WEAKLY INHOMOGENEOUS TURBULENCE THEORY WITH APPLICATIONS TO GEOPHYSICAL FLOWS

by

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ABSTRACT

In this study, we investigate the problems of inhomogeneous turbulence. The main goal has been to find the connections between analytic turbulence theories, notably the formalism of the direct interaction approximation (DIA), and turbulence phenomenology. To simplify the results from an inhomogeneous DIA model, we have employed a diagonalization procedure and an assumption of weak inhomogeneity. The diagonalization procedure expands the off-diagonal correlations in terms of the diagonal correlations and the mean shear, and the weakly inhomogeneous assumption assumes a scale separation between the eddies and mean field. As a consequence, the DIA diagonalization process becomes an asymptotic expansion. It also leads to the recovery of some inhomogeneous quantities in the configuration space. Therefore useful information about spatial transfer is extracted from what is basically a spectral transfer model.

We consider a simple two-dimensional shear flow and a passive tracer flow as practical examples of applications of the theory. In both cases, we find expressions almost identical to those of classical eddy diffusion equation. The eddy diffusivities in our results are positive definite and can be obtained without recourse to empirical constants. We use an abridged form of the DIA to calculate eddy diffusivities for large-scale atmospheric motions in which the energy spectrum is prescribed. Under the assumption that the inhomogenous eddy diffusivities represent the total eddy diffusivities, these calculated coefficients agree well with observed values.

A directly simulated two-dimensional, barotropic turbulence model is designed to verify various assumptions made in weakly inhomogeneous theory. The experiment is especially concerns in the calculation of the averaged Green's function defined in the DIA. From this experiment the theoretical prediction of the fluctuation-dissipation theorem is tested against simulated results. The agreement is excellent. The simulated energy spectrum and decorrelation rates are also compared with the theoretical results. The analytic turbulence theories hold convincingly for all our experiments.

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Edward N. Lorenz — Professor
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DEDICATED TO

DR. JACKSON HERRING

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Chapter 1: Introduction

This study was inspired by the following three considerations,

(1) the common treatment for a climate model and a turbulence model;

(2) the importance of inhomogeneity in a climate model, which is largely neglected in current turbulence studies;

(3) the very terse expression for eddy diffusion postulate, in contrast with generally intricate results from modern turbulence theories under the same physical situation.

Each of the above topics will be discussed in a section of this chapter. The object of this thesis is mainly to explore for the third topic. Our study starts by assuming the simple physical situation of a weakly inhomogeneous shear embedded in a homogeneous turbulent flow. A stochastic model, based on the method called the direct interaction approximation (Kraichnan, 1959) is subsequently developed to solve this problem. After a series of simplifications, the eddy diffusion equation finally emerges. It is hoped that the very convenient form for inhomogeneous dynamic fluxes may lend to this study a wide range of applicability. The remaining part of this thesis is divided into two subjects: the first, an example which illustrates how to use our method to calculate atmospheric tracer transfer; the second, a critical investigation of a fundamental concept in modern turbulence theories--namely the fluctuation-dissipation theorem which is assumed in our model. We have used direct numerical simulations to conduct a verifying experiment.

In this introductory chapter we will emphasize the historical background which motivates our study, leaving most of the mathematical details for later chapters.

1.1 Statistical Mechanical View of Climate and Turbulence Models

The two inseparable keywords connecting climate and turbulence problems are "nonlinear" and "statistical." It is the nonlinearity of flow systems which makes the output signals (future states) so intricate and unpredictable that a statistical description is necessary. We can formulate the common goal of both climate and turbulence models as follows: assume an ensemble is a collection of infinite realizations of the same flow system but evolving in the various dynamic states. Each state can then be described by a set of dynamic variables and denoted by a point in the phase space spanned by the same set of dynamic variables. At any given moment an ensemble is represented by a cloud of points in phase space. A probability density function can be used to measure their distribution. From one time to another each point moves in a unique trajectory according to the same deterministic law (perhaps in the general form of the Navier-Stokes equations). Consequently, the probability distribution function also changes. The time evolution function for the probability distribution is called the Liouville equation. An important problem of statistical mechanics is to find the stationary solution of the Liouville equation or the equilibrium probability distribution of the ensemble. Thereafter, all the other ensemble averaged variables can be deduced by integration of the product of the variable and the joint probability density function over the phase space.

