WAVE-INDUCED CENTRIFUGAL INSTABILITY IN A STRATIFIED SHEAR LAYER

by

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ABSTRACT

An approximate analytic small amplitude analysis is applied to a mixed layer transition zone model with simple velocity shear and density gradient in order to assess the likelihood of the development of a centrifugal instability due to the streamline curvature induced by a long internal gravity wave supported by the density difference across the transition layer. It is determined that this instability may occur only for Richardson numbers well within the range of the Kelvin-Helmholz instability, and it is argued that the latter instability will be dominant.

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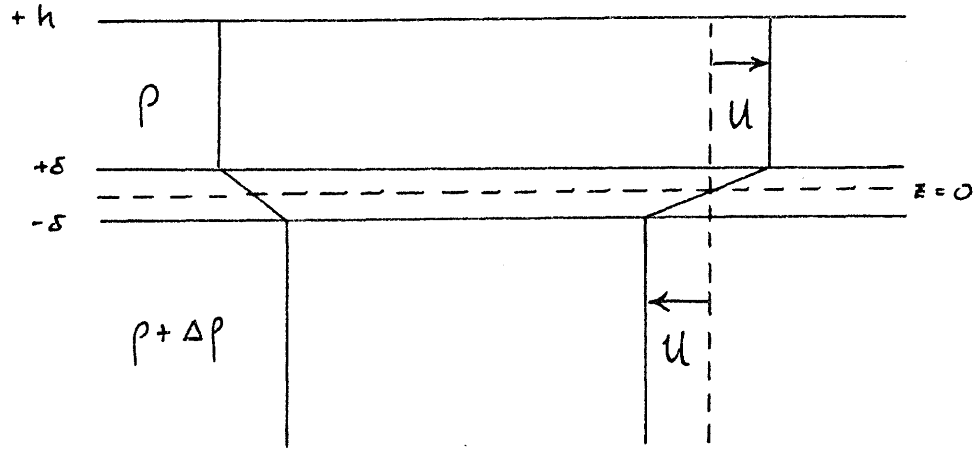
1. Introduction

The depth of the surface mixed layer in both the atmosphere and the ocean depends upon sea surface conditions and the stability of the upper boundary in the atmospheric case, and the lower boundary in the oceanic case. In this paper I will use the spatial orientation of the oceanic surface layer and consider the possible effect of long internal waves on the stability of a finite pycnocline in the presence of mean shear. In particular, a three-layer model is considered as shown in Figure 1, with the middle layer thickness assumed to be substantially smaller than the depth of the upper layer. An interfacial gravity wave is postulated to travel along the shear layer with a sufficiently large horizontal scale that its dynamics may be approximated by the two-layer model corresponding to shrinking the shear layer thickness to zero. In the neighborhood of the shear layer the initially parallel flow is now curved to follow the wave-associated undulations, and it is suggested that this flow is similar to the boundary layer flow along a curved wall (Figure 2) with the exceptions that there are no truly rigid walls, that the shear layer is stratified, and that viscosity (turbulent or otherwise) is assumed to have little effect upon the perturbation dynamics.

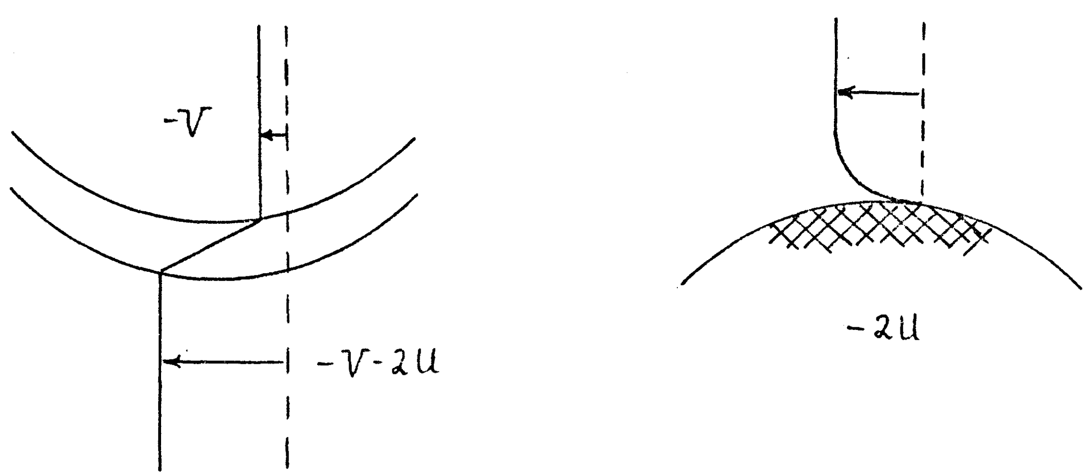
It was in 1940 that H. Görtler showed that a viscous boundary layer flow along a curved wall was destabilized by centrifugal effects with the perturbations describing longitudinal vortices (Görtler 1940). His explanation of the instability is reflected in the following particle analysis found in Betchov and Criminale (1967). They consider a two-dimensional laminar flow U_0 along a concave wall of constant curvature with a boundary layer of thickness δ. Within the boundary layer the shear of the mean flow is taken to be approximately equal to U_0/δ. Since the centrifugal forces act to press the fluid against the wall, a pressure gradient is established such that the pressure gradient balances the centrifugal force U^2/R, where R is the radius of curvature. If a fluid particle is perturbed from a position 1 vertically to a position 2 without any dissipative loss of energy, then by Bernoulli's equation:

$$\frac{p_1}{\rho} + \frac{1}{2} U_1^2 = \frac{p_2}{\rho} + \frac{1}{2} U_2^2 \qquad (1.1)$$



Three-Layer Model of the Mean Flow

FIGURE 1



Corresponding Shear Layer and Wall Layer Flows

FIGURE 2

If $u_1 = U$ and $u_2 = U - \Delta u$, then with the neglect of the term quadratic in Δu:

$$\Delta P + U \Delta u = 0 \qquad (1.2)$$

where I have set $\rho = 1$ for convenience. Recognizing that :

$$\Delta P = -\frac{U^2}{R}\Delta h \qquad \text{and} \qquad \Delta U = \frac{u_0}{\delta}\Delta h \quad , \qquad (1.3)$$

then

$$\Delta u - \Delta U = \left(\frac{U}{R} - \frac{u_0}{\delta}\right)\Delta h \quad . \qquad (1.4)$$

This shows that the velocity change experienced by the particle (Δu) will be smaller than the velocity change of the ambient flow (ΔU) since the radius of curvature is assumed to be significantly larger than the boundary layer thickness. Therefore the displaced particle velocity will be less than the local mean velocity which implies that the centrifugal force developed at the particle will be less than the local mean, so the local pressure gradient force will overcome the particle centrifugal force, accelerating the particle further away from its initial position. Hence, the flow is unstable. If the wall had been convex, then the effect is reversed and the particle is returned toward its initial position.

Later, A.M.O. Smith (1955) solved the Görtler problem with fewer simplifying assumptions and obtained a more refined stability diagram. Witting (1958) however, extended the domain of occurence of this instability by demonstrating its appearance as a secondary instability on the Tollmein-Schlicting waves in a boundary layer over a flat plate. As reported by Betchov and Criminale (1967), they found the disturbance above the critical level in regions of curvature of one sign, and in regions of curvature of opposite sign the disturbance was below the critical level. The direction of flow relative to the Tollmein-Schlicting wave changes sign across this level.

In the geophysical context, Scorer and Wilson (1963) have invoked the Taylor-Görtler mechanism as a possible explanation for the appearance of patches of clear air turbulence in the lee waves of mountains. The argument is general and demonstrates that curvature induced forces can overcome the restoring forces of a stably stratified atmosphere under circumstances where streamline curvature and velocity shear effects are large enough to balance the component of the gravity forces perpendicular to the streamlines. This means that a disturbance can only be amplified while it passes through one of these generating regions. The saving grace is that this is an inviscid instability, so between amplifying regions the disturbance does not decay, but just "rattles" along being advected by the mean flow, so if it passes through enough such regions, its amplitude can be expected to become finite, hence overturning and turbulence may occur. With this background, it seems attractive to consider the role of the Taylor-Görtler instability in the transition layer at the boundary of a mixed layer.

There has been a great deal of work done on the stability of stratified shear layers, and an excellent review of this material is the Drazin and Howard (1967) paper. Two important points which I have siezed upon to divorce the realm of the Kelvin-Helmholz and Holmboe instabilities from that of the Taylor-Görtler instability are: (1) the Kelvin-Helmholz instability can occur only for Richardson numbers less than some critical value, so a base flow can be constructed which is stable with respect to this form of instability; and (2) although the Holmboe instability can occur for all Richardson numbers (Holmboe, 1962) it is dependent upon there being a sharper density transition than velocity transition (in terms of the thickness of the transition layers), so if the base flow contains no such density "steps", then again, the flow is considered stable. Therefore, the question arises as to whether or not a long interfacial gravity wave is able to alter the initial flow through the mechanism of the Taylor-Görtler instability under circumstances where the flow is otherwise stable. If this is so, then the mechanism must be included in any geophysical problems concerned with the stability of transition regions. In fact, the process might be viewed as an intermediate stage between the initial state and a Holmboe instability driven breakdown and mixing of the flow.

Consider the initial postulated three-layer flow of Richardson number greater than ($\frac{1}{4}$), with density and velocity transition layers of the same thickness. A growing Görtler-type disturbance will eventually reach large enough amplitude to cause overturning within the shear layer. The ensuing mixing will create a new distribution of both density and velocity, possibly leading to new transition layers that satisfy the requirements of the Holmboe instability, thereby promoting further turbulent mixing in the vicinity of the of the transition zone. Furthermore, if the layer above is a mixed layer near equilibrium, the possibility of the Görtler instability provides a mechanism for the further deepening of the layer in the presence of internal waves. Under such a development, the critical Richardson number criteria derived on the basis of the Kelvin-Helmholz instability would have to be revised upward to account for this newly proposed mechanism for the production of turbulence.

2. Development of the Equations

Since the underlying flow to be perturbed has periodic curvature, the most convenient coordinate system for following the development of the perturbations is the wavy system shown in Figure 3 after Smith (1955). Let the center of the shear layer $(\bar{\zeta}=0)$ be described by:

$$z = a \sin kx \tag{2.1}$$

where a is the amplitude of the underlying wave, and k is its wave number. The scale factors for the transformation from cartesian coordinates (x, y, z) to curvilinear coordinates $(\bar{x}, \bar{y}, \bar{\zeta})$ are:

$$h_x = 1 - K\bar{\zeta} \quad ; \quad h_y = 1 \quad ; \quad h_z = 1 \tag{2.2}$$

where K is the curvature and is taken as positive for the case shown in Figure 4. With this convention, the system of equations is:

$$\frac{\partial u}{\partial t} + \frac{u}{1-K\bar{\zeta}} \frac{\partial u}{\partial \bar{x}} + v \frac{\partial u}{\partial \bar{y}} + w \frac{\partial u}{\partial \bar{\zeta}} - \frac{K}{1-K\bar{\zeta}} u w + \frac{1}{\rho} \frac{1}{1-K\bar{\zeta}} \frac{\partial P}{\partial \bar{x}} + g \sin\alpha = 0 \tag{2.3}$$

$$\frac{\partial v}{\partial t} + \frac{u}{1-K\bar{\zeta}} \frac{\partial v}{\partial \bar{x}} + v \frac{\partial v}{\partial \bar{y}} + w \frac{\partial v}{\partial \bar{\zeta}} + \frac{1}{\rho} \frac{\partial P}{\partial \bar{y}} = 0 \tag{2.4}$$

