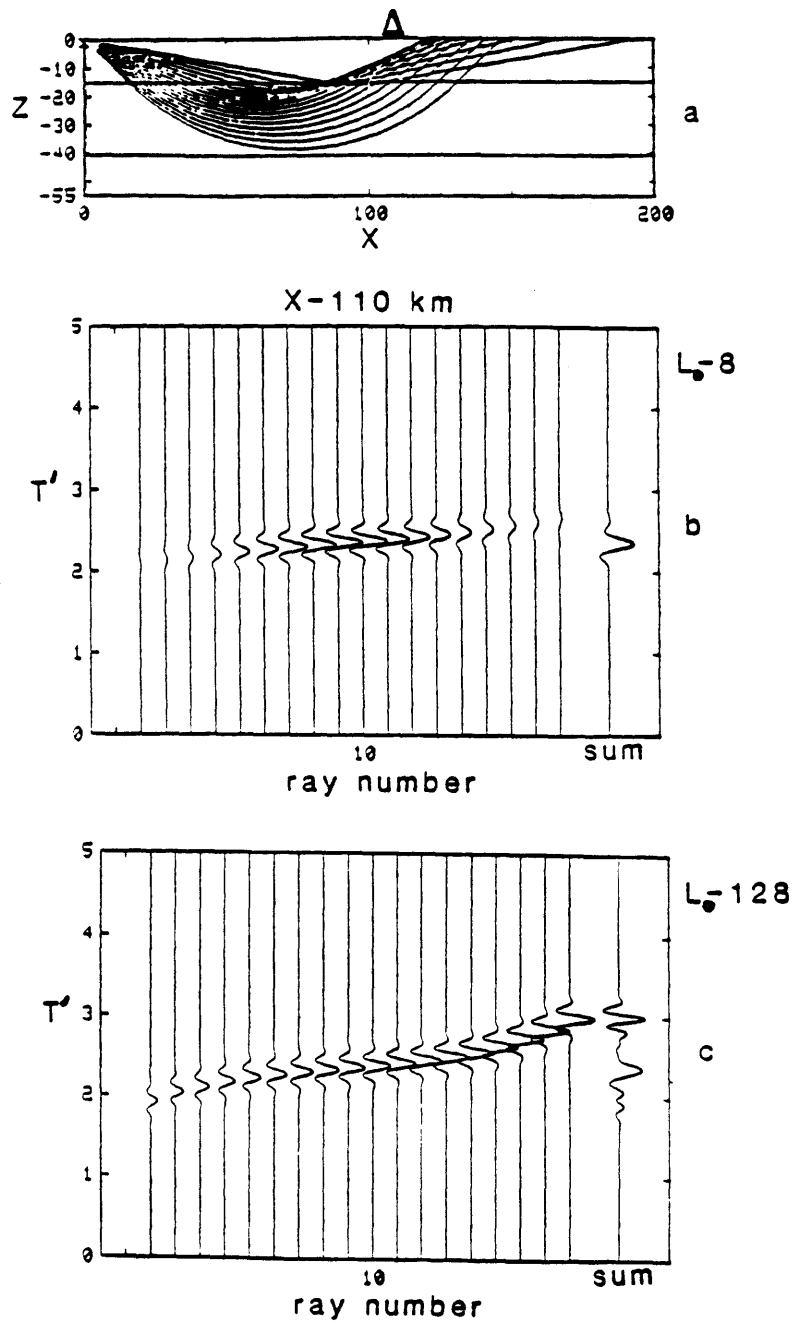


Figure 11

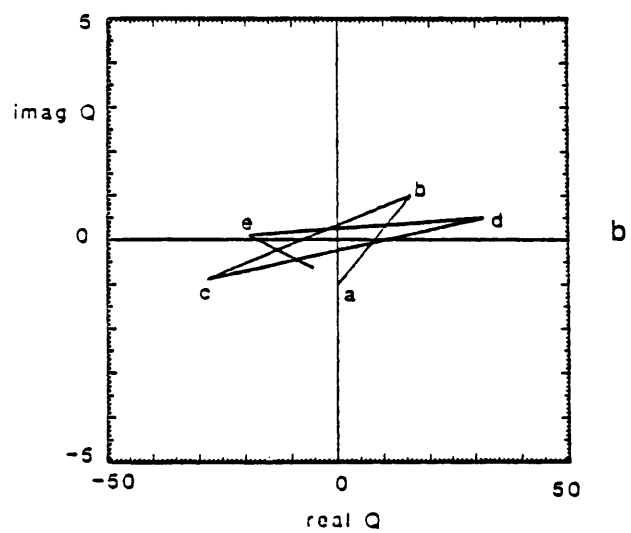
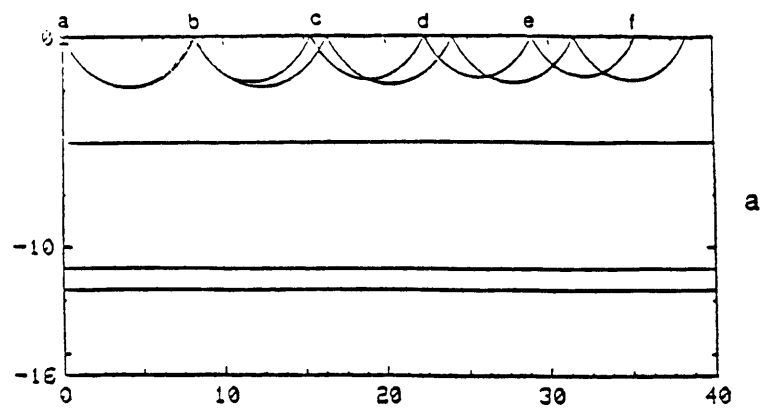


Figure 12

SINGLE INHOMOGENEITY

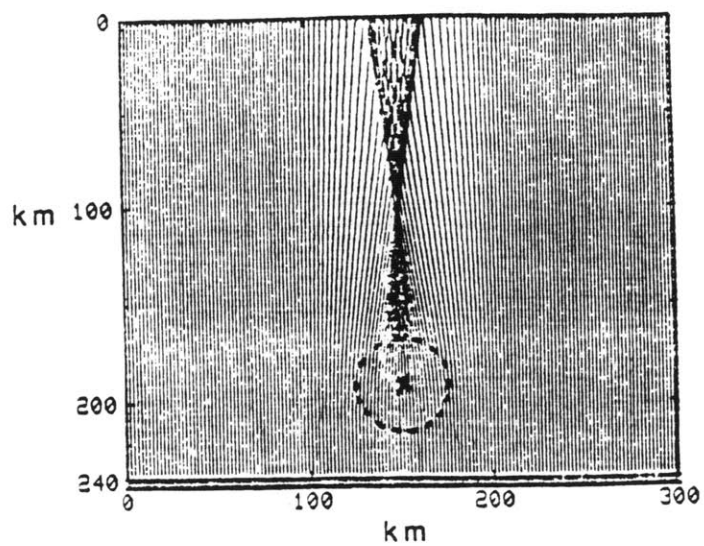
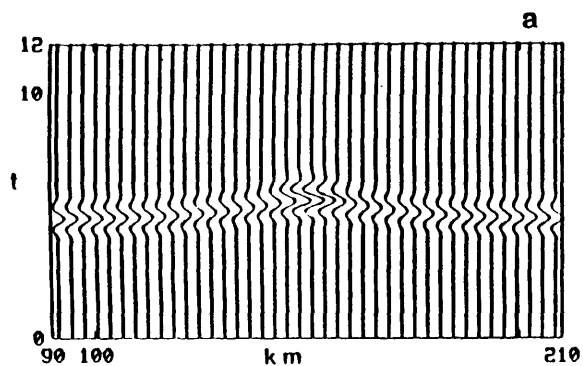


Figure 13

SINGLE INHOMOGENEITY

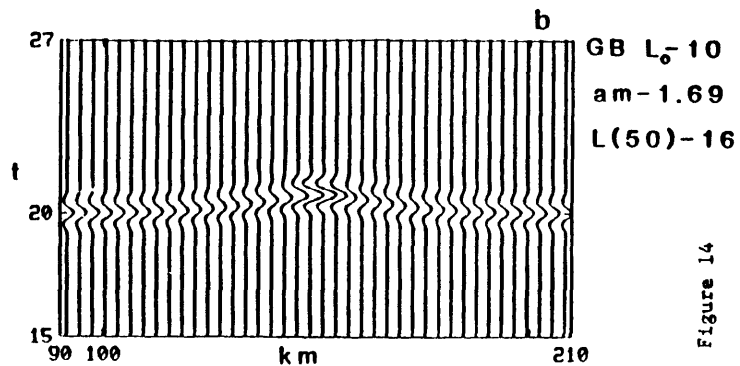
a-25 km A1

v 6-.6 km/sec



PARAB45

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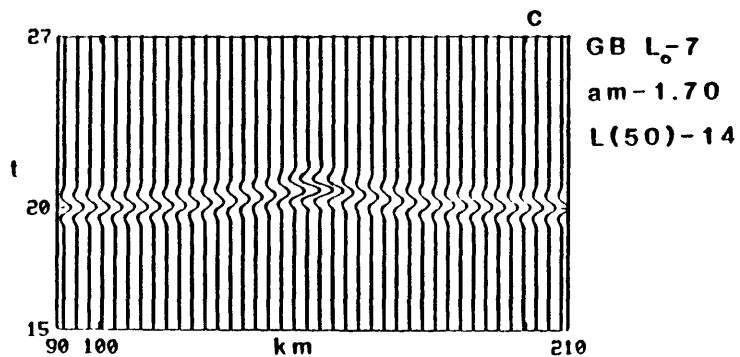


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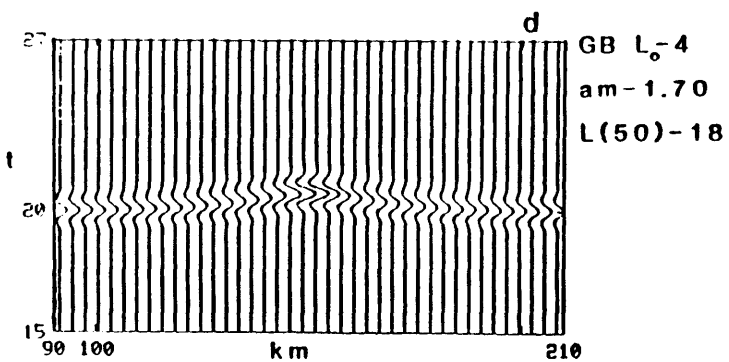
Figure 14



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L(50)-14



GB L_o-4

am-1.70

L(50)-18

SINGLE INHOMOGENEITY

a-25 km B1

v 6-.6 km/sec

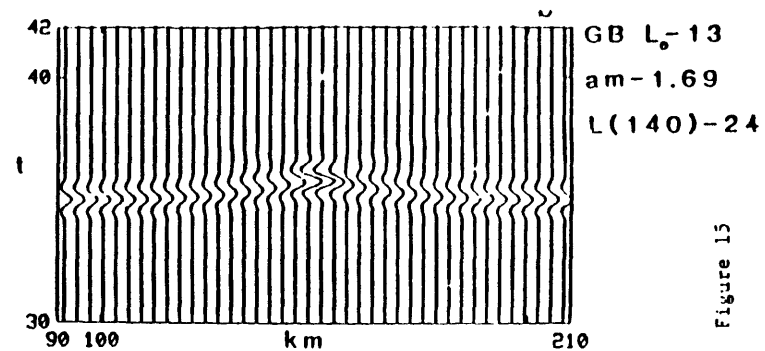
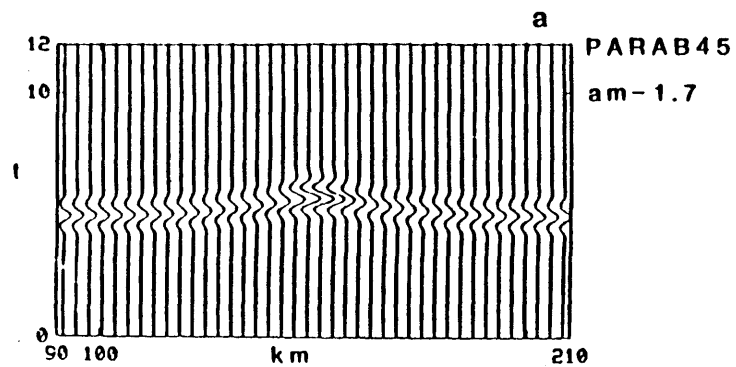
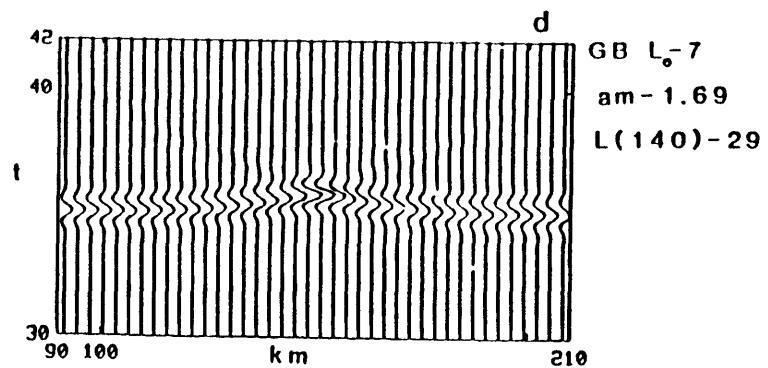
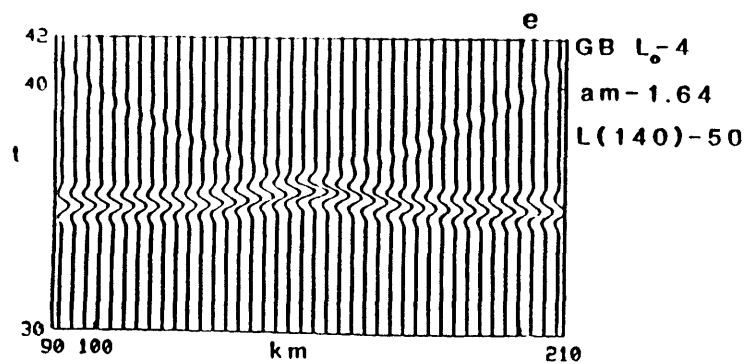
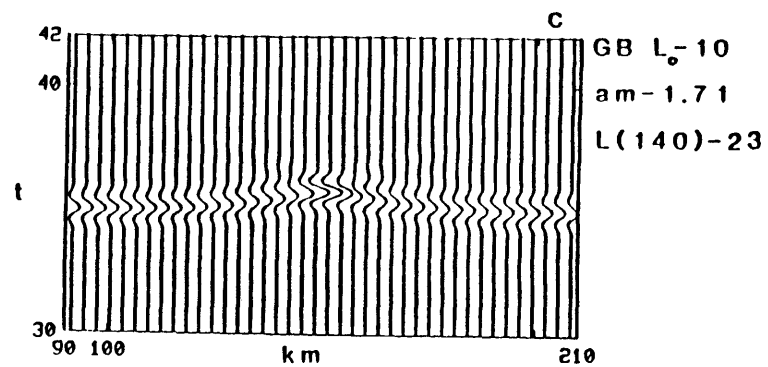


Figure 15



SINGLE INHOMOGENEITY

a-15 km A2

v-6.-.6 km/sec

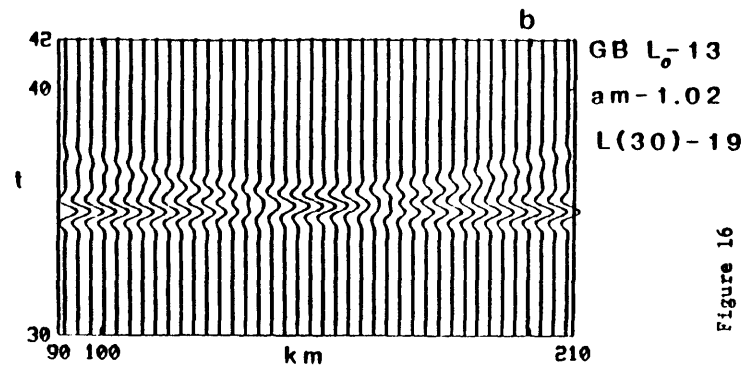
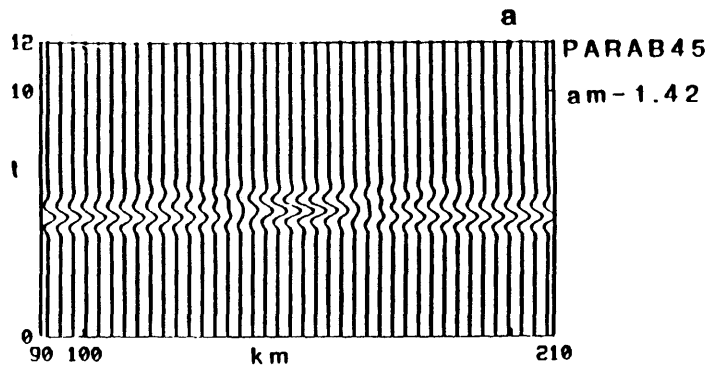
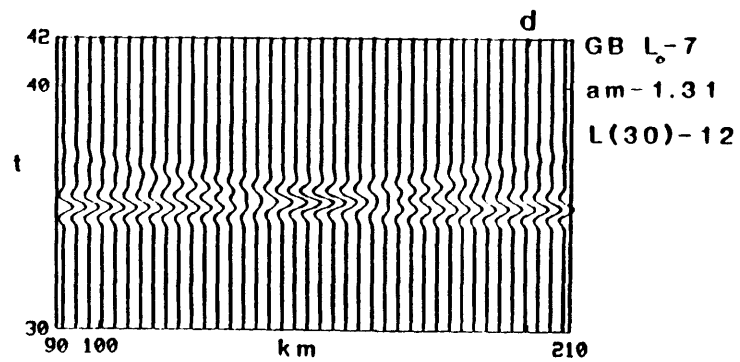
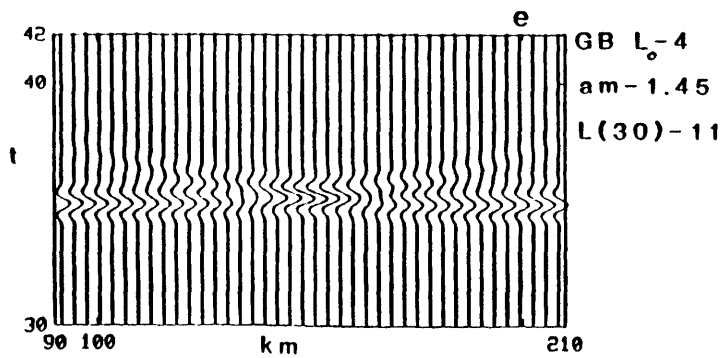
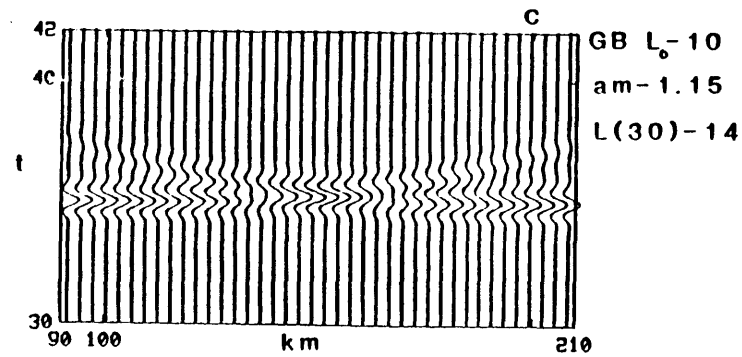


Figure 16



BASIN STRUCTURE

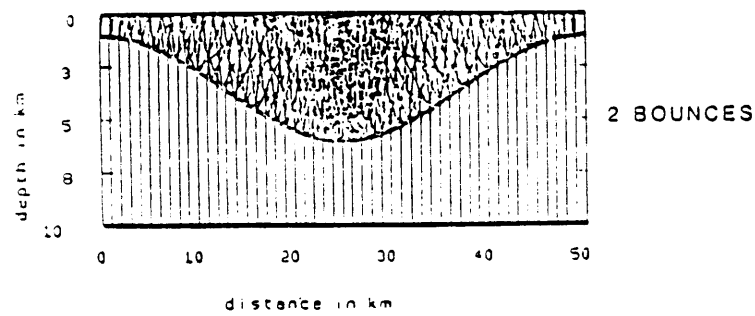
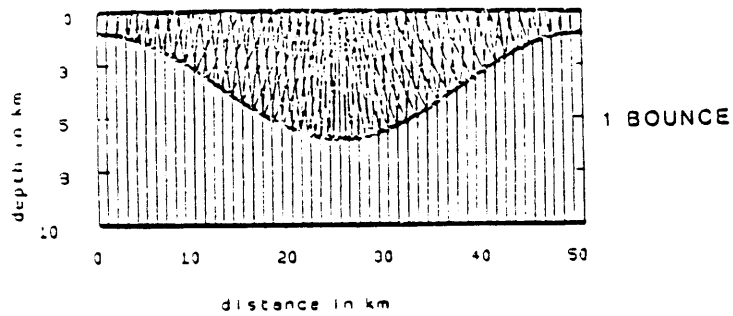
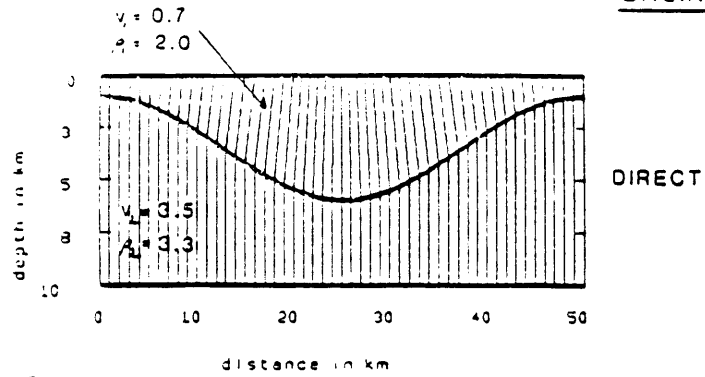


Figure 17

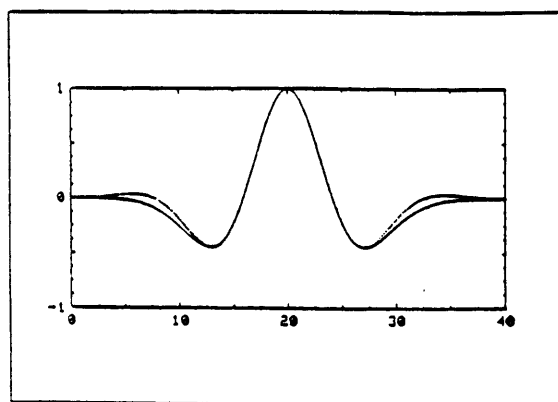


Figure 18

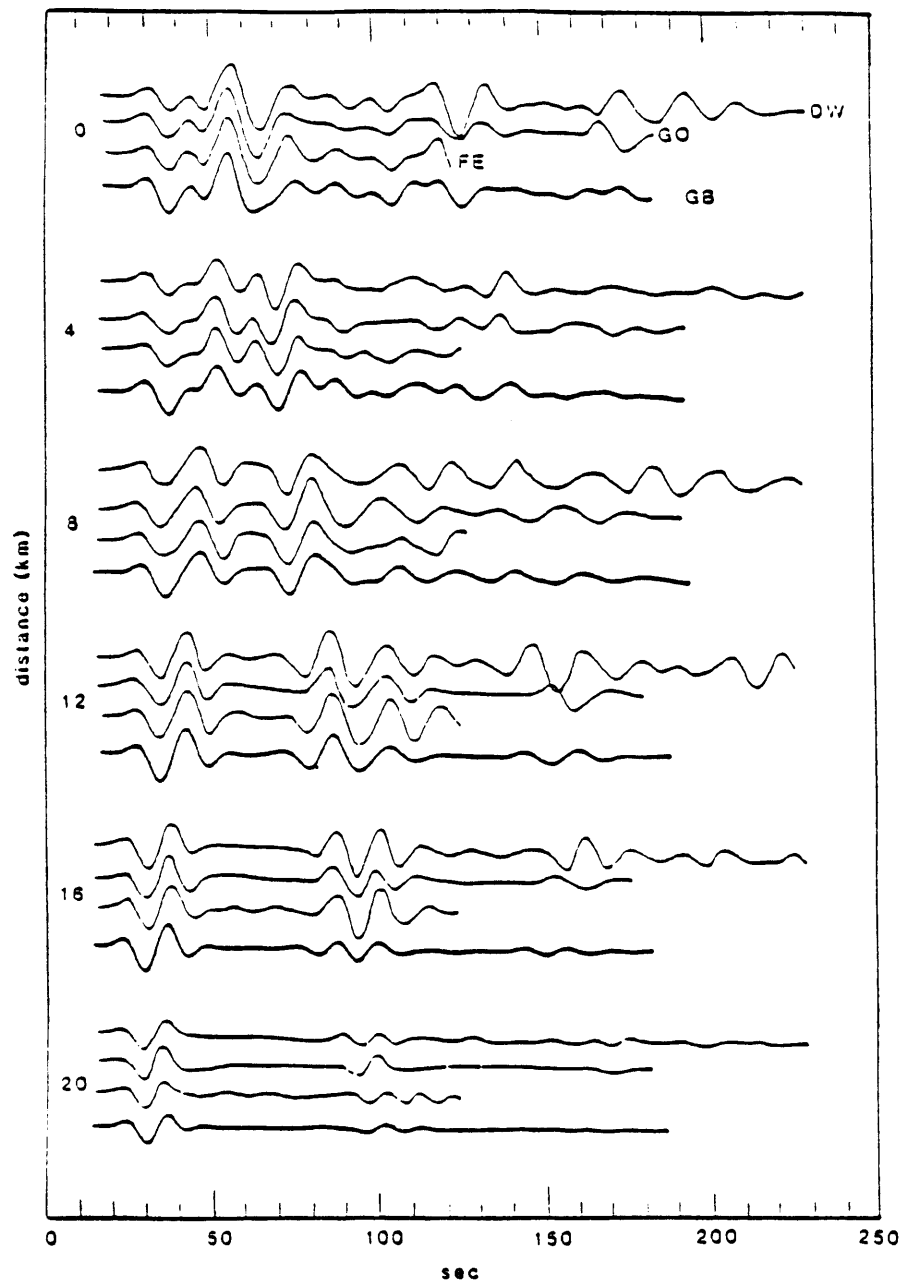
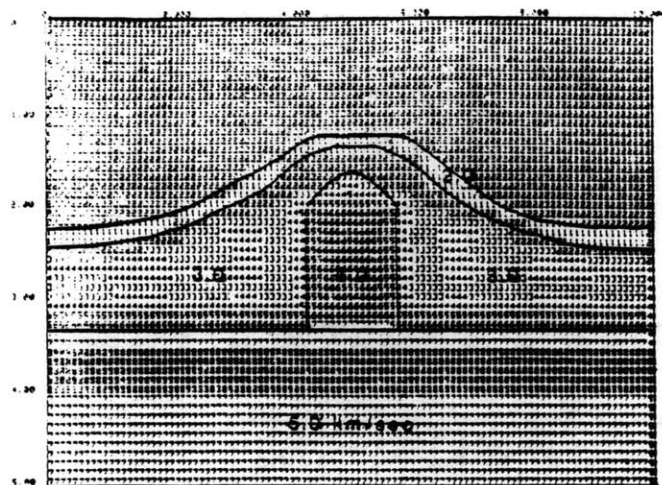
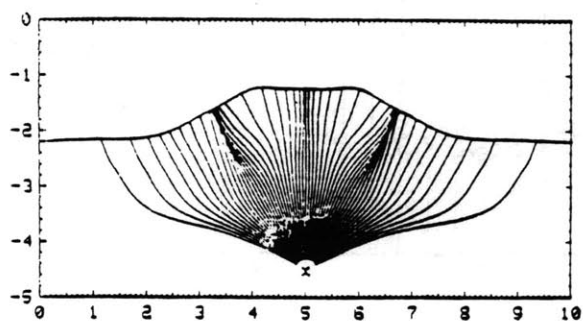


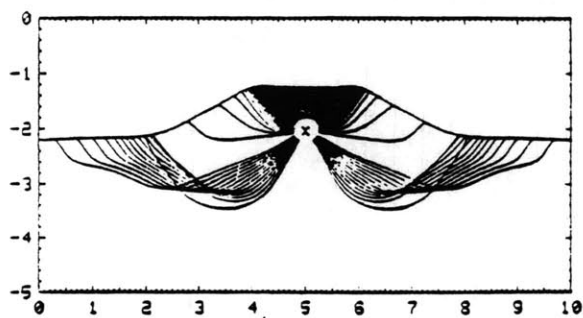
Figure 19



MT ST HELENS STRUCTURE



deep source
3.5 km
beneath summit



shallow source
1.0 km
beneath summit

Figure 20

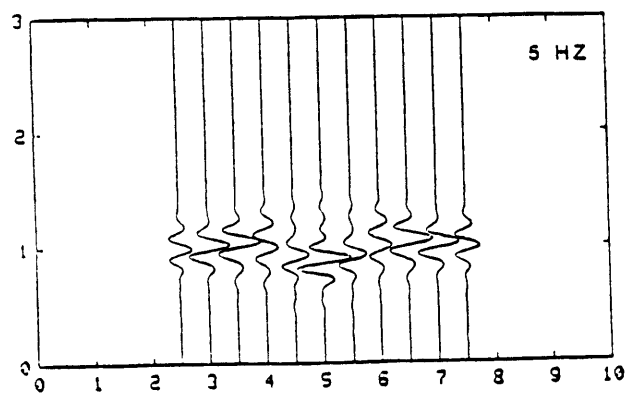
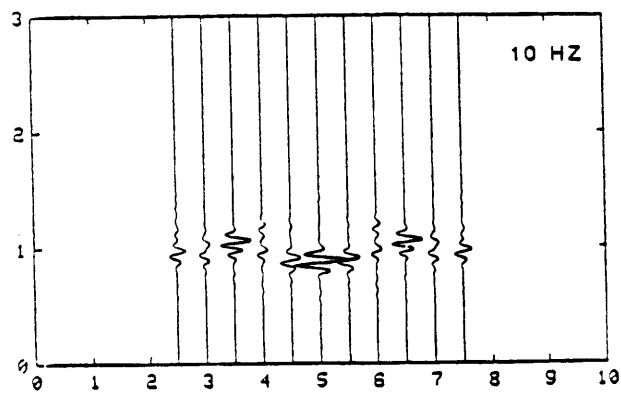
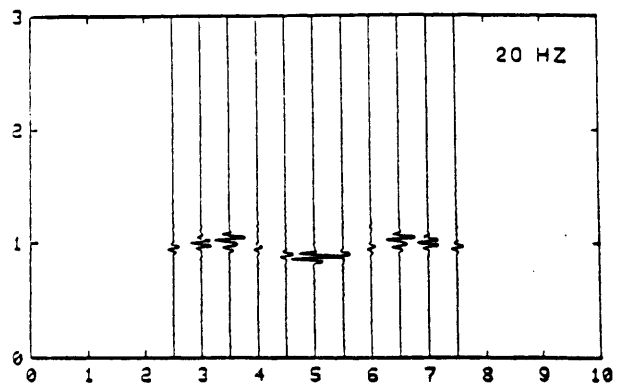


Figure 21

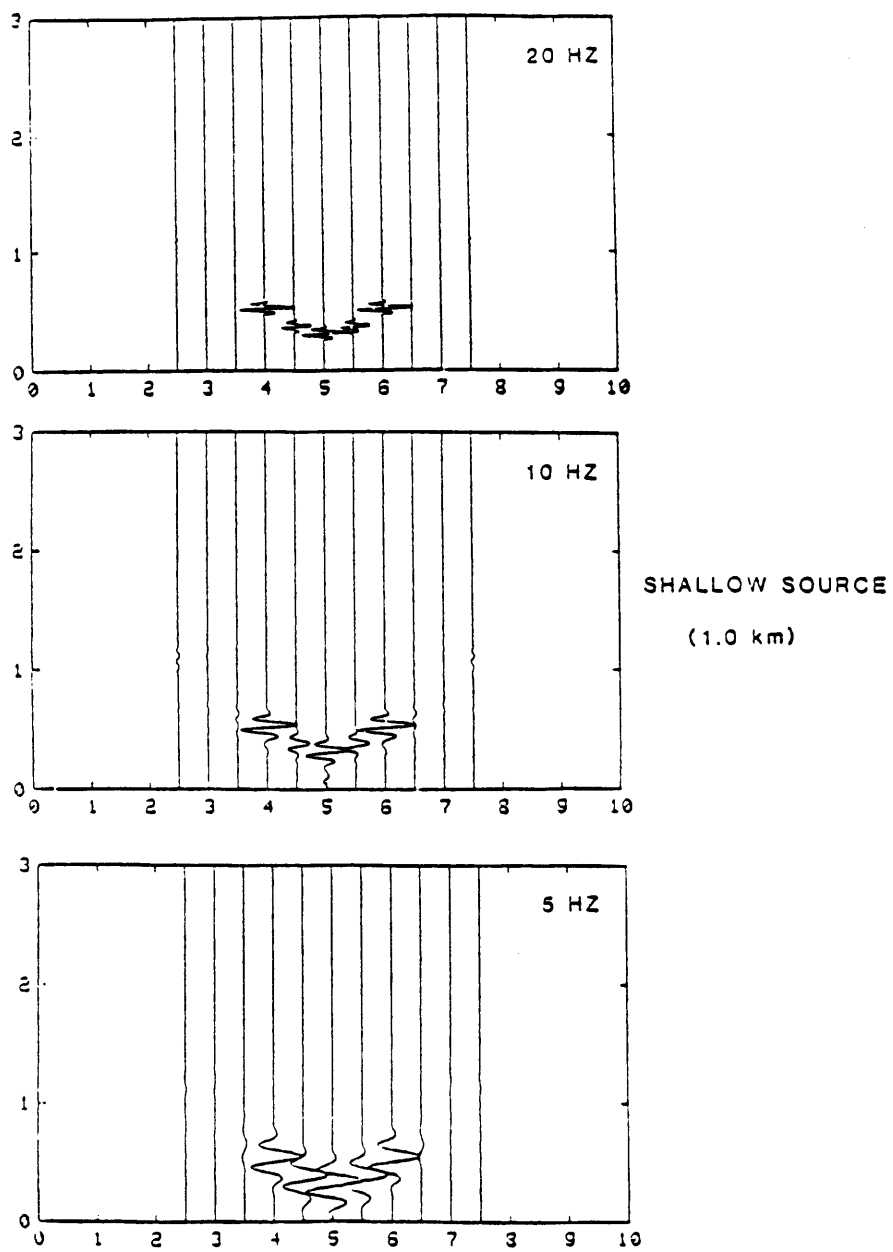


Figure 22

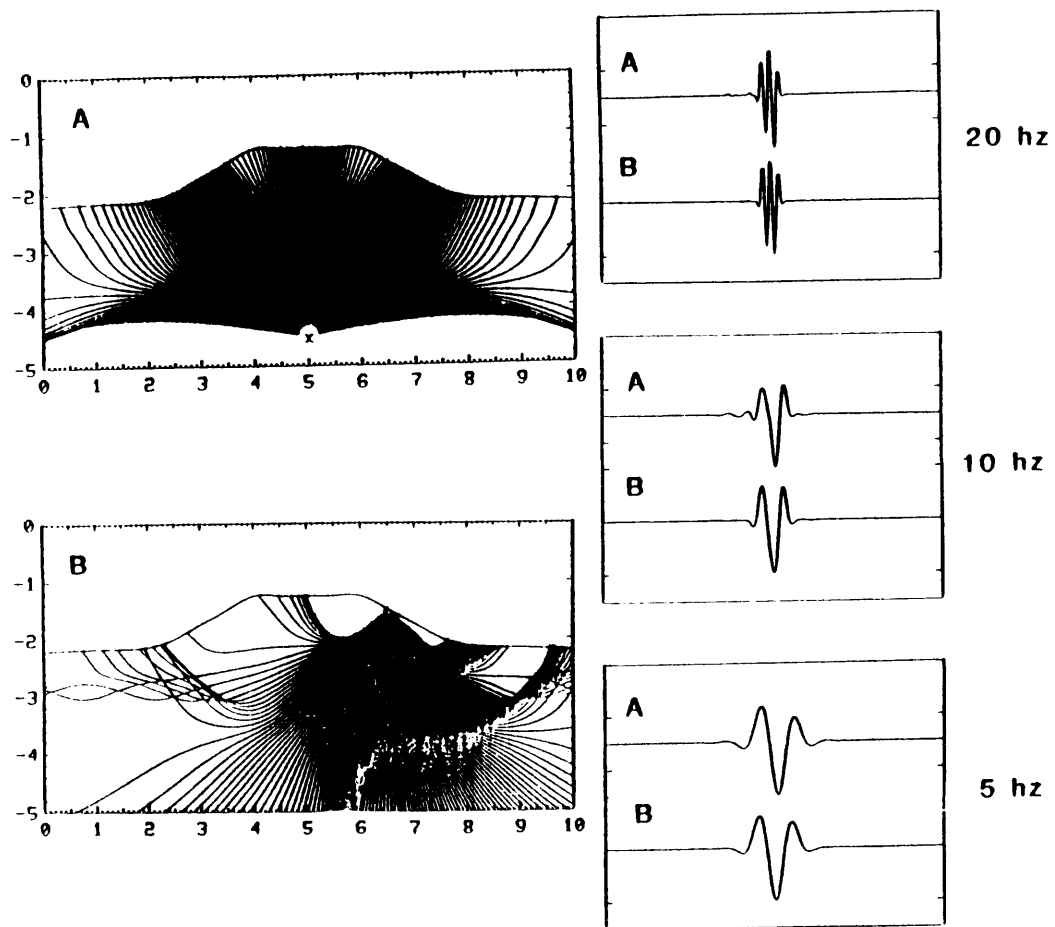
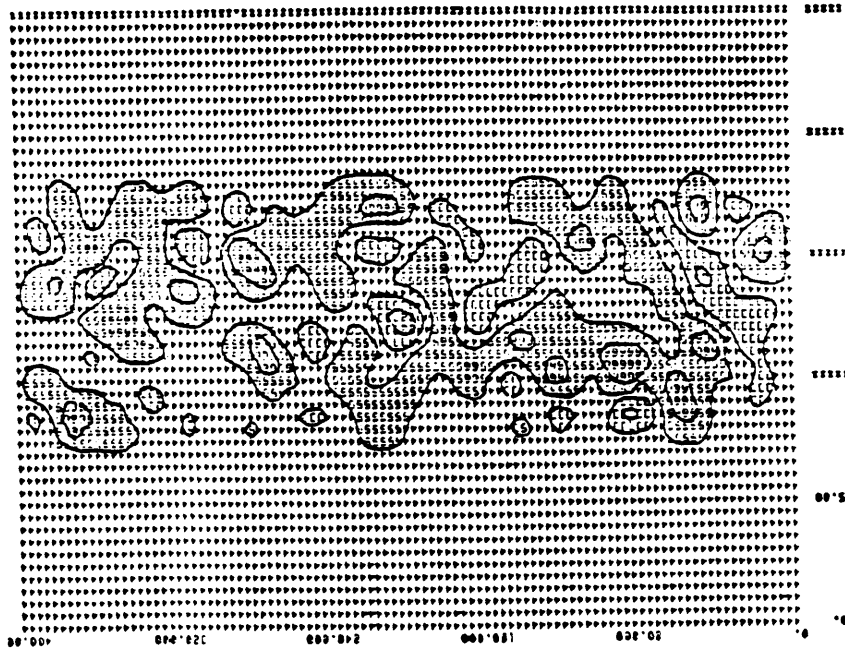