As elegant a concept as it is, the Liouville equation cannot be solved without formidable mathematical complexity, although a few attempts have been made along these lines with various degrees of approximation and success, notably by Edwards (1964), Herring (1965), and Salmon et al. (1976).

Kraichnan (1958) indicated that the equation of motion would impose constraint in the sense of least square on the probability distribution function only at the level of second-, third- and fourth-order moments. (A moment is defined as an ensemble averaged correlation of velocity, e.g., $\langle V^n \rangle$ is the n^{th} moment). This means that a reduction of dimensionality of the Liouville equation into a system of low-order moment equations appears to be a feasible approach. The equation for the moments can be derived from a deterministic governing equation without showing the probability density function explicitly. The ultimate goal is then to find the stationary solutions of low-order moments, which after interpretation correspond to elements like energy spectrum, eddy fluxes etc., in climate models.

Unfortunately, the equations in this moment hierarchy always introduce higher order moments as unknowns. Hence, it is necessary to supply additional conditions, called closures, to solve the problem. The traditional closure is just to find a relationship between higher and lower order moments. In more complicated physical problems, the closure becomes more difficult to find. The same mathematical incompleteness on leads to, for example, an early turbulence closure scheme by Millionshtichikov (1941), or a modern day closure for climate models by Stone (1972).

A completely revolutionary thinking, yet deeply rooted in the above mentioned tradition, was proposed by Kraichnan (1958, 1959) and then elaborated in a series of studies by Kraichnan, Herring and Leith, among others (see Leslie, 1973). The archetypal model of this approach is called the direct interaction approximation (DIA). It seeks a reexpansion (renormalization) of cumulants, which are the residual parts of moments unrelated to the lower-order moments according to the rules of the normal distribution, at the lowest possible level (see appendix B and section 2.2). Among the many versions of DIA and related models, one of the physically more illuminating interpretations is that the analytic turbulence model of this type corresponds to the exact solutions of certain stochastic equations, in which a random forcing and a linear eddy drag term take the place of the usual nonlinear interaction term in the deterministic formalism. Since the stochastic model leads to a system of inhomogeneous linear differential equations, all the second-order statistical elements can be calculated by incorporation of an ensemble-averaged Green's function. Therefore, the closure problem has been circumvented automatically (see Leith, 1971; Salmon, 1976).

The stochastic equation of the DIA is a special case of the generalized Langevin equation, which also describes various nonlinear phenomena. A famous example is Brownian motion (see chapter 2). A great reward in connecting DIA to a stochastic model is to let some fundamental features of nonlinear systems become apparent. These properties were obscure in the Navier-Stokes equation. A large part of Chapters 2 and 5 will discuss this subject.

1.2 The Importance of Inhomogeneity

It comes as no surprise that the same Langevin equation also appears in some climate models (Hasselman, 1976; Lemke, 1977), though a direct extension from analytic turbulence theories to climate models is still beyond reach. The bottleneck for further direct application may be the difficulty in implementing inhomogeneity in the context of modern turbulence theories. Except for a few preliminary studies (e.g. Kraichnan, 1964a; 1972 Leslie, 1973), the major concern in modern turbulence theories is the spectral transfer between eddies with different scales. Hence some spatial symmetry, which is either a homogeneity (the invariance of statistical properties upon shifting positions along an axis) or an isotropy (the invariance of statistical properties upon turning different directions), is always assumed. Yet in the real world nature seldom attains such simplicity. More often,

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* Intuition often betrays meteorologists. Here we idealize the long scale atmosphere motion as turbulent, mixing process. The description of a diffusive nature caused by nonlinear interactions should only apply on the very restricted cases, for example, the fully-realized, isotropic turbulence. Any time when flow system consists of more than one dynamic variable (e.g., a thermal effect) the diffusion law will be in jeopardy.

We should note here that even the DIA suggests strongly a tendency back to isotropy for the N-S equation, it is not clear that the same DIA will lead automatically to a diffusion law for more generalized flow systems. The eddy diffusion equation we derived in this study will only be a special case of the DIA model.