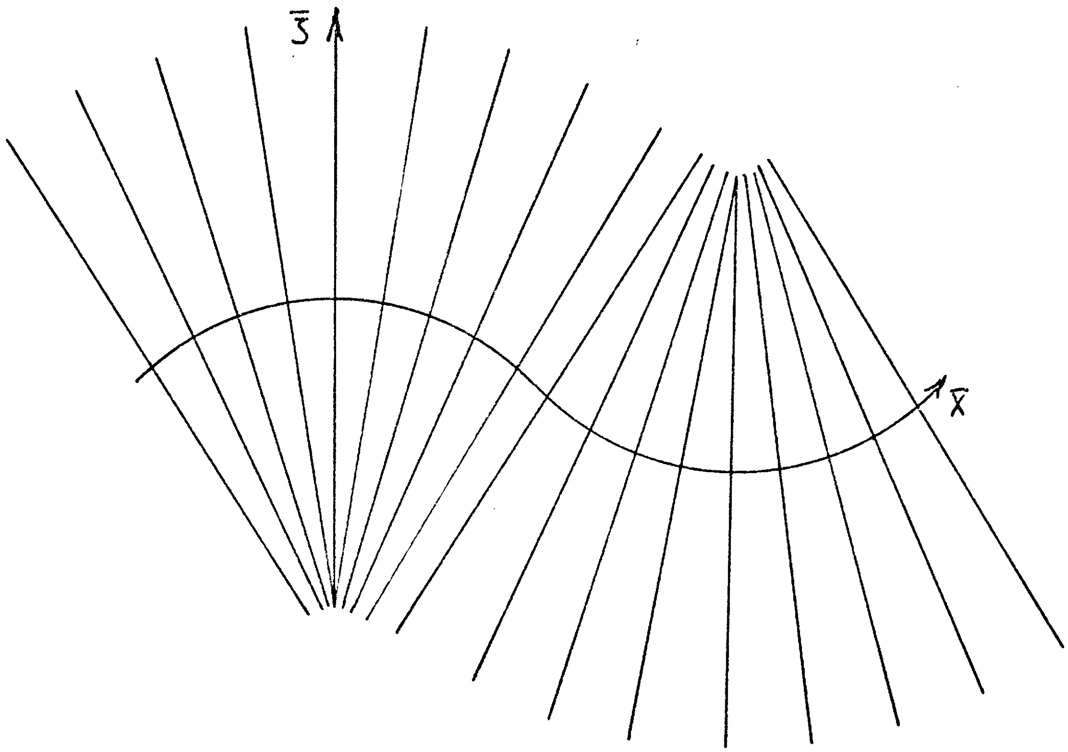
$$\frac{\partial w}{\partial t} + \frac{u}{1-K\bar{\zeta}} \frac{\partial w}{\partial \bar{x}} + v \frac{\partial w}{\partial \bar{y}} + w \frac{\partial w}{\partial \bar{\zeta}} + \frac{K}{1-K\bar{\zeta}} u^2 + \frac{1}{\rho} \frac{\partial P}{\partial \bar{\zeta}} + g \cos\alpha = 0 \tag{2.5}$$

$$\frac{\partial \rho}{\partial t} + \frac{u}{1-K\bar{\zeta}} \frac{\partial \rho}{\partial \bar{x}} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial \bar{\zeta}} = 0 \tag{2.6}$$

$$\frac{1}{1-K\bar{\zeta}} \frac{\partial u}{\partial \bar{x}} + \frac{\partial v}{\partial \bar{y}} + \frac{\partial w}{\partial \bar{\zeta}} - \frac{K}{1-K\bar{\zeta}} w = 0 \tag{2.7}$$

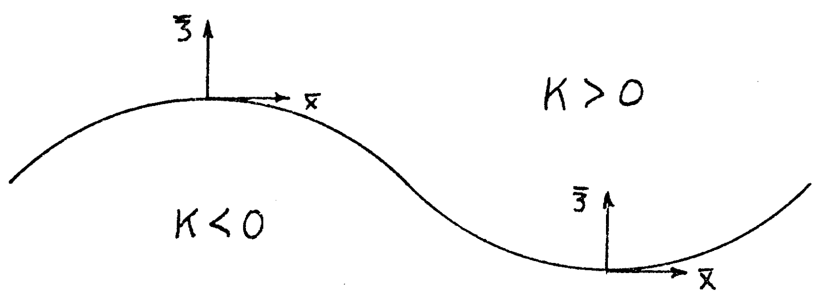
where

$$\alpha = \mathrm{Tan}^{-1} \frac{\partial}{\partial x}(a \sin kx) \tag{2.8}$$



Curvilinear Coordinate System

FIGURE 3



Definition of Curvature

FIGURE 4

In making this transformation it is assumed that the surface $\bar{\zeta}=0$ does not vary in time with respect to the (x, y, z) frame of reference. Therefore, the physical implication is that the original frame is moving with the wave phase speed relative to the fluid body, so the velocities indicated in the equations of motion are those measured from the wave's reference frame.

The hydrostatic assumption is made and the Boussinesq approximation follows with $\rho=\rho_0+\varrho$, so that:

$$\frac{1}{\rho}\,\frac{1}{1-K\bar{\zeta}}\,\frac{\partial P}{\partial \bar{x}} + g\sin\alpha \longrightarrow \frac{1}{\rho_0}\,\frac{1}{1-K\bar{\zeta}}\,\frac{\partial p}{\partial \bar{x}} + \varrho\,\frac{g}{\rho_0}\sin\alpha \tag{2.9}$$

and

$$\frac{1}{\rho}\,\frac{\partial P}{\partial \bar{\zeta}} + g\cos\alpha \longrightarrow \frac{1}{\rho_0}\,\frac{\partial p}{\partial \bar{\zeta}} + \varrho\,\frac{g}{\rho_0}\cos\alpha \quad . \tag{2.10}$$

Furthermore, it is assumed that the variables may be broken down into their "underlying" and "perturbation" parts as follows, where $\epsilon \ll 1$:

$$\begin{aligned}
u &= U(\bar{x},\bar{\zeta}) + \epsilon\, u'(\bar{x},\bar{y},\bar{\zeta},t)\\
v &= V(\bar{\zeta}) + \epsilon\, v'(\bar{x},\bar{y},\bar{\zeta},t)\\
w &= W(\bar{x},\bar{\zeta}) + \epsilon\, w'(\bar{x},\bar{y},\bar{\zeta},t)\\
p &= P(\bar{x},\bar{\zeta}) + \epsilon\, p'(\bar{x},\bar{y},\bar{\zeta},t)\\
\varrho &= \rho(\bar{\zeta}) + \epsilon\, \rho'(\bar{x},\bar{y},\bar{\zeta},t)\ .
\end{aligned} \tag{2.11}$$

with the $\bar{x}$-dependency of the underlying part coming from the underlying wave-induced velocity and pressure. Therefore after substituting the above changes and expansions into equations (2.3 - 2.7), and equating all the terms of order ϵ , the system of perturbation equations is written:

$$\frac{\partial u'}{\partial t} + \frac{U}{1-K\bar{\zeta}}\frac{\partial u'}{\partial \bar{x}} + \frac{u'}{1-K\bar{\zeta}}\frac{\partial U}{\partial \bar{x}} + V\frac{\partial u'}{\partial \bar{y}} + W\frac{\partial u'}{\partial \bar{\zeta}} + w'\frac{\partial U}{\partial \bar{\zeta}} - \frac{K}{1-K\bar{\zeta}}\left(Wu'+Uw'\right) + \frac{1}{\rho_0}\frac{1}{1-K\bar{\zeta}}\frac{\partial p'}{\partial \bar{x}} + \frac{g}{\rho_0}\sin\alpha\,\rho' = 0 \tag{2.12}$$

$$\frac{\partial v'}{\partial t} + \frac{U}{1-K\bar{\zeta}}\frac{\partial v'}{\partial \bar{x}} + V\frac{\partial v'}{\partial \bar{y}} + W\frac{\partial v'}{\partial \bar{\zeta}} + w'\frac{\partial V}{\partial \bar{\zeta}} + \frac{1}{\rho_0}\frac{\partial p'}{\partial \bar{y}} = 0 \tag{2.13}$$

$$\frac{\partial w'}{\partial t} + \frac{U}{1-K\bar{\zeta}}\frac{\partial w'}{\partial \bar{x}} + \frac{u'}{1-K\bar{\zeta}}\frac{\partial W}{\partial \bar{x}} + V\frac{\partial w'}{\partial \bar{y}} + W\frac{\partial w'}{\partial \bar{\zeta}} + w'\frac{\partial W}{\partial \bar{\zeta}} + \frac{2K}{1-K\bar{\zeta}}Uu' + \frac{1}{\rho_0}\frac{\partial p'}{\partial \bar{\zeta}} + \frac{g}{\rho_0}\cos\alpha\,\rho' = 0 \tag{2.14}$$

$$\frac{\partial \rho'}{\partial t} + \frac{U}{1-K\bar{\zeta}}\frac{\partial \rho'}{\partial \bar{x}} + V\frac{\partial \rho'}{\partial \bar{y}} + W\frac{\partial \rho'}{\partial \bar{\zeta}} + w'\frac{\partial \rho}{\partial \bar{\zeta}} = 0 \tag{2.15}$$

$$\frac{1}{1-K\bar{\zeta}}\frac{\partial u'}{\partial \bar{x}} + \frac{\partial v'}{\partial \bar{y}} + \frac{\partial w'}{\partial \bar{\zeta}} - \frac{K}{1-K\bar{\zeta}}w' = 0 \tag{2.16}$$

Most of the terms in the system above need no comment, but it is worth noting that $\frac{K}{1-K\xi}(Wu'+Uw')$ is analogous to the coriolis force in a rotating system, and the term $\frac{2K}{1-K\xi}Uu'$ represents the centrifugal force. The factor $1-K\xi$ that occurs in denominators just represents the divergence of the coordinate system, and the term $\left(\frac{Kw'}{1-K\xi}\right)$ in the continuity equation also derives from this source. Although it was not pointed out earlier, the coordinate system chosen for this problem breaks down if ξ is allowed to be of the order of $|K|^{-1}$; this need not cause any concern, since the suppositions that follow in setting up the underlying flow will disallow a large amplitude wave, and the analysis shall be restricted to the "inner" domain of the region surrounding the shear layer.

3. The Underlying Flow

As stated earlier, the supposition entertained here is that the scale of the underlying wave is much greater than the thickness of the shear layer, and so I approximate the three layer system with two layers of constant velocity and constant density with a common interface across which these two quantities are discontinuous. Also, of considerable influence is the depth of the upper layer since I will consider the wave to be "long" with respect to this layer (see Figure 5).

Matching the two regions across the interface by linearized boundary conditions yields this relation (see, for example, Lamb 1932):

$$\frac{\rho_2-\rho_1}{k}g = \rho_1(c+u)^2\coth(kh) + \rho_2(c-u)^2 \tag{3.1}$$

where c is the wave phase speed in the $-x^*$ direction. The perturbations have been assumed to take the form $\exp ik(x^*+ct)$. Approximating the $\coth(kh)$ term for $kh<1$, I find:

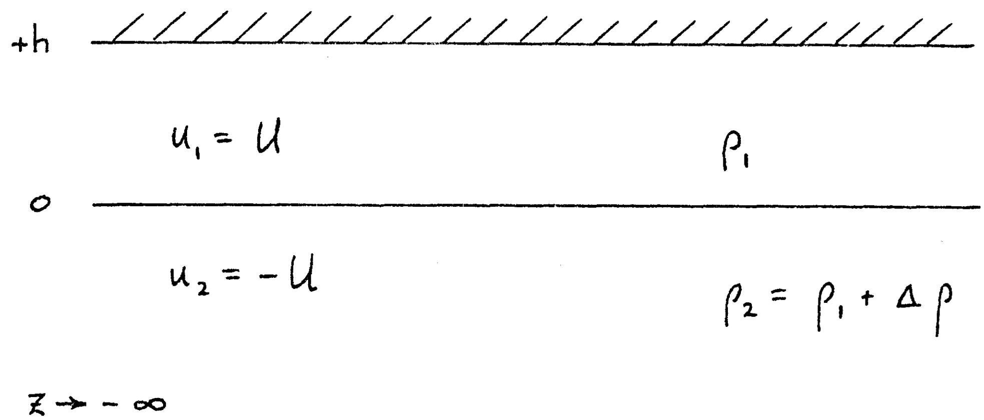
$$c = -u\,\frac{\rho_1-\rho_2 kh}{\rho_1+\rho_2 kh} \pm \sqrt{gh\,\frac{\rho_2-\rho_1}{\rho_1+\rho_2 kh} - \frac{4u^2\rho_1\rho_2}{(\rho_1+\rho_2 kh)^2}kh} \tag{3.2}$$

This may be simplified considerably by defining a quantity

$$ri \equiv \frac{g\,\Delta\rho/\rho}{k\,4\,u^2} \quad ; \quad \Delta\rho = \rho_2-\rho_1 \quad , \quad \rho\sim\rho_1\sim\rho_2 \tag{3.3}$$