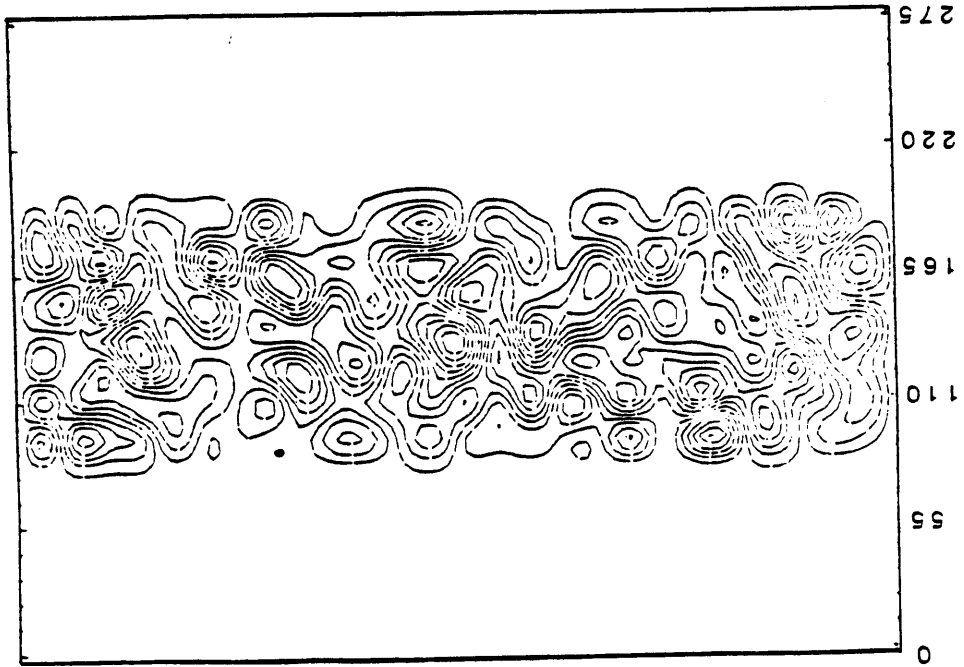
Figure 23

Figure 24

contour
int .3



contour
int .1



random velocity layer
8 km/sec $\pm$ 3%, a = 15 km
seed = 0

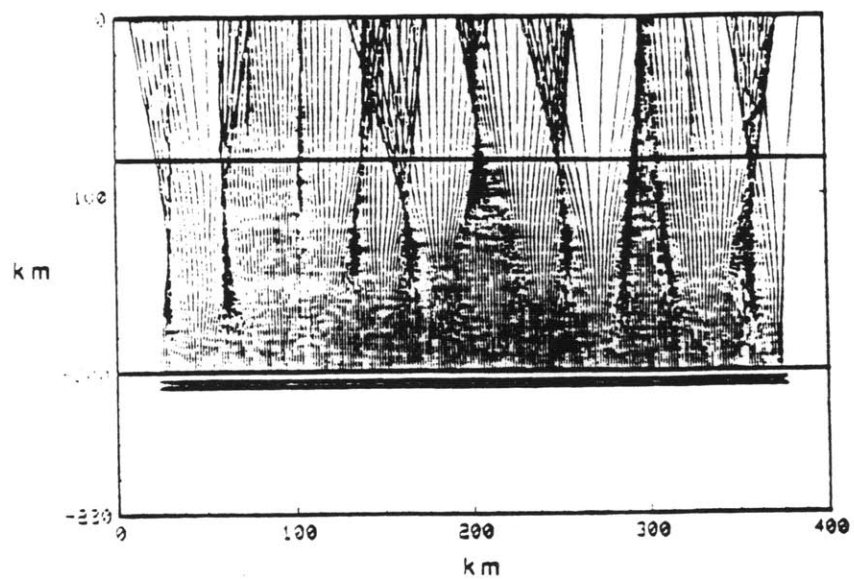


Figure 25

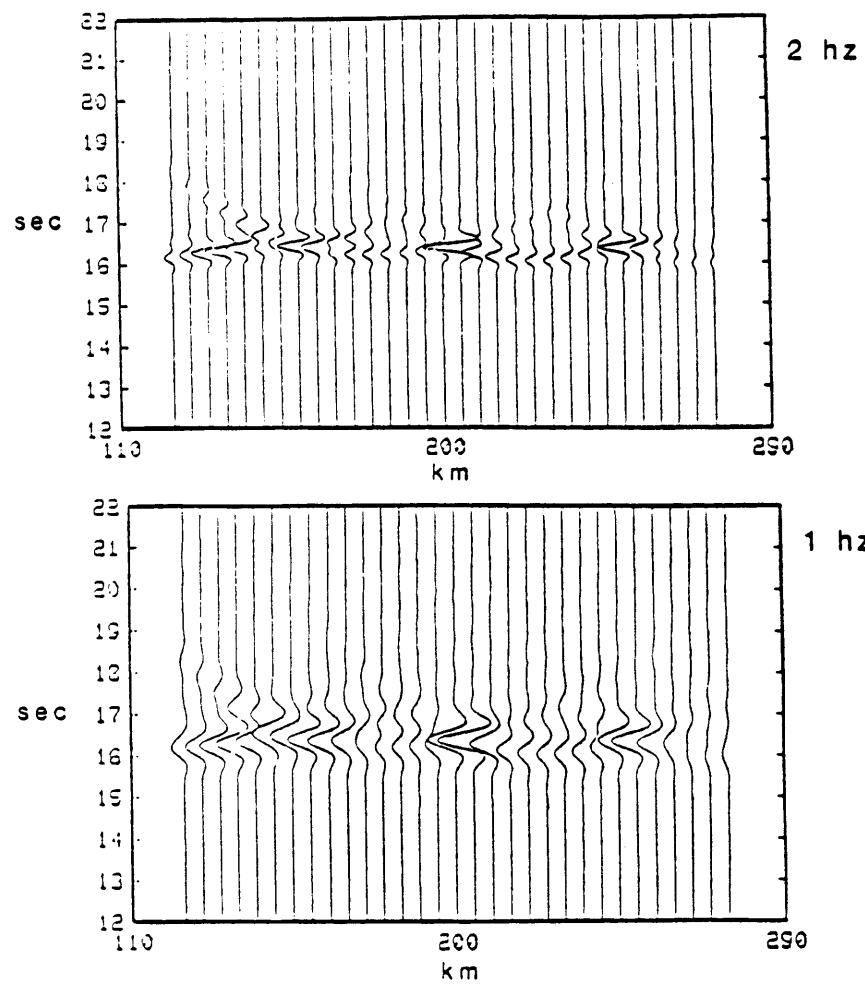
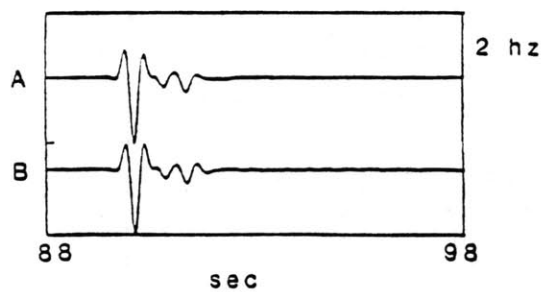
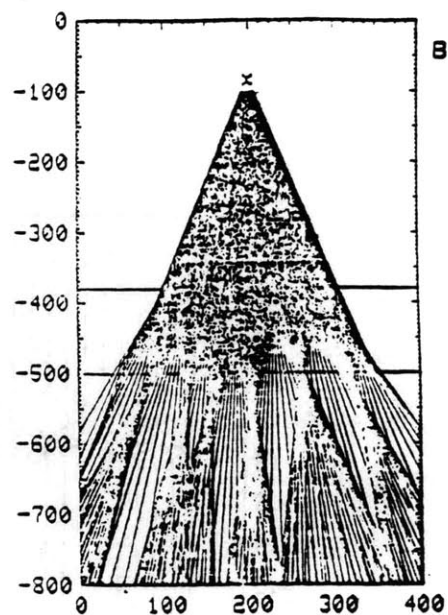
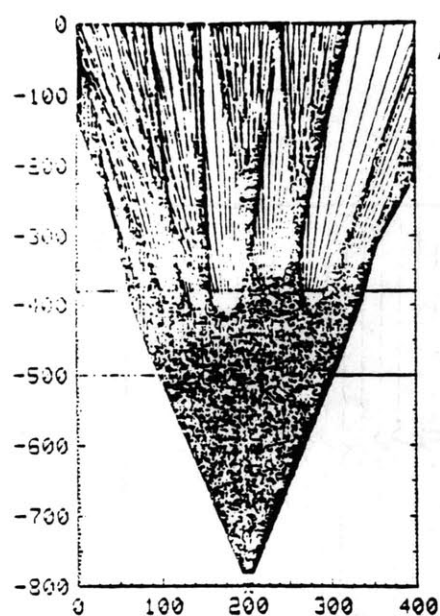
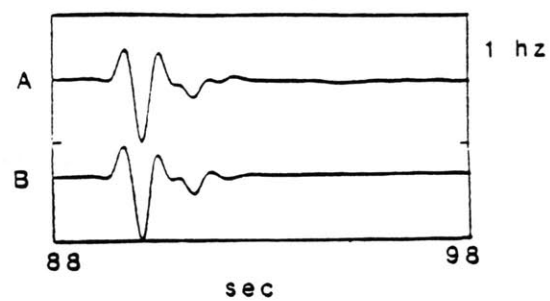


Figure 26



A) S-(200,800)
R-(200,80)



B) S-(200,80)
R-(200,800)

Figure 27

RANDOM LAYER

iseed=0

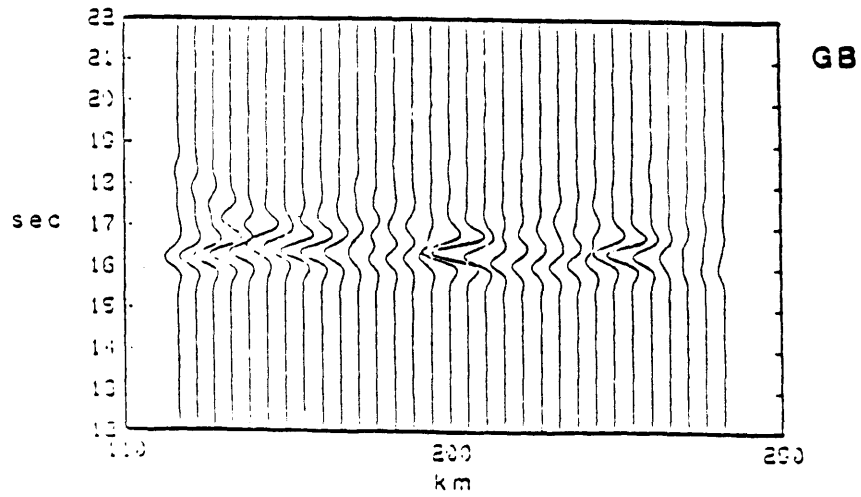
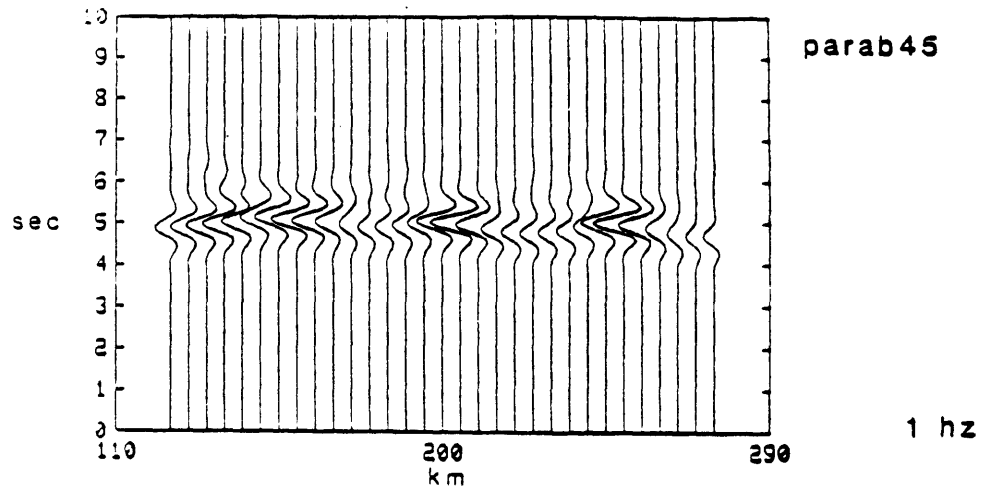
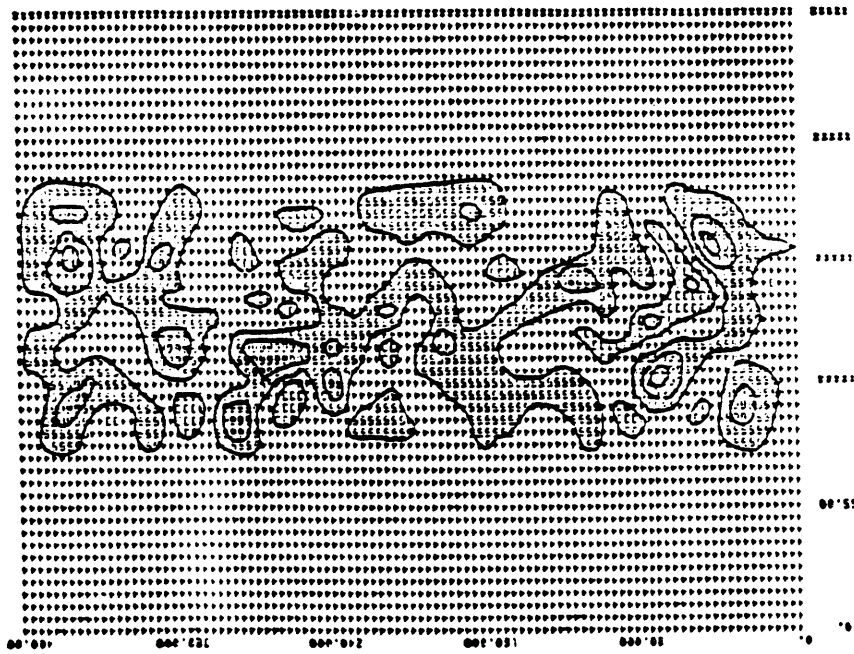


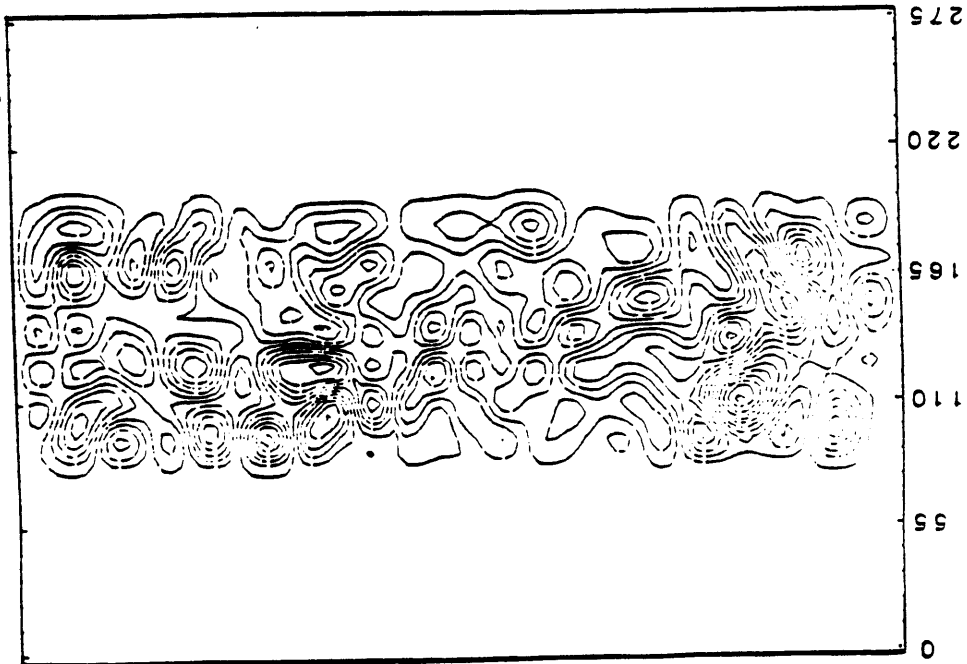
Figure 23

Figure 29

contour
int .3



contour
int .1



8 km/sec±3%, a-15 km
isec-6

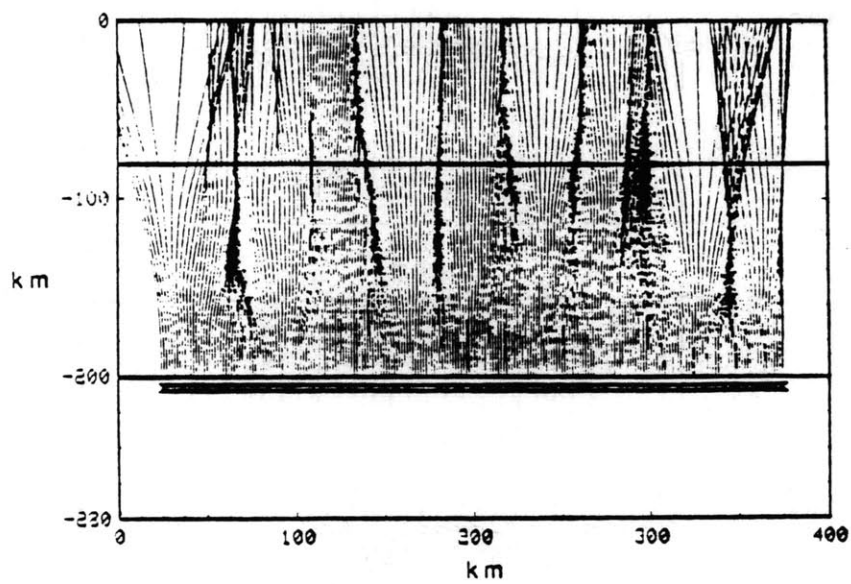


Figure 30

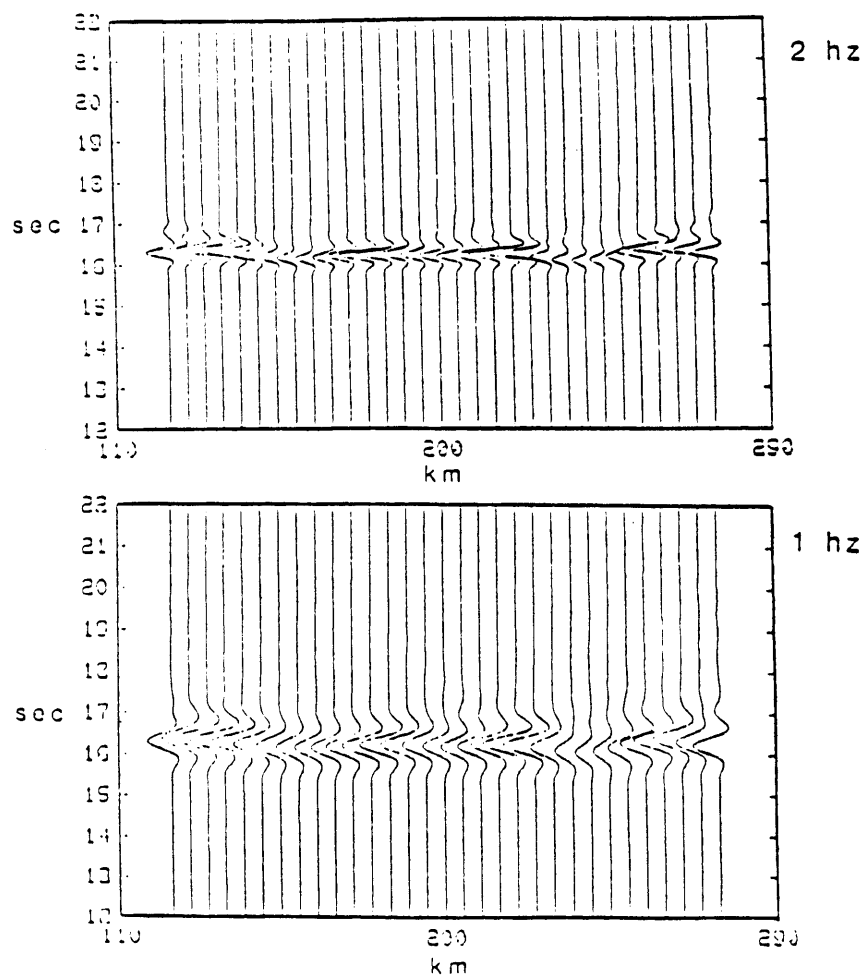
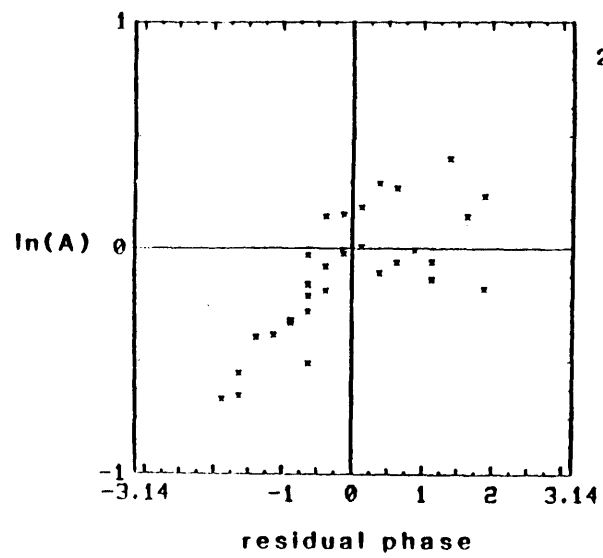
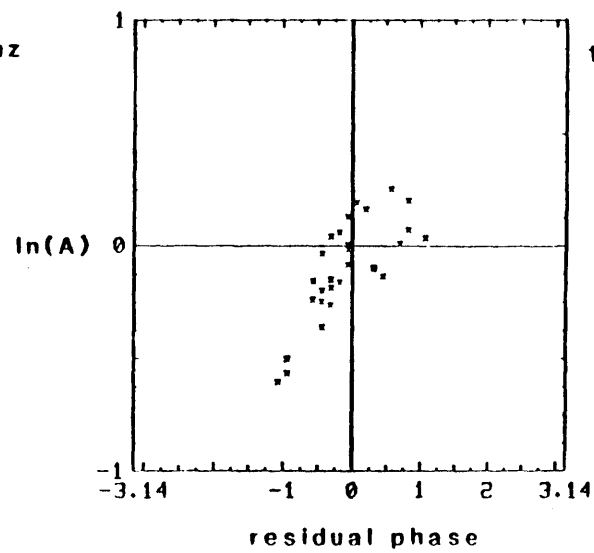


Figure 31



2 Hz



1 Hz

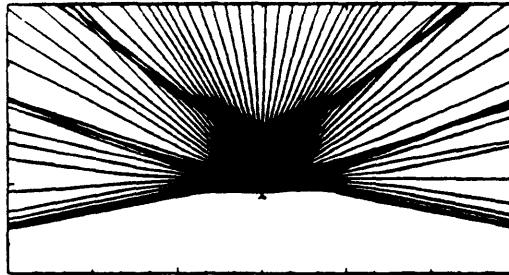
Figure 32

Chapter 3

Topics in Gaussian beam synthesis

by

Robert Nowack



Abstract

In this paper, several topics in Gaussian beam synthesis are described. This includes the choice of the complex beam parameter, ϵ , and its interpretation in terms of a complex source point. Asymptotic and exact evaluations of the Gaussian beam integral for a line source are then investigated. Next, the problem connected with an initial point in a velocity gradient is discussed, which in a vertically varying media reduces to the difference between an initial plane wave and an initial Snell wave. A comparison is done with the reflectivity method for the PP phase in the Imperial Valley structure. Then, a comparison is then done between the slant stack of a wavefield and the Gaussian beam decomposition using large beam widths, showing good agreement. Beams reflected at an interface are then investigated using a first order correction to the plane wave reflection coefficient typically used in the Gaussian beam method. Finally, the finite frequency scattering effects from an indented interface are computed using the Gaussian beam method and compared with those computed using the Aki-Larner method.

Introduction

Many decompositions of wavefields are possible in terms of local, global, or intermediate building blocks. These building blocks include rays, modes, and beams. In this paper we investigate several topics in the approximate decomposition of wavefields in terms of beams. These topics include the specification of the beam parameter, ϵ , and its interpretation in terms of a complex source point. Asymptotic and exact evaluations of the Gaussian beam integral for a line source are then investigated. Then, the problem connected with the decomposition of a point source in a velocity gradient is discussed, and reduces to the difference between an initial plane wave and an initial Snell wave. A comparison with the PP phase in the Imperial Valley structure is then made. Next, a slant stack of a wavefield is computed and compared with the beam decomposition for a large beam width, showing good agreement. A first order correction to the plane wave reflection coefficient typically used at interfaces in the Gaussian beam method is next described. Finally, the finite frequency scattering effects from a dented interface are computed using the Gaussian beam method and compared with the effects computed using the Aki-Larner method.

Choice of the complex parameter ϵ

Each Gaussian beam in the Gaussian beam superposition requires the choice of a complex parameter, ϵ . To be consistent within the superposition, one value of ϵ is usually chosen for all beams in the sum. The real part of ϵ shifts the beam waist away from the start point on the ray. The imaginary part of ϵ specifies the beam width at the beam waist. The appropriate range of values for ϵ was shown by Nowack and Aki [1984] to depend on the type of initial wavefield to be approximated by Gaussian beams. When decomposing an initial plane wave into Gaussian beams, useful results with a finite aperture occur for small $|\epsilon|$. In the limit as $|\epsilon| \rightarrow 0$, the Gaussian beam superposition reduces to a Kirchhoff integral with a paraxially approximated Green's function. When decomposing a point source into Gaussian beams, useful results occur for large $|\epsilon|$ (see Nowack and Aki [1984], example C). In the limit as $|\epsilon| \rightarrow \infty$, the Gaussian beam superposition reduces to a plane wave decomposition or a Weyl integral for a point or line source. A finite value of $|\epsilon|$ in the above cases has the result of smoothing over diffraction effects (see Nowack and Aki [1984], example B for the diffraction effects of a half plane and example D for the generation of a head wave). A finite $|\epsilon|$ compresses the energy to a vicinity of the stationary points. The choice of ϵ depends on what part of the wavefield one is interested in modelling. The critical value of $\text{Im}(\epsilon)$ was used by Nowack and Aki [1984] (or the optimal value in the terminology of Cerveny et. al. [1982]) as a soft cutoff between large and

small values of $|\epsilon|$ for a given source and receiver geometry. The above results are summarized in Figure 1. Nowack and Aki [1984] also found in their Example G that the beam width should be smaller than the heterogeneity scale. Moving the beam waist by changing the real part of ϵ may be useful particularly when a localized heterogeneity exists. The beam waist can then be moved to the region of the heterogeneity. For heterogeneities distributed along the whole path, the beam waist is typically put at the source or the receiver.

The complex source point approach to Gaussian beams

In this section, Gaussian beams resulting from a complex source coordinate is described. The complex parameter, ϵ , for each Gaussian beam in the Gaussian beam synthesis can be considered as a complex distance from a source point in complex space to a start point along the ray in real space. Thus

$$\epsilon = d - ib \quad (1)$$

where d is the real distance between the source point, or the beam waist, and the start point and corresponds to S_0 in the notation of Cerveny et. al [1984]. When ϵ is real, it corresponds to the radius of curvature of the wavefront at the start point. For a complex ϵ , d can be used to shift the beam waist along the ray. Putting the source point in complex space by

specifying $b \neq 0$ results in an evanescent wavefield which attenuates along the wavefront.

The complex source point approach to constructing evanescent wavefields, in particular Gaussian beams, has been described by Keller and Streifer [1969], Deschamps [1971], Shin and Felsen [1977], and Wu [1985]. Here, the 2-D Helmholtz equation is considered where the Green's function is specified from

$$(\nabla^2 + k^2) G(r, r') = -\delta(r, r')$$

and

$$G(r, r') = i/4 H_0^{(1)}(kD)$$

where $D = ((x - x')^2 + (z - z')^2)^{1/2}$. For $kD \gg 1$,

$$G(r, r') \approx 1/4 (2/\pi kD)^{1/2} e^{ikD - i\pi/4} \quad (2)$$

The Green's function is now analytically extended by using a complex source point. Thus $r' \rightarrow \hat{r}'$. This can be done on either the exact or an approximate Green's function. For example, let $r' \rightarrow \hat{r}' = (0, 0 + ib)$, then

$$D \rightarrow \hat{D} = [x^2 + (z - ib)^2]^{1/2} \quad \text{Re } \hat{D} > 0$$

In the paraxial region of the z axis

$$\hat{D} \approx (z - ib) \left[1 + 1/2 \left(\frac{x}{z - ib} \right)^2 \right]$$

or

$$\hat{D} = z - ib + \frac{z x^2}{2(z^2 + b^2)} + \frac{i b x^2}{2(z^2 + b^2)}$$

The analytically extended Green's function in the paraxial region is then

$$\hat{G}(r, r') \approx C/4 \frac{e^{ik(z-ib)}}{\sqrt{\pi k (z-ib)}} e^{\frac{ikxz^2}{2(z^2+b^2)}} e^{\frac{-kbz^2}{2(z^2+b^2)}} \quad (3)$$

This is a Gaussian beam where C is a normalizing constant. The beam curvature is

$$K(z) = \frac{1}{z(1 + b^2/z^2)}$$

and the beam width is

$$L(z) = (2b/k)^{1/2} (1 + z^2/b^2)^{1/2}$$

Comparing with equations (A2.3) and (A2.4) of Nowack and Aki [1984] gives

$$b = \frac{k}{2} L_M^2 = L_0^2$$

where L_M is the initial beam width at the beam waist and L_0 is a normalized initial beam width independent of frequency used by Nowack and Aki [1984]. The complex parameter can then be written in two ways

$$\varepsilon = S_0 - i \frac{k}{2} L_M^2 = d - ib$$

where in the second case ε refers to the complex distance from the source to the start point on the ray.

The fact that an analytically extended Green's function is a Gaussian beam has been used by a number of authors (see for example Felsen [1976], Ra et. al. [1973], Pott and Harris [1984]) to model the effects of beam scattering. The exact or approximate scattering results for a point

source is found and then extended to a complex source point to give the scattering results for a beam. Thus the scattering results for certain types of extended sources, Gaussian beams, can be gotten from the analogous point source problem without performing any integration. In addition the plane wave response can be gotten from the point source response by letting $b \rightarrow \infty$.

In the Gaussian beam synthesis method, a quite different approach is taken where approximate Gaussian beams are traced through the medium and superposed to simulate the scattering effects for a point source or an arbitrary extended source. The complex source point is expected to be useful in evaluating how appropriate the approximate Gaussian beams are in different complicated geologic media.

The use of complex phase

Introducing a finite imaginary part to ϵ results in a complex phase for the component beams in the Gaussian beam synthesis as in eqn. (3). This imaginary phase part will act to modify the integrand of the Gaussian beam synthesis. In order to represent a point source by beam superposition, a value of b less than infinity will act to filter the plane wave components in a Gaussian manner.

The use of complex phase in wave superposition integrals has been used by several authors in order to obtain various computational advantages. In the discrete wavenumber method (see Larner [1970], Bouchon and Aki [1977]) a complex frequency is used to reduce the effects of artificially induced periodic sources resulting from discretizing the wavenumber integral. Thus

$$e^{i\omega t} \rightarrow e^{i\omega_r t} e^{-\omega_i t}$$

This results in an exponentially damped time domain trace and smoothing and broadening in the frequency domain since

$$e^{-\omega_i \operatorname{sgn}(t)t} \leftrightarrow \frac{2\omega_i}{\omega_i^2 + \omega^2}$$

are a Fourier transform pair. Since $\omega = vk$, adding a small complex part to the frequency and requiring v to be real, adds a complex part to the wavenumber, k , also. This has the effect of moving singularities off the real axis. Thus, adding a small complex part to the frequency in this case can be used to gain numerical stability, economize on computation time, and reduce effects of artificial periodic sources when sampling the wavenumber integral. The integral is slightly distorted, but this is corrected after the synthesis by multiplying by $e^{i\omega_i t}$.

An interesting twist on this strategy was given recently by Madariaga and Papadimitriou [1985] to numerically compute the Gaussian beam impulse response having singularities at the geometric arrivals. A small

complex part is added to the phase time on the order of the time spacing of the impulse response to be computed. Thus

$$\vartheta \rightarrow \hat{\vartheta} + i\Delta t$$

The result of this is to filter the high frequencies and smooth the transient signal by convolving with a function which has a width on the order of the sample spacing Δt , since

$$e^{-|\omega|\Delta t} \leftrightarrow \frac{2\Delta t}{t^2 + \Delta t^2}$$

are also a Fourier transform pair. Thus singularities at geometric arrivals are slightly smoothed to give a stable sampled impulse response without the need to use an explicit source time function such as the Gabor wavelet as was used by Nowack and Aki [1984]. A particular source time function can then be incorporated in a convenient manner at a later point.

asymptotic and exact solution of the Gaussian beam synthesis for a line source

In this section, the asymptotic and exact solutions to the Gaussian beam superposition for a line source are given. The geometry for a line source at the origin and a receiver at (x, z) is shown in figure (2a), where $R^2 = (x^2 + z^2) = (s^2 + n^2)$ and

$$s = R \cos(\varphi_r - \varphi) = z \cos \varphi + x \sin \varphi = v (xp_x + zp_z)$$

$$n = R \sin(\varphi_r - \varphi) = x \cos \varphi - z \sin \varphi = v (x p_x - z p_z)$$

p_x and p_z are the horizontal and vertical ray parameters where

$$p_x = \sin \varphi / v \quad p_z = \cos \varphi / v$$

The wavefield generated by a line source can be represented in terms of the 2-D Weyl integral as

$$u(x, z, \omega) = i / 4\pi \int_{-\infty}^{+\infty} e^{i\omega(p_x x + p_z z)} \frac{dp_x}{p_z} = i / 4 H_0^{(1)}(\omega R / v) \quad (5a)$$

where $\text{Im } p_z \geq 0$ and $z \geq 0$. Changing the integration variable to angle, φ , one obtains

$$u(x, z, \omega) = i / 4\pi \int_L e^{i\omega/v(\sin \varphi x + \cos \varphi z)} d\varphi \quad (5b)$$

where the contour L goes from $-\pi/2 - i\infty$ to $+\pi/2 + i\infty$. The time domain solution due to a line source is given by

$$u(x, z, t) = 1 / 2\pi \frac{H(t - R/v)}{\sqrt{t^2 - R^2/v^2}} \quad (5c)$$

The Gaussian beam synthesis in terms of angle can be written

$$u_{gb}(x, z, \omega) = i / 4\pi \int_{-\pi/2}^{+\pi/2} \left[\frac{\varepsilon}{\varepsilon + s} \right]^{1/2} e^{i\omega/v \left[s + \frac{\pi^2}{2(\varepsilon + s)} \right]} d\varphi \quad (6)$$

where ε is the complex beam parameter described in a previous section. Usually only homogeneous central rays are considered. As $\varepsilon \rightarrow \infty$ and extending the contour to include the inhomogeneous waves, then this reduces to the exact solution given in (5b).

