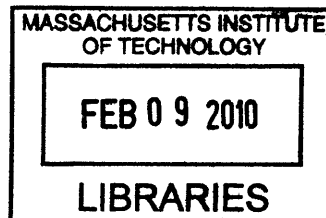


Essays in Asset Pricing and Market Imperfections

by

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Submitted to the Alfred P. Sloan School of Management
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2010

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Abstract

The first part of the thesis studies the impact of liquidity crashes on asset prices. In financial markets, liquidity could have large downward jumps. The thesis proposes a dynamic model where investors face the risk of potential liquidity crises. We find that investors choose optimal portfolios not only to hedge the risk of asset fundamentals, but also to hedge the risk of potential liquidity crashes. The potentially illiquid assets tend to have a lower price, a higher volatility, and a lower volume turn-over. Liquidity hedging could induce high return premium and asset returns could have excess volatility over the fundamentals. The risk of potential liquidity crises will also generate rich patterns in return dynamics and the expected asset returns could be driven by risks that are not systematic.

The second part of the thesis analyzes the effect of illiquidity on the extreme risk of hedge funds. Hedge funds' returns often exhibit positive autocorrelations, which suggests illiquidity in their asset holdings. In this part, using a data set containing monthly returns of over 5,600 hedge funds, I study how illiquidity affects the extreme risk of hedge funds. I use $MA(q)$ processes to model hedge funds' returns and use smoothing coefficients as proxies for liquidity. The tail risks are estimated using the extreme value theory and the generalized Pareto distribution. We find that illiquidity in general has a negative impact on the tail risk of hedge funds' returns. In particular, the true Value-at-Risk (VaR) of hedge funds could be much higher when illiquidity is taken into consideration.

The third part of the thesis studies asset pricing under heterogeneous information. In an asset market where agents have heterogeneous information, asset prices not only depend their expectations of the true fundamentals but also depend on their expectations of the expectations of others. Iterations of such expectations lead to the

so-called “infinite regress” problem, which makes the analysis of asset pricing under heterogeneous information challenging. In this part, we solve the infinite-regress problem in a simple economic setting under a fairly general information structure. This allows us to examine how different forms of information heterogeneity impacts the behavior of asset prices, their return dynamics, trading volume as well as agents’ welfare.

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Acknowledgments

I am very grateful to members of my thesis committee: Professors Jiang Wang, Leonid Kogan, and Jun Pan, for their invaluable supervision and constant encouragement. Without their support, this thesis would have never come into existence. I would like to thank Professors Andrew Lo, Steve Ross, and other faculty members of the Sloan Finance Department. I would also like to express my gratitude to Manuel Adelino, Jack Bao, Tara Bhandari, Alejandro Drexler, Kan Huang, Apurv Jain, Jiro Kondo, Dimitris Papanikolaou, Mary Tian, Ngoc-Khanh Tran, and Jialan Wang for many useful conversations.

I am deeply indebted to my parents for all their support. Finally, this thesis is dedicated to Fei.

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Chapter 1

Introduction

This thesis studies asset prices under market imperfections. There are many forms of market imperfections. In this thesis, I focus on illiquidity and information asymmetry.¹

Chapter 2 studies the impact of the risk of potential liquidity crashes on asset prices and trading behaviors. Recent financial crisis starting in 2007 and several other incidences in the past have echoed the importance of the risk of infrequent but severe liquidity crises. During the crisis, liquidity suddenly becomes low and even freeze up in some market segments. One of the fundamental functions for financial markets is to provide liquidity so that investors can buy and sell assets. During liquidity crisis, investors are forced to hold their positions or face high transaction costs, if they are able to trade at all. In such circumstance, investors' ability to move consumption over different time periods and cross different economic states is greatly reduced. The arrival of such liquidity crisis is often unexpected. Investors may have little or no clue as to when market will seize up. The fear that market liquidity could

¹There is a large literature on the relation between market imperfections and asset prices (See Vayanos and Wang (2009) for a comprehensive survey).

suddenly dry up could have significant impact on investors's trading behaviors and on equilibrium asset prices, even before the materialization of such events.

In this chapter, I develop a dynamic model where liquidity has some probability to experience a sudden downward jump and I address the asset pricing and trading implication of such liquidity events. In the model, risk-averse investors receive non-traded income which is correlated with the payoff of a risky asset so that investors trade the risky asset to hedge against their income risks.

In the bad liquidity state, investors' ability of risk sharing is greatly reduced. Therefore, during normal times, in anticipation of such a disastrous event, investors trade assets, not only to hedge against the risk of future non-traded income (income hedging), but also to hedge against the risk of future liquidity crashes (liquidity hedging). There are important asset allocation and risk-sharing consequences of investors' liquidity hedging. First, the need to hedge against the liquidity shock reduces investors ability to share income risks in normal times. Even if there is no aggregate risk, levels of investors' idiosyncratic risk will affect prices so that the asset price could be time-varying. This will generate excess *ex ante* equity volatility in normal times. Second, investors with different levels of income shock will put different weights on their demand for liquidity hedging. In particular, due to limited diversification in the bad liquidity state, investors whose income shock positively correlates with the asset payoff tend to experience a larger loss in the event of a liquidity dry-up than investors whose income shock negatively correlates with the asset payoff .

Investors' liquidity hedging have important asset pricing implications. The potentially illiquid asset tends to have a lower price and there tends to be a positive liquidity premium. The size of the liquidity premium is increasing in the probability of crash, but decreasing in the probability of recovery. Moreover, a positive proba-

bility of liquidity crash will generate excess asset price volatility, because, investors could not share their idiosyncratic risk optimally and the level of idiosyncratic risk will affect the asset price. The asset price picks up extra volatility from idiosyncratic risks.

The presence of the liquidity crash risk will also change the return dynamics. The risk premium in the asset return tends to be higher than that in the frictionless market since investors now demand extra premium to hold the potentially illiquid asset. Moreover, the asset return will be more volatile since the asset price has extra movement from idiosyncratic risks.

Chapter 3 studies the impact of illiquidity on the extreme risks of hedge funds. The study of hedge funds has become one of the major topics in the finance literature. Hedge funds are privately organized, lightly regulated asset management firms that employ a variety of strategies, including short sale and options. One principal difference between hedge funds and traditional asset management firms is that hedge fund managers have the primary goal of an “absolute” return, or a target rate of return, irrespective of the market performance. Managers were rarely willing to spend the time or money for active risk controls. However this absence of risk management has changed in the past years. In particular, the collapse of the Long Term Capital management (LTCM) and the financial crisis in 2007 forced many investors to weight the risk against the high returns, and to demand transparency of risk profiles of hedge funds. Today, risk management has become one of the essential part of hedge funds.

It has been argued in the previous literature that there might be illiquidity in hedge funds’ asset holdings (Getmansky, Lo, and Makarov (2004)). This illiquidity could affect the risks faced by hedge funds, in particular, their extreme risks. In-

tuitively, when a large negative event occurs, the hedge funds whose asset holdings are illiquid could potentially suffer larger loss since it is more difficult for them to unwind their position.

In this chapter, using a data set containing monthly returns of over 5,600 hedge funds, I study how illiquidity affects the extreme risk of hedge funds. I use $MA(q)$ processes to model hedge funds' returns and use smoothing coefficients as proxies for liquidity. The tail risks are estimated using the extreme value theory and the generalized Pareto distribution. We find that illiquidity in general has a negative impact on the tail risk of hedge funds' returns. In particular, the true Value-at-Risk (VaR) of hedge funds could be much higher when illiquidity is taken into consideration.

Chapter 4 studies asset prices under heterogenous information. Information asymmetry is another major source of market imperfection. In an asset market where agents have heterogeneous information, investors' private information is impounded into asset prices through their trades, and prices will then reflect the average beliefs cross investors. As a result, investors not only care about the fundamental values of a security, but also pay attention to other investors' beliefs on the fundamental value, a situation which Keynes (1936) refers to as the "Beauty Contest". Since private beliefs or their averages are not observable, beliefs of beliefs and their higher iterations also matter. Capturing these higher order beliefs, especially in an intertemporal setting, makes the formal analysis of the market behavior under asymmetric information a challenging task, which is also known as the infinite regress problem.

In Chapter 4, we solve the infinite-regress problem in a simple economic setting under a fairly general information structure. This allows us to examine how different forms of information heterogeneity impacts the behavior of asset prices, their return dynamics, trading volume as well as agents' welfare.

We show that the infinite-regress problem yields the long-range history dependence of the current market behavior. In general, current asset prices and their dynamics depend on the whole history of past shocks. In particular, revelations of past underlying shocks can influence current prices more than concurrent shocks. Information heterogeneity increases the divergence in investors' beliefs about economic fundamentals. Such a divergence tends to reduce the amount of risk sharing among investors and their effective risk tolerance. Consequently, stock prices become lower and more volatile. In addition, information heterogeneity reduces the level of liquidity and consequently the amount of trading in the market. We further show that the effect of information heterogeneity is non-monotonic in the amount of private information agents have. It is maximized when agents receive moderate amount of private information.

Moreover, we show that information heterogeneity tends to reduce investors welfare. In particular, investors are typically made worse off by possessing private information—they can be made better off by either revealing or abandoning all their private information collectively. We also find that investors with superior information does not necessarily enjoy higher welfare. The adverse selection problem makes it very costly for them to trade with less informed investors for risk sharing. Such a cost can out weight the potential gain they make from speculating on their private information.

Chapter 2

Market Shutdowns and Liquidity Premium

2.1 Introduction

In the recent financial crisis, the issue of illiquidity has more than ever attracted the attention of the public. The liquidity crisis triggered by the sub-prime event deeply impacted financial markets. During the crisis, liquidity suddenly became low and even froze up in some market segments. For example, during the later part of 2007, the market of the Structured Investment Vehicles(SIVs) almost completely collapsed. A SIV invests in long term securities, financed by selling short-term debt like commercial papers and rolling them over. The liquidity crunch caused by the sub-prime crisis in the commerce paper market made it very difficult for SIVs to borrow short term and they were not able to pay off the previous issued short-term debt. As a consequence, many SIVs defaulted and the market froze up. Another example is the LIBOR market. After the shock of the bankruptcy filing of Lehman

Brothers on September 15, 2008, the spread of the 3-month LIBOR rate over the Overnight Index Swaps (the Libor/OIS spread), reached more than 200 basis points, up from only 80 basis points in the beginning of the month. The interbank lending market was effectively seized up.

In the existing literature, many authors have documented similar incidences of sudden market liquidity dry-ups (e.g. Chordia, Roll, and Subrahmanyam (2001), Jones (2002), and Pastor and Stambaugh (2003)). For example, the liquidity measure of Pastor and Stambaugh had a very large downward spike in October 1987, in the month of the stock market crash. It also jumped down in November 1973 (the Mideast Oil crisis), and in September 1998 (the LCTM crisis), among other times.

One of the fundamental functions for financial markets is to provide liquidity so that investors can buy and sell assets. During liquidity crisis, investors are forced to hold their positions or face high transaction costs, if they are able to trade at all. In such circumstance, investors' ability to move consumption over different time periods and cross different economic states is greatly reduced. The arrival of such liquidity crisis is often unexpected. Investors may have little or no clue as to when market will seize up. The fear that market liquidity could suddenly dry up could have significant impact on investors's trading behaviors and on equilibrium asset prices, even before the materialization of such events.

In this paper, we develop a dynamic model where liquidity has some probability to experience a sudden downward jump and we address the asset pricing and trading implication of such liquidity events. In the model, risk-averse investors receive non-traded income which is correlated with the payoff of a risky asset so that investors trade the risky asset to hedge against their income risks. In normal times, investors can trade both the risk free asset and the risky asset with no cost. However, during normal times, there is a positive probability that the liquidity for the risky asset will

suddenly dry up in the next period. The liquidity crash is assumed to be reversible so that in the liquidity dry-up state, there is some probability that the liquidity will jump back to its normal level in the next period.

In the bad liquidity state, investors' ability of risk sharing is greatly reduced. Therefore, during normal times, in anticipation of such a disastrous event, investors trade assets, not only to hedge against the risk of future non-traded income (income hedging), but also to hedge against the risk of future liquidity crashes (liquidity hedging). There are important asset allocation and risk-sharing consequences of investors' liquidity hedging. First, the need to hedge against the liquidity shock reduces investors ability to share income risks in normal times. The liquidity hedging pushes investors' risky asset holdings away from the optimal income risk-sharing position. Therefore, even if there is no aggregate risk, levels of investors' idiosyncratic risk will affect prices so that the asset price could be time-varying. This will generate excess *ex ante* equity volatility in normal times. Second, investors with different levels of income shock will put different weights on their demand for liquidity hedging. In particular, investors whose income shock positively correlates with the asset payoff tend to experience a larger loss in the event of a liquidity dry-up than investors whose income shock negatively correlates with the asset payoff . This is because, with everything else equal, a higher income correlation with the stock dividend means a higher exposure to the dividend risk. A materialized liquidity crash next period will prevent these investors from re-balancing the asset holding in the future. Hence they tend to incur larger utility loss.

Investors' liquidity hedging have important asset pricing implications. The potentially illiquid asset tends to have a lower price and there tends to be a positive liquidity premium. The asset price implication comes from the difference in the impact magnitude of realized liquidity crashes among different investors. As discussed

in the previous paragraph, in the event of a liquidity dry-up, investors whose idiosyncratic shocks positively co-move with the asset payoff will have a larger loss, and therefore they care more about the liquidity crash state and they will put more weight on the liquidity hedging demand. Moreover, assuming the income shock is persistent, it will be better for investors to build up more position in the risky asset when entering the bad liquidity state. Therefore, compared with frictionless market, the selling need from investors whose income shock positively correlates with the asset dividend will be larger than the buying need from investors whose income shock negatively correlates with the asset dividend. Hence the asset price falls compared with that in the frictionless market and we have a positive liquidity premium. The size of the liquidity premium is increasing in the probability of crash, but decreasing in the probability of recovery. Moreover, a positive probability of liquidity crash will generate excess asset price volatility, because, investors could not share their idiosyncratic risk optimally and the level of idiosyncratic risk will affect the asset price. The asset price picks up extra volatility from idiosyncratic risks.

The presence of the liquidity crash risk will also change the return dynamics. The risk premium in the asset return tends to be higher than that in the frictionless market since investors now demand extra premium to hold the potentially illiquid asset. Moreover, the asset return will be more volatile since the asset price has extra movement from idiosyncratic risks.

Numerically results in the paper show that the impact of the liquidity crash risk on asset prices could be economically large. In our benchmark parametrization, if we assume a 3% liquidity crash probability per year and an expected recovery period of 5 years, the potentially illiquid asset will be traded in the normal times at a 3.8% discount below its fully liquid counterpart. This will help to explain the high

empirical liquidity premium.¹

One interesting result, which echoes the findings in Vayanos (1998) and Longstaff (2009), is that, although a small probability of liquidity crash could generate large positive liquidity discount, there are scenarios when the chance of a liquidity dry-up will in deed increase the asset price and lead to a negative liquidity premium. Assuming that the income shocks are very transitory, the liquidity hedging component may have less loading on income shocks than the income hedging component, because, due to the transitory nature of the income shock, it may not be optimal to build up large position into the lock-up period. In this case, the *ex post* impact asymmetry of liquidity crashes among investors with different income loadings on the asset dividend will cause the selling demand to be smaller than the buying demand and hence stock price increases compared with that in the frictionless market.

This paper is organized as following. Section 2.2 discusses the related literature. Section 2.3 describes the model in the general framework. Section 2.4 characterizes the solution of the equilibrium. Section 2.5 analyzes the equilibrium with focus on the optimal portfolio choice. Section 2.6 studies the asset price implications. Section 2.8 discusses the case where the probability of crash is state dependent. Section 3.7 concludes. The numerical procedures and all the proofs are given in the appendix.

¹Some papers argue that the model generated liquidity premium is low relative to the empirical observed premium (Amihud and Mendelson (1986a), Brennan and Subrahmanyam (1996b), Aiyagari and Gertler (1991), Constantinides (1986), Heaton and Lucas (1996), Vayanos (1998), and Huang (2003)). On the other hand, Lo, Mamaysky, and Wang (2004) finds that when investors have frequent trading need, the existence of a small transaction cost can generate a large liquidity premium.

2.2 Related Literature

This paper is closely related to Longstaff (2009). In his model, liquid asset can be traded at all time, but the illiquid asset can only be traded initially, and then there is a lock-up period of deterministic length right after the initial trading. Investors differ by their time discount factor. In this framework, the paper shows that the “lock-up” period could have significant impact on asset valuation and trading volume. Our paper also studies the liquidity premium, but through a somewhat different channel. In our paper, both the arrival of the lock-up period and the arrival of recovery are random, and liquidity discount is a consequence of the fear of the liquidity crashes, rather than the actual realization of such crisis.

This paper is also closely related to Lo, Mamaysky, and Wang (2004). They analyze the effect of fixed transaction cost when investors have continuous trading need. Investors receive non-traded income continuously over time so that in the frictionless market, they trade continuously to share risks. However, the existence of transaction costs prevents the continuous risk-sharing trading. They find that even a small transaction cost can lead to a large non-trading zone and generate a significant liquidity premium in asset prices. In the presence of transaction cost, the trading volume is finite (in contrast to the infinite trading volume in the frictionless continuous time model). The increase of transaction cost, however, only has marginal effects on trading volume. Our paper compliments theirs by introducing a random arrival of liquidity dry-up.

Hong and Wang (2000) studies the impact of the periodic market closure. In their model, investors have continuous trading needs from both risk-sharing motives and speculative motives, while the market closes and reopens periodically (with deterministic timing). During the market closure, investors are not able to share

risks, and furthermore, investors' private information will not be reflected by the price since no trading happens. These effects of time-varying hedging trade and time-varying information asymmetry can generate a rich pattern in asset returns and trading volume. Our paper differs from theirs in that in our model, the arrival of the liquidity crash/ recovery is random. The results in this model are mostly generated by investors' fear of such liquidity events rather than the realization of such events.

This paper is also closely related to Huang (2003). In his paper, overlapping-generations of investors face sudden surprise shock to liquidate their assets and exit the economy. Investors buy or sell consol bonds to smooth consumption over time. However, they may need to pay a (proportional) transaction cost to trade assets. The paper finds that without additional constraints (e.g. limited borrowing), the liquidity premium is low. Though, both study the impact of liquidity shocks, our model differs from that in Huang (2003) in a few dimensions. In our model, investors are infinitely-lived and they trade securities to hedge against their income shocks. In the event of a liquidity shock, investors still remain in the economy and the liquidity will revert back with positive probability.

Our paper compliments the literature on market frictions and transaction costs (Amihud and Mendelson (1986a), Constantinides (1986), Brennan and Subrahmanyam (1996b), Aiyagari and Gertler (1991), Heaton and Lucas (1996), Vayanos (1998), Huang (2003), Acharya and Pedersen (2005b), Huang and Wang (2009), etc). This paper is also broadly related to the literature of market crashes and the Peso problem (Rietz (1988), Liu, Longstaff, and Pan (2003), Liu, Pan, and Wang (2005) Barro (2006), Wachter (2008), Gabaix (2008), Drechsler and Yaron (2008), Gourio (2008), Ghosh and Julliard (2008), Barro (2009), etc).

2.3 The Model

In this section, we develop a dynamic model to study the impact of potential liquidity crashes.

Securities. There is one consumption good, which will serve as numeraire. There are two traded assets: the risky asset and the risk-free bond. The risk-free bond has an exogenous return of r . The risky asset pays a cumulative dividend D_t , which follows:

$$dD_t = \mu_D dt + \sigma_D dB_{D,t},$$

where $\mu_D > 0$, $\sigma_D > 0$, and $B_{D,t}$ is a one-dimensional Brownian motion. The number of shares of the risky asset is normalized to be 2. The risky asset in this model is not necessarily a stock, and it could refer to any asset which has a risky payoff. But for notational convenience, henceforward in this paper, we use the term “risky asset” and “stock” interchangeably.

Agents. There are two agents $i = 1, 2$. Each agent is endowed with θ_0^i shares of stocks at time $t = 0$, with $\sum_{i=1}^2 \theta_0^i = 2$. In the following, we assume that $\theta_0^1 = \theta_0^2 = 1$ for symmetry. At each time t , agent i receives a non-traded income. The cumulative income process N_t^i is given by:

$$dN_t^i = (-1)^i Y_t dB_{N,t},$$

where $B_{N,t}$ is a Brownian motion with $dB_{N,t} \cdot dB_{D,t} = \rho_{ND} dt$ (without loss of generality, we assume $\rho_{ND} \geq 0$). The process Y_t follows:

$$dY_t = -\kappa Y_t dt + \sigma_Y dB_{Y,t},$$

where $\kappa \geq 0$, $\sigma_Y > 0$, and $B_{Y,t}$ is a Brownian motion independent of $B_{D,t}$ and $B_{N,t}$. The assumption that the non-traded incomes of the two agents offset each other perfectly ensures that there is no aggregate income risk, which allows us to focus on the impact of market frictions. However, we can readily generalize the model to include non-trivial aggregate risks. We allow different persistence levels of the non-traded income shocks. The level of persistence of the income shocks has important consequence in the impact of liquidity dry-up on asset prices. Intuitively speaking, if the non-traded income is very transitory (e.g. close to i.i.d.), investors can basically do very little today to hedge against the income risk in the future, and therefore the un-realized risk of liquidity jump would have little impact on investors' trading today. Whereas, if the non-traded risk is very persistent (e.g. close to a random walk), the dynamic hedging would be important to investors, and therefore, the un-realized risk of liquidity jump would have large impact on investors' trading today.

Utility. Both agents have the CARA utility with the same time discount ρ and the same risk aversion α . They choose consumption $\{c_t^i\}_{t=0}^{\infty}$ and stock holding $\{\theta_t^i\}_{t=0}^{\infty}$ to maximize:

$$E_0 \left[\int_{t=0}^{\infty} -e^{-\rho t - \alpha c_t^i} dt \right].$$

To ensure finite utility, we impose the parameter constraint:

$$|\alpha \sigma_Y| < \frac{1}{2} + \frac{\kappa}{r}. \quad (2.1)$$

Liquidity Crash and Recovery. There are two liquidity states: good and bad. In good liquidity state, investors can trade the risk free bond and the risky asset at no cost. At each time in the good state, there is a small probability that the liquidity

will dry up, with jump intensity λ_L (we allow λ_L to be state dependent). In the event of liquidity dry-up, the stock market completely freezes up and investors can only trade bonds. However, liquidity crash is recoverable. At each time in the bad liquidity state, there is a probability that the liquidity will revert back to normal, with jump intensity λ_H .

The assumption that the market closes in the event of liquidity crash is somewhat strong. But this could be justified if we assume a quick recovery rate (i.e. large λ_H). A more realistic assumption is that when liquidity crashes, investors will face a sudden spike in transaction cost when trading assets. The model in this paper is a limiting case of this scenario, where the transaction cost jumps to infinity, which provides an upper bound of the impact of the liquidity crash.

2.4 The Solution of the Equilibrium

We solve the model in two steps. In the first step we take the price process as given and solve the investors' optimization problems and find the optimal asset holding under the given price process. In the second step, we find the price process that clears the market.

The case of the frictionless market, i.e. no probability of liquidity crash ($\lambda_L = 0$), can be easily solved in closed form. The following proposition characterizes the equilibrium in this case.

Proposition 1. *In the case of the frictionless market ($\lambda_L = 0$), the stock price is a constant P^* , given by:*

$$P^* = \frac{\mu_D}{r} - \alpha \sigma_D^2,$$

the optimal stock holding of investor i is

$$\theta_t^{*i} = 1 - \frac{\rho_{ND}}{\sigma_D} (-1)^i Y_t,$$

and the value function of investor i is of the form $e^{-\rho t} e^{-\alpha(\tau W_t^i + v_L^{*i}(Y_t))}$, where W_t^i is the total wealth in bond and stock, and $v_L^{*i}(Y) = \frac{1}{2} a_L Y^2 + b_L Y + c_L$, with a_L , b_L , and c_L constants.

The proof of this proposition is standard. The stock price can be decomposed into two terms. The term $\frac{\mu_D}{r}$ is the present value of future cash flow discounted at the risk-free rate, and the term $\alpha \sigma_D^2$ is the risk premium for the diffusion shock $dB_{D,t}$ (henceforward referred to as the dividend risk premium). The optimal stock holding of investor i is $\theta_t^{*i} = 1 - \frac{\rho_{ND}}{\sigma_D} (-1)^i Y_t$. Investors whose income shock positively correlates with the stock dividend will hold less stock than investors whose income shock negatively correlates with the stock dividend. In this case, the level or the volatility of the non-traded income shock Y_t has no effect on the stock price since it can be perfectly eliminated through investors' risk sharing. Consequently, the stock price is a constant over time, independent of the level of Y_t . The dollar stock return volatility is the same as the volatility of the dividend, and there is no excess return volatility.

If $\lambda_L > 0$, the stock price will not be independent of the level of the idiosyncratic risk, even though at the aggregate level, the total non-traded risk is zero. The intuition is that, out of the fear for a liquidity crash, investors not only need to hedge the future income risk, but also need to hedge the future liquidity risk. The liquidity crash, however, will have different impacts on investors with different levels of income risks. In particular, investors whose income positively correlates with stock dividend will be affected differently from investors whose income negatively correlates

with the stock dividend. This impact asymmetry will lead to imperfect risk sharing in normal times and therefore the market clearing price will depend on the level of idiosyncratic shocks (we will discuss this intuition in detail later in Section 2.5).

This intuition suggests that the price process is given by $P_{L,t} = h_L(Y_t)$, where $h_L(Y)$ is a smooth function. We conjecture that the price is in deed of this form and will later check the market clearing condition. With the conjectured price function $h_L(Y)$, we define the liquid wealth to be part of the investors holding that can be transfer into consumption goods without any transaction cost. In this specification, the liquid wealth in the good liquidity state is just the sum of wealth in bonds and wealth in stocks. In the bad liquidity state, the liquid wealth is only the wealth in bond. Therefore, in the event of a liquidity jump, the liquid wealth will also experience a jump. We can write the liquid wealth process as:

$$dW_t^i = rW_t dt + \theta_t^i(dP_t + dD_t - rP_t dt) + dN_t^i - c_t^i dt. \quad (2.2)$$

Purely for notional reason, we assume that P_t is zero in the bad liquidity state (Investors can not trade stocks in bad liquidity state, and the zero stock price is only for the convenience of writing the wealth process). Notice that under this notional assumption P_t may have jumps. The following proposition characterizes the equilibrium.

Proposition 2. (i) *The equilibrium price in the good liquidity state is a function of Y_t : $P_t = h_L(Y_t)$.*

(ii) *The value function of investor i in the good liquidity state is of the form $-e^{-\rho t - \alpha(rW_t^i + v_L^i(Y_t))}$, and the value function in the bad liquidity state is of the form $-e^{-\rho t - \alpha(rW_t^i + v_H^i(Y_t, \theta_t^i))}$. The functions $v_L^i(\cdot)$, and $v_H^i(\cdot, \cdot)$ satisfy the following Bellman*

equations:²

$$\begin{aligned}
0 = \min_{\theta_t^i} & \quad r - \rho - r \ln r + r\alpha v_L^i - \alpha r \theta_t^i \{\mu_D - \kappa Y_t \frac{dh_L}{dY} + \frac{1}{2} \frac{d^2 h_L}{dY^2} \sigma_Y^2 - r h_L(Y_t)\} \\
& + \alpha \kappa Y_t \frac{dv_L^i(Y_t)}{dY} - \frac{1}{2} \alpha \frac{d^2 v_L^i(Y_t)}{dY^2} \sigma_Y^2 \\
& + \frac{1}{2} \alpha^2 \left\{ \left(r \frac{dh_L}{dY} \sigma_Y \theta_t^i + \frac{dv_L^i(Y_t)}{dY} \sigma_Y \right)^2 + r^2 Y_t^2 + r^2 \sigma_D^2 (\theta_t^i)^2 + 2(-1)^i r^2 Y_t \sigma_D \rho_{ND} \theta_t^i \right\} \\
& + \lambda_L \{ e^{-\alpha \{v_H^i(\theta_t^i, Y_t) - v_L^i(Y_t) - r \theta_t^i h_L(Y_t)\}} - 1 \},
\end{aligned}$$

and

$$\begin{aligned}
0 = & \quad r - \rho - r \ln r + r\alpha v_H^i - \alpha r \theta_t^i \mu_D + \alpha \kappa Y_t \frac{\partial v_H^i(Y_t, \theta_t^i)}{\partial Y} - \frac{1}{2} \alpha \frac{\partial^2 v_H^i(Y_t, \theta_t^i)}{\partial Y^2} \sigma_Y^2 \\
& + \frac{1}{2} \alpha^2 \left\{ \left(\frac{\partial v_H^i(Y_t, \theta_t^i)}{\partial Y} \sigma_Y \right)^2 + r^2 Y_t^2 + r^2 \sigma_D^2 (\theta_t^i)^2 + 2(-1)^i r^2 Y_t \sigma_D \rho_{ND} \theta_t^i \right\} \\
& + \lambda_H \{ e^{-\alpha \{v_L^i(Y_t) - v_H^i(Y_t, \theta_t^i) + r \theta_t^i h_L(Y_t)\}} - 1 \}.
\end{aligned}$$

(iii) The market clearing condition $\theta_t^1(Y) + \theta_t^2(Y) = 2$ determines the stock pricing function $h_L(Y)$.

In general, due to the non-Gaussian nature of the jump shocks, the equilibrium price function $h_L(Y)$ and the Bellman equations for $v_L^i(\cdot)$ and $v_H^i(\cdot, \cdot)$ will not have close form solutions and numerical techniques have to be employed to find the value functions and the equilibrium price function. Appendix A.3 gives a detail account for the numerical procedure.

²Notice that due to the symmetry of investor 1 and 2, the value functions and the price function satisfy the following symmetry conditions:

$$h_L(Y) = h_L(-Y), \quad v_L^1(Y) = v_L^2(-Y), \quad v_H^1(Y, \theta) = v_H^2(-Y, \theta).$$

2.5 Analysis of Equilibrium

The presence of a positive probability of liquidity crash prevents investors from perfect risk-sharing in the good state. The first order condition for the Bellman equation implies that, in normal times, the optimal stock holding θ_t^i of investor i is given by:

$$\theta_t^i = \frac{\sigma_{Q,t}^2}{\frac{\lambda_L e^{-\alpha\eta^i(Y_t, \theta_t^i)} \sigma_D^2}{r} + \sigma_{Q,t}^2} \theta_t^{i,I} + \frac{\frac{\lambda_L e^{-\alpha\eta^i(Y_t, \theta_t^i)} \sigma_D^2}{r}}{\frac{\lambda_L e^{-\alpha\eta^i(Y_t, \theta_t^i)} \sigma_D^2}{r} + \sigma_{Q,t}^2} \theta_t^{i,L}, \quad (2.3)$$

where $\theta_t^{i,I}$, $\theta_t^{i,L}$, $\sigma_{Q,t}$, and $\eta^i(Y, \theta)$ are given by:

$$\begin{aligned} \theta_t^{i,I} &= \frac{1}{\alpha r \sigma_{Q,t}^2} \{ \mu_{Q,t} - \alpha r \rho_{ND} \sigma_D (-1)^i Y_t - \alpha \sigma_Y^2 \frac{dv_L^i}{dY}(Y_t) \frac{dh_L}{dY}(Y_t) \}, \\ \theta_t^{i,L} &= \frac{1}{\alpha r \sigma_D^2} \{ \frac{\partial v_H^i}{\partial \theta} - r h_L(Y_t) \}, \\ \sigma_{Q,t} &= \sqrt{\sigma_D^2 + \sigma_Y^2 \left(\frac{dh_L}{dY}(Y_t) \right)^2}, \\ \eta^i(Y, \theta) &= v_H^i(Y, \theta) - v_L^i(Y) - r \theta h_L(Y). \end{aligned}$$

Equation (2.3) has a very intuitive interpretation. The optimal stock holding is a weighted average of $\theta_t^{i,I}$ and $\theta_t^{i,L}$. The quantity $\theta_t^{i,I}$ can be viewed as the optimal stock holding conditional on the event that there will not be a liquidity crash next period, which we refer to as the income hedging component. The quantity $\theta_t^{i,L}$ can be viewed as the optimal stock holding conditional on the event that there will be a liquidity crash next period, which we referred to as the liquidity hedging component. The quantity $\eta^i(Y_t, \theta_t^i)$ measures the utility difference before and after the realization of a liquidity crash. The optimal stock holding is the weighted average of the income hedging component and the liquidity hedging component. The optimal weights reflect the probability of the liquidity crash and the size of the utility loss due

to the crash event. Intuitively, in the presence of a possible liquidity crash, investors choose their optimal stock holding to balance utility in good future states and in bad future states. There are two issues. Firstly, investors need to determine the optimal stock holding, $\theta_t^{i,L}$, conditional on the event that there will be a liquidity crash next period, and the optimal stock holding, $\theta_t^{i,I}$, conditional on the event that there will not be a liquidity crash next period. Secondly, investors need to find an optimal stock holding to balance these two possibilities. That is, investors mix $\theta_t^{i,L}$ and $\theta_t^{i,I}$ with suitable weights to optimize his utility.

It is easy to see from equation (2.3) that the presence of a potential liquidity crash will push investors's stock holding away from the otherwise optimal level: $\theta_t^{*,i} = 1 - \frac{\rho ND}{\sigma_D} (-1)^i Y_t$. This confirms the intuition that the fear of a market shutdown will lead to sub-optimal risk sharing in normal times.

Using the optimal stock holding equation (2.3), and the market clearing condition, we can write down the stock price at $Y = 0$.

Proposition 3. *At $Y = 0$, the stock price in the good liquidity state is:*

$$h_L(0) = \frac{\mu_D}{r} - \alpha \sigma_D^2 - \left\{ -\frac{\frac{1}{2} h_L''(0) \sigma_Y^2}{r + \lambda_L e^{-\alpha \{ \mu_D - \frac{1}{2} \alpha r \sigma_D^2 - r h_L(0) + c_H - v_L(0) \}}} \right\}, \quad (2.4)$$

where c_H is a constant given in Appendix A.2.

Proof. This is a direct consequence of the optimal asset holding equation (2.3), together with the market clearing condition and the fact that $h_L(Y)$ is an even function. \square

The term $\frac{\mu_D}{r} - \alpha \sigma_D^2$ is the price P^* in the frictionless market. The term in the parenthesis on the right hand side of equation (2.4) can be viewed as a measure of (a lower bound of) liquidity jump premium. The liquidity jump premium is related

to the convexity (concavity) of the pricing function in terms of Y . The sign of the liquidity premium depends on the sign of the second derivative of h_L . If $h_L''(0) < 0$, the liquidity premium is positive, and if $h_L''(0) > 0$, the liquidity premium is negative (we will later show that both cases could in deed happen and liquidity premium could be either positive or negative).³ The magnitude of the liquidity premium is related to the volatility of the idiosyncratic risks. More volatile idiosyncratic risks tend to be associated with larger magnitudes of liquidity premium. This is quite intuitive since a larger volatility of idiosyncratic risk implies a greater risk-sharing needs for investors, and a potential market shutdown will thus have a larger impact on the asset price.⁴

Equation (2.3) suggests that the risk of liquidity crash impacts the portfolio choice of investor i through the income hedging demand $\theta_t^{i,I}$, the liquidity hedging demand $\theta_t^{i,L}$, and the potential utility loss in the event of a crash $\eta^i(Y_t, \theta_t^i)$.