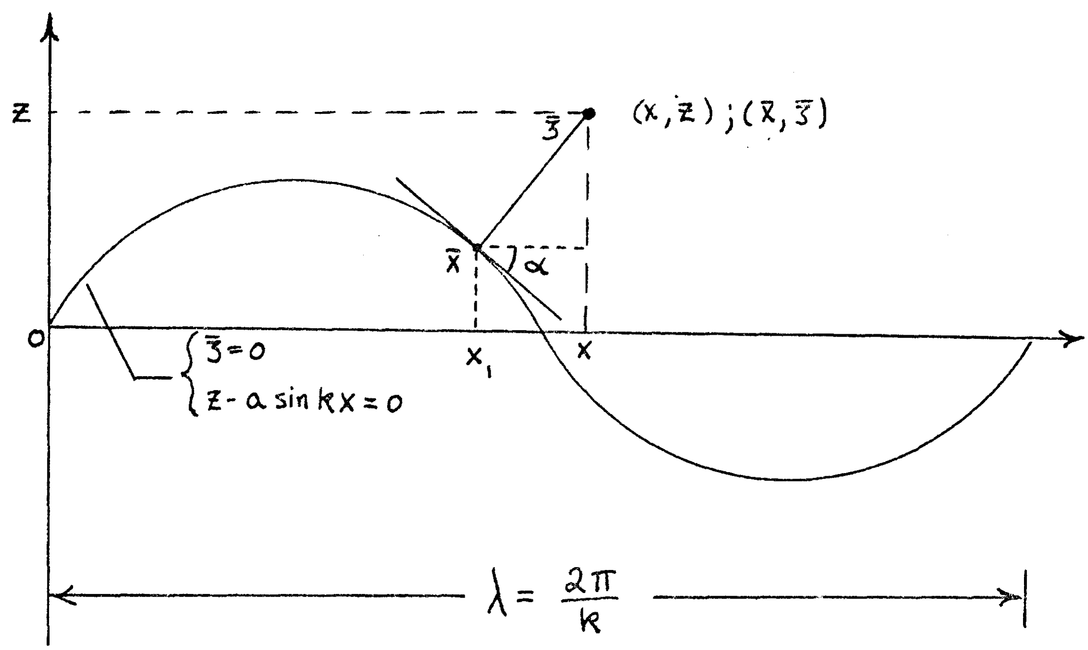
then

$$c = u\left\{-\left(\frac{1-kh}{1+kh}\right) \pm 2\sqrt{(ri-1)\frac{kh}{1+kh}}\right\} \tag{3.4}$$



Two-Layer Wave Model

FIGURE 5



Relationship Between (x,z) and $(\bar{x},\bar{\zeta})$ Labels

FIGURE 6

Thus it is evident that ri is the Richardson number for a wave on the interface between shearing fluids, and $ri < 1$ indicates that the wave grows and is therefore unstable. Furthermore, if $kh \ll 1$, then equation (3.2) has the approximate form:

$$c = -u \pm \sqrt{g \frac{\Delta\rho}{\rho} h} \tag{3.5}$$

which indicates that the wave behaves like a shallow water wave and propagates at a phase speed of $\pm\sqrt{g'h}$ with respect to the upper layer, where g' is the reduced gravity. This has an added attraction for the problem being dealt with here, since the phase speed does not depend upon the wave number provided all of the waves are long enough. Therefore several waves may be combined without violating the steadiness restriction imposed during the transformation of coordinates. In fact, so long as one is careful not to create regions of extreme curvature that would be difficult to handle with the chosen coordinate system, the underlying flow could be composed of several waves to produce regions of higher curvature and greater wave slope where the instability would be more likely to occur.

Using equation (3.4) the velocity field associated with the wave as seen from the frame of the mean velocity (as in Figure 5) is written:

$$u_1 = akU\left\{2 \pm 2\sqrt{(ri-1)\frac{1+kh}{kh}}\right\} e^{ik(x^*+ct)} \tag{3.6}$$

$$u_2 = -akU\left\{-\frac{2}{1+kh} \pm 2\sqrt{(ri-1)\frac{kh}{1+kh}}\right\} e^{kz} e^{ik(x^*+ct)} \tag{3.7}$$

$$w_1 = iakU\left\{\frac{2kh}{1+kh} \pm 2\sqrt{(ri-1)\frac{kh}{1+kh}}\right\} \frac{h-z}{h} e^{ik(x^*+ct)} \tag{3.8}$$

$$w_2 = iakU\left\{-\frac{2}{1+kh} \pm 2\sqrt{(ri-1)\frac{kh}{1+kh}}\right\} e^{kz} e^{ik(x^*+ct)} \tag{3.9}$$

where the subscripts refer to the appropriate layer. Translate the measuring frame along at the phase speed of $-c$, then the complete velocity field as seen from this frame can be written:

$$u_{1_T} = U\left\{\frac{2kh}{1+kh} \pm 2\sqrt{(ri-1)\frac{kh}{1+kh}}\right\} + akU\left\{2 \pm 2\sqrt{(ri-1)\frac{1+kh}{kh}}\right\} e^{ikx} \tag{3.10}$$

$$u_{2_T} = U\left\{ \frac{-2}{1+kh} \pm 2\sqrt{(\alpha i-1)\frac{kh}{1+kh}} \right\} - akU\left\{ \frac{-2}{1+kh} \pm 2\sqrt{(\alpha i-1)\frac{kh}{1+kh}} \right\} e^{kz} e^{ikx} \quad (3.11)$$

$$w_{1_T} = iakU\left\{ \frac{2kh}{1+kh} \pm 2\sqrt{(\alpha i-1)\frac{kh}{1+kh}} \right\} \frac{h-z}{h} e^{ikx} \quad (3.12)$$

$$w_{2_T} = iakU\left\{ \frac{-2}{1+kh} \pm 2\sqrt{(\alpha i-1)\frac{kh}{1+kh}} \right\} e^{kz} e^{ikx} \quad (3.13)$$

where X is the same cartesian coordinate used in section 2, so that now these velocities must be transformed from (x, y, z) space to $(\bar{x}, \bar{y}, \bar{z})$ space.

Figure 6 indicates the geometric relationship between the (x, z) and the $(\bar{x}, \bar{z})$ labels of a given point in space with respect to the surface $z = a \sin kx$. From this I have determined that $\bar{x}$ is related to x_1 by means of an elliptic integral of the second kind, but for small slope it can be approximated by:

$$\bar{x} = x_1 + \left(\frac{ak}{2}\right)^2 \left[x_1 + \frac{\sin 2kx_1}{2k} \right] + \mathcal{O}(ak)^4 \quad (3.14)$$

Furthermore, the relations between x_1, x, z, and $\bar{z}$ are:

$$\bar{z} = (z - a \sin kx_1)\sqrt{1 + (ak)^2 \cos^2 kx_1} \quad (3.15)$$

$$x = x_1 + (ak)^2 \frac{\sin 2kx_1}{2k} - z(ak \cos kx_1) \quad (3.16)$$

In order to proceed, it seems reasonable to transform the velocities with errors of order $(ak)^2$, and so it is straightforward to show that:

$$e^{ikx} = e^{ik\bar{x}} + \mathcal{O}(ak)^2 \qquad (\text{for } \bar{z} \sim a) \quad (3.17)$$

and

$$e^{kz} = e^{k\bar{z}} + \mathcal{O}(ak)^2 \quad (3.18)$$

However, the exchange $\bar{z} = z$ in the expression $\frac{h-z}{h}$ cannot be justified by these arguments. Instead I make this identification by recognizing that the perturbations will have virtually no effect far from the shear layer since there exists no mechanism for propagation, so the velocity field pertinent to the stability problem will be a localized "small z" field where the significance of the neglected term is thought to be small.

The unit vector interrelationship is the final piece of information needed for the transformation:

$$\hat{i} = \frac{\hat{e}_1 - ak\cos kx_1\, \hat{e}_2}{\sqrt{1 + (ak)^2\cos^2 kx_1}} \quad ; \quad \hat{k} = \frac{ak\cos kx_1\, \hat{e}_1 + \hat{e}_2}{\sqrt{1 + (ak)^2\cos^2 kx_1}} \tag{3.19}$$

where $(\hat{i}, \hat{k})$ are the unit vectors of the (x, z) axes, and $(\hat{e}_1, \hat{e}_2)$ are the unit vectors of the $(\bar{x}, \bar{z})$ axes. Therefore, if the overbar denotes the velocity as measured in the $(\bar{x}, \bar{z})$ coordinates, then:

$$\bar{u}_T = u_T + w_T\, ak\cos k\bar{x} + \mathcal{O}(ak)^2 \tag{3.20}$$

$$\bar{w}_T = w_T - u_T\, ak\cos k\bar{x} + \mathcal{O}(ak)^2 \tag{3.21}$$

and so the complete velocity field fully transformed in the curvilinear coordinate system is:

$$\bar{u}_{1_T} = U\left[\frac{2kh}{1+kh} \pm 2\sqrt{(Ri-1)\frac{kh}{1+kh}}\right]\left[1 + \frac{a}{h}(1+kh)\sin k\bar{x}\right] \tag{3.22}$$

$$\bar{u}_{2_T} = U\left[\frac{-2}{1+kh} \pm 2\sqrt{(Ri-1)\frac{kh}{1+kh}}\right]\left[1 - ak\, e^{k\bar{z}}\sin k\bar{x}\right] \tag{3.23}$$

$$\bar{w}_{1_T} = U\left[\frac{2kh}{1+kh} \pm 2\sqrt{(Ri-1)\frac{kh}{1+kh}}\right]\frac{\bar{z}}{h}\, ak\cos k\bar{x} \tag{3.24}$$

$$\bar{w}_{2_T} = U\left[\frac{-2}{1+kh} \pm 2\sqrt{(Ri-1)\frac{kh}{1+kh}}\right]\left(e^{k\bar{z}} - 1\right) ak\cos k\bar{x} \,. \tag{3.25}$$

I will use the above expressions for the velocity in layers one and three of the shear layer model, and postulate a linear transition zone for layer two such that the entire velocity profile is piecewise continuous. The shear across this layer is:

$$\Delta\bar{u} = U\left\{2 \pm 2\sqrt{(Ri-1)\frac{kh}{1+kh}}\;\; \frac{a}{h}(1+2kh)\sin k\bar{x}\right\} \tag{3.26}$$

and it is found that locally cartesian continuity can be satisfied in constructing a $\overline{w}$ field within the layer which is matched to the field outside without altering $\overline{w}$ in either layer one or three.

The immediate use of this carefully considered underlying flow approximation is to evaluate the size of the various terms in the equations of motion, but first it is necessary to introduce the scaling assumptions.

4. Scaling and Simplification

I pick the following scales for non-dimensionalization:

$$\text{velocity: } U_0 \ ; \ (\bar{x},\bar{y},\bar{z}) : \delta \ ; \ \rho : \Delta\rho \ ; \ p : \pi \ ; \ t : \varpi^{-1} \qquad (4.1)$$

where $U_0 = \frac{1}{2}$ the mean flow velocity difference across the layer, excluding the long wave flow field.

$\delta = \frac{1}{2}$ the shear layer thickness.

$\Delta\rho$ = the total density change across the shear layer.