The steepest descent method is now applied to (6) in order to obtain the asymptotic solution for large ω , following Cerveny et. al. [1982]. Equation (6) can be written in the form

$$I(\omega) = \int_C F(\varphi) e^{\omega f(\varphi)} d\varphi \quad \text{for } \omega \rightarrow \infty$$

where

$$F(\varphi) = i/4\pi \left| \frac{\varepsilon}{\varepsilon + s} \right|^{1/2} e^{i/2 \text{ARG} \frac{\varepsilon}{\varepsilon + s}}$$

and

$$f(\varphi) = i/v \left[s + \frac{n^2}{2(\varepsilon + s)} \right]$$

This can be written as

$$f(\varphi) = i/v \left[R \cos \Delta\varphi + \frac{R^2 \sin^2 \Delta\varphi}{2(\varepsilon + R \cos \Delta\varphi)} \right]$$

where $\Delta\varphi = \varphi_r - \varphi$. The saddle point is defined by $f'(\varphi) = 0$, where

$$f'(\varphi) = \frac{i}{v} R \sin \Delta\varphi \left[1 - \frac{R \cos \Delta\varphi}{(\varepsilon + R \cos \Delta\varphi)} - \frac{R^2 \sin^2 \Delta\varphi}{2(\varepsilon + R \cos \Delta\varphi)^2} \right]$$

A saddle point occurs at $\varphi = \varphi_r$, the same as for the exact line source representation. There is an additional saddle point coming from equating the bracket to zero which will be discussed subsequently. At the saddle point, $\varphi = \varphi_r$, in the Gaussian beam case

$$f''(\varphi)|_{\varphi=\varphi_r} = \frac{-i\varepsilon R}{v(\varepsilon + R)}$$

The exponent can then be approximately written as

$$f(\varphi) \approx f(\varphi_r) + 1/2 f''(\varphi)|_{\varphi=\varphi_r} \Delta\varphi^2$$

or

$$f(\varphi) \approx iR/v - \frac{i\varepsilon R}{2v(\varepsilon + R)} (\varphi_r - \varphi)^2$$

Let $(\varphi_r - \varphi)^2 = A e^{i2\vartheta}$ then

$$f(\varphi) \approx iR/v - R/2v \left| \frac{\varepsilon}{\varepsilon + R} \right| A e^{i(2\vartheta + \pi/2 + \text{ARG}(\frac{\varepsilon}{\varepsilon + R}))}$$

or

$$f(\varphi) \approx iR/v - R/2v \left| \frac{\varepsilon}{\varepsilon + R} \right| A [\cos\beta + i\sin\beta]$$

where $\beta = 2\vartheta + \pi/2 + \text{ARG}\left(\frac{\varepsilon}{\varepsilon + R}\right)$. Now choose ϑ such that $\sin\beta = 0$

which gives the constant phase or steepest contour. Thus $\beta = n\pi$. This

results in $\vartheta = -\frac{\pi}{4} - \frac{1}{2}\text{ARG}\left(\frac{\varepsilon}{\varepsilon + R}\right)$ for steepest descent. The integral can

then be approximated as (see for example Aki and Richards [1980], box 6.3)

$$I(\omega) = \left| \frac{2\pi}{\omega f''(\varphi_r)} \right|^{1/2} F(\varphi_r) e^{i\vartheta} e^{\omega f(\varphi_r)}$$

or

$$u_{\varphi_0}(x, z, \omega) \approx \frac{1}{4\pi} \left| \frac{2\pi v}{\omega R} \right|^{1/2} e^{i\omega R/v + i\pi/4} \approx \frac{i}{4} H_0^{(1)}(\omega R/v) \quad (7)$$

which is also the asymptotic expansion for the exact solution. Thus for large ω , the Gaussian beam synthesis for the saddle point $\varphi = \varphi_r$ gives the correct response. This will be invalid when $\varepsilon = 0$ for the Gaussian beam synthesis since then $f''(\varphi_r) = 0$. In order to investigate this behavior, an

exact transient evaluation of the Gaussian beam integral is done.

The Cagniard method is first used to evaluate the exact transient response (5c) from (5a). The substitution $\bar{s} = -i\omega$ is made in (5a) where $\bar{s}$ is real and positive. Equation (5a) can then be written

$$u(x, z, \bar{s}) = i/4\pi \int_{-\infty}^{\infty} e^{-\bar{s}(p_z x + p_z z)} \frac{dp_z}{p_z}$$

This allows the branch cuts to be changed to those in figure (2b) since now $\text{Re } p_z \geq 0$. Now let $t = p_z x + p_z z$, and deform the contour in the complex p-plane so that t is real. Thus

$$u(x, z, \bar{s}) = i/4\pi \int_C e^{-\bar{s}t} \frac{dp_z}{p_z}$$

The Cagniard contour C_+ and C_- is shown in figure (2b) where

$$p_{z(+/-)} = \frac{xt}{R^2} (+/-) \frac{i|z|}{R^2} \left[t^2 - R^2/v^2 \right]^{1/2} \quad t > R/v$$

According to Jordan's Lemma the contributions on the large arcs in figure (2b) go to zero leaving the result

$$u(x, z, \bar{s}) = i/4\pi \int_{C_+} \frac{e^{-\bar{s}t}}{p_z} dp_z + i/4\pi \int_{C_-} \frac{e^{-\bar{s}t}}{p_z} dp_z$$

and

$$u(x, z, \bar{s}) = -i/4\pi \int_{R/v}^{\infty} \frac{e^{-\bar{s}t}}{p_{z+}} \left(\frac{dp_{z+}}{dt} \right) dt + i/4\pi \int_{R/v}^{\infty} \frac{e^{-\bar{s}t}}{p_{z-}} \left(\frac{dp_{z-}}{dt} \right) dt$$

where

$$\left(\frac{dp_z(+/-)}{dt} \right) = \frac{x}{R^2} (+/-) \frac{i|z|t}{R^2} \left[t^2 - R^2/v^2 \right]^{-1/2}$$

and

$$\left(\frac{dp_z(+/-)}{dt} \right) \frac{1}{p_z(+/-)} = \frac{(+/-)i}{\left[t^2 - R^2/v^2 \right]^{1/2}}$$

Then

$$u(x, z, \bar{s}) = 1/2\pi \int_{R/v}^{\infty} \frac{1}{\left[t^2 - R^2/v^2 \right]^{1/2}} e^{-\bar{s}t} dt$$

Using the fact that $L^{-1} \left\{ \int_{t_1}^{\infty} f(t) e^{-\bar{s}t} dt \right\} = f(t) H(t - t_1)$, the exact time

domain response is given by (5c).

The Cagniard method is now used to obtain the transient response of the Gaussian beam integral in (6). First, the Gaussian beam integral is converted to p_z and extended from $-\infty$ to $+\infty$ to include the full range of ray parameter. The substitution $\bar{s} = -i\omega$ is then made, where $\bar{s}$ is real and positive. Thus

$$u_{gb}(x, z, \bar{s}) = i/4\pi \int_{-\infty}^{\infty} \left(\frac{\varepsilon}{\varepsilon + s} \right)^{1/2} e^{-\bar{s} \left\{ \frac{s}{v} + \frac{n^2}{2v[\varepsilon + s]} \right\}} \frac{dp_z}{p_z}$$

where the branch cuts are defined by $\text{Re } p_z \geq 0$. Now the substitution is

made $t = \left\{ \frac{s}{v} + \frac{n^2}{2v[\varepsilon + s]} \right\}$. Since $s^2 + n^2 = R^2$, then

$t = \left\{ \frac{s}{v} + \frac{R^2 - s^2}{2v[\varepsilon + s]} \right\}$. We now want to deform the contour in the complex

p-plane so that t is real where $s = p_z x + p_z z$. First, s is solved in terms of t , giving

$$s = (vt - \epsilon) (+/-) \sqrt{v^2 t^2 + \epsilon^2 - R^2}$$

The two solutions are shown in figure (2c) for $\epsilon = -ib$ and $R = b/4$. For $\epsilon = -ib$, then the solution that reduces to $s = vt$ for $b \rightarrow \infty$ is

$$s(t) = vt + ib - iX \quad (R^2 + b^2) \geq v^2 t^2$$

where $X = \left\{ (R^2 + b^2) - v^2 t^2 \right\}^{1/2}$, and

$$s(t) = vt + ib + X \quad v^2 t^2 \geq (R^2 + b^2)$$

where $X = \left\{ v^2 t^2 - (R^2 + b^2) \right\}^{1/2}$. The factor $p = p(s)$ can be written as

before as

$$p_z(+/-) = \frac{xs}{vR^2} (+/-) \frac{i|z|}{vR^2} \left\{ s^2 - R^2 \right\}^{1/2} \quad |s| > R$$

and

$$p_z(+/-) \approx \frac{xs}{vR^2} (+/-) \frac{i|z|s}{vR^2} \quad |s| \gg R$$

The contour C_+ and C_- are shown in figure (2d). The original contour is deformed into this contour in order to make t real. Then

$$u_{gb}(x, z, \bar{s}) = i/4\pi \int_{C_+} \left(\frac{-ib}{-ib + s} \right)^{1/2} e^{-st} \frac{dp_z}{p_z} + i/4\pi \int_{C_-} \left(\frac{-ib}{-ib + s} \right)^{1/2} e^{-st} \frac{dp_z}{p_z}$$

Converting this to an integration in time

$$u_{gb}(x, z, \bar{s}) = -i/4\pi \int_{r/v}^{\infty} \left(\frac{-ib}{-ib + s} \right)^{1/2} \left\{ \frac{1}{p_{z+}} \frac{dp_{z+}}{ds} \right\} \left\{ \frac{ds}{dt} \right\} dt$$

$$+ i/4\pi \int_{r/v}^{\infty} \left\{ \frac{-ib}{-ib + s} \right\}^{1/2} \left\{ \frac{1}{p_{z-}} \frac{dp_{z-}}{ds} \right\} \left\{ \frac{ds}{dt} \right\} dt$$

where $\left\{ \frac{1}{p_{z(+/-)}} \frac{dp_{z(+/-)}}{ds} \right\} = \frac{(+/-)i}{[s^2 - R^2]^{1/2}}$

$$\left\{ \left[\frac{-ib}{-ib + s} \right] \frac{ds}{dt} \right\} = \frac{v}{X} \sqrt{b} \left\{ X + ivt \right\}^{1/2} \quad (R^2 + b^2) > v^2 t^2$$

$$\left\{ \left[\frac{-ib}{-ib + s} \right] \frac{ds}{dt} \right\} = \frac{v}{X} \sqrt{-ib} \left\{ \bar{X} + vt \right\}^{1/2} \quad v^2 t^2 > (R^2 + b^2)$$

also

$$[s^2 - R^2]^{-1/2} = \left[2 (X + ivt)(b - X) \right]^{-1/2} \quad (R^2 + b^2) > v^2 t^2$$

$$[s^2 - R^2]^{-1/2} = \left[2 (\bar{X} + vt)(ib + \bar{X}) \right]^{-1/2} \quad v^2 t^2 > (R^2 + b^2)$$

The final response is

$$\left\{ \left[\frac{-ib}{-ib + s} \right]^{1/2} \frac{ds}{dt} [s^2 - R^2]^{-1/2} \right\} = \frac{H(t - R/v) \chi}{2\pi [t^2 - R^2/v^2]^{1/2}}$$

where χ is a modification factor to the exact solution when using the Gaussian beam superposition, where

$$\chi = \frac{\sqrt{b} (v^2 t^2 - R^2)^{1/2}}{X [2(b - X)]^{1/2}} \quad (R^2 + b^2) > v^2 t^2$$

$$\chi = \frac{\sqrt{-ib} (v^2 t^2 - R^2)^{1/2}}{X [2(ib + \bar{X})]^{1/2}} \quad v^2 t^2 > (R^2 + b^2)$$

This modification factor is shown in figure (1e). The modification factor has a singularity at $v^2 t^2 = (R^2 + b^2)$. After this time the modification factor becomes complex, but is bounded. This additional singularity moves to infinity as $b \rightarrow \infty$. To check the existence of this singularity, we note

that the function $f'(\varphi) = 0$ given previously has an additional zero when the bracket term is zero, giving $s = R \cos \varphi = ib (+/-) (R^2 + b^2)^{1/2}$. For the plus sign and using the formula for $s(t)$, this corresponds to $t = (R^2 + b^2)^{1/2}/v$, which is what is shown here. For a critical value of b , this secondary singularity occurs at 1.41 times the geometric arrival. In the p-plane shown in figure (2d) this corresponds to the two points, A_1 and A_2 . The problem of this secondary singularity occurs when extending the original contour from $-\infty$ to $+\infty$ to include the evanescent beam waves. It would be difficult to define central rays for these. The usual Gaussian beam formulation only includes central rays with real takeoff angles, and the additional singularity won't in general be included in the usual Gaussian beam synthesis. The usual Gaussian beam integral thus does not correspond to the exact transient solution for a line source. It approaches the exact solution as $\varepsilon \rightarrow \infty$. For large ω and real takeoff angles, the Gaussian beam integral is asymptotically equivalent to the exact integral representation as long as $\varepsilon \neq 0$. In the general case, the Gaussian beam response is good near the geometric arrival but deteriorates for larger times.

Initial point in a medium with a velocity gradient

In this section, the initial conditions for the variables $p(s)$ and $q(s)$ that solve the dynamic ray equations are discussed. The variable $q(s)$ has the interpretation of a spreading and $\frac{1}{v} \frac{p}{q}$ has the interpretation of a wavefront curvature. Gaussian beams can be formed as a linear combination of two independent sets of solutions to the dynamic ray equations with a complex weighting factor. Thus

$$q(s) = q_2(s) + \varepsilon q_1(s)$$

$$p(s) = p_2(s) + \varepsilon p_1(s)$$

and $\varepsilon = d - ib$ is described in an earlier section. With b strictly greater than zero, then a beam is defined in which $q(s) \neq 0$. This uses the fact that the determinant of the matrix formed by the two solutions, $\begin{bmatrix} q_1 & q_2 \\ p_1 & p_2 \end{bmatrix}$, is an invariant of the motion, and is sometimes called the Lagrange invariant. Cerveny et. al. [1982] uses two independent solutions with initial conditions, $\begin{bmatrix} 1 & 0 \\ 0 & 1/v_0 \end{bmatrix}$. The initial conditions, $q_1(0) = 1$ and $p_1(0) = 0$, correspond to an initial planar wavefront with, $p_1/q_1 = 0$. The initial conditions, $q_2(0) = 0$ and $p_2(0) = 1/v_0$ correspond to an initial point source, with $p_2/q_2 \rightarrow \infty$. The source region is usually assumed to be locally homogeneous.

Madariaga [1984] constructed a beam solution in terms of a small perturbation of a constant Snell wave having a constant horizontal ray parameter. For a source imbedded in a vertically varying medium, he found that the (q_1, p_1) solution must be modified in order to specify a constant Snell wave, thus

$$q_1(0) = 1 \quad p_1(0) = \frac{-p_x^2 v_{,z}(0)}{\cos \varphi_0} \quad (8)$$

where φ_0 is the initial takeoff angle of the ray from the vertical and $p_x = \sin \varphi_0 / v_0$ is the horizontal ray parameter which in a vertically varying medium is constant along a ray. The (q_2, p_2) solution is the same in each case. For a homogeneous source region, where $v_{,z}(0) = 0$, Madariaga's constant Snell wave solution is the same as the previous plane wave solution. When $\varepsilon \rightarrow \infty$, Madariaga's beam superposition in a vertically varying media reduces to the standard WKBJ result.

Madariaga's initial conditions in a vertically varying medium is shown by imbedding a thin homogeneous layer within the gradient as in figure (3a). A source is put in the constant velocity layer. Boundary conditions are then matched on the top and bottom interfaces and the layer thickness shrunk to zero. Boundary conditions for p and q at an interface are given by Cerveny and Hron [1979] derived using the phase matching principle. The 2-D results are given in a compact form by Cerveny [1981] as

$$\begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix} = (+/-) \frac{\cos \tilde{\vartheta}}{\cos \vartheta} \begin{bmatrix} 1 & 0 \\ C + I_n & \frac{\cos^2 \vartheta}{\cos^2 \tilde{\vartheta}} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$$

where (q, p) are the incident values and $(\tilde{q}, \tilde{p})$ are the transmitted or reflected values. The upper sign is for the transmitted case and the lower sign for the reflected case. The angles are defined in figure (3b). The effects due to interface curvature, represented by C , and the effects due to velocity heterogeneity, represented by I_n , are additive, where

$$C = G \left\{ \frac{\cos \vartheta}{v} (-/+) \frac{\cos \tilde{\vartheta}}{\tilde{v}} \right\} \frac{1}{\cos^2 \tilde{\vartheta}}$$

and

$$I_n = \frac{-\sin \vartheta}{v \cos^2 \tilde{\vartheta}} \left\{ 2 \left(\frac{v_{,m}}{v} \cos \vartheta (-/+) \frac{\tilde{v}_{,m}}{\tilde{v}} \cos \tilde{\vartheta} \right) + \frac{\sin \vartheta}{v} (v_{,l} - \tilde{v}_{,l}) \right\}$$

where G is the local interface curvature at the point of incidence, where G is positive if the interface is convex as seen by the incident ray. $\vec{l}$ is the local tangent to the ray and $\vec{m}$ is perpendicular to it as seen in figure (3b).

In the case of a ray striking the lower horizontal interface as in figure (3b), then

$$\tilde{v}_{,m} = -\tilde{v}_{,z} \sin \vartheta \quad \tilde{v}_{,l} = \tilde{v}_{,z} \cos \vartheta$$

Also, since $v = \tilde{v}$ at the interface, then $\vartheta = \tilde{\vartheta}$ by Snell's law. The interface curvature term is zero, $C = 0$, and the velocity inhomogeneity

term is $I_n = \frac{-v_{,z} \sin^2 \vartheta}{v^2 \cos \vartheta}$ then

$$\tilde{q} = q \quad \tilde{p} = p - \frac{v_z \sin^2 \vartheta}{v^2 \cos \vartheta} q$$

Shrinking the layer thickness to zero results in the modified initial conditions for (q_1, p_1) as in eqn (8). For a vertical ray, these conditions reduce to Cervený's initial conditions. For a horizontal grazing ray, the boundary conditions at the interface break down.

Part of the discrepancy in the two initial conditions for (q_1, p_1) is that an initial plane wave imbedded in a medium with a vertical velocity gradient is not a wave with a constant horizontal ray parameter. Madariaga's initial conditions for (q_1, p_1) construct an initial constant Snell wave. A constant Snell wave has a caustic or focus at its bottoming point. This results in a problem when using Madariaga's beam superposition for an initial ray that is exactly horizontal for which both the (q_2, p_2) solution and the (q_1, p_1) solution are singular and thus no longer independent. In the WKB method, the contributions from horizontal rays are typically windowed out (see Sinton and Frazier [1982]).

By using Cervený's initial conditions, two independent solutions are always guaranteed. The disadvantage of this approach is that the (q_1, p_1) may no longer correspond to a constant Snell wave in the global coordinates. Also, since the weight function in the beam superposition is derived by matching to a known solution in a homogeneous source region, then in an arbitrary source region, the weight function will only be

approximate. Beydoun and Ben Menahem [1985] used Cervený's initial conditions in various gradients and compared with exact analytic solutions. They showed that good agreement occurred provided that several high frequency conditions were satisfied. In addition, the Gaussian beam superposition was found to have more restrictions than the ray theoretical solution. This may be in part due to the approximate weight function used in the Gaussian beam superposition.

Several examples are now done using Cervený's initial conditions, C.i.c., and Madariaga's initial conditions, M.i.c., in a vertical gradient. In the first example a surface source is used. The medium has a linear gradient with velocities from 1.5 km/sec to 5.6 km/sec from 0 to 10 km. A Gabor wavelet with $f = 4\text{hz}$ and $\gamma = 3$ was used (see Nowack and Aki [1984] eqn. (22)). Results in the time-distance domain are shown in figure (4) with the C. i.c. shown on the left and the M. i.c. shown on the right. Two initial beam widths are used, $L_0 = 5 \text{ km}^{1/2}$ and $L_0 = 15 \text{ km}^{1/2}$, where $L_0 = 5 \text{ km}^{1/2}$ is near an average critical value. The C. i.c. for $L_0 = 15 \text{ km}^{1/2}$ has noise predominantly in front of the geometric arrival. The M. i.c. for $L_0 = 15 \text{ km}^{1/2}$ has noise later in time than the geometric arrival.

In figure (5), the contributions from each beam is shown for a station at 12 km. The same model as in figure (4) is used. The rays are numbered from 1 to 20, from shallow to steeply dipping at the source. The ray number-phase time plots are shown for both initial conditions and initial

beam widths, $L_0 = 5, 15 \text{ km}^{1/2}$. The resultant seismogram in each case is shown by the scaled sum on the right of each sub-plot in figure (5). In all sub-plots, the geometric ray to the station, ray number 12, is a stationary point. A significant difference between the C. i.c. and the M. i.c. is the curvature of the beam arrivals near the geometric arrival. Using the C. i.c., the beams near the geometric arrival come in earlier, while using the M. i.c., the beams near the geometric arrival come in later than the geometric arrival. This can be understood by noting that using the C. i.c. no q_1 or q_2 caustics occur along the ray. Thus the curvature of the beams should be in the same sense as in a homogeneous medium, in which the surrounding beams come in earlier than the geometric arrival (see Nowack and Aki [1984] figure (5b)). Several later arrivals are seen for C. i.c., $L_0 = 15 \text{ km}^{1/2}$, but this is due to an artificial q_1 caustic caused by projecting these rays above the free surface. Using the M. i.c., all beams have passed through a q_1 caustic at the bottoming point of each ray which changes the curvature of the beams. The C. i.c. has no bottoming point caustic.

Another difference between the two initial conditions, C. i.c. and M. i.c., is the phase of the individual beam wavelets. There is a phase delay for the C. i.c. and a phase advance for the M. i.c.. In each case, the phase of the individual beam wavelets depends on the initial beam width used. In all cases, the resulting sum is approximately zero phase. Thus in each case, the individual beam phase shifts is offset by the curvature of the beams

away from the geometric arrival. This occurs in such a way as to give a seismogram with no resulting phase shift. This results from using a weight function in the superposition which was derived by matching to a known solution.

In figure (6), the time-distance wavefields are shown for a source at 4. km depth in a linear gradient. The velocity is 1.8 km/sec to 5.6 km/sec from 0. to 20. km in depth. A Gabor wavelet with $f = 4 \text{ hz}$ and $\gamma = 4$ For the M. i.c. there is now a small second arrival for distances greater than 16 km, using both initial beam widths, 4 and 20 $\text{km}^{1/2}$. Also, the amplitude is slightly larger in the 14-16 km range for the M. i.c. case. In order to investigate this, ray number-phase time plots are constructed for several station locations.

In figure (7), the ray number-phase time plots are shown for a station range of 22 km, for both initial conditions and initial beam widths. The rays are numbered from 1 to 20 from shallow to steeply dipping at the source, as shown in the above ray diagram. The geometric arrival is specified by ray number 17. For the C. i.c., the beams near the geometric beam arrive earlier in time. The noise in front of the geometric arrival in the scaled sum for $L_0 = 20 \text{ km}^{1/2}$ can be reduced by using more beams in the superposition. For the M. i.c., the beams nearby the geometric arrival, ray number 17, come in later in time. But, another stationary point occurs at ray number 9. This is a maximum stationary point, and has lower

amplitudes. It corresponds to the ray that takes off horizontally at the source. There is a separation between rays that are upgoing at the source and those that go through a bottoming point and pass through a q_1 caustic. Since the contributions don't quite go to zero at this secondary stationary point and don't phase cancel, an artifact occurs in the resulting sum. This artifact occurs for both $L_0 = 4 \text{ km}^{1/2}$ and $L_0 = 20 \text{ km}^{1/2}$. Again, in the WKB result these ray numbers are typically windowed out. This would not be a useful remedy if the geometric arrival is near this horizontal ray as in the Imperial Valley example to follow.

In figure (8), a similar set of ray number-phase time plots are shown for a station at 15 km. In this case, the geometric arrival is at ray number 13. This is much closer to the horizontal ray at ray number 9. For the M. i.e., the secondary stationary point doesn't cause a spurious arrival, but now interferes directly with the main pulse. Modifying the initial beam width doesn't alleviate the problem.

Madariaga [1984] considered a beam superposition as a small perturbation of a WKB superposition with respect to horizontal ray parameter. This approach loses the advantage of using two independent solutions for (p,q) for a horizontal ray. Madariaga's initial conditions cause a spurious phase for the rays near horizontal which can't be eliminated by changing the initial beam width. Cerveny's initial conditions are guaranteed to give two independent solution for (p,q) , but doesn't

correspond to a small perturbation of the WKBJ solution in an inhomogeneous source region. In addition, since the weight function is derived for a homogeneous source region, it will only be approximate using the Cerveny initial conditions in an inhomogeneous region. The major difference between the two initial conditions is in defining a local plane wave in the ray centered coordinates versus defining a constant Snell wave in the global coordinates.

Seismograms in the Imperial Valley velocity structure

In this example, Gaussian beam seismograms are computed using the Imperial Valley velocity structure. An interesting feature of the recorded strong ground motion for the 1979 Imperial Valley earthquake was the large vertical accelerations at some but not all sites (see Archuleta [1982]). For example, at the station, E06, the peak acceleration was 1705 cm/sec^2 . Several explanations for this localized region of high acceleration have been considered, including the source rupture process, and a path effect due to the velocity gradient in the upper 5 km of the Imperial Valley. Here we explore only the path effect. A likely candidate for the large acceleration resulting from a path effect is the PP phase which was a prominent arrival recorded during the 1979 refraction survey of the Imperial Valley (see McMechan and Mooney [1980], and Fuis et. al. [1982]). Because of the focusing effect due to the multiple bounce, the PP

phase has a localized high amplitude over a limited distance range. Archuleta [1982] found that the travel times of the observed high accelerations were consistent with a PP phase originating from a source point at 4.0 km depth and 5.5 km north of the hypocenter, although he now attributes the main cause of localized high acceleration to a source effect.

The Imperial Valley structure derived from the refraction survey is shown in figure (9). The ray diagram for the PP phase is also shown. In figure (10), seismograms computed using the Gaussian beam method are compared to seismograms computed using the Reflectivity method, without attenuation, by R. Archuleta. The Gaussian beam method uses the Cerveny initial conditions described in the previous section. A significant portion of the energy comes from nearly horizontal rays at the source. The source is modelled using a point double couple radiation pattern with a dip of 75 degrees, a 0.0 degree slip angle, and the seismic section 13 degrees off strike. A Gabor wavelet with a frequency of 6 hz was used in the Gaussian beam seismograms. To obtain the Gaussian beam result for a point source, the GB results were multiplied by a ray method correction given by Cerveny et. al. [1982]. The value of L_0 was set at $2.0 \text{ km}^{1/2}$ resulting in average beam widths at the receivers of about 2 km. Both seismic sections show the predominant high amplitudes of the PP phase in the distance range of 15-18 km, with the Gaussian beam result being slightly closer. Significant S wave energy is seen in the Reflectivity section. The Reflectivity section shown in Archuleta [1982] included the effects of

attenuation which diminished the S waves, and further enhanced the PP arrival.

Plane wave decomposition

In this section, a simple comparison is given between the individual beam contributions in the Gaussian beam superposition and a slant stack of a time-distance wavefield. The operation of slant stacking is kinematically equivalent to a plane wave or τ - p decomposition. In mathematical notation this is also known as an inverse Radon transform.

The plane wave decomposition holds a central role in the velocity analysis in vertically varying media. The τ -method for velocity analysis was used for example by Bessonova et. al. [1974, 1976]. McMechan and Ottolini [1980] showed that τ - p curves can be directly observed in slant stacked wavefield data. Clayton and McMechan [1981] performed an iterative migration to convert the τ - p wavefield to a velocity depth estimate. This was applied to the upper mantle structure by Walck and Clayton [1984].

Using the τ - p wavefield has the advantage that triplications are unwrapped and thus made more easily interpretable. Also, in a vertically varying medium, each τ - p trace corresponding to a ray parameter, p , satisfies a 1-D wave equation. The relationship between the τ - p wavefield

and seismogram synthesis has been investigated by Chapman [1978,1981] and Treitel et.al. [1982]. For a line source in a vertically varying medium, Chapman [1981] gives the Radon transform formulas for wavefield decomposition and synthesis as

$$\tilde{u}(\tau, p) = \int_{-\infty}^{\infty} u(\tau + px, x) dx \quad (9a)$$

and

$$u(t, x) = \int_{-\infty}^{\infty} \frac{-1}{2\pi} \tilde{u}^+(t - px, p) dp \quad (9b)$$

where $\tilde{u}^+$ is the time derivative, Hilbert transform of $\tilde{u}$. These are reversible linear transformations which map a line in one domain to a point in the other domain. Each trace of $\tilde{u}(\tau, p)$ represents an individual plane wave, or in a vertically varying medium a constant Snell wave since in general they won't be planar.

An intermediate step in the Gaussian beam method is the evaluation of each beam contribution along a specific ray arriving at a given station. Each beam satisfies a local wave equation along each central ray. The final GB solution using the slowness method is a superposition of the beam contributions. Thus

$$u(t, x) = \int_{-1/v}^{1/v} u_{gb}(s, n, t) dp \quad (10)$$

where u_{gb} is an individual beam contribution and the integral is over the ray number measured at the source. In the Gaussian beam method this

integral is usually only over real angles. Using large initial beam widths in a vertically varying medium, then each beam will approach an initial plane wave contribution as in equation (9b).

A simple comparison is now done between a slant stacked wavefield and the individual beam contributions in equation (9b). The velocity model consists of a 15 km homogeneous layer with a velocity of 5.6 km/sec over a linear gradient from 5.6 to 8.0 km/sec. A Gabor wavelet is used with $f = 4 \text{ Hz}$ and $\gamma = 3$. The ray diagram is shown in figure (11a). A reduced time wavefield is shown in figure (11b). In order to use the same format as the Gaussian beam decomposition, each ray number is simply counted from 1 to 18 from shallow to steeply dipping at the source as in figure (12a). A station distance of 120 km is used. Equation (9a) is then applied to wavefield in figure (11b) resulting in a $\tilde{u}(\tau, p)$ wavefield. This plane wave decomposition is shown in figure (12b) in the ray number format for a station at 120 km. The slight phase shift shown on the first several traces may be the result of using a finite x aperture in the slant stack. Filtering this plot with ray number corresponds to a beam forming operation in which energy from certain directions is enhanced with respect to other directions.

Figure (12c) shows the ray number-phase time plot for the Gaussian beam synthesis using a large initial beam width of $L_0 = 128 \text{ km}^{1/2}$. For this initial beam width, some amplitude filtering is still seen for the first

several ray number traces. The resulting scaled sum for a range of 120 km is shown on the right of figure (12c) corresponding to the synthesis operation in eqn (10). Comparing figures (12b) and (12c) shows that the locus of maximum amplitudes map out the same curve for the slant stack and the Gaussian beam decomposition. It is interesting that the individual wavelets in figures (12b) and (12c) have similar phase shifts. This results from the approximations in using the Gabor wavelet in the Gaussian beam method, in particular for the Hilbert transform.

Since the slant stacking operation is a linear beam forming operation, an important extension of this is to investigate the variations of the ray parameter-phase time wavefields in laterally varying media. This has been investigated for several structures by Treitel et. al. [1982]. Because of its usefulness in vertically varying media, plane wave or beam decompositions are anticipated to be important data representations in more complicated media. This representation is straightforward to generate as a by product of many seismogram synthesis routines, in particular, as ray centered beam responses in the Gaussian beam method.

Q1 caustics

Previous results have indicated that in decomposing a point source into beams that stable results occur for large initial beam widths. Nowack and Aki [1984] suggested that the critical value of L_0 be used to separate

large from small initial beam widths, where for $S_0 = 0$, $L_{cr} = |q_2/q_1|^{1/2}$. When initial beam widths smaller than critical are used, then phase interference and amplitude decay only weakly restrict the number of beams contributing at a given station. A finite aperture will then result in a reduced amplitude of the resulting seismogram.

A q_1 caustic is defined where $q_1 = 0$ in the wavefield. For a station at a q_1 caustic, then $L_{cr} \rightarrow \infty$. Thus the initial beam width at the source resulting in the narrowest beam width at the receiver approaches infinity, i.e. a plane wave. However, in this case using a slightly smaller initial beam width should still give a satisfactory amplitude, since the aperture of rays will still be large compared to the focused energy at the q_1 caustic. Typically an average value of L_0 is used for the suite of rays in the vicinity of the station. This would then smooth over the beam solutions near the station. Thus at a q_1 caustic, the finite aperture should not pose a problem for a suitably large but finite L_0 . However, a finite L_0 will average over nearby rays and lose some spatial resolution.

In surface wave ray tracing, Yomogida and Aki [1985] found a q_1 caustic along the equator for a high latitude source when using a Mercator mapping from spherical to cartesian coordinates. They found that as long as L_0 was large enough to avoid truncation errors from the aperture, then the amplitude was accurate to 1%. But, an increasing amount of averaging of surrounding beams, without the aide of phase interference, is seen in

figure (4) of Yomogida and Aki [1985]. A slight increase in the beam width at the q_1 caustic to be several times the ray spacing should stabilize the solution. Another possible way of avoiding the problems associated with a q_1 caustic is to use the real part of ϵ , S_0 , to shift the beam waist along the ray. But, this will also introduce an additional free parameter.

Gaussian beams incident on a planar interface

In this section, the asymptotic analysis of a beam reflected on a planar interface is given, following the development of Ra et. al. [1973]. The saddle point contribution to the reflected field is given by

$$u(x_s, z_s, \omega) \approx \frac{R(\Theta)}{\sqrt{k_1 D_r}} e^{ik_1 D_r - i\pi/4} \quad (11)$$

where $k_1 D_r \gg 1$, D_r is the distance from source to receiver, and $R(\Theta)$ is a plane wave reflection coefficient. As shown in a previous section, the reflected beam contribution can be gotten by using a complex source point. Using the image source point as in figure (13), then

$$x' \rightarrow ib \sin \alpha$$

$$z' \rightarrow z_0 - ib \cos \alpha$$

and the angle Θ is defined from $\sin \Theta = \frac{x_s - x'}{D_r}$. When $b \rightarrow 0$, then

$D_r \rightarrow \rho_r$ and $\Theta \rightarrow \vartheta_r$ in figure (13).

For $\vartheta > \vartheta_c = \sin^{-1}(v_1/v_2)$, then in addition to the reflected field there will also be a diffracted head wave field. In the vicinity of ϑ_c , a uniform approximation must be used. Going to a Gaussian beam by using a complex source point requires using the appropriate complex angles in these formulas. Here only the reflected field will be investigated. In the Gaussian beam method as currently implemented, the real reflection coefficient for the beam axis is used, and all ray tracing is done along the beam axis. Thus in the reflection coefficient, Θ is replaced by α .