Firstly, we analyze the relative size of the loadings of $\theta_t^{i,L}$ and $\theta_t^{i,I}$ on the idiosyncratic shock Y_t . The relative size of the loadings is important in that it determines whether a potential liquidity crash risk will push investors to buy stocks or push them to sell stocks, compared with the optimal income hedging level. The direction of this trade will affect the sign of the liquidity premium. It can be shown that the relative size of the loadings of $\theta_t^{i,L}$ and $\theta_t^{i,I}$ on Y_t depends on the persistence of the shock Y_t . Assuming the income shock Y_t is persistent, a high current Y_t will imply a high future Y_t over a long period of time. In the presence of a possible next period liquidity crash, since investors will not be able to trade in the future, they will build up more position immediately before the crash so that their position in

³Longstaff (2009) and Vayanos (1998) found similar results that liquid assets could actually be less valuable than illiquid assets.

⁴Notice that equation (2.4) is an endogenous relation since $h''(0)$ and $v_L(0)$ are both endogenously determined.

stock will help them hedge their persistent income shock in the period of market shutdowns. Hence their optimal liquidity hedging demand should load more on Y than the optimal income hedging demand.

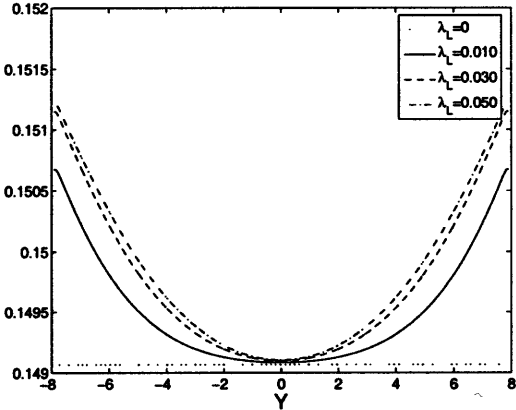
Secondly, we analyze the relative size of the weights different investors put on incoming hedging and liquidity hedging. Equation (2.3) shows that these weights are determined by the potential utility loss in the event of a crash $\eta^i(Y_t, \theta_t^i)$. One can show that, in general, at the optimal, a liquid crash leads to a large utility damage to those investors whose non-traded income shock loads positively on the dividend shocks than to those investors whose non-traded income shock loads negatively on the dividend shocks. This is quite intuitive. A negative correlation between the income shocks and the dividend shocks provides a natural diversification. Therefore, in the event of a liquidity crash, investors whose income shocks negatively correlates with asset dividend tend to be better off due to the diversification effect. An immediate consequence is that, in the equilibrium, the investor whose non-traded income loads positively on dividend shocks will put more weight on $\theta_t^{i,L}$ because they incur larger loss in the event of a crash and hence care more about the optimal stock holding in the crash state.

Finally, we put these two elements together and analyze the asset price implication. Assuming the non-traded shock is persistent, the optimal stock holding conditional on crash next period, $\theta_t^{i,L}$ has more loadings on Y_t than the optimal stock holding conditional on no crash next period, $\theta_t^{i,I}$. Therefore, compared with the frictionless market, those investors whose non-traded income loads positively on dividend shocks will further reduce the stock holding per unit of Y_t , and those investors whose non-traded income loads negatively on dividend shocks will further increase the stock holding per unit of Y_t . However, since those investors whose non-traded income loads positively on dividend shocks will put more weight on the crash possi-

bility, the desired amount of stock holding increase will be smaller than the desired amount of stock holding decrease, and therefore, the overall effect is a net stock sell and the stock price falls.

An important message from the analysis in the previous paragraphs is that in our model, the asset price implication of the liquidity crash risk is a consequence of the asymmetry among the impact of the realized liquidity event on investors with different level of idiosyncratic shocks. This asymmetry lead to different liquidity hedging among investors, which lead to different buying/selling demand and the price changes.

Figure 2-1: Equilibrium Price for Transitory Income Shocks
 $h_L(Y)$



Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $r = 0.15$, $\kappa = 0.30$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$, $\lambda_H = 0.00$. $P^* = 0.149$. P^* is the stock price in the frictionless market.

Although in the above analysis, we conclude that the risk of liquidity crash could reduce asset price and lead to a positive liquidity premium, interestingly, the opposite could also occur and the liquidity crash risk may increase the asset price and lead

to a negative liquidity premium⁵. This could happen when the idiosyncratic shocks are very transitory. In this case, building up positions right before the liquidity crash will not be very helpful for investors to hedge the future income risks, since future income shocks have little to do with current income shocks. Therefore, in this case, the magnitude of the loading of the liquidity hedging $\theta_t^{i,L}$ on Y_t could be less than the loading of the income hedging $\theta_t^{i,I}$. Investors with positive income loading on dividend shocks will decrease their stock holding loadings on Y_t and hence want to buy stocks, and investors with positive income loading on dividend shocks will increase their stock holding loadings on Y_t and hence want to sell stocks. The asymmetry of impact of liquidity crash on these two types of investors will then lead to a net stock buy and hence the stock price increases.⁶ Figure 2-1 illustrate this point. The mean-reverting parameter of the idiosyncratic shock is set to be $\kappa = 0.3$ so the shock is fairly transitory. The recovery probability is set to be zero to maximize the price impact. The figure plots the price function against Y for different levels of crash probability. The price in normal times is higher than that in the frictionless market and is increasing in the crash probability.

2.6 Asset Price Implications

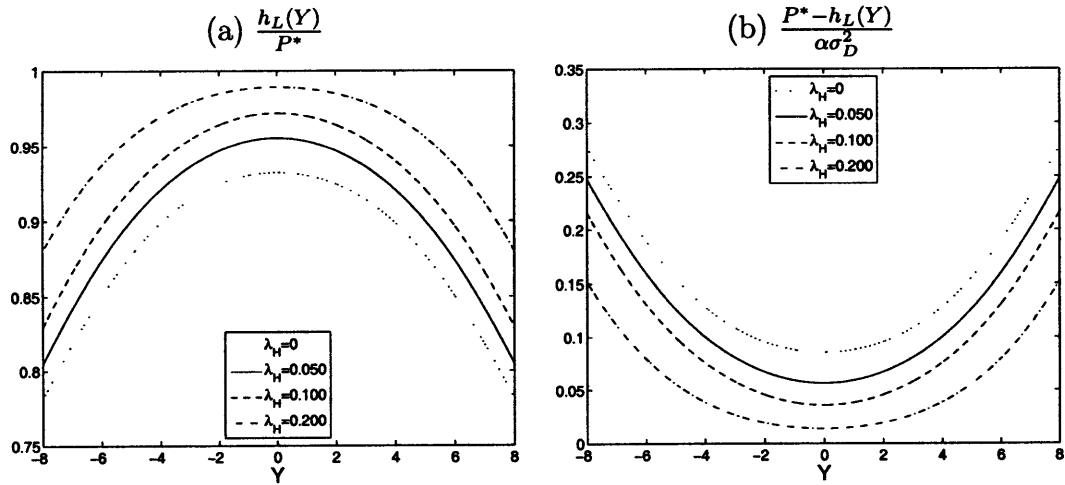
A. Price Discount

Panel (a) in Figure 2-2 plots the equilibrium stock price $h_L(Y)$ normalized by the frictionless price P^* against the level of the income shock Y_t when the probability of

⁵Longstaff (2009) and Vayanos (1998) found similar results that liquid assets could actually be less valuable than illiquid assets under certain conditions.

⁶Lemma 2 in the Appendix A.2 gives a proof for the above discussion in the case where there is no recovery ($\lambda_H = 0$).

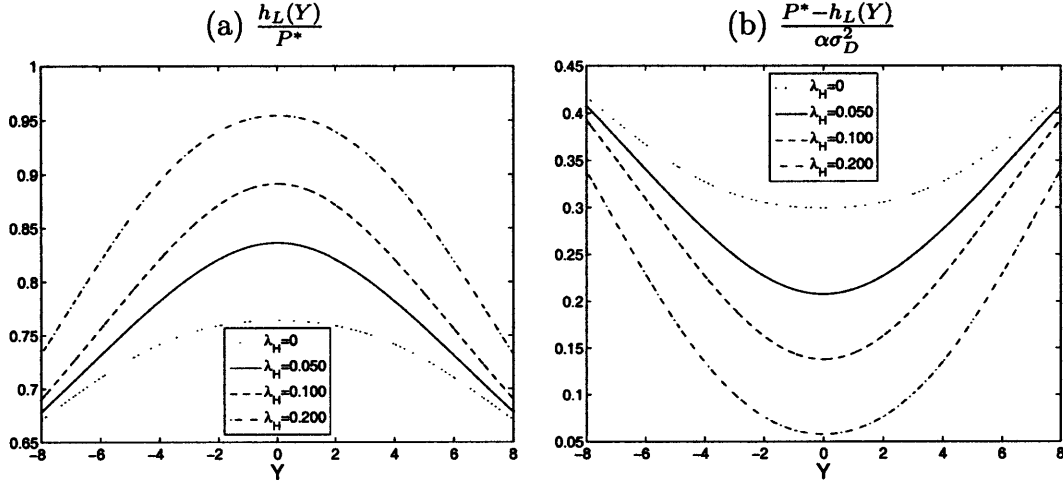
Figure 2-2: Equilibrium Price: $\lambda_L = 0.01$



Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $\tau = 0.15$, $\kappa = 0.04$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$, $\lambda_L = 0.010$. $P^* = 0.149$. P^* is the stock price in the frictionless market. The standard deviation of Y_t in its stationary distribution is $\sqrt{Var[Y_t]} = 1.768$.

crash is small $\lambda_L = 0.01$, for several values of the recovery probability. And Panel (a) in Figure 2-3 provides similar graph when the probability of crash is $\lambda_L = 0.05$. The stock price is lower than then the price in the frictionless market. The price decreases as the probability of liquidity crash (λ_L) increases, and increases as the probability of recovery (λ_H) increases. The stock price depends on the level of the idiosyncratic shock and is thus time-varying. The price as a function of the magnitude of the income shocks $|Y_t|$ is decreasing. The larger the size of the idiosyncratic shock, the lower the stock price. This reflects the fact that the liquidity crash risk has more impact when the need for risk-sharing is higher. The liquidity discount could be economically large, especially when the idiosyncratic shock is large. Even for a small probability of crash with $\lambda_L = 0.01$ and a relative large probability of recovery $\lambda_H = 0.2$, the stock can be traded more than 2% below its frictionless value when

Figure 2-3: Equilibrium Price: $\lambda_L = 0.05$



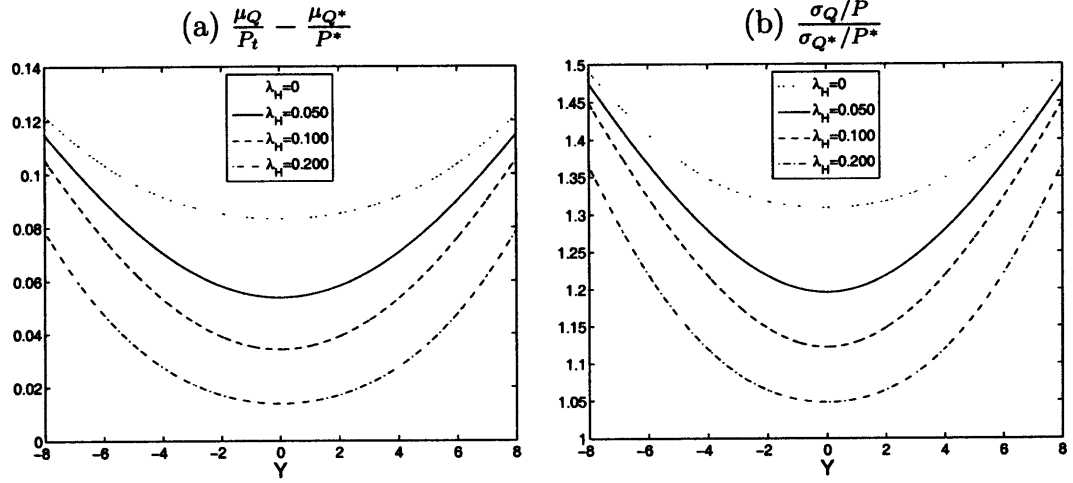
Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $r = 0.15$, $\kappa = 0.04$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$, $\lambda_L = 0.050$. $P^* = 0.149$. P^* is the stock price in the frictionless market. The standard deviation of Y_t in its stationary distribution is $\sqrt{Var[Y_t]} = 1.768$.

the idiosyncratic shock is 1 standard deviation away from its mean. This discount increases to 5% when the probability of crash is $\lambda_L = 0.05$. Since the level of μ_D only affects the level of the fundamental price but not risk premiums, to get a better sense the size of the liquidity premium, we can compare the liquidity risk premium with the dividend risk premium (which is given by $\alpha\sigma_D^2$). Panel (b)s of Figure 2-2 and Figure 2-3 plot the ratio of the liquidity risk premium and the dividend risk premium. With $\lambda_L = 0.01$, the liquidity discount can be as high as 5% of the dividend risk premium. And with $\lambda_L = 0.05$, this ratio increases to 15%.

B. Time-Varying Expected Return and Volatility

In contrast to the frictionless market, the stock price is time-varying in the presence of a possible liquidity crash, which leads to a time-varying expected return. We define

Figure 2-4: Conditional Mean and Volatility of Excess Asset Return



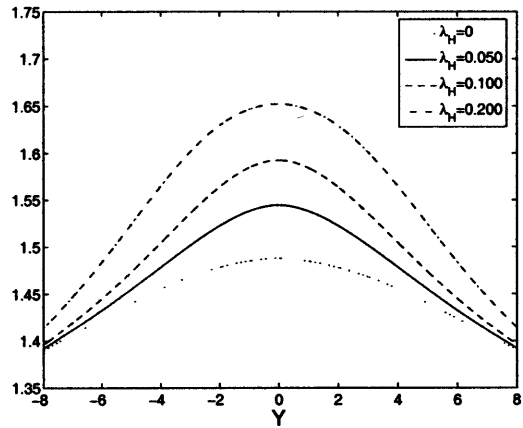
Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $r = 0.15$, $\kappa = 0.04$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$, $\lambda_L = 0.050$. $P^* = 0.149$. P^* is the stock price in the frictionless market. The standard deviation of Y_t in its stationary distribution is $\sqrt{Var[Y_t]} = 1.768$.

the dollar excess stock return as $dQ_t = dD_t + dP_t - rP_t dt$, and define the percentage excess return as $\frac{dQ_t}{P_t}$. We use μ_Q to denote the drift of dQ_t and σ_Q to denote the conditional volatility of dQ_t . The dollar conditional expected excess return is defined to be the drift of dQ_t and the percentage conditional expected excess return is defined to be the drift of $\frac{dQ_t}{P_t}$.

In the case of frictionless market, the conditional expected excess return in dollar amount and in percentage are both constant over time: $\mu_{Q^*} = \mu_D - rP^* = \alpha r \sigma_D^2$, and $\frac{\mu_{Q^*}}{P^*} = \frac{\alpha r^2 \sigma_D^2}{\mu_D - \alpha r \sigma_D^2}$. The conditional volatility of the excess return is the volatility of the dividend σ_D .

In the presence of a potential liquidity crash, the conditional expected excess return will no longer be constant over time. The level of idiosyncratic risk will drive the expected return. A higher level of idiosyncratic risk means a larger impact of

Figure 2-5: Trading Activity



Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $r = 0.15$, $\kappa = 0.04$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$, $\lambda_L = 0.050$. $P^* = 0.149$. P^* is the stock price in the frictionless market. The standard deviation of Y_t in its stationary distribution is $\sqrt{Var[Y_t]} = 1.768$. Trading activity is proxied by the conditional volatility of the optimal stock holding. The trading activity in the frictionless case is 1.69.

the potential liquidity crash and investors thus demand higher risk premium in stock return. Moreover, the variation of stock price will add an extra source of volatility in the stock return other than the dividend variation so that the stock return becomes more volatile. Panel (a) of Figure 2-4 plots the difference between the conditional percentage expected stock return in the friction case and in the frictionless case, against the level of the idiosyncratic shock Y_t , and panel (b) plots the conditional volatility of the percentage return normalized by the conditional volatility in the frictionless case. Both the conditional mean and volatility are increasing in the level of the idiosyncratic shock $|Y|$. They are decreasing in the probability of recovery. The liquidity premium in the stock return could be large, especially when the idiosyncratic shock is high. For example, assuming a expected recovery period of 5 years, the liquidity premium in the stock return could be more than 2% when the idiosyncratic

shock is one standard deviation from its mean.

C. Trading Activity

The risk of liquidity crash will also have trading volume implications. In our continuous time model, we use the conditional volatility of investors' stock holding as a proxy for trading activity (In equilibrium, the two investors hold the opposite excessive position so the conditional volatility of optimal holdings of the two agents are the same). Figure 2-5 plots the trading activity against Y_t . We see that in the presence of a potential liquidity shock, investors trade less stocks (in the graph, the value of the trading activity in the frictionless market is a constant at 1.69). The trading activity reduction is more with a smaller probability of recovery. Moreover, investors are expected to trade less when their income shock is large. This is consistent with the intuition that when there is the risk of liquidity crash, investors could not achieve optimal risk sharing in normal times and they react less to their idiosyncratic income shocks.

2.7 Impact of Probabilities of Crash and Recovery

The effect of liquidity crash risks depends heavily on the two probabilities: the probability of liquidity crash (λ_L) and the probability of liquidity recovery (λ_H). Intuitively, as the probability of crash increases, investors fear more of the market shutdown. As the probability of recovery increases, the expected duration of the liquidity crash decreases and the effect of the risk is smaller.

Table 2.1 reports the simulated moments for different crash probabilities and recovery probabilities. The simulated moments include: the liquidity price premium normalized by the frictionless price, the price volatility normalized by the frictionless

price, and the average of the liquidity premium in percentage stock return (defined as the percentage excess stock return in the friction case minus the percentage excess stock return in the frictionless market).

We can see from the table that impact of the crash is increasing in λ_L and decreasing in λ_H . The simulated quantities are increasing along the λ_L direction and decreasing along the λ_H direction. The liquidity impact can be economically significant. We could focus on the two scenarios: very infrequent but very severe liquidity crashes, and more frequent but less severe liquidity crashes. For the first scenario, we could look at the crash probability of $\lambda_L = 0.03$ so that the liquidity event happens on average once in every 33 years. In this case, if the expected recovery period is 5 years ($\lambda_H = 0.2$), the stock on average will be traded at 3.8% below the frictionless price, the liquidity premium can be 4.8% of the risk premium coming from the fundamentals, and the liquid premium in stock return can be over 1%. For the second scenario, we could look at the crash probability of $\lambda_L = 0.1$ so that the liquidity event happens on average once in every 10 years. In this case, the average impact of the liquidity event of asset prices and returns is fairly small if we assume that the recovery period is less than one year ($\lambda_H \geq 1$). In this respect, investors fear more the infrequent but severe liquidity crash than a more frequent but mild event. Furthermore, under the constant jump intensity assumption, the impact of the liquidity jump risk on the price volatility and the return volatility seems to be small for reasonable numbers of crash and recovery probabilities.

The impact of the risk of liquidity crash also crucially depends on the persistence of the idiosyncratic shock Y_t . Investors fear the market shutdown because it will prevent them from future risk sharing. A more persistent idiosyncratic shock will amplify this fear because once the liquidity event happens, a persistent shock will have a longer impact on the future income. Therefore, investors react more to the

persistence shock and the asset price implication is more salient. Table 2.2 reports the same simulated moments as in Table 2.1, but with a less persistence income shock where $\kappa = 0.08$. The magnitude of the liquidity crash impact is much smaller in this case than in the case of $\kappa = 0.04$. For example, when $\lambda_L = 0.03$ and $\lambda_H = 0.2$, the stock is traded on average only 1.3% below the frictionless market price (vs. 3.8% when $\kappa = 0.04$), and the liquidity premium in stock return is only 0.35% (vs. 1.02% when $\kappa = 0.04$).

2.8 State Dependent Probability of Crash

The probability of crash is not necessarily constant over time. It is likely that the crash probability depends on the level of the income shocks: when the shocks from income are low, the likelihood of liquidity crash could be small, but when the shocks from income are high, the likelihood of liquidity crash could be large. In this section, we discuss the case of the state dependent crash probabilities. For simplicity we assume that the probability of recovery is constant, but the probability of the crash is a function of the size of Y_t , i.e.

$$\lambda_L = \lambda_L(|Y_t|),$$

where $\lambda_L(\cdot)$ is a non-negative function. To capture the feature that the probability of crash is high when the income shock is large, we assume that the function $\lambda_L(\cdot)$ is increasing.

The assumption that the probability of crash is increasing in the level of the income shock has asset allocation implications. From equation 2.3, in the case of the state dependent jump intensity, we see that when $|Y_t|$ is large, investors put larger

Table 2.1: Simulated Moments for Constant Probability of Crash and Recovery:
 $\kappa = 0.04$

		λ_H						
		0.050	0.100	0.200	0.500	1.000	1.500	2.000
λ_L	Mean of Normalized Liquidity Premium							
	$E[\frac{P^*-P}{P^*}]$ (%)							
0.010		5.129	3.430	1.439	0.153	0.048	0.023	-0.018
0.030		12.326	8.524	3.803	0.459	0.125	0.058	0.001
0.050		17.530	12.357	5.734	0.745	0.198	0.092	0.020
0.100		26.387	19.096	9.401	1.383	0.368	0.174	0.075
λ_L	Normalized Price Volatility							
	$\sqrt{Var[P]}/P^*$ (%)							
0.010		0.977	0.897	0.566	0.043	0.001	0.004	0.006
0.030		1.461	1.641	1.264	0.117	0.001	0.004	0.004
0.050		1.497	1.941	1.698	0.181	0.001	0.004	0.003
0.100		1.287	2.164	2.305	0.309	0.004	0.004	0.003
λ_L	Mean of Return Liquidity Premium							
	$E[\frac{\mu_Q}{P}] - E[\frac{\mu_Q^*}{P^*}]$ (%)							
0.010		1.399	0.919	0.377	0.040	0.012	0.006	-0.005
0.030		3.641	2.417	1.025	0.119	0.032	0.015	0.000
0.050		5.503	3.658	1.580	0.193	0.051	0.024	0.005
0.100		9.276	6.124	2.701	0.362	0.096	0.045	0.019

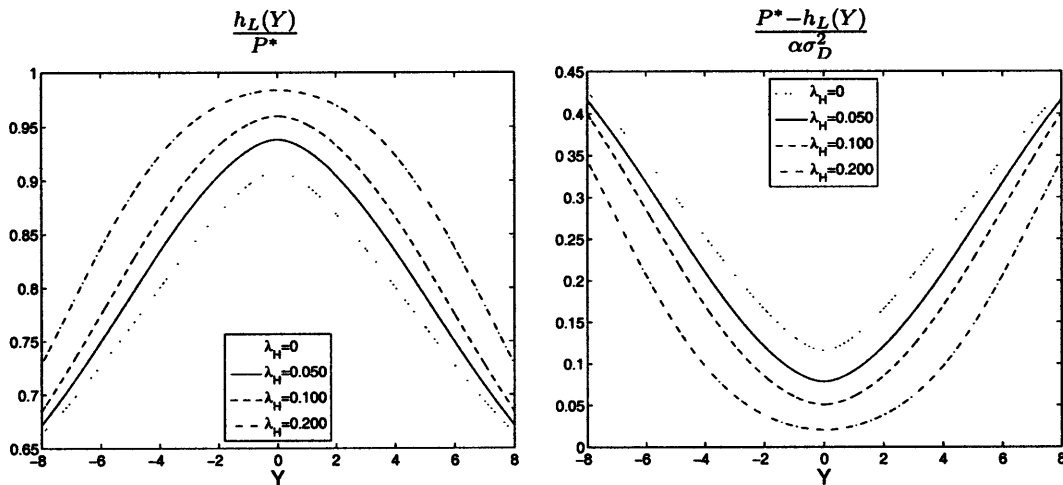
Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $r = 0.15$, $\kappa = 0.04$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$. $P^* = 0.149$. P^* is the stock price where there is no possibility of crash.

Table 2.2: Simulated Moments for Constant Probability of Crash and Recovery:
 $\kappa = 0.08$

		λ_H						
		0.050	0.100	0.200	0.500	1.000	1.500	2.000
λ_L	Mean of Normalized Liquidity Premium $E[\frac{P^*-P}{P^*}]$ (%)							
	0.010	1.696	1.099	0.468	0.097	0.048	-0.003	0.004
	0.030	4.218	2.841	1.316	0.293	0.118	0.016	0.014
	0.050	6.110	4.212	2.043	0.477	0.178	0.041	0.025
	0.100	9.391	6.706	3.492	0.891	0.307	0.108	0.059
λ_L	Normalized Price Volatility $\sqrt{Var[P]}/P^*$ (%)							
	0.010	0.288	0.230	0.106	0.004	0.001	0.007	0.004
	0.030	0.564	0.494	0.252	0.015	0.001	0.005	0.007
	0.050	0.687	0.643	0.352	0.023	0.003	0.004	0.008
	0.100	0.797	0.823	0.503	0.040	0.006	0.003	0.007
λ_L	Mean of Return Liquidity Premium $E[\frac{\mu_Q}{P}] - E[\frac{\mu_{Q^*}}{P^*}]$ (%)							
	0.010	0.459	0.296	0.125	0.026	0.013	-0.001	0.001
	0.030	1.171	0.778	0.354	0.078	0.031	0.004	0.004
	0.050	1.731	1.170	0.554	0.127	0.047	0.011	0.007
	0.100	2.756	1.913	0.962	0.238	0.082	0.029	0.016

Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $r = 0.15$, $\kappa = 0.08$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$. $P^* = 0.149$. P^* is the stock price where there is no possibility of crash.

Figure 2-6: Equilibrium with State Dependent Crash Probability: $\lambda_L(Y) = 0.01 + 0.01 \frac{|Y|}{\sqrt{2\kappa}}$



Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $r = 0.15$, $\kappa = 0.04$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$. $P^* = 0.149$. P^* is the stock price in the frictionless market.

weight on the liquidity hedging demand (compared with the case of constant λ_L). The intuition in section 2.5 suggests that the liquidity crash affects the stock price through the different impact level of the realized crash among investors with different level of idiosyncratic incomes. In particular, the liquidity downward jump has a larger utility affect on investors whose income shock is positively related to the stock dividend. A state dependent probability of crash will in fact amplify this impact asymmetry, which will lead to a larger price change when the size of the idiosyncratic shock is larger. The intuition is as following. If the crash probability is increasing in the magnitude of Y_t , a larger income shock means a higher probability of liquidity crash. Therefore, compared with the constant jump intensity case, investors whose income shock positively correlates with the stock dividend will put even more weight on the liquidity hedging demand than investors whose income shock negatively correlates

Table 2.3: Simulated Quantities in the Case of State Dependent Probability of Crash:
 $\lambda_L(Y) = 0.01 + 0.01 \cdot |Y| / \frac{\sigma_Y}{\sqrt{2\kappa}}$

λ_H	$E[\frac{P^*-P}{P^*}]$ (%)	$E[\frac{P^*-P}{\alpha\sigma_D^2}]$ (%)	$\sqrt{Var[\frac{P}{P_0^*}]}$ (%)	$E[\frac{\mu_D}{P} - \frac{\mu_D}{P_0^*}]$ (%)
0.000	11.851	15.023	2.969	3.645
0.050	8.561	10.851	2.844	2.543
0.100	5.996	7.600	2.561	1.734
0.200	2.740	3.473	1.686	0.765

Model Parameters: $\alpha = 1.50$, $\rho = 0.10$, $r = 0.15$, $\kappa = 0.04$, $\sigma_Y = 0.50$, $\sigma_D = 0.28$, $\rho_{ND} = 0.95$, $\mu_D = 0.04$. $P^* = 0.149$. P^* is the stock price in the frictionless market.

with the stock dividend. This in turn implies that, assuming a fairly persistent income shock, the selling demand is even larger than the buying demand which leads to a lower stock price.

An immediate corollary of the above discussion is that the state dependent probability of crash will increase the concavity of the pricing function as Y_t . It will amplify the size of the liquidity premium, as well as the magnitude of the price volatility. Figure 2-6 provides the graphs of equilibrium price when the probability of crash is a function of Y : $\lambda_L(Y) = 0.01 + 0.01 \frac{|Y|}{\frac{\sigma_Y}{\sqrt{2\kappa}}}$. The specification of this function is somewhat ad hoc. Notice that $\frac{\sigma_Y}{\sqrt{2\kappa}}$ is the standard deviation of Y_t in its stationary distribution. This specification implies that for every extra standard deviation away from its mean, the probability of crash increases by 0.01. We can compare Figure 2-6 with Figure 2-2 where the probability of crash is constant at 0.01. The graph of stock price in the state dependent crash probability is significantly thinner than that of constant probability. This suggest both a higher liquidity premium and a higher stock price volatility.

Table 2.3 reports the simulated moments in the case of state dependent crash

probability. We can compare these numbers with numbers in Table 2.1. For the same recovery probability, the liquidity premium is significantly higher than the case of constant crash probability. For example, with the recovery probability at $\lambda_H = 0.20$, the mean of the normalized liquidity premium ($E[\frac{P^* - P}{P^*}]$) in the state dependent probability case is 2.75% as opposed to 1.43% in the constant probability case. The impact of state dependent crash probability on the price volatility is more salient. In the table, the normalized price volatilities are almost three times larger in the state dependent probability case than those in the constant probability case, with the same level of recovery speed. For example, at $\lambda_H = 0.2$, the simulated normalized volatility is 1.68% in the the state dependent probability case vs 0.52%, in the constant probability case. This suggests that a state dependent probability of crash could be a useful channel to generate excess volatility.

2.9 Conclusion

Financial markets have experienced sudden liquidity shocks in the past. During the liquidity crisis, it becomes difficult to trade assets, and sometimes certain markets even freeze up and investors find themselves difficult to get out of their positions. For example, during the crisis starting in 2007, the market for the Structured Investment Vehicles (SIVs) seized up in later part of 2007 and never recovered. The LIBOR spread was so high after the collapse of Lehman Brothers that the interbank lending market effectively shut down temporarily.

Such liquidity crashes impact people's investment decisions, trading behaviors, and consequently affect asset prices, even if there is little change in the asset fundamentals. During such liquidity events, many of the important functions of financial markets are interrupted. Investors may find it very difficult to share risks, the infor-

mation flow may stop, and the price discovery process may be disrupted. Moreover, the surprise nature of the liquidity dry-ups also impacts the market before the materialization of the crisis and has asset price and trading consequences in normal times when the market liquidity is good. Investors fear the disastrous liquidity crash and in anticipation of these potential crisis, they adjust their portfolios accordingly before crashes happen.

This paper proposes a continuous time model where investors face the risk of potential market liquidity crises. We focus our analysis on the *ex ante* impact of such liquidity crash risks on asset prices and trading behavior in normal times before the crash happens. The paper finds that investors choose optimal portfolio not only to hedge the risk of the asset fundamentals, but also to hedge the risk of the liquidity crash. The liquidity hedging demand reflects the size of *ex post* damage in the event of the liquidity dry-up. Such liquidity hedging reduces investors' ability to share risks in normal times and investors will bear idiosyncratic risks that otherwise could be perfectly eliminated. Consequently, asset prices are not only affected by the level of aggregate risks, but also by the level of idiosyncratic risks. This will generate a time-varying asset price and generate a rich pattern of return dynamics even if there are no aggregate risks. The conditional moments of asset returns could be driven by risks that are not systematic. The impact of the liquid crash risk depends on the persistence level of investors income shocks. In general, the more persistent the income shock, the larger the price impact. Interestingly, when investors income risks are very transitory, the liquidity hedging could even generate a negative liquidity premium in asset prices (similar to results in Vayanos (1998) and Longstaff (2009)).

This paper is a pilot project and can be extended in a number of directions. The assumption that in the event of a liquidity crash, the market completely shuts down is in general strong. A more realistic assumption would be to impose a large but

finite transaction cost when the liquidity is in the crash state. Also, in this paper, we assume that the arrival of the liquidity crash is purely random and exogenous, while in reality, such arrival could be linked to certain economic conditions. Therefore it would be very interesting to develop a model where the crash happens endogenously. These will be directions for future research.

Chapter 3

How Does Illiquidity Affect the Extreme Risk of Hedge Funds

3.1 Introduction

The study of the hedge funds has become one of the major topics in the finance literature. Hedge funds are privately organized, lightly regulated asset management firms that employ a variety of strategies, including short sale and options. One principal difference between hedge funds and traditional asset management firms is that hedge fund managers have the primary goal of an “absolute” return, or a target rate of return, irrespective of the market performance. For many years, risk management seemed to be of secondary importance. Managers were rarely willing to spend the time or money for active risk controls. However this absence of risk management has changed in the past years. In particular, the collapse of the Long Term Capital management (LTCM) forced many investors to weight the risk against the high returns, and to demand transparency of risk profiles of hedge funds. Today,

risk management has become one of the essential part of hedge funds.

Value at risk (VaR) is one of the most important measures of risk for traditional investment vehicle. It reports the maximum possible loss for a given confidence interval and reflects the tail risk of asset returns. More precisely, assume an asset has stochastic return \tilde{R} . Given a confidence level, say, 99%, the VaR at 99% is defined to be

$$VaR_{99\%} := R_{99\%},$$

where $R_{99\%}$ satisfies

$$Prob(\tilde{R} \leq R_{99\%}) = 1 - 99\%.$$

Though only recently introduced, VaR has become one of the most widely accepted measures for risk management (e.g. Jorion (1997)). Various authors have been using VaR to analyze the return of hedge funds. Jorion (2000) uses VaR to study a single fund. Gupta and Liang (2005) computes the VaR of hedge funds using a extreme value approach and finds that VaR is a better proxy for hedge fund risks than traditional measures like volatility.

However, Lo (2001) questions the validity of VaR in hedge fund returns. He argues that due to the unique risk aspect of hedge fund, e.g. the survivorship bias, the dynamic risk analytics, nonlinearities, and liquidity and credit, the use of VaR is problematic. Lo outlines an ambitious research agenda to address these issues. In this paper, we aim to address one of these issues and study the impact of illiquidity on the tail risk of hedge fund returns.

Getmansky, Lo, and Makarov (2004) documents that monthly return of hedge funds are highly autocorrelated with an average first order autocorrelation coefficient of around 0.3. The existence of this high autocorrelation itself suggests that either there are liquidity constraints (like non-synchronous recording), or there are return

smoothing. They suggest that the autocorrelation can be viewed as a proxy for illiquidity.

In this paper we propose an approach to compute the VaR taking the liquidity risk into account. We use the smoothing model developed by Getmansky, Lo, and Makarov (2004) to estimate the "real" and "liquid" month return. Then we use the extreme value theory method to compute the VaR. Our approach is different from existing methods (e.g. Gupta and Liang (2005)) in that existing VaR methods do not take into account of the liquidity risk. Laporte (2003) recently suggested an liquidity adjusted VaR, which decompose the total VaR into market Var, liquidity VaR and correlation effects. Our model differs from Laporte's model, partly in that Laporte uses hedge fund index, which is not quite so reliable due to the diversification of hedge fund strategies. The existing VaR methods have the possibility of underestimating the true risk of hedge fund. Our model emphasizes the role of the "true return" that may not be reflected in the observed returns due to smoothing or non-synchronous trading.

This paper is generally related to the literature of liquidity and asset pricing (Diamond and Dybvig (1983), Glosten and Milgrom (1985), Kyle (1985), Amihud and Mendelson (1986b), Glosten and Harris (1988), Grossman and Miller (1988), Amihud and Mendelson (1993), Shleifer and Vishny (1992), Brenna and Subrahmanyam (1995), Hart (1995), Brennan and Subrahmanyam (1996a), Brennan, Chordia, and Subrahmanyam (1998), Chordia, Roll, and Subrahmanyam (2000), Hasbrouck (2001), Huberman and Halka (2001), Amihud (2002), Gromb and Vayanos (2002), Duffie and Pederson (2003), Acharya and Pedersen (2005a), and Brunnermeier and Pederson (2005)).