Upon substituting (4.1) into (2.12 - 2.16) and rearranging, it becomes advantageous to define some non-dimensional combinations of the scale factors. Let:

$$\frac{\varpi\delta}{U_0} = S \quad ; \quad \frac{g\Delta\rho}{\rho_0}\frac{\delta}{U_0^2} = Ri \quad ; \quad \frac{\pi}{\rho_0 U_0^2} = 1 \quad . \qquad (4.2)$$

Furthermore, let V be constant, since all of the shear was taken to lie along $\bar{x}$, and let:

$$(\tilde{u}, \tilde{v}, \hat{w}) = \left(\frac{U}{U_0}, \frac{V}{U_0}, \frac{W}{U_0}\right) \quad , \qquad (4.3)$$

then the system of equations in non-dimensional form is:

$$S\frac{\partial u'}{\partial t} + \frac{\tilde{u}}{1-\delta K\bar{z}}\frac{\partial u'}{\partial \bar{x}} - u'\frac{\partial \tilde{W}}{\partial \bar{z}} + \tilde{V}\frac{\partial u'}{\partial \bar{y}} + \tilde{W}\frac{\partial u'}{\partial \bar{z}} + w'\frac{\partial \tilde{u}}{\partial \bar{z}} - \frac{\delta K}{1-\delta K\bar{z}}w'\tilde{u} + \frac{\partial p'/\partial \bar{x}}{1-\delta K\bar{z}} + Ri\sin\alpha\,\rho' = 0 \qquad (4.4)$$

$$S\frac{\partial v'}{\partial t} + \frac{\tilde{u}}{1-\delta K\bar{z}}\frac{\partial v'}{\partial \bar{x}} + \tilde{V}\frac{\partial v'}{\partial \bar{y}} + \tilde{W}\frac{\partial v'}{\partial \bar{z}} + \frac{\partial p'}{\partial \bar{y}} = 0 \qquad (4.5)$$

$$S\frac{\partial w'}{\partial t} + \frac{\tilde{u}}{1-\delta K\bar{z}}\frac{\partial w'}{\partial \bar{x}} + \frac{u'}{1-\delta K\bar{z}}\frac{\partial \tilde{W}}{\partial \bar{x}} + \tilde{V}\frac{\partial w'}{\partial \bar{y}} + \tilde{W}\frac{\partial w'}{\partial \bar{z}} + w'\frac{\partial \tilde{W}}{\partial \bar{z}} + \frac{2\delta K}{1-\delta K\bar{z}}\tilde{u}u' + \frac{\partial p'}{\partial \bar{z}} + Ri\cos\alpha\,\rho' = 0 \qquad (4.6)$$

$$S\frac{\partial \rho'}{\partial t} + \frac{\tilde{u}}{1-\delta K\bar{z}}\frac{\partial \rho'}{\partial \bar{x}} + \tilde{V}\frac{\partial \rho'}{\partial \bar{y}} + \tilde{W}\frac{\partial \rho'}{\partial \bar{z}} + w'\frac{\partial \rho}{\partial \bar{z}} = 0 \qquad (4.7)$$

$$\frac{1}{1-\delta K\bar{z}}\frac{\partial u'}{\partial \bar{z}} + \frac{\partial v'}{\partial \bar{y}} + \frac{\partial w'}{\partial \bar{z}} - \frac{\delta K}{1-\delta K\bar{z}}w' = 0 \quad . \qquad (4.8)$$

There are three immediate approximations to the system that are of great help. First, I totally neglect the variation of the scale factor $(1-\delta K \bar{z})^{-1}$ and set it equal to one since the curvature K is of order ak^2 thereby indicating the error to be of order $a\delta k^2$. Second, I assume that a local analysis in $\bar{x}$ is valid since I consider perturbations that change over a scale δ, while the flow changes over a scale of k^{-1}. Therefore the perturbation quantities (named "pert") take the assumed form:

$$\text{pert}\,(\bar{x},\bar{y},\bar{z},t) = \text{pert}\,(\bar{z})\, e^{i(\beta\bar{x} + \ell\bar{y} + \omega t)} \tag{4.9}$$

where all quantities are dimensionless. And third, I assume that $\sin\alpha$ in (4.4) may be neglected under the small slope assumption used throughout this analysis. Therefore, the system of equations now becomes:

$$i\left[\delta\omega + \tilde{u}\beta + \tilde{v}\ell + i\frac{\partial\tilde{w}}{\partial\bar{x}}\right]u' + \tilde{w}\frac{\partial u'}{\partial\bar{z}} + w'\frac{\partial\tilde{u}}{\partial\bar{z}} - \delta K w'\tilde{u} + i\beta p' = 0 \tag{4.10}$$

$$i\left[\delta\omega + \tilde{u}\beta + \tilde{v}\ell\right]v' + \tilde{w}\frac{\partial v'}{\partial\bar{z}} + i\ell p' = 0 \tag{4.11}$$

$$i\left[\delta\omega + \tilde{u}\beta + \tilde{v}\ell - i\frac{\partial\tilde{w}}{\partial\bar{z}}\right]w' + \tilde{w}\frac{\partial w'}{\partial\bar{z}} + u'\frac{\partial\tilde{w}}{\partial\bar{x}} + 2\delta K\tilde{u}u' + \frac{\partial p'}{\partial\bar{z}} + Ri\cos\alpha\,\rho' = 0 \tag{4.12}$$

$$i\left[\delta\omega + \tilde{u}\beta + \tilde{v}\ell\right]\rho' + \tilde{w}\frac{\partial\rho'}{\partial\bar{z}} + w'\frac{\partial\rho}{\partial\bar{z}} = 0 \tag{4.13}$$

$$i\beta u' + i\ell v' + \frac{\partial w'}{\partial\bar{z}} - \delta K w' = 0 \tag{4.14}$$

As one attempts to combine these equations into one equation in one of the unknowns, it becomes apparent that the complexity of the underlying velocity field thwarts all attempts at obtaining a manageable equation, and so it is desirable to find some justification for simplifying it. Therefore I return to the underlying flow for magnitude estimates of its various components.

First of all, identify the velocity scale U_0 with the mean flow velocity U so that:

$$\left(\frac{\bar{U}_T}{U}, \frac{\bar{W}_T}{U}\right) \equiv (\tilde{u}, \tilde{w}) \tag{4.15}$$

then in terms of magnitudes only with $k \sim 1/L$, I find:

Top Layer:

$$\tilde{u} \sim 1 + a/h \quad ; \quad \frac{\partial \tilde{u}}{\partial \bar{x}} \sim \frac{a\delta}{hL} \quad ; \quad \frac{\partial \tilde{u}}{\partial \zeta} \sim 0$$
$$\tilde{w} \sim \frac{a\delta}{hL} \quad ; \quad \frac{\partial \tilde{w}}{\partial \bar{x}} \sim \frac{a}{h}\left(\frac{\delta}{L}\right)^2 \quad ; \quad \frac{\partial \tilde{w}}{\partial \zeta} \sim \frac{a\delta}{hL} \tag{4.16}$$

Middle Layer:

$$\tilde{u} \sim 1 + \frac{a}{L} + \frac{a}{h} \quad ; \quad \frac{\partial \tilde{u}}{\partial \bar{x}} \sim \frac{\delta}{L}\left(\frac{a}{L} + \frac{a}{h}\right) ; \quad \frac{\partial \tilde{u}}{\partial \zeta} \sim 1 + \frac{a}{h}$$
$$\tilde{w} \sim \frac{\delta}{L}\left(\frac{a}{L} + \frac{a}{h}\right) \quad ; \quad \frac{\partial \tilde{w}}{\partial \bar{x}} \sim \left(\frac{\delta}{L}\right)^2\left(\frac{a}{L} + \frac{a}{h}\right) ; \quad \frac{\partial \tilde{w}}{\partial \zeta} \sim \frac{\delta}{L}\left(\frac{a}{L} + \frac{a}{h}\right) \tag{4.17}$$

Bottom Layer:

$$\tilde{u} \sim 1 + \frac{a}{L} \quad ; \quad \frac{\partial \tilde{u}}{\partial \bar{x}} \sim \frac{a\delta}{L^2} \quad ; \quad \frac{\partial \tilde{u}}{\partial \zeta} \sim \frac{a\delta}{L^2}$$
$$\tilde{w} \sim \frac{a\delta}{L^2} \quad ; \quad \frac{\partial \tilde{w}}{\partial \bar{x}} \sim \frac{a}{L}\left(\frac{\delta}{L}\right)^2 \quad ; \quad \frac{\partial \tilde{w}}{\partial \zeta} \sim \frac{a\delta}{L^2} \quad . \tag{4.18}$$

Taking each of the layers separately, the following pattern emerges:

$$\tilde{u} \sim 1 + e \quad ; \quad \frac{\partial \tilde{u}}{\partial \bar{x}} \sim \frac{\delta}{L} e \quad ; \quad \frac{\partial \tilde{u}}{\partial \zeta} \sim \left(0 ; \frac{\delta}{L} e ; 1 + e\right)^*$$
$$\tilde{w} \sim \frac{\delta}{L} e \quad ; \quad \frac{\partial \tilde{w}}{\partial \bar{x}} \sim \left(\frac{\delta}{L}\right)^2 e \quad ; \quad \frac{\partial \tilde{w}}{\partial \zeta} \sim \frac{\delta}{L} e \tag{4.19}$$

where $e = \frac{a}{h}$ or $\frac{a}{L}$ depending upon what layer is under discussion. I will discuss the balance of terms in the system (4.10 - 4.14) using equation (4.10) since it is representative of the others. Noting that $K \sim ak^2 = \frac{a}{L^2}$, I insert the appropriate magnitude in front of the terms to obtain:

$$0 = i\left[S\omega + (1+e)\tilde{u}\beta + \tilde{v}\ell + i\,\frac{\delta}{L}e\,\frac{\partial \tilde{W}}{\partial \zeta}\right]u' + \frac{\delta}{L}e\,\frac{\partial u'}{\partial \zeta} + (1+e)^* w'\frac{\partial \tilde{u}}{\partial \zeta} - (1+e)\frac{\delta a}{L^2}\,w'\tilde{u}K + i\beta p' \tag{4.20}$$

If I allow that $\frac{a}{L} \sim \frac{a}{h} \sim \frac{\delta}{L} \ll 1$ for the purpose of this argument, then the terms underlined above are of the same order of magnitude, and they are characteristically small when compared with the other terms of the equation. However, it is essential that the last underlined term be retained since it is a manifestation of the effect of the curved flow which is the topic of this paper. Therefore I cannot systematically neglect the other small terms while retaining this one.

It is on this point that I must drop all pretense of upholding the quantitative integrity of the problem and instead settle for a qualitatively valid understanding of the process. So I proceed to neglect the $\tilde{W}$ terms altogether, and retain only that part of $\tilde{u}$ which corresponds to the mean shear. In effect, the velocity field present in the three-layer idealization before the underlying wave was introduced is assumed to have been diverted such that it now follows the undulatory profile. The system of equations is then:

$$i\left[S\omega + \tilde{u}\beta + \tilde{V}\ell\right] u' + w' \frac{\partial \tilde{u}}{\partial \zeta} - \delta K \tilde{u} w' + i\beta p' = 0 \tag{4.21}$$

$$i\left[S\omega + \tilde{u}\beta + \tilde{V}\ell\right] v' + i\ell p' = 0 \tag{4.22}$$

$$i\left[S\omega + \tilde{u}\beta + \tilde{V}\ell\right] w' + 2\delta K \tilde{u} u' + \frac{\partial p'}{\partial \zeta} + Ri \cos\alpha\, \rho' = 0 \tag{4.23}$$

$$i\left[S\omega + \tilde{u}\beta + \tilde{V}\ell\right] \rho' + w' \frac{\partial \rho}{\partial \zeta} = 0 \tag{4.24}$$

$$i\beta u' + i\ell v' + \frac{\partial w'}{\partial \zeta} - \delta K w' = 0 \tag{4.25}$$

Finally, $\tilde{V} \equiv 0$, since a steady drift of the reference frame in the $\bar{y}$-direction adds nothing revealing to the problem. Now, the system is reduced to one equation in w' and its first and second derivatives.

$$w'_{\zeta\zeta} - w'_{\zeta}\, \delta K \frac{S\omega + 2\tilde{u}\beta}{S\omega + \tilde{u}\beta} - w'\left\{\beta^2 + \ell^2 + \frac{Ri \frac{\partial \rho}{\partial \zeta}(\beta^2+\ell^2)}{(S\omega + \tilde{u}\beta)^2} + \frac{\beta \tilde{u}_{\zeta\zeta}}{S\omega + \tilde{u}\beta} + \frac{2\ell^2 \delta K \tilde{u}\tilde{u}_{\zeta}}{(S\omega + \tilde{u}\beta)^2}\right\} = 0 \tag{4.26}$$

If $\delta K = 0$, this equation is seen to reduce to the usual stability equation for the finite shear layer Kelvin-Helmholz instability. But the addition of curvature to the problem changes it in a very fundamental way. With $w'_{\zeta}(\,) \neq 0$, it represents more than just the convergence or divergence of the coordinate system, and attempts to derive general criteria for stability from this equation proved fruitless.