The reflection coefficient in the acoustic case has a slightly different form than that for the SH case. The reflection coefficient in the acoustic case is given by

$$R_a(\vartheta) = \frac{\rho_2 v_2 \cos \vartheta - \rho_1 v_1 \left(1 - \frac{v_2^2}{v_1^2} \sin^2 \vartheta\right)^{1/2}}{\rho_2 v_2 \cos \vartheta + \rho_1 v_1 \left(1 - \frac{v_2^2}{v_1^2} \sin^2 \vartheta\right)^{1/2}}$$

For the SH case, the reflection coefficient is given by

$$R_{SH}(\vartheta) = \frac{\rho_1 v_1 \cos \vartheta - \rho_2 v_2 \left(1 - \frac{v_2^2}{v_1^2} \sin^2 \vartheta\right)^{1/2}}{\rho_1 v_1 \cos \vartheta + \rho_2 v_2 \left(1 - \frac{v_2^2}{v_1^2} \sin^2 \vartheta\right)^{1/2}}$$

This difference in reflection coefficient is related to the different type of impedance in the acoustic and the SH cases (see for example Aki and Richards [1980], Box 5.4, or Aki [1970], lecture notes pp. 60-61)

It is assumed that the beam is well defined with $k_1 b \gg 1$. In addition, the paraxial region is considered with $\vartheta_r \approx \alpha$. The reflection coefficient is then approximated as

$$\begin{aligned} R(\Theta) &\approx R(\alpha) + R'(\vartheta)|_{\vartheta=\alpha} \Delta\vartheta \\ R(\Theta) &\approx R(\alpha) \left\{ 1 + \frac{R'(\vartheta)|_{\vartheta=\alpha}}{R(\vartheta)} \Delta\vartheta \right\} \\ R(\Theta) &\approx R(\alpha) e^{\nu} \end{aligned}$$

where ν is a complex phase correction which has the effect of relocating the maximum of the reflected field (see Ra et. al. [1973], Felsen [1984]).

In the acoustic case, then

$$\frac{R'(\vartheta)|_{\vartheta=\alpha}}{R(\alpha)} \approx \frac{2 \sin \alpha \chi_a}{\left[1 - \frac{v_2^2}{v_1^2} \sin^2 \alpha \right]^{1/2}}$$

$$\text{where } \chi_a = \frac{\frac{\rho_2 v_2}{\rho_1 v_1} \left[\frac{v_2^2}{v_1^2} - 1 \right]}{\left[\frac{\rho_2^2 v_2^2}{\rho_1^2 v_1^2} \cos^2 \alpha + \frac{v_2^2}{v_1^2} \sin^2 \alpha - 1 \right]} \quad \text{For } \rho_1 = \rho_2, \text{ then } \chi_a = \frac{v_2}{v_1} \text{ in}$$

the acoustic approximation. In a similar manner, for the SH case

$$\frac{R'(\vartheta)|_{\vartheta=\alpha}}{R(\alpha)} \approx \frac{2 \sin \alpha \chi_{SH}}{\left[1 - \frac{v_2^2}{v_1^2} \sin^2 \alpha \right]^{1/2}}$$

$$\text{where } \chi_{SH} = \frac{\frac{\rho_1 v_1}{\rho_2 v_2} \left[\frac{v_2^2}{v_1^2} - 1 \right]}{\left[\frac{\rho_1^2 v_1^2}{\rho_2^2 v_2^2} \cos^2 \alpha + \frac{v_2^2}{v_1^2} \sin^2 \alpha - 1 \right]}. \quad \text{The complex beam shift is}$$

then

$$\nu = \frac{R'(\vartheta)|_{\vartheta=\alpha}}{R(\alpha)} \Delta\vartheta_r \approx \frac{2\sin\alpha \chi}{\left[1 - \frac{v_2^2}{v_1^2} \sin^2\alpha\right]^{1/2}} \Delta\vartheta_r$$

For a real source point

$$\Delta\vartheta_r = \sin^{-1}\left(\frac{n}{\rho_r}\right) \approx \frac{n}{\rho_r}$$

For a complex source point, $\Delta\vartheta_r \rightarrow \Delta\hat{\vartheta}_r$ and $\rho_r \rightarrow D_r \approx (\rho_r - ib)$, then

$$\Delta\hat{\vartheta}_r \approx \frac{n}{\rho_r - ib} = \frac{\rho_r}{\rho_r - ib} \sin\Delta\vartheta_r$$

The formula for the complex beam shift correction, ν , is then given by

$$\nu = 2\sin\alpha \chi \left\{ \frac{\rho_r + ib}{\left[1 - \frac{v_2^2}{v_1^2} \sin^2\alpha\right]^{1/2}} \right\} \frac{\rho_r \sin\Delta\vartheta_r}{(\rho_r^2 + b^2)}$$

As $b \rightarrow 0$, this is a real perturbation. For $b \rightarrow \infty$, then the beam approaches a plane wave and ν goes to 0. The Gaussian beam result is modified by a first order correction to the beam axis reflection coefficient at an interface by a factor, e^ν .

In order to see how the factor e^ν shifts the maximum of the reflected beam amplitude, the maximum of the factor $e^{\frac{-n^2}{L^2} + nB}$ is found where

$$L^2 = \frac{2b}{k} \left[1 + \frac{\rho_r^2}{b^2} \right] \text{ and } B = \frac{2\sin\alpha \chi}{(\rho_r^2 + b^2)} \operatorname{Re} \left\{ \frac{\rho_r + ib}{\left[1 - \frac{v_2^2}{v_1^2} \sin^2\alpha\right]^{1/2}} \right\}. \text{ Then}$$

$$n_{\max} = +L^2 B = \frac{2 \sin \alpha \chi}{kb} \operatorname{Re} \frac{\rho_r + ib}{\left[1 - \frac{v_2^2}{v_1^2} \sin^2 \alpha\right]^{1/2}}$$

or

$$n_{\max} = \frac{2 \sin \alpha \chi}{k} \left(\frac{\rho_r}{b} \right) \frac{1}{\left[1 - \frac{v_2^2}{v_1^2} \sin^2 \alpha\right]^{1/2}} \quad \alpha < \vartheta_c$$

$$n_{\max} = \frac{2 \sin \alpha \chi}{k} \frac{1}{\left[\frac{v_2^2}{v_1^2} \sin^2 \alpha - 1\right]^{1/2}} \quad \alpha > \vartheta_c$$

This is analogous to the formulas given in Ra et. al. [1973]. Note that the beam shift goes to zero for $k \rightarrow \infty$. This is sometimes used to justify neglecting this factor in high frequency beam propagation.

As a simple example, consider a surface source in a 35 km thick crust with $\beta_1 = 3.5 \text{ km/sec}$ and $\rho_1 = 2.74 \text{ g/cm}^3$ and $\beta_2 = 4.6 \text{ km/sec}$ and $\rho_2 = 3.3 \text{ g/cm}^3$. The critical angle, ϑ_c , is 49.54 degrees and a critical distance of 82. km. Let angle, α of the beam axis be 20. degrees with a range of 25.47 km. The angle, ϑ_r , corresponding to a range of 40 km is 29.74 degrees. ρ_r is 80.62 km. The critical value of $b = L_0^2 = |q_2/q_1| = \rho_r$ is used. For $f = 5 \text{ hz}$, $n_{\max} = -.09 \text{ km}$, or about an eighth of a wavelength. For larger b , this value would be correspondingly less in the pre-critical range.

Finite frequency effects of an indentation on an interface

In this final section, the scattering effects due to an indentation on an interface is investigated. This was first studied by Larner [1970] for a dented Moho under the Montana large aperture seismic array (LASA). Larner [1970] used the discrete wavenumber technique to compute the amplitude and phase perturbations of a single frequency signal impinging at different angles on various interfaces indentations. He found a significant frequency variation in the scattered signal. Bard and Bouchon [1980] extended this to the analysis of transient signals.

The model geometry is shown in figure (14a). W and h are the width and height of the indentation. The observation plane is 25 km above the interface. The upper half space has $v_1 = 3.0 \text{ km/sec}$ and $\rho_1 = 2.8 \text{ km/sec}$, and the lower half space has $v_2 = 4.0 \text{ km/sec}$ and $\rho_2 = 3.3 \text{ g/cm}^3$. An SH wave is incident from below with an angle of 32 degrees from the vertical. A Gabor wavelet is assumed with $\gamma = 4$ and several central frequencies. The peak amplitude of the time domain trace is then picked along the observation plane.

The results using the Gaussian beam method are compared with the time domain implementation of the Aki-Larner discrete wavenumber technique (see Bard and Bouchon [1980]). The peak amplitudes along the observation plane are shown in figure (14). Figures (14b-e) show the amplitude patterns for an increasing indentation height h from 0 km to 5

km for a pulse with a 20 km incident wavelength. The solid line is the amplitude pattern computed using the Gaussian beam method and the dotted line is the amplitude computed using the Aki-Larner discrete wavenumber method. The center of the indentation is located at 70 km. As the indentation height increases, the amplitude pattern increases with an asymmetrical appearance. The Gaussian beam solution consistently shows larger side lobes than the discrete wavenumber solution. The difference may be due to the fact that the Gaussian beam method is a high frequency method and the Aki-Larner method is a low frequency method. The differences in the solutions is probably an artifact of using the Gaussian beam method for the low frequencies of this example. However, the width of the main peak of the amplitude pattern is comparable for the two methods. In figures (14f-g), the amplitude anomaly for the two methods is shown for indentation heights of 2.5 and 5.0 km with an incident wavelength of 10 km. The main amplitude peak is narrower for this case than for the 20 km wavelength case and is also higher in amplitude. There is thus a frequency dependence of the scattering anomaly. This was also observed by Larner [1970]. A simple use of ray theory for an example such as this would never reveal the frequency dependent nature of the scattering. The high but still finite frequency methods like the Gaussian beam method can be used to bring out some of these finite frequency effects.

Conclusions

Several topics in Gaussian beam synthesis have been investigated in this paper including the choice of the beam parameter, ϵ , and its relation to a complex source point. The asymptotic and exact evaluations to the Gaussian beam integral shows an additional singularity which goes to infinity as $\epsilon \rightarrow \infty$. But, this typically won't be computed in the usual Gaussian beam synthesis. An initial point source in a vertically varying gradient distinguishes between an initial Snell wave, which is singular for a horizontal ray, and an initial plane wave. In a homogeneous source region these are the same, but in an inhomogeneous source region each has different advantages. A first order correction to the plane wave reflection coefficients are then described. This results in a beam shift which goes to zero for high frequency beam propagation. A simple example of a 35 km layer shows a modest beam shift for a 5 hz signal. Finally, the finite frequency scattering effects for an indented interface are computed using the Gaussian beam method and shows reasonable agreement with the effects computed using the Aki-Larner method.

Further work using the Gaussian beam method include investigation of the real part of the complex beam parameter, ϵ , which can be used to shift the beam waist presumably to where the strong heterogeneity is located. For small, continuously distributed heterogeneities, a multiple beam Kirchhoff scheme would be one possible approach. This would involve

decomposing the wavefield at regular intervals. Finally, the development of a practical implementation of the full 3-D Gaussian beam formalism will be important in synthetic seismogram modelling.

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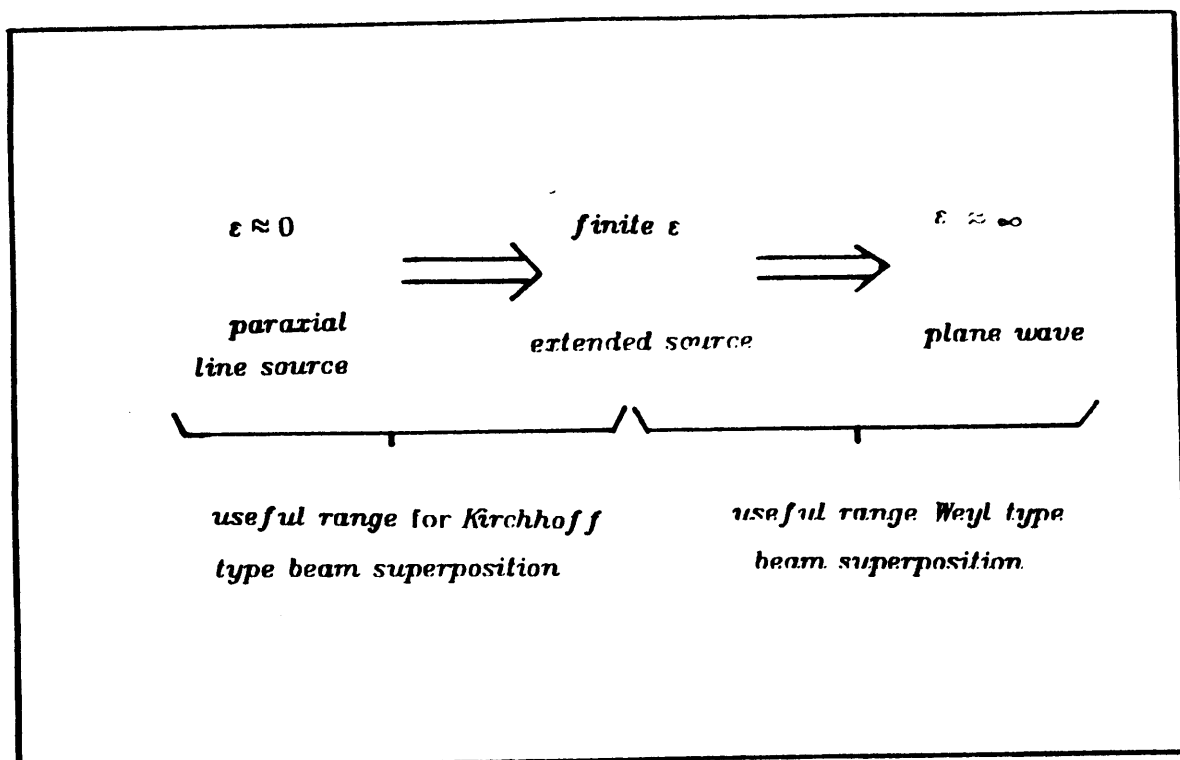
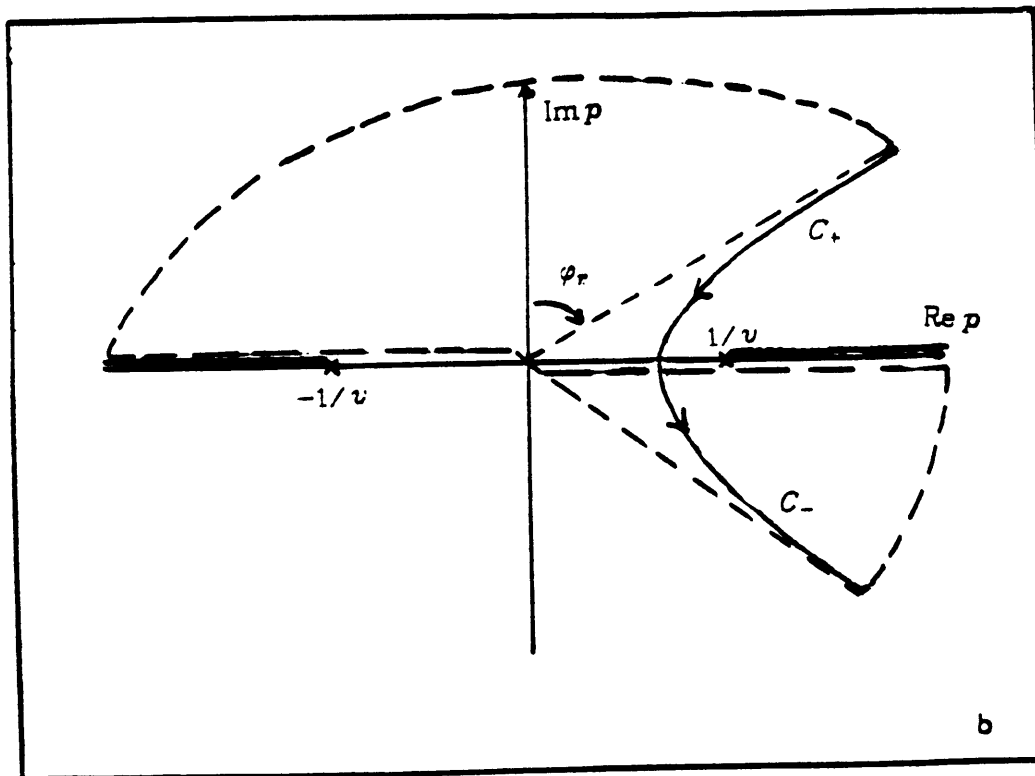
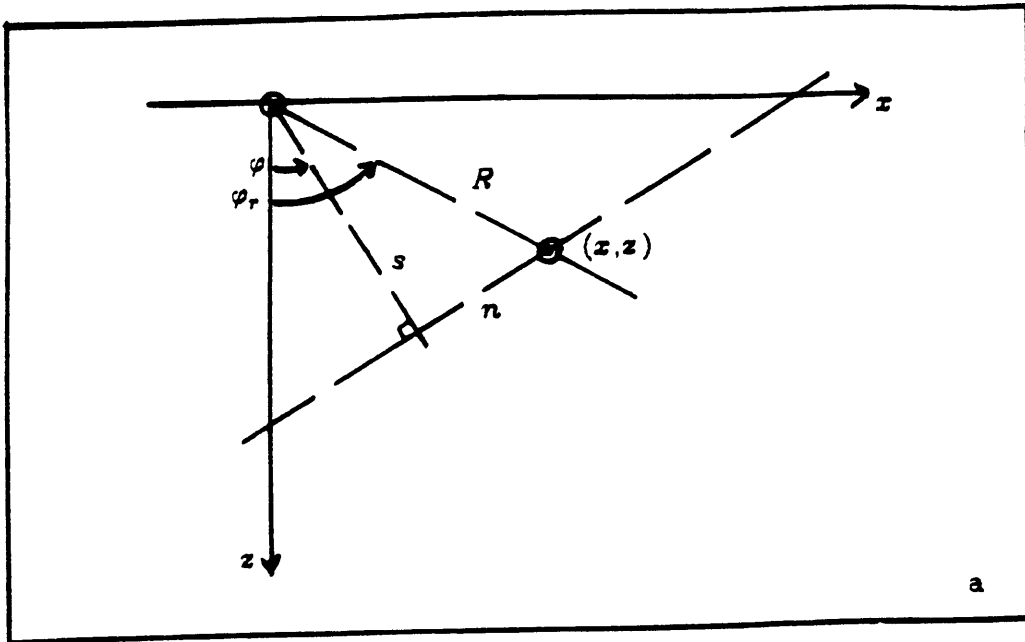


Figure 2



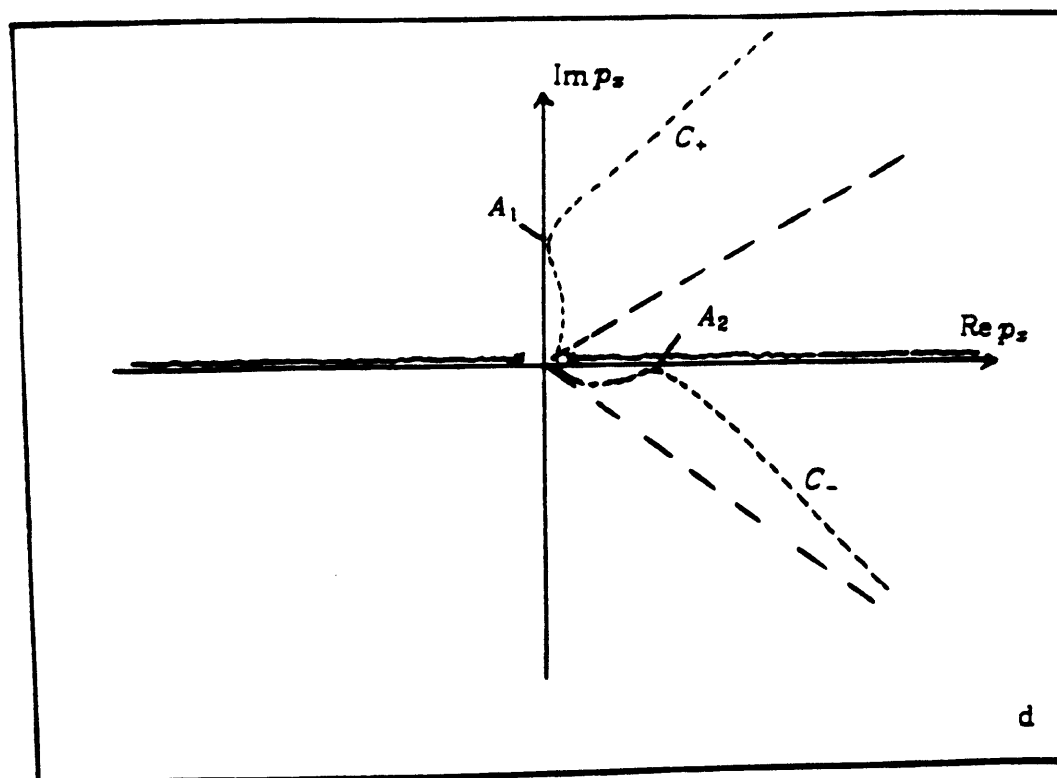
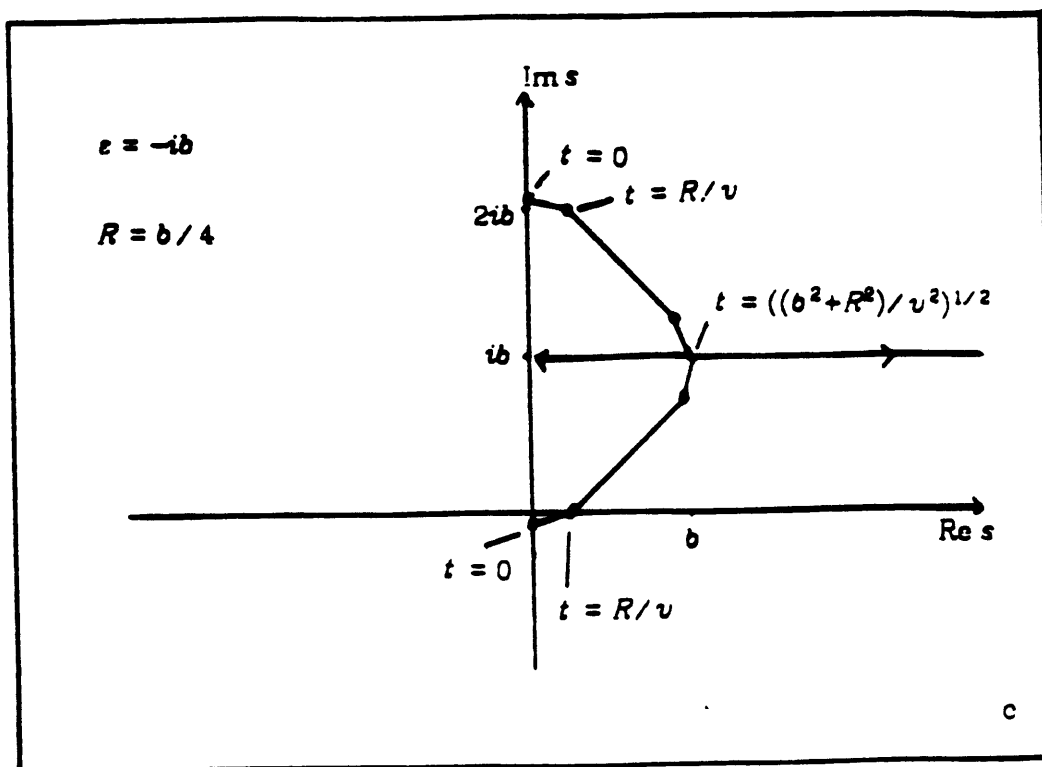


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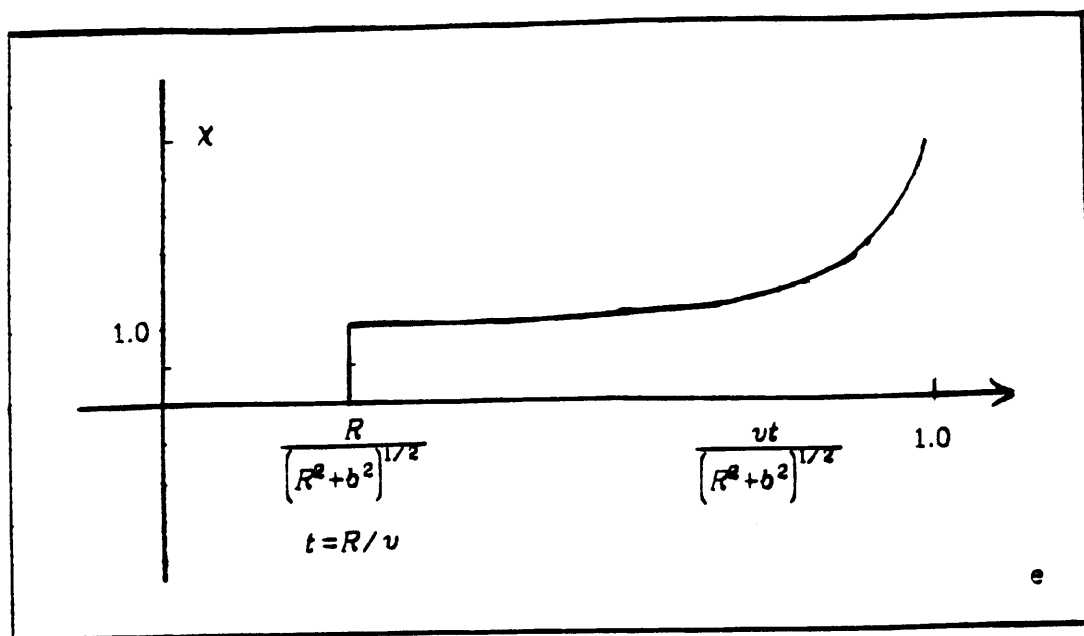


Figure 3

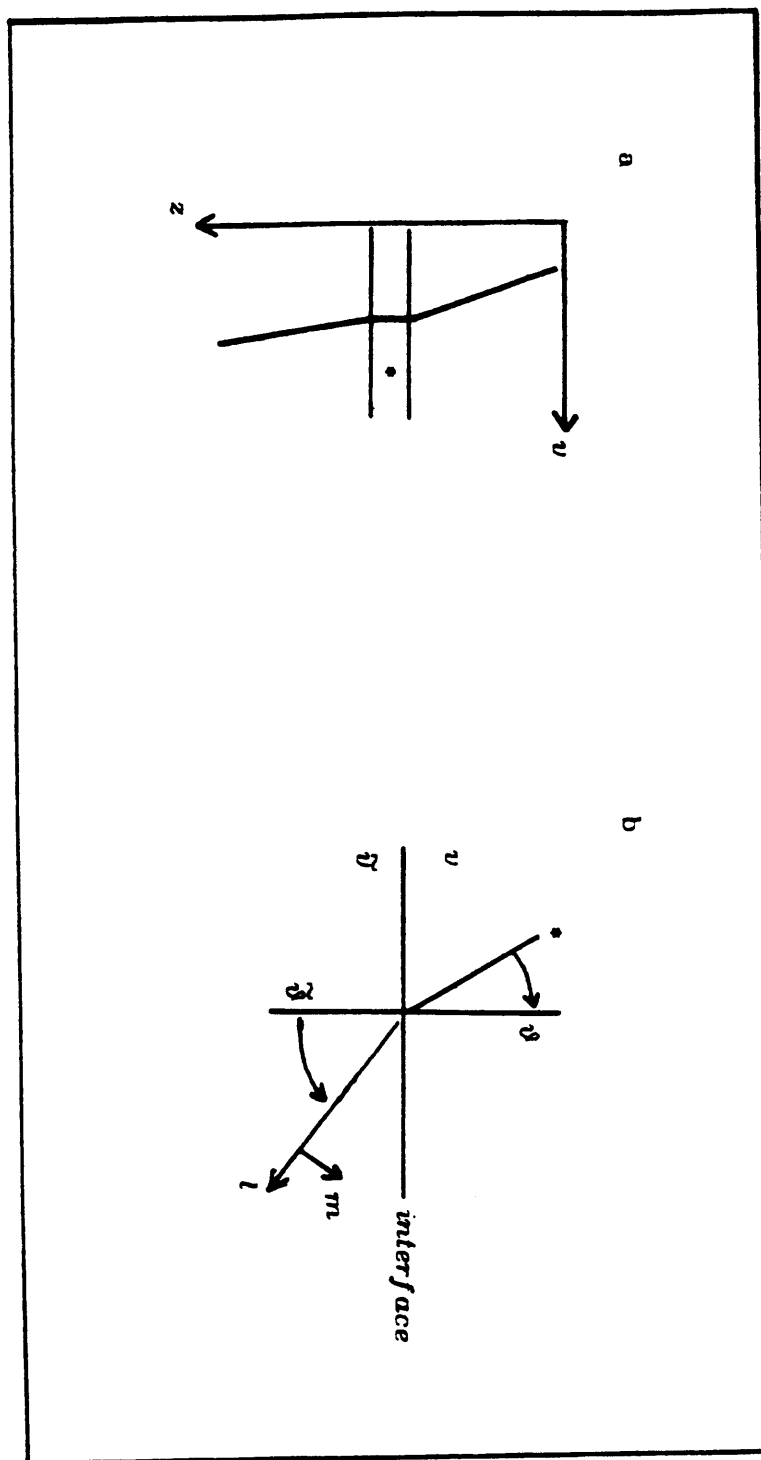
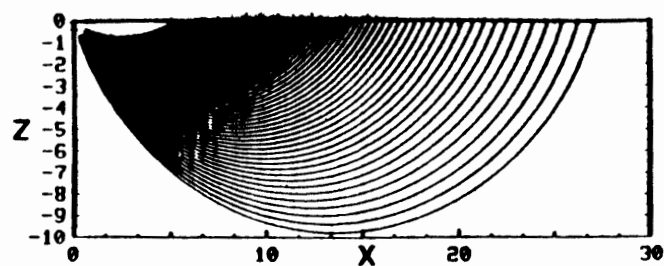


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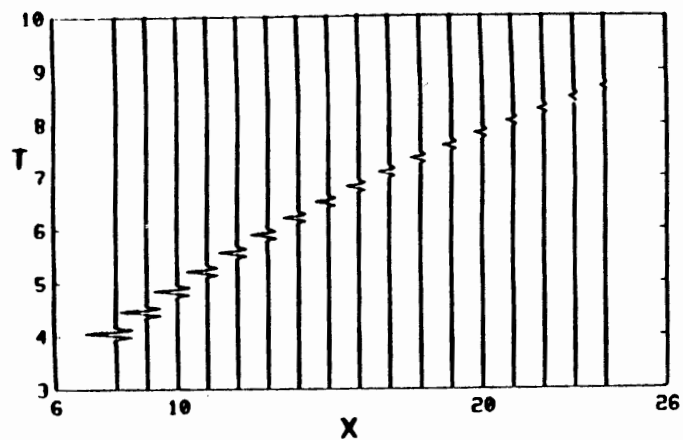


LINEAR GRADIENT

1.8 to 5.65 km/sec

from 0-10 km

C. l.c.



M. l.c.

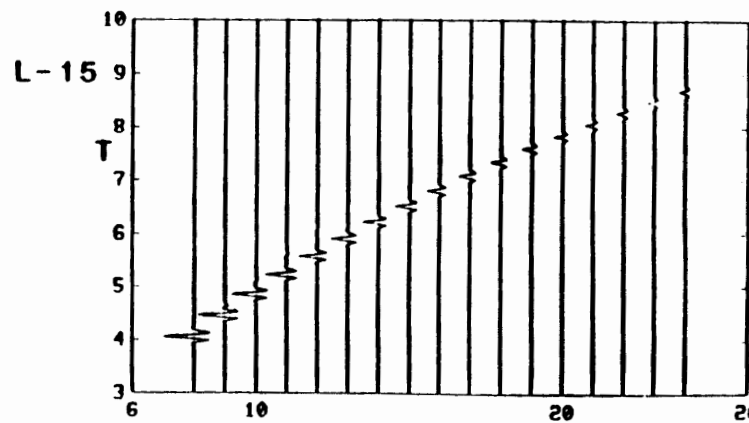
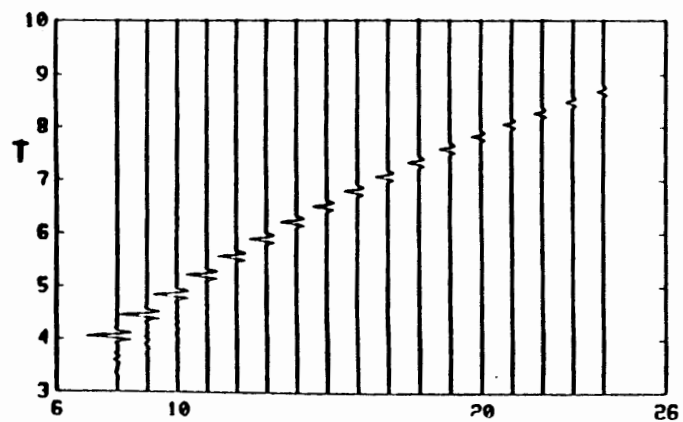
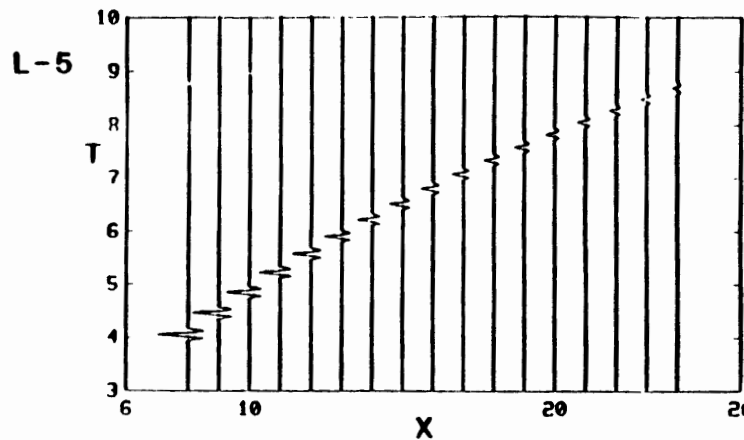


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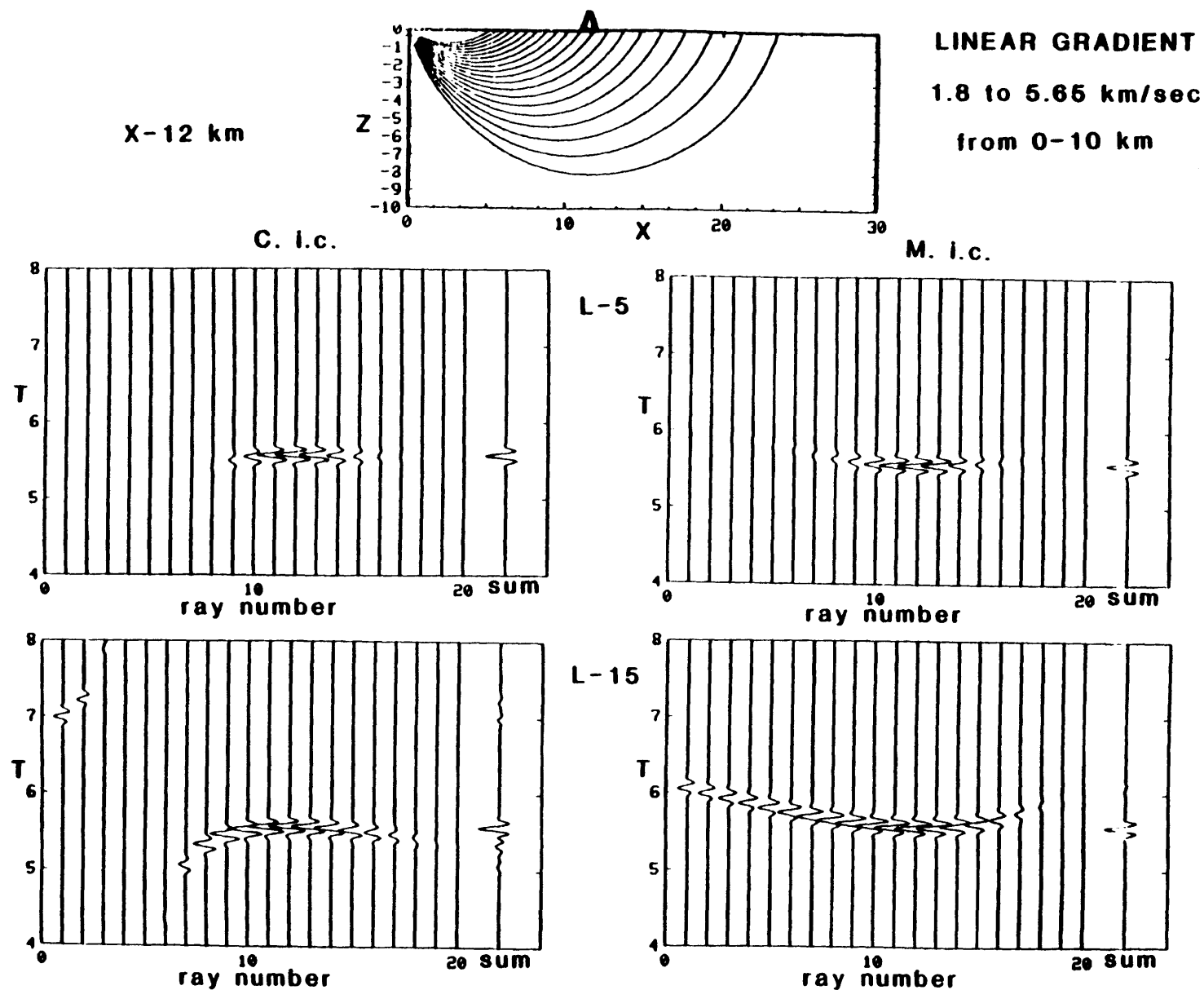