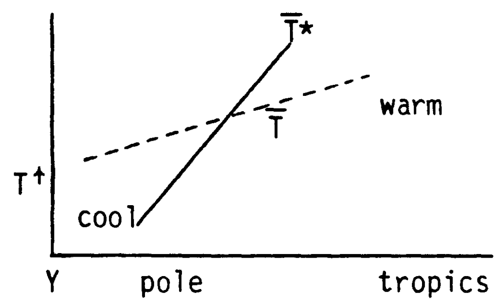
as in climate problems, it is the inhomogeneity of flow systems which demand explicit treatment.

Intuitively*, one may expect that the nonlinear interaction between eddies tends to reduce the inhomogeneity in the flow, analogous to the effect that the random collisions between molecules equalize the local temperature in a room. If this is true we then consider a hypothetical situation in which the energy diagram is idealized as the following:

[External forcing ⇄ mean] flow ⇄ eddies ⇄ eddies...⇄ eddies → friction

Inside the box the inhomogeneity is introduced into the flow by the asymmetric external forcing. Being slow and small in magnitude, but energetically significant, the link between external forcing and mean flow can be called a "physical process." Outside the box it is the more rapid, fluctuating "dynamic" process that smooths out the inhomogeneity and prevails perhaps on the higher level (see the next section where we classify the spatial symmetry: the isotropy as 1, homogeneity as 2 and inhomogeneity as 3) of symmetry. We expect that the symmetric statistical elements (e.g., the diagonal correlations of the Fourier components...) can describe quite faithfully the major energetic cascade of flow lying symbolically outside the box.

The above picture can neatly fit within the notion of the atmospheric general circulation. A simple zonally averaged climate model can be pictured as:

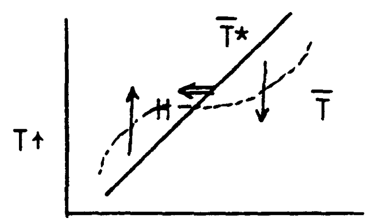


The solid line displays the radiative equilibrium temperature $\overline{T}^*$ and the dotted line $\overline{T}$ is the true temperature. By Newtonian heating $Q = \overline{T}^* - \overline{T}$, zonal available potential energy ZPE will be generated as heating occurs in the tropics and cooling near the pole. To balance this heating, a climate model demands $Q(Y) \approx \partial H/\partial Y$ where H is the eddy heat flux. Obviously the physical process ([Q] $\rightarrow$ ZPE) is mainly maintained by local unevenness (inhomogeneity in north-south direction). Note that only the divergence of H, instead of H appears explicitly in this type of climate model.

On the other hand, the eddy energy, EE, is produced by internal deformation of the mean flow through baroclinic instability and various paths of nonlinear cascade. As effectively argued by Salmon (1978) and Haidvogel et al. (1980) the dynamic process (APE $\rightarrow$ EE) can be described by a baroclinic but horizontally homogeneous turbulence model. Such flow is characterized by a constant mean temperature gradient and constant heat flux H.

But the constant heat flux H, by using up APE, tends to flatten the mean temperature profile, which cannot be reflected in a fixed mean temperature gradient model like Salmon's (1978) or Haidvogel's

(1980). Graphically:



Further departure of $\overline{T}$ from $\overline{T}^*$ will be enhanced unevenly along the Y-coordinate. Both radiative heating and cooling processes and the inhomogeneity will increase until a balance (not necessarily equal in magnitude) between the constant (homogeneous) part and the Y-varying (inhomogeneous) part of the heat flux has been achieved.

This distortion of the constant mean temperature gradient, a form of dehomogenization has been long known in thermal convection problems. The interesting part of the above simple scheme is that the radiative heating appears in the interior of the atmosphere, hence the impact of lateral boundaries (another source of inhomogeneity) can be minimized. Nevertheless, a complete analytical radiative-dynamic turbulence model is still a few steps away from realization.

In this study we have to settle for a less ambitious goal, i.e., a weakly inhomogenous model. A spatially slowly varying, time-independent mean shear is superposed upon an existing homogeneous turbulent flow. With periodic boundary conditions the whole problem is reduced to the calculation of diagonal correlations of Fourier decomposed dynamic variables. A DIA expansion is used, but the results are excessively complicated. The bulk of our effort is to draw useful information from a large number of inhomogeneous statistical elements. This task cannot be accomplished without recourse to the classical turbulence theories. The clue comes from the familiar eddy diffusion equation.