5. Stability Analysis

In looking for the precursors of longitudinal rolls, the expectation is that the perturbation will exhibit most of its structure in the cross-stream direction and therefore any rapid changes along the flow are quite unlikely. Since this is a local analysis in $\bar{x}$, the strong restriction that β is zero poses no problems. Physically, I imagine a sequence of local analyses where it is probable that the $\bar{y}-\bar{\zeta}$ structure varies along the flow, and hence, the dependence of the perturbations upon $\bar{X}$ develops in this manner. With setting $\beta = 0$, it is no longer necessary to think in terms of an advective time scale for the local problem, therefore the Strouhal number can be set to unity, and the equation becomes:

$$w'_{\bar{\zeta}\bar{\zeta}} - \delta K w'_{\bar{\zeta}} - w' \frac{\ell^2}{\omega^2} \left\{ \omega^2 + Ri \cos\alpha \; \rho_{\bar{\zeta}} + 2\delta K \tilde{u} \left[\tilde{u}_{\bar{\zeta}} - \delta K \tilde{u} \right] \right\} = 0 \quad (5.1)$$

I will proceed to establish a stability criterion through the use of integral techniques. It is assumed that w' and ω may be complex, and that w', $w'_{\bar{\zeta}}$ and their complex conjugates vanish far from the shear layer. In terms of the scaled vertical coordinate, this occurs at $|\bar{\zeta}| \to \infty$. Furthermore, let $Ri \cos\alpha \; \rho_{\bar{\zeta}} + 2\delta K \tilde{u} (\tilde{u}_{\bar{\zeta}} - \delta K \tilde{u}) = G(\bar{\zeta})$, and take $\int_{-\infty}^{+\infty} w'^* (5.1) \, d\bar{\zeta}$. After integration by parts and use of the boundary conditions:

$$-\omega \int_{-\infty}^{+\infty} |w'_{\bar{\zeta}}|^2 d\bar{\zeta} - \omega \int_{-\infty}^{+\infty} w'^* w'_{\bar{\zeta}} \, \delta K \, d\bar{\zeta} - \int_{-\infty}^{+\infty} |w'|^2 \left\{ \omega \ell^2 + \frac{\ell^2}{\omega} G(\bar{\zeta}) \right\} d\bar{\zeta} = 0 \quad (5.2)$$

The complex conjugate operation yields:

$$-\omega^* \int_{-\infty}^{+\infty} |w'_{\bar{\zeta}}|^2 d\bar{\zeta} + \omega^* \int_{-\infty}^{+\infty} w'^* w'_{\bar{\zeta}} \, \delta K \, d\bar{\zeta} - \int_{-\infty}^{+\infty} |w'|^2 \left\{ \omega^* \ell^2 + \frac{\ell^2}{\omega^*} G(\bar{\zeta}) \right\} d\bar{\zeta} = 0 \quad (5.3)$$

Subtraction of these two equations gives:

$$i \, \mathrm{Im}\,\omega \int_{-\infty}^{+\infty} |w'_{\bar{\zeta}}|^2 d\bar{\zeta} + \mathrm{Re}\,\omega \int_{-\infty}^{+\infty} w'^* w'_{\bar{\zeta}} \, \delta K \, d\bar{\zeta} + i \, \mathrm{Im}\,\omega \, \ell^2 \int_{-\infty}^{+\infty} |w'|^2 \left(1 - \frac{G(\bar{\zeta})}{|\omega|^2} \right) d\bar{\zeta} = 0 \quad (5.4)$$

and addition gives:

$$\mathrm{Re}\,\omega \int_{-\infty}^{+\infty} |w'_{\bar{\zeta}}|^2 d\bar{\zeta} + i \, \mathrm{Im}\,\omega \int_{-\infty}^{+\infty} w'^* w'_{\bar{\zeta}} \, \delta K \, d\bar{\zeta} + \mathrm{Re}\,\omega \, \ell^2 \int_{-\infty}^{+\infty} |w'|^2 \left(1 + \frac{G(\bar{\zeta})}{|\omega|^2} \right) d\bar{\zeta} = 0 \quad (5.5)$$

These equations may now be combined to cancel the $\int_{-\infty}^{+\infty} w^* w'_{\xi} \cdots$ term by multiplying equations (5.4) by $i\,\mathrm{Im}\,\omega$, and equation (5.5) by $\mathrm{Re}\,\omega$, then subtracting to leave:

$$|\omega|^2 \int_{-\infty}^{+\infty} \left(|w'_{\xi}|^2 + \ell^2 |w'|^2 \right) d\xi \;+\; \frac{(\mathrm{Re}\,\omega)^2 - (\mathrm{Im}\,\omega)^2}{|\omega|^2} \int_{-\infty}^{+\infty} \ell^2 |w'|^2 G(\xi)\, d\xi = 0 \quad (5.6)$$

This relation is not very helpful since the sign of $G(\xi)$ depends upon the sign of $\mathrm{Re}\,(\omega)^2$. What is needed is a side condition that would produce an equation in which $\mathrm{Re}\,(\omega)^2$ is replaced by a definite quantity of either sign without undue restriction on the validity of the problem. In particular, if the expression above could be reduced to the special case where ω cannot be complex, then $G(\xi)$ would possess a critical value where ω changes from real to imaginary, thus indicating the regions of stability and instability. Such a side condition is the requirement that the vertical structure of the perturbation be modal.

This assumption means that I will be considering perturbations that possess no time dependent phase relationship in the vertical, and therefore I exclude the possibility of propagating vertical waves in favor of standing waves. Mathematically, I say the only complex quantity in the perturbation equation (5.1) is $\omega = \Lambda_r + i\,\Lambda_i$, so I may separate the real and imaginary parts of this equation and integrate as before:

$$\int_{-\infty}^{+\infty} w'\, \mathrm{Re}\,(5.1)\, d\xi = \Lambda_r \int_{-\infty}^{+\infty} (w'_{\xi})^2 d\xi \;+\; \Lambda_r \ell^2 \int_{-\infty}^{+\infty} (w')^2 \left(1 + \frac{G(\xi)}{|\omega|^2} \right) d\xi = 0 \quad . \quad (5.7)$$

Similarly,

$$\int_{-\infty}^{+\infty} w'\, \mathrm{Im}\,(5.1)\, d\xi = \Lambda_i \int_{-\infty}^{+\infty} (w'_{\xi})^2 d\xi \;+\; \Lambda_i \ell^2 \int_{-\infty}^{+\infty} (w')^2 \left(1 - \frac{G(\xi)}{|\omega|^2} \right) d\xi = 0 \quad . \quad (5.8)$$

Assuming that neither Λ_r or Λ_i are zero, I cancel them out of their respective equations, and add the two to obtain:

$$\int_{-\infty}^{+\infty} (w'_{\xi})^2 d\xi \;+\; \ell^2 \int_{-\infty}^{+\infty} (w')^2 d\xi = 0 \quad (5.9)$$

For ℓ^2 greater than zero (ie, ℓ real), this demands that the perturbation vanishes everywhere. Therefore, for a non-trivial result, ω must be either real, or imaginary, but not both. Consider the equation for ω_i not equal to zero:

$$0 = \int (w'_\xi)^2 d\xi + \frac{\ell^2}{|\omega|^2} \int (w')^2 \left\{ |\omega|^2 - Ri \cos\alpha \, \rho_\xi - 2\delta K \tilde{u} \tilde{u}_\xi + 2(\delta K \tilde{u})^2 \right\} d\xi . \quad (5.10)$$

Since $\rho_\xi < 0$ for stable stratification, it is evident that the only term available for driving the second integral negative is the shear-curvature term, $2\delta K \tilde{u} \tilde{u}_\xi = \delta K (\tilde{u})^2_\xi$. A necessary condition for instability is:

$$2\delta K \tilde{u} \tilde{u}_\xi > \omega_i^2 - Ri \cos\alpha \, \rho_\xi + 2(\delta K \tilde{u})^2 > 0 \quad (5.11)$$

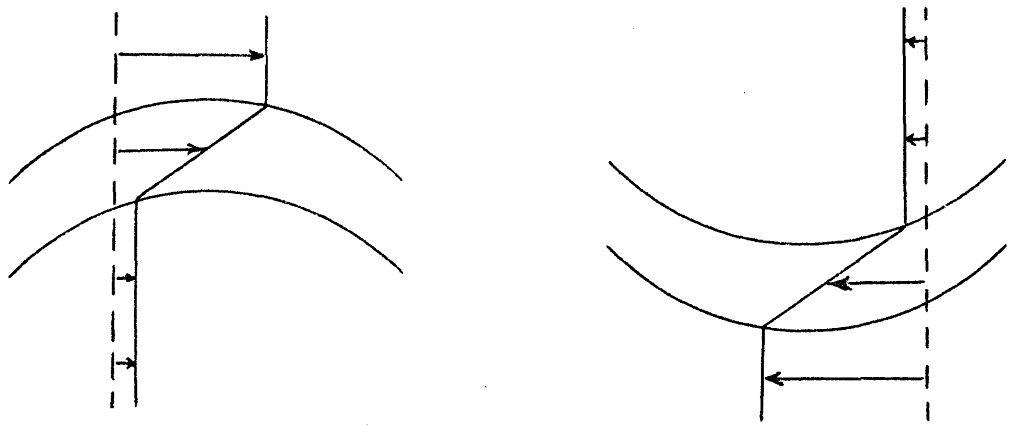
over a portion of the domain. Furthermore, the last inequality may be considered the weakest necessary condition for the problem. It implies that the curvature and the velocity field must be of the same sign if the shear is positive, and of opposite sign if the shear is negative (Figure 7).

The second curvature term may be considered to be the pseudo-inertial frequency as seen from the underlying wave frame of reference. This term acts in the same direction as the buoyancy frequency term in much the same way as does the inertial frequency term in large scale flows on a rotating planet, except here the vorticity vector is perpendicular to gravity. The net result is that the destabilizing force developed through the interaction of shear and curvature must become greater than the stabilizing forces of gravity <u>and</u> "rotation" before the perturbation can grow locally. More will be said later about the pseudo-inertial frequency, but now I return to the perturbation equation (5.1) for an approximate solution.

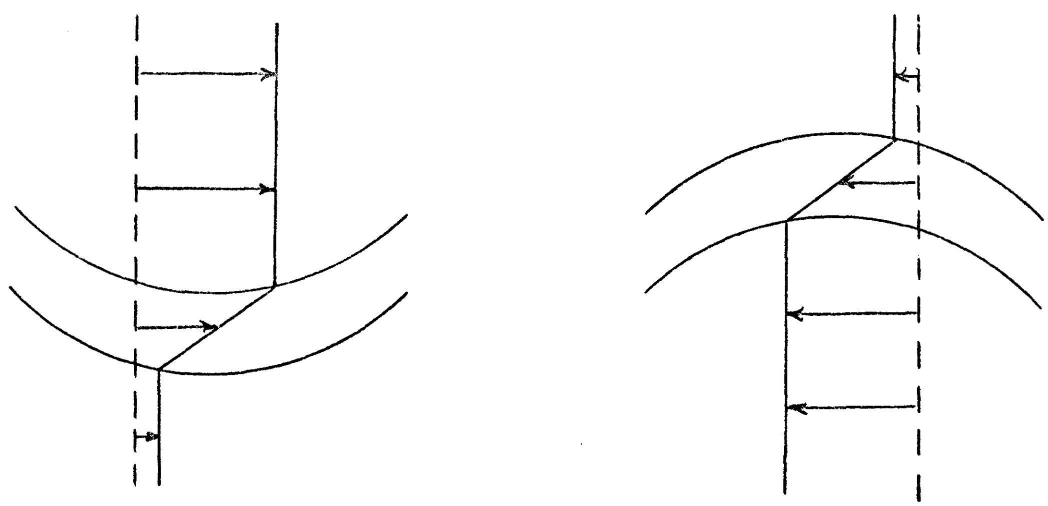
I will consider a piecewise continuous profile of density and velocity for the underlying flow of the following form:

Top Layer:	$\rho_\xi = 0$	;	$\tilde{u} = -\varphi$
Middle Layer:	$\rho_\xi = -1/2$	;	$\tilde{u} = \xi - \varphi - 1$
Bottom Layer:	$\rho_\xi = 0$	;	$\tilde{u} = -\varphi - 2$

(5.12)



Stability



Instability

Stability-Instability Region for Positive Shear

FIGURE 7

These particular numerical values arise from the choice of the nondimensionalization scales used in the analysis, and ϕ depends upon the phase speed of the underlying wave. Inserting the values of (5.12) into (5.1), the equation for the middle layer becomes:

$$w'_{\bar{\zeta}\bar{\zeta}} - w'_{\bar{\zeta}}\,\delta K - w'\frac{\ell^2}{\omega^2}\left\{\omega^2 - \tfrac{1}{2}Ri\cos\alpha + 2\delta K(\bar{\zeta}-\phi-1)\left[1-\delta K(\bar{\zeta}-\phi-1)\right]\right\} = 0 . \qquad (5.13)$$

The method of solution of this kind of equation requires an infinite power series in $\bar{\zeta}$ and therefore is not well suited to the purpose of this analysis. Furthermore, the WKB approximate solution would require the $\bar{\zeta}$ -dependent terms to be small compared to the Richardson number terms or the most differentiated w' term, which seems difficult to imagine unless ϕ were much larger than one to maintain the importance of the curvature terms. Therefore I must again seek qualitative insight and consider only the mean structure in the layer.