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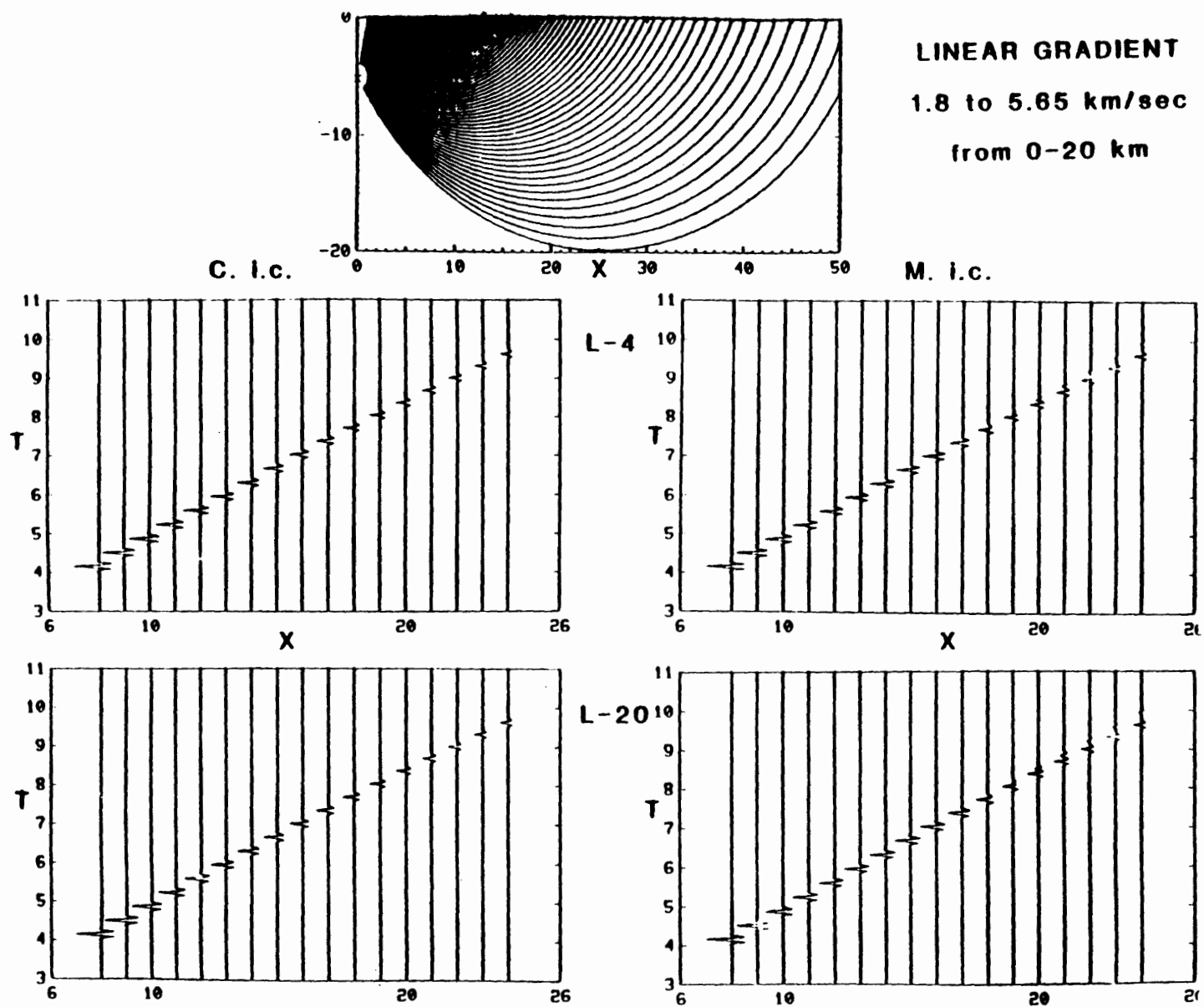


Figure 7

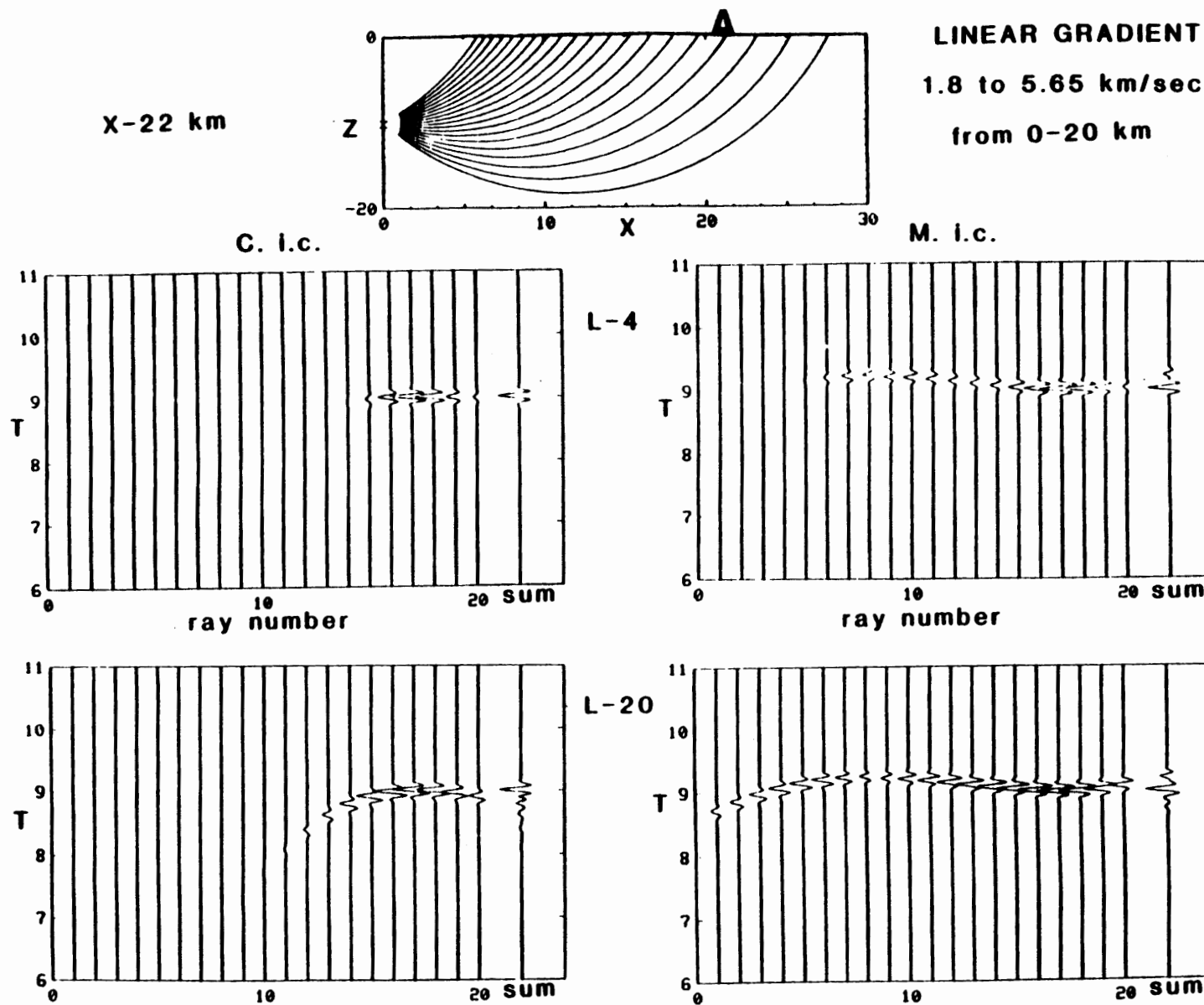


Figure 8

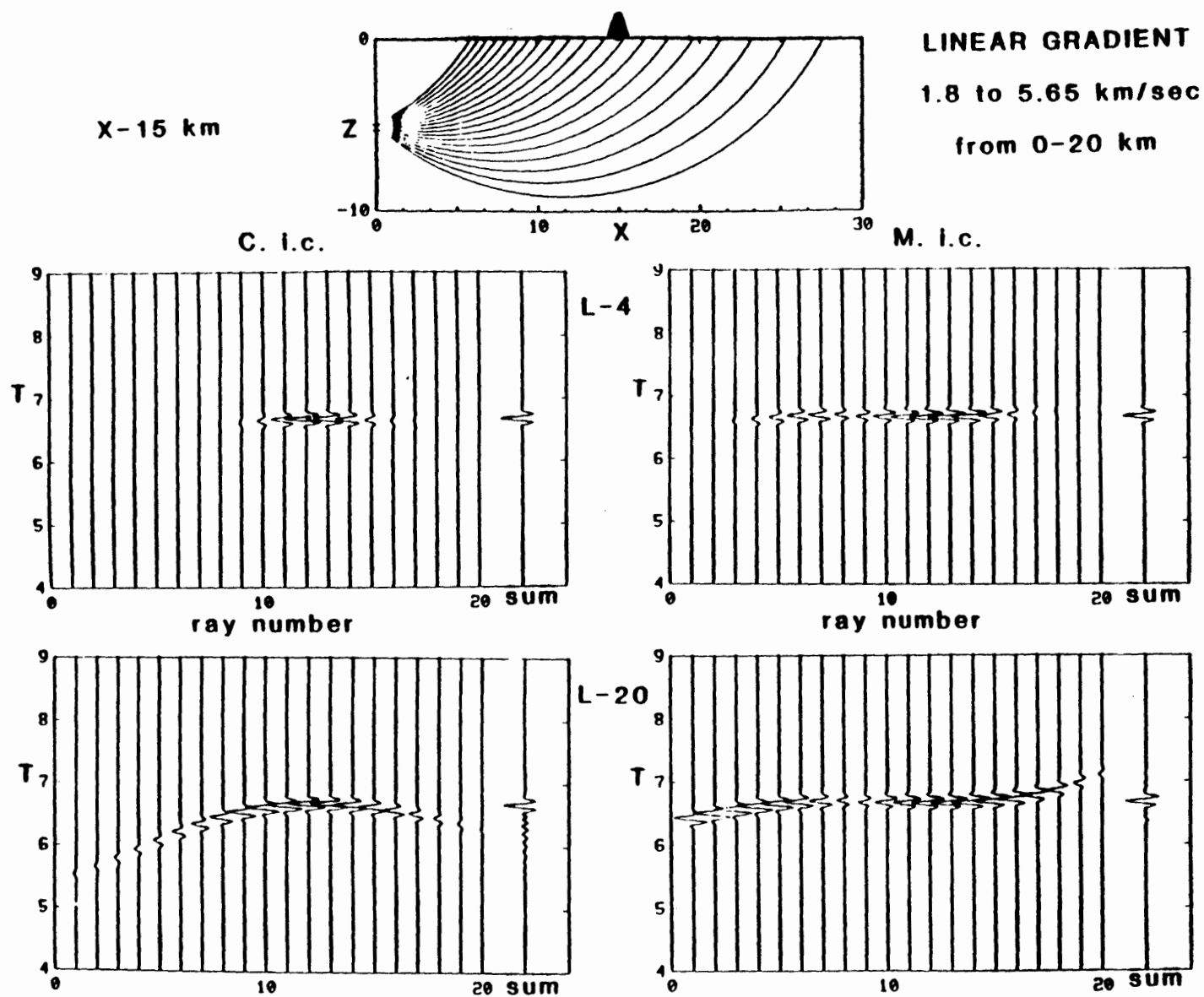


Figure 9

PP PHASE IN IMPERIAL VALLEY STRUCTURE

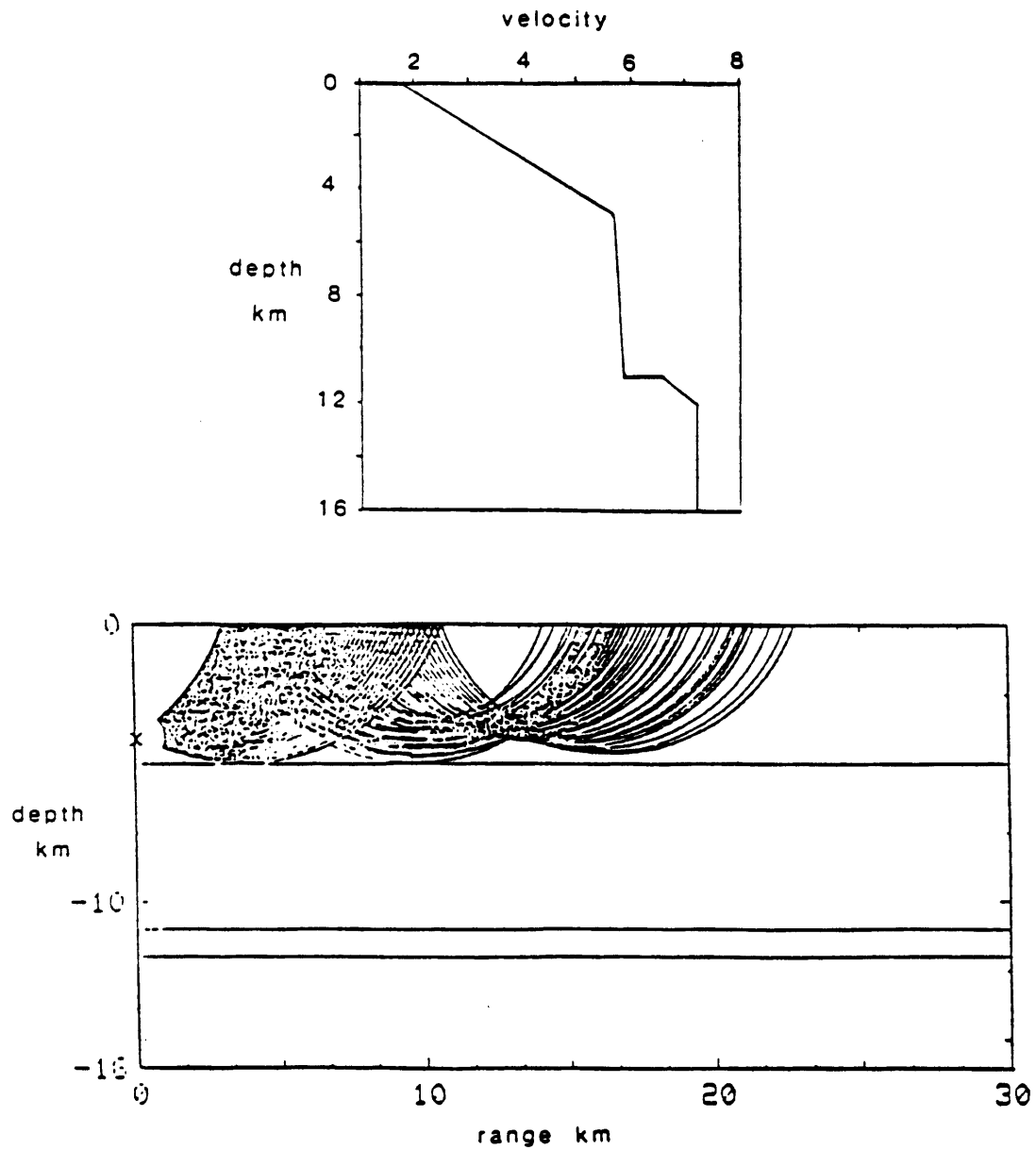


Figure 10

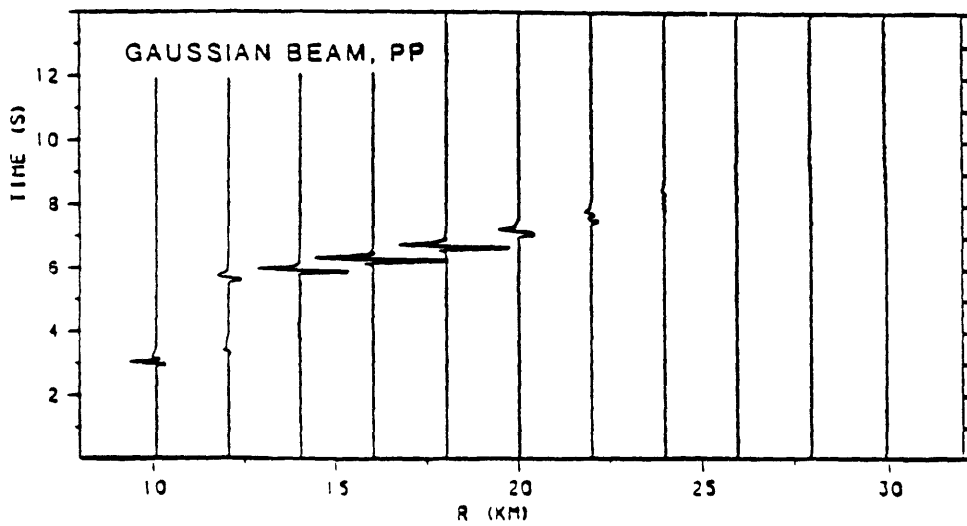
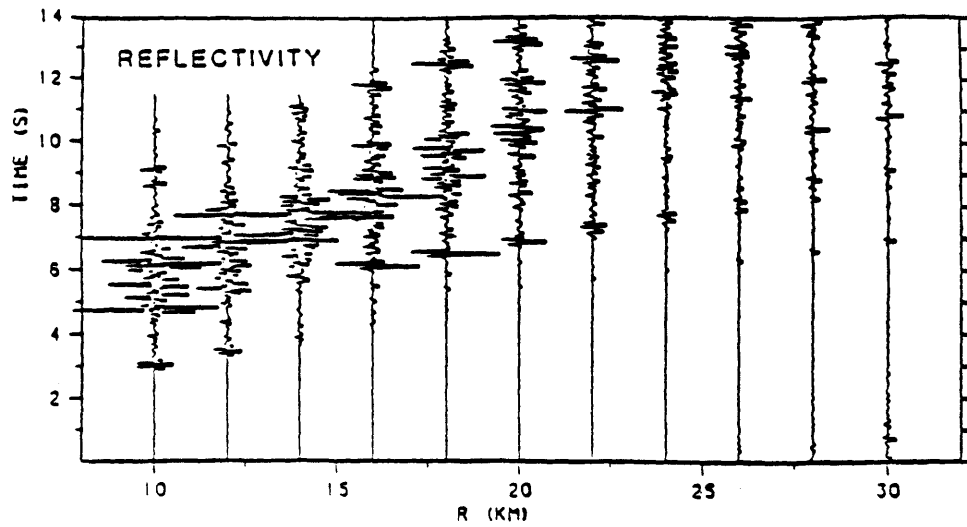


Figure 11

LAYER OVER GRADIENT

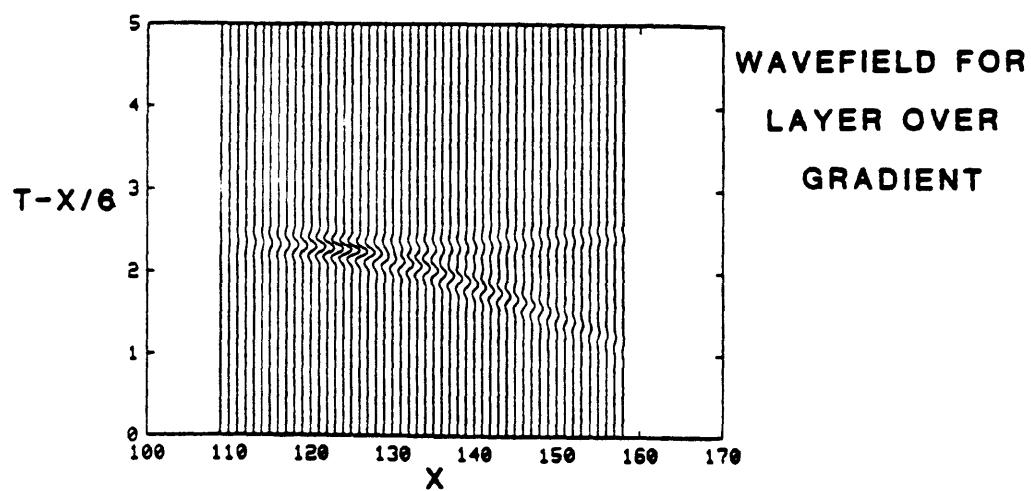
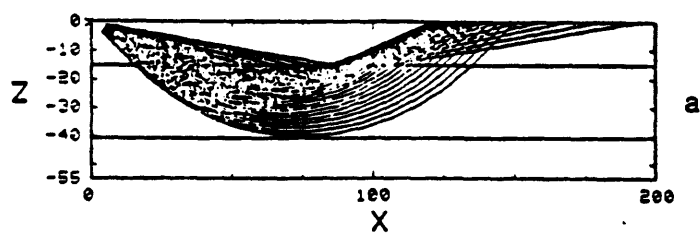


Figure 12

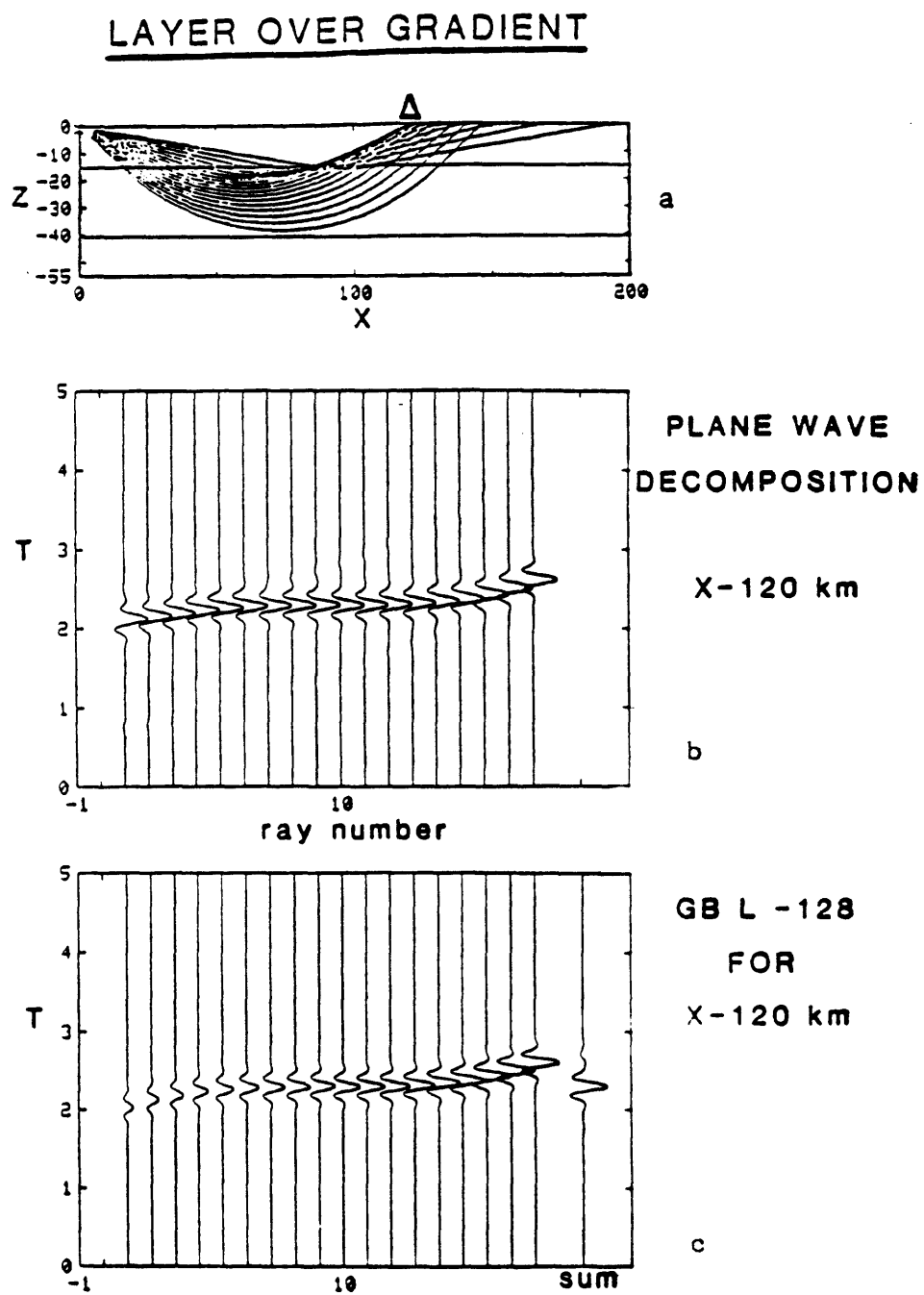


Figure 13

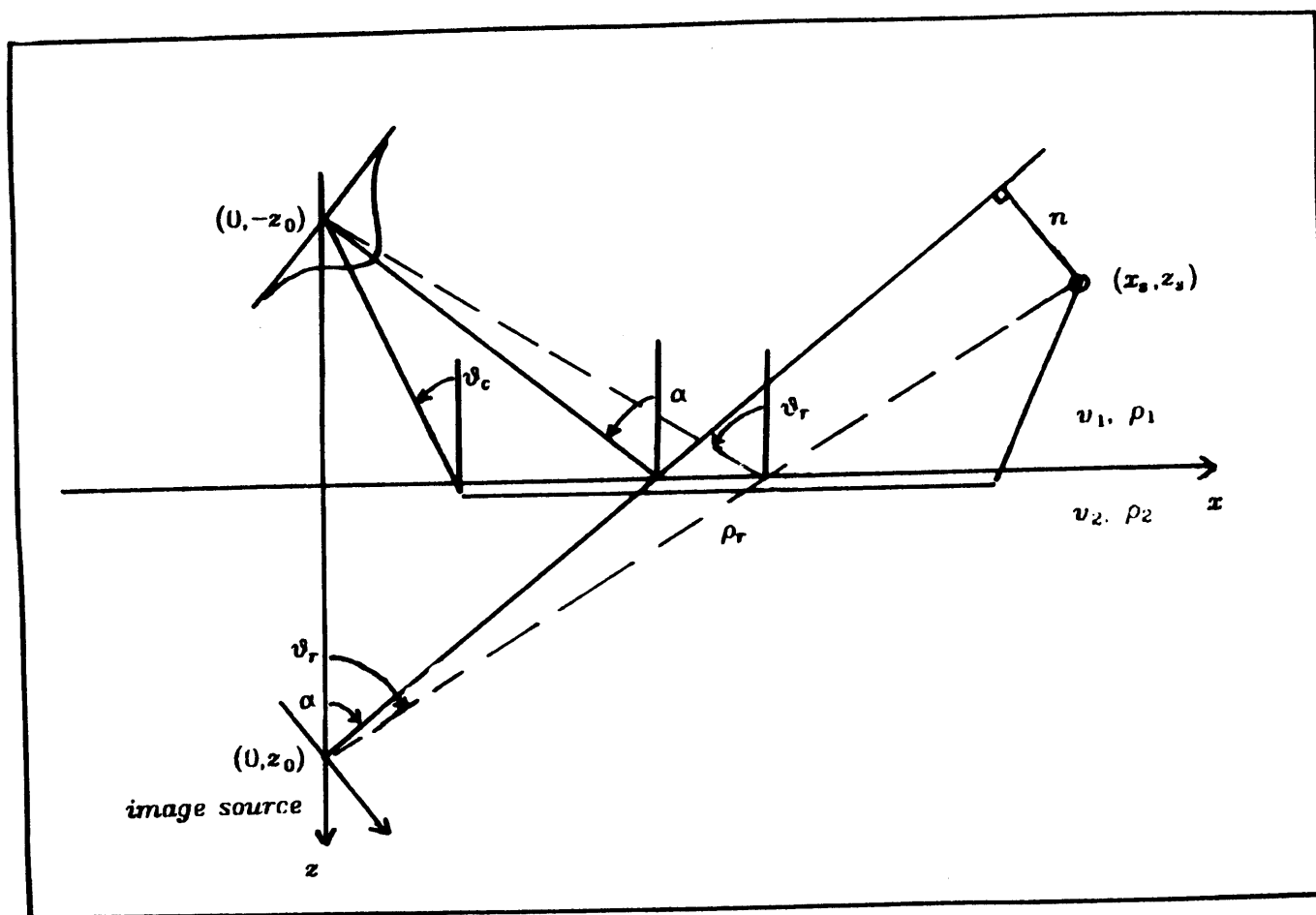
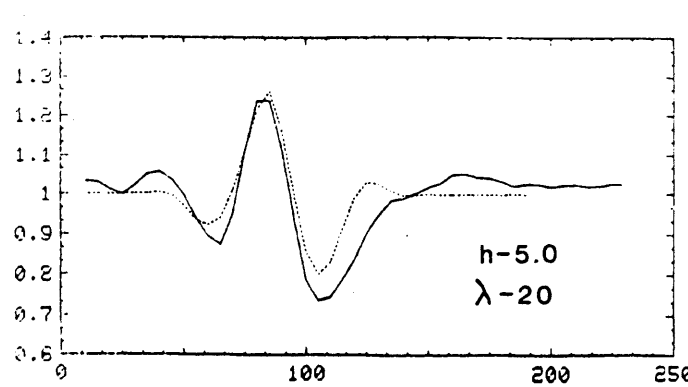
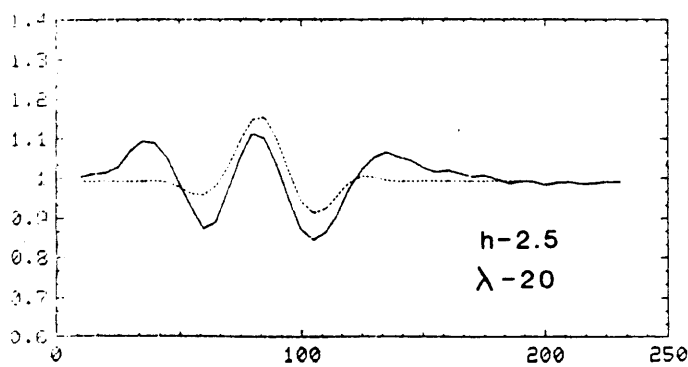
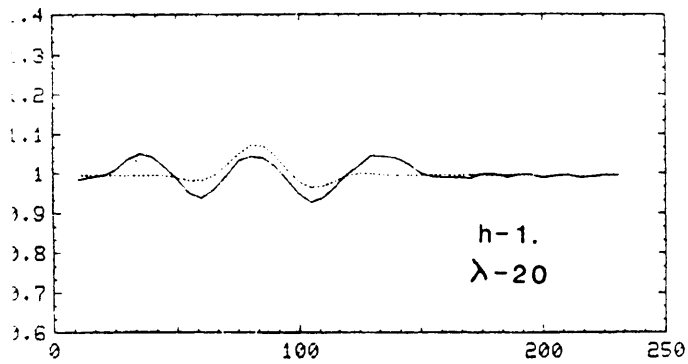
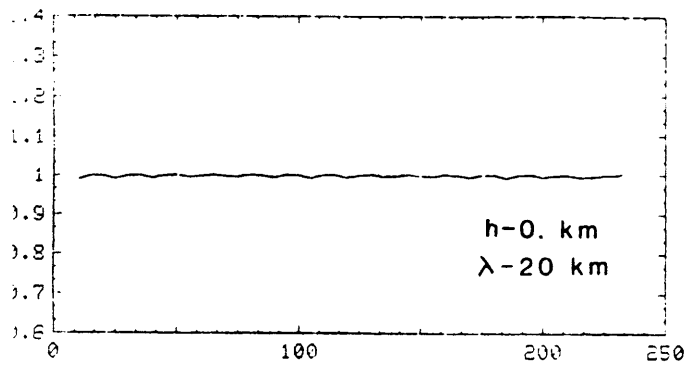
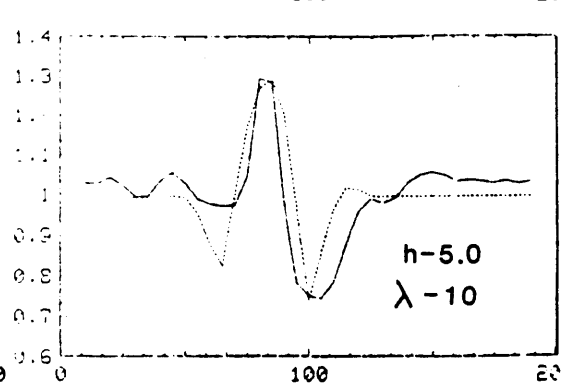
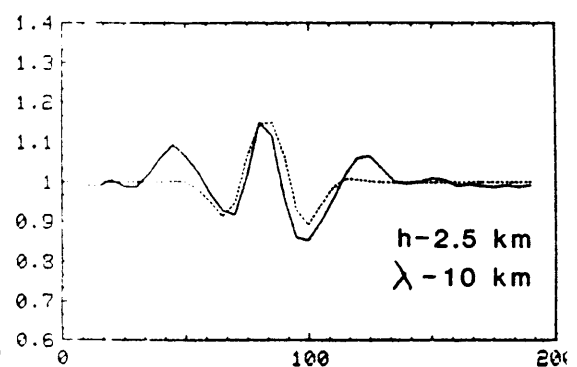
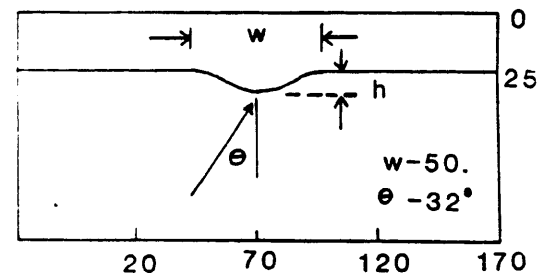


Figure 14



INDENTATION ON AN INTERFACE

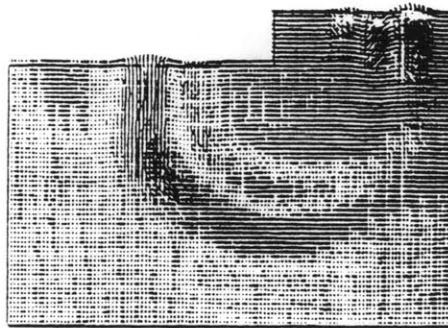


Chapter 4

Iterative inversion for velocity using waveform data

by

Robert Nowack and Keiiti Aki



Abstract

In this paper, inversion for material parameters using waveform data is investigated. An iterative linearized approach is used in which a linear sensitivity operator must be derived. This can be done economically by using reciprocity of the Green's function. In order to avoid a large matrix inversion, several descent algorithms are described. Data errors and a priori model information are incorporated using covariance operators. A fast and reasonably accurate forward modelling scheme is required, and here we make use of the Gaussian beam method in a slowly varying heterogeneous medium. Different types of linearizations can be done including a linearization in terms of the field, the Born approximation, and a linearization in terms of the log field, the Rytov approximation. The relative merits are discussed. Finally several numerical examples are performed using a field linearization in order to test the method. Transmitted body waves through a model with sources beneath a heterogeneity and surface stations are used. The results using the waveform data identify the trial structures. A comparison is done with the travel time inversion results.

Introduction

In this paper, an iterative inversion method for earth structure is investigated. Previous inversions for velocity using array data have mostly used travel times in constructing the tomography problem (see Aki [1977], Aki et.al. [1977]). Recently iterative methods, such as the algebraic reconstruction method, have been applied to travel time inversions (see McMechan [1983], Clayton and Comer [1983], Humphreys et.al. [1984]). Several of these iterative algorithms are described by Censor [1981]. The iterative methods have advantages over generalized inversion methods when dealing with large sparse matrices, but can have convergence problems when small eigenvalues are present (Ivansson [1983]). Still, there is much more information in seismic data than just first arrival travel times, including amplitudes and waveforms. Figure 1 shows a sketch of a heterogeneous region with surrounding sources and receivers. A simple transmitted waveform is shown.

A number of studies have incorporated amplitude and waveform data in seismic inversion and interpretation. For example, spatial variations of phase time and amplitude from teleseismic body waves recorded at the Montana LASA were interpreted by Larner [1970] in terms of a dented Moho. Haddon and Husebye [1978] and Thomson and Gubbins [1982] did joint interpretations of travel time and amplitude to infer the structure beneath NORSAR. Waveform inversion using refraction data assuming a

vertically varying structure has been done by Shaw [1983]. Brown [1984] did a comparison of travel time inversions and waveform inversions using synthetic data also in a vertically varying structure. Migration of reflection seismic data is a form of inversion (see Schneider [1978]), and it has been recently shown to be kinematically similar to a linearized Born inversion (see Tarantola [1984b], Miller et.al. [1984]). Direct inversions based on the linearized model have been investigated by Cohen and Bleistein [1979] and Raz [1981]. There has also been some progress in generalizing exact 1-D inversion methods to higher dimensions (see Newton [1983]).