1.3 Eddy Diffusivity and Diagonalization

To a climate modeler, the most urgent question is to find the local values of various eddy fluxes in terms of the mean field or mean gradient. It is common to write the eddy transport process by a gradient diffusion hypothesis, $\langle q'v'\rangle = -D\,\partial\langle q\rangle/\partial y$ where q' is the fluctuating part of a transportable property $q = \langle q\rangle + q'$, and v' and y are the respective eddy velocity and spatial coordinate in which the mean field $\langle q\rangle$ is varying. D with a dimension of $(\text{velocity})^2$ (time), is called the eddy diffusivity. In classical turbulence theory D has a phenomenological form (Taylor, 1921, see chapter 4.1):

$$D \approx \int_0^\infty \langle V'(0)\;V'(\tau)\rangle\, d\tau \quad .$$

The integrand is a time-lag velocity correlation following marked flow particles. The Lagrangian aspect of D will not be emphasized here for the time being. The eddy transport equation can be written approximately as:

$$\langle V'q'\rangle \approx -\;[\langle V'V'\rangle\tau']\;\frac{\partial\langle q\rangle}{\partial Y} \quad .$$

where $\langle V'V' \rangle$ is a zero-time separation velocity variance and τ' is a correlation time. The above form has only been derived with additional assumptions. The rationale for its applicability is that when the eddy correlations in D are more "global" than the local value of $\langle q'V' \rangle$, D can be treated as a near-constant. We can even generalize the above notion a little further in the following.

We assign a numerical value to each state of spatial symmetry following turbulence terminology:

(1) isotropy,

(2) homogeneity,

(3) inhomogeneity.

The usefulness of the eddy diffusion law will depend on the rule of thumb:

$$b < a \quad \text{for} \quad a,b = 1,2,3$$

where

$$\langle q'V' \rangle^{(a)} = - D^{(b)} \frac{\partial \langle q \rangle^{(a)}}{\partial Y}$$

The superscripts in above expression mark the state of spatial symmetry.

In this study, we look particularly for the form

$$\langle q'V' \rangle^{(3)} = - D^{(1) \text{ or } (2)} \frac{\partial \langle q \rangle^{(3)}}{\partial Y}$$

whose derivation is called the diagonalization process, for the reason that the eddy flux $\langle V'q' \rangle^{(3)}$, composed solely of the off-diagonal correlation (see appendix C), will be represented in terms of the diagonal correlations in $D^{(2),(1)}$, as well as the inhomogeneous mean shear $\partial \langle q \rangle^{(3)} / \partial Y$. To achieve this form, a crucial assumption has to be made. A weakly inhomogeneous assumption assumes that a scale separation exists between eddies and inhomogeneous mean shear. The implication of this assumption is that the spectral transfer (energy) problem becomes completely independent of the spatial transfer problem, since a weakly inhomogeneous shear cannot produce enough eddy energy, say, to appear within eddy diffusivity D. This is consistent with our previous recognition of a globally uniform dynamic process. We speculate that no other assumption will be able to simplify a spectral expression into a useful form like the eddy diffusion equation.

1.4 Outline of the Study

In chapter 2, we will describe briefly the structure of turbulent stochastic models and how modern turbulence theories relate to them. The characteristic features of the nonlinear interaction term will be presented by a generalized Langevin equation, under condition of homogeneity. Chapter 3 will give an extension from homogeneous DIA models to inhomogeneous DIA models. There is no attempt to restrict the magnitude of inhomogeneity at the beginning. The DIA diagonalization procedure is subsequently applied. To recover the eddy diffusion equation, the further assumption of weak inhomogeneity is made. An example of a simple inhomogeneous shear flow is chosen to illustrate the expansion method. This chapter ends when a positive-definite eddy diffusivity is found for the vorticity flux.

To compare new and old turbulence theories, an inhomogeneous tracer model is set in Chapter 4 under a similar physical situation as those in classical turbulence theory, namely Taylor's eddy diffusion theorem. The result derived from the DIA formalism yields a compatible, if not identical, expression to Taylor's result. The equation of eddy tracer diffusivity is used to examine some aspects of atmospheric tracer problems. The observed atmospheric energy spectrum gives an estimation of the eddy damping rate through various closure schemes. The resulting eddy tracer diffusivity seems to be within the observed values.