The paper is organized as follows. Section 3.2 describes the data set used in the paper. Section 3.3 discusses summary statistics and properties of hedge fund return

distributions. Section 3.4 specifies the model, summarizes the extreme value theory, and states the method for estimating the VaR. Section 3.5 reports and analyzes the estimation results. Section 3.6 discusses the impact of fat tails on extreme risks. Section 3.7 concludes.

3.2 Data

We use the hedge fund dataset from TASS Management Limited. The TASS data set contains monthly returns for 3470 live funds, and 2225 dissolved funds, as of June 30, 2005. The first reported monthly return was in 1977 and 1978 for some dead fund. The total asset under management for live funds listed in TASS accounts for about 1/2 of the total asset under management across all hedge funds.

There are several listed reason why a fund dropped out of the TASS data base: Fund closed to New Investment(0.32%), Fund Dormant(4%), Fund has Merged into another entity(3.07%), Fund liquidated (52.75%), Fund no longer reporting to TASS (30%), TASS has been unable to contact the man (7.53%), and unknown (6%). In this paper we will use “dead fund”, or “graveyard fund” to refer to all the funds which have dropped out of the TASS database, and use “liquidated fund” to refer to fund which has been dropped out of the TASS because of liquidation.

We use funds with at least 60 reported monthly returns. This leaves us with 1458 live funds and 747 dead funds. The 5 year filtering criterion may introduce some survival bias in our analysis, but the existence of dead fund data will help to mitigate this problem to some degree.

TASS also categorizes all hedge funds into 11 styles: Convertible Arbitrage, Dedicated Short Bias, Emerging Markets, Equity Market Neutral, Event Driven, Fixed Income Arbitrage, Fund of Funds, Global Macro, Long/Short Equity Hedge, Man-

Table 3.1: Descriptive statistics for Hedge Fund Returns: Live Funds

Live Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005									
Style	No.	Perct.	Mean	Std.	Skew	Kurt	ρ_1	ρ_2	ρ_3
Convertible Arbitrage	59	4.05	0.79	1.84	-0.27	5.12	0.41	-0.03	0.07
Dedicated Short Bias	14	0.96	0.02	7.13	-0.49	6.95	0.06	-0.01	0.01
Emerging Markets	102	7.00	1.24	7.11	-0.72	10.80	0.19	-0.03	-0.00
Equity Market Neutral	61	4.18	0.72	2.51	0.29	5.86	0.05	0.09	0.08
Event Driven	142	9.74	0.95	2.33	-0.41	9.16	0.21	0.03	0.06
Fixed Income Arbitrage	41	2.81	0.78	2.28	-2.30	23.10	0.22	0.04	0.02
Fund of Funds	320	21.95	0.73	2.09	-0.19	8.78	0.23	0.00	0.00
Global Macro	52	3.57	0.95	3.98	0.37	6.09	0.05	-0.01	0.04
Long/Short Equity Hedge	486	33.33	1.11	4.85	0.29	6.88	0.14	0.01	0.02
Managed Futures	126	8.64	0.82	5.90	0.05	4.73	0.03	-0.07	-0.00
Other	55	3.77	0.90	2.90	-0.13	12.86	0.20	0.07	0.06

ρ_1 , ρ_2 and ρ_3 are the autocorrelation coefficients.

aged Futures and Other.

3.3 Descriptive Statistics: Fat Tails and Serial Correlations

Table 3.1, and Table 3.2 give the basic statistics of live and dead fund in TASS with at least 5 years of month return data.

We can see from the summary statistics that that hedge fund monthly returns in general have very "fat tails". The average kurtosis of funds far exceeds 3(the kurtosis of standard normal distribution). The Fixed Income Arbitrage has a particular high average kurtosis: 23.10 for live funds and 21.85 for graveyard funds. Emerging market(10.85 for live fund and for 11.91 graveyard fund), Event Driven (9.16 for live funds and 8.38 for graveyard funds) and Fund of Funds(8.78 for live funds and

Table 3.2: Descriptive statistics for Hedge Fund Returns: Graveyard Funds

Graveyard Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005									
Style	No.	Perct.	Mean	Std.	Skew	Kurt	ρ_1	ρ_2	ρ_3
Convertible Arbitrage	23	3.08	0.75	2.41	-0.55	11.27	0.27	0.07	0.02
Dedicated Short Bias	9	1.20	0.07	5.74	-0.03	4.48	0.08	-0.03	-0.07
Emerging Markets	54	7.23	0.40	7.97	-1.38	11.91	0.16	-0.00	0.00
Equity Market Neutral	24	3.21	0.67	2.67	0.09	4.93	0.13	-0.02	0.06
Event Driven	56	7.50	0.76	3.09	-0.78	8.38	0.21	0.03	0.05
Fixed Income Arbitrage	26	3.48	0.70	3.07	-2.60	21.89	0.21	0.10	0.05
Fund of Funds	139	18.61	0.51	4.19	-0.37	7.43	0.15	-0.01	-0.00
Global Macro	51	6.83	0.44	5.60	0.01	6.24	0.07	-0.06	-0.03
Long/Short Equity Hedge	183	24.50	1.00	6.39	-0.32	6.49	0.11	0.02	0.01
Managed Futures	130	17.40	0.51	6.99	-0.16	7.00	-0.02	-0.04	-0.01
Other	6	0.80	0.78	3.80	-1.15	11.71	0.06	0.01	0.01

ρ_1 , ρ_2 and ρ_3 are the autocorrelation coefficients.

7.43 for graveyard funds) all have significantly high kurtosis.

This suggests that the distribution of hedge fund monthly returns is not normal. The tails are fat. If we use the assumption that returns are normal to compute tail risks (e.g. VaR), we will in general have underestimates.

We also see from the summary statistics that monthly returns are highly autocorrelated. For the live funds, the average first order autocorrelation of Convertible Arbitrage is 41%, which is very high in any standard. Even Drive, Fixed Income Arbitrage and Fund of Funds all have average first order autocorrelation above 20%. Hedge fund managers are generally regarded as one of the smartest group of managers in financial industry. This high level of serial correlation suggests that the autocorrelation of hedge fund returns could not be easily arbitrated away.

Getmansky, Lo, and Makarov (2004) documents this high serial correlation of hedge funds. They suggest that this level of high serial correlation may be the result of liquidity constraint or return smoothing. Their paper points out that one of the

possible source of liquidity constraints is the non-synchronous trading. Due to the illiquidity of the underlying assets, the recorded price does not reflect the true value of the asset (the lag effect). Lo and Mackinlay (1990) proposes a model to study the impact of non-synchronous trading on autocorrelation. They show that under factor risk assumptions, the n -th order monthly autocorrelation of a well-diversified portfolio is given by:

$$\rho_n = \frac{(1 - p^q)p^{nq-q+1}}{q(1 - p^2) - 2p(1 - p^q)},$$

where q is the number of trading days in a month, p is the probability of non-trading each day. Table 3.3 gives the estimate of the daily probability of non-trading as a function of monthly autocorrelation.

From the Table 3.3, we see that for an monthly autocorrelation of 0.20, the daily non-trading probability will have to be 0.86, with an expected non-trading duration of 6.26 days! Therefore, non-synchronous trading alone may not be able to generate such high autocorrelation in hedge fund returns.

3.4 The Model

3.4.1 The Underlying Return

In this paper, we use moving averages to model hedge fund returns. In particular, we assume that the observed hedge fund return is a moving average ($MA(q)$) of the underlying “real” or “true” returns.¹ Let R_t^0 be the observed monthly return for a hedge fund. We assume that the observed return R_t^0 is the moving average of the

¹We adopt this specification from Getmansky, Lo and Makarov (2004) Getmansky, Lo, and Makarov (2004).

Table 3.3: Daily non-trading probability and monthly autocorrelation

Monthly Autocorrelation	Daily Prob. of Non-trading	Days of non-trading
0.05	0.62	1.62
0.10	0.76	3.25
0.15	0.83	4.76
0.20	0.86	6.26
0.25	0.89	7.81
0.30	0.90	9.48
0.35	0.92	11.33
0.40	0.93	13.41
0.45	0.94	15.82
0.50	0.95	18.65
0.55	0.96	22.05
0.60	0.96	26.27
0.65	0.97	31.63
0.70	0.97	38.73
0.75	0.98	48.63
0.80	0.98	63.41
0.85	0.99	87.97
0.90	0.99	136.99
0.95	1.00	278.20

underlying “real” return R_t :

$$R_t^0 = \theta_0 R_t + \theta_1 R_{t-1} + \dots + \theta_q R_{t-q}, \quad (3.1)$$

where $\sum_{i=1}^q \theta_i = 1$. We assume that the underlying returns R_t are i.i.d.

Using the observed data on R_t^0 , we can use the standard MA(q) model to estimate the coefficients θ_i . We then can calculate the underlying return R_t using the observed data on R_t^0 and the estimated coefficients $\theta_1, \dots, \theta_q$.²

3.4.2 VaR

Value at Risk (VaR) is a common measure in practice to estimate the maximal loss a fund can have. Let α be a confidence level (e.g. $\alpha = 0.99$), the VaR at α level is a number $x(\alpha)$ such that

$$Prob(Return \leq x(\alpha)) = 1 - \alpha.$$

²To calculate the underlying return R_t is equivalent to estimate the white noise in the MA(q) model. Given the observed data $\{R_t^0\}_{t=1}^T$, we first use maximal likelihood method to estimate coefficients $\theta_1, \dots, \theta_q$. Then setting R_{-1}^0 and R_0^0 to their long term mean $E(R_t^0)$, we can solve for R_t^0 , for $t = 1, \dots, T$ from the following T linear equations:

$$R_t^0 = \theta_0 R_t + \theta_1 R_{t-1} + \dots + \theta_q R_{t-q}, \quad \text{for } t = 1, \dots, T$$

There are two types of error in this calculation of R_t^0 . First is the error from setting R_{-1}^0 and R_0^0 to their long term mean $E(R_t^0)$. We can easily show that as long as the underlying MA(q) is invertible, this type of error decays exponentially as t becomes large. The second type of error arise from the estimation error of the coefficients $\theta_1, \dots, \theta_q$. One can show that if the MA(q) process is invertible, the second type of error is of the same order as the estimation error of $\theta_1, \dots, \theta_q$. In the model in this paper, we find that almost all (greater than 99%) estimated MA(2) models are invertible, and the most of the asymptotic variance of the estimated values are also small (less than 10% of the estimated value of θ_i).

In other words, $|x(\alpha)|$ is the cutoff loss at a α confidence level. VaR captures the maximal loss a fund can incur.

If returns are normal with mean zero and standard deviation σ_R . Then the 99% VaR is

$$2.36 \times \sigma_R. \tag{3.2}$$

If returns are not normal, as is suggested by the real data, the above standard deviation based formula will not be accurate. Usually financial returns have fat tails, therefore the above standard deviation based formula will underestimate the true Value-at-Risk.

A modified moment-based VaR approach proposes the following formula for VaR (e.g. Favre and Galeano (2002)):

$$VaR_\alpha = E(R) + [z + \frac{z^2 - 1}{6}S - \frac{z^3 - 3z}{24}K - \frac{2z^3 - 5z}{36}S^2]\sigma,$$

where $z = Normalcdf(-\alpha)$, $S = skewness$, and $K = kurtosis$. This modified VaR works well for modest value of skewness and kurtosis, but fails if the distribution of returns is far away from normal distribution. The better approach will be the extreme value theory.

There is a large literature on how to compute the VaR of various financial returns (e.g. Bali (2003) and Longin (2000) , etc). The key idea is to use the extreme value theory. Here we briefly summarize the central results in this area.

3.4.3 Extremely Value Theory and VaR

Let $F(x)$ be the cumulative distribution function of a random variable R and $\omega(F)$ be the right end point of F (i.e. $\omega(F) = \sup\{x \in \mathbb{R} | F(x) < 1\}$). For a $u < \omega(F)$, define $F^{[u]}(x)$ be the conditional cdf for R conditional on the event $R \geq u$, i.e.

$$F^{[u]}(x) = \text{Prob}(R < x | R \geq u), \quad x \geq u.$$

Bayes' Law implies that

$$F^{[u]}(x) = \frac{F(x) - F(u)}{1 - F(u)}, \quad x \geq u.$$

We have the following main result from extremely value theory (Leadbetter, Lindgren and Rootzen (1983) Leadbetter, Lindgren, and Rootzen (1983) and Reiss and Thomas (2001) Reiss and Thomas (2001)).

Theorem 1 (Balkema-de Haan-Pickands). *If F belongs to the domain of attraction, then*

$$|F^{[u]}(x) - W_{\gamma, u, \sigma}(x)| \rightarrow 0, \quad \text{as } u \rightarrow \omega(F)$$

for some parameters γ , u and σ , where

$$W_{\gamma, u, \sigma}(x) = 1 - \left(1 + \gamma \frac{x - u}{\sigma}\right)^{-1/\gamma}$$

belongs to the Generalized Pareto Distribution.

Notice that by this theorem, if u is closed to $\omega(F)$, then the distribution $F(x)$ can be approximated by:

$$F(x) \approx (1 - F(u))W_{\gamma, u, \sigma}(x) + F(u). \tag{3.3}$$

There are many methods to estimate the parameters in the generalized Parato distribution, e.g. Hill's estimator, Moment estimator, etc. We here use the Maximal likelihood approach. Let R_1, \dots, R_N be observed i.i.d data. Consider the ordered statistics $R_{1:N} \leq R_{2:N} \leq \dots \leq R_{N:N}$. Fixed a integer $k \leq N$, let $u = R_{N-k:N}$. Now we can write down the joint likelihood function of $(R_{N-k:N}, \dots, R_{N:N})$ using the approximation (3.3). We then can use maximal likelihood method to estimate the parameters γ and σ .

The final question is how to choose the cutoff k . Many authors have investigate the optimal k (Beirlant, Vynckier, and Teugels (1996), Drees and Kaufmann (1998) , etc). In this paper we use the ad hoc automatic choice described in Section 5.1 of Reiss and Thomas (2001).

3.5 Empirical Results

3.5.1 “True” Returns and Moving Average Estimates

In this section, we report the result of the MA(q) Model (3.1) from Section 3.4.1.

From Table 3.4 and Table 3.5, we see that the liquidity coefficient $\hat{\theta}_0$ is in general less than 1 (the smaller $\hat{\theta}_0$, the less liquid the fund is). For live funds, the average of the liquidity coefficient $\hat{\theta}_0$ is only 0.66 for Convertible Arbitrage , which to some degree implies that only 66% percent of the current ”true” return is reflected in the reported return data. the average of the liquidity coefficient $\hat{\theta}_0$ is also low for Fixed Income Arbitrage (0.78), Event Driven (0.79), Fund of Funds(0.80) and Emerging Markets (0.84). This result is similar to the findings of Getmansky, Lo, and Makarov (2004).

Table 3.4: Results for Moving Average Return Model MA(2): Live Funds

Live Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005.								
Style	No.	Perct.	$\hat{\theta}_0$	$std(\hat{\theta}_0)$	$\hat{\theta}_1$	$std(\hat{\theta}_1)$	$\hat{\theta}_2$	$std(\hat{\theta}_2)$
Convertible Arbitrage	59	4.05	0.66	0.14	0.26	0.11	0.08	0.09
Dedicated Short Bias	14	0.96	0.95	0.16	0.05	0.10	-0.00	0.13
Emerging Markets	102	7.00	0.84	0.15	0.15	0.10	0.01	0.10
Equity Market Neutral	61	4.18	0.87	0.20	0.02	0.20	0.11	0.14
Event Driven	142	9.74	0.79	0.13	0.16	0.10	0.06	0.10
Fixed Income Arbitrage	41	2.81	0.78	0.14	0.15	0.10	0.07	0.10
Fund of Funds	320	21.95	0.80	0.23	0.15	0.21	0.05	0.15
Global Macro	52	3.57	0.98	0.19	0.04	0.13	-0.02	0.16
Long/Short Equity Hedge	486	33.33	0.88	0.20	0.10	0.13	0.02	0.16
Managed Futures	126	8.64	1.10	0.22	0.01	0.13	-0.12	0.16
Other	55	3.77	0.78	0.21	0.14	0.19	0.08	0.11

We estimate the model $R_t^0 = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$, where R_t^0 is the observed hedge fund monthly return, and R_t is the “true” monthly return. Averages of the coefficient θ_i in each fund category are reported in this table. The sample standard deviation of the estimated $\hat{\theta}_i$ in each category is also reported.

Table 3.5: Results for Moving Average Return Model MA(2): Graveyard Funds

Graveyard Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005.								
Style	No.	Perct.	$\hat{\theta}_0$	$std(\hat{\theta}_0)$	$\hat{\theta}_1$	$std(\hat{\theta}_1)$	$\hat{\theta}_2$	$std(\hat{\theta}_2)$
Convertible Arbitrage	23	3.08	0.71	0.14	0.19	0.08	0.10	0.13
Dedicated Short Bias	9	1.20	1.13	0.67	0.01	0.23	-0.14	0.45
Emerging Markets	54	7.23	0.87	0.19	0.12	0.12	0.01	0.12
Equity Market Neutral	24	3.21	0.91	0.19	0.10	0.13	-0.01	0.11
Event Driven	56	7.50	0.77	0.24	0.17	0.16	0.05	0.14
Fixed Income Arbitrage	26	3.48	0.75	0.24	0.15	0.11	0.10	0.16
Fund of Funds	139	18.61	0.90	0.24	0.10	0.14	-0.00	0.15
Global Macro	51	6.83	1.01	0.22	0.05	0.16	-0.06	0.16
Long/Short Equity Hedge	183	24.50	0.89	0.15	0.08	0.11	0.03	0.12
Managed Futures	130	17.40	1.14	0.35	-0.06	0.24	-0.08	0.20
Other	6	0.80	0.95	0.19	0.02	0.19	0.03	0.07

We estimate the model $R_t^0 = \theta_0 R_t + \theta_1 R_{t-1} + \theta_2 R_{t-2}$, where R_t^0 is the observed hedge fund monthly return, and R_t is the “true” monthly return. Averages of the coefficient θ_i in each fund category are reported in this table. The sample standard deviation of the estimated $\hat{\theta}_i$ in each category is also reported.

Table 3.6: Results for Loss VaR Using Observed Returns: Live Funds

Live Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005.				
Style	No.	Perct.	$ \hat{R}_{99\%} $	$std(\hat{R}_{99\%})$
Convertible Arbitrage	59	4.05	5.00	5.78
Dedicated Short Bias	14	0.96	22.34	15.18
Emerging Markets	102	7.00	20.40	16.08
Equity Market Neutral	61	4.18	5.85	5.21
Event Driven	142	9.74	6.23	5.34
Fixed Income Arbitrage	41	2.81	13.83	35.41
Fund of Funds	320	21.95	5.16	4.28
Global Macro	52	3.57	9.35	6.19
Long/Short Equity Hedge	486	33.33	11.87	8.26
Managed Futures	126	8.64	14.18	8.61
Other	55	3.77	7.80	8.14

We report the results of estimated loss VaR at 99% level, using the extreme value theory approach discussed in section 3.4.3, for the observed returns. Averages of the absolute value of $\hat{R}_{99\%}$ (VaR at 99% level) for each category are reported in the table.

3.5.2 VaR of Observed Returns and “True” Returns

In this section, we report the result of VaR for both the observed return and the “true” underlying return calculated from MA(2) model estimated in section 3.5.1. Here we compute the VaR at a 99% level.

In the next two tables we report the results of the estimation of VaR at 99% level using the “true” underlying return.

From Tables 3.6, 3.7, 3.8, and 3.9, we can see that:

(1). The loss VaR of the graveyard funds is in general greater than the loss VaR of the live funds. On average, the VaR of graveyard funds is 5% higher than live funds in the same category (for most of the categories). This suggests that VaR is to some degree a good proxy for the risk of hedge fund (e.g. Gupta and Liang (2005)).

(2). The loss VaR using the “True” return data is higher than the loss VaR using the observed data. This is consistent in both live funds and graveyard funds.

Table 3.7: Results for Loss VaR Using Observed Returns: Graveyard Funds

Graveyard Fund in TASS data base with at least 5 year monthly return data as of June 30, 2005.

Style	No.	Perct.	$ \hat{R}_{99\%} $	$std(\hat{R}_{99\%})$
Convertible Arbitrage	23	3.08	8.26	8.35
Dedicated Short Bias	9	1.20	16.21	17.37
Emerging Markets	54	7.23	25.47	19.53
Equity Market Neutral	24	3.21	6.81	3.99
Event Driven	56	7.50	10.32	10.50
Fixed Income Arbitrage	26	3.48	10.92	10.25
Fund of Funds	139	18.61	11.51	12.24
Global Macro	51	6.83	15.39	12.25
Long/Short Equity Hedge	183	24.50	18.32	14.77
Managed Futures	130	17.40	20.18	21.41
Other	6	0.80	9.90	5.04

We report the results of estimated loss VaR at 99% level, using the extreme value theory approach discussed in section 3.4.3, for the observed returns. Averages of the absolute value of $\hat{R}_{99\%}$ (VaR at 99% level) for each category are reported in the table.

Table 3.8: Results for Loss VaR Using “True” Returns: Live Funds

Live Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005.

Style	No.	Perct.	$ \hat{R}_{99\%} $	$std(\hat{R}_{99\%})$
Convertible Arbitrage	59	4.05	6.53	7.24
Dedicated Short Bias	14	0.96	21.04	12.62
Emerging Markets	102	7.00	23.34	17.52
Equity Market Neutral	61	4.18	6.52	6.27
Event Driven	142	9.74	7.47	6.50
Fixed Income Arbitrage	41	2.81	10.40	14.05
Fund of Funds	320	21.95	6.03	4.62
Global Macro	52	3.57	9.65	7.29
Long/Short Equity Hedge	486	33.33	13.43	8.55
Managed Futures	126	8.64	12.77	8.25
Other	55	3.77	8.33	7.90

We report the results of estimated loss VaR at 99% level, using the extreme value theory approach discussed in section 3.4.3, for the “true” returns. Averages of the absolute value of $\hat{R}_{99\%}$ (VaR at 99% level) for each category are reported in the table.

Table 3.9: Results for Loss VaR Using “True” Returns: Graveyard Funds

Graveyard Fund in TASS data base with at least 5 year monthly return data as of June 30, 2005.				
Style	No.	Perct.	$ \hat{R}_{99\%} $	$std(\hat{R}_{99\%})$
Convertible Arbitrage	23	3.08	9.94	8.63
Dedicated Short Bias	9	1.20	10.85	6.11
Emerging Markets	54	7.23	28.68	18.64
Equity Market Neutral	24	3.21	6.83	4.42
Event Driven	56	7.50	11.48	10.76
Fixed Income Arbitrage	26	3.48	11.89	11.48
Fund of Funds	139	18.61	14.69	17.88
Global Macro	51	6.83	17.25	16.78
Long/Short Equity Hedge	183	24.50	21.24	17.21
Managed Futures	130	17.40	20.13	27.25
Other	6	0.80	9.86	4.75

We report the results of estimated loss VaR at 99% level, using the extreme value theory approach discussed in section 3.4.3, for the “true” returns. Averages of the absolute value of $\hat{R}_{99\%}$ (VaR at 99% level) for each category are reported in the table.

But the difference is not very large. However this small difference may be the result of category averaging. So we next look at the difference of VaR between observed return and “true” return at the fund level.

From Table 3.10 and Table 3.11, we see that on the fund level, the loss VaR using the “True” return are on average greater than the loss VaR using the observed data. There are a few exceptional categories. The most noticeable one is the live Fixed Income Arbitrage, with a difference of -3.43 . But when we look at the graveyard Fixed Income Arbitrage, the difference becomes 2.24 . The next noticeable categories with negative differences are live Managed Futures and Dedicated Short Bias, with about -1.5 . But those two categories have high liquidity coefficient, suggesting that liquidity may not be a serious issue for those two types at the first place.

Table 3.10: Results of Difference between Loss VaR Using “True” Returns and Loss VaR Using Observed Returns: Live Funds

Live Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005.				
Style	No.	Perct.	$ \hat{R}_{99\%}^{True} - \hat{R}_{99\%}^{Obs} $	$std(\hat{R}_{99\%}^{True} - \hat{R}_{99\%}^{Obs})$
Convertible Arbitrage	59	4.05	1.64	2.16
Dedicated Short Bias	14	0.96	-1.29	4.35
Emerging Markets	102	7.00	3.03	4.64
Equity Market Neutral	61	4.18	0.78	3.28
Event Driven	142	9.74	1.28	2.19
Fixed Income Arbitrage	41	2.81	-3.43	24.94
Fund of Funds	320	21.95	0.90	1.92
Global Macro	52	3.57	0.29	2.47
Long/Short Equity Hedge	486	33.33	1.56	4.46
Managed Futures	126	8.64	-1.40	3.70
Other	55	3.77	0.54	3.47

We report the results of estimated loss VaR at 99% level, using the extreme value theory approach discussed in section 3.4.3. We report the difference between the loss VaR $R_{99\%}^{True}$ using the “true” returns and the loss VaR $R_{99\%}^{Obs}$ using the observed returns.

Table 3.11: Results of Difference between Loss VaR Using “True” Returns and Loss VaR Using Observed Returns: Graveyard Funds

Graveyard Fund in TASS data base with at least 5 year monthly return data as of June 30, 2005.				
Style	No.	Perct.	$ \hat{R}_{99\%}^{True} - \hat{R}_{99\%}^{Obs} $	$std(\hat{R}_{99\%}^{True} - \hat{R}_{99\%}^{Obs})$
Convertible Arbitrage	23	3.08	1.07	3.31
Dedicated Short Bias	9	1.20	0.26	4.16
Emerging Markets	54	7.23	5.89	9.72
Equity Market Neutral	24	3.21	0.59	1.83
Event Driven	56	7.50	0.96	4.95
Fixed Income Arbitrage	26	3.48	2.24	2.94
Fund of Funds	139	18.61	2.14	4.52
Global Macro	51	6.83	-0.15	8.29
Long/Short Equity Hedge	183	24.50	3.71	5.28
Managed Futures	130	17.40	-1.91	8.67
Other	6	0.80	-0.28	2.07

We report the results of estimated loss VaR at 99% level, using the extreme value theory approach discussed in section 3.4.3. We report the difference between the loss VaR $R_{99\%}^{True}$ using the “true” returns and the loss VaR $R_{99\%}^{Obs}$ using the observed returns.

3.6 Fat Tails and Extreme Risks

In this section, we briefly discuss the impact of fat tails on the value-at-risk.

The summary statistics (Table 3.1) suggest that the observed returns of hedge funds are generally not normally distributed. For example, the kurtosis of most categories are much higher than that of normal distribution. As discussed in Section 3.4.2, the fat tails of the returns will entail more sophisticated approaches than using only standard deviation to estimate the extreme risks. If we assume normal distribution, the estimated loss VaR will understate the extreme risks. The third column of Table 3.13 reports the loss VaRs of the observed returns of live funds under the assumption that the return distributions are normal. We can compare these numbers with those in the third column of Table 3.6, where the VaRs are computed using the extreme value theory. Assuming normal distribution will indeed underestimate the VaR, and generally, the larger the kurtosis, the higher the degree of underestimation. For example, the estimated VaR for the category of Fixed Income Arbitrage funds (with an average kurtosis of 23.10) using the extreme value theory, is 13.83, much larger than the estimated VaR of 5.31, assuming normal distribution.

The linear un-smoothing procedure (3.1) will preserve the fat tails of the return. Table 3.12 reports the standard deviation, the skewness, and the kurtosis of the “true” returns for live funds (The “true” returns have the same mean as the observed returns due to the requirement of $\sum_i \theta_i = 1$). The “true” returns also have high kurtosis, similar to the observed returns.

The fourth column of Table 3.13 reports the VaRs of the “true” returns under the assumption that returns are normal. The last column of Table 3.13 reports the difference between the VaR of the “true” returns and the observed returns, under the assumption that both returns are normal. Two inferences can be drawn from this

Table 3.12: Higher Moments for “True” Returns: Live Funds

Live Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005					
Style	No.	Perct.	Std.	Skew	Kurt
Convertible Arbitrage	59	4.05	2.44	-0.05	5.18
Dedicated Short Bias	14	0.96	7.45	-0.55	7.22
Emerging Markets	102	7.00	8.28	-0.81	10.87
Equity Market Neutral	61	4.18	2.73	0.32	6.01
Event Driven	142	9.74	2.83	-0.42	9.17
Fixed Income Arbitrage	41	2.81	2.75	-2.01	21.86
Fund of Funds	320	21.95	2.49	-0.25	8.59
Global Macro	52	3.57	4.11	0.35	5.78
Long/Short Equity Hedge	486	33.33	5.44	0.18	6.54
Managed Futures	126	8.64	5.38	0.05	4.58
Other	55	3.77	3.34	-0.17	12.08

table. First, for “true” returns, the estimated VaRs without taking the fat tails into consideration will understate the extreme risks (see Table 3.8). Second, comparing the last column of Table 3.13 and the third column of Table 3.10, we see that the volatility increase from the observed returns to the “true” returns explains a large portion of the VaR increase from the observed returns to the “true” returns.

3.7 Conclusion

Hedge funds’ returns often exhibit positive autocorrelation, which suggests illiquidity in their asset holdings. In this paper, using a data set containing monthly returns of over 5,600 hedge funds, we study how the illiquidity affects the extreme risk of hedge funds. We use $MA(q)$ processes to model hedge funds’ returns and use smoothing coefficients as proxies for liquidity. The tail risks are estimated using the extreme value theory and the Generalized Pareto distribution. We find that illiquidity in general has a negative impact on the tail risk of hedge funds’ returns. In particular,

Table 3.13: Loss VaR Assuming Normal Distributions: Live Funds

Live Funds in TASS data base with at least 5 year monthly return data as of June 30, 2005.

Style	No.	Perct.	$ \hat{R}_{99\%}^{Obs} $	$ \hat{R}_{99\%}^{True} $	$ \hat{R}_{99\%}^{True} - \hat{R}_{99\%}^{Obs} $
Convertible Arbitrage	59	4.05	4.35	5.78	1.43
Dedicated Short Bias	14	0.96	16.61	17.36	0.75
Emerging Markets	102	7.00	16.74	19.57	2.83
Equity Market Neutral	61	4.18	5.93	6.48	0.55
Event Driven	142	9.74	5.45	6.40	1.17
Fixed Income Arbitrage	41	2.81	5.31	6.40	1.09
Fund of Funds	320	21.95	4.89	5.81	0.93
Global Macro	52	3.57	9.28	9.58	0.30
Long/Short Equity Hedge	486	33.33	11.27	12.64	1.37
Managed Futures	126	8.64	13.79	12.56	-1.23
Other	55	3.77	6.64	7.79	1.14

We report the results of estimated loss VaR at 99% level, using the standard deviation of the returns (equation 3.2), for the observed returns and for the “true” returns. Averages of the absolute value of $\hat{R}_{99\%}$ (VaR at 99% level) for each category are reported in the table.

the true Value-at-Risk (VaR) of hedge funds could be much higher when illiquidity is taken into consideration.

Chapter 4

Asset Pricing Under Heterogeneous Information

(This is joint work with Jiang Wang)

4.1 Introduction

The impact of information asymmetry has long been of key interest in the study of financial markets.¹ Investors' private information is impounded into asset prices through their trades, and prices will then reflect the average beliefs across investors. As a result, investors not only care about the fundamental values of a security, but also pay attention to other investors' beliefs on the fundamental value, a situation which Keynes (1936) refers to as the "Beauty Contest." Since private beliefs or their averages are not observable either, beliefs of beliefs and their higher iterations also matter. Capturing these higher order beliefs, especially in an intertemporal setting,

¹See, for example, Hayek (1945), Grossman (1977), and Grossman and Stiglitz (1980).

makes the formal analysis of the market behavior under asymmetric information a challenging task, which is also known as the infinite regress problem.² Most of the existing work focuses on situations in which the dimensionality of the problem can be reduced to be finite, either by limiting the dimension of underlying shocks or the form of information asymmetry.³ In this paper, we consider a model which allows for general forms of information heterogeneity and directly solves the infinite regress problem and the resulting market equilibrium. This allows us to examine in detail how different forms of information heterogeneity affects the behavior of asset prices, trading activity and investor welfare.

We show that the infinite-regress problem yields the long-range history dependence of the current market behavior. In general, current asset prices and their dynamics depend on the whole history of past shocks. In particular, revelations of past underlying shocks can influence current prices more than concurrent shocks. Information heterogeneity increases the divergence in investors' beliefs about economic fundamentals. Such a divergence tends to reduce the amount of risk sharing among investors and their effective risk tolerance. Consequently, stock prices become lower and more volatile. In addition, information heterogeneity reduces the level of liquidity and consequently the amount of trading in the market. We further show that the effect of information heterogeneity is non-monotonic in the amount of private information agents have. It is maximized when agents receive moderate amount of private information.

Moreover, we show that information heterogeneity tends to reduce investors wel-

²For earlier discussions of the infinite-regress problem, see Townsend (1983). Makarov and Rytchkov (2007) formally shows the infinite dimensionality of the problem in a setting similar to ours.

³See, for example, Wang (1993), He and Wang (1995), Bacchetta and Van Wincoop (2005), and Allen, Morris, and Shin (2006).

fare. In particular, investors are typically made worse off by possessing private information—they can be made better off by either revealing or abandoning all their private information collectively. We also find that investors with superior information does not necessarily enjoy higher welfare. The adverse selection problem makes it very costly for them to trade with less informed investors for risk sharing. Such a cost can out weight the potential gain they make from speculating on their private information.

We consider a continuous-time model in which fundamentals of the economy change stochastically over time. The economy is populated by long-lived agents receiving both endowment shocks and private information about the fundamentals. They trade competitively in a security market to share their endowment risks and to speculate on the future movements of security prices with their private information. The risk-sharing trading motive allows us to avoid the introduction of noise traders, which is necessary in examining the welfare implications of information heterogeneity. The information structure we consider is quite general—different agents can receive different private signals about the underlying shocks of the economy.

In such a setting, asset prices are affected by the average beliefs of the agents about the fundamentals. Therefore, they need to forecast the average beliefs of other agents as well as how the average beliefs evolve over time. These forecasts will be based on all the information they have received in the past. As a result, agents' forecasts, optimal trading policies and the equilibrium asset prices all become dependent on the whole history of the economy. The Markovian structure typically assumed in economic modeling is no longer valid. An infinite number of the state variables, in particular, the whole history of the economy is needed to characterize the economy and its equilibrium. Under Gaussian assumption on the underlying shocks of the economy and constant absolute risk aversion of the agents, we are able

to solve the agents non-Markovian forecasting and optimization problems and then the market equilibrium.

The paper is organized as follows. Section 4.2 describes the economy. In Section 4.3, we present the equilibrium of the economy. Section 4.4 considers the special case of homogeneous information, which serves as a benchmark in analyzing the impact of information heterogeneity. In Sections 4.5 and 4.6, we consider how information heterogeneity influences asset prices, return dynamics and agents' welfare. We first consider in Section 4.5 a particular form of information heterogeneity in which agents have diffuse private information on the fundamentals of the economy and examine its impact on the market behavior. We then extend our analysis to the other forms of information heterogeneity. Section 4.7 concludes the paper. All proofs are provided in the appendix.

4.2 The Economy

We consider a pure exchange economy with a single, perishable consumption good.

4.2.1 Securities Market

There is a competitive securities market with two traded securities, a risk-free bond and a risky stock. The bond yields a constant return of $r > 0$. The stock pays a cumulative dividend D_t , which is given by

$$dD_t = M_t dt + b_D dB_t, \tag{4.1}$$

where $\{B_t : 0 < t < \infty\}$ is a n -dimensional Brownian motion, b_D is a non-zero constant vector and M_t follows a mean reversion process

$$dM_t = -a_M M_t dt + b_M dB_t, \quad (4.2)$$

with b_M being a non-zero constant vector and $a_M > 0$ a constant. The initial values of D_t and M_t at time $t = 0$ are D_0 and M_0 , respectively. We will use σ_D and σ_M to denote the length of the vector b_D and b_M , respectively. Similar notation will be used later for other processes.