A recent formulation by Tarantola [1984a] uses an iterative linearized approach and is described in the next section. A linear sensitivity operator must be derived, and this can be done economically by using reciprocity of the Green's function. Several descent algorithms are described which avoid a large matrix inversion. A fast and reasonably accurate forward modelling scheme is required, and here we make use of the Gaussian beam method in a slowly varying medium. Data errors and a priori information are incorporated using covariance operators. Different types of linearizations are possible including a linearization in terms of the field, the Born approximation, and a linearization in terms of the log of the field, the Rytov approximation. Their relative merits are discussed. Finally several numerical examples are performed using a linearization in terms of the field and transmitted waveforms in order to test the method.

Linearized inversion

In this section a review of the linearized inverse approach is given following Madden [unpublished lecture notes], Tarantola [1984a, 1984b, 1984c], and Lailly [1983]. In the linearized approach, first a linear sensitivity or Frechet derivative operator must be derived and then used in an iterative procedure for the model parameters. In addition, several descent algorithms are described which don't require the inversion of a large matrix.

The forward problem can be written in the form

$$L (p(\vec{r}, t)) = S(\vec{r}, t) \quad (1)$$

where L is a model dependent differential operator, $p(\vec{r}, t)$ is the field variable, and $S(\vec{r}, t)$ is the source term. For example, for the scalar wave equation

$$L = \left[\frac{1}{v(\vec{r})^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \quad (2)$$

The solution of (1) can be written

$$p(\vec{r}, t) = \int dV(\vec{r}') g(\vec{r}, t; \vec{r}') * S(\vec{r}', t) + \text{boundary terms}$$

where $g(\vec{r}, t; \vec{r}')$ is the Green's function from $\vec{r}'$ to $\vec{r}$, and $*$ is a time convolution. This can be derived from the bilinear identity [see Lanczos, 1961]. For the scalar wave equation the Kirchhoff integral is the boundary term. In the following, homogeneous boundary conditions will be assumed

giving zero for the boundary term. Finally, for a self adjoint operator L with homogeneous boundary conditions, the Green's function is reciprocal with respect to source and receiver location. Thus

$$g(\vec{r}, t; \vec{r}_g) = g(\vec{r}_g, t; \vec{r})$$

In order to obtain a linear sensitivity operator, a perturbed problem is constructed

$$\left[L + \delta L \right] \left[p + \delta p \right] = S + \delta S \quad (3)$$

or

$$L(\delta p) = -\delta L \left[p + \delta p \right] + \delta S$$

Assuming that $\delta L \delta p$ is small is equivalent to the Born approximation which requires that the perturbed field δp be much smaller than the unperturbed primary field p . To first order, (3) can be written

$$L(\delta p) = -\delta L \left[p \right] + \delta S \quad (4)$$

where $-\delta L(p)$ is an equivalent source term for the medium perturbations and δS is the term for the source perturbations. Here we will investigate the "inverse medium problem" for a given source, thus $\delta S = 0$. The solution for the perturbed field can then be written as a space integration over secondary equivalent sources

$$\delta p(\vec{r}_g, t; \vec{r}_s) = -\int dV(\vec{r}) g(\vec{r}, t; \vec{r}_g) * \delta L \left[p(\vec{r}, t; \vec{r}_s) \right] \quad (5)$$

where δL includes the model perturbations, and p is the incident primary field. Equation (5) can be written as

$$\delta p(\vec{r}_g, t; \vec{r}_s) = \frac{\partial p}{\partial L} \delta L = F \delta L \quad (6)$$

where F is a linear sensitivity or Frechet derivative operator, and the operator δL has imbedded within it the model variations. Assuming that the primary field can be written as $p(\vec{r}, t; \vec{r}_s) = g(\vec{r}, t; \vec{r}_s)$, then

$$\delta p(\vec{r}_g, t; \vec{r}_s) = - \int dV(\vec{r}) g(\vec{r}, t; \vec{r}_g) * \delta L \left[g(\vec{r}, t; \vec{r}_s) \right]$$

Thus the linear sensitivity operator can be constructed by computing $g(\vec{r}, t; \vec{r}_s)$ which propagates the field from the source $\vec{r}_s$ to each model point $\vec{r}$, and $g(\vec{r}, t; \vec{r}_g)$ which backpropagates the field back from the geophones into the model at $\vec{r}$. This requires the computation of $N_g + N_s$ forward problems evaluated at the interior points of the model where N_g is the number of geophones and N_s is the number of sources.

For a given L , the perturbed operator, δL , can be separated into operations on the forward field, $p(\vec{r}, t; \vec{r}_s)$, and perturbations of the model parameters. For example for the scalar wave equation

$$\delta p(\vec{r}_g, t; \vec{r}_s) = \int dV(\vec{r}) \left\{ \frac{2}{v^3(\vec{r})} \ddot{p}(\vec{r}, t; \vec{r}_s) * g(\vec{r}, t; \vec{r}_g) \right\} \delta v(\vec{r}) \quad (7)$$

where the term in the brackets is the linear sensitivity operator, $F = \frac{\partial p}{\partial v}$,

$\delta v(\vec{r})$ is the velocity model perturbation at $\vec{r}$, and $\ddot{p}(\vec{r}, t; \vec{r}_s)$ is the second time derivative of the primary field computed from the source point to the point $\vec{r}$ in the medium. The linear sensitivity operators for the elastic wave equation with respect to the field are given by Tarantola

[1984c]. The linear sensitivity operators for the acoustic wave equation with respect to the field are given by Tarantola [1984a], and used in a simple 1-D example in Appendix 3.

In the following, the linearized problem (5) will be imbedded within an iterative procedure. The effect of ignoring the $\delta L \delta p$ term in the linearization will result in no multiple interactions with the perturbations within δL , or no "cross talk" between perturbations during each iteration. Multiple interactions from previous iterations are included via the Green's functions. In principle, an iterative strategy should be able handle large velocity contrasts. This approach requires the calculation of Green's functions in inhomogeneous media inexpensively and reasonably accurately. Various forward modelling approaches could be considered including ray theoretical methods, the discrete wavenumber method, or the finite difference or finite element methods. Any inversion method is only as good as the forward modelling scheme on which it is based.

The forward problem as a function of model parameters is in general nonlinear function and can be written

$$\vec{p} = f(\vec{m}) \quad (8)$$

where $\vec{p}$ is the field and $\vec{m}$ is the model parameters. In general, $\vec{p}_{obs} \neq f(\vec{m}_{prior})$, where $\vec{p}_{obs}$ is the observed field and $\vec{m}_{prior}$ is the a priori model based on previous information. The objective is to find the combined vector $\begin{bmatrix} \vec{p}, \vec{m} \end{bmatrix}$ that satisfies $\vec{p} = f(\vec{m})$ and is closest to

$\left[\vec{p}_{obs}, \vec{m}_{prior} \right]$ in some sense. This distance could be defined in various ways depending on the noise structure of the problem, including the L_2 norm, the L_1 norm, or the L_∞ norm. Here the weighted L_2 norm will be considered with it's associated induced inner product. This results in a least squares formulation. Thus we want to minimize the functional [see Tarantola and Valette, 1982, Tarantola, 1984a]

$$S(\vec{m}) = \frac{1}{2} \left\{ \left| \vec{p}_{obs} - f(\vec{m}) \right|^2 + \left| \vec{m} - \vec{m}_{prior} \right|^2 \right\} \quad (9)$$

where the $1/2$ has been introduced for later convenience. The linear sensitivity operator, F_k at $\vec{m}_k$ can then be defined as

$$f(\vec{m}_k + \delta \vec{m}) = f(\vec{m}_k) + F_k \delta \vec{m} + O(|\delta \vec{m}|^2)$$

The negative of the gradient of $S(\vec{m}_k)$ gives the local direction of maximum descent at $\vec{m}_k$. To first order this can be written

$$\vec{\gamma}_k = -\nabla S(\vec{m}_k) = \left\{ F_k^* \delta \vec{p} - (\vec{m}_k - \vec{m}_{prior}) \right\} \quad (10)$$

where $\delta \vec{p}_k = \vec{p}_{obs} - f(\vec{m}_k)$, and F_k^* is the adjoint of the linear sensitivity operator, F_k . The adjoint is defined from the bilinear identity

$$\langle \delta \vec{p}, F \delta \vec{m} \rangle_{\delta p} = \langle F^* \delta \vec{p}, \vec{m} \rangle_{\delta m} = \text{boundary term} \quad (11)$$

where $\langle, \rangle$ is a defined inner product. The boundary term is assumed to be zero using homogeneous boundary conditions. Thus, F is a linear mapping from model to data space, and the adjoint, F^* , is a linear

mapping from data space back to model space. Inverse operators are in general difficult to construct, but adjoint operators through the bilinear identity are straightforward to obtain. In addition, adjoint operators have very useful properties which aide in the construction of inverse and generalized inverse operators.

Consider a linear problem, $F\vec{m} = \vec{p}_{obs}$, with noisy observations but where the nullspace of F is zero, $N(F) = \vec{0}$. The nullspace, $N(F)$, represents the subspace of the model space that satisfies $F\vec{m} = \vec{0}$. The standard least squares solution results by using the adjoint theorem, $R^\perp(F) = N(F^*)$, where R is the range of $F\vec{m}$ in data space. Thus mapping $F\vec{m}$ and $\vec{p}_{obs}$ through F^* annihilates the component of $\vec{p}_{obs}$ in $R(F)$ resulting in a consistent set of equations. Then, $F^*F\vec{m} = F^*\vec{p}_{obs}$ or $\vec{m} = (F^*F)^{-1}F^*\vec{p}_{obs}$. For $N(F) = 0$, (F^*F) is full rank and an inverse exists. The standard least squares solution thus results from two operations on the data, $\vec{p}_{obs}$. First, the data is operated on by the adjoint operator, F^* , which projects $\vec{p}_{obs}$ from data space to model space. This blurred image in model space is then filtered with the operator $(F^*F)^{-1}$. When $N(F) \neq 0$, a generalized inverse can be computed by using additional adjoint theorems. This is equivalent to the Lanczos formulation which results in a minimum-norm least-squares solution [Aki and Richards, 1980]. An alternative is to stabilize the (F^*F) operator using a maximum likelihood procedure which reduces to minimizing a functional like (9) with appropriate data and model covariance operators defined. Any a priori

knowledge about the component of the model in the null space can be incorporated into the final solution. This might include smoothness of the solution.

The maximum likelihood algorithm of Tarantola and Valette [1982] is a linear iterative procedure for solving the nonlinear inverse problem, (8), by minimizing the functional (9). It can be written

$$\vec{m}_{k+1} = \vec{m}_k + [H_k]^{-1} \left\{ F_k^* \delta \vec{p} - (\vec{m}_k - \vec{m}_{prior}) \right\} \quad (12)$$

where the term in the brackets is the negative gradient of the functional $S(\vec{m})$, and

$$H_k = [I + F_k^* F_k] \quad (13)$$

Covariance operators can be introduced by a suitable definition of the data and model space scalar products [see Tarantola, 1984c]. Thus

$$\begin{aligned} \langle \vec{m}_1, \vec{m}_2 \rangle_m &= \vec{m}_1^T C_m^{-1} \vec{m}_2 \\ \langle \vec{p}_1, \vec{p}_2 \rangle_p &= \vec{p}_1^T C_p^{-1} \vec{p}_2 \end{aligned}$$

where $\vec{m}^T$ is the transpose of $\vec{m}$, C_m is the model covariance operator, and C_p is the data covariance operator. From this definition of the scalar products, the adjoint operator, F_k^* , of F_k is related to the transpose by

$$F_k^* = C_m F_k^T C_p^{-1} \quad (14)$$

The adjoint is equal to the transpose when $C_m = I$ and $C_p = I$. For a complex operator F , the transpose is replaced by the conjugate transpose operator.

A major problem with the iterative scheme (12) is that in discretized form it requires the inversion of a possibly large matrix. Since the goal is to iteratively solve the original problem, (8), only an approximate solution of the the imbedded linearized problem may be required. Thus a simple iterative scheme could be written

$$\vec{m}_{k+1} = \vec{m}_k + \alpha_k W_k \vec{\gamma}_k \quad (15)$$

where $\vec{\gamma}_k$ is the negative gradient of the functional $S(\vec{m})$, W_k is called a preconditioner, and α_k is a scalar that approximately solves a 1-D line search of $S(\vec{m})$ in the direction $W_k \vec{\gamma}_k$. α_k is thus chosen to minimize $S(\vec{m}_k + \alpha_k W_k \vec{\gamma}_k)$ with respect to α_k giving

$$\alpha_k = \frac{\langle W_k \vec{\gamma}_k, \vec{\gamma}_k \rangle}{\langle W_k \vec{\gamma}_k, H_k W_k \vec{\gamma}_k \rangle} \quad (16)$$

where H_k is given in (13). For the maximum likelihood algorithm, (11), $W_k = H_k^{-1}$ giving $\alpha_k = 1$. This would then be similar to a modified Newton's method for solving the nonlinear problem, (8), dependent on the choice of the covariance operators used.

Various simplifying choices for W_k could be considered. Choosing $W_k = I$ results in the steepest descent method. Thus

$$\vec{m}_{k+1} = \vec{m}_k + \alpha_k \vec{\gamma}_k \quad (17)$$

where

$$\alpha_k = \frac{\langle \tilde{\gamma}_k, \tilde{\gamma}_k \rangle}{\langle \tilde{\gamma}_k, H_k \tilde{\gamma}_k \rangle}$$

with H_k given in (13). For a quadratic $S(\mathbf{m})$, $\tilde{\gamma}_{k+1}$ and $\tilde{\gamma}_k$ will be perpendicular. One thus moves in the direction $\tilde{\gamma}_k$ until tangent to a level curve of $S(\tilde{\mathbf{m}})$. At this point, $\tilde{\gamma}_{k+1}$ is chosen perpendicular to the level curve in a direction of maximum descent.

Another possible choice for W_k is [see Tarantola, 1984a]

$$W_k = \left[\text{DIAG} (H_k) \right]^{-1} \quad (18)$$

This is similar to a modified Jacobi method with suitably defined covariance operators.

In any of the preconditioning strategies, the single step convergence properties for quadratic functionals, $S(\tilde{\mathbf{m}})$, are governed by the difference in the smallest and largest eigenvalues of $W_k H_k$ with H_k given in (13). From Luenberger(1984),

$$S(\tilde{\mathbf{m}}_k + 1) \leq \left[\frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} \right]^2 S(\tilde{\mathbf{m}}_k)$$

Thus the closer W_k is to H_k^{-1} , the better the single step convergence. A general overall strategy is to construct a preconditioning scheme that is easy to compute and possesses a favorable eigenvalue structure at each step.

Instead of using the gradient directions in the steepest descent approach, improved convergence for very little extra effort can be gotten by using so called conjugate gradient directions defined with respect to a new inner product, $\langle x, x \rangle = x^T H_k x$ where H_k is given in (13). The Fletcher-Reeves implementation of the conjugate gradient method to nonquadratic problems is briefly outlined. First, given $\vec{m}_{prior}$ and $\vec{\gamma}_0 = -\nabla S(\vec{m}_{prior})$, set $\vec{d}_0 = \vec{\gamma}_0$. Now let $\vec{m}_{k+1} = \vec{m}_k + \alpha_k \vec{d}_k$ where α_k minimizes $S(\vec{m}_k + \alpha_k \vec{d}_k)$. The next direction of descent, $\vec{d}_{k+1}$ is chosen to be H_k -orthogonal to the previous direction $\vec{d}_k$. This is done by setting $\vec{d}_{k+1} = \vec{\gamma}_{k+1} + \beta_k \vec{d}_k$, where $\beta_k = \langle \vec{\gamma}_{k+1}, \vec{\gamma}_{k+1} \rangle / \langle \vec{\gamma}_k, \vec{\gamma}_k \rangle$. $\vec{d}_{k+1}$ is thus a direction modified from the steepest descent direction. In a similar fashion to steepest descent, $\vec{d}_{k+1}$ and $\vec{d}_k$ are orthogonal but with respect to H_k . The conjugate gradient method has the important property that the estimate $\vec{m}_k$ minimizes a quadratic functional $S(m)$ over a subspace spanned by all the previous directions $(\vec{d}_0, \dots, \vec{d}_{k-1})$ [see Luenberger, 1984].

All of the above methods require the the linear sensitivity operator F_k and the adjoint of this operator F_k^* . For example for the scalar wave equation (2)

$$F_k = \left\{ \frac{2}{v^3(r')} \ddot{p}(r', t, r_s) * g(r', t; r_g) \right\} = \frac{\partial p}{\partial v} \quad (19)$$

where velocity v is the model parameter. Thus to first order $\delta p = \frac{\partial p}{\partial v} \delta v$.

Using the bilinear identity, (11), with $C_m = I$ and $C_p = I$ and homogeneous boundary conditions, then $F^* = F^T$ and

$$\begin{aligned} \langle \delta p, F \delta v \rangle_p &= \sum_{r_s} \sum_{r_g} \int dt \delta p F(\delta v) \\ &= \sum_{r_s} \sum_{r_g} \int dt \delta p(r_g, t; r_s) \int dV(r') \left\{ \frac{2}{v^3(r')} \ddot{p}(r', t; r_s) * g(r', t; r_g) \right\} \delta v(r') \end{aligned}$$

Then,

$$\begin{aligned} \langle F^* \delta p, \delta v \rangle_v &= \int dV(r') F^*(\delta p) \delta v \\ &= \int dV(r') \left[\sum_{r_s} \sum_{r_g} \int dt \left\{ \frac{2}{v^3(r')} \ddot{p}(r', t; r_s) * g(r', t; r_g) \right\} \delta p(r_g, t; r_s) \right] \delta v(r') \end{aligned}$$

Thus F_k^* is just F_k with summations now over geophone and source locations and an integration in time.

The final adjoint operator includes the data and model covariance operators. Assuming uncorrelated errors over source and receiver locations and time, then

$$C_p(\vec{r}_g, t; \vec{r}_s | \vec{r}_g', t'; \vec{r}_s') = \sigma_{gs}^2 \delta_{gg'} \delta_{ss'} \delta_{tt'} \quad (20)$$

where σ_{gs} represents the estimated error in the trace corresponding to the g^{th} receiver and the s^{th} source. For the model covariance operator, a spatial Gaussian correlation is a commonly used choice [see Aki and Richards, 1980; Tarantola, 1984a]. Thus

$$C_v(\vec{r}, \vec{r}') = (2\pi)^{-3/2} \frac{\sigma_v^2}{L^3} \exp \left\{ -\frac{1}{2} \frac{(\vec{r} - \vec{r}')^2}{L^2} \right\} \quad (21)$$

where v is the model parameter $\tilde{m}$, L is the medium correlation length for the Gaussian correlation function, and σ_v represents the estimated departures of $\tilde{m}(\vec{r})$ from $\tilde{m}_{prior}(\vec{r})$. As L goes to zero, then $C_v(\vec{r}, \vec{r}') = \sigma_v^2 \delta(\vec{r} - \vec{r}')$. The model covariance operator will constrain the iterated model parameters as well as act like a spatial smoothing filter. The complete adjoint can be written, $F_k^* = C_v F_k^T C_p^{-1}$, as in (14).

For the scalar wave equation, the preconditioned descent method can be written

$$\vec{v}_{k+1} = \vec{v}_k + \alpha_k W_k \left[C_v F_k^T \delta \vec{p}' - (\vec{v}_k - \vec{v}_{prior}) \right] \quad (22)$$

where

$$\delta \vec{p}' = C_p^{-1} \delta \vec{p}$$

are the weighted data residuals. Now let

$$\delta \mathcal{V} = C_v F_k^T \delta \vec{p}' \quad (23)$$

$$\delta \mathcal{V} = C_v \sum_{\vec{r}_g} \sum_{\vec{r}_s} \int dt \left\{ \frac{2}{v^3(\vec{r})} \ddot{p}(\vec{r}, t; \vec{r}_s) * g(\vec{r}, t; \vec{r}_g) \right\} \delta p'(\vec{r}_g, t; \vec{r}_s) \quad (24)$$

This can be rewritten as

$$\delta \mathcal{V}(\vec{r}) = C_v \sum_{\vec{r}_s} \sum_{\vec{r}_g} \int dt \frac{2}{v^3(\vec{r})} \ddot{p}(\vec{r}, t; \vec{r}_s) \left\{ g(\vec{r}, -t; \vec{r}_g) * \delta p'(\vec{r}_g, t; \vec{r}_s) \right\} \quad (25)$$

where the term in the brackets corresponds to the propagation of the weighted residuals from the geophones into the model backwards in time. This is then compared with the computed primary field from the sources to

the model point. A summation over all source and receiver locations is then done and an integration over time. Finally, the a priori model covariance operator is applied. This is reminiscent of an imaging principle in which the reflector (velocity perturbation) exists where the downgoing and upgoing waves coincide in time (Claerbout, 1976, Tarantola, 1984b).

In order to implement this algorithm, $p(\vec{r}, t; \vec{r}_s)$ and $g(\vec{r}, t; \vec{r}_g)$ must be computed where $\vec{r}$ is a point in the model (see Figure 2). Since for an inhomogeneous medium an inexpensive forward modelling scheme is required, one possibility is that paraxial ray theory or the Gaussian beam method be used to calculate p and g (see Cervený et al., 1982, Nowack and Aki, 1984). An advantage of using these methods is that no two-point ray tracing is required. In addition, the Gaussian beam method produces a smoothed field with no unphysical singularities in amplitude resulting from caustics which may have adverse effects on an inversion.

Using the ray approximation in 3-D,

$$g(\vec{r}, t; \vec{r}_g) \approx A_{rr_g} \delta(t - T_{rr_g})$$

$$\ddot{p}(\vec{r}, t; \vec{r}_s) \approx A_{rr_s} \ddot{S}(t - T_{rr_s})$$

where A_{rr_i} and T_{rr_i} are the amplitude and travel time computed by ray theoretical methods and $\ddot{S}(t)$ is the second derivative of the source time pulse. For the 2-D case, an additional $\pi/4$ phase shift in the far field must be included. Equation (24) is then

$$\delta v_k = C_v F_k^T \delta p' = C_v \sum_{r_s, r_g} \frac{2}{v^3(\vec{r})} A_{rr_s} A_{rr_g} \int dt \ddot{S}(t - T_{rr_s}) \delta p'(r_g, t + T_{rr_g}; r_s) \quad (26)$$

This can be rewritten as

$$\delta v_k = C_v \sum_{r_s, r_g} \frac{2}{v^3(\vec{r})} A_{rr_s} A_{rr_g} \delta \tilde{p}_k(r_g, t = T_{rr_g} + T_{rr_s}; r_s) \quad (27)$$

where

$$\delta \tilde{p}_k = \ddot{S}(-t) * \delta p'_k(r_g, t; r_s) \quad (28)$$

$\delta \tilde{p}$ is the cross correlation of the weighted residuals with the second derivative of the source time pulse. δv_k is the result of the weighted sum over the phase surface through the data defined by $t = T_{rr_g} + T_{rr_s}$ for each model point $\vec{r}$.

When the a priori starting model is close enough to the true velocity model so that the Born approximation is strictly valid, then $F_k^* = F_0^*$ for all iterations. In addition, if the a priori model is homogeneous, then

$$\delta v_k = C_v F_0^T \delta p' = C_v \sum_{r_s, r_g} \frac{1}{v^3 8\pi^2 R_{rr_s} R_{rr_g}} \delta \tilde{p}(r_g, t = \frac{R_{rr_s} + R_{rr_g}}{v}; r_s) \quad (29)$$

where $\delta \tilde{p}$ is defined above, R_{rr_s} is the distance between source and model point, and R_{rr_g} is the distance between the geophone and the model point. This is similar to a Kirchhoff migration and a cross correlation with the source time pulse [Schneider, 1978; Tarantola, 1984b]. Thus assuming the Born approximation is strictly valid, then a velocity inversion using a descent approach is equivalent to an iterative sequence of Kirchhoff migrations.

A complete iterative step involves a preconditioner, W_k , and a scalar α_k , which solves a 1-D line search. W_k is assumed to be some approximation to $(I + F_k^* F_k)^{-1}$. Approximating F_k by a matrix, then

$$F = \left\{ B_{lj} \right\}$$

where

$$B_{lj} = \left\{ \frac{2}{v^3(r_j)} \ddot{p}(r_j, t; r_s) * g(r_j, t; r_g) \right\}$$

with $l = 1, N_p$ where N_p is the number of data values $= N_t * N_s * N_g$, and $j = 1, N_v$ where N_v is the number of velocity model values. Assuming $F_k^* = \frac{\sigma_v^2}{\sigma_p^2} F_k^T$ with $C_p = \sigma_p^2 I$ and $C_v = \sigma_v^2 I$, then

$$(H_k)_{ij} = (I + F_k^* F_k)_{ij} = \left\{ \delta_{ij} + \frac{\sigma_v^2}{\sigma_p^2} \sum_{m=1}^{N_p} B_{mi} B_{mj} \right\}$$

Thus

$$(I + F_k^* F_k)_{ij} = \delta_{ij} + \sum_{r_s, r_g} \int dt \frac{4\sigma_v^2}{\sigma_p^2 v^3(r_i) v^3(r_j)} \left\{ \ddot{p}(r_i, t; r_s) * g(r_i, t; r_g) \right\} \left\{ \ddot{p}(r_j, t; r_s) * g(r_j, t; r_g) \right\} \quad (30)$$

In the ray approximation this can be written

$$(I + F_k^* F_k)_{ij} = \delta_{ij} + \frac{4\sigma_v^2}{\sigma_p^2 v^3(r_i) v^3(r_j)} \sum_{r_s, r_g} A_{r_i, r_s} A_{r_i, r_g} A_{r_j, r_s} A_{r_j, r_g} \tilde{S}(t = T_{r_i, r_s} + T_{r_i, r_g} - T_{r_j, r_s} - T_{r_j, r_g}) \quad (31)$$

where

$$\tilde{S}(t) = \tilde{S}(-t) * \tilde{S}(t)$$

With a high frequency source, this matrix will be sparse. For a diagonally dominant operator, H_k , then an improved single step convergence can be gotten using $W_k = (DIAG(H_k))^{-1}$, where

$$DIAG(H_k)_{ii} = \left\{ 1 + \frac{4\sigma_v^2 \tilde{S}(0)}{\sigma_p^2 v^6(r_i)} \sum_{r_s, r_g} A_{r_i r_s}^2 A_{r_i r_g}^2 \right\} \quad (32)$$

Thus a simple preconditioned iterative scheme can be written

$$v_{k+1}(r_i) = v_k(r_i) + \alpha_k \left\{ 1 + \frac{4\sigma_v^2 \tilde{S}(0)}{\sigma_p^2 v^6(r_i)} \sum_{r_s, r_g} A_{r_i r_s}^2 A_{r_i r_g}^2 \right\}^{-1} \quad (33)$$

$$C_v \sum_{r_s, r_g} \frac{2}{v^3(\neq)} A_{r_i r_s} A_{r_i r_g} \int dt \tilde{S}(t - T_{r_i r_s}) \delta p'(r_g, t + T_{r_i r_g}, r_s)$$

When the a priori model is homogeneous, the starting model is sufficiently close to the true model so that the Born approximation is strictly valid. and with $\sigma_v \rightarrow \infty$, then

$$v_{k+1}(r_i) = v_k(r_i) + \alpha_k 8\pi^2 v^3(r_i) \left\{ \frac{C_v}{\sigma_v^2} \right\} \left[\sum_{r_s, r_g} \left\{ \frac{1}{R_{r_i r_s}^2} \frac{1}{R_{r_i r_g}^2} \right\} \right]^{-1} \quad (34)$$

$$\sum_{r_s, r_g} \left\{ \frac{1}{R_{r_i r_s}^2} \frac{1}{R_{r_i r_g}^2} \right\} \frac{\sigma_p^2}{\tilde{S}(0)} \delta \tilde{p}(r_g, t = \frac{R_{r_i r_s} + R_{r_i r_g}}{v}, r_s)$$

where $\left\{ C_v / \sigma_v^2 \right\}$ includes the spatial filtering part of the Covariance operator, $\delta \tilde{p}$ is given in equation (28) and is normalized by $\tilde{S}(0) / \sigma_p^2$, and α_k is gotten from equation (16). The factor $\left[\sum_{r_s, r_g} \left\{ \frac{1}{R_{r_i r_s}^2} \frac{1}{R_{r_i r_g}^2} \right\} \right]^{-1}$

approximately corrects for geometric spreading. A simple variant of this formula will be used in the examples to follow.

Alternative linearizations

The formulation in the previous section was applied to the linearization of the field variables. General limitations to this approach are summarized in appendix 1. A simple example of the limitations in using a linearization in terms of the field can be seen as follows. Consider a field in an unperturbed homogeneous medium (see Brown, [1984]).

$$u_0(r, t) = \frac{\delta(t - R/v_0)}{4\pi R}$$

where $R = |\mathbf{r} - \mathbf{r}_0|$. In a slightly perturbed homogeneous model with velocity $v_1 = v_0 + \delta v$, the field is

$$u_1(r, t) = \frac{\delta(t - R/v_1)}{4\pi R}$$

The perturbed minus unperturbed field is then approximated by the linearization

$$u_1 - u_0 = \delta u \approx \frac{\partial u}{\partial v} \delta v \quad (34)$$

or

$$\frac{\delta(t - R/v_1)}{4\pi R} - \frac{\delta(t - R/v_0)}{4\pi R} \approx \frac{\partial}{\partial v} \left[\frac{\delta(t - R/v_0)}{4\pi R} \right] \delta v$$

which reduces to

$$\delta(t - R/v_1) - \delta(t - R/v_0) \approx \delta'(t - R/v_0) \left(\frac{R}{v_0^2} \right) \delta v$$

Thus the actual differential seismogram is approximated by a constant times the derivative of a delta function. As seen from Figure 3, depending on the frequency content this will be a poor approximation when

$\delta t = \left(\frac{R}{v_1} - \frac{R}{v_0} \right)$ is large. For a single harmonic frequency in a homogeneous medium, then $u = \frac{e^{i\omega R/v}}{4\pi R}$. The linearization in terms of the field can then be written

$$\frac{e^{i\omega R/v_1}}{4\pi R} - \frac{e^{i\omega R/v_0}}{4\pi R} \approx \frac{\partial u}{\partial v} \delta v = \left(\frac{-i\omega R}{v_0^2} \right) \frac{e^{i\omega R/v_0}}{4\pi R} \delta v$$

or

$$e^{i\omega R \left(\frac{1}{v_1} - \frac{1}{v_0} \right)} - 1 \approx \left(\frac{-i\omega R}{v_0^2} \right) \delta v$$

for small $\omega R \left(\frac{1}{v_1} - \frac{1}{v_0} \right) \ll 1$, then the exponential can be approximated as $e^{i\omega \delta t} \approx 1 + i\omega R \left(\frac{1}{v_1} - \frac{1}{v_0} \right)$. Thus for sufficiently low frequencies, such that $\omega \delta t \ll 1$, then the linearization will be a good approximation to the actual differential field.

An alternative linearization could be performed in terms of the log of the field variables. This has been successfully applied electromagnetic problems by Madden (personal communication). By perturbing the velocity in a homogeneous medium, we again consider the extreme case of

$ka \gg 1$. With $\ln(u_0) = \ln \left(\frac{e^{i\omega R/v_0}}{4\pi R} \right)$, then

$$\ln(u_1) - \ln(u_0) \approx \frac{\partial(\ln(u))}{\partial(\ln v)} \delta(\ln v)$$

or

$$i\omega R\left\{\frac{1}{v_1} - \frac{1}{v_0}\right\} \approx i\omega R\left\{\frac{-1}{v_0^2}\right\} (v_0) \delta(\ln v)$$

$$\left\{\frac{1}{v_1} - \frac{1}{v_0}\right\} \approx \frac{-1}{v_0^2}(v_1 - v_0)$$

Thus for this simple example of a homogeneous medium perturbation there is no dependence on frequency for the complex phase linearization. The region of validity for the log field parameterization is more involved in the general case where there is an amplitude perturbation as well. In any event, this type of linearization has been used successfully applied in travel time inversions (see Aki et al. [1977], Clayton and Comer [1983]), where only the imaginary part of the complex phase is used.

The linear sensitivity operator in terms of the log field is now derived and compared with the Born linearization. The log field linearization is additive in the phase, and is equivalent to Rytov's method of smooth perturbations (see Tatarskii [1971], section 45; Ishimaru [1978]). Rytov's approximation is compared in Appendix 2 with other asymptotic, multiplicative approximations including geometric optics and the parabolic equation method. We start with the Helmholtz equation

$$\nabla^2 u + \frac{\omega^2}{v^2(\vec{x})} u = 0$$

and let $\psi = \ln(u(\vec{x}, \omega))$, then the following nonlinear Riccati equation is obtained

$$\nabla^2 \psi + (\nabla \psi)^2 + \frac{\omega^2}{v(\vec{x})^2} = 0$$

Now a perturbed problem is derived with $\psi = \psi_0 + \varepsilon \psi_1$ and $v = v_0(\vec{x}) + \varepsilon v_1(\vec{x})$, assuming $\varepsilon \ll 1$, $\psi_0 \approx \psi_1$, and $v_0 \approx v_1$. Then

$$\begin{aligned} & \left\{ \nabla^2 \psi_0 + (\nabla \psi_0)^2 + \frac{\omega^2}{v_0^2(\vec{x})} \right\} \\ & + \varepsilon \left\{ \nabla^2 \psi_1 + 2 \nabla \psi_0 \nabla \psi_1 - \frac{2 \omega^2 v_1(\vec{x})}{v_0^3(\vec{x})} \right\} \\ & + \varepsilon^2 \left\{ (\nabla \psi_1)^2 + \frac{3 \omega^2 v_1^2(\vec{x})}{v_0^4(\vec{x})} \right\} = 0 \end{aligned}$$

The first bracket is equal to zero by solving the forward problem in the unperturbed medium, $v_0(\vec{x})$. Dropping the third bracket in ε^2 constitutes the Rytov approximation which is now linear in ψ_1 and v_1 . Letting $\delta v(\vec{x}) = \varepsilon v_1$ and $\delta \psi = \varepsilon \psi_1$ gives

$$\nabla^2(\delta \psi) + 2 \nabla \psi_0 \nabla(\delta \psi) = \frac{2 \omega^2 \delta v(\vec{x})}{v_0^3(\vec{x})}$$

This is a linear equation for $\delta \psi$ which can be simplified by the substitution, $\delta \psi = \bar{\psi} e^{-\psi_0}$

$$\nabla^2 \bar{\psi} + \frac{\omega^2}{v_0^2(\vec{x})} \bar{\psi} = \frac{2 \omega^2 \delta v}{v_0^3} e^{\psi_0}$$

which has a solution

$$\bar{\psi}(r_g, r_s) = -\int dV(r') g(r_g, r') e^{\psi_0(r', r_s)} \frac{2\omega^2 \delta v}{v_0^3}$$

or

$$\delta\psi(r_g, r_s) = -\int dV(r') g(r_g, r') e^{\psi_0(r', r_s) - \psi_0(r_g, r_s)} \frac{2\omega^2 \delta v}{v_0^3} \quad (35)$$

This gives the linear sensitivity operator in terms of the log field variables in the form $\delta\psi = \frac{\partial\psi}{\partial v} \delta v$.

To see how this compares to the Born approximation or a linearization in terms of the field, equation (35) is exponentiated

$$e^{\delta\psi} = e^{\psi - \psi_0} = e^{-\int dV(r')(\dots)}$$

For small $-\int dV(r')(\dots)$

$$e^{\psi} \approx e^{\psi_0} - \int dV(r') p(r', r_s) g(r_g, r') \frac{2\omega^2 \delta v(r')}{v_0^3(r')}$$

which is equivalent to the frequency transformed equation (7). Thus equation (7) can be derived from (35) assuming that the exponentiated argument is small. It appears that the Rytov approximation includes some multiple scattering effects not included in the Born approximation. However, the Rytov approximation was originally derived assuming $\delta\psi \ll 1$. Thus, there has been an ongoing controversy as to whether the Rytov is in fact any better than the Born approximation (see Tatarskii [1971], Barabatenko et. al. [1971], Aki [1973], Brown [1966, 1967], Taylor [1967], Yura et. al. [1983]). In fact, Aki and Richards [1980] derive their

expressions for the log amplitude and phase from the exponentiated Born series (EBS).