In chapter 5 we present numerical simulations of a spectral model in order to examine the validity of the fluctuation-dissipation theorem, since we have made several critical simplifications based on it. The results fit theoretical predictions precisely.

The last chapter is a summary and some conclusions. It also includes a discussion of suggested further extensions of this research.

Chapter 2: Homogeneous Turbulence Theories--A Brief Review

Modern turbulence theories are often labeled as "analytic", not because they can provide analytic solutions for the Navier-Stokes (N-S) equation, but because they can construct statistical solutions of the N-S equation without referring to empirical values. In this chapter we try to give a brief review of the physical reasonings and mathematical consistancy which underlie the two most fundamental modern turbulence theories--the direct interaction approximation (DIA) and the eddy-damped Markovian model (EDM). We start by introducing a stochastic model which not only shares some statistical features with the N-S equation but also can be solved exactly. The DIA and EDM are found to relate respectively to certain types of stochastic models. Our choice of topics will favor those relevant to this study. More detailed accounts can be found in the comprehensive monographs by Orszag (1974) and Rose and Sulem (1978).

2.1 Langevin Equation

Consider a small macroscopic particle immersed in a liquid at thermal equilibrium. When the particle is sufficiently small, it will move perpetually in a random manner. This phenomena, first observed by the botanist Brown in the last century, is known as Brownian motion. In a simplified form we can describe the motion of this particle by a one-dimensional form of Newton's second law:

$$m \frac{dv}{dt} = f(t) \qquad (2.1.1)$$

It says the particle of mass m has a velocity $v \equiv dy/dt$ at time t, where y is the coordinate of center of mass. f(t) is the forcing which represents the interaction between the noted particle and other environmental particles. Since f(t) represents many other degrees of freedom in the system, it will be impossible to write f(t) in a precise form at a specified moment. More properly, a statistical description, i.e., an ensemble average, is taken for the above equation of motion. f(t) appears to fluctuate randomly and rapidly, and is expected to contribute nothing as we average over a long period of time. Suppose erdodicity holds for the system, i.e., the time average = ensemble average. Hence $\langle f(t) \rangle = 0$ and:

$$m \frac{d\langle V \rangle}{dt} = 0 \qquad (2.1.2)$$

But the above argument is obviously too crude for the following reason; since the ultimate equilibrium value of motion $\langle v \rangle$ can only be zero, if given an initial state $\langle v \rangle \neq 0$, Eq. (2.1.2) fails to describe the gradually vanishing trend of $\langle v \rangle$, the so-called irreversible relaxation toward equilibrium.

The conjecture is this: a part in a system which interacts (adjusts) vigorously and rapidly with many other parts of the system is very "reluctant" to deviate from its statistical equilibrium state. It seems that a nonlinear system possesses almost a "self-cured" mechanism to restore the equilibrium.

The designated recovering force $\langle f \rangle$ must be a function of $\langle v \rangle$ and satisfy the condition that $\langle f \rangle = 0$ when $\langle V \rangle = 0$. If $\langle V \rangle$ is not very far from the value zero, $\langle f \rangle$ can then be expanded in a Taylor series whose first nonvanishing term will give:

$$\langle f \rangle = -\eta \langle V \rangle$$

The negative sign (assume $\eta > 0$) is put to emphasize that $\langle f \rangle$ acts to reduce $\langle v \rangle$ to zero as time progresses. Let us go back to the original equation of motion. It can now be written as:

$$m \frac{dv}{dt} + \eta v = f'(t) \qquad (2.1.3)$$

where the approximation $\eta v \approx \eta \langle v \rangle$ is expected to introduce negligible error. The equation (2.1.3) is called the Langevin equation (see, for example, Srinivasan and Vasudevan, 1971). It states explicitly that the environmental force f consists of two parts: a slow varying part $-\eta v$ and a rapid fluctuating part f'. The linear friction form of $-\eta v$ is especially deceptive. As the analogy between Brownian motion and turbulence will soon reveal, it can be found:

(1) This term represents the total effect of nonlinear interaction within the system on the noted particle,

(2) It does not represent friction in the general sense since no energy will be lost if all parts of the system are added together,

(3) η depends on the dynamic