4.2.2 Agents

The economy is populated by a continuum set of agents A . The measure of A is normalized to be one. The set A consists of subsets A_i , $i \in I$, i.e.,

$$A = \bigcup_{i \in I} A_i. \quad (4.3)$$

For convenience, we also use I to denote the total number of subsets. Agents in each subset A_i are assumed to be identical. We use m_i to denote the measure of set A_i .

Each agent $i \in A_i$ ($i \in I$) is initially endowed with one unit of the stock. In addition, he receives a non-traded cumulative income of G_t^i at time t , where

$$G_t^i = \int^t Y_t^i dN_s = \int^t (Y_t + Z_t^i) dN_s, \quad (4.4)$$

where N_t is Brownian motion, Y_t and Z_t^i follow standard O-U processes, respectively.

In particular,

$$dN_t = b_N dB_t \quad (4.5a)$$

$$dY_t = -a_Y Y_t dt + b_Y dB_t \quad (4.5b)$$

$$dZ_t^i = -a_Z^i Z_t^i dt + b_Z^i dB_{Z,t}^i \quad (4.5c)$$

with a_Y and a_Z^i being positive constants, b_F , b_Y and b_Z^i being constant vectors of proper order, and $B_{Z,t}^i$ a one-dimensional Brownian motion independent of B_t . The initial values of Y_t and Z_t^i at $t = 0$ are given by Y_0 and Z_0^i , respectively. The instantaneous income flow for agent i is $(Y_t + Z_t^i)dN_t$ at time t . Thus, $Y_t dN_t$ represents the common component of the income flow and $Z_t^i dN_t$ gives the idiosyncratic component. Since dN_t characterizes the random shock to the non-traded income, Y_t and Z_t^i define the common and idiosyncratic exposure to risk in their non-traded income, respectively. Summing up all agents' non-traded cash flow we have

$$G_t = \int_i G_t^i di = \int^t Y_t dN_s + \int^t \left(\int_i Z_t^i di \right) dN_s. \quad (4.6)$$

To induce the allocational trading, we assume that the shocks to N_t and the shocks to D_t are positively correlated, i.e.,

$$b_D b_N^\top = \sigma_{DN} = \rho_{DN} \sigma_N \sigma_D > 0. \quad (4.7)$$

4.2.3 Information Structure

The underlying state of the aggregate economy is given by three variables, D_t , M_t and Y_t . In general, agents do not have perfect information on the underlying state of the economy. Instead, they have private information on some of these state variables.

Below we consider a rich information structure in which agents receive a mixture of public and private information.

Each agent $i \in A_i$ ($i \in I$) observes market prices of traded assets, in particular the stock price P_t and its dividend D_t . He also observes his total exposure to the non-traded income risk, $Y_t^i = Y_t + Z_t^i$. In addition, he receives a signal S_t^i on M_t , given as follows

$$dS_t^i = M_t dt + b_S^i dB_{S,t}^i, \quad (4.8)$$

where b_S^i is a constant, $B_{S,t}^i$ is a one dimensional Brownian motion, and $S_0^i = 0$. Moreover, we assume that public announcements reveal the aggregate state of the economy with a lag of T . In other words, M_{t-T} and Y_{t-T} will be known to all agents at time t .

Let \mathcal{F}_t^i denote the information set of agent i at time t . Then we have

$$\mathcal{F}_t^i = \{P_s, D_s, Y_s^i, S_s^i : s \leq t\} \vee \{M_s, Y_s : s \leq t - T\} \quad (4.9)$$

where \vee represents the union of two information sets.

For expositional convenience, we use the following convention: If $\|b_S^i\|^2 = \infty$, then agents receive no signal S_t^i . Thus, by our definition for $\|b_S^i\|^2 = \infty$, we have

$$\mathcal{F}_t^i = \{P_s, D_s, Y_s^i : s \leq t\} \vee \{M_s, Y_s : s \leq t - T\}.$$

4.2.4 Preferences

Each agent $i \in A_i$ has a time-separable, constant absolute risk aversion (CARA) utility over his life time consumption. Let c_t denotes his consumption rate at t , $\alpha > 0$ his risk aversion and $\rho > 0$ his time discount coefficient. Agents choose their

consumption and trading policies to maximize the expected utility of the form

$$E_t \left[\int_t^\infty -e^{-\rho s - \alpha c_s} ds \middle| \mathcal{F}_t^i \right]. \quad (4.10)$$

CARA preferences also impose the following condition on the non-traded income to guarantee that agents' lifetime expected utility is finite:

$$\left(\frac{r\alpha\sigma_Y}{r + 2a_Y} \right)^2 + \frac{1}{I} \sum_{i=1}^I \left(\frac{r\alpha\sigma_Z^i}{r + 2a_Z^i} \right)^2 < 1. \quad (4.11)$$

4.2.5 Simplifications

For notational convenience, we let

$$B_t = \left(B_{D,t} \quad B_{M,t} \quad B_{Y,t} \quad B_{N,t} \right)^\top \quad (4.12)$$

where $B_{D,t}$, $B_{M,t}$, $B_{Y,t}$ and $B_{N,t}$ are all one-dimensional Brownian motions. Although our analysis allows for general correlation structure for the underlying shocks, to simplify exposition we assume that $B_{D,t}$, $B_{M,t}$, $B_{Y,t}$, $B_{N,t}$, $B_{Z,t}^i$ and $B_{S,t}^i$ (for each $i \in A$) are mutually independent.⁴

Except when explicitly stated otherwise, we will consider the case when $I = A$, i.e., when there are large number of agent groups. Then we have

$$G_t = \int_i G_t^i di = \int^t Y_t dN_s + \int^t \left(\int_i Z_t^i di \right) dN_s = \int^t Y_t dN_t. \quad (4.13)$$

Thus, the aggregate exposure to the non-traded risk depends only on Y_t . In this

⁴Correlation of $B_{Z,t}^i$ and $B_{S,t}^i$ across agents can be non-zero. In general, we assume that $B_{Z,t}^i$ is mutually independent for different subsets of agents (i.e., for $i, j \in I$ and $i \neq j$). But $B_{S,t}^i$ can be correlated across different subsets of agents.

case, the underlying state of the economy is fully determined by D_t , M_t and Y_t .

4.2.6 Discussion of the Model

The information structure defined above is fairly general. It contains situations considered in the literature as special cases but extends to more general situations. Although our solution will be given for the general information structure, our analysis will focus on several simple cases in order to develop the intuition on the effects of different aspect of information heterogeneity. These simple cases are listed below.

A. Homogeneous Information

Homogeneous information refers to the case where all agents have the same information. This is achieved in our setting by letting $S_t^i = S_t, \forall i \in A$. That is

$$\mathcal{F}_t^i = \mathcal{F}_t^{HI} = \{D_s, N_s, S_s, P_s : s \leq t\} \vee \{M_s, Y_s : s \leq t - T\}. \quad (4.14)$$

Identical private signals across all agents are equivalent to a public signal. The quality of the signal then determines the total amount of information the agents have. In order to examine how the amount of information in the economy affect its behavior, we consider two extreme cases.

First, suppose that $\sigma_S = 0$, i.e., the public signal is exact. Then all agents actually observe M_t at t . The market price of the stock further reveals Y_t . We then have the case of *Full Information*, where all agents effectively observe the underlying state of the aggregate economy, especially, M_t and Y_t . Formally, this is equivalent to stating that

$$\mathcal{F}_t^i = \mathcal{F}_t^{FI} = \{D_s, M_s, Y_s, N_s : s \leq t\}. \quad (4.15)$$

Second, suppose that $\sigma_S = \infty$, i.e., the public signal is completely uninformative. This is the case where agents have no information on the fundamental of the stock, in particular M_t , other than what is revealed by realized dividends. The market price of the stock, which will only depend on Y_t , in effect reveals it to the agents. We refer to this as the case of *No Information* and have

$$\mathcal{F}_t^i = \mathcal{F}_t^{NI} = \{D_s, N_s, P_s : s \leq t\} \vee \{M_s, Y_s : s \leq t - T\}. \quad (4.16)$$

It should be clear that agents have more information in the full information case than in the no information case. Thus, comparing the equilibrium in these two cases allow us to gauge the impact of the amount of information in the market, as opposed to the difference in information between agents.

B. Diffuse Information

Diffuse information refers to the situation in which agents receive different private information, with comparable but independent noise. In other words, the overall information is diffusely distributed among all agents, without abnormal concentration. We can further assume perfect symmetry between them, i.e., $\sigma_S^i = \sigma_S$, $a_Z^i = a_Z$ and $\sigma_Z^i = \sigma_Z$. For the behavior of the aggregate market, such a symmetry between agents is not essential. It merely simplifies the exposition.

The case of diffuse information was considered in Hellwig (1980) and Diamond and Verrecchia (1981) in static settings and in He and Wang (1995) and Allen, Morris and Shin (2006) in finite horizon, discrete time settings. Discrete time and finite horizon allow the state space to be limited to finite dimensions and thus avoid the “infinite regress” problem. Our continuous-time setting does not limit the dimension of the state space and allows us to solve the “infinite regress” problem and examine

its implications.

C. Asymmetric Information

The case of asymmetric information allows for more concentrated private information among different agents. Such a situation arises in our setting when there are only finite number of groups of investors, i.e., I is a finite set. Analyzing such a situation allows us to examine how concentrated private information, as opposed to diffused private information, influences market behavior.

The case of hierarchic information (where $I = 2$ and σ_S is zero for one of the two groups) was considered by Grossman and Stiglitz (1980) in a static setting and by Wang (1993) in a dynamics setting. The asymmetric information case with two groups is considered by Makarov and Rytchkov (2007) under the additional assumption that agents behave myopically and have the same information precision. In this paper, we study in detail the simple case where there are two groups (i.e. $I = 2$) under the general information structure and without the myopic assumption.

In our model, in addition to heterogenous information, we also explicitly model agents' allocational trades, i.e., trades from motives unrelated to private information. We do so by introducing non-traded income shocks, as in Diamond and Verrecchia (1981) and Wang (1994). These income shocks give rise to trading needs for risk sharing. Endogenizing the allocational trades rather than inserting them as exogenous noise allows us to further examine the impact of information on welfare.

4.3 Market Equilibrium

For the economy defined above, we now consider the solution to its equilibrium. Given the Gaussian nature of the primary state variables and the CARA preference,

we are interested in linear equilibria in which the stock price is a linear function of past shocks. In particular, we have the following result:

Proposition 4. *The equilibrium stock price process is of the form:*

$$\begin{aligned}
P_t = & -p_0 + \int_{t-T}^t h_P(t-s)dB_s + g_{PM}M_{t-T} + g_{PY}Y_{t-T} \\
& + \int^i \left[\int_{t-T}^t h_{PZ}^i(t-s)dB_{Z,s}^i ds + g_{PZ}^i Z_{t-T}^i \right] \\
& + \int^i \left[\int_{t-T}^t h_{PN}^i(t-s)dB_{S,s}^i ds + g_{PS}^i S_{t-T}^i \right]
\end{aligned} \tag{4.17}$$

where $h_P(t)$ is a (1×4) vector of deterministic functions of t .

In the various special cases examined in the literature (e.g., Wang (1993, 1994), He and Wang (1995) and Allen, Morris and Shin (2006)), the equilibrium is shown to take such a linear form in the underlying shocks. The general case we consider here makes an substantial extension: It allows an infinite number of state variables. As evident from (4.17), the price may depend on the the whole path of past underlying shocks. In continuous time, this implies that the state space is infinite-dimensional. Such a large state space is necessary to capture the nature of the so-called “infinite regress” problem that arises under general heterogenous information.

The full solution for the general case is provided in the Appendix. To simplify exposition, here we only present the solution in the case of diffuse information with symmetric parameters as described in Section 4.2.6 (Case C). It captures the basic nature of the equilibrium under heterogenous information and the intuition behind it.

In the diffuse information case, the shocks to the idiosyncratic endowment Z_t^i and the noise of the individual signal are independent. As evident from (4.17),

they will not appear in the stock price function due to the Law of Large Numbers. Consequently, the stock price should only depend on the past shocks of aggregate variables:

$$P_t = -p_0 + \int_{t-T}^t h_P(t-s)dB_s + g_{PM}M_{t-T} + g_{PY}Y_{t-T}. \quad (4.18)$$

In the remainder of this section, we outline the key steps in obtaining the solution to the equilibrium as described in Proposition 4 (details can be found in the Appendix for the general solution).

4.3.1 Filtering Problem

In addition to the stock price P_t , each agent i are concerned about six state variables, D_t , M_t , N_t , Y_t , Z_t^i and S_t^i . Let

$$X_t^i = \left(D_t \quad M_t \quad Y_t \quad Z_t^i \quad N_t \quad S_t^i \right)^\top, \quad B_t^i = \left(B_{D,t} \quad B_{M,t} \quad B_{Y,t} \quad B_{Z,t}^i \quad B_{N,t} \quad B_{S,t}^i \right)^\top.$$

X_t^i then defines the vector of state variables for agent i and is governed by the following dynamics

$$dX_t^i = -a_X X_t^i dt + b_X dB_t^i, \quad (4.19)$$

where a_X and b_X can be easily constructed from (4.1), (4.2), (4.5), and (4.8). Note that symmetry among the agents implies that a_X and b_X are the same for all i .

Applying the Non-Markovian Filtering technique, we obtain the following theorem:

Theorem 2. *Assume that the stock price is given by the form in (4.17). Let*

$$\hat{M}_t^i = E[M_t | \mathcal{F}_t^i], \quad \hat{Y}_t^i = E[Y_t | \mathcal{F}_t^i], \quad \text{and} \quad \hat{X}_t^i = E[X_t^i | \mathcal{F}_t^i].$$

Then, under filtration \mathcal{F}_t^i ,

(i) the stock price can be expressed as

$$P_t = -p_0 + \int_{t-T}^t \hat{h}_P(t-s) d\hat{B}_s^i + g_{PM} \hat{M}_{t-T}^i + g_{PY} \hat{Y}_{t-T}^i, \quad (4.20)$$

where \hat{B}_s^i is a 6-dimensional Brownian motion with respect to \mathcal{F}_t^i ;

(ii) the dynamics of \hat{X}_t^i is given by

$$d\hat{X}_t^i = -a_X \hat{X}_t^i + \hat{b}_X d\hat{B}_t^i \quad (4.21)$$

where \hat{b}_X is a 6×6 constant matrix.

4.3.2 Optimization

Under the price process (4.17), we can express the instantaneous excess return on the stock as

$$dQ_t = dP_t + dD_t - rP_t dt \quad (4.22)$$

Furthermore, conditioning on the information of agent i , we can re-express the excess stock return as

$$dQ_t = \hat{\mu}_{Q,t}^i dt + \hat{b}_Q d\hat{B}_t^i, \quad (4.23)$$

where

$$\hat{\mu}_{Q,t}^i = \int_{t-T}^t \hat{h}_Q(t-s) d\hat{B}_s^i + \hat{g}_Q \hat{X}_{t-T}^i,$$

and $\hat{h}_Q(\cdot)$ is a 1×6 -vector-valued function and \hat{g}_Q is a constant 1×6 vector.

Let c_t^i denote agent i 's consumption rate at t and x_t^i his holdings of the stock. His wealth evolves according to the following process:

$$dW_t^i = rW_t^i dt - c_t^i dt + x_t^i dQ_t + Y_t^i dN_t, \quad (4.24)$$

where dQ_t is given in (4.23). The optimization problem of agent i is

$$\max_{\{c_t^i, x_t^i\}} E_0^i \left[\int_{t=0}^{\infty} -e^{-\rho t - \alpha c_t^i} dt \right] \quad (4.25)$$

subject to his budget constraint (4.24). We have the following solution:

Theorem 3. *Under the return process (4.23) and the wealth dynamics (4.24),*

(i) *the value function of agent i is given by:*

$$J_t^i \triangleq -e^{-\rho t - \alpha(rW_t^i + V_t^i)},$$

where V_t^i is an Ito process with respect to \mathcal{F}^i , given by

$$\begin{aligned} V_t^i &= \int_{t-T}^t \left[\int_{t-T}^s v_{11}(t-s, t-s') d\hat{B}_{s'}^i \right]^\top d\hat{B}_s^i + 2\hat{X}_{t-T}^{i\top} \int_{t-T}^t v_{12}(t-s) d\hat{B}_s^i \\ &+ \hat{X}_{t-T}^{i\top} v_{22} \hat{X}_{t-T}^i + \int_{t-T}^t v_1(t-s) d\hat{B}_s^i + v_2 \hat{X}_{t-T}^i + v_0 \end{aligned}$$

with the functions $v_{11}(\cdot, \cdot)$, $v_{12}(\cdot)$, $v_1(\cdot)$, and constants v_{22} , v_2 , and v_0 given in the Appendix;

(ii) *his optimal stock holding x_t^i is given by*

$$x_t^i = \frac{1}{r\alpha} \frac{\hat{\mu}_{Q,t}^i}{\hat{b}_Q \hat{b}_Q^\top} - \frac{Y_t^i \hat{b}_Q \hat{b}_N^\top}{\hat{b}_Q \hat{b}_Q^\top} - \frac{1}{r} \frac{\hat{b}_Q (b_{V,t}^i)^\top}{\hat{b}_Q \hat{b}_Q^\top}, \quad (4.26)$$

where

$$b_{V,t}^i = \left[\int_{t-T}^t v_{11}(0, t-s) d\hat{B}_s^i \right]^\top + 2\hat{X}_{t-T}^i{}^\top v_{12}(0) + v_1(0).$$

The optimal stock holding x_t^i given by equation (4.37) is decomposed into three components. The first term is the usual myopic demand, which is determined by the instantaneous mean and volatility of the excess stock return. The second term is the static hedging demand (hedging against the income shock Y_t^i). Our specification about D_t and N_t allows us to re-write this term as $-\frac{\rho_{DN}\sigma_D}{b_Q b_Q^\top} Y_t^i$. Notice that without loss of generality, we assume $\rho_{DN} > 0$. If agent i receives a positive shocks on Y_t^i , then he will reduce his stock holdings so as to reduce his overall risk exposure. The third term is the dynamic hedging demand (see, Merton (1971)). The sign of this term is determined by the instantaneous correlation between the stock return and the value process.

4.3.3 Market Clearing

From Theorem 3, we know that the optimal stock holding of agent i is of the form

$$x_t^i = \int_{t-T}^t \hat{h}_x(t-s) d\hat{B}_s^i + \hat{g}_x \hat{X}_t^i. \quad (4.27)$$

Moreover, we can re-write the optimal stock holding as a linear combination of shocks under the objective information set. The following lemma formalizes this idea:

Lemma 1. *The optimal stock holding of agent i is of the form:*

$$\begin{aligned} x_t^i &= \int_{t-T}^t h_x(t-s) dB_s + \int_{t-T}^t h_{xZ}(t-s) dB_{Z,s}^i + \int_{t-T}^t h_{xS}(t-s) dB_{S,s}^i \\ &+ g_{xM} M_{t-T} + g_{xY} Y_{t-T} + g_{xZ} Z_{t-T}^i + x_0. \end{aligned}$$

By the Law of Large Numbers, we have the following market clearing condition:

Theorem 4. *The market clears if and only if*

$$\begin{aligned} h_x(s) &= 0, \\ g_{xM} &= 0, \quad g_{xY} = 0, \\ x_0 &= 1. \end{aligned}$$

The equilibrium price is given by a 1×4 vector-valued functions $h_P(s)$, and three constants g_{PM} , g_{PY} , and p_0 . Therefore the market clear condition in Theorem 4 gives a well-identified system for $h_P(s)$, g_{PM} , g_{PY} , and p_0 (for a formal statement of equilibrium condition, see the Appendix). Its solution describes an equilibrium of the economy in the form of Proposition 4.

4.4 Special Case: Homogeneous Information

We first consider the special case of homogeneous information. This case serves a benchmark in our analysis of the impact of heterogeneous information on asset prices.

We define the expected future dividends of the stock, discounted at the risk-free rate, by

$$F_t \triangleq E \left[\int_t^\infty e^{-rs} dD_{t+s} | M_t \right] = \frac{1}{r + a_M} M_t. \quad (4.28)$$

For simplicity, we will refer to F_t as the “fundamental value” of the stock. Of course, the actual price of the stock in general differs from its fundamental value for two reasons. First, in general, agents do not observe F_t but have to rely on their own information to form expectations, which can be different from the true value of F_t . Second, actual dividends can be different from their expected value and a

corresponding risk premium will arise in equilibrium. Our analysis will focus on these two factors.

As stated in Section 4.2.6, under homogeneous information we have

$$\mathcal{F}_t^i = \mathcal{F}_t^{HI} = \{D_s, N_s, S_s, P_s : s \leq t\} \vee \{M_s, Y_s : s \leq t - T\} \quad \forall i \in A.$$

Thus, all agents have the same forecast about future dividends, which is determined by M_t . In particular,

$$E[M_t | \mathcal{F}_t^i] = E[M_t | \mathcal{F}_t^{HI}] = E[M_t | \{D_s, N_s, S_s, P_s : s \leq t\} \vee \{M_s, Y_s : s \leq t - T\}] = \hat{M}_t$$

and $E[Y_t | \mathcal{F}_t^i] = Y_t$. The following proposition characterizes the equilibrium.

Theorem 5. *Under homogeneous information, in equilibrium, (i) the stock price P_t is given by*

$$P_t = E_t [F_t | \mathcal{F}_t^{HI}] - (p_0 + h_Y Y_t) = \frac{1}{r + a_M} \hat{M}_t - p_0 - h_Y Y_t, \quad (4.29)$$

where $\hat{M}_t = E [M_t | M_{t-T} \vee (D, S)_{[t-T, t]}]$,

$$p_0 = \alpha \left(\hat{\sigma}_F^2 + h_Y \sigma_Y^2 + \frac{1}{r} h_Y \sigma_Y \phi_{11} \right), \quad h_Y = \frac{\alpha r \sigma_{DN}}{r + a_Y - \alpha \sigma_Y^2 \phi_{10}}, \quad (4.30)$$

$\hat{\sigma}_F^2 = \left(\frac{1}{r + a_M} \hat{b}_M + \bar{b}_D \right) \left(\frac{1}{r + a_M} \hat{b}_M + \bar{b}_D \right)^\top$, and ϕ_{11} and ϕ_{10} are two positive constants;

(ii) the optimal stock holding of agent i is

$$x_t^i = 1 - h_{xZ} Z_t^i, \quad (4.31)$$

where

$$h_{xZ} = \frac{1}{\hat{\sigma}_F^2 + h_Y^2 \sigma_Y^2} (\sigma_{DN} + h_Y \sigma_Y^2 \phi_{12})$$

and ϕ_{12} is a positive constant.

From the first equality in (4.29), we see that equilibrium stock price consists of two components. The first component is the expected value future dividends discounted at the risk-free rate, i.e., the “fundamental value” of the stock. The second component gives the risk discount in the stock price, arising from the risk in its future dividends. In particular, as agents are risk averse, the stock sells below its fundamental value.

The risk discount has two terms. The constant term, p_0 , gives the unconditional risk discount. The stochastic term, $h_Y Y_t$, arises from the aggregate exposure to the non-traded risk, which is determined by Y_t . Since the non-traded risk is positively correlated with the risk of the stock (i.e., $\sigma_{DN} > 0$), an increase in the non-traded risk, i.e., an increase in Y_t , will cause an increase in agents’ overall risk, which includes both the risks of their stock position and non-traded income. In particular, they will choose to sell the stock in order to reduce their overall risk exposure. This gives rise to their static hedging demand. In equilibrium, the aggregate decrease in stock demand causes a decrease in the stock price, which is linear in Y_t .

We also note that the stochastic nature of Y_t further leads to agent’s dynamic hedging needs. Especially, changes in Y_t over time lead to changes in the expected returns on the stock. The desire of the agents to hedge the changes in their investment opportunities will further modify their stock demand, both on average and over time as Y_t varies, which will also influence the equilibrium price.⁵ In particular, the term

⁵See Merton (1971) for a general discussion of dynamic hedging when agents face time-varying returns. Wang (1993) provides a detailed discussion on the dynamic hedging demand and its impact on equilibrium prices in a setting similar to ours here.

$\alpha\sigma_Y^2\phi_{11}$ in the denominator of h_Y comes from the dynamic hedging effect. In other words, if agents behavior myopically, we will have $\phi_{11} = 0$ and $h_Y = \alpha r \sigma_{DN} / (r + a_Y)$.

We now can give a clear interpretation of the unconditional risk discount p_0 . Understandably, it is proportional to α , agent's risk aversion. The first term, $\hat{\sigma}_F^2$ simply gives the perceived risk in the stock's fundamental value. The second term, $h_Y^2\sigma_Y^2$, gives the additional price risk due to the time-variation of the aggregate non-traded risk. The third term comes from the additional decrease in stock demand due to its changing expected returns, generated by the time variation in Y_t .

Agents' stock holding in equilibrium depends only on their heterogeneous shocks; common shocks such Y_t only shift prices. In the absence of private information, the only source of heterogeneity comes from their exposure to non-traded risks, which is given by Z_t^i . Indeed, as stated in (4.31), their equilibrium stock holding is linear in Z_t^i . The proportionality coefficient h_{xZ} gives the optimal response of stock holdings to one unit increase in idiosyncratic non-traded risk. In particular, a positive idiosyncratic non-traded risk cause an agent to reduce his stock holding. Trading among agents, however, has no price impact since idiosyncratic shocks are washed out in aggregation.

Although agents have the same information in the case of homogenous information, the total amount of information they have can vary, depending on the precision of the common signal S_t , which is measured by the volatility of the signal σ_S . As σ_S increases, the common signal becomes noisier and agents have less information about the future dividends of the stock. A natural question is how does the amount of information in the market affect the market equilibrium, in particular, market prices and welfare? We have the following result:

Theorem 6. *As σ_S increases, p_0 and h_Y increase, and h_{xZ} decreases. Moreover, the*

variance of stock prices $Var[P_t]$ decreases as σ_S increases.

We first analyze the effect of the information amount on h_Y , the sensitivity of the stock price to the aggregate endowment. h_Y is determined by the relative size of the myopic demand, the static hedging demand, and the dynamic hedging demand (The investors' demand decomposition is given by Equation (4.37)). Equation (4.30) suggests that the information amount impacts h_Y only through dynamic hedging (In the appendix, we show that ϕ_{10} is positively related to the loading on Y_t of the change in the value function due to the shock $dB_{Y,t}$). The investment opportunities (Y_t and Z_t) are changing over time. Since the idiosyncratic shock $dB_{Z,t}$ does not affect the stock return, agents only use the stock to hedge against the risk of the investment opportunity changes induced by $dB_{Y,t}$. To analyze the impact of information amount on dynamic hedging, we first fix the signal noisiness σ_S . Assume a positive shock $dB_{Y,t}$ hits. This will positively impact the next period aggregate endowment, which reduces the next period stock price and hence reduces the realized excess return dQ_t . On the other hand, an increase in the next period aggregate endowment will have two effects on the next period value function. First is the expected return effect. When the next period aggregate endowment is higher, so will be the next period expected excess return and the next period myopic demand, which is determined by the risk adjusted expected excess return. Therefore, the next period value function will increase by an amount which is positively related to the next period aggregate endowment, scaled by the perceived risk of the stock return. Second is the endowment risk effect. An increase in the next period Y will increase the aggregate risk of the non-traded income. Therefore, the value function will incur a loss, which is positively related to a combination of the next period aggregate endowment and idiosyncratic endowment (in our linear equilibrium setting, the combination will be linear). The

two effects work against each other, but we can show (see the Appendix) that in the case of homogenous information, the endowment risk effect always dominates the expected return effect, and when a positive shock $dB_{Y,t}$ hits, the value function incur a loss, by the amount which is positively related to a (linear) combination of the next period aggregate endowment and idiosyncratic endowment. Hence the loadings of the dynamic hedging demand on Y_t and Z_t are negative. Now assume that the signal becomes more noisy. Since the information amount about the fundamentals decreases, the cash flow is more risky to investors and the perceived risk of the stock return increases. When investors consider the stock return to be more risky, they decrease the stock demand, which will weakens the expected return effect discussed above. On the other hand, the endowment risk effect remains almost the same since the perceived risk of the future endowment is unchanged. Therefore, the endowment risk effect becomes even more dominating over the expected return effects, and the loading on Y_t of the changes in the value function due to the shock $dB_{Y,t}$ becomes more negative. Hence, relative to the myopic and static hedging demand, the sensitivity of the dynamic hedging demand on Y_t becomes more negative. When Y_t increases, the dynamic hedging demand decreases more relative to the myopic and static hedging demand, and the stock prices falls more (that is, h_Y increases as σ_S increases).

Second, we discuss the effect of information amount on the discount level p_0 (given by Equation (4.30)). A decrease of information amount will have three effects on p_0 . Firstly, when information about cash flow decreases, the perceived risk of the fundamental value increases, which will increase the risk premium. Secondly, as discussed in the paragraph above, a reduction of information amount of cash flow will lead to a increase of the sensitivity of the stock price on the aggregate endowment shock Y_t . Therefore the risk of the aggregate endowment will have more impact on

the risk of the stock return, and agents demand higher risk premium. Thirdly, a decrease of information amount of cash flow will alter the perceived risk of the changes in future investment opportunities. A reduction of information on cash flow weakens the expected return effect and the endowment risk effect becomes more pronounced. Therefore, the perceived risk of the changes in the future investment opportunity will increase, and agents demand higher risk premium. All the three effects work in the same direction and the price discount level p_0 is an increasing function of the noisiness of the signals.

Finally, we discuss the impact of information amount of price variance:

$$Var[P] = \frac{\hat{\sigma}_M^2}{2a_M(a_M + r)^2} + \frac{h_Y^2 \sigma_Y^2}{2a_Y}. \quad (4.32)$$

There are two offsetting effects. Firstly, when information amount about cash flow decreases, the unconditional variance of the perceived M_t will decrease,⁶ which will lower the stock price variance. Secondly, as the signal becomes more noisy, the endowment shocks will have larger impact on the stock price (h_Y increases), which will raise the stock price variance. The two effects offset each other. We can show that when information is homogeneous, the former effect dominates and overall, the stock price variance falls.

To summarize, when information is homogeneous, the reduction in the information amount on cash flow impacts the stock price through the channels of (i) the perceived risk of the fundamental values, and (ii) investors' dynamic hedging behavior. When information amount about cash flow decreases, the perceived risk of fundamentals increases. Moreover, agents lower the sensitivity of their optimal

⁶An extreme case is when there is no information about M at all (even without the observation of D_t), then the perceived M_t will be constant. Hence the variance of the perceived M_t will be zero.

stock holding on the expected excess return, and hence the changes in the future expected return will become less important than the changes in the future endowment risk. Therefore, when information amount decrease, the changes of value function will react more negatively to the change of the aggregate endowment shock. Hence, relative to the myopic and static hedging demand, a unit increase of the aggregate endowment will lead to a higher decrease in investors' the dynamic hedging demand. Consequently, a unit increase of the aggregate endowment will induce a higher drop in the stock price. Through the channel of dynamic hedging, a reduction in the amount of information on cash flow will increase the impact of non-traded endowment on the stock price. The magnitude of the effect of both channels increases as the amount of information about cash flow decreases and reaches maximum when there is no private signal (i.e. $\sigma_S = \infty$).

To end this section, we make a remark on the impact of parameters. The persistence parameter a_Y is of interest. The smaller a_Y is, the more persistent the aggregate endowment, and the larger the impact of the dynamic hedging. Therefore the impact of the amount of information on stock price will be larger when the parameter a_Y is smaller.

4.5 The Impact of Information Heterogeneity: The Case of Diffuse Information

We now analyze how the presence of heterogenous information affects the market equilibrium. In this section, we consider the case of diffuse private information as described in Section 4.2.6, Part B. In particular, we will examine how private information affects the behavior of asset prices, trading activity and agents' welfare.

We return in Section 4.6 to consider how our results extend to the cases with other forms of heterogeneous information.

For notational convenience, we define the expectation operator E_t^i and the average expectation operator \bar{E}_t as:

$$E_t^i[\cdot] = E[\cdot | \mathcal{F}_t^i], \quad \bar{E}_t[\cdot] = \int^i E_t^i[\cdot]. \quad (4.33)$$

4.5.1 Information Heterogeneity

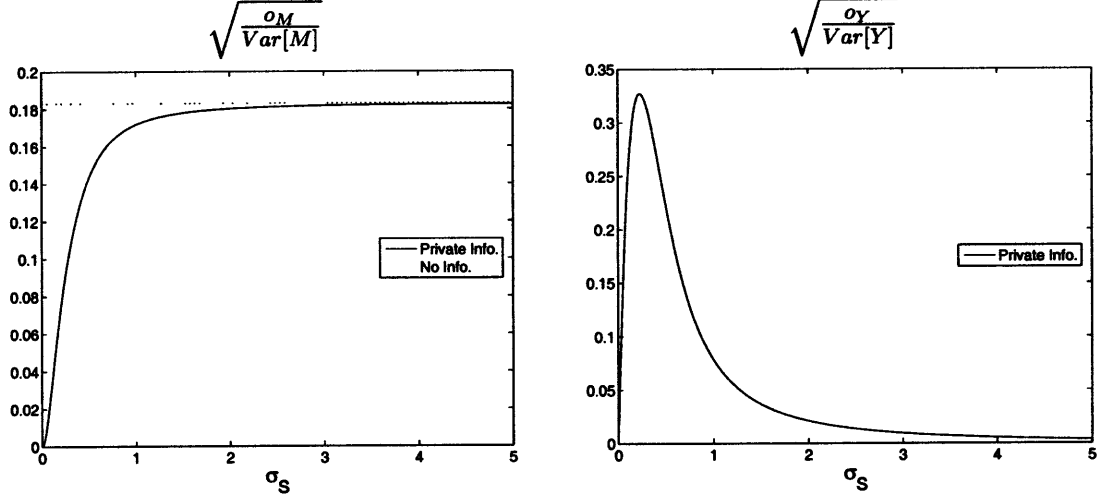
There are two aspects of information structures: (i) the information amount: the amount of information available to investors; (ii) the information heterogeneity: the differentiation of information among different investors. Information amount available to one investor is determined by the precision of his private signal and by the information content of public signals. One can measure the information amount by the forecasting error. More precisely, for each investor $i \in A$, we define the forecasting error to be

$$o_M^i = \text{Var}[E_t^i[M_t] - M_t]. \quad (4.34)$$

The symmetry in parameters implies that o_M^i is the same for all $i \in A$. The left panel of Figure 4-1 plots the forecasting error of M_t against the precision of private signals. The results are very intuitive. The forecasting error is zero in the full information case and is largest in the no information case. The error increases as the private signals become noisier.

When investors receive different private signals about fundamentals, they will have different expectations about future stock payoff. The stock price aggregates these expectations and investors extract useful information about fundamentals and

Figure 4-1: Information Efficiency $\sqrt{\frac{\sigma_M}{\text{Var}[M]}}$ and $\sqrt{\frac{\sigma_Y}{\text{Var}[Y]}}$ against σ_S : Diffuse Information



Information Efficiency $\sqrt{\frac{\sigma_M}{\text{Var}[M]}}$ and $\sqrt{\frac{\sigma_Y}{\text{Var}[Y]}}$. Model Parameters: $r = 0.05$, $\alpha = 10$, $T = 20$, $a_M = 0.4$, $a_Y = 0.5$, $a_Z = 0.7$, $\sigma_D = 0.7$, $\rho_{DN} = 0.5$, $\sigma_M = 0.2$, $\sigma_Y = 1$, $\sigma_Z = 0.4$.