The mean shear-curvature term is the exact shear in the layer (a constant = 1) times the average velocity times the curvature, and the mean pseudo-inertial term involves only the square of the average velocity. Therefore if I model equation (5.13) by an equation with constant coefficients, I find:

$$w'_{\bar{\zeta}\bar{\zeta}} - w'_{\bar{\zeta}}\,\delta K - w'\frac{\ell^2}{\omega^2}\left\{\omega^2 - \tfrac{1}{2}Ri\cos\alpha - 2\,\delta K(\phi+1) - 2\left[\delta K(\phi+1)\right]^2\right\} = 0 . \qquad (5.14)$$

Having proposed such an equation, the solution will be of the form:

$$w' \approx e^{\gamma\bar{\zeta}} \qquad (5.15)$$

where γ is complex. Substitution of (5.15) into (5.14) gives

$$\omega^2\left(\gamma^2 - \delta K\gamma - \ell^2\right) = \ell^2\left\{-\tfrac{1}{2}Ri\cos\alpha - 2\,\delta K(\phi+1) - 2\left[\delta K(\phi+1)\right]^2\right\} . \qquad (5.16)$$

Assuming that $\omega = \Lambda_r + i\Lambda_i$ and $\gamma = \gamma_r + i\gamma_i$, the real and the imaginary parts of (5.16) are respectively

$$\begin{aligned}(\Lambda_r^2-\Lambda_i^2)(\gamma_r^2-\gamma_i^2-\delta K\gamma_r-\ell^2) - 4\Lambda_r\Lambda_i\gamma_r\gamma_i + 2\Lambda_r\Lambda_i\,\delta K\gamma_i = \\ \ell^2\left\{-\tfrac{1}{2}Ri\cos\alpha - 2\,\delta K(\phi+1) - 2\left[\delta K(\phi+1)\right]^2\right\}\end{aligned} \qquad (5.17)$$

$$2\Lambda_i\Lambda_r\left(\gamma_r^2-\gamma_i^2-\delta K\gamma_r-\ell^2\right) + (\Lambda_r^2-\Lambda_i^2)(2\gamma_r\gamma_i - \delta K\gamma_i) = 0 . \qquad (5.18)$$

The imaginary part demonstrates the link between the complex behavior of ω and γ. If $\Lambda_i \Lambda_r = 0$, then either $\gamma_i = 0$ or $\gamma_r = \frac{1}{2}\delta K$, and of course, the reverse logic holds as well. So again, the justifiable modal assumption may be used to good advantage in isolating this special case.

Let $w' = e^{\gamma\xi} \cos(\Gamma + \theta)\xi$, where θ is just a constant, then

$$\omega^2\left(-\gamma^2 - \Gamma^2 - \ell^2 + (2\gamma - \delta K)(\gamma - \Gamma \tan[\Gamma+\theta]\xi)\right) = \ell^2\left(-\tfrac{1}{2}Ri\cos\alpha - 2\delta K(\phi+1) - 2[\delta K(\phi+1)]^2\right). \quad (5.19)$$

Recall that $\omega^2 = \Lambda_r^2 - \Lambda_i^2 + 2i\Lambda_r\Lambda_i$; so the real and imaginary parts of (5.19) may be separated to give:

$$(\Lambda_r^2 - \Lambda_i^2)(-\gamma^2 - \Gamma^2 - \ell^2 + (2\gamma - \delta K)(\gamma - \Gamma\tan[\Gamma+\theta]\xi)) = \ell^2\left(-\tfrac{1}{2}Ri\cos\alpha - 2\delta K(\phi+1) - 2[\delta K(\phi+1)]^2\right). \quad (5.20)$$

$$2\Lambda_i\Lambda_r\left(-\gamma^2 - \Gamma^2 - \ell^2 + (2\gamma - \delta K)(\gamma - \Gamma\tan[\Gamma+\theta]\xi)\right) = 0 \quad (5.21)$$

Clearly a function of ξ in a relation between constants is unacceptable; even though the constants were derived by a non-rigorous approximation, the presence of $\tan(\Gamma+\theta)\xi$ still represents an internal inconsistency. Removal demands that:

$$\gamma = \frac{\delta K}{2} \quad (5.22)$$

or rather, that the only non "trigonometric" behavior of the solution be that which is needed to remove the w'_ξ term in equation (5.14). The term "trigonometric" is understood to include whatever power series is necessary to satisfy the original equation with variable coefficients. And so, recognizing that $2\gamma - \delta K = 0$, equation (5.21) gives:

$$\Lambda_i \Lambda_r = 0 \quad (5.23)$$

in agreement with the modal integral result following equation (5.9). If I take

$\Lambda_r = 0$, then (5.20) becomes:

$$\Lambda_i^2 = \frac{\ell^2}{\left(\frac{\delta K}{2}\right)^2 + \Gamma^2 + \ell^2} \left[-\frac{1}{2} Ri \cos\alpha - 2\delta K(\Phi+1) - 2\left[\delta K(\Phi+1)\right]^2 \right] \tag{5.24}$$

and again, instability depends upon the shear-curvature term being of the right sign and of sufficient magnitude. Furthermore, this result shows that the disturbances which have the highest growth rates, also have the lowest vertical wave number. To be more precise about this the solutions in the three regions must be matched, so I return to the solution in the other two layers.

Inserting the values of (5.12) into (5.1), the equations for the top and bottom layers become respectively:

$$w'_{\bar{\zeta}\bar{\zeta}} - \delta K w'_{\bar{\zeta}} - w' \ell^2 \left[1 - 2 \left(\frac{\delta K \Phi}{\omega} \right)^2 \right] = 0 \tag{5.25}$$

$$w'_{\bar{\zeta}\bar{\zeta}} - \delta K w'_{\bar{\zeta}} - w' \ell^2 \left[1 - 2 \left(\frac{\delta K (\Phi+2)}{\omega} \right)^2 \right] = 0 \tag{5.26}$$

Let $w_1' = e^{\gamma_1 \bar{\zeta}}$ be the perturbation velocity in the top layer, and $w_2' = e^{\gamma_2 \bar{\zeta}}$ be that for the bottom layer, then:

$$\gamma_1 = \frac{\delta K}{2} \pm \sqrt{\left(\frac{\delta K}{2}\right)^2 + \ell^2 \left[1 - 2 \left(\frac{\delta K \Phi}{\omega} \right)^2 \right]} \tag{5.27}$$

$$\gamma_2 = \frac{\delta K}{2} \pm \sqrt{\left(\frac{\delta K}{2}\right)^2 + \ell^2 \left[1 - 2 \left(\frac{\delta K (\Phi+2)}{\omega} \right)^2 \right]} \tag{5.28}$$

If ω is imaginary, then γ_1 and γ_2 are real; if ω is real, then γ_1 and γ_2 may be complex if $\omega^2 \sim (\delta K \Phi)^2$. Therefore, if the perturbation frequency is small enough to fall into the pseudo-inertial range, then waves are possible in the upper and lower layers. But recall that the model of the underlying flow was inviscid with no vorticity in the upper and lower layers, so how can inertia waves exist in a flow that possesses no absolute vorticity? They cannot, so this pseudo-inertial term may be thought to arise from the various approximations involved in setting up the system equations and in defining the underlying flow. Clearly, the approximate system loses its validity away from the near vicinity of the shear layer, so it is reasonable to treat the inertial term as being spurious in the top and bottom layers so that:

$$\gamma_1 = \frac{\delta K}{2} - \sqrt{\left(\frac{\delta K}{2}\right)^2 + \ell^2} \tag{5.29}$$

$$\gamma_2 = \frac{\delta K}{2} + \sqrt{\left(\frac{\delta K}{2}\right)^2 + \ell^2} \tag{5.30}$$

where the appropriate sign has been chosen to insure that the perturbation decays away from the shear layer. Now I must match the three solutions together.

The requirement that the interface displacement be continuous across the interfaces located at $\bar{\zeta} = \pm 1$ leads to the continuity of w' across these interfaces as one of the linearized matching conditions. The other condition, namely that w'_{ζ} be continuous is found by integrating equation (5.1) over a vanishing region ϵ about each interface. Then after working through the application of these conditions at both $\zeta = \pm 1$, I find:

$$\Gamma \tan \Gamma = \sqrt{\left(\frac{\delta K}{2}\right)^2 + \ell^2} \tag{5.31}$$

This result may now be included in equation (5.24) to yield:

$$\Lambda_i^2 = \frac{\ell^2 \cos^2 \Gamma}{\Gamma^2} \left[-\tfrac{1}{2} Ri \cos\alpha - 2\delta K(\phi+1) - 2\left[\delta K(\phi+1)\right]^2 \right] \tag{5.32}$$

or, if the approximation is made that $\ell^2 + \left(\frac{\delta K}{2}\right)^2 \simeq \ell^2$, then $\ell^2 \simeq \Gamma^2 \tan^2 \Gamma$ by (5.31), so that (5.32) becomes:

$$\Lambda_i^2 = \sin^2 \Gamma \left[-\tfrac{1}{2} Ri \cos\alpha - 2\delta K(\phi+1) - 2\left[\delta K(\phi+1)\right]^2 \right] \tag{5.33}$$

which shows that the fastest growing disturbances have vertical wavelengths near $\frac{2}{n+1/2}$ multiples of the half shear layer depth δ. Therefore, since ℓ^2 can be no larger than Γ^2, the dominant instability mode is presumed to have both a vertical and a cross-stream wavelength of approximately twice the shear layer thickness so far as this crude estimate is able to determine. Also, since the trigonometric part of w' goes like $\cos \Gamma \bar{\zeta}$, then it is even in $\bar{\zeta}$, and since the lowest mode $n=0$ is assumed to be the most unstable, then w' does not change sign in the interval as shown in Figure 8. This is precisely the mode that one would expect to see overturn in counter-rotating longitudinal cells with height and width on the order of the shear layer thickness.

6. Skewed Underlying Wave

The analysis up to this point has been concerned with the special case where the underlying wave propagates along the direction of the shearing plane. This section seeks to discover what changes are involved when the more general case is encountered. Figure 9 shows the geometry under consideration.

The frame of reference takes $\vec{k}$ to lie along the $\bar{X}$ -axis, where again, the underlying wave is stationary. $W(z)$ as defined in the figure is in cartesian space from a frame in which $W(0) = 0$. This frame is translated $(-\tilde{u}_0)$ and then transformed so that in the new curvilinear system:

$$u(\bar{\zeta}) = \tilde{u}_0 + \tilde{u}(\bar{\zeta}) \quad ; \quad v(\bar{\zeta}) = \tilde{v}(\bar{\zeta}) \tag{6.1}$$