However, asymptotic equivalence may not correspond to a practical equivalence between the two approaches as shown in the previous simple example. Keller [1969] showed that the Born and the Rytov approximations are equally accurate in their dependence on the small velocity parameter, ϵ , but show quite different asymptotic properties as a function of range. Mueller et al. [1979] noted that both the Born and the Rytov approximations required that some norm of the velocity perturbations over the volume be small compared to unity. In addition, the Born approximation required small ka , where a is the scale of the heterogeneity, in agreement with the range of validity for the single scattering formulation given in Appendix 1. Thus low spatial wavenumber, band limited velocity perturbations appears to favor the Rytov approximation. From the context of an inverse problem, it is natural to estimate the low spatial wavenumbers of the velocity first via a multiple scale type of procedure. For small scale residual heterogeneities, the Born approximation to the field may have an advantage.

Experiments in diffraction tomography comparing the Born and the Rytov formulations have been conducted by Kaveh et. al. [1982,1983] who showed that when a large forward scattering component exists, the linearized reconstructed images based on the Born approximation were

propagation in the atmosphere show closer agreement with the lognormal distribution (see Tatarskii [1971]).

In order to check the linearization in terms of the log field for a large scale heterogeneity, a homogeneous velocity perturbation, δv , is again considered. Equation (35) can then be written

$$\delta\psi(r_g, r_s) = \frac{-2\omega^2\delta v}{v_0^3} \int dV(r') \frac{e^{ikR_{r_g r'}}}{4\pi R_{r_g r'}} \frac{e^{ikR_{r' r_s}}}{4\pi R_{r' r_s}} 4\pi R_{r' r_s} e^{-ikR_{r_g r_s}}$$

Using the Fresnel approximation with z axis oriented from the source to receiver

$$\delta\psi(r_g, r_s) = \frac{-\omega^2\delta v z}{2\pi v_0^3} \int_0^z \frac{dz'}{(z-z')z'} \int_{-\infty}^{\infty} dx' e^{\frac{ikx'^2}{2z'(z-z')}} \int_{-\infty}^{\infty} dy' e^{\frac{iky'^2}{2z'(z-z')}}$$

Using the relation $\int_{-\infty}^{\infty} e^{-ax^2} = (\pi/a)^{1/2}$, then

$$\delta\psi(r_g, r_s) = \frac{-\omega^2\delta v z}{2\pi v_0^3} \int_0^z \frac{dz'}{z'(z-z')} \frac{2\pi z'(z-z')}{-ikz}$$

or

$$\delta\psi(r_g, r_s) = \frac{-i\omega z \delta v}{v_0^2}$$

Now let $\delta\psi = i\omega\delta t = i\omega z \left(\frac{1}{v_1} - \frac{1}{v_0} \right) \approx \frac{-i\omega z \delta v}{v_0^2}$. Thus, the linearization in terms of log variables will retrieve a global velocity change in one iteration without the apparent low frequency requirement of the Born approximation of equation (7).

more in error than the Rytov, but the Rytov required the determination of the phase of the scattered waves. There are numerical difficulties in obtaining the log of the field since the phase of the complex log must be unwrapped. Still, numerical algorithms for this exist (see Tribolet [1978]). Also, algorithms based on the smoothness of the phase at adjacent spatial points have had some initial success (see Kaveh et.al. [1982], Kaveh et.al. [1984]).

Another consideration as to whether to use a linearization based on the Born (field) or the Rytov (log of the field) approximations is the noise structure of the problem (see Tatarskii [1971]). If the data is expressed as a sum of signals plus independent Gaussian noise, N , then the linearized inversion should be applied to the waveform or to the real and imaginary Fourier components. If the noise structure is multiplicative and proportional to the signal, then $\log N$ will behave in a Gaussian manner around $\log u$. In this case, the log should be taken (see Aki and Richards [1980], pp 639). Detailed work on the effects of additive and multiplicative noise for moment tensor inversion of surface wave data was done by Patton and Aki [1979].

If the noise is Gaussian and additive in the field, then the fluctuations in the amplitude should follow a Rayleigh distribution. If the noise is Gaussian and additive in the complex phase, then the fluctuations in the amplitude should follow a lognormal distribution. Experiments of light

There are several basic linearizations that can be used in inversion studies, including those based on the additive nature of the field, Born, or the log field, Rytov. Travel time inversions are more similar to the Rytov approximation except that only the imaginary part of the complex phase is utilized and usually only first arrival times. Keller [1969] noted that any advantages that the Rytov approximation may have over the Born approximation may be lost when the waveform contains more than one wave. This would occur when there is high wavenumber velocity fluctuations causing significant backscatter. This may be the reason for the success of simple migration algorithms which map small scale reflectors and which are kinematically equivalent to Born inversion (see Miller et.al. [1984]). Another consideration as to whether to use the log field data, or the amplitude and phase, is the numerical difficulties in unwrapping the phase when taking the complex log, although work on this has been done (Tribolet [1978], Kaveh [1982,1984]).

One strategy for inversion of material parameters is to use a multiple scale procedure in which the large scale fluctuation are inverted for first, and compensated for. This might include several Rytov iterations followed by a series of Born iterations. A compromise approach would be to use an iterative travel time inversion first to adjust for the major phase differences and then use the iterative Born inversion on the waveforms to incorporate more of the data and fine tune the model. In the following numerical examples, The Born linearization is used on residual waveform

data assuming small velocity perturbations.

Examples

In this section, several examples are given using a linearization in terms of the field as described previously. A simple descent method is used

$$v_{k+1} = v_k + \alpha W_k (F_k^* \delta p_k) \quad (36)$$

where $F_k^* = C_v F_k^T C_p$, W_k is a preconditioner, and α is a scaling factor. For an initial homogeneous model close to the true model, then $F_k^* \approx F_0^*$. Operating on the residuals by the adjoint is then given in (29) as

$$\begin{aligned} \delta \hat{v}_k &= F_k^* \delta p_k \approx \\ & \frac{C_v}{v^3 8 \pi^2} \sum_{\tau_s} \sum_{\tau_g} \left[\frac{1}{R_{\tau\tau_s} R_{\tau\tau_g}} \right] \delta \tilde{p}(r_g, l = \frac{R_{\tau\tau_s} + R_{\tau\tau_g}}{v}, \tau_s) \end{aligned}$$

where $\delta \tilde{p}$ is the cross correlation of the second derivative of the source time function with the weighted data residuals, C_v is the model covariance operator and includes a spatial smoothing function. Here, a Gaussian smoothing function as in (21) is used. Operating on the residuals by the adjoint is kinematically similar to a Kirchhoff migration. If the preconditioner, W_k , is chosen as I , then the velocity inversion in (36) with $F_k^* = F_0^*$ is equivalent to a sequence of Kirchhoff migrations. An alternative choice for the preconditioner, W_k , is the inverse of the diagonal of $(I + F^* F)$ as in (32). The part of this preconditioner that approximately corrects for geometric spreading is

$$W_k = \left\{ \sum_{\tau_s} \sum_{\tau_g} \left[\frac{1}{R_{\tau\tau_s}^2 R_{\tau\tau_g}^2} \right] \right\}^{-1} \quad (37)$$

This simple preconditioner will be used in addition to the identity in the next example.

The geometry of the first example is shown in Figure 4. There are nine sources marked by x's and nineteen receivers marked by triangles. The axes are in km to give specific units and the initial velocity is 8 km/sec. For this example, a single smooth heterogeneity with a radius of 10 km is reduced in velocity by -.01 km/sec. This is shown by the circle in Figure 4. The interior box in Figure 4 shows the region where the velocity is to be inferred.

Figure 5 shows the seismograms computed using the Gaussian beam method for the single smooth heterogeneity shown in Figure 4 for three of the nine source locations. The source wavelet is the Gabor wavelet

$$f(t) = e^{-(2\pi f_0 t / \gamma)^2} \cos(2\pi f_0 t + \varphi_0)$$

where $f_0 = 4 \text{ Hz}$, $\gamma = 3$, $\varphi_0 = 0$. This results in a 2 km wavelength. The correction given by Cervený et.al. [1982] is used to approximately compute the 3-D response. In Figure 5, a slightly larger amplitude can be seen on some seismograms for different source locations. For example, for $S = (30., 140.)$, the amplitude increases due to focusing at a range of 180 km and a time of 24 sec.

The residual, perturbed minus unperturbed, seismograms are shown in Figure 6 for three source locations. The amplitudes of the residuals are

about ten times smaller than the amplitudes of the unperturbed seismograms. This satisfies the Born requirement that the scattered field be small compared to the primary field. The wavelet cross correlation with the data results in a symmetric wavelet centered on the arrival time, and the second derivative approximately results in a factor, $-\omega_0^2$, assuming a Gabor wavelet.

Figure 7 shows the result of back propagating the residuals into model space using the adjoint operator. This would be the first iteration in a descent procedure with $W_k = I$ and $C_v = \sigma_v \delta(\vec{r} - \vec{r}')$. A grid of 100 by 100 points has been evaluated in model space. The result in Figure 7 is scaled from -1. to +1.. The contours are at (-.75, -.25, .25, .75) with the -.75 contour in the center. The parallelogram and the small plus represent the location of the true heterogeneity. The data residuals appear to be streaked along lines from the receivers to the stations in model space. The response to the simple transpose operator for a single heterogeneity thus has a streaked appearance in model space.

There are additional operations that can be included in an individual velocity inversion step. This includes a more involved preconditioner as in equation (37) which approximately corrects for geometric spreading, and is an approximation to filtering the backprojection. Figure 8 shows the result of using this preconditioner. It has the effect of reducing some of the streaks in the backprojection. Again, the true heterogeneity is

represented by the parallelogram.

Another operation that can be applied to the backprojection is spatial smoothing by the model covariance operator based on a priori information. Figure 9 shows the result of smoothing the backprojection by a Gaussian smoothing filter with a 5 km radius. This filtering operation eliminates the streaks and localizes the heterogeneity. This heterogeneity is slightly elongated in the z direction compared to the true heterogeneity represented by the parallelogram. This results from sources below and receivers above the heterogeneous region.

To check the sign of the derived heterogeneity, the example was redone with a positive instead of negative velocity heterogeneity, and the inferred velocity was also positive. A final step in an iteration is the choice of the scalar, α , which specifies how far in a particular descent direction one moves. A particular value for the scaling factor can be computed from equation (16), although other values can be used to over or under relax the problem. The scaling factor gotten to match the single heterogeneity case will be used to relax each iteration step in the next examples.

In the next example, three smooth heterogeneities are used, each with a velocity heterogeneity of -0.01 km/sec lower than the background. The geometry is shown in Figure 10. The sources are below the heterogeneity and the stations above. The result of backprojecting the residuals into model space and applying a 4 km Gaussian covariance operator is shown in

Figure 11. The contour interval goes from -1. to +1., and the contours are at (-.75, -.25, .25, .75). The true locations of the heterogeneities are represented by the parallelograms. Using the scaling factor gotten from the single heterogeneity case, then the lower heterogeneity has a velocity perturbation of -.0109 which is approximately accounted for in this iteration. The upper two heterogeneities have about half that perturbation.

Figure 12 shows the second iteration result of backprojecting and smoothing with a 4 km Gaussian model covariance operator. The upper two heterogeneities can be seen along with side lobes. Using the scaling factor gotten from the single heterogeneity case, the upper two heterogeneities have a velocity perturbation of -.0052 km/sec. When combined with the perturbation from the first iteration, then all three heterogeneities have been accounted for in the inversion.

In the next example, a ring heterogeneity is considered. The model geometry is shown in Figure 13. This is an idealization of the structure derived by Thurber [1984] under the Kilauea volcano using a travel time inversion. He found a high velocity core of the volcano surrounding an interior lower velocity region, which was inferred to be a magma complex. For this numerical example, just the ring is reduced in velocity by -.01 km/sec. Figure 14 shows the result of smoothing the backprojection with a Gaussian model covariance operator with a 4 km radius. The contours go

from -1. to +1. with a .25 contour interval. The velocity perturbation forms a ring with the largest perturbations on the sides of the ring. Using a scaling factor from the single heterogeneity case gives an amplitude of -.02 on the sides of the ring, or twice as large as the true heterogeneity. Smaller perturbations form the the top and bottom of the ring. In order to examine the next iteration, a scaling factor of half of this is used so as to compensate for the sides of the ring. The bottom of the ring is half compensated for. The backprojection for the second iteration with a 4 km Gaussian covariance operator applied is shown in Figure 15. Using the scaling factor for the single heterogeneity for this iteration gives a velocity of -.011, which approximately compensates for the top and bottom of the ring.

When applying this iteration procedure to actual data, the scaling factor, α , must be approximately derived using equation (16), or an over relaxed version of it. In these numerical examples, only a rough value was used to see how convergence proceeds. Nonetheless, choice of this scaling factor is essential part of the problem.

Finally, a simple comparison of these results with travel time inversions using the same model geometries is done. The model is divided into 10 by 10 blocks or 100 model parameters. 171 travel times are used, not all of which give new information about the model. A damped inversion is used. The result for a single heterogeneity is shown in Figure 16a. Using

a damping of .001, a velocity of $-.0102$ km/sec is gotten. This result compares with the waveform inversion shown in Figure 8. The travel time inversion for three heterogeneities is shown in Figure 16b. This is again similar to the two iteration waveform result. Finally, the travel time inversion result for the ring structure is shown in Figure 16c. In this case the waveform result may do slightly better. As discussed in a previous section, for large scale heterogeneities, a complex phase linearization may be more robust. Travel time inversions are in this class, in which only the imaginary part of the phase is used with first arrivals. However, for small velocity perturbations the results are comparable. In actual cases, waveform inversions allow for a more complete use of the data when the phase perturbations are small, or have been previously compensated for.

Conclusions

In this paper, a linearized inversion procedure for material parameters has been investigated. In this procedure a linear sensitivity operator must be derived and can be computed economically by using reciprocity of the Green's function. Data errors and a prior model information are included via covariance operators. Large matrix inversions are avoided by using descent algorithms. A fast forward modelling scheme is required and here the Gaussian beam method for a laterally varying medium is used. This allows for inhomogeneous initial models.

Different linearizations are possible including a linearization in terms of the field, the Born approximation, and a linearization in terms of the log field, the Rytov approximation. Travel time inversions are in the class of linearizations for the log field, where only the imaginary part of the complex phase is used with first arrivals. Log field linearizations may be more robust than field linearizations for large scale heterogeneities where forward scattering predominates, but phase unwrapping may be difficult numerically. The linearization in terms of the field for small perturbations from a homogeneous background is kinematically equivalent to a sequence of Kirchhoff migrations. Field linearizations are expected to be useful for small scale heterogeneities which result in scattering effects that are additive in the field. Several numerical examples are performed using a field linearization which identify the trial structures. A comparison is then

done with travel time inversion results.

Appendix 1

In this appendix a brief review is given for the conditions of validity of the Born approximation in a continuous fluctuating medium characterized by a spatial Gaussian correlation function where

$$N(\mathbf{r}') = \frac{E[\mu(\mathbf{r})\mu(\mathbf{r}+\mathbf{r}')] }{E[\mu^2]} = e^{-|\mathbf{r}|^2/a^2}$$

where $\mu(\mathbf{r}) = -\delta v(\mathbf{r})/v(\mathbf{r})$, and $E[\dots]$ represents the expectation over an ensemble of random media, and a is the correlation length. If the process is ergodic then this expectation can be replaced by a spatial average over one realization. The fractional loss of energy to the primary wave for a medium with velocities with a Gaussian correlation function is (see Aki and Richards [1980])

$$\frac{\Delta I}{I} = \sqrt{\pi} E[\mu^2] k a k L (1 - e^{-k^2 a^2})$$

For the Born approximation to be valid then $\frac{\Delta I}{I} \ll 1$, or

$$\sqrt{\pi} E[\mu^2] k a k L \ll 1 \quad k a \gg 1$$

$$\sqrt{\pi} E[\mu^2] k^3 a^3 k L \ll 1 \quad k a \ll 1$$

where L is the propagation distance and a is the correlation distance. Thus, $k a \gg 1$ refers to large scale scatterers and $k a \ll 1$ refers to small scale scatterers with respect to the wavelength. $k a \ll 1$ corresponds to Rayleigh scattering. For large scale scatterers, $k a \gg 1$, then the mean

square velocity variation must be small for the strict validity of the Born approximation. For small scale scatterers, relatively large mean square velocity fluctuations can be accommodated by the Born approximation. Thus

$$ka = 10. \quad E[\mu^2]kL \ll .056$$

$$ka = 2.0 \quad E[\mu^2]kL \ll .280$$

$$ka = 0.5 \quad E[\mu^2]kL \ll 4.50$$

$$ka = 0.1 \quad E[\mu^2]kL \ll 564.$$

For a given propagation distance, L , linearizations based on the Born approximation will favor small scale fluctuations.

Appendix 2

In this appendix a comparison is done between perturbation solutions which to first order are additive in the complex phase or multiplicative in the field. These include the first order geometric optics approximation, the parabolic equation method, and the Rytov approximation. These types of approximations work best in a smoothly varying media with predominantly forward scattering. First, the geometric optics approximation is outlined and compared with the standard parabolic approximation. These are then compared to the Rytov approximation.

Starting with the Helmholtz equation, the first order geometric optics approximation has the following general trial solution

$$u = e^{\bar{k}\varphi_0 + \varphi_1} \quad (\text{A.2.1})$$

where $\bar{k}$ is a large parameter and $\varphi_0 \approx \varphi_1$. In this approximation $\bar{k}$ is usually identified with an average wavenumber and large $\bar{k}$ implies a small wavelength. By substituting this into the Helmholtz equation then

$$\begin{aligned} & \bar{k}^2 \left\{ \nabla \varphi_0 \nabla \varphi_0 + k^2 / \bar{k}^2 \right\} \\ & + \bar{k} \left\{ \nabla^2 \varphi_0 + 2 \nabla \varphi_0 \nabla \varphi_1 \right\} \\ & + \left\{ \nabla^2 \varphi_1 + \nabla \varphi_1 \nabla \varphi_1 \right\} = 0 \end{aligned} \quad (\text{A.2.2})$$

where $k = \omega / v(\vec{x})$. The particular trial solution usually used for geometric optics is $u = A(\vec{x}) e^{i\bar{k}S(\vec{x})}$ where $\bar{k} = \omega / \langle v \rangle$. Thus $\varphi_0 = iS(\vec{x})$

and $\varphi_1 = \ln A(\vec{x})$ and (A.2.2) can then be written as

$$\begin{aligned} & \bar{k}^2 \left\{ -\left[\nabla S \right]^2 + k^2 / \bar{k}^2 \right\} \\ & + 2i\bar{k} \left\{ \frac{1}{2} \nabla^2 S + \nabla S \nabla \ln A \right\} \\ & + \left\{ \nabla^2 A / A \right\} = 0 \end{aligned} \quad (\text{A.2.3})$$

For large $\bar{k}$, the first geometric optics solution involves dropping the $\left\{ \nabla^2 A / A \right\}$ term. The Eikonal equation results from equating the first bracket to zero and the transport equation results from equating the second bracket to zero. The transport equation relates the imaginary part of the phase to the real part of the phase.

The standard parabolic equation method is derived by using for (A2.2.1) the particular trial solution $u = Ue^{i\bar{k}x}$ where U is assumed to be a slowly varying function, then $\varphi_0 = i\bar{k}x$ and $\varphi_1 = \ln U(\vec{x})$. (A.2.2) can then be written

$$\begin{aligned} & \bar{k}^2 \left\{ -1 + k^2 / \bar{k}^2 \right\} \\ & + 2i\bar{k} \left\{ 0 + \frac{U_{,x}}{U} \right\} \\ & + \left\{ \frac{U_{,xx} + U_{,yy} + U_{,zz}}{U} \right\} = 0 \end{aligned} \quad (\text{A.2.4})$$

For large $\bar{k}$, the Eikonal equation give $k = \bar{k}$, and the transport equation

gives $U_{,x} = 0$. The parabolic approximation results from neglecting the $U_{,xx}/U$ in the final bracket but retaining the transverse part of the Laplacian, $\nabla_{\text{pert}}^2 = U_{,yy} + U_{,zz}$. This is not entirely consistent within the perturbation scheme used but it gives a one way operator in x ,

$$2i\bar{k}U_{,x} + \nabla_{\text{pert}}^2 U + \left\{k^2 - \bar{k}^2\right\}U = 0 \quad (\text{A.2.5})$$

The transverse Laplacian is called the diffraction term and the term $\left\{k^2 - \bar{k}^2\right\}$ is called the thin lens term and locally adjusts for the refractive index.

The Rytov approximation results by using the trial solution

$$u = e^{\psi_0 + \varepsilon\psi_1} \quad (\text{A.2.6})$$

where ε is a small parameter, and $\psi_0 \approx \psi_1$. The velocity is also expanded as $v(\vec{x}) = v_0(\vec{x}) + \varepsilon v_1(\vec{x})$, where $v_0 \approx v_1$. Substituting this into the Helmholtz equation gives

$$\begin{aligned} & \left\{ \nabla^2 \psi_0 + (\nabla \psi_0)^2 + \omega^2 / v_0^2(\vec{x}) \right\} \\ & + \varepsilon \left\{ \nabla^2 \psi_1 + 2\nabla \psi_0 \nabla \psi_1 - 2\omega^2 v_1(\vec{x}) / v_0^3(\vec{x}) \right\} \\ & + \varepsilon^2 \left\{ (\nabla \psi_1)^2 + 3\omega^2 v_1^2(\vec{x}) / v_0^4(\vec{x}) \right\} = 0 \end{aligned} \quad (\text{A.2.7})$$

where the first term in the brackets is a solution of the Helmholtz equation with $v = v_0(\vec{x})$ and $u = e^{\psi_0}$. In the first Rytov approximation, the ε^2

term is dropped. Equating the second term to zero gives

$$\nabla^2 \psi_1 + 2\nabla \psi_0 \nabla \psi_1 = 2v_1 \omega^2 / v_0^3 \quad (\text{A.2.8})$$

In order to compare with the previous approximations let $\psi_0 = \bar{k} \varphi_0$ and $\psi = \varphi_1 / \varepsilon = \bar{k} \varphi_1$, then (A.2.7) can be written

$$\begin{aligned} & \left\{ \bar{k} \nabla^2 \varphi_0 + \bar{k}^2 (\nabla \varphi_0)^2 + \bar{k}^2 \left(\frac{k_0^2}{\bar{k}^2} \right) \right\} \\ & + \left\{ \nabla^2 \varphi_1 + \bar{k} 2 \nabla \varphi_0 \nabla \varphi_1 + \bar{k} \left(\frac{k_0^2}{\bar{k}^2} \frac{2v_1}{v_0} \right) \right\} \\ & + \left\{ (\nabla \varphi_1)^2 + \left(\frac{k_0^2}{\bar{k}^2} \frac{3v_1^2}{v_0^2} \right) \right\} = 0 \end{aligned} \quad (\text{A.2.9})$$

where $\bar{k} = \omega / \langle v_0(\vec{x}) \rangle$, $k_0 = \omega / v_0(\vec{x})$, and $\bar{k} \approx k_0$. The Rytov approximation again retains the first two brackets, where the first bracket is assumed known. The $\bar{k}^2$ terms relates to the Eikonal equation in equation (A.2.2), only now related to the velocity model $v_0(\vec{x})$. The $\bar{k}$ terms correspond to the transport equation in (A.2.2) plus a correction term to the velocity linear in $v_1(\vec{x})$. Finally, the term $\nabla^2 \varphi_1$ is retained, but not the term $(\nabla \varphi_1)^2$ and the higher order terms containing the velocity correction $v_1(\vec{x})$. This is similar to the parabolic approximation where again only a part of the third bracket in (A.2.2) is retained. In that case the transverse Laplacian perpendicular to the predominant direction of the incident wave is retained. For example if we let v_0 be a homogeneous velocity, and $\psi_0 = i\bar{k}x$, then by dropping terms in ε^2 , (A.2.7) can be written in terms of $U = e^{\varepsilon \psi_1}$

$$2i\bar{k}U_{,x} + \left\{ \nabla^2 U - \frac{(\nabla U)^2}{U} \right\} - \frac{2\varepsilon v_1 \omega^2}{v_0^3} U = 0$$

This is similar to the parabolic equation given in (A.2.5) with the thin lens term approximated to first order in $\varepsilon v_1(\vec{x})$ and the term $\nabla_{\vec{p}\vec{\sigma}\vec{p}}^2 U$ replaced by $\nabla^2 U - (\nabla U)^2 / U$. Using the Transverse Laplacian results in a simpler equation but the Rytov equation above is consistent to order ε .

Appendix 3

In this appendix, the acoustic wave equation is investigated in a simple 1-D example, where

$$L = \left\{ \frac{1}{K(\tau)} \frac{\partial^2}{\partial t^2} - \nabla \left(\frac{1}{\rho(\tau)} \nabla \right) \right\}$$

The transposed linear sensitivity operators for a perturbation in density and bulk modulus from the field are given in Tarantola [1984a] where

$$\begin{aligned} \delta K'_k &= U_k^T \delta p = \\ & \frac{1}{K^2(\tau)} \sum_{\tau_s} \sum_{\tau_g} \int dt \left\{ \ddot{p}(\tau, t; \tau_s) * g(\tau, t; \tau_g) \right\} \delta p(\tau_g, t; \tau_s) \end{aligned}$$

$$\begin{aligned} \delta \rho'_k &= V_k^T \delta p = \\ & \frac{1}{\rho^2(\tau)} \sum_{\tau_s} \sum_{\tau_g} \int dt \left\{ \nabla p(\tau, t; \tau_g) * \nabla g(\tau, t; \tau_g) \right\} \delta p(\tau_g, t; \tau_s) \end{aligned}$$

or

$$\delta K'_k = \frac{1}{K^2(\tau)} \int dt \sum_{\tau_s} \ddot{p}(\tau, t; \tau_s) \left\{ \sum_{\tau_g} g(\tau, -t; \tau_g) * \delta p(\tau_g, t; \tau_s) \right\} \quad (\text{A.3.1})$$

$$\delta \rho'_k = \frac{1}{\rho^2(\tau)} \int dt \sum_{\tau_s} \nabla p(\tau, t; \tau_s) \left\{ \sum_{\tau_g} \nabla g(\tau, -t; \tau_g) * \delta p(\tau_g, t; \tau_s) \right\} \quad (\text{A.3.2})$$

In the 1-D case, a coincident source and receiver are considered where $\tau_g = \tau_s = z = 0$. Small perturbations of a homogeneous starting model to the true model will be assumed. The observed reflection seismogram is

$$P_{\text{obs}}(\tau_g, t; \tau_s) = P_{\text{obs}}(t)$$

and the predicted seismogram is

$$P_k(r_g, t; r_s) = P_k(t)$$

The differential seismogram is then

$$\delta p_k(r_g, t; r_s) = P_{obs}(t) - P_k(t)$$

The operation in (A.3.1) can then be written

$$\delta K'_k = \frac{1}{K_k^2(r)} \int dt \ddot{p}_k(r, t; r_s) \langle g(z, t; 0) \delta p_k(t) \rangle$$

where $\langle \dots \rangle$ denotes a cross correlation. The analogous operation on

$\delta \rho'_k$ in (A.3.2) can be written

$$\delta \rho'_k = \frac{1}{\rho_k^2(z)} \int dt \frac{\partial}{\partial z} p_k(z, t; 0) \frac{\partial}{\partial z} \langle g(z, t; 0) \delta p_k(t) \rangle$$

With an initial homogeneous model with velocity v then

$$g(z, t; 0) = \delta(t - z/v)$$

and

$$\langle g(z, t; 0) \delta p_k(t) \rangle = \delta p_k(t + z/v)$$

Then

$$\delta K'_k(z) = \frac{1}{K_k^2(z)} \int dt \frac{\partial^2}{\partial t^2} p_k(z, t; 0) \delta p_k(t + z/v)$$

$$\delta \rho'_k(z) = \frac{1}{\rho_k^2(z)} \int dt \frac{\partial}{\partial z} p_k(z, t; 0) \frac{\partial}{\partial z} \delta p_k(t + z/v)$$

Now with $p_k(z, t; 0) = \delta(t - z/v)$ then

$$\delta K'_k = \frac{1}{K_k^2(z)} \delta \ddot{p}_k(t = 2z/v)$$

$$\delta \rho'_k(z) = \frac{1}{v^2 \rho_k^2(z)} \delta \ddot{p}_k(t = 2z/v)$$

where $t = 2z/v$ places the model perturbation at $z = vt/2$. Since $v = \sqrt{K/\rho}$ and $\eta = \rho v = \text{impedance}$, then

$$\frac{\delta v}{v} = \left\{ \frac{\delta K}{K} - \frac{\delta \rho}{\rho} \right\}$$

$$\frac{\delta \eta}{\eta} = \left\{ \frac{\delta K}{K} + \frac{\delta \rho}{\rho} \right\}$$

Including the model covariances this can be written assuming,

$$C_K \approx \sigma_K^2 \approx K_k^2(z)$$

$$\delta K_k(z) = C_K \delta K'_k(z) \approx \frac{\sigma_K^2}{K_k^2(z)} \delta \ddot{p}(2z/v) \approx \delta \ddot{p}(2z/v)$$

and with $\sigma_\rho^2 \approx \rho_k^2(z)$,

$$\delta \rho_k = C_\rho \delta \rho'_k \approx \frac{\sigma_\rho^2}{v^2 \rho_k^2(z)} \delta \ddot{p}(2z/v) \approx \frac{1}{v^2} \delta \ddot{p}(2z/v)$$

Perturbations in velocity and impedance can then be written

$$\frac{\delta v}{v} = \frac{1}{2} \left[\frac{1}{K} - \frac{1}{v^2 \rho} \right] \delta \ddot{p}(2z/v) = 0$$

and

$$\frac{\delta \eta}{\eta} = \frac{1}{2} \left[\frac{1}{K} + \frac{1}{K} \right] \delta \ddot{p}(2z/v) = \frac{1}{K} \delta \ddot{p}(2z/v)$$

Thus for the 1-D reflection seismogram modelled using the acoustic wave equation, there is no velocity perturbation, but there is an impedance perturbation. This is as expected from the 1-D exact inversion results.

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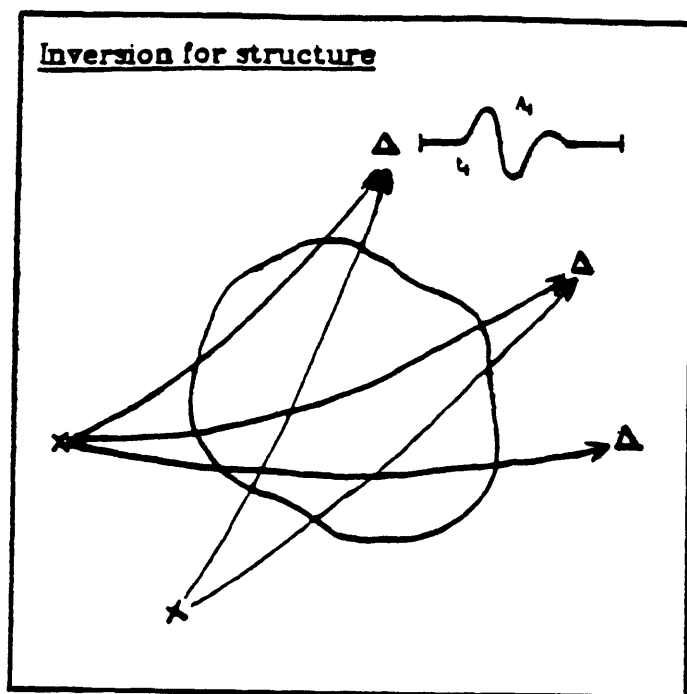


Figure 1

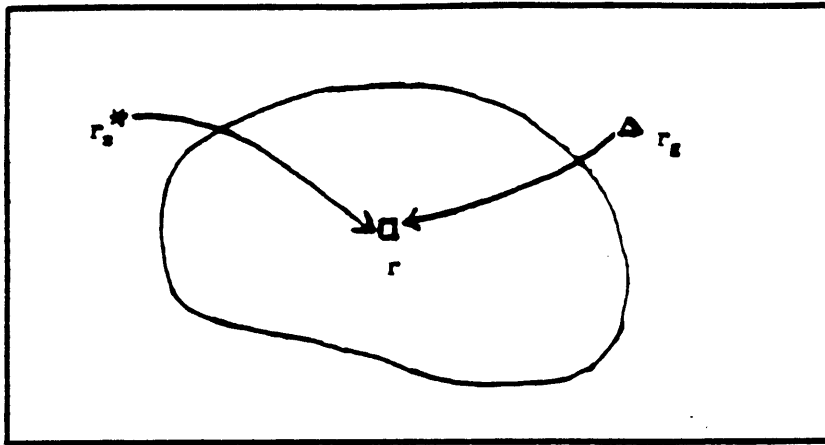


Figure 2

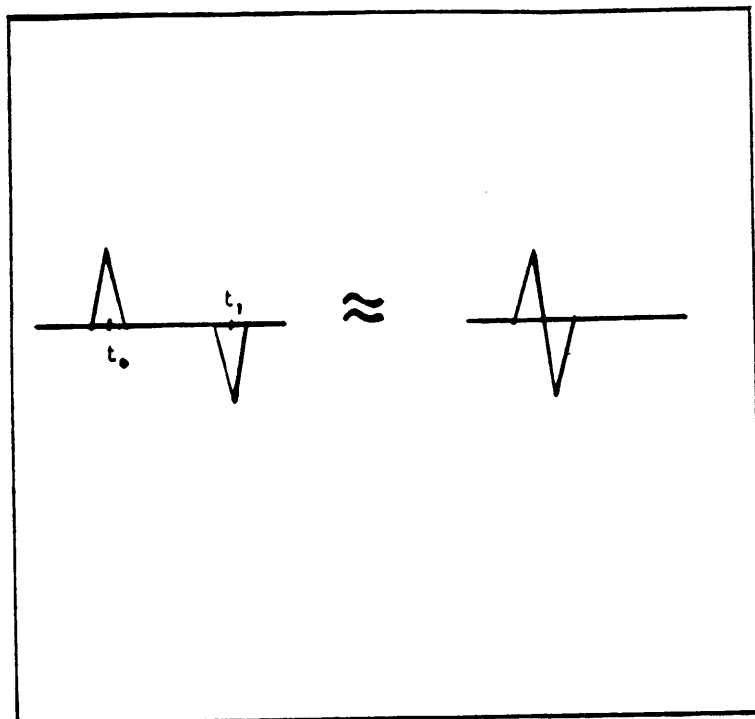


Figure 3

single smooth heterogeneity

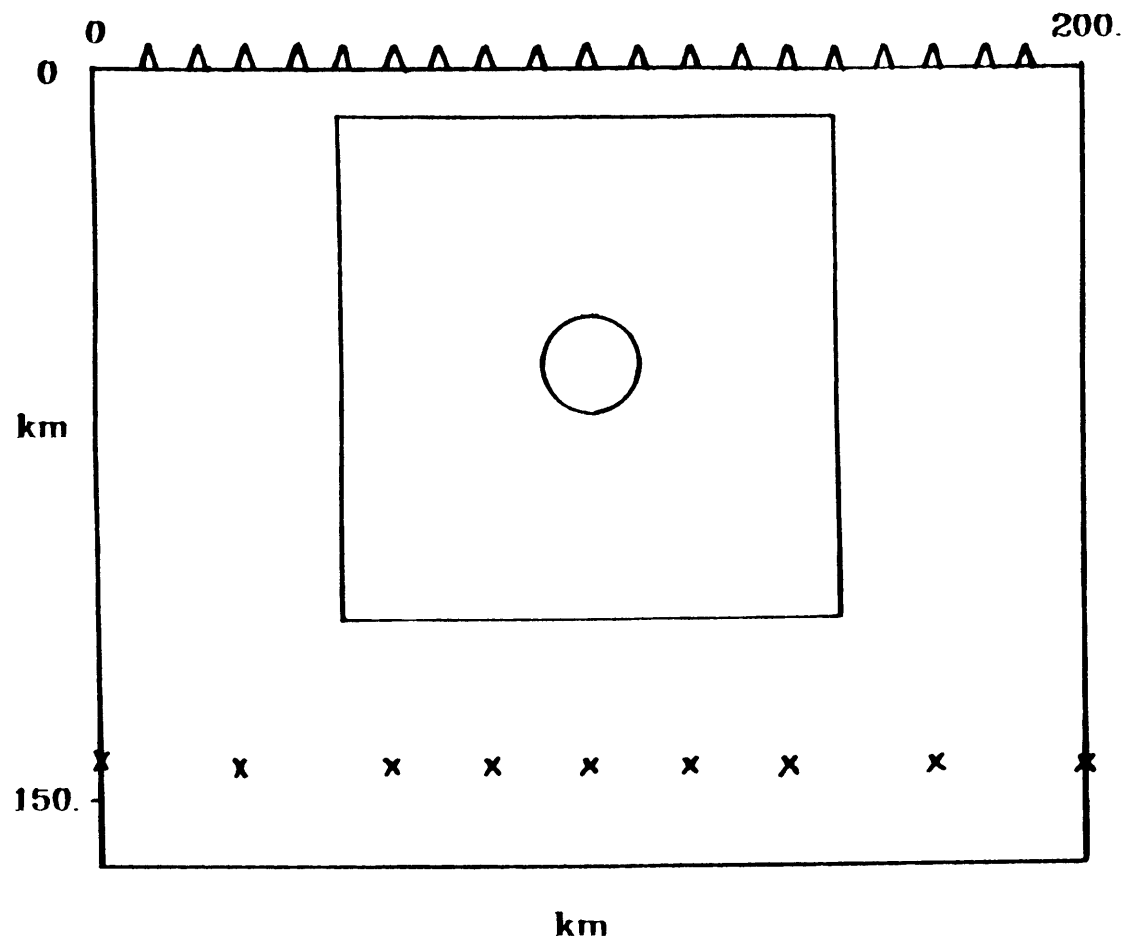


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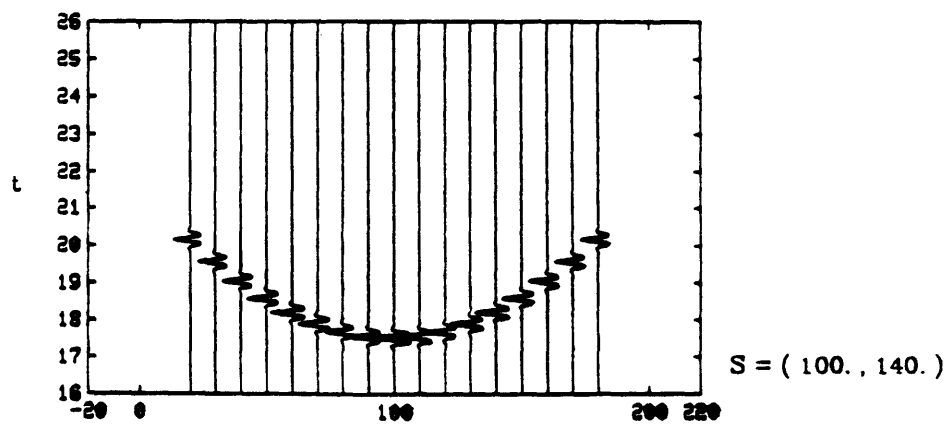
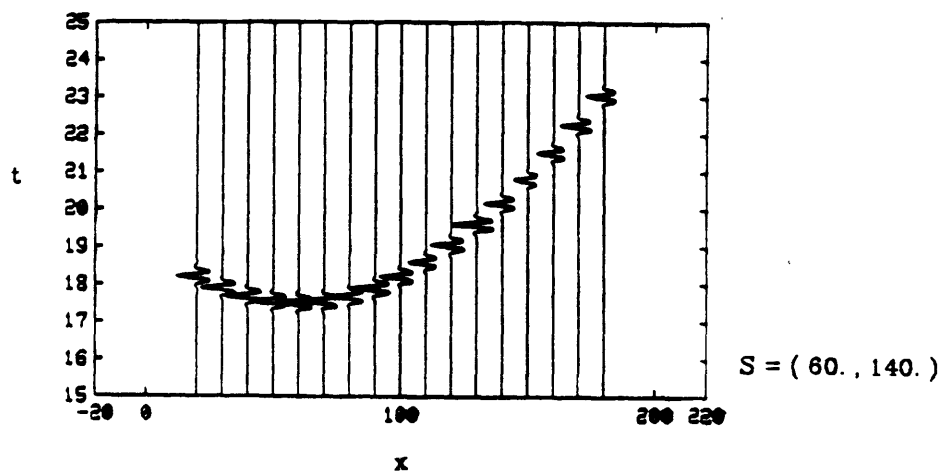
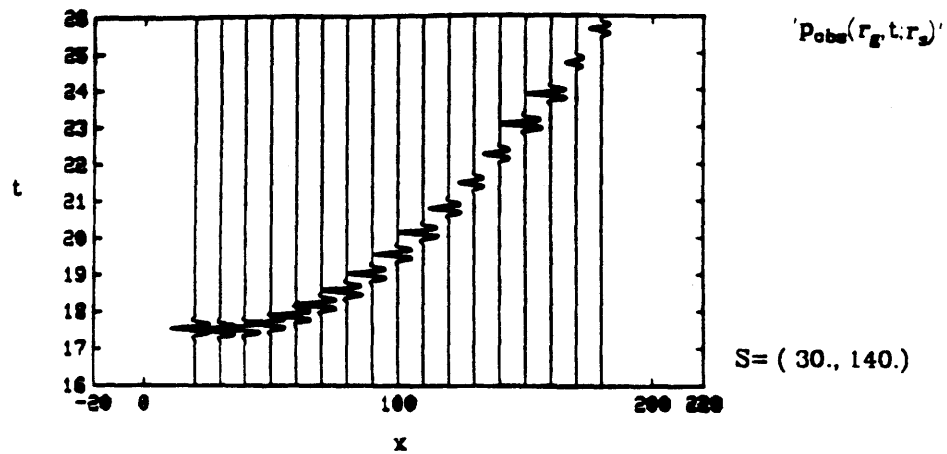