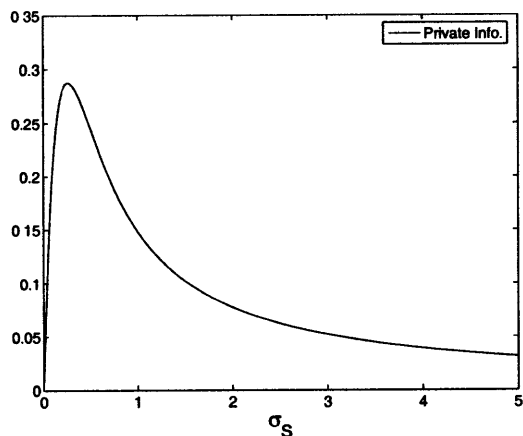
about other investors' expectations. We can quantify the degree of information heterogeneity using the average of the difference between the perceived M_t of each agent and the average perceived M_t :

$$\delta = \int_i \{ \text{Var}[E_t^i[M_t] - \bar{E}_t[M_t]] \}. \quad (4.35)$$

In our model, there are two sources of information heterogeneity: their private signals S_t^i about M_t , and their total non-traded income shocks $Y_t + Z_t^i$. When information is heterogenous, the stock price does not fully reveal the aggregate income shock Y_t . Therefore their total non-traded income shocks $Y_t + Z_t^i$ will be informationally valuable. This will contribute to the degree of information heterogeneity.

When investors only use their private signals and ignore the information value

Figure 4-2: Normalized Information Heterogeneity against σ_S : Diffuse Information



Normalized Information Heterogeneity: $\sqrt{\int^i \frac{\text{Var}[E_t^i[M_t] - \bar{E}_t[M_t]]}{\text{Var}[M_t]}}$. Model Parameters: $r = 0.05$, $\alpha = 10$, $T = 20$, $a_M = 0.4$, $a_Y = 0.5$, $a_Z = 0.7$, $\sigma_D = 0.7$, $\rho_{DN} = 0.5$, $\sigma_M = 0.2$, $\sigma_Y = 1$, $\sigma_Z = 0.4$.

of the stock prices, the degree of information heterogeneity will exhibit hump-shape against the signal precision. If the precision of private signals is high, investors have highly accurate estimate of the fundamental and the difference between investors' estimates will be small. If the precision of private signals is bad, without using stock prices as an additional source of information, investors forecast of M_t will converge to $E[M_t|D_s, s \leq t]$ and the degree of information heterogeneity will go to zero.

When investors use both private signals and public signals to forecast fundamentals, the relation between the information heterogeneity and signal precision could still be hump-shaped. Figure 4-2 confirms the hump-shape relation between the degree of information heterogeneity and the precision of private signal and its asymptotic behavior as $\sigma_S \rightarrow \infty$.

An interesting corollary of the above discussion is that, the forecasting error of the aggregate income shock Y_t may also be hump-shaped against the precision of

private signals (the second graph of Figure 4-1.) The stock price serves as signals to both the stock fundamental and the aggregate income shock. When investors agree more on the perceived stock fundamentals, the stock price becomes a better signal for Y_t . Therefore the forecasting error of Y_t is directly related to the degree of information heterogeneity.

4.5.2 Stock Price

In this part, we discuss the impact of information heterogeneity on asset prices, in particular, the level of the stock price discount p_0 in Equation (4.18) and the unconditional variance of the stock price. According to Equation (4.17), the stock price can be written as:

$$P_t = -p_0 + \int_{t-T}^t h_{PM}(t-s)dB_{M,s} + \int_{t-T}^t h_{PD}(t-s)dB_{D,s} + \int_{t-T}^t h_{PY}(t-s)dB_{Y,s} + g_{PM}M_{t-T} + g_{PY}Y_{t-T}.$$

In the case of heterogeneous information, the information structure effect the stock prices through the amount of information and the degree of information heterogeneity. As discussed in section 4.4, the amount information impact the stock price through the channel of the perceived risk of the fundamentals, and the channel of the changes in the investors' dynamic hedging behavior. The hedging of the changing investment opportunities will amplify the effect of the non-traded income on the stock price, when the information amount is reduced.

An increase of the degree of information heterogeneity will also amplify the price impact of the non-trade income, however, through a different channel. The amplification comes from the role of the public signal (e.g. the stock price) in the formation

of investors' expectation on the stock fundamentals.⁷ This effect still exists even if investors are myopic. As we have pointed out, unlike the homogeneous information case, when information is heterogeneous, the stock price does provide investors with *additional* informational value beyond their private signals to forecast M_t . Investors put non-zero weight both on their private signals and on the stock price when forming forecast about the stock fundamental. The current stock price will reflect (i) the average of investors' perceived current fundamental, (ii) the average of investor's forecasted future stock price, and (iii) the aggregate demand for both static hedging and dynamic hedging. When investors receive different private signals, the stock price become valuable to forecast M_t . On the other hand, the stock price is negatively correlated with the aggregate income shock due to investors' hedging motive. Therefore, the aggregate income shock is negatively correlated with the average of investors' perceived fundamental. This negative correlation will reinforce the impact of the aggregate income shock on the stock price due to hedging trades. One interesting observation is that the amplification will be persistent over time, even if the shock to the aggregate income itself is not persistent. This is because in the case of independent income shock, the past shocks of income shocks will have impact on current and future expectation of the fundamentals, even if it has no impact on current or future non-traded income itself.

Moreover, in a dynamic setting, this effect of information heterogeneity will reinforce the dynamic hedging effect to further amplify the impact of the non-traded income on the stock price. When the degree of information heterogeneity increase, the sensitivity of the stock price on the non-traded income increases. This has two effects: the perceived risk of stock return rises and the sensitivity of the expected

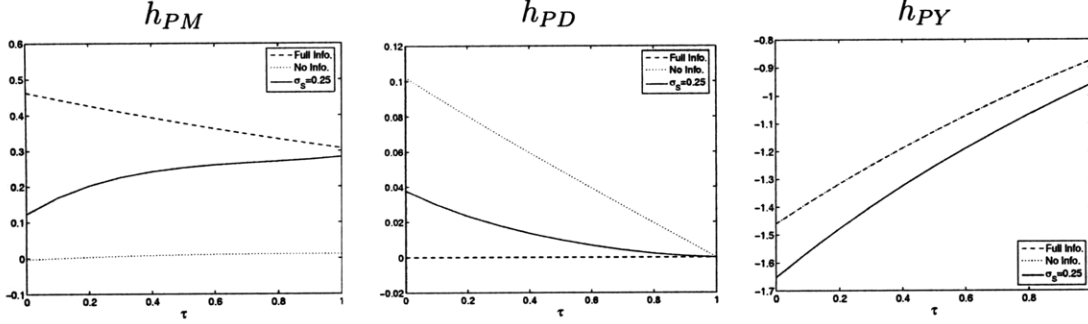
⁷Bacchetta and Van Wincoop (2005) finds similar magnification effect of information dispersion on the non-informational trade in the foreign exchange rate market.

return on the aggregate income shocks increases. These two effect offset each other in the changes of the investment opportunities related to expected return. Firstly, since the stock is more risky, the myopic demand is less sensitive to the expected return, which reduces the positive loading on Y_t of the changes in the future value function. Secondly, One unit increase in Y_t will induce more increase in the expected return, which will increase the positive loading on Y_t of the changes in the future value function. These two effects each other and in general the first effect will dominate. Overall, these two effect reduce the positive loading on Y_t of the changes in the value function. Since the realized return is negatively related to the shocks to Y_t , the investors increase the magnitude of their negative position in the dynamic hedging demand as the degree of information heterogeneity increase. This future increase the negative impact of Y_t on stock price.

Figure 4-3 shows the graphs of the three functions $h_{PM}(t)$, $h_{PD}(t)$, and $h_{PY}(t)$, for the cases of full information, no information, and diffuse information with a fixed $\sigma_S = 0.25$. The shape $h_{PM}(t)$ gives the impact of shocks to M_t at lag t , the fundamental of the stock, on the current price. What is intriguing is that $h_{PM}(t)$ is increasing in t . That is, the impact of current shock to the fundamental, which is given by $h_{PM}(0)$ is actually smaller than the impact of a past shock with lag t , which is given by $h_{PM}(t)$. This clearly demonstrate the long range dependence of current prices on past shocks. In comparison, $h_{PM}(t)$ is decreasing in t under full information and flat under no information. Under full information, the impact of past shocks die off exponentially. Under no information, the true fundamental is never learned—agents only rely on realized dividends to learn about it. As a result, shocks to the fundamental has practically not impact on prices.

We can also see that the amplification of $h_{PY}(t)$ through the channel of information amount (as reflected by the difference between the full information case and the

Figure 4-3: Impulse Response of Price $P_{t+\tau}$ to Aggregate Shocks against Horizon τ : Diffuse Information



Impulse Response Function $h_{PM}(\cdot)$, $h_{PD}(\cdot)$ and $h_{PY}(\cdot)$ of Price $P_{t+\tau}$ to Aggregate Shocks $dB_{M,t}$, $dB_{D,t}$ and $dB_{Y,t}$. Model Parameters: $r = 0.05$, $\alpha = 10$, $T = 1$, $a_M = 0.40$, $a_Y = 0.5$, $a_Z = 0.7$, $\sigma_D = 0.7$, $\rho_{DN} = 0.5$, $\sigma_M = 0.2$, $\sigma_Y = 1$, $\sigma_Z = 0.4$.

no information case) is very small—the curve of full information case and no information case are nearly indistinguishable. But the amplification of $h_{PY}(t)$ through the channel of information heterogeneity is quantitatively large.

The level of price discount p_0 is

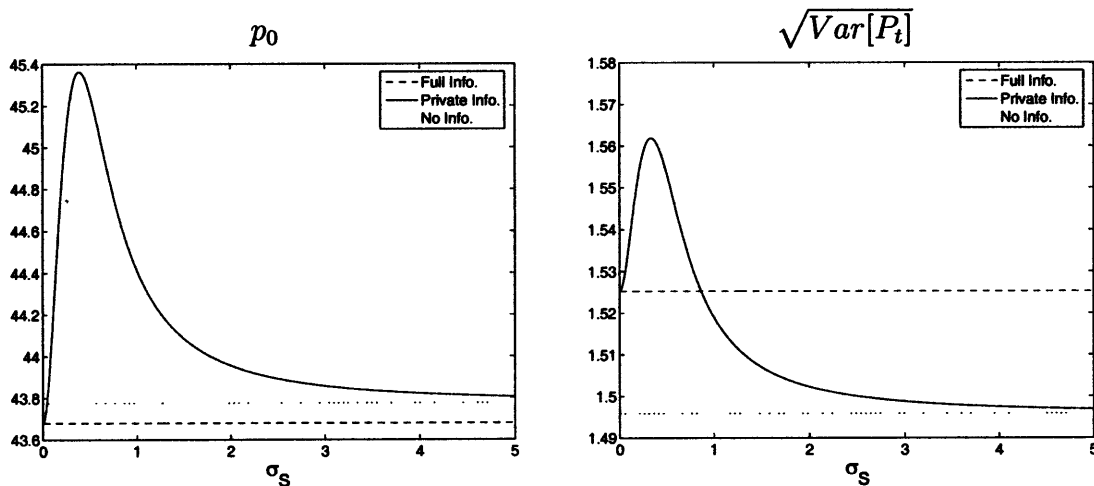
$$p_0 = \alpha \left\{ \hat{b}_Q \hat{b}_Q^\top + \frac{1}{r} \hat{b}_Q v_1^\top(0) \right\}, \quad (4.36)$$

where \hat{b}_Q is given in Equation (4.23) and $v_1(0)$ is given in Theorem 3. The price discount contains two component: (i) the total amount of perceived risk for the stock return, $\hat{b}_Q \hat{b}_Q^\top$; (ii) and the total amount of perceived risk for dynamic changing of investment opportunities, $\frac{1}{r} \hat{b}_Q v_1^\top(0)$. And understandably, p_0 is proportional to the risk aversion α . The impact of information heterogeneity on the two components of price discount level can be understood as follows. In the presence of information heterogeneity, the sensitivity of price to the aggregate income shocks is magnified.

The stock return reacts more to the income shocks and the perceived return volatile increases. Investors then demand higher risk premium and p_0 rises. Furthermore, as discussed before, the amplification effect on the impact of Y_t due to the information heterogeneity reinforces the amplification effect on the impact of Y_t due to the dynamic hedging effect. This further increases the sensitivity of the price of the aggregate income shocks, which in turn further increase the both the perceived risk of stock return and the perceived risk of the changes of investment opportunities. Risk premium increases. The two effects reinforce each other. Overall, p_0 increases as the degree of information heterogeneity increases. The first part of Figure 4-4 confirms this intuition. As σ_S increases, the information amount reduces, which increases the price discount level. On the other hand, when σ_S is small, as σ_S increases, the degree of information heterogeneity increases (see Figure 4-2). Therefore the effect of heterogeneity increases. The effect of information amount and the effect of information heterogeneity work together and price discount level decreases. When σ_S is large, an increase in σ_S still reduce information amount about the fundamental, but it will reduce the degree of information heterogeneity. Therefore the effect of heterogeneity increases. The effect of information amount and the effect of information heterogeneity work in different direction and overall, the price discount level decreases. This gives a hump-shaped curve of p_0 against σ_S .

The variance of the stock price is determined by the response of price to the three aggregate shocks: $dB_{M,t}$, $dB_{D,t}$, and $dB_{Y,t}$. The amplification effect on the price response of income shocks will increase the contribution of $dB_{Y,t}$ in the price variance, which tends to raise the price variability. Theorem 6 shows that in the homogeneous information case, although a reduction of information amount will amplify the sensitivity of price to income shock through the dynamic hedging channel, this amplification effect is dominated by the reduction of the unconditional variance

Figure 4-4: Price Level p_0 and Volatility against σ_S : Diffuse Information



Expected Price Level p_0 and price volatility $\sqrt{Var[P_t]}$. Model Parameters: $r = 0.05$, $\alpha = 10$, $T = 20$, $a_M = 0.4$, $a_Y = 0.5$, $a_Z = 0.7$, $\sigma_D = 0.7$, $\rho_{DN} = 0.5$, $\sigma_M = 0.2$, $\sigma_Y = 1$, $\sigma_Z = 0.4$.

of the perceived fundamental, and thus the price variability falls as the information amount decreases. However, when information is heterogenous, the amplification effect could be so large that the price variance could be even higher than that in the full information case. The second part of Figure 4-4 graphs the price variance against the private signal precision. When the private signal is not exact, information amount is reduced, compared with full information case. The information amount effect tends to lower price variance. However, when $\sigma_S > 0$, there exists information heterogeneity, which tends to increase the price variance. These two effect offset each other and the overall effect could be of both signs. The stock price variance could be higher than that in the full information case, especially when the degree of information heterogeneity is high. This could potentially be helpful to explain the excess stock volatility puzzle. The extra price volatility could be coming from

information heterogeneity among investors. Finally, the *U*-shape of both curves is a consequence of the non-monotonic relation between the degree of information heterogeneity and the signal precision(See Part A. of this section).

4.5.3 Discuss of the Impact of Dynamic Hedging

In this section, we discuss the role of dynamic hedging on the impact of information heterogeneity on the stock price. As discussed before in the section, the dynamic hedging behavior will amplify the impact on the aggregate income on the stock price when the degree of information heterogeneity increases. In fact, this effect of dynamic hedging could be so large to cause qualitatively difference between a dynamic model and a static or myopic model. Qiu and Wang (2009) solves a static model with diffuse information. In the static diffuse information model, the price discount level is strictly increasing as the private signals get more noise. The unconditional variance of price, on the other hand, is strictly decreasing as signals precision gets poorer. Furthermore, the price discount level and the unconditional price variance in the diffuse information case are all between those quantities in the full information case and the no information case. However, Figure 4-4 show that in the dynamic diffuse information model: (i) price level and price variance could exhibit non-monotonicity against the precision of private signals; (ii) price level and price variance could be outside the interval of those quantities in the full information and no information case. In particular, when information is diffuse, the price discount level could be even higher than that of the no information case, and the price variance could exceed that of the full information case.⁸ Figure 4-5 illustrate this point by comparing the price impact of information heterogeneity in the fully dynamic case and in the

⁸This discussion suggests in some cases, static or myopic models could miss qualitatively some of the important features of fully dynamic models.

myopic case.⁹ The graphs show that the price impact of information is much larger when investors fully hedge the changes in the future investment opportunity. The information heterogeneity can cause as much as 8% increase in the price discount level in the dynamic model, while there is only a maximum of 1% increase in the myopic model. Furthermore, in the myopic case, even though the information asymmetry could amplify the impact of the non-traded income on the price variability, the effect of the reduced unconditional risk of the perceived fundamental values dominate and the overall effect of information asymmetry actually lead to a lower price volatility. However, in the dynamic case, the amplification effect could dominate and the price variance could increase as the signal precision gets worse.

4.5.4 Return Dynamics

Previous discussion indicates that information heterogeneity has an effect of increasing the perceived conditional volatility of instantaneous excess stock return. In this section, we analyze the impact of information structure on returns over positive horizon. Fix a horizon $\tau > 0$, the excess stock return from t to $t + \tau$ is:

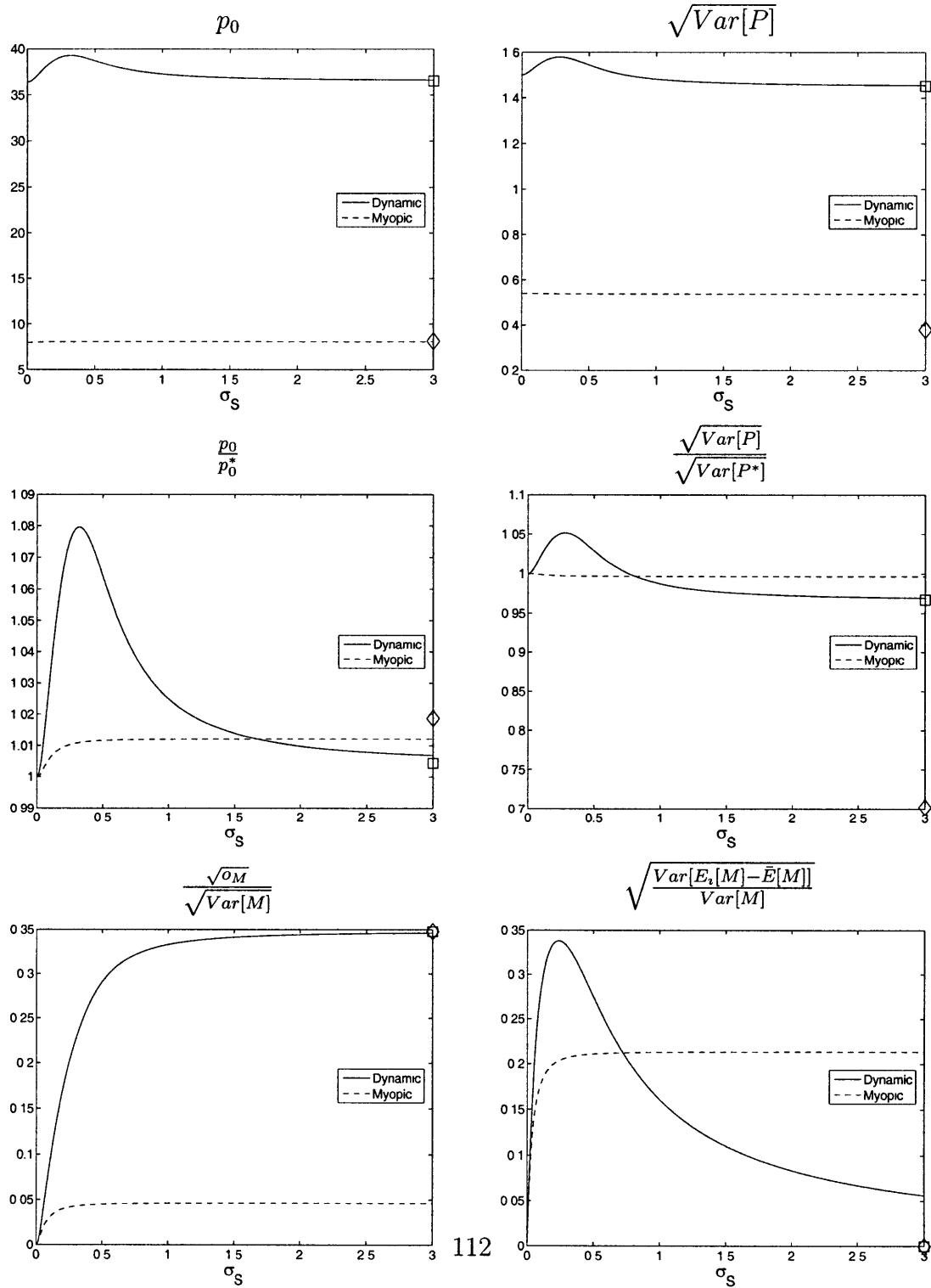
$$Q_{t,t+\tau} = \int_t^{t+\tau} \{dP_s - rP_s ds + dD_s\}. \quad (4.38)$$

We can study the decay rate of the lagged auto-correlation of excess return $Corr[Q_{t-\tau_0:t}, Q_{t+\tau:t+\tau+\tau_0}]$ as τ goes to infinity. Figure 4-6 plots the decay rate of the lagged auto-correlation of excess return, measured by $-\frac{d \log Corr[Q_{t-\tau_0:t}, Q_{t+\tau:t+\tau+\tau_0}]}{d\tau}$.

⁹In the myopic case, we assume that investors' optimal stock holding is:

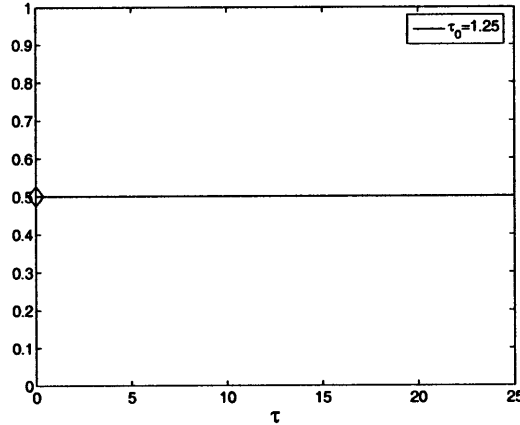
$$x_t^i = \frac{1}{r\alpha} \frac{\hat{\mu}_{Q,t}^i}{\hat{b}_Q \hat{b}_Q^\top} - \frac{Y_t^i \hat{b}_Q \hat{b}_N^\top}{\hat{b}_Q \hat{b}_Q^\top}. \quad (4.37)$$

Figure 4-5: Equilibrium Quantities against σ_S : Myopic vs. Dynamic



It is the same as the persistence parameter a_Y of the aggregate income shocks. This suggests that information heterogeneity does not slow down the decay rate of the lagged return auto-correlation.

Figure 4-6: Decay Rate of Lagged Auto-Correlation of Excess Stock Return against τ



Decay Rate of Lagged Auto-Correlation of Excess Stock Return: $\frac{d \log \text{Corr}[Q_{t-\tau_0,t}, Q_{t+\tau, t+\tau+\tau_0}]}{d\tau}$. Model Parameters: $r = 0.05$, $\alpha = 10.00$, $T = 10.00$, $a_M = 0.400$, $a_Y = 0.500$, $a_Z = 0.700$, $\sigma_D = 0.700$, $\rho_{DN} = 0.500$, $\sigma_M = 0.200$, $\sigma_Y = 1.000$, $\sigma_Z = 0.400$. The decay rate of shocks of Y_t is marked as a diamond at the vertical axis.

4.5.5 Trading Activities

Next we analyze trading activities. For agent i , we use the instantaneous volatility of dx_t^i as a proxy to measure the trading volume. So by definition, the total trading volume is

$$Vol_t = \frac{1}{\sqrt{2\pi}} \int_i \sqrt{\frac{\text{Var}[dx_t^i | \mathcal{F}_t^i]}{dt}}. \quad (4.39)$$

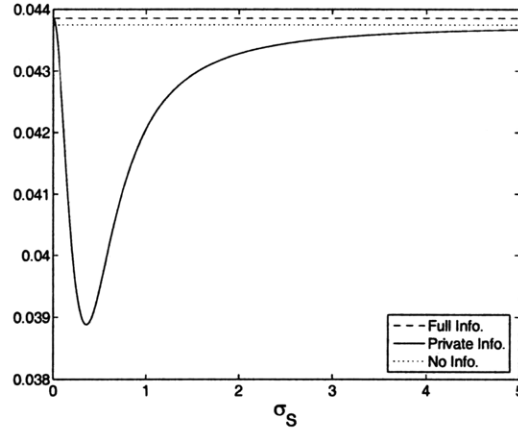
Theorem 6 implies that a reduction of information amount will lower the trading

volume because the perceived stock return becomes more volatile and the stock becomes less attractive as an hedging instrument. So the optimal stock holding will be less sensitive to investors' idiosyncratic income shocks and trading activity decreases.

When information is heterogenous, investors have both re-balancing trading motive and informational trading motive. They trade according to their total income shock $Y_t + Z_t^i$ and their private signal. The optimal stock holding can be written as the sum of three parts. The first is the risk adjusted informational component, which is determined by the difference between investors' own forecast about future stock returns and investors' forecast about the average belief about future stock returns. The second is the static hedging component, which is determined by investor's perceived idiosyncratic income shocks. The third is the dynamic hedging component, which reflects the difference between the change of agents' investment opportunity and their estimate of the investment opportunity changes of other investors. All the three components are adjusted by investors' perceived risk about the stock return. The effect of an increase in the degree of informational heterogeneity can be analyzed as follows. On one hand, when information becomes more diverse, the difference between the investor's perceived fundamental value and his forecast of the average believe increases. Therefore, the informational trading motive tend to increase and trading activity tends to increase. Furthermore, the perceived difference between investors own dynamic hedging demand and the average dynamic hedging demand tend to increase when the information becomes more diverged, which also tends to increase the trading volume. On the other hand, as the degree of information heterogeneity increase, the perceived return volatility increases, the stock becomes a less attractive hedging device, and therefore the trading activities tend to fall. Overall, the effect of the increase of perceived stock return risk tends to

dominate and the overall trading volume tends to fall as information becomes more diverged. Figure 4-7 shows that the impact of information heterogeneity on trading volume could significant and could be much larger than the impact of information amount.

Figure 4-7: Total Trading Volume against σ_S : Diffuse Information



Total Trading Volume. Model Parameters: $r = 0.05$, $\alpha = 10$, $T = 20$, $a_M = 0.4$, $a_Y = 0.5$, $a_Z = 0.7$, $\sigma_D = 0.7$, $\rho_{DN} = 0.5$, $\sigma_M = 0.2$, $\sigma_Y = 1$, $\sigma_Z = 0.4$.

4.5.6 Welfare

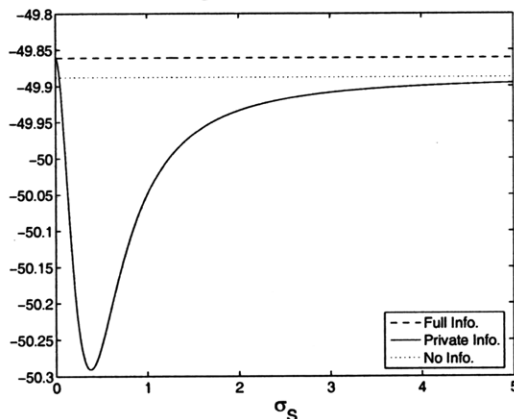
An important feature of our model is that stock trading is endogenized, which allows us to perform welfare study. Welfare can be defined through the certainty equivalence of the value function. More precisely, welfare \bar{V} is defined as:

$$-e^{-\alpha r \bar{V}} = E[J_0], \quad (4.40)$$

where J_0 is the value function at $t = 0$. The value function at time 0 depends on the initial bond holding and initial stock holding. To make meaningful comparison, we

assume that the initial bond holdings are the same for all investors, and the initial stock holdings are also the same across investors.

Figure 4-8: Welfare against σ_S : Diffuse Information



Welfare. Model Parameters: $r = 0.05$, $\alpha = 10$, $T = 20$, $a_M = 0.40$, $a_Y = 0.5$, $a_Z = 0.7$, $\sigma_D = 0.7$, $\rho_{DN} = 0.5$, $\sigma_M = 0.2$, $\sigma_Y = 1$, $\sigma_Z = 0.4$.

In our model, since we assume that agents only have information on the component of the dividend which is un-correlated with the endowment shocks, an reduction of information amount will only decrease the welfare. In particular, the Hirshleifer effect does not exist in our model. An increase in the amount of information tends to lower stock return volatility and help investors better hedge the risk of the non-traded income over time and investors tend to be better off.

Information heterogeneity will magnify the welfare loss. When the degree of information heterogeneity increases, investors become less certain about the average beliefs of fundamentals. This will reinforce the welfare loss due to the uncertainty about the fundamentals and lead to further welfare loss. Figure 4-8 illustrates this point in the case of diffuse information. Welfare of each agent decreases at first when σ_S increase from zero, when both the effect of the information amount and

the information heterogeneity work in the same direction(the information amount decreases and the degree of information heterogeneity increases in the case). The welfare reaches a minimum for some intermediate value of σ_S^* and increases as σ_S increase after that point. This U-shaped welfare function against the signal precision is consistent with the observation that while the information amount always decrease as σ_S increases, the degree of information heterogeneity in fact is a decreasing function of σ_S when σ_S is large.¹⁰

One phenomenon is of particular interest in Figure 4-8. When private signals are noisy enough, the welfare loss associated with information heterogeneity could be so large that it would be better off for all agents just to discard completely their private signal. In deed, the welfare in the heterogeneous information case falls below the welfare in the no information case when precision of signals is small enough, and the no information case Pareto dominates the heterogenous information case.

4.5.7 The Sensitivity of Results to Parameters

In this section, we brief discuss the sensitivity of the above results to the parameter choices. We are particular interests in the persistence parameters a_M , a_Y , and a_Z . For each parameter, we first discuss the impact of the stock price when M_t is perfectly observable. Then we discuss the impact of the parameter on the amount of information and the degree of information heterogeneity, assuming heterogenous information structure. We leave most of the graphs of the sensitivity analysis in the

¹⁰Dow and Rahi (1971) studies the impact of information asymmetry on welfare in a static set-up. They also show that the welfare could exhibit a U-shaped pattern against the signal precision. However, tn their model, an increase of information amount could lead to a reduction of the risk-sharing opportunity and hence Hirshleifer effect exits. Hence the U-shaped curve in their model is a result of the offsetting effects of the less risk sharing opportunities due to the earlier resolution of the hedging uncertainty from the better information and a better hedging instrument due to the less risky stock return from the better information.

Appendix.

Assume the aggregate income shock become less persistent, i.e. a_Y increases. In the case of perfectly observable M_t , a less persistent Y_t means less impact of dynamic hedging demand, since the change of future investment opportunities is less predictable. This will lead to a smaller sensitivity of the aggregate income shock on the stock price. Both the perceived excess return risk and the unconditional price variability falls. Therefore, the discount level and the price variance decreases. When investors have different private information, the smaller sensitivity of the non-traded income due to the less persistence of Y_t will increase the informational value of the stock price to forecast the fundamentals. Therefore, the public signal contains more information about M_t , and hence the total information available to agents increases. Furthermore, investors put more weight on the public signal and the information heterogeneity tends decrease.¹¹ However, an increasing in a_Y has another effect on investors' expectation. Since investors can not observe the aggregate income shock Y_t directly, they need to make forecast. One piece of information they use to forecast Y_t is their total income shock $Y_t + Z_t^i$. The less persistent Y_t is, the more valuable the total income shock $Y_t + Z_t^i$ is to forecast Y_t . Therefore, an increasing of a_Y will again increase the information amount of M_t . On the other hand, since an increasing in a_Y makes the total income shock $Y_t + Z_t^i$ a better forecaster of Y_t , investors put more weight on $Y_t + Z_t^i$ in their estimation of Y_t and hence on M_t as well. This lead to a higher impact of the idiosyncratic shocks Z_t^i on investors' forecast, which increases the degree of information heterogeneity. Moreover, this effect is more pronounced when the precision of the private signals on M_t is low, because in this case, investors

¹¹This is different from the effect caused by a reduction of persistence of M_t . In that case, an increase in a_M render the private signal a less informational valuable. Investors then put more weight on public signals. Hence the degree of information heterogeneity decreases while the total amount of information *decreases*.

will rely more on the public signal and a better forecast of Y_t from $Y_t + Z_t^i$ will be more beneficial. To sum up, an increasing of a_Y will always lead to a better forecast of the fundamentals, but have two offsetting effects on the information heterogeneity. When σ_S is small, the effect through the public signal dominates and information heterogeneity decreases as a_Y decreases. When σ_S is large, the effect through the information value of $Y_t + Z_t^i$ dominates and the information heterogeneity increases as a_Y increases. Although, the information amount always increases as a_Y increases, the degree of information heterogeneity can go either way. Therefore, the overall price impact of the information structure(measured by ratio of the equilibrium quantities to the full information quantities) could also go either way.

The effect of the persistence coefficient a_Z can be analyzed as follows. First discuss the full information case. The idiosyncratic shocks Z_t^i do not move the stock price. However, they have impact on investors' dynamic hedging demand. Intuitively, a less persistence idiosyncratic endowment shock will make the hedge-able dynamic risk to be less sensitive to the current Y_t . Hence the price impact of Y_t decreases. Consequently both the excess return and the stock price become less risky, and the discount level and the price variable fall. Now assume information is heterogenous. As a_Z increases, the stock price co-move less with the aggregate endowment, and the price becomes a better forecaster of the fundamental values. Therefore, the information amount about M_t increases. Furthermore, since public signals become more accurate, investors put more weight on the public signals, and the information heterogeneity decreases. On the other hand, similar to the affect of an increasing in a_Y , an increasing in a_Z will increase the information value of the total income shock $Y_t + Z_t$ to forecast Y_t . Therefore, in forming their forecast on M_t , investors will put more weight on $Y_t + Z_t^i$ and the information heterogeneity will increase. Also this effect is larger when the precision of the private signal is low(because, public

information is more important in forecasting M_t , and a better estimate of Y_t will make the stock price a less noise signal of M_t). To sum up, overall, a less persistent Z_t^i will lead to a smaller forecasting error of M_t , but will have offsetting effects on the information heterogeneity.

When a_M gets larger, the dividend process M_t become less persistent. First we discuss the case where investors perfectly observe M_t . When investors perfectly observe M_t , an increase in a_M will decrease the loading of stock price on the M_t . The excess return react less to the shocks of dividend and hence the perceived stock return becomes less risky. Therefore the price discount level falls. Furthermore, a lower perceived stock return risk will lower the sensitivity of dynamic hedging demand on the changes of non-traded income shocks and therefore the stock price moves less as Y_t changes. This further lowers the return risk and the discount level falls more. Moreover, the unconditional price variance will fall because both loadings of the stock price on dividend and non-traded income falls. Now we discuss the effect of a_M on the information amount and on the degree of information heterogeneity. In the presence of information heterogeneity, a less persistent M_t will reduce the forecasting power of the private signal. In fact the normalized forecasting error $\frac{o_M}{Var[M]}$ is increasing as a function of a_M (Though the absolute forecasting error o_M in decreasing in a_M since $Var[M]$ it self is decreasing). Hence the information amount decreases. Furthermore, investors thus put more weight on public signals. Therefore the degree of information heterogenous decreases.

We next discuss the impact of the conditional volatilities: σ_M , σ_Y , and σ_Z .

The volatility σ_M of M_t measures the risk of the asset fundamentals. When the risk of the fundamental increases, both the risk premium and the price variance increases. σ_M also affects the investors' forecasting error. A large σ_M yields a smaller absolute forecasting power of the private signals. This will increase both the absolute

forecasting error and the absolute information heterogeneity, which further reinforce the impact on the discount level and the price variance. σ_Y measures the volatility of the aggregate endowment shock. An increase in σ_Y will increase the conditional risk of the stock return, and raise the sensitivity of dynamic hedging demand on the aggregate income, and therefore enhance the price impact of Y_t . This will lead to an increase in risk premium and an increase in the price variance. Furthermore, an increase in the price impact of Y_t will make the price less informative about the fundamentals, the price informativeness drops, and the investors' forecasting error rises.