From the figure it is evident that:

$$\frac{\tilde{v}(\bar{\zeta})}{\tilde{u}(\bar{\zeta})} = \tan\theta \tag{6.2}$$

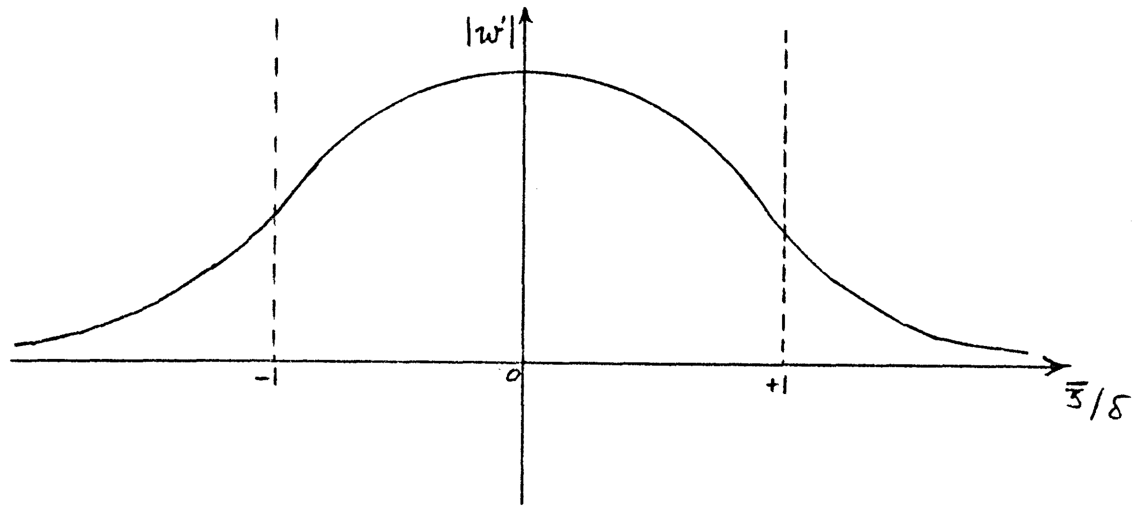
For longitudinal roll behavior, the disturbance should be characterized by an extremely small wave number in the direction of the absolute shear. Let $\hat{\gamma}$ be the unit vector in this direction, and let $\vec{\eta} = \beta\hat{\imath} + \ell\hat{\jmath}$, where the perturbations have the form pert$(\bar{\zeta})\exp(i\beta\bar{x} + i\ell\bar{y} + i\omega t)$; then

$$\vec{\eta}\cdot\hat{\gamma} = 0 \tag{6.3}$$

so

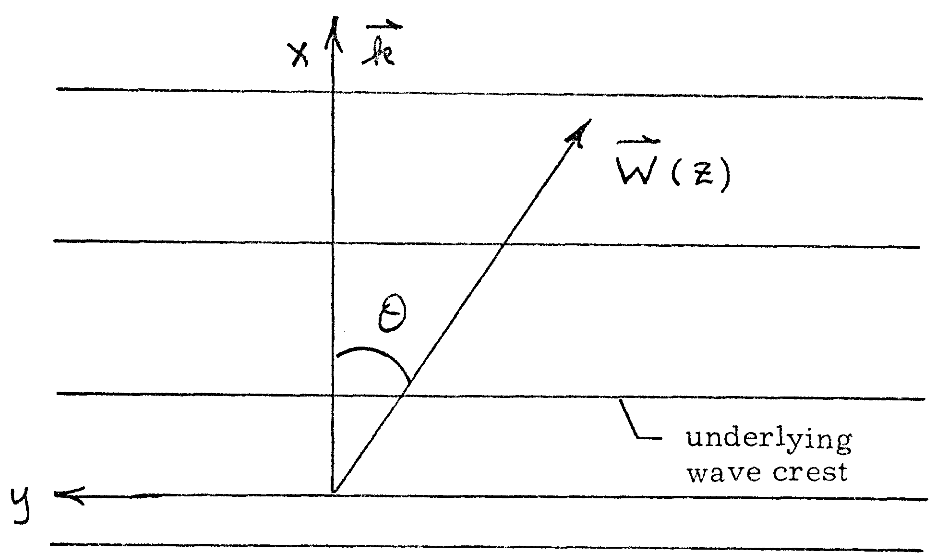
$$\beta\cos\theta + \ell\sin\theta = 0 \tag{6.4}$$

With the assumption of a local analysis in $\bar{X}$, the arguments displayed earlier



Most Unstable Mode

FIGURE 8



Skewed Underlying Wave Model

FIGURE 9

in this paper lead to the following perturbation equations:

$$i(\omega+\beta\tilde{u}_o+\beta\tilde{u}+\ell\tilde{v})u' + w'\tilde{u}_\xi - \delta K w'(\tilde{u}_o+\tilde{u}) + i\beta p' = 0 \tag{6.5}$$

$$i(\omega+\beta\tilde{u}_o+\beta\tilde{u}+\ell\tilde{v})v' + w'\tilde{V}_\xi + i\ell p' = 0 \tag{6.6}$$

$$i(\omega+\beta\tilde{u}_o+\beta\tilde{u}+\ell\tilde{v})w' + 2\delta K(\tilde{u}_o+\tilde{u})u' + p'_\xi + Ri\cos\alpha\,\rho' = 0 \tag{6.7}$$

$$i(\omega+\beta\tilde{u}_o+\beta\tilde{u}+\ell\tilde{v})\rho' + w'\rho_\xi = 0 \tag{6.8}$$

$$i\beta u' + i\ell v' + w'_\xi - \delta K w' = 0 \tag{6.9}$$

Combining equations (6.2) and (6.4) gives:

$$\beta\tilde{u} = -\ell\tilde{V} \tag{6.10}$$

which can be used to good advantage in simplifying the equations. After some algebra, I find the system reduces to:

$$\begin{aligned} w'_{\xi\xi} - \delta K\,\frac{\omega+2\beta\tilde{u}_o+\beta\tilde{u}}{\omega+\beta\tilde{u}_o}\,w'_\xi - w'\Big\{\frac{\ell^2}{\cos^2\theta} + \frac{\ell^2}{(\omega+\beta\tilde{u}_o)^2\cos^2\theta}\Big[Ri\cos\alpha\,\rho_\xi \\ + 2\delta K\,\tilde{u}_\xi(\tilde{u}_o+\tilde{u}) - 2\big(\delta K\cos\theta[\tilde{u}_o+\tilde{u}]\big)^2\Big] \\ + \frac{\ell\delta K}{\omega+\beta\tilde{u}_o}\tan\theta\Big(2\delta K[\tilde{u}_o+\tilde{u}] + \tilde{u}_\xi\Big)\Big\} = 0 \end{aligned} \tag{6.11}$$

Based on the special case treated earlier, I make the modal assumption from the start, and let $\omega+\beta\tilde{u}_o = \Lambda_r + i\Lambda_i$. Then performing the integration process on the imaginary part of the equation, I find:

$$0=\int_{-\infty}^{+\infty}(w'_\xi)^2 d\xi + \int_{-\infty}^{+\infty}(w')^2\frac{\ell^2}{\cos^2\theta}\left[1-\frac{Ri\cos\alpha\,\rho_\xi + 2\delta K(\tilde{u}_o+\tilde{u})\left[\tilde{u}_\xi - \delta K\cos^2\theta(\tilde{u}_o+\tilde{u})\right]}{\Lambda_r^2+\Lambda_i^2}\right]d\xi \tag{6.12}$$

and indeed this compares favorably with equation (5.10). The same operation on the real part of the equation yields little helpful information.

A better method for obtaining information concerning constraints on Λ_r when $\Lambda_i \neq 0$ is to perform the $\int_{-\infty}^{+\infty} w'(\ )\, d\xi$ operation on the equations found by taking the sum and the difference of the real and imaginary parts of equation (6.11). $\int_{-\infty}^{+\infty} w'(Re + Im)\, d\xi$ gives:

$$\int_{-\infty}^{+\infty} (w'_\xi)^2\, d\xi + \int_{-\infty}^{+\infty} (w')^2 \left[\frac{\ell^2}{\cos^2\theta} - \frac{\beta\, \delta K}{\Lambda_r} \left(\delta K[\tilde{u}_0 + \tilde{u}] + \tfrac{3}{4}\, \tilde{u}_\xi \right) \right] d\xi = 0 \qquad (6.13)$$

and $\int_{-\infty}^{+\infty} w'(Re - Im)\, d\xi$ gives:

$$\int_{-\infty}^{+\infty} (w')^2 \left\{ \frac{\ell^2}{\cos^2\theta}\ \frac{Ri \cos\alpha\, \rho_\xi + 2\delta K[\tilde{u}_0 + \tilde{u}][\tilde{u}_\xi - \delta K \cos^2\theta[\tilde{u}_0 + \tilde{u}]]}{\Lambda_r^2 + \Lambda_i^2} - \frac{\beta \delta K}{\Lambda_r}\left(\delta K[\tilde{u}_0 + \tilde{u}] + \tfrac{3}{4}\, \tilde{u}_\xi \right) \right\} d\xi = 0 \qquad (6.14)$$

If Λ_i is to be different from zero then equation (6.12) indicates that $\delta K[\tilde{u}_0 + \tilde{u}]\tilde{u}_\xi$ must be positive over a significant portion of the domain. This in turn requires the last bracketed term in (6.14) to be positive (the minus sign is not included), and equation (6.13) independently demands the same. Therefore, if the shear is positive, then $\delta K[\tilde{u}_0 + \tilde{u}]$ must be positive, so:

$$\frac{\beta\, \delta K}{\omega_r + \beta \tilde{u}_0} > 0 \qquad (6.15)$$

If the shear is negative, then $\delta K[\tilde{u}_0 + \tilde{u}]$ is negative and so:

$$\frac{\beta\, \delta K}{\omega_r + \beta \tilde{u}_0} < 0 \qquad (6.16)$$

Recall that $\tilde{u}$ was taken to have an average value of zero over the shear layer, therefore $\tilde{u}_0$ may have the dominant force in determining the sign of the integrated $\tilde{u}_0 + \tilde{u}$ term, which implies that $\tilde{u}_0$ must have the same sign as δK in (6.15), and the opposite sign of δK in (6.16). This in turn allows me to limit the possible range of $c_r = -\omega_r/\beta$ (the phase speed of the perturbation in the $\bar{X}$ -direction from the wave coordinate frame) to:

$$|c_r| < |u_0| \qquad (6.17)$$

where I use the assumption that c_r should be equally likely to assume either sign. I admit that the conclusion (6.17) is very tenuous, but it is presented as an imperfect inference of this analysis in lieu of a more definitive statement.

Look again at equation (5.10) and compare it with equation (6.12) as it is rewritten below:

$$\int (w'_\xi)^2 d\xi + \int (w')^2 \frac{\ell^2}{\cos^2\theta} \left\{ 1 - \frac{Ri \cos\alpha \, \rho_\xi + 2\,\delta K \cos\theta [\tilde{u}_0 + \tilde{u}]\, \tilde{W}_\xi - 2(\delta K \cos\theta)^2 (\tilde{u}_0 + \tilde{u})^2}{\lambda_r^2 + \lambda_i^2} \right\} d\xi = 0 . \qquad (6.18)$$

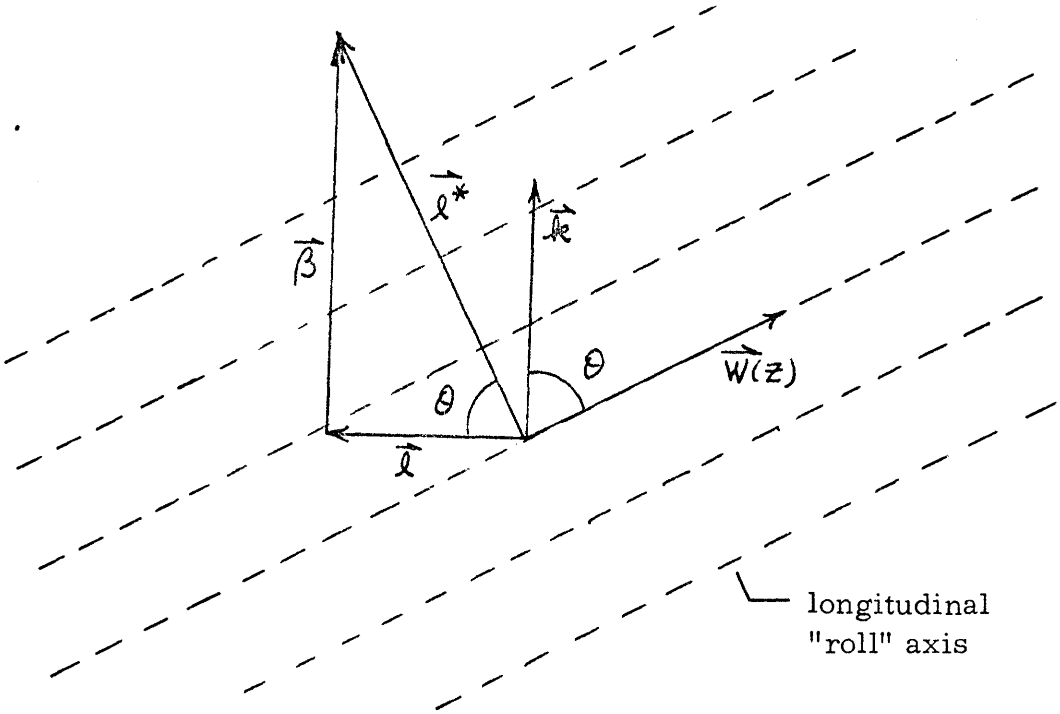
The factor $\ell^2/\cos^2\theta$ replaces the previous ℓ^2 factor, but they are really equivalent. Figure 10 shows that the cross-axis wave number of the disturbance, designated ℓ^*, is equal to $\ell^2/\cos^2\theta$, and so the spacing of the "rolls" in section 5 corresponds exactly to the spacing of the "rolls" in this section. Changes in the curvature terms can also be easily explained, for now it is the projection along the shear axis of the product of curvature and that part of the velocity field which flows in the direction of the underlying wave that is important. As displayed in Figure 11 the projection of $\delta K[\tilde{u}_0 + \tilde{u}]$ along $\hat{\gamma}$ is $\delta K \cos\theta [\tilde{u}_0 + \tilde{u}]$, and this quantity multiplies the absolute shear in the problem to give rise to the "instability" term, while it multiplies itself to produce the pseudo-inertial frequency term. As a result then it can be said that for a given value of absolute shear and curvature, the presence of a non-zero angle between the wave vector and the shear axis will tend to weaken the instability (and stabilize it altogether if θ is large enough) since:

1) $|\tilde{u}_0|$, the phase speed of the wave along the x-axis relative to the frame in which the flow is zero at $z = 0$, is reduced,

2) the curvature-shear term is reduced while the stratification term remains unaltered, and

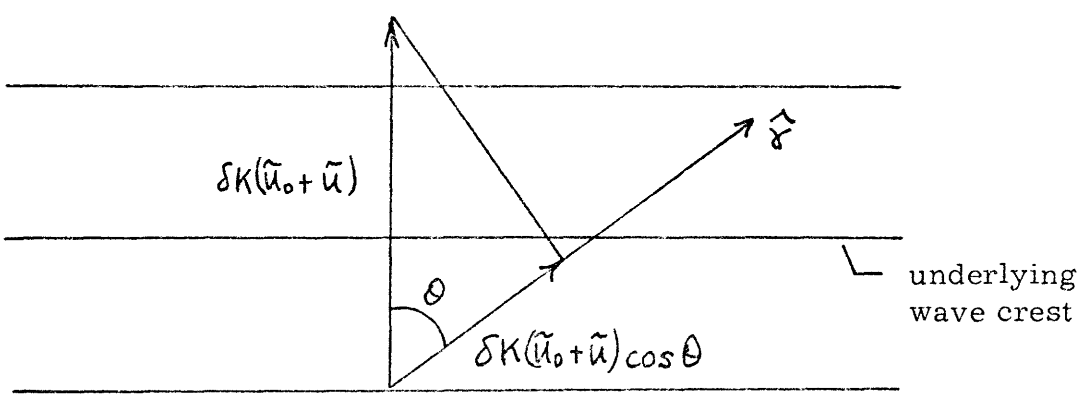
3) the denominator, $\lambda_r^2 + \lambda_i^2$ appears to be greater since λ_r^2 can no longer be definitely set equal to zero.

Furthermore, it is believed that the reduction in the pseudo-inertial term, while promoting instability, does not have the magnitude to offset reason number 2 above.



Cross-stream Wave Number of Perturbation

FIGURE 10



Projection of $\delta K[\tilde{u}_0 + \tilde{u}]$ Along Shear Axis

FIGURE 11

7. Interpretation of Analysis

In order to understand how this instability of certain regions of the underlying wave affects the entire flow in the shear layer, a certain amount of speculation is necessary. From the necessary conditions for instability derived in section 5, the term which may be said to drive the instability is $\delta K(\phi+1)$. If the underlying wave is a long wave travelling in the direction of the upper layer, then the wave will be overtaking all parts of the flow and the velocity field as seen from the wave frame will have no zero point. Placing an observer in a frame such that he sees the velocity at the center of the shear layer as zero, he is expected to see a disturbance grow slightly whenever an unstable portion of the wave propagates past him. For the sake of the argument, let this region be a small section in the center of the wave crest, then he sees a ripple strengthen slightly as the crest passes. After this occurence, the ripples propagate in the cross-stream direction until one period passes and the next crest intensifies the ripples a little more. In effect, the underlying wave transports a series of generating lines through the flow which give a selective "kick" to the disturbances that are present, somewhat like a parade of inverse steam rollers. With no dissipation, even an extremely weak series of amplifications will eventually build up under the action of a long train of waves.

As a side comment on the instability problem, there is also a question as to how the propagation of these ripples is affected due to the curvature induced forces in regions where $\mathcal{N}_i = 0$. Setting $\mathcal{N}_i = 0$ in equation (5.20) produces:

$$\mathcal{N}_r^{\,2} = \frac{\ell^2}{\left(\frac{\delta K}{2}\right)^2 + \Gamma^2 + \ell^2} \left[\tfrac{1}{2} Ri \cos\alpha + 2\,\delta K(\phi+1) + 2\left[\delta K(\phi+1)\right]^2 \right] \quad (7.1)$$

Therefore the phase speed of the disturbance will change as the wave propagates by, so a certain degree of ray bending is to be expected. But since the underlying wave field is assumed to be periodic, whatever its shape, there should be no net focusing of the energy of these disturbances. These remarks point up what may be considered as the basic effect of the centrifugal forces in this problem: they alter the restoring force of the stratification. Where they overcome this force, the disturbance intensifies, and does not propagate;

otherwise they merely alter the restoring force and thereby change the propagation characteristics of the disturbances. Also, since it is assumed that the curvature induced forces become comparable to the stratification related forces at some points to produce the instability, then it is evident that changes in the propagation characteristics may be substantial, and a properly formulated global analysis should replace the speculation in which I have indulged here.

This brings me back to the main question of this paper. Can this instability exist for Richardson numbers greater than those which indicate the Kelvin-Helmholz instability? The reason that I am not so interested in the possibility of both of these processes occuring simultaneously lies within the foregoing arguments of this section. The intensification period for the Görtler type of instability is a small fraction of the period of the underlying wave, while the Kelvin-Holmholz mechanism operates over nearly the entire period. I say nearly because I have not established how that instability is affected in regions of significant curvature. Therefore, I would expect the Görtler instability mode to be swiftly overwhelmed in the debris of the Kelvin-Helmholz initiated breakdown.

So, for the purpose of answering the proposed question I will take equation (5.24) and set $\lambda_i = 0$. While it is true that my approximate system of equations predicts that $w' = 0$ when $\omega = 0$ with $\beta = \tilde{V} = 0$, this is because the smaller order advective terms were discarded. It is reasonable to go ahead with $\lambda_i \sim 0$ and use (5.24) as a first test to see if a necessary condition for instability can be satisfied:

$$\left[\delta K(\phi+1)\right]^2 + \delta K(\phi+1) + \frac{1}{4} Ri \cos\alpha = 0 \tag{7.2}$$

This may be solved to give:

$$\delta K(\phi+1) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - Ri \cos\alpha} \tag{7.3}$$

$(\phi+1)$ represents the scaled speed of the wave in the original coordinate frame

X^*, therefore from equation (3.4)

$$\phi + 1 = \frac{1-kh}{1+kh} \pm 2\sqrt{(ri-1)\frac{kh}{1+kh}} \qquad (7.4)$$

or

$$\phi \simeq \pm 2\sqrt{(ri-1)\,kh} \qquad (7.5)$$

From the definition of Ri and ri I may write:

$$ri = \frac{Ri}{4\delta K} \qquad (7.6)$$

and when this relation is substituted into equation (7.5) the result is:

$$\phi \simeq \pm \sqrt{\frac{Ri\,h}{\delta}} \qquad (7.7)$$

where it has been assumed that $\delta K << \frac{1}{4}$. Now, in order to satisfy (7.2) I wish to use the circumstances which maximize $|\delta K(\phi+1)|$. Clearly, $\phi > 0$ is the most helpful situation and leads to the relation:

$$2\delta K = \frac{-1 \pm \sqrt{1 - Ri\cos\alpha}}{1 + \sqrt{Ri\,h/\delta}} \qquad (7.8)$$

Furthermore, if I insert the maximum value of $|K|$ with a minus sign, and neglect $\cos\alpha$, then:

$$2a\delta k^2 = \frac{1 \pm \sqrt{1 - Ri}}{1 + \sqrt{Ri\,h/\delta}} \qquad (7.9)$$

The numerator must be real, therfore $Ri \leq 1$ in this approximation. So let $Ri = 1$ and solve for the wave amplitude a with $\delta = 5m$, $h = 30m$, $k^{-1} = 10^2 m$, which are reasonable values for an oceanic transition layer. I find this leads to:

$$a \sim 3 \times 10^2\ m$$

which is certainly absurd.

Next consider the case where $Ri = \overline{Ri_c}$, the value of the maximum Richardson number for the Kelvin-Helmholz instability. By the usual definition:

$$\overline{Ri} = \frac{g\Delta\rho}{\rho}\frac{2\delta}{4U^2} = \frac{1}{2}Ri \qquad . \qquad (7.10)$$

Therefore $\overline{Ri}_c = 1/4$ corresponds to $Ri = 1/2$, and using this in (7.9) with the same parometers as before I find a wave amplitude of:

$$a \sim 10^2 m$$

still an absurd result. It is on this basis then that I find it impossible for the Gortler instability to exist without the Kelvin-Helmholz instability, and therefore, it has no chance of developing in the type of flows inwhich this analysis applies.

One final observation may be advanced to try to bring this Görtler instability into its own. In (7.2) the pseudo-inertial term acts with the stratification term and therefore places a limit on the size of $\phi+1$ in that:

$$\left[\delta K(\phi+1)\right]^2 < \delta K(\phi+1) \qquad (7.11)$$

Since this term was considered of dubious validity in section 5 where the outer layers were considered, might it also be questioned in the middle layer? I have been unable to generate decisive arguments from this analysis, but since the question has been posed I have calculated the inferred wave amplitude for $Ri = 1/2$ after neglecting the squared term in equation (7.2) to find:

$$\begin{aligned} \delta K(\phi+1) &= -\frac{1}{4} Ri \cos\alpha \\ a &\sim 10^2 m \end{aligned} \qquad (7.12)$$

which clearly indicates the non-existence of the instability for Richardson numbers greater than $\overline{Ri}_c$ even when the pseudo-inertial term is dropped.

8. Conclusion

In this paper I have developed a model of a stratified shear layer with large scale undulations for the purpose of investigating the role of centrifugal instability of the Taylor-Görtler variety in the dynamics of the interaction of long internal waves with the transition region at the boundary of a surface mixed layer. The system equations are valid only for linear long waves with wave slope much less than one, and the underlying wave velocity field has been all but neglected in the final form of the model.

A necessary condition for instability was found to be:

$$\delta K \tilde{u} \tilde{u}_{\bar{\xi}} > \frac{\mathcal{N}_i^2}{2} - \frac{1}{2} Ri \cos\alpha \, \rho_{\bar{\xi}} + (\delta K \tilde{u})^2 > 0 \tag{8.1}$$

over some region of the flow field, and therefore the signs of the curvature, velocity, and velocity shear must be related in such a way that:

$$\delta K \tilde{u} \tilde{u}_{\bar{\xi}} > 0 \tag{8.2}$$

Solving the system for a simple linear distribution of density and velocity in the shear layer, with the further substitution of mean values for $\bar{\xi}$ -dependent terms, a relation for the disturbance growth rate was found. The zero growth rate balance of terms (neutral stability case) demanded that:

$$2\delta K = \frac{-1 \pm \sqrt{1 - Ri}}{1 + \sqrt{Ri\, h/\delta}} \tag{8.3}$$

Upon the substitution of reasonable mean flow parameters h, δ, and k, it was found that Richardson numbers greater than or equal to the critical value for Kelvin-Helmholz instability required a long wave amplitude one order of magnitude greater than the depth of the shallow layer. This result was obtained from the configuration most favorable to the instability since the wave crests were aligned perpendicular to the plane of the mean shear, and the propagation of this wave was chosen such that the velocity as measured from the wave frame was the maximum, and possessed no zero points. If there had been a zero relative velocity in the shear layer, then the flow would have had to have been considered in two pieces; that above, and that below the zero level. However, it is easy to see that each of these layers would have had a smaller mean velocity than the case considered, and therefore would have been less favorable to the instability. Therefore, I conclude that the Taylor-Görtler instability is simply not competitive with the well known Kelvin-Helmholz instability in cases where this analysis is applicable.

A possibility that still remains is that there may exist conducive regions of high curvature on finite amplitude, non-linear interfacial waves. However, this situation would require a careful consideration of the finite amplitude dynamics, and this is clearly beyond the scope of this paper.

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