Figure 5

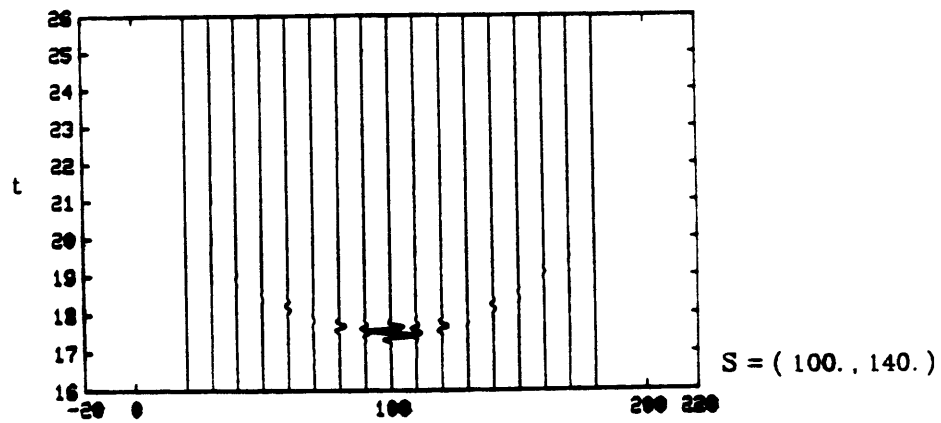
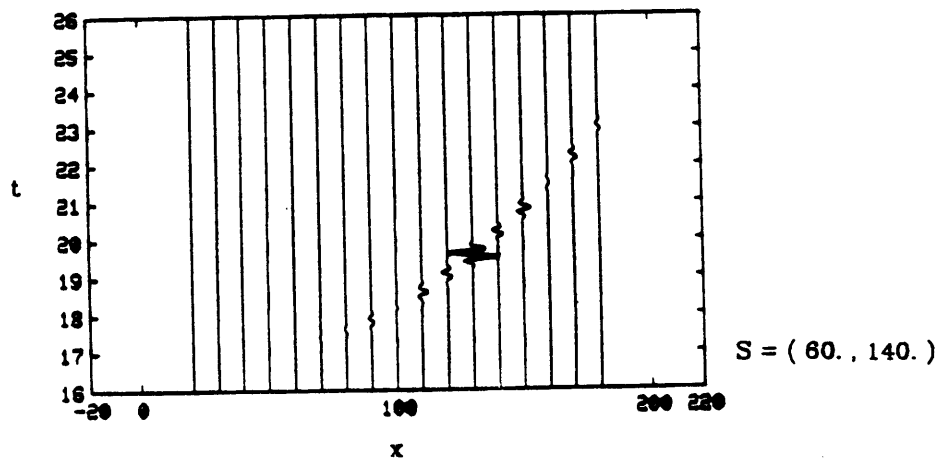
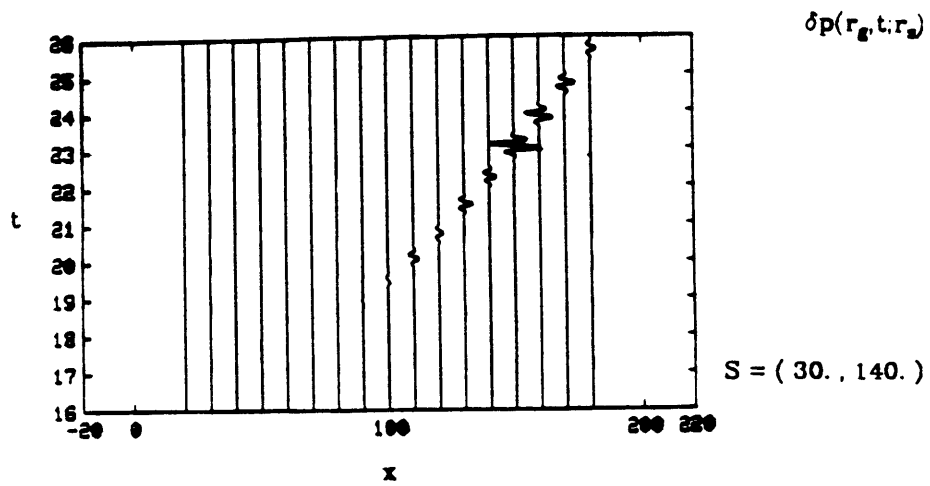


Figure 6

single heterogeneity

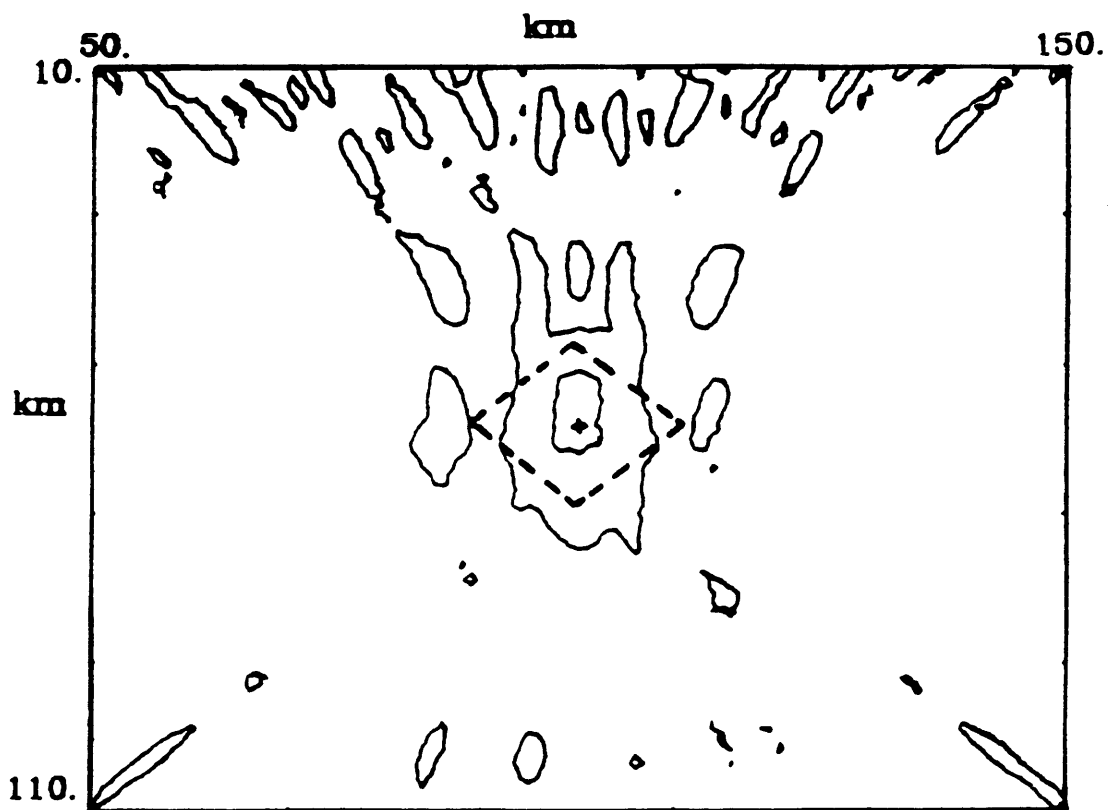


Figure 7

single heterogeneity

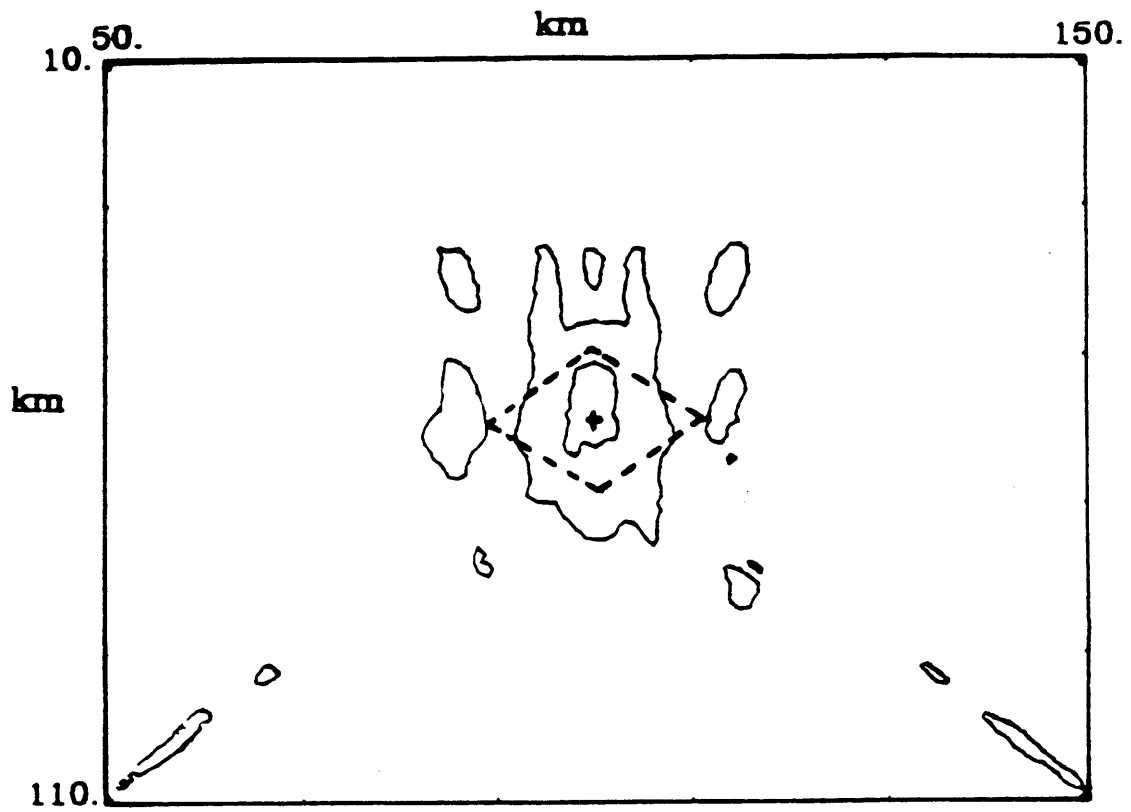


Figure 8

single heterogeneity

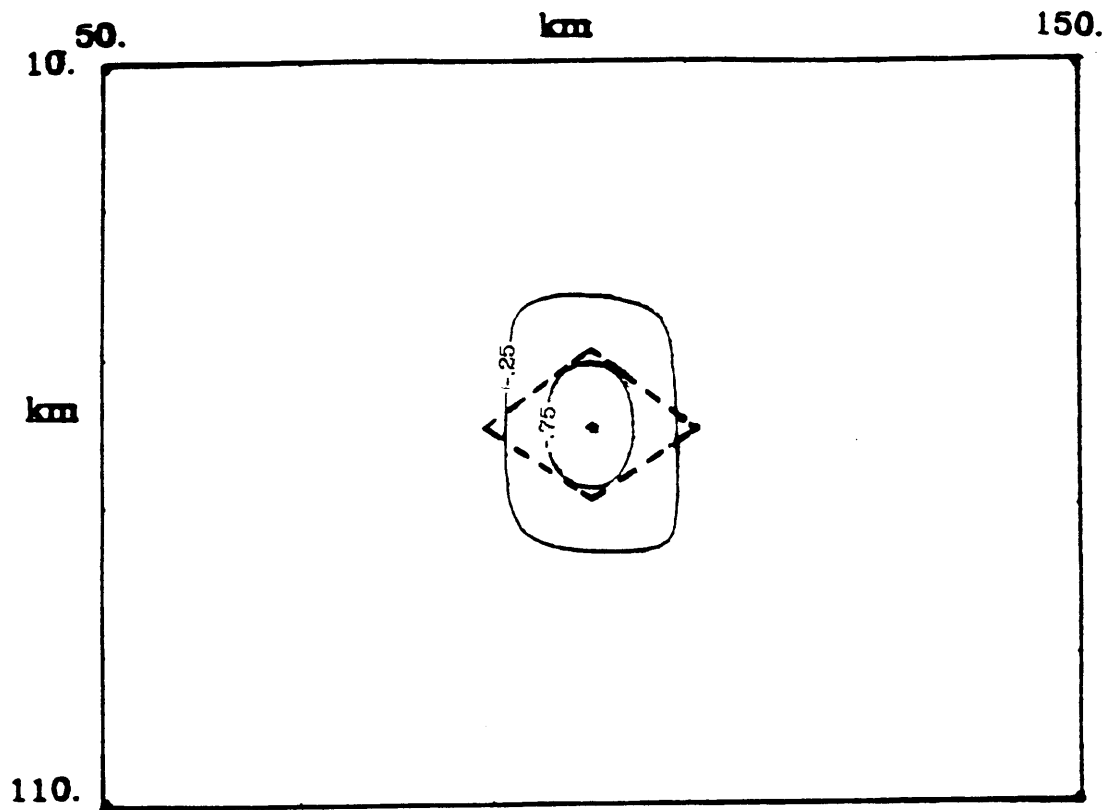


Figure 9

three heterogeneities

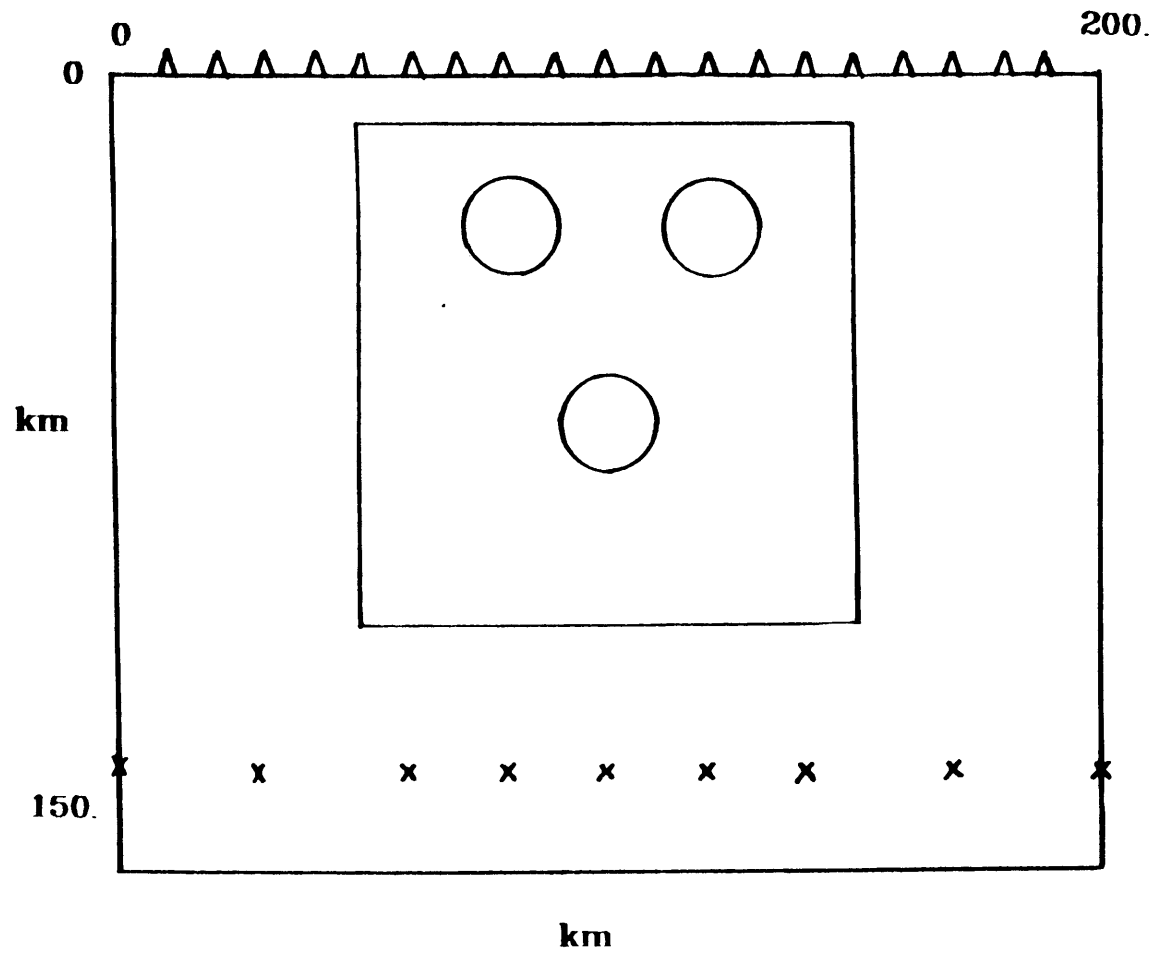


Figure 10

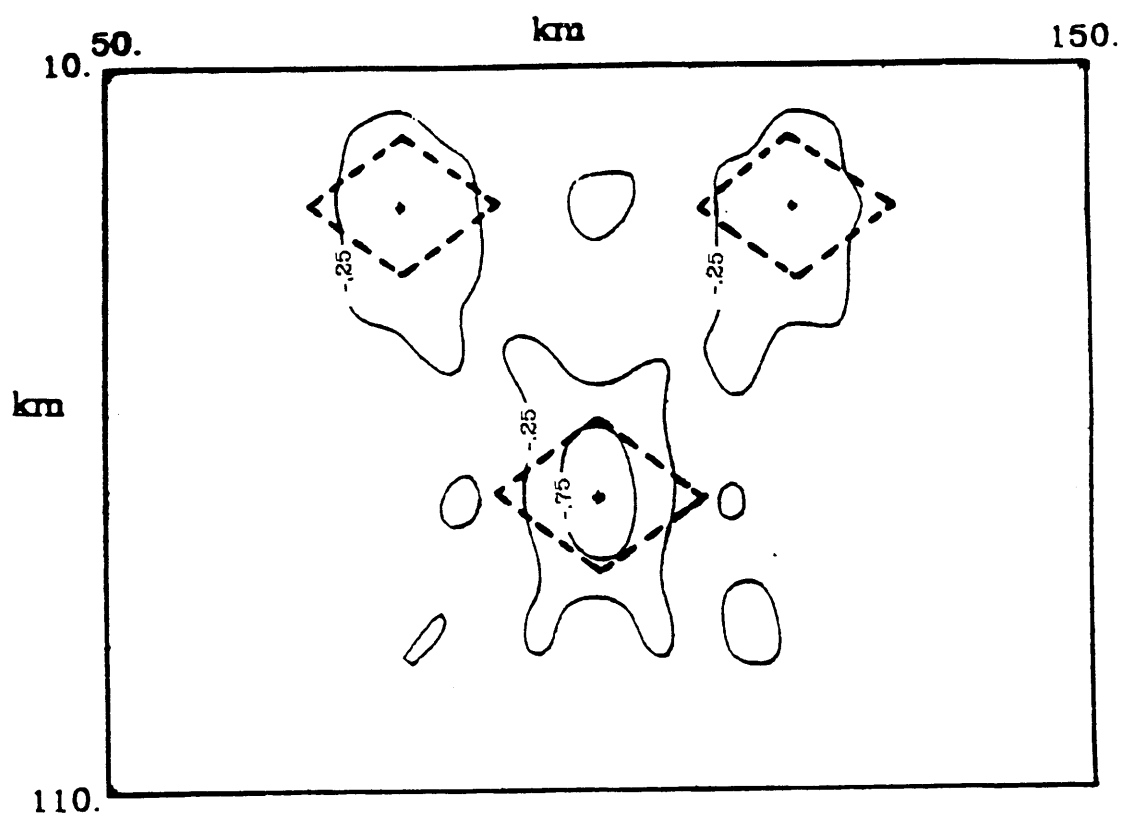


Figure 11

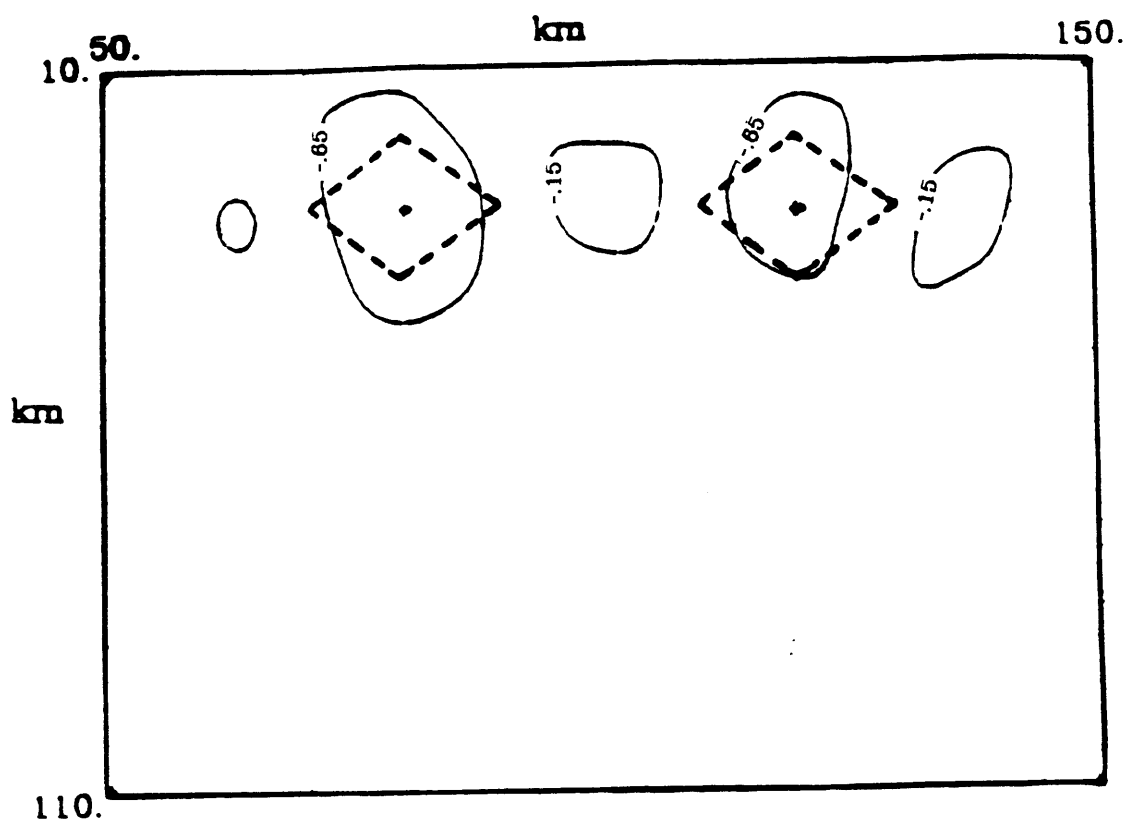


Figure 12

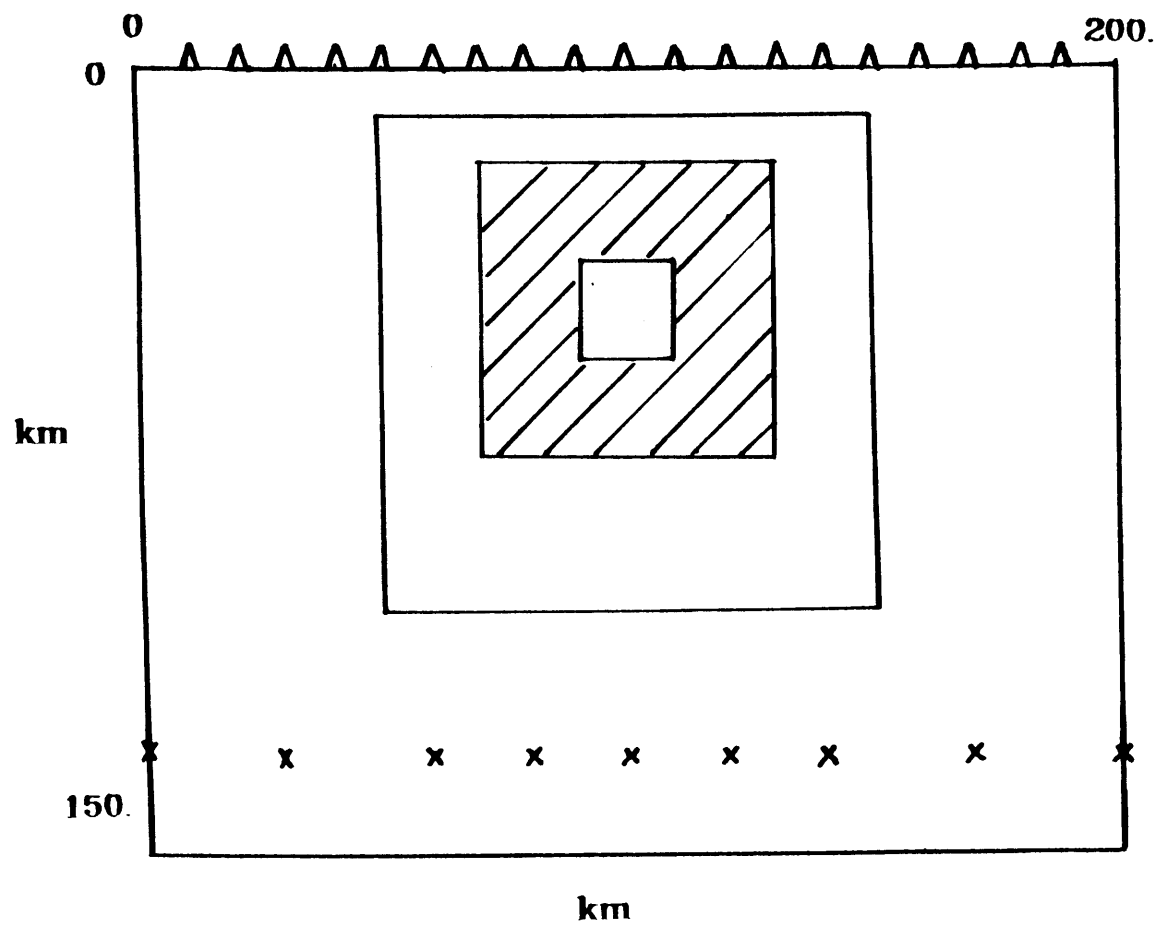


Figure 13

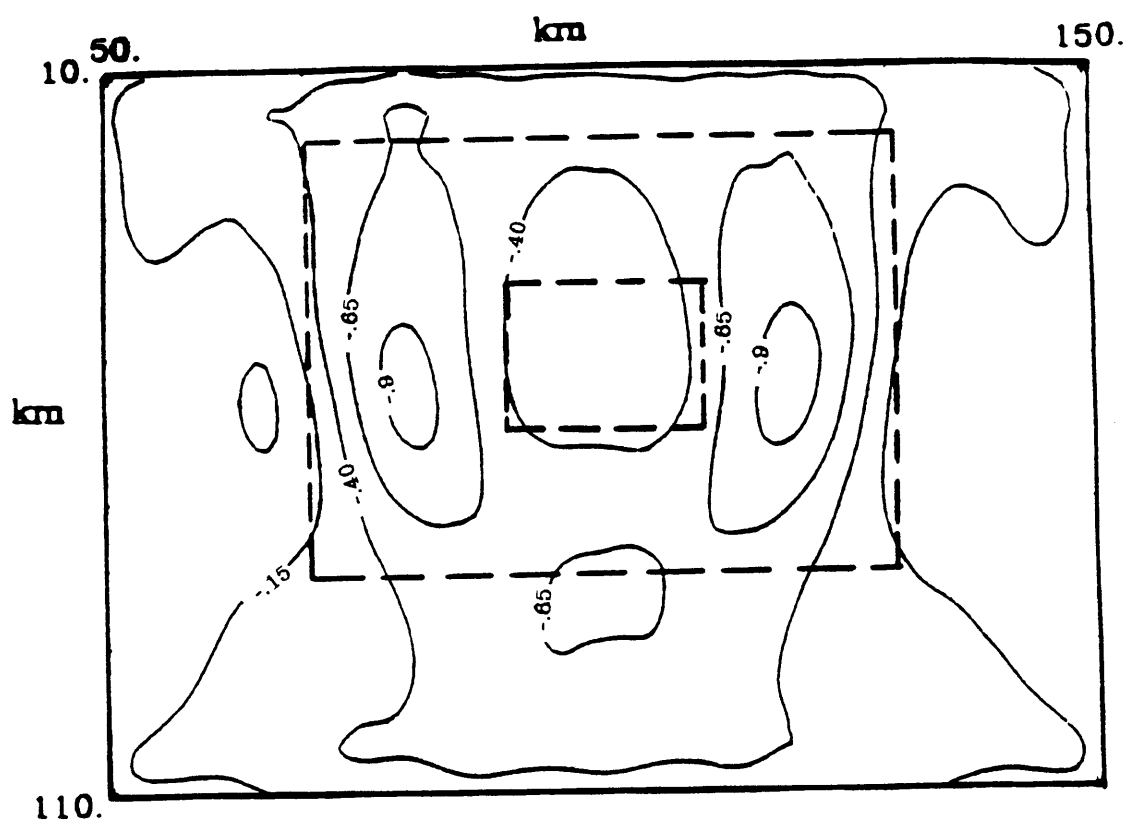


Figure 14

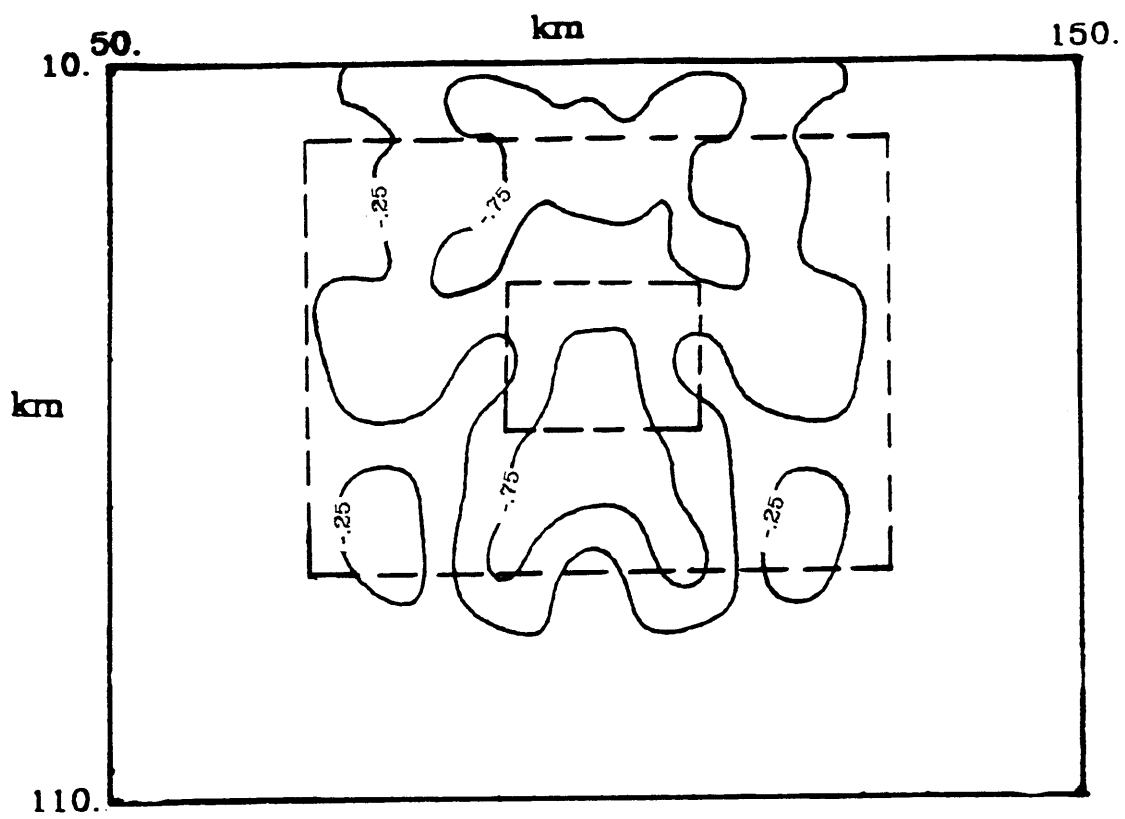


Figure 15

Figure 16