4.6 Other Forms of Information Heterogeneity

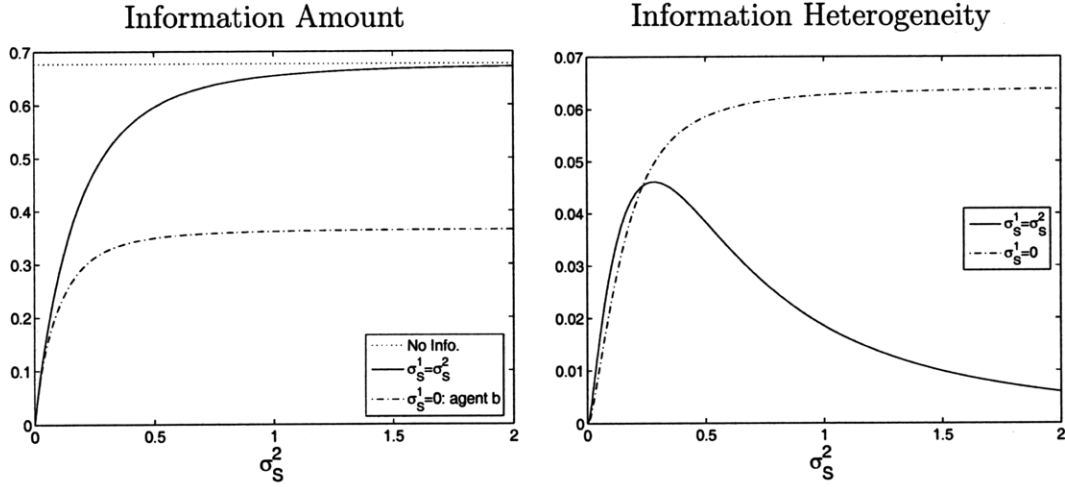
In the diffuse information case discussed before, we assume that information is distributed among a large number of investors and we focus on the impact of information dispersion. To better understand how heterogeneity in information quality affects asset prices, we further study the asymmetric information case where information is concentrated among a few groups of investors (Section 4.2.6). There are two groups of investors, i.e. $I = \{1, 2\}$. To focus on the information heterogeneity, unless otherwise explicitly stated, we assume that income shocks of the two groups are un-correlated, i.e. $Y \equiv 0$, and the idiosyncratic income shocks have the same volatility and the same mean reversion rate, i.e. $\sigma_Z^1 = \sigma_Z^2 = \sigma_Z$, $a_Z^1 = a_Z^2 = a_Z$, and $m_1 = m_2 = \frac{1}{2}$.

A special case of the asymmetric information is the hierarchical information case, where one group observes M_t perfectly ($\sigma_S^1 = 0$) and the other group does not receive any private signal about M_t ($\sigma_S^2 = \infty$). This setting has been studied by various authors, e.g. Grossman and Stiglitz (1980) and Wang (1993). One feature of the hierarchical information is that it has the maximum degree of information

heterogeneity among all the information structures. Therefore we will use it as a bench mark.

4.6.1 Asset Price Implications

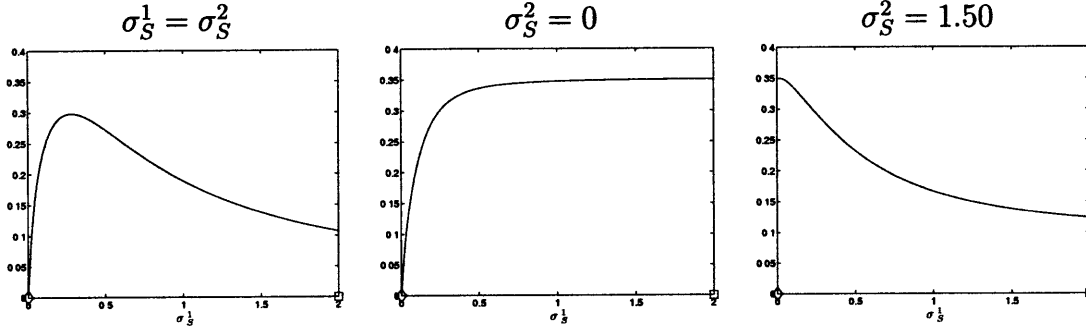
Figure 4-9: Information Amount and Information Heterogeneity in Symmetric Precision and Ordered Precision Cases



Information Amount and Information Heterogeneity. Model Parameters: $r = 0.05$, $\alpha = 7.00$, $T = 1.0000$, $a_M = 1.500$, $a_Z^1 = 0.900$, $a_Z^2 = 0.900$, $\sigma_M = 1.200$, $\sigma_D = 0.500$, $\rho_{DN} = 0.500$, $\sigma_Z^1 = 2.500$, $\sigma_Z^2 = 2.500$, $m_1 = 0.500$.

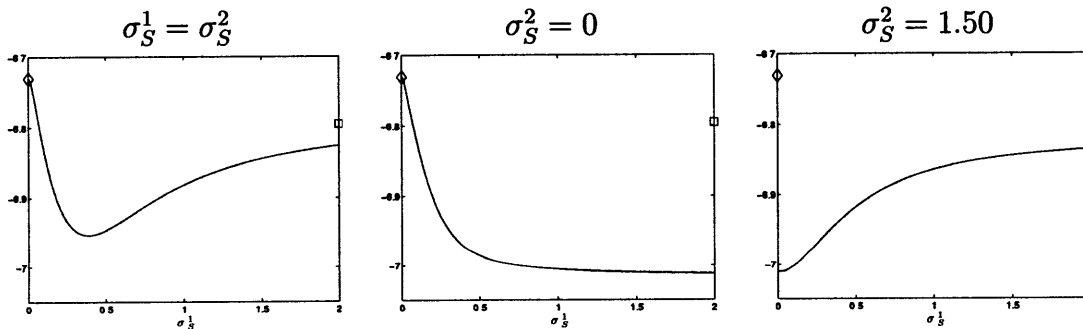
To help organize thoughts, in the section, we analyze two cases of information asymmetry: (i) the case of symmetric signal precision, where signals received by the two groups have the same precision, so that no group has a clear informational advantage; (ii) the case of ordered signal precision, where group one receives a perfect signal of M_t ($\sigma_S^1 = 0$) and group two receives a signal of some precision ($\sigma_S^2 \geq 0$). The hierarchical information can be viewed as the limiting case of the ordered signal precision when $\sigma_S^2 \rightarrow \infty$.

Figure 4-10: Normalized Information Heterogeneity against σ_S^1 : Two Groups



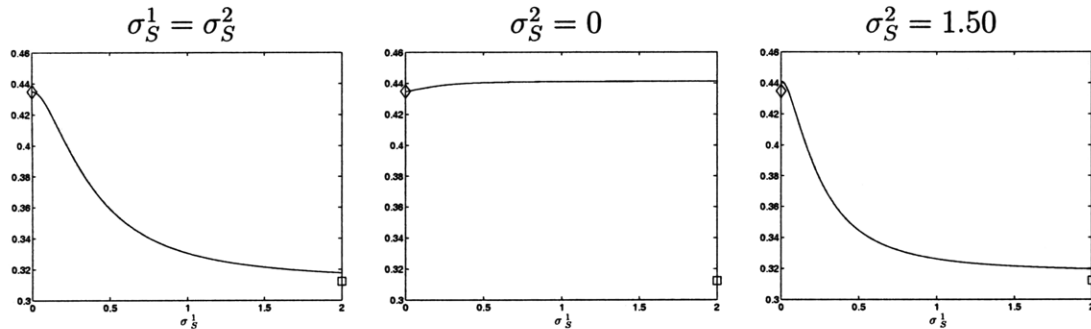
Normalized Information Heterogeneity. Model Parameters: $r = 0.05$, $\alpha = 7.00$, $T = 1.0000$, $a_M = 1.500$, $a_Z^1 = 0.900$, $a_Z^2 = 0.900$, $\sigma_M = 1.200$, $\sigma_D = 0.500$, $\sigma_F = 1.000$, $\rho_{DN} = 0.500$, $\sigma_Z^1 = 2.500$, $\sigma_Z^2 = 2.500$, $m_1 = 0.500$. The full information quantity is marked by a diamond at the left side of the graph. The no information quantity is marked by a square at the right side of the graph.

Figure 4-11: Expected Price Level p_0 : Asymmetric Information



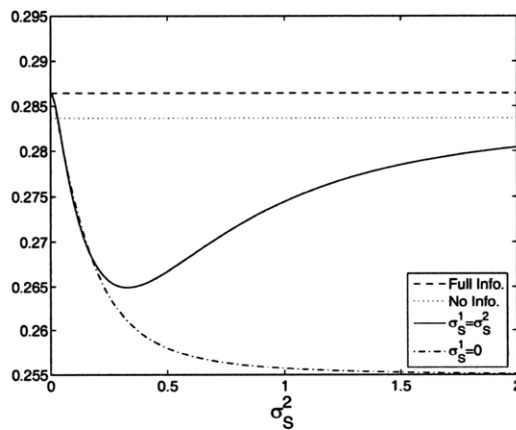
Model Parameters: $r = 0.05$, $\alpha = 7.00$, $T = 1.0000$, $a_M = 1.500$, $a_Z^1 = 0.900$, $a_Z^2 = 0.900$, $\sigma_M = 1.200$, $\sigma_D = 0.500$, $\sigma_F = 1.000$, $\rho_{DN} = 0.500$, $\sigma_Z^1 = 2.500$, $\sigma_Z^2 = 2.500$, $m_1 = 0.500$. The full information quantity is marked by a diamond at the left side of the graph. The no information quantity is marked by a square at the right side of the graph.

Figure 4-12: Price Volatility $\sqrt{Var[P]}$: Asymmetric Information



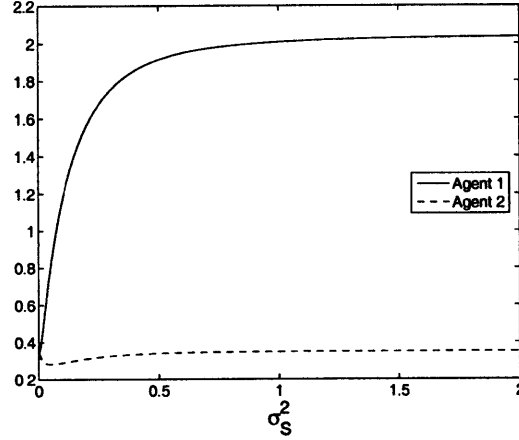
Model Parameters: $r = 0.05$, $\alpha = 7.00$, $T = 1.0000$, $a_M = 1.500$, $a_Z^1 = 0.900$, $a_Z^2 = 0.900$, $\sigma_M = 1.200$, $\sigma_D = 0.500$, $\sigma_F = 1.000$, $\rho_{DN} = 0.500$, $\sigma_Z^1 = 2.500$, $\sigma_Z^2 = 2.500$, $m_1 = 0.500$. The full information quantity is marked by a diamond at the left side of the graph. The no information quantity is marked by a square at the right side of the graph.

Figure 4-13: Total Trading Volume in Symmetric Precision and Ordered Precision Cases



Total Trading Volume. Model Parameters: $r = 0.05$, $\alpha = 7.00$, $T = 1.0000$, $a_M = 1.500$, $a_Z^1 = 0.900$, $a_Z^2 = 0.900$, $\sigma_M = 1.200$, $\sigma_D = 0.500$, $\rho_{DN} = 0.500$, $\sigma_Z^1 = 2.500$, $\sigma_Z^2 = 2.500$, $m_1 = 0.500$.

Figure 4-14: Price Impact of Income Shocks in Ordered Precision Case with $\sigma_S^1 = 0$.



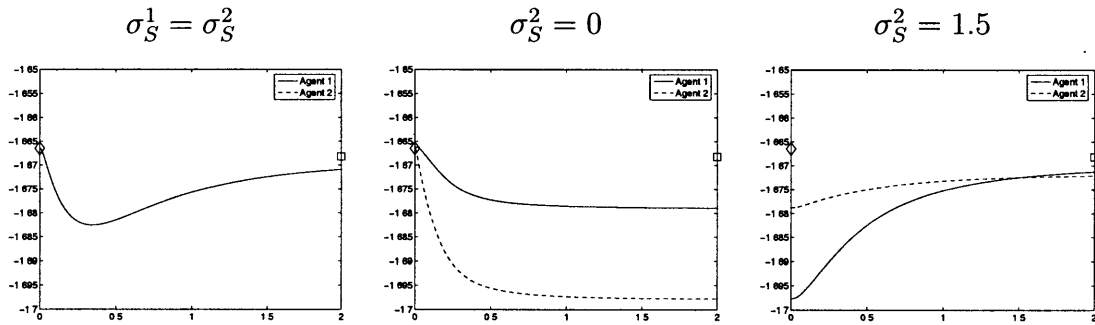
Price Impact of Trades from Income Shocks $\frac{h_{PZ1}(0)}{h_{xZ1}^1(0)}$ and $\frac{h_{PZ2}(0)}{h_{xZ2}^2(0)}$. Model Parameters: $r = 0.05$, $\alpha = 7$, $T = 1$, $a_M = 1.5$, $a_Z^1 = 0.9$, $a_Z^2 = 0.9$, $\sigma_M = 1.2$, $\sigma_D = 0.5$, $\rho_{DN} = 0.5$, $\sigma_Z^1 = 2.5$, $\sigma_Z^2 = 2.5$, $m_1 = 0.5$.

Figure 4-9 shows information amount and information heterogeneity in cases of symmetric precision and ordered precision respectively. With the same σ_S^2 , the forecasting error of group 2 in the ordered precision case is always smaller than that in the symmetric precision case. The information heterogeneity in the symmetric precision case exhibits hump-shape against σ_S^2 . The measure of information heterogeneity in the ordered precision case is increasing as σ_S^2 increases. When σ_S^2 goes to infinity, the ordered precision case converges to hierarchical information and the information heterogeneity is largest.

The intuition that information heterogeneity will amplify the income shocks still holds in the asymmetric information case. The stock price provides *additional* informational value for some investors to estimate stock fundamentals, and therefore investors' income shocks will be negatively correlated with investors' expectation

about the fundamental through the channel of the stock price. This will enhance the impact of income shocks on the stock price due to hedging demand. Consequently, the price discount level and the price variance both tend to rise. Figure ?? and Figure 4-13 re-confirm these intuitions in the asymmetric information set-up. The trading volume is lowest and the price discount level and the price volatility is highest when information is hierarchical, even though the information amount in the hierarchical case is larger than that in the symmetric precision case.

Figure 4-15: Welfare against σ_S^1



Welfare. Model Parameters: $r = 0.05$, $\alpha = 7$, $T = 1$, $a_M = 1.5$, $a_Z^1 = 0.9$, $a_Z^2 = 0.9$, $\sigma_M = 1.2$, $\sigma_D = 0.5$, $\rho_{DN} = 0.5$, $\sigma_Z^1 = 2.5$, $\sigma_Z^2 = 2.5$, $m_1 = 0.5$.

When investors have signals of different precision, the perceived stock return volatility of the better informed investors will be lower than the perceived stock return volatility of the less informed investors. Therefore, when forming his stock holding, the better informed investor will response more to change in his perceived stock return, and hence the belief of the better informed group will have a larger weight in the stock price. This has two consequences. First, since the stock price is more affected by the perceived fundamentals of the better informed investor, the less informed investor can then learn more from stock prices about M_t and prices

partially narrow the information advantage of the better informed trader. Secondly, the less informed investor put more weight on stock price to estimate the stock fundamental than the better informed investor. Therefore, through the expectation of the less informed agent, the price impact of the income shock of the better informed investors will be amplified more than that of the less informed investors.

Figure 4-14 plots the price impact of both investors hedging trades. The hedging trades of better informed investors have a much higher price impact than that of less informed investors. And the price impact of the better informed investors hedging trades is increasing as the signal of the less informed investor get noisier. When the better informed investor makes a selling hedging trade, the price will fall. But the less informed investor will partially interpret the price fall as a bad news for fundamentals and they will also tend to make selling trades based on this perception. The price then falls further.

Therefore, the information advantage of the better informed investor is in deed a double-edged sword. On one hand, better information allow him to better trade against the less informed investor and get positive informational rent. On the other hand, he will find himself harder to hedge against his non-traded income shocks since his hedging trade will have a larger price impact.

Figure 4-15 plots the welfare of both agents as a function of σ_S^1 , in three situations: (i) $\sigma_S^1 = \sigma_S^2$; (ii) $\sigma_S^2 = 0$, and $\sigma_S^2 = 1.5$. There are a few very interesting observations from the graphs. First, when the private signals of the two agents have the same precision, the investor's welfare is *U*-shaped against the signal precision, and the welfare can be even lower than that of the no information case. This is consistent with the intuition that an increase in the degree of information heterogeneity or a decrease in information amount will lead to a welfare loss. Second, when the signal precisions of the two agents are different, the more informed investor has a lower

welfare than the less informed investor. This seemingly counter-intuitive result is a consequence of the price impact effect of the more informed investor's hedging trades discussed above. The more informed investor has greater difficulty hedging his non-traded income due to its price impact and the welfare loss outweighs the welfare gain due to the speculative trading. Therefore, the more informed investor could have lower welfare than the less informed investor.

4.6.2 Momentum and Reversal

Many empirical papers have documented that stock returns exhibit short-run/medium-run Momentum and long-run reversal (e.g. Jegadeesh and Titman (1993)). Behavioral explanations usually interpret momentum as under-reaction to information (for example, Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), and Hong and Stein (1999)). Other papers try to give rational explanations. For example, Liu and Zhang (2008) suggests that a macro-economic risk factor (the growth rate of industrial production) may explain a large portion of the momentum profits. Albuquerque and Miao (2008) explores the role of information in the momentum-reversal phenomenon. In the paper, they are able to generate both momentum and reversal in a model where informed agents have some advanced information about future fundamentals.

In this section, we study the momentum/reversal effect through the auto-correlation of the excess stock return in the asymmetric information setting. The instantaneous excess stock return Q_t follows

$$dQ_t = dP_t - rP_t dt + dD_t. \quad (4.41)$$

Fixed a time horizon $\tau > 0$, the excess return from time t to $t + \tau$ is given by

$$Q_{t:t+\tau} := Q_{t+\tau} - Q_t.$$

We consider the case where investors have full information about M_t , but the income shocks of the two agents are correlated.

Proposition 5. *Assume that there are two groups of investors, i.e. $I = \{1, 2\}$, and investors have full information. Further assume $Y_t \equiv 0$, but $b_1 \cdot b_2 = \rho_{12}\sigma_1\sigma_2$. Then the equilibrium stock price is given by*

$$P_t = -p_0 + \frac{1}{a_M + r} M_t - h_1 Z_t^1 - h_2 Z_t^2, \quad (4.42)$$

where p_0 , h_1 , and h_2 are constants.

The auto-covariance of excess stock return $Cov[Q_{t-\tau:t}, Q_{t:t+\tau}]$ is given by

$$Cov[Q_{t-\tau:t}, Q_{t:t+\tau}] \quad (4.43)$$

$$= -\frac{1}{2a_1} \cdot \{h_1\sigma_1(1 - e^{-a_1\tau})\}^2 \left(1 - \left(\frac{r}{a_1}\right)^2\right) \quad (4.44)$$

$$-\frac{1}{2a_2} \cdot \{h_2\sigma_2(1 - e^{-a_2\tau})\}^2 \left(1 - \left(\frac{r}{a_2}\right)^2\right) \quad (4.45)$$

$$-\frac{\rho_{12}\sigma_1\sigma_2}{a_1 + a_2} h_1 h_2 \left\{ \left(1 - \left(\frac{r}{a_1}\right)^2\right)(1 - e^{-a_1\tau})^2 + \left(1 - \left(\frac{r}{a_2}\right)^2\right)(1 - e^{-a_2\tau})^2 \right\}. \quad (4.46)$$

The first two terms in Equation (4.43) are both negative if $\alpha_1 > r$ and $\alpha_2 > r$. The sign of the third term depends on the sign of the correlation ρ_{12} . It will be positive if $\rho_{12} < 0$. The auto-covariance of stock returns can be decomposed into two components: (i) the covariance between current expected returns and future expected returns; (ii) the covariance between next period un-expected returns and future expected returns. The covariance between current expected returns and future expected returns tends to be positive since income shock are persistent. When the income shocks cross agents are un-correlated, the covariance between next period

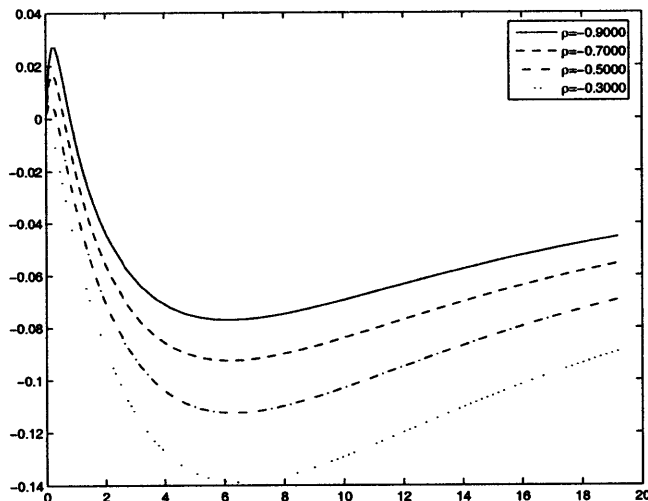
un-expected returns and future expected returns tends to be negative because a current positive income shock will decrease the current return but will raise next period expected return. One can show that when income shocks are not too persistent, in particular, when $a_1 > r$ and $a_2 > r$, the second effect dominate the first effect. Therefore, stock return exhibits negative auto-correlation when $\alpha_1 > r$ and $\alpha_2 > r$. Now assume income shocks of agent 1 and agent 2 are negatively correlated, and the persistence of two shocks are different. The negative correlation between the two shocks will induce a negative unconditional correlation between the two income shocks and thus reduce the the magnitude of the positive covariance between current expected returns and future expected returns by diversification. The negative correlation will also reduce the magnitude of the negative covariance between next period un-expected returns and future expected returns through the same diversification channel. Therefore, the introduction of the negative correlation will push the return auto-correlation higher, and possibly above zero.

Figure 4-16 plots the autocorrelation against the time horizon τ for different correlation between income shocks of both groups. The autocorrelation is positive for small and medium τ , which indicates that the stock return exhibits short-run and medium-run momentum. The autocorrelation is negative for large τ , which indicates that the stock return exhibits long-run reversal.

4.7 Conclusion

This paper studies the asset pricing and welfare impact of information asymmetry. We propose a fairly general model in continuous-time, which incorporates many existing models of Markovian nature (e.g. full information models and hierarchical models), and models on non-Markovian nature(infinite-regress models). Analytical

Figure 4-16: Auto-Correlation of Excess Stock Returns $Corr[Q_{t:t+\tau}, Q_{t-\tau:t}]$ against τ



Auto-Correlation of Excess Stock Returns: $Corr[Q_{t:t+\tau}, Q_{t-\tau:t}]$. Model Parameters: $\tau = 0.05$, $\alpha = 8$, $T = 1$, $a_M = 3$, $a_Z^1 = 2.8$, $a_Z^2 = 0.25$, $\sigma_M = 0.1$, $\sigma_D = 0.5$, $\rho_{DN} = 0.9$, $\sigma_Z^1 = 3.42$, $\sigma_Z^2 = 0.95$, $\sigma_S^1 = 0$, $\sigma_S^2 = 0$, $m_1 = 0.45$.

frameworks are developed to solve the non-Markovian filtering problem and the non-Markovian optimization problem in the continuous-time CARA-Gaussian set-up.

We focus on the impact of information heterogeneity on asset prices and trading activities. We find that information heterogeneity tends to increase the level of price discount, increase the stock price variance, and decrease the trading volume. When investors have different private signals, they form estimate based on both private and public signals. In the presence of information heterogeneity, the stock price provide investors additional informational value beyond their private signal to estimate the stock fundamentals. Since the stock price is also a noisy signal of investors income shocks, the forecasted fundamentals will be negatively correlated with the income shock and hence the price impact of income shocks is amplified. The amplification will raise investors' perceive stock return volatility. Consequently the risk premium

increase, the stock variance increases, and the trading volume falls. Information heterogeneity also tends to magnify the negative auto-correlation of stock returns.

The welfare impact of information heterogeneity is particularly interesting. Information heterogeneity could lead to a welfare loss due to the wealth loss associated with differential information. Moreover the welfare loss associated with the wealth reduction could be so large that it could be better off for all agents if they discard their private information about fundamentals and only use public information, even though their private information is valuable to forecast the fundamentals. Information heterogeneity is costly. Finally, when there is differentiation among investors' signal precision, high information quality could be a double-edged sword for the better informed investors. On one hand, better informed investors could have informational rent when trading with less informed traders. On the other hand, the re-balancing trade of better informed investors will have a larger price impact and investors with better information will find themselves in a difficult position to hedge their non-traded income shocks, which will make them worse off.

Appendix A

Appendix for Chapter Two

A.1 Proof of Proposition 1

We conjecture that the price is a constant P . Standard arguments (e.g. Wang (1993)) imply that the Bellman equation for v_L^i is

$$\begin{aligned} 0 = & \min_{\theta_t^i} r - \rho - r \ln r + r\alpha v_L^i - \alpha r \theta_t^i (\mu_D - rP) \\ & + \alpha \kappa Y_t \frac{\partial v_L^i(Y_t)}{\partial Y} - \frac{1}{2} \alpha \frac{\partial^2 v_L^i(Y_t)}{\partial Y^2} \sigma_Y^2 \\ & + \frac{1}{2} \alpha^2 \left\{ \left(\frac{\partial v_L^i(Y_t)}{\partial Y} \sigma_Y \right)^2 + r^2 Y_t^2 + r^2 \sigma_D^2 (\theta_t^i)^2 + 2(-1)^i r^2 Y_t \sigma_D \rho_{ND} \theta_t^i \right\}. \quad (\text{A.1}) \end{aligned}$$

The first order condition for the optimal stock holding is

$$0 = -(\mu_D - rP) + \alpha r \rho_{ND} \sigma_D (-1)^i Y + \alpha r (\sigma_D^2) \theta_t^i.$$

This together with the market clearing condition immediately gives the optimal stock holding and the equilibrium stock price in the proposition. For the value function

v_L^i , we conjecture that it is of the form

$$v_L^i(Y) = \frac{1}{2}a_L Y^2 + b_L(-1)^i Y + c_L,$$

where a_L , b_L , and c_L are constants. Plug back into the Bellman equation (A.1), and compare coefficients of the quadratic function of Y , we get

$$\begin{aligned} \alpha\sigma_Y^2 a_L^2 + (r + 2\kappa)a_L + ar^2(1 - \rho_{ND}) &= 0, \\ b_L &= \frac{-\alpha r^2 \sigma_D \rho_{ND}}{r + \kappa + \alpha\sigma_Y^2 a_L}, \\ c_L &= \frac{-r + \rho + r \ln r}{\alpha r} + \frac{1}{2r}(a_L - \alpha b_L^2)\sigma_Y^2 + \frac{1}{2}\alpha r \sigma_D^2. \end{aligned}$$

A.2 Case Where $\lambda_H = 0$

The Bellman equations of functions $v_L^i(Y)$ and $v_H^i(\theta, Y)$ and the solution could be quite complicated for general value of the jump intensity. To simplify the Bellman equations and obtain some insights, we consider the special case where there is no recovery after the liquidity crash, i.e. $\lambda_H = 0$. In this case, the impact of the liquidity crash reaches its maximum. The simpler mathematics in the case would allow us to obtain some intuition which would apply to more generally. In particular, the function $v_H^i(Y, \theta)$ can be solved in close form and is in fact quadric in Y_t and θ_t .

Lemma 2. *Assume $\lambda_H = 0$. The function $v_H^i(Y, \theta)$ is given by:*

$$v_H^i(Y, \theta) = c_H + \frac{1}{2}a_H Y^2 + (\mu_D + b_H(-1)^i Y)\theta - \frac{1}{2}\alpha r \sigma_D^2 \theta^2,$$

where a_H , b_H , and c_H are constants satisfying the following equations:

$$\begin{aligned} \frac{1}{2}\alpha\sigma_Y^2(a_H)^2 + \left(\frac{1}{2}r + \kappa\right)a_H + \frac{1}{2}\alpha r^2 &= 0, \\ b_H = \frac{-\alpha r^2 \sigma_D \rho_{ND}}{r + \kappa + \alpha \sigma_Y^2 a_H}, \quad c_H = \frac{1}{r\alpha} \left\{ -r + \rho + r \ln r + \frac{1}{2}\alpha\sigma_Y^2 a_H - \frac{1}{2}\alpha^2\sigma_Y^2 b_H^2 \right\}. \end{aligned}$$

This result can be proved similarly as Proposition 1. Please see Appendix A.1. As a corollary, in this case, the liquidity hedging component $\theta_t^{i,L}$ (see Equation 2.3) is

$$\theta_t^{i,L} = \frac{1}{\alpha r \sigma_D^2} \{ \mu_D - r h_L(Y_t) + b_H (-1)^i Y_t \}.$$

Lemma 3. Assume $\lambda_H = 0$ and $\lambda_L > 0$ small. Then

- (i) $\left| \frac{\partial \theta_t^{i,I}}{\partial Y_t} \right| < \left| \frac{\partial \theta_t^{i,L}}{\partial Y_t} \right|$, if the idiosyncratic shock Y_t is persistent (i.e. κ small), and $\left| \frac{\partial \theta_t^{i,I}}{\partial Y_t} \right| > \left| \frac{\partial \theta_t^{i,L}}{\partial Y_t} \right|$, if the idiosyncratic shock Y_t is transitory (i.e. κ large).
- (ii) $\eta^1(Y_t, \theta_t^1) > \eta^2(Y_t, \theta_t^2)$ if $Y_t > 0$, and, $\eta^1(Y_t, \theta_t^1) < \eta^2(Y_t, \theta_t^2)$ if $Y_t < 0$.

Proof. (i) One can easily show that $|b_H| < \alpha r \rho_{ND} \sigma_D$ if $\kappa < r(\sqrt{\alpha^2 \sigma_Y^2 + \frac{1}{4}} - \frac{1}{2})$, and $|b_H| > \alpha r \rho_{ND} \sigma_D$ if $\kappa > r(\sqrt{\alpha^2 \sigma_Y^2 + \frac{1}{4}} - \frac{1}{2})$. This provides an continuity-based proof when λ_L is very small.

(ii) The continuity-based proof goes as following. In the equilibrium, by market clearing, the optimal stock holding $\theta_t^1 = \bar{\theta} - \hat{\theta}_t$, and $\theta_t^2 = \bar{\theta} + \hat{\theta}_t$. One can easily calculate that

$$\eta^2(Y_t, \theta_t^2) - \eta^1(Y_t, \theta_t^1) = 2\hat{\theta}_t \{ \mu_D - \alpha \sigma_D^2 - r h_L(Y_t) \} + 2\bar{\theta} b_H Y_t - (v_L^2(Y_t) - v_L^1(Y_t)).$$

Notice that if $Y_t > 0$, then $\hat{\theta}_t < 0$. Intuitively, when $Y_t > 0$, $v_L^2(Y_t) - v_L^1(Y_t) < 0$ because investor 2 will have a smaller position in the stock and hence forgo some

of the dividend gains. Assume now λ_L is very small. Then $h_L(Y)$ is very close to P^* , and v_L^i is close to the value function in the frictionless market. It is easy to show that in the frictionless market, v_L^i is quadratic in Y taking the form $\frac{1}{2}a_L Y^2 + b_L(-1)^i Y + c_L$, where $b_L > b_H$ (see proposition 1). Hence if λ_L is very small, the difference $\eta^2(Y_t, \theta_t^2) - \eta^1(Y_t, \theta_t^1)$ is close to $2(b_H - b_{L,0})Y_t$ (notice that the total stock outstanding is normalized to be two so that $\bar{\theta} = 1$). Hence if $Y_t > 0$, then the difference is negative, and vice versa. \square

A.3 Numerical Computation of the Equilibrium

The stock price is given by a function $h_L(Y)$, and for Bellman equations, we need to solve v_L and v_H together. We use standard techniques of discrete Markovian approximation to the continuous time process Y_t . We fix a finite grid \mathcal{S}_Y^h for values of Y , where h is the step size in the grid. We use the following expectation approximation for interior points y on the grid \mathcal{S}_Y^h (e.g. Kushner and Dupuis (2001))

$$Q_h(y) = |\kappa y|h + \sigma_Y^2, \quad \Delta t(y) = \frac{h^2}{Q_h(y)},$$

$$p_+(y) = \frac{(-\kappa y)^+ h + \frac{1}{2}\sigma_Y^2}{Q_h(y)}, \quad p_-(y) = \frac{(-\kappa y)^- h + \frac{1}{2}\sigma_Y^2}{Q_h(y)}.$$

Here the interior point y only communicate with the two nearby points on the grid, $y+h$ and $y-h$, with probability p_+ and p_- respectively. For points at the boundary of \mathcal{S}_Y^h , we use standard reflecting boundary condition so that boundary points are reflected with 100% probability to its nearest interior point.

With this discretization, one can readily re-write the Bellman equations (using the fact that Y_t is independent of $B_{D,t}$ and $B_{N,t}$). With a fixed price process $h_L(Y)$,

the Bellman equations for v_H^i and v_L^i are of the form:

$$e^{-\alpha v_H^i(Y_t, \theta_t^i)} = \frac{1}{1 + \lambda \Delta t} \delta(\Delta t) e^{f_H^i(\theta_t^i, Y_t)} \{p_+ e^{-\alpha v_H^i(x + h_{e1}, \theta_t^i)} + p_- e^{-\alpha v_H^i(x - h_{e1}, \theta_t^i)}\}^{\frac{1}{1+r\Delta t}} + \frac{\lambda \Delta t}{1 + \lambda \Delta t} e^{g_H(\theta, Y_t)},$$

$$e^{-\alpha v_L^i(Y_t)} = \min_{\theta} \frac{1}{1 + \lambda \Delta t} \delta(\Delta t) e^{f_L^i(\theta_t^i, Y_t, P_t)} \cdot \{p_+ e^{-\alpha(v_L^i(x + h_{e1}) + r\theta_t^i(h_L(x + h_{e1}) - h_L(x)))} + p_- e^{-\alpha(v_L^i(x - h_{e1}) + r\theta_t^i(h_L(x - h_{e1}) - h_L(x)))}\}^{\frac{1}{1+r\Delta t}} + \frac{\lambda \Delta t}{1 + \lambda \Delta t} e^{g_L(\theta, Y_t, P_t)},$$

where:

$$\begin{aligned} f_H^i(\theta_t, Y_t) &= \frac{\alpha r \Delta t}{1 + r \Delta t} \left\{ \frac{1}{2} \alpha r \sigma_D^2 \theta_t^2 - (\mu_D - (-1)^i \alpha r Y_t \sigma_D \rho_{ND}) \theta_t + \frac{1}{2} \alpha r Y_t^2 \right\} \\ g_H(\theta_t, Y_t) &= -\alpha(v_L^i(Y_t) + r\theta_t^i h_L(Y_t)), \\ \delta(\Delta t) &= \frac{1 + r \Delta t}{1 + \rho \Delta t} r^{-\frac{r \Delta t}{1+r\Delta t}}, \end{aligned}$$

and

$$\begin{aligned} f_L^i(\theta_t, Y_t, P_t) &= \frac{\alpha r \Delta t}{1 + r \Delta t} \left\{ \frac{1}{2} \alpha r \sigma_D^2 \theta_t^2 - (\mu_D - r P_t - (-1)^i \alpha r Y_t \sigma_D \rho_{ND}) \theta_t + \frac{1}{2} \alpha r Y_t^2 \right\} \\ g_L(\theta_t, Y_t, P_t) &= -\alpha(v_H^i(Y_t, \theta_t^i) - r\theta_t^i h_L(Y_t)). \end{aligned}$$

Since investors 1 and 2 differ only by their sign of the income shock, this symmetry leads to the symmetry between the value functions: $v_H^1(\theta, Y) = v_H^2(\theta, -Y)$ and $v_L^1(Y) = v_L^2(-Y)$. And the market clear condition becomes $\theta^1(Y) + \theta^1(-Y) = 2\bar{\theta}$, where $2\bar{\theta}$ is the total share outstanding. With this symmetry in mind, we only need to solve for the Bellman equation for investor 2. And from now on, we simply use

quantities without investor index to denote quantities of investor 2.

The first order condition for the optimization of stock holding yields

$$\begin{aligned}
h_L(x) = & \frac{\frac{E_x[h_L(y)e^{-\alpha(v_L(y)+r\theta h_L(y))}]}{E_x[e^{-\alpha(v_L(y)+r\theta h_L(y))}]} - (\alpha r \sigma_D^2 \theta - (\mu_D - \alpha r \sigma_D \rho_{ND} x)) \Delta t}{(1 + r \Delta t) + \frac{\lambda_L \Delta t e^{\hat{g}_L - \hat{f}_L}}{\hat{\delta} E_x[e^{-\alpha(v_L(y)+r\theta h_L(y))}]^{\frac{1}{1+r\Delta t}}}} \\
& + \frac{\frac{\lambda_L \Delta t}{r} v'_H e^{\hat{g}_L - \hat{f}_L} (\hat{\delta} E_x[e^{-\alpha(v_L(y)+r\theta h_L(y))}]^{\frac{1}{1+r\Delta t}})^{-1}}{(1 + r \Delta t) + \frac{\lambda_L \Delta t e^{\hat{g}_L - \hat{f}_L}}{\hat{\delta} E_x[e^{-\alpha(v_L(y)+r\theta h_L(y))}]^{\frac{1}{1+r\Delta t}}}} \quad (A.2)
\end{aligned}$$

where

$$\begin{aligned}
\hat{f}_L(\theta_t, Y_t, P_t) &= \frac{\alpha r \Delta t}{1 + r \Delta t} \left\{ \frac{1}{2} \alpha r \sigma_D^2 \theta_t^2 - (\mu_D - \alpha r Y_t \sigma_D \rho_{ND}) \theta_t + \frac{1}{2} \alpha r Y_t^2 \right\} \\
\hat{g}_L(\theta_t, Y_t, P_t) &= -\alpha (v_H(Y_t, \theta_t)), \\
\hat{\delta} &= \frac{1}{1 + \rho \Delta t} r^{-\frac{r \Delta t}{1+r\Delta t}}.
\end{aligned}$$

Note if $\lambda_L = 0$, this equation becomes standard and the price is then the present value of the future cash flow with proper discount rate. The market clearing is $\theta(Y) + \theta(-Y) = 2\bar{\theta}$.

The general idea of solving the equilibrium is to first fix the price function $h_L(Y)$, and compute the solutions v_L and v_H to the Bellman equations, and the optimal stock holdings $\theta(Y)$. Then find the pricing function such that market clears. In practice, the fixed point problem of the pricing function will be time consuming. So we use a modified iteration method, where we update the stock price, value functions and the stock holdings simultaneously at each iteration. The idea is as follows. Notice that if we fix the next period stock price function and value function, equation (A.2) can be used to compute the market clearing price (price on the left hand side of equation (A.2)) and quantities, given the next period quantities (on the right hand side of

equation (A.2)). This way, we could use iterations more efficiently.

More precisely, we use the following iteration scheme.

Step 0: Guess an initial pricing function $P = h_L^{(0)}(Y)$ and value function $v_L^{(0)}(Y)$, and set $n = 0$.

Step 1. For the pricing function $P = h_L^{(n)}(Y)$ and value function $v_L^{(n)}(Y)$, compute the value function $v_H^{(n)}(\theta, Y)$ using the value iteration for v_H (we could use $v_H^{(n-1)}$ as an initial data for the iteration).

Step 2. Find the market clearing price $\hat{P} = \hat{h}_L^{(n)}(Y)$ given the next period price and value function, and use this price to update the value function to get $\hat{v}_L^{(n)}(Y)$. Set $h_L^{(n+1)}(Y) = \hat{h}_L^{(n)}(Y)$, and $v_L^{(n+1)}(Y) = \hat{v}_L^{(n)}(Y)$, and then $n = n + 1$.

Step 3. If $\|v_L^{(n)}(Y) - v_L^{(n+1)}(Y)\|$ and $\|h_L^{(n)}(Y) - h_L^{(n+1)}(Y)\|$ are smaller than some pre-specified error, stop. Otherwise go to step 1.

Appendix B

Appendix for Chapter Four

B.1 Non-Markovian Filtering

In this appendix, we study a general non-Markovian filtering problem. Let (Σ, \mathcal{F}, P) be a probability space and $\{B_t\}_{t=-\infty}^{+\infty}$ be a standard m -dimension Brownian motion with standard augmented filtration $\{\mathcal{F}_t\}_{t=-\infty}^{+\infty}$. X_t is a m -dimensional adapted process with the following dynamics:

$$dX_t = \mu_X X_t dt + b_X dB_t,$$

where μ_X is a $m \times m$ constant matrix, and σ_X is a $m \times m$ non-degenerate constant matrix.

Assume $T > 0$ is a fixed number and S_t is a n -dimensional ($n < m$) adapted process w.r.t. $\{\mathcal{F}_t\}_{t=-\infty}^{+\infty}$ with the following dynamics:

$$S_t = \int_{t-T}^t h_S(t-s) dB_s + g_S X_{t-T},$$

where $h_S(z)$ (for $0 \leq z \leq T$) is a $n \times m$ matrix-valued functions and g_S is a $n \times m$ constant matrix.¹

Let $\hat{\mathcal{F}}_t$ be the filtration generated by S_t and X_{t-T} , i.e.²

$$\hat{\mathcal{F}}_t \equiv \sigma(\mathcal{F}_t^S \cup \mathcal{F}_{t-T}^X).$$

Define the filtered state vector:

$$\hat{X}_t = E[X_t | \hat{\mathcal{F}}_t]. \tag{B.1}$$

Before we state the main theorem in this section, we need to define a few constants first. First define

$$b = h_S(0)^\top \{h_S(0)h_S(0)^\top\}^{-1} h_S(0). \tag{B.2}$$

Let I_n be the identity matrix of dimension n . Then $I_n - b$ is symmetric and admits the following decomposition

$$I_n - b = k^\top k, \tag{B.3}$$

where k is a $\text{rank}(I_n - b) \times n$ matrix with full row rank. Also define

$$b_S = h_S(0)h_S^\top(0), \tag{B.4}$$

¹It is easily seen that S_t is an Ito Process if $h_S(T) = g_S b_X$. We will assume this equation throughout this section.

²We use \mathcal{F}^Y to denote the filtration generated by a process Y .

and

$$b_k = kk^\top. \quad (\text{B.5})$$

Now we state the main result of this section.

Theorem 7. *Assume the matrix b_S defined in (B.4) has full rank. Then*

(a). *The dynamics of the filtered state vector \hat{X}_t (defined in (B.1)) under the filtration $\{\hat{\mathcal{F}}_t\}_{t=-\infty}^{+\infty}$ is*

$$d\hat{X}_t = \mu_X \hat{X}_t dt + \hat{b}_X d\hat{B}_t,$$

where $\{\hat{B}_t\}_{t=-\infty}^{\infty}$ is an $n + \text{rank}(k)$ dimensional Brownian Motion adapted to the filtration $\{\hat{\mathcal{F}}_t\}_{t=-\infty}^{+\infty}$, and \hat{b}_X is a constant $m \times (n + \text{rank}(k))$ matrix.

(b). *The dynamics of S_t under the filtration $\{\hat{\mathcal{F}}_t\}_{t=-\infty}^{+\infty}$ is*

$$S_t = \int_{t-T}^t \hat{h}_S(t-s) d\hat{B}_s + \hat{g}_S \hat{X}_{t-T},$$

where $\hat{h}_S(\cdot)$ is a $n \times (n + \text{rank}(k))$ matrix valued function, and $\hat{g}_S = g_S$.

In the rest of the section, we sketch the proof to the theorem above.

Lemma 4. (a). *The process S_t can be written as an Ito form:*

$$dS_t = \mu_{S_t} dt + h_S(0) b_X dB_t,$$

where

$$\mu_{S_t} = \int_{t-T}^t \dot{h}_S(t-v) dB_v + g_S \mu_X X_{t-T}.$$

(b). Fixed initial time T_0 , define the process

$$\eta_t^S = S_t - S_{T_0} - \int_{T_0}^t E[\mu_{S_u} | \hat{\mathcal{F}}] du. \quad (\text{B.6})$$

Then $\{\eta_t^S\}_{t=T_0}^\infty$ is a $\hat{\mathcal{F}}$ -martingale.

Proof. Part (a) is a direct consequence Ito's lemma. Using the Law of iterated expectations, we can show that $E[\eta_{t_2}^S - \eta_{t_1}^S | \hat{\mathcal{F}}_{t_1}] = 0$, for any $T_0 \leq t_1 \leq t_2$, which readily establishes part (b). \square

Lemma 5. For a fixed initial time T_0 . There exists a $\hat{\mathcal{F}}$ -adapted martingale $\{\eta_t^B\}_{t=T_0+T}^\infty$ such that (heuristically)

$$d\eta_t^B = dB_{t-T} - E[dB_{t-T} | \hat{\mathcal{F}}_t]. \quad (\text{B.7})$$

Proof. Equation (B.7) gives only a heuristical definition. More rigorously, let $\{\{t_i^{(n)}\}_{i=0}^\infty\}_{n=0}^\infty$ be a set of nested partitions of $[T_0, \infty)$. For a fixed $n \geq 0$, define the process $\{\eta_t^n\}_{t=T_0}^\infty$ by

$$\begin{aligned} \eta_t^n &= \sum_{i=0}^{N-1} \{\Delta B_{t_i^{(n)}-T} - E[\Delta B_{t_i^{(n)}-T} | \hat{\mathcal{F}}_{t_i^{(n)}}]\} \\ &\quad + B_{t-T} - B_{t_i^{(N)}-T} - E[B_{t-T} - B_{t_i^{(N)}-T} | \hat{\mathcal{F}}_{t_i^{(n)}}], \end{aligned} \quad (\text{B.8})$$

where N is the largest integers smaller than $(t - T_0)/\Delta t^{(n)} - 1$, and $\Delta B_{t_i^{(n)}-T} = B_{t_{i+1}^{(n)}-T} - B_{t_i^{(n)}-T}$. One can prove that as $n \rightarrow +\infty$, the sequence of processes $\{\eta_t^n\}_{n=1}^\infty$ converges to a $\hat{\mathcal{F}}$ -adapted process $\{\eta_t^B\}_{t=T_0}^\infty$ and $\{\eta_t^B\}_{t=T_0+T}^\infty$ is a $\hat{\mathcal{F}}$ -adapted martingale. \square

Remark 8. From now on, we will let T_0 goes to $-\infty$. Therefore we have two $\hat{\mathcal{F}}$ -martingale processes $\{\eta_t^S\}_{t=-\infty}^{+\infty}$ and $\{\eta_t^B\}_{t=-\infty}^{+\infty}$, which, together, generate the filtration $\hat{\mathcal{F}}$.³

One of the key to the solution of the filtering problem is a linear Fredholm integral equation of second kind. We define the $n \times n$ matrix-valued kernel function $K(\cdot, \cdot)$ below (Note: All the functions are defined either on $[0, T]$, or $[0, T] \times [0, T]$):

$$K(s, s') = \begin{cases} \int_{t-T}^{s'} \dot{h}_S(s-v) \dot{h}_S^\top(s'-v) dv + \dot{h}_S(s-s') \dot{h}_S^\top(0) & \text{if } s > s' \\ \int_{t-T}^s \dot{h}_S(s-v) \dot{h}_S^\top(s'-v) dv + h_S(0) \dot{h}_S(s'-s) & \text{if } s' \geq s. \end{cases} \quad (\text{B.9})$$

Notice that the kernel $K(s, s')$ is continuous away from the diagonal line $s = s'$. Under mild technical condition, the Fredholm integral equation of second kind with $K(\cdot, \cdot)$ as kernel has a solution.

Let $G_S(s)$ ($s \in [0, T]$) be the solution to the following Fredholm integral equation of second kind:

$$\begin{aligned} & \int_0^T b_S^{-1} K(s', s) G_S^\top(T-s) ds + G_S^\top(t-s') \\ &= b_S^{-1} \left\{ \int_{t-T}^{s'} \dot{h}_S(t-v) \dot{h}_S^\top(s'-v) dv + \dot{h}_S(t-s') h_S(0)^\top \right\}^\top. \end{aligned} \quad (\text{B.10})$$

And let $G_B(s)$ ($s \in [0, T]$) to be the solution to the following integral equation:

$$\begin{aligned} & \int_0^T b_S^{-1} K(s', s) G_B^\top(T-s) ds + G_B^\top(T-s) \\ &= b_S^{-1} \dot{h}(s' - (t-T)) k^\top \end{aligned} \quad (\text{B.11})$$

³More rigorously, we should specify the initial data $\eta_{T_0}^S$ and $\eta_{T_0}^B$ and the initial σ -algebra at time T_0 . But if T_0 is negative enough, this will not affect the resulting filtration.

We now are in the position to state one of the key results of this section.

Proposition 6. (a). *The $\hat{\mathcal{F}}$ -martingales $\{\eta_t^S\}_{t=-\infty}^{+\infty}$ and $\{\eta_t^B\}_{t=-\infty}^{+\infty}$ defined in Lemma 4, Lemma 5 and Remark 8, satisfy the following equation:*

$$d\eta_t^S = \left\{ \int_{t-T}^t \Lambda_S(t-s) dB_s \right\} dt + h_S(0) dB_t, \quad (\text{B.12})$$

$$d\eta_t^B = k^\top \left\{ \int_{t-T}^t \Lambda_B(t-s) dB_s \right\} dt + k dB_{t-T}. \quad (\text{B.13})$$

where $\Lambda_S(u)$ and $\Lambda_B(u)$ ($u \in [0, T]$) are given by

$$\begin{aligned} \Lambda_S(u) &= h_{\mu_S}(u) - G_S(u) h_S(0) \sigma_X - \int_0^u G_S(v) \dot{h}(u-v) dv, \\ \Lambda_B(u) &= -G_B(u) h_S(0) \sigma_X - \int_0^u G_B(v) \dot{h}(u-v) dv. \end{aligned}$$

(b) Define $\{\hat{B}_t^S\}_{t=-\infty}^{+\infty}$ and $\{\hat{B}_t^B\}_{t=-\infty}^{+\infty}$ by

$$\begin{aligned} \hat{B}_t^S &= b_S^{-\frac{1}{2}} \cdot \eta_t^S, \\ \hat{B}_t^B &= b_k^{-\frac{1}{2}} \cdot \left\{ \int_{t-T}^t \Lambda_B(t-s) dB_s \right\} dt + k dB_{t-T}. \end{aligned}$$

Then the processes

$$\hat{B}_t = \begin{pmatrix} \hat{B}_t^S \\ \hat{B}_t^B \end{pmatrix}.$$

is a $(n + \text{rank}(k))$ -dimensional Brownian motion under the filtration $\hat{\mathcal{F}}$.

The proof is a straightforward computation, using the fact that all the processes

are Gaussian (for a reference, see Liptster and Shiryaev (2004)). A detailed proof is available upon request.

To get the filtered process for a general process with T -lag-revelation, we notice that X_{t-T} is $\hat{\mathcal{F}}_t$ measurable, not $\hat{\mathcal{F}}_{t-T}$ measurable. So we need to decompose X_{t-T} into the sum of a $\hat{\mathcal{F}}_{t-T}$ measurable part and the $\hat{\mathcal{F}}_{t-T}$ -orthogonal part. The following lemma gives the decomposition result.

Lemma 6. *We can decompose X_{t-T} as*

$$X_{t-T} = \hat{X}_{t-T} + \int_{t-T}^t \hat{h}_{XS}(t-s) d\hat{B}_s^S + \int_{t-T}^t \hat{h}_{XB}(t-s) d\hat{B}_s^B, \quad (\text{B.14})$$

where $\{\hat{X}_t\}$ is filtered state vector given by (B.1), and $\hat{h}_{XS}(\cdot)$ and $\hat{h}_{XB}(\cdot)$ are defined as:

$$\hat{h}_{XS}(s) = \int_0^s e^{\mu_X v} b_X \Lambda_S^\top (T-s+v) b_S^{-\frac{1}{2}} dv \quad (\text{B.15})$$

$$\hat{h}_{XB}(s) = \left\{ \int_0^s e^{\mu_X v} b_X \Lambda_B^\top (T-s+v) dv + e^{\mu_X s} b_X k^\top \right\} b_k^{-\frac{1}{2}}. \quad (\text{B.16})$$

Proof. Since \hat{X}_{t-T} is orthogonal to $\{d\hat{B}_s^S\}_{s=t-T}^t$, the coefficients in (B.14) are given (heuristically) by:

$$\hat{h}_{XS}(t-s) ds = \text{cov}(X_{t-T}, d\hat{B}_s^S), \quad (\text{B.17})$$

$$\hat{h}_{XB}(t-s) ds = \text{cov}(X_{t-T}, d\hat{B}_s^B). \quad (\text{B.18})$$

The result of the lemma then follows. □

The following is then straightforward.

Lemma 7. *The dynamic of S_t under $\{\hat{\mathcal{F}}_t\}_{t=-\infty}^{\infty}$ is given by*

$$S_t = \int_{t-T}^t \hat{h}_{SS}(t-s) d\hat{B}_s^S + \int_{t-T}^t \hat{h}_{SB}(t-s) d\hat{B}_s^B + g_S \hat{X}_{t-T}, \quad (\text{B.19})$$

where $\hat{h}_{SS}(\cdot)$, $\hat{h}_{SB}(\cdot)$ are given by:

$$\hat{h}_{SS}(u) = \int_u^T h_S(v) \Lambda_S^\top(v-u) b_S^{-1/2} dv + h_S(u) b_X^\top h_S^\top(0) b_S^{-1/2} + g \hat{h}_{XS}(u).$$

$$\hat{h}_{SB}(u) = \int_u^T h_S(v) \Lambda_B^\top(v-u) b_k^{-1/2} dv + g \hat{h}_{XB}(u).$$

The above lemma gives the proof of Part (b) of the main result Theorem 7.

The following lemma will prove the Part (a) of Theorem 7.

Lemma 8. *Under the filtration $\{\hat{\mathcal{F}}_t\}_{t=-\infty}^{+\infty}$, the filtered state vector \hat{X}_t is still Markovian. It satisfies the linear SDE:*

$$d\hat{X}_t = \mu_X d\hat{X}_t dt + \hat{b}_{XS} d\hat{B}_t^S + \hat{b}_{XB} d\hat{B}_t^B.$$

where

$$\hat{b}_{XS} = b_X b_X^\top h_S^\top b_S^{-\frac{1}{2}} + \int_0^T e^{\mu_X(v)} b_X \Lambda_S^\top(v) b_S^{-\frac{1}{2}} dv$$

$$\hat{b}_{XB} = e^{\mu_X T} k^\top b_k^{-\frac{1}{2}} + \int_0^T e^{\mu_X(v)} b_X \Lambda_B^\top(v) b_k^{-\frac{1}{2}} dv$$

Proof. Since $\hat{X}_t = E[X_t | \hat{\mathcal{F}}_t]$. We can compute the dynamic of $\{\hat{X}_t\}$ using the standard

Kalman filtering method. Notice

$$\begin{aligned}
d(E[X_t|\hat{\mathcal{F}}_t]) &= E[dX_t|\hat{\mathcal{F}}_t] + E[X_{t+dt}|d\hat{\mathcal{F}}_t] \\
&= \mu_X \hat{X}_t dt + E[X_{t+dt}|d\hat{\mathcal{F}}_t] \\
&= \mu_X \hat{X}_t dt + E[X_{t+dt}|d\hat{B}_t^S, d\hat{B}_t^B]
\end{aligned}$$

The second term now can be computed using standard linear projection. \square

B.2 A General Optimization Theorem

In this section, we solve a general optimization problem with non-Markovian price process. The agent has exponential utility over consumption stream $\{c_t\}_{t=0}^\infty$, with time discount ρ . In particular, the agent solves:

$$\max_{\{c_t, x_t\}_{t=0}^\infty} E_0\left[\int_{t=0}^\infty -e^{-\rho t - \alpha c_t} dt\right], \tag{B.20}$$

and the wealth of the agent evolves according to:

$$dw_t = rw_t dt - c_t dt + x_t dQ_t + \bar{\Theta}_t dF_t, \tag{B.21}$$

where x_t is the number of stock that agent i holds at time t , $\bar{\Theta}_t$ is the income process, dQ_t is the excess return and dF_t is the value of a unit income at time t .

Let \hat{B}_t be the Brownian motions of the agent's filtration, and assume

$$\begin{aligned}
dQ_t &= \mu_{Q,t} dt + b_{Q,t} d\hat{B}_t, \\
dF_t &= \mu_{F,t} dt + b_{F,t} d\hat{B}_t.
\end{aligned} \tag{B.22}$$

We state one of the main result of this section.

Theorem 9. *Assume agent has CARA utility (B.20) with risk aversion α and time discount ρ . Assume the wealth process is governed by (B.21), where the dynamics of dQ_t , and dF_t are given by (B.22). Then the following hold:*

(i) *The value function can be written as:*

$$-e^{-\rho t - \alpha(rw_t + V_t)}.$$

The process V_t follows:

$$dV_t = \mu_{V,t}dt + b_{V,t}dB_t,$$

where V_t , $\mu_{V,t}$, and $b_{V,t}$ satisfy the Bellman equation:

$$0 = -\frac{1}{2b_{Q,t}b_{Q,t}^\top}(\alpha r \bar{\Theta}_t b_{Q,t} b_{F,t}^\top - \mu_{Q,t} + \alpha b_{Q,t} b_{V,t}^\top)^2 \quad (\text{B.23})$$

$$+\frac{1}{2}\alpha^2(r\bar{\Theta}_t b_{F,t} + b_{V,t})(r\bar{\Theta}_t b_{F,t} + b_{V,t})^\top \quad (\text{B.24})$$

$$+r\alpha V_t - \alpha r \bar{\Theta}_t \mu_{F,t} - \alpha \mu_{V,t} \quad (\text{B.25})$$

$$+r - \rho - r \ln r. \quad (\text{B.26})$$

(ii) *The optimal consumption c_t is*

$$c_t = rw_t + V_t - \frac{\ln r}{\alpha}.$$

(iii) *The optimal stock holding x_t is*

$$x_t = \frac{\mu_{Q,t}}{r\alpha b_{Q,t}b_{Q,t}^\top} - \frac{\bar{\Theta}_t b_{Q,t}b_{F,t}^\top}{b_{Q,t}b_{Q,t}^\top} - \frac{b_{Q,t}b_{V,t}^\top}{rb_{Q,t}b_{Q,t}^\top}. \quad (\text{B.27})$$

Proof. Let $-\hat{J}_t$ denote the value function, then

$$\hat{J}_t = \min_{c,x} E_t[e^{-\rho t - \alpha c} dt + \hat{J}_{t+dt}].$$

Therefore

$$0 = \min_{c,x} E_t^i[e^{-\rho t - \alpha c} dt + d\hat{J}_t].$$

Normalize \hat{J}_t : let

$$J_t = e^{\rho t} \hat{J}_t.$$

Then

$$0 = \min_{c,x} E_t[e^{-\alpha c} dt - \rho J_t dt + dJ_t].$$

The exponential nature of utility suggests that the value function can be written as:

$$J_t = e^{-\alpha(rw_t + V_t)},$$

where V_t does not contain w_t . And for general purpose, write

$$dw_t = (rw_t - c_t)dt + d\hat{w}_t.$$

where $d\hat{w}_t$ does not contain w_t .

Using these specifications, we have

$$\begin{aligned} dJ_t &= J_t(-r\alpha dw_t - \alpha dV_t) + \frac{1}{2}J_t d\langle -r\alpha w_t - \alpha V_t \rangle \\ &= -\alpha r(rw_t - c_t)J_t dt + J_t(-\alpha r d\hat{w}_t - \alpha dV_t + \frac{1}{2}\alpha^2 d\langle r\hat{w}_t + V_t \rangle). \end{aligned}$$

Bellman equation is now

$$\begin{aligned} 0 = \min_{c,x} \{ & e^{-\alpha c} dt + r\alpha(c_t - rw_t)J_t dt \\ & + J_t E_t(-\rho - \alpha r d\hat{w}_t - \alpha dV_t + \frac{1}{2}\alpha^2 d\langle r\hat{w}_t + V_t \rangle). \end{aligned}$$

First order condition for c gives:

$$\alpha e^{-\alpha c} = r\alpha J_t.$$

So

$$c = rw_t + V_t - \frac{\ln r}{\alpha}.$$

and

$$e^{-\alpha c} + r\alpha(c_t - rw_t)J_t = J_t(r + r\alpha V_t - r \ln r).$$

And the Bellman equations becomes:

$$0 = \min_x E_t[r + r\alpha V_t - r \ln r - \rho - \alpha r d\hat{w}_t - \alpha dV_t + \frac{1}{2}\alpha^2 d\langle r\hat{w}_t + V_t \rangle].$$

Now under the specification of $d\hat{w}_t$:

$$d\hat{w}_t = x_t dQ_t + \bar{\Theta}_t dN_t,$$

we have

$$d\hat{w}_t = (x_t \mu_{Q,t} + \bar{\Theta}_t \mu_{N,t}) dt + (x_t b_{Q,t} + \bar{\Theta}_t b_{N,t}) d\hat{B}_t.$$

Now Bellman equation is

$$\begin{aligned} 0 = & \min_x \{ r - r \ln r - \rho + r\alpha V_t \\ & - \alpha r x_t \mu_{Q,t} - \alpha r \bar{\Theta}_t \mu_{N,t} - \alpha \mu_{V,t} \\ & + \frac{1}{2} \alpha^2 (r x_t b_{Q,t} + r \bar{\Theta}_t b_{N,t} + b_{V,t}) (r x_t b_{Q,t} + r \bar{\Theta}_t b_{N,t} + b_{V,t})^\top \}. \end{aligned}$$

Re-write in terms of polynomial of x ,

$$\begin{aligned} 0 = & \min_{x_t} \frac{1}{2} \alpha^2 r^2 (x_t)^2 b_{Q,t} b_{Q,t}^\top + r x_t (\alpha^2 b_{Q,t} (r \bar{\Theta}_t b_{N,t} + b_{V,t})^\top - \alpha \mu_{Q,t}) \\ & + \frac{1}{2} \alpha^2 (r \bar{\Theta}_t b_{N,t} + b_{V,t}) (r \bar{\Theta}_t b_{N,t} + b_{V,t})^\top \\ & + r\alpha V_t - \alpha r \bar{\Theta}_t \mu_{N,t} - \alpha \mu_{V,t} \\ & + r - \rho - r \ln r. \end{aligned}$$

First order condition for x_t implies

$$x_t = \frac{\mu_{Q,t}}{\alpha r b_{Q,t} b_{Q,t}^\top} - \frac{b_{Q,t} (r \bar{\Theta}_t b_{N,t} + b_{V,t})^\top}{r b_{Q,t} b_{Q,t}^\top}.$$

The first term is the myopic demand, the second term is the hedging demand.

Plug it back into the Bellman equation, we get

$$\begin{aligned}
0 &= -\frac{1}{2b_{Q,t}b_{Q,t}^\top}(\alpha b_{Q,t}(r\bar{\Theta}_tb_{N,t} + b_{V,t})^\top - \mu_{Q,t})^2 \\
&\quad + \frac{1}{2}\alpha^2(r\bar{\Theta}_tb_{N,t} + b_{V,t})(r\bar{\Theta}_tb_{N,t} + b_{V,t})^\top \\
&\quad + r\alpha V_t - \alpha r\bar{\Theta}_t\mu_{N,t} - \alpha\mu_{V,t} \\
&\quad + r - \rho - r \ln r.
\end{aligned}$$

This complete the proof. □

B.3 Optimization Problem of Agent i

In this section, we study the optimization problem of agent i .

Agent i solves:

Optimization Problem (P):

$$\begin{aligned}
&\max_{\{x_t^i, c_t^i\}_{t=0}^\infty} E\left[\int_0^\infty -e^{-\rho t - c_t^i} dt \mid \mathcal{F}_0^i\right], \\
s.t. \quad &dw_t^i = (rw_t^i - c_t^i)dt + x_t^i dQ_t + \Theta_t^i dF_t,
\end{aligned} \tag{B.28}$$

where x_t^i is the stock holding, $\Theta_t^i = Y_t + Z_t^i$ is the income, dQ_t is the excess stock return, and dF_t is unit income value.

Now we state the key linearity assumptions about Θ_t^i , dQ_t , and dF_t . Assume:

$$\begin{aligned} dQ_t &= \left\{ \int_{t-T}^t \hat{h}_Q(t-s) d\hat{B}_s^i + g_Q \hat{X}_{t-T}^i \right\} dt + b_Q d\hat{B}_t^i, \\ dF_t &= \left\{ \int_{t-T}^t \hat{h}_F(t-s) d\hat{B}_s^i + g_F \hat{X}_{t-T}^i \right\} dt + b_F d\hat{B}_t^i, \\ \Theta_t^i &= \int_{t-T}^t \hat{h}_{\Theta^i}(t-s) d\hat{B}_s^i + g_{\Theta^i} \hat{X}_{t-T}^i, \end{aligned}$$

where \hat{X}_t^i is a Markovian state vector adapted to the filtration $\{\mathcal{F}_t^i\}_{t=-\infty}^{+\infty}$ with dynamics:

$$d\hat{X}_t^i = \mu_X \hat{X}_t^i dt + \hat{b}_X d\hat{B}_t^i. \quad (\text{B.29})$$

Now we apply the general optimization result Theorem 9. Conjecture that the process V_t takes the form: ⁴

$$\begin{aligned} V_t &= \int_{t-T}^t \left(\int_{t-T}^s v_{11}(t-s, t-s') d\hat{B}_{s'}^i \right)^\top d\hat{B}_s^i \\ &\quad + 2\hat{X}_{t-T}^{i\top} \int_{t-T}^t v_{12}(t-s) d\hat{B}_s^i + \hat{X}_{t-T}^{i\top} v_{22} \hat{X}_{t-T}^i \\ &\quad + \int_{t-T}^t v_1(t-s) d\hat{B}_s^i + v_2 \hat{X}_{t-T}^i + v_0. \end{aligned} \quad (\text{B.30})$$

Lemma 9. (a). *The process V_t is a Ito process with respect to $\{\mathcal{F}_t^i\}_{t=-\infty}^{+\infty}$ if and only if*

$$2\hat{b}_X^\top v_{12}(s) = v_{11}^\top(s, T), \quad \forall s \in [0, T] \quad (\text{B.31})$$

⁴All the functions in this equations are defined either on $[0, T]$ or $[0, T] \times [0, T]$.

$$v_{22}\hat{b}_X = v_{12}(T) \quad (\text{B.32})$$

$$v_2\hat{b}_X = v_1(T) \quad (\text{B.33})$$

(b). Under the assumption in part (a),

$$dV_t = \mu_{V,t}dt + b_{V,t}d\hat{B}_t^i. \quad (\text{B.34})$$

where $\mu_{V,t}$ and $b_{V,t}$ are given by

$$\begin{aligned} \mu_{V,t} = & \int_{t-T}^t \left(\int_{t-T}^s \dot{v}_{11}(t-s, t-u) d\hat{B}_u^i \right)^\top d\hat{B}_s^i \\ & + 2(\hat{X}_{t-T}^i)^\top \cdot \int_{t-T}^t \{ \mu_X^\top v_{12}(t-s) + \dot{v}_{12}(t-s) \} d\hat{B}_s^i \\ & + 2(\hat{X}_{t-T}^i)^\top \mu_X^\top v_{22} \hat{X}_{t-T}^i \\ & + \int_{t-T}^t \dot{v}_1(t-s) d\hat{B}_s^i + v_2 \mu_X \hat{X}_{t-T}^i \\ & + \text{tr}(\hat{b}_X^\top v_{22} \hat{b}_X) - 2\text{tr}(\hat{b}_X v_{12}^\top(T)) \end{aligned}$$

$$b_{V,t} = \left\{ \int_{t-T}^t v_{11}(0, t-s) d\hat{B}_s^i \right\}^\top + 2(\hat{X}_{t-T}^i)^\top v_{12}(0) + v_1(0), \quad (\text{B.35})$$

where (with a slight abuse of notation) $\dot{v}_{11}(x, y)$ is defined as:

$$\dot{v}_{11}(x, y) = \frac{\partial v_{11}}{\partial x}(x, y) + \frac{\partial v_{11}}{\partial y}(x, y).$$

Proof. Part (b) is a direct consequence of Ito's calculation. \square

Now apply the Bellman equation in Theorem 9 to the conjectured form of V_t , we

can get the following characterization of the process V_t :

Proposition 7. *The functions $v_{11}(\cdot, \cdot)$, $v_{12}(\cdot)$, $v_1(\cdot)$, and the constant vectors v_{22} , v_2 , and v_0 satisfies the following system: S1-S6 and boundary condition B1-B3.*

(S1): For all $0 \leq x \leq y \leq T$:

$$\begin{aligned}
& \frac{1}{b_Q b_Q^\top} \{ \hat{h}_Q(x) - \alpha b_Q (r b_F^\top h_{\Theta^i}(x) + v_{11}(0, x)) \}^\top \{ \hat{h}_Q(y) - b_Q (r b_F^\top h_{\Theta^i}(x) + v_{11}(0, y)) \} \\
& - \alpha^2 \{ r b_F^\top h_{\Theta^i}(x) + v_{11}(0, x) \}^\top \{ r b_F^\top h_{\Theta^i}(y) + v_{11}(0, y) \} \\
& + \alpha \dot{v}_{11}(x, y) + \alpha r \{ h_{\Theta^i}^\top(x) h_F(y) + h_F^\top(x) h_{\Theta^i}(y) \} - r \alpha v_{11}(x, y) \\
= & 0. \tag{B.36}
\end{aligned}$$

(S2): For any $0 \leq x \leq T$:

$$\begin{aligned}
& \frac{1}{b_Q b_Q^\top} \{ g_Q - \alpha b_Q (2v_{12}(0)^\top + r b_F^\top g_{\Theta^i}) \}^\top \{ \hat{h}_Q(x) - \alpha b_Q (v_{11}(0, x) + r b_F^\top h_{\Theta^i}(x)) \} \\
& - \alpha^2 \{ 2v_{12}(0) + r g_{\Theta^i}^\top b_F \} \{ v_{11}(0, x) + r b_F^\top h_{\Theta^i}(x) \} + 2\alpha \{ \mu_X^\top v_{12}(x) + \dot{v}_{12}(x) \} \\
& + \alpha r (g_{\Theta^i}^\top h_F(x) + g_F^\top h_{\Theta^i}(x)) - 2\alpha r v_{12}(x) \\
= & 0. \tag{B.37}
\end{aligned}$$

(S3):

$$\begin{aligned}
& \frac{1}{2b_Q b_Q^\top} \{ g_Q - \alpha b_Q (2v_{12}(0) + r g_{\Theta^i}^\top b_F)^\top \}^\top \{ g_Q - \alpha b_Q (v_{12}(0) + r g_{\Theta^i}^\top b_F)^\top \} \\
& - \frac{1}{2} \alpha^2 (2v_{12}(0) + r g_{\Theta^i}^\top b_F) (2v_{12}(0) + r g_{\Theta^i}^\top b_F)^\top \\
& + \alpha \{ \mu_X^\top v_{22} + v_{22}^\top \mu_X \} + \frac{1}{2} \alpha r \{ g_{\Theta^i} g_F + g_F^\top g_{\Theta^i} \} - r \alpha v_{22} \\
= & 0. \tag{B.38}
\end{aligned}$$

(S4): For any $0 \leq x \leq T$

$$\begin{aligned}
& \frac{1}{b_Q b_Q^\top} (-b_Q v_1^\top(0) - r p_0) \{ \hat{h}_Q(x) - \alpha b_Q (v_{11}(0, x) + r b_F^\top h_{\Theta^i}(x)) \} \\
& - \alpha^2 v_1(0) \{ v_{11}(0, x) + r b_F^\top h_{\Theta^i}(x) \} + \alpha \dot{v}_1(x) - r \alpha v_1(x) \\
& = 0.
\end{aligned} \tag{B.39}$$

(S5):

$$\begin{aligned}
& \frac{1}{b_Q b_Q^\top} (-b_Q v_1^\top(0) - r p_0) (g_Q - \alpha b_Q (2v_{12}(0) + r g_{\Theta^i}^\top b_F)^\top) \\
& - \alpha^2 v_1(0) (2v_{12}(0) + r g_{\Theta^i}^\top b_F)^\top + \alpha v_2 \mu_X - r \alpha v_2 = 0.
\end{aligned} \tag{B.40}$$

(S6):

$$\begin{aligned}
& \frac{1}{2b_Q b_Q^\top} \int_{t-T}^t \text{tr} \{ (\hat{h}_Q(x) - \alpha b_Q (v_{11}(0, x) + r b_F^\top h_{\Theta^i}(x)))^\top \\
& \cdot (\hat{h}_Q(x) - \alpha b_Q (v_{11}(0, x) + r b_F^\top h_{\Theta^i}(x))) \} dx + \frac{1}{2b_Q b_Q^\top} (b_Q v_1^\top(0) + r p_0)^2 \\
& - \frac{1}{2} \alpha^2 \int_{t-T}^t \text{tr} \{ (v_{11}(0, x) + r b_F^\top h_{\Theta^i}(x))^\top (v_{11}(0, x) + r b_F^\top h_{\Theta^i}(x)) \} ds \\
& - \frac{1}{2} \alpha^2 v_1(0) v_1^\top(0) + \alpha \text{tr}(\hat{b}_X^\top v_{22} \hat{b}_X) \\
& - 2\alpha \text{tr}(\hat{b}_X v_{12}^\top(T)) + \alpha r \int_0^t \text{tr}(h_{\Theta^i}^\top(s) h_F(s)) ds - r \alpha v_0 - r + \ln r + \rho \\
& = 0.
\end{aligned} \tag{B.41}$$

Boundary Conditions:

(B1): For any $0 \leq x \leq T$

$$2\hat{b}_X^\top v_{12}(x) = v_{11}^\top(x, T) \tag{B.42}$$

(B2):

$$v_{22}\hat{b}_X = v_{12}(T) \tag{B.43}$$

(B3):

$$v_2\hat{b}_X = v_1(T) \tag{B.44}$$

Proof. The proposition follows from direct computation using Lemma 9 and the equation (B.23) in Theorem 9. \square

S1-S6, and the boundary condition B1-B3, give a complete system for $v_{11}(\cdot, \cdot)$, $v_{12}(\cdot)$, v_{22} , $v_1(\cdot)$, v_2 , and v_0 . Below we outline how to solve this system using a fixed point method.

Step 0: Set an initial guess of the value of $v_{12}(0)$.

Step 1: Given $v_{12}(0)$, use B1 to get the value of $v_{11}(0, T)$. Then use this value of $v_{11}(0, T)$ and S1 to solve for the function $v_{11}(0, x)$ (notice that S1 can be reduced to a delayed integral equation about $v_{11}(0, x)$). At the same time, use $v_{12}(0)$ and S3 to solve for v_{22} (S3 is a linear Lyapunov equation about v_{22}).

Step 2: Use the value of $v_{12}(0)$ and $v_{11}(0, x)$ from Step 1, and S2, to solve for the function $v_{12}(x)$ (notice that S2 is a linear ODE for $v_{12}(x)$).

Step 3: Use the function $v_{12}(x)$ obtained in Step 2, to get $v_{12}(T)$. Notice this value of $v_{12}(T)$ and the value of v_{22} obtained in Step 1 are completely determined by the initial guess of $v_{12}(0)$ in Step 0. Now use B1 to get a equation for $v_{12}(0)$. And solve for $v_{12}(0)$ (one could use a iteration scheme here). Then with the true value of $v_{12}(0)$. use Step 1, 2 to get the true function $v_{11}(0, x)$ and v_{22} . In this Step, we get the true value of $v_{12}(0)$, v_{22} , $v_{11}(0, x)$, and $v_{12}(x)$.

Step 4: Use $v_{11}(0, x)$ obtained in Step 3, and S1, to solve for $v_{11}(x, y)$ (notice that the function $v_{11}(x, y)$ is completely determined by the value of $v_{11}(0, x)$).

Step 5: Guess an initial guess of $v_1(0)$.

Step 6: Use the guessed value $v_1(0)$, the function $v_{11}(0, x)$ obtained in Step 3, and S4, to solve for the function $v_1(x)$ (note that S4 gives an ODE for $v_1(x)$). Also use $v_1(0)$ and S5 to solve for v_2 (S5 is a linear equation for v_2).

Step 7: Use the $v_1(x)$ obtained in Step 6 to get $v_1(T)$. Notice both $v_1(T)$ and v_2 in Step 6 are completely determined by the guess of $v_1(0)$. We then use B3 to get an equation on $v_1(0)$. Solve for $v_1(0)$ using this equation. Use this true value of $v_1(0)$ to get the true function $v_1(x)$ and true value of v_2 from Step 6. Finally use S6 to obtain v_0 .

B.4 Proof of Theorem 5 and 6

Since information is homogeneous among agents, price is fulling revealing for Y_t . Let $\hat{M}_t = E[M_t | \hat{\mathcal{F}}_t]$, where $\hat{\mathcal{F}}_t$ is the information available to agent at time t .

B.4.1 Investors' Filtering Problem

We first consider the investor's filtering problem.

Lemma 10. *Assume the agents have information set $\hat{\mathcal{F}}_t = \mathcal{F}_t^{D,S}$, where S_t is the common signal. Then the filtered process $\hat{M} := E[M_t | \mathcal{F}_t^{D,S}]$, together with the dividend process D_t follows:*

$$d\hat{M}_t = -\alpha_M \hat{M}_t dt + \hat{b}_M d\hat{B}_t \quad (\text{B.45})$$

$$dD_t = \hat{M}_t dt + \hat{b}_D d\hat{B}_t, \quad (\text{B.46})$$

where \hat{B}_t is a two-dimensional Brownian Motion adapted to the filtration \hat{F}_t , and \hat{b}_M and \hat{b}_D are constant vectors given by

$$\hat{b}_M = o[\sigma_D^{-1} \ \sigma_S^{-1}], \quad \hat{b}_D = [\sigma_D \ 0].$$

Here $o = E[(\hat{M}_t - M_t)^2]$ is the forecasting error, given by

$$o = \frac{1}{\sigma_D^{-2} + \sigma_S^{-2}} \{ \sqrt{\alpha_M^2 + \sigma_M^2(\sigma_D^{-2} + \sigma_S^{-2})} - \alpha_M \}. \quad (\text{B.47})$$

Furthermore, the quantity $\hat{\sigma}_F^2$ defined by:⁵

$$\hat{\sigma}_F^2 = \left(\frac{1}{\alpha_M + r} \hat{b}_M + \hat{b}_D \right) \cdot \left(\frac{1}{\alpha_M + r} \hat{b}_M^\top + \hat{b}_D^\top \right).$$

is an increasing function of σ_S and satisfies

$$\hat{\sigma}_F^2(\sigma_S) - \hat{\sigma}_F^2(0) = \frac{2r}{(\alpha_M + r)^2} \cdot o. \quad (\text{B.48})$$

In particular, $\hat{\sigma}_F^2(\sigma_S)$ is an increasing function of σ_S . The unconditional price variance associated with the fundamental value is an decreasing function of σ_S and satisfies

$$\frac{1}{(r + \alpha_M)^2} \left\{ \frac{\hat{\sigma}_M^2(\sigma_S)}{2\alpha_M} - \frac{\sigma_M^2}{2\alpha_M} \right\} = -\frac{o}{(r + \alpha_M)^2}. \quad (\text{B.49})$$

Proof. Standard continuous time Markovian filtering (e.g. Wang (1993)) implies

⁵ $\hat{\sigma}_F^2$ is indeed the conditional variance of excess return that are associated with the fundamental value.

(B.45). The forecasting error o satisfies

$$0 = -2\alpha o + \sigma_M^2 - o^2(\sigma_D^{-2} + \sigma_N^{-2}). \quad (\text{B.50})$$

which gives (B.47). □

B.4.2 Optimization

Under the price process:

$$P_t = -p_0 + \frac{1}{\alpha_M + r} \hat{M}_t - h_{PY} Y_t,$$

for some constants p_0 and h_{PY} , the excess return process is given by

$$dQ_t = \{h_{PY}(\alpha_Y + r)Y_t + rp_0\}dt + \hat{b}_Q d\hat{B}_t,$$

We can decompose the volatility of return into two component:

$$\hat{b}_Q \hat{b}_Q^2 = h_{PY}^2 \sigma_Y^2 + \hat{\sigma}_F^2,$$

The investor's wealth W_t evolves overtime: The wealth process follows:

$$dW_t^i = rW_t^i + x_t^i dQ_t + (Y_t + Z_t^i) dF_t - c_t^i dt. \quad (\text{B.51})$$

Define the related state vector: $X^i = [Y \ Z^i]^\top$ so that we can express the excess return as:

$$dQ_t = (g_Q X_t^i + rp_0)dt + \hat{b}_Q d\hat{B}_t.$$

Standard argument (see Wang (1993)) yields that the value function will be of the form

$$J_t = -e^{-\rho t - \alpha \{rW_t + \frac{1}{2}(X_t^i)^\top \Phi_2 X_t^i + \Phi_1 X_t^i + \Phi_0\}},$$

where Φ_2 , Φ_1 , and Φ_0 are constant matrices of the proper order. The optimal stock holding is

$$x_t^i = \frac{1}{\alpha r \hat{b}_Q \hat{b}_Q^\top} (h_Y (r + a_Y) + r p_0) - \frac{1}{\hat{b}_Q \hat{b}_Q^\top} (Y_t + Z_t^i) - \frac{1}{r \hat{b}_Q \hat{b}_Q^\top} (\hat{b}_Q b_X^\top \Phi_2 + \hat{b}_Q b_X^\top \Phi_1) \quad (\text{B.52})$$

We denote Φ_2 and Φ_1 as:

$$\Phi_2 = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \quad \Phi_1 = (\Phi_{1,Y}, \quad \Phi_{1,Z}).$$

B.4.3 Market Clearing and Equilibrium

Market clearing implies that the optimal stock holding is of the form

$$x_t^i = 1 - h_{xZ} Z_t^i.$$

Now applying the optimal stock holding equation (B.52) from the optimization argument, we get

$$\begin{aligned} h_Y (r + a_Y) + \alpha h_Y \sigma_Y^2 \Phi_{11} - \alpha r \sigma_{DN} &= 0, \\ \alpha h_Y \sigma_Y^2 \Phi_{12} - \alpha r \sigma_{DN} &= h_{xZ} \alpha r \hat{b}_Q \hat{b}_Q^\top. \end{aligned}$$

and

$$p_0 = \alpha \{ \hat{b}_Q \hat{b}_Q^\top - \frac{1}{r} h_Y \sigma_Y^2 \Phi_{1,Y} \}.$$

For notational convenience, define

$$y = h_{xZ}^2 \hat{b}_Q \hat{b}_Q^\top.$$

Lemma 11. $\hat{\sigma}_F^2$ is a decreasing function of y .

This lemma is the key to the proof of the proposition. Using this lemma, and the basic theory of two-dimensional Riccati equation, we can show the following result, which conclude the proof of the theorems.⁶

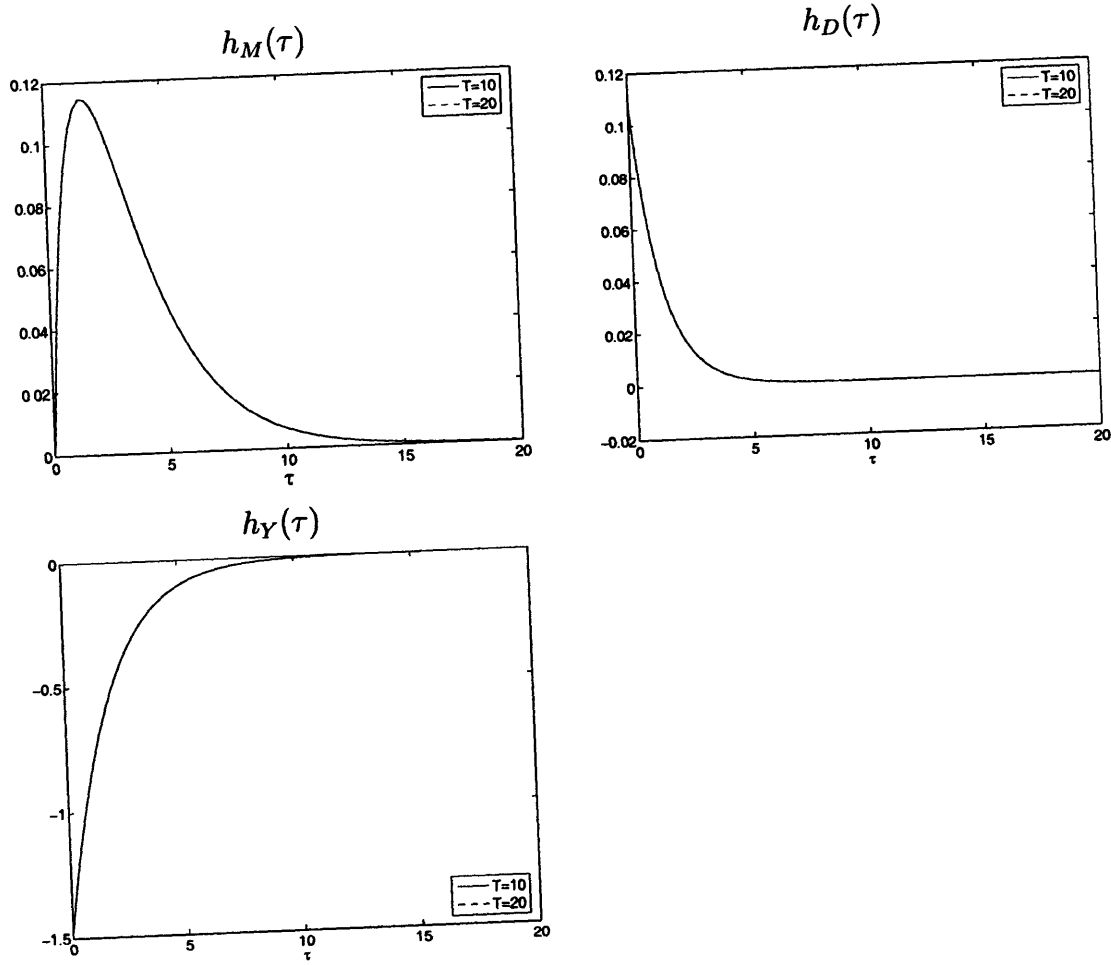
Lemma 12. (i) $\Phi_{11} < 0$ and $\Phi_{1,Y} < 0$. (ii) $|\Phi_{11}|$ is decreasing in $\hat{\sigma}_F^2$ and $|\Phi_{1,Y}|$ is increasing in $\hat{\sigma}_F^2$. Therefore h_Y and p_0 are increasing as information amount decreases. (iii) The stock price variance is decreasing in $\hat{\sigma}_F^2$.

B.5 Choice of the revelation T

When T is large (compared with the half life of the shocks of M_t , Y_t , and Z_t), the effect of T on asset price will be very small. Figure B-1 shows that in the benchmark parametrization in the paper, when we increase T from 10 to 20, the stock price changes very little.

⁶The detailed proof of both lemmas are available upon request.

Figure B-1: Effect of T



$r = 0.05, \alpha = 10.00, a_M = 0.400, a_Y = 0.500, a_Z = 0.700, \sigma_D = 0.490, \rho_{DN} = 0.500,$
 $\sigma_M = 0.200, \sigma_Y = 1.000, \sigma_Z = 0.400. \sigma_S = 0.500$

B.6 Diffuse Information: Static Case

B.6.1 The Economy

We consider a pure exchange economy with a single, perishable consumption good.

Time Period. There are only two periods: $t = 0$ and $t = 1$.

Securities markets. There are two traded assets in the securities market, one risk-free bond and one risky stock. The risk-free bond yields a constant return $r \geq 0$. At time $t = 1$, the risk stock pays a dividend $M + \epsilon$, where M and ϵ are independent normal random variables. The total stock supply is normalized to be one. The stock price at time $t = 0$ is p .

Agents. There are (countable) infinitely many agents in the economy, indexed by $i \in I$. Each agent i is endowed with γ_i share on stock at $t = 0$. Moreover, agents have non-tradable income at $t = 1$. The non-tradable income of agent i is $(Y + Z_i)e$, where Y , Z_i , and e are normal random variables. We assume that (i) Y , e , M , and ϵ are mutually independent, except for the pair (e, ϵ) where $Cov[e, \epsilon] \geq 0$, and (ii) Z_i are independent of Z_j for $i \neq j$, and they are independent of all other variables. For simplicity, we assume Z_i have the same variance: $Var[Z_i] = Var[Z]$. Agents have CARA utility and maximize:

$$\max E_0[-e^{-\alpha W^i} | \mathcal{F}_0^i],$$

where W^i is the wealth of agent i at time $t = 1$, and \mathcal{F}_0^i is the information available to i at time $t = 0$. Agent i have initial wealth W_0^i at time $t = 0$.

Information structure. At $t = 0$, each agent i receives a noisy signal $S_i = M + \xi_i$, where ξ_i is independent of M and all other variables. Moreover, ξ_i and ξ_j are independent for any $i \neq j$. For simplicity, we assume ξ_i have the same variance:

$Var[\xi_i] = Var[\xi]$. The information set of agent i at time $t = 0$ is:

$$\mathcal{F}_0^i = \{p, Y + Z_i, M + \xi_i\}.$$

B.6.2 Equilibrium

We are interested in linear equilibria. In a linear equilibrium, the stock price is of the form:

$$p = -p_0 + h_M M - h_Y Y, \tag{B.53}$$

where p_0 , h_M , and h_Y are constant.

It is straightforward to show that under the price function equation (B.53), the optimal stock holding of agent i is

$$x_i = \frac{E_i[M] - (1+r)p - \alpha(Y + Z_i)Cov[e, \epsilon]}{\alpha(Var_i[M] + Var[\epsilon])}.$$

The conditional variance $Var_i[M]$ is a constant for all agents due to symmetry. We use $\hat{Var}[M]$ to denote this conditional variance.

Market clearing implies (and use the law of large numbers)

$$p = \frac{\int^i E_i[M]}{1+r} - \frac{\alpha Cov[e, \epsilon]}{1+r} Y - \frac{\alpha(\hat{Var}[M] + Var[\epsilon])}{1+r}.$$

The following lemma characterizes the linear equilibrium.

Proposition 8. *There exists a linear equilibrium and the equilibrium price is*

$$p = \frac{1}{1+r} \left\{ \left(1 - \frac{\hat{Var}[M]}{Var[M]}\right) M - \left(\alpha Cov[e, \epsilon] + \frac{\hat{Var}[M]}{Var[Y]\theta} \right) Y - \alpha (\hat{Var}[M] + Var[\epsilon]) \right\}.$$

where $\theta \geq \alpha Cov[e, \epsilon]$ satisfies

$$\frac{1}{Var[\xi]} + \frac{1}{Var[Z]} \frac{1}{\theta^2} = \alpha Cov[e, \epsilon] \frac{1}{\theta} \left\{ \frac{1}{Var[\xi]} + \frac{1}{Var[M]} + \left(\frac{1}{Var[Y]} + \frac{1}{Var[Z]} \right) \frac{1}{\theta^2} \right\} \quad (\text{B.54})$$

and $\hat{Var}[M]$ is the variance of M conditional on each agent's information set (the same cross agents due to symmetry). Moreover, $\hat{Var}[M]$ is between 0 and $Var[M]$, and given by:

$$\hat{Var}[M] = \frac{1}{\frac{1}{Var[M]} + \frac{1}{Var[\xi]} + \frac{1}{Var[Y]\theta^2} + \frac{1}{Var[Z]\theta^2}}.$$

We consider two bench mark cases: (i) full information case where $Var[\xi] = 0$; and (ii) the no information case where agents do not receive any private signals about M . We will use p^* and p^{**} to denote the price in the full information case and the price in no information case respectively. Straightforwardly, we have the following results

Lemma 13. *The equilibrium prices in the full information case p^* and no information case p^{**} are given by:*

$$p^* = \frac{1}{1+r} M - \frac{\alpha Cov[e, \epsilon]}{1+r} Y - \frac{\alpha}{1+r} Var[\epsilon],$$

$$p^{**} = -\frac{\alpha Cov[e, \epsilon]}{1+r} Y - \frac{1}{1+r} \alpha (Var[M] + Var[\epsilon]).$$

The following proposition compare the equilibrium price level and volatility with the two benchmark cases.

Proposition 9. (i) *The price discount level p_0 is*

$$p_0 = \alpha(\hat{Var}[M] + Var[\epsilon]).$$

The the price discount level p_0 of the diffuse information case is in between those of full information and no information case: $p_0^ < p_0 < p_0^{**}$.*

(ii) *The price variance can be written as:*

$$\begin{aligned} Var[p] &= \frac{1}{(1+r)^2} \{Var[M] + \alpha^2 Cov[e, \epsilon]^2 Var[Y] - \hat{Var}[M] (1 - \frac{\alpha Cov[e, \epsilon]}{\theta})\} \\ &= Var[p^*] - \frac{1}{(1+r)^2} \{ \hat{Var}[M] (1 - \frac{\alpha Cov[e, \epsilon]}{\theta}) \}. \end{aligned}$$

*The price variance of diffuse information case is between those of full information and no information case: $Var[p^{**}] < Var[p] < Var[p^*]$.*

Finally we consider the comparative statics of prices as the noisiness of the private signal changes.

Proposition 10. *As the noisiness of the private signal (measured by $Var[\xi]$) increases, (i) the conditional variance $\hat{Var}[M]$ increases; (ii) the price discount level p_0 increases; and (iii) the price variance $Var[p]$ decreases.⁷*

⁷For some parameters, there might be multiple equilibria when $Var[\xi]$ is large. In the presence of multiple equilibria, to study the comparative statics, we choose the equilibrium to be on the continuous path of equilibria starting from the solution where $Var[\xi] = 0$. Section B.6.3 gives a detailed discussion of multiple equilibria.

B.6.3 Discussion of Multiple Linear Equilibria

When the signal precision is low, there might be multiple linear equilibria.⁸ Basically, there exist constants $0 < c_1 < c_2$ (independent of $Var[M]$ and $Var[\xi]$) such that: (i) if $Var[M] \in [0, c_1)$, then there is a unique solution for all signal precision and the equilibrium monotonically converges to the no information case as the noisiness of signal goes to infinity, (ii) if $Var[M] \in [c_2, c_2]$, then for small noisiness of signal, there will be unique equilibrium, but there will be multiple equilibria when the noisiness of the signal is large, and moreover we are not able to extend the solution *continuously* from full information case to the case where the noisiness of the signal goes to infinity. The continuous branch stops at a finite level of $Var[\xi]$; and (iii) if $Var[M] \in [c_2, \infty)$, then for small noisiness of signal, there will be unique equilibrium, but there will be multiple equilibria when the noisiness of the signal is large. However, we are able to extend the solution *continuously* from full information case to the case where the noisiness of the signal goes to infinity. But this continuous branch will *NOT* converge to the no information case as the noisiness of the signal goes to infinity.

⁸Mathematically, the linear equilibrium is determined by the solution to a cubic algebraic equation. For some parameter choices, the cubic equation may have multiple real solutions.

Bibliography

- Acharya, V.V., and L.H. Pedersen, 2005a, Asset pricing with liquidity risk, *Journal of Financial Economics* 77, 375–410.
- Acharya, Viral V., and Lasse Heje Pedersen, 2005b, Asset pricing with liquidity risk, *Journal of Financial Economics* 77, 375 – 410.
- Aiyagari, S. Rao, and Mark Gertler, 1991, Asset returns with transactions costs and uninsured individual risk, *Journal of Monetary Economics* 27, 311–331.
- Albuquerque, Rui, and Jianjun Miao, 2008, Advance information and asset prices, *Working Paper*.
- Allen, Franklin, Stephen Morris, and Hyum Song Shin, 2006, Beauty contests and iterated expectations in asset markets, *The Review of Financial Studies* 19, 719–752.
- Amihud, Y., 2002, Illiquidity and stock returns: cross-section and time-series effects, *Journal of Financial Markets* 5, 31–56.
- Amihud, Yakov, and Haim Mendelson, 1986a, Asset pricing and the bid-ask spread, *Journal of Financial Economics* 17, 223–249.
- Amihud, Y., and H. Mendelson, 1986b, Liquidity, maturity, and the yield on u.s. treasury securities, *Journal of Financial Economics* 17, 223–249.
- , 1993, The effects of beta, bid-ask spread, residual risk and size on stock returns, *Journal of Finance* 44, 478–486.
- Bacchetta, Philippe, and Eric Van Wincoop, 2005, Can Information Heterogeneity Explain the Exchange Rate Determination Puzzle?, *SSRN eLibrary*.
- Bali, T.G., 2003, An extreme value approach to estimating volatility and value at risk, *Journal of Business* 76, 83–108.

- Barberis, Nicholas, Andrei Shleifer, and Robert Vishny, 1998, A model of investor sentiment., *Journal of Financial Economics* 49, p307 – 343.
- Barro, Robert J, 2006, Rare disasters and asset markets in the twentieth century*, *Quarterly Journal of Economics* 121, 823–866.
- Barro, Robert J., 2009, Rare disasters, asset prices, and welfare costs, *The American Economic Review* 99, 243–264(22).
- Beirlant, Jan, Petra Vynckier, and Jozef Teugels, 1996, Tail index estimation, pareto quantile plots, and regression diagnosis, *Journal of the American Statistical Association* 91, 1659–1667.
- Brenna, M.J., and A. Subrahmanyam, 1995, Investment analysis and price formation in securities markets, *Journal of Financial Economics* 38, 361–382.
- Brennan, M.J., T. Chordia, and A. Subrahmanyam, 1998, Alternative factor specifications, security characteristics, and the cross-section of expected stock returns, *Journal of Financial Economics* 49, 345–373.
- Brennan, M.J., and A. Subrahmanyam, 1996a, Market microstructure and asset pricing: on the compensation for illiquidity in stock returns, *Journal of Financial Economics* 41, 441–464.
- Brennan, Michael J., and Avanidhar Subrahmanyam, 1996b, Market microstructure and asset pricing: On the compensation for illiquidity in stock returns, *Journal of Financial Economics* 41, 441 – 464.
- Brunnermeier, M., and L. H. Pederson, 2005, Market liquidity and fund liquidity, *Working Paper*.
- Chordia, T., R. Roll, and A. Subrahmanyam, 2000, Commonality in liquidity, *Journal of Financial Economics* 56, 3–28.
- Chordia, Tarun, Richard Roll, and Avanidhar Subrahmanyam, 2001, Market liquidity and trading activity, *The Journal of Finance* 56, 501–530.
- Constantinides, George M., 1986, Capital market equilibrium with transaction costs, *The Journal of Political Economy* 94, 842–862.
- Daniel, Kent, David Hirshleifer, and Avanidhar Subrahmanyam, 1998, Investor psychology and security market under- and overreactions., *Journal of Finance* 53, p1839 – 1885.

- Diamond, D., and P. Dybvig, 1983, Bank runs, deposit insurance, and liquidity, *Journal of Political Economy* 91, 401–419.
- Diamond, Douglas W., and Robert E. Verrecchia, 1981, Information aggregation in a noisy rational expectations economy, *Journal of Financial Economics* 9, 221–235.
- Dow, James, and Rohit Rahi, 1971, Informed trading, investment and welfare, *forthcoming The Journal of Business*.
- Drechsler, Itamar, and Amir Yaron, 2008, What's Vol Got to Do With It, *SSRN eLibrary*.
- Drees, H., and E. Kaufmann, 1998, Selection of the optimal sample fraction in univariate extreme value estimation., *Stoch. Proc. Appl.* 75, 149–172.
- Duffie, D., N. Garleanu, and L.H. Pederson, 2003, Over-the-counter markets, *Econometrica (forthcoming)*.
- Favre, L., and J.A. Galeano, 2002, Mean-modified value at risk optimization with hedge funds, *Journal of Alternative Investments* 5, 21–25.
- Gabaix, Xavier, 2008, Variable rare disasters: An exactly solved framework for ten puzzles in macro-finance, NBER Working Papers 13724 National Bureau of Economic Research, Inc.
- Getmansky, M., A.W. Lo, and I. Makarov, 2004, An econometric model of serial correlation and illiquidity in hedge fund returns, *Journal of Financial Economics* 74, 529–609.
- Ghosh, Anisha, and Christian Julliard, 2008, Can rare events explain the equity premium puzzle?, FMG Discussion Papers dp610 Financial Markets Group.
- Glosten, L.R., and L. Harris, 1988, Estimating the components of the bid/ask spread, *Journal of Financial Economics* 21, 123–142.
- Glosten, L.R., and P.R. Milgrom, 1985, Bid, ask and transaction prices in a specialist market with heterogeneously informed traders, *Journal of Financial Economics* 14, 71–100.
- Gourio, Francois, 2008, Disasters and recoveries, *The American Economic Review* 98, 68–73(6).

- Gromb, D., and D. Vayanos, 2002, Equilibrium and welfare in markets with financially constrained arbitrageurs, *Journal of Financial Economics* 66, 361–407.
- Grossman, S.J., and M.H. Miller, 1988, Liquidity and market structure, *Journal of Finance* 43, 617–633.
- Grossman, Sanford J., and Joseph E. Stiglitz, 1980, On the impossibility of informationally efficient markets, *The American Economic Review* 70, 393–408.
- Gupta, A., and Bing Liang, 2005, Does hedge funds have enough capital? a value-at-risk approach, *Journal of Financial Economics* 77, 219–253.
- Hart, O.D., 1995, *Firm, contracts, and financial structure* (Oxford University Press, Oxford UK).
- Hasbrouck, J., and D. Seppi, 2001, Common factors in prices, order flows and liquidity, *Journal of Financial Economics* 59, 383–411.
- He, Hua, and Jiang Wang, 1995, Differential information and dynamic behavior of stock trading volume, *The Review of Financial Studies* 8, 919–972.
- Heaton, John, and Deborah J. Lucas, 1996, Evaluating the effects of incomplete markets on risk sharing and asset pricing, *The Journal of Political Economy* 104, 443–487.
- Hong, Harrison, and Jeremy C. Stein, 1999, A unified theory of underreaction, momentum trading, and overreaction in asset markets., *Journal of Finance* 54, p2143 – 2184.
- Hong, Harrison, and Jiang Wang, 2000, Trading and returns under periodic market closures, *The Journal of Finance* 55, 297–354.
- Huang, Jennifer, and Jiang Wang, 2009, Liquidity and Market Crashes, *Rev. Financ. Stud.* 22, 2607–2643.
- Huang, Ming, 2003, Liquidity shocks and equilibrium liquidity premia, *Journal of Economic Theory* 109, 104 – 129.
- Huberman, G., and D. Halka, 2001, Systematic liquidity, *Journal of Financial Research* 24, 161–178.

- Jegadeesh, Narasimhan, and Sheridan Titman, 1993, Returns to buying winners and selling losers: Implications for stock market efficiency., *Journal of Finance* 48, p65 – 91.
- Jones, Charles M., 2002, A Century of Stock Market Liquidity and Trading Costs, *SSRN eLibrary*.
- Jorion, P., 1997, *Value at risk: the new benchmark for controlling market risk* (University of California, Irvine, CA).
- , 2000, Risk management lessons for long term captial management, *European Financial Management* 6, 277–300.
- Keynes, John Maynard, 1936, *The General Theory of Employment, Interest and Money* (Macmillan, London).
- Kushner, Harold, and Paul Dupuis, 2001, *Numerical Methods for Stochastic Control Problems in Continuous Time* (Springer, New York).
- Kyle, A.S., 1985, Continuous auctions and insider trading, *Econometrica* 53, 1315–1335.
- Laporte, N., 2003, Modeling liquidity risk in a var model, *MSCI working paper*.
- Leadbetter, M.R., G. Lindgren, and H. Rootzen, 1983, *Extremes and related properties of random sequences and processes* (Springer-Verlag: New York Heidelberg Berlin).
- Liptster, Robert, and Albert Shiryaev, 2004, *Statistics of Random Processes I* (Springer, New York).
- Liu, Jun, Francis A. Longstaff, and Jun Pan, 2003, Dynamic asset allocation with event risk, *The Journal of Finance* 58, 231–259.
- Liu, Jun, Jun Pan, and Tan Wang, 2005, An Equilibrium Model of Rare-Event Premia and Its Implication for Option Smirks, *Rev. Financ. Stud.* 18, 131–164.
- Liu, Xiaolei, and Lu Zhang, 2008, Momentum profits, factor pricing, and macroeconomic risk, *forthcoming, Review of Financial Studies*.
- Lo, A.W., 2001, Risk management for hedge funds: Introduction and overview, *Financial Analysts Journal* 57, 16–33.

- Lo, Andrew W., Harry Mamaysky, and Jiang Wang, 2004, Asset prices and trading volume under fixed transactions costs, *Journal of Political Economy* 112, 1054–1090.
- Longin, F.M., 2000, From value at risk to stress testing: The extreme value approach, *Journal of Banking and Finance* 24, 1097–1130.
- Longstaff, Francis A., 2009, Portfolio claustrophobia: Asset pricing in markets with illiquid assets, *American Economic Review* forthcoming.
- Makarov, Igor, and Oleg Rytchkov, 2007, Forecasting the forecasts of others, *Working Paper*.
- Pstor, ubo, and Robert F. Stambaugh, 2003, Liquidity risk and expected stock returns, *Journal of Political Economy* 111, 642–685.
- Qiu, Weiyang, and Jiang Wang, 2009, Information heterogeneity: A static model, *Memo*.
- Reiss, R.D., and M. Thomas, 2001, *Statistical analysis of extreme values* (Birkhäuser: Basel Boston Berlin) second edn.
- Rietz, Thomas A., 1988, The equity risk premium a solution, *Journal of Monetary Economics* 22, 117–131.
- Shleifer, A., and R.W. Vishny, 1992, Liquidation values and debt capacity: A market equilibrium approach, *Journal of Finance* 47, 1343–1366.
- Townsend, Robert M., 1983, Forecasting the forecasts of others, *The Journal of Political Economy* 91, 546–588.
- Vayanos, D, 1998, Transaction costs and asset prices: a dynamic equilibrium model, *Rev. Financ. Stud.* 11, 1–58.
- Wachter, Jessica, 2008, Can time-varying risk of rare disasters explain aggregate stock market volatility?, NBER Working Papers 14386 National Bureau of Economic Research, Inc.
- Wang, Jiang, 1993, A model of intertemporal asset prices under asymmetric information, *The Review of Economic Studies* 60, 249–282.
- , 1994, A model of competitive stock trading volume, *The Journal of Political Economy* 102, 127–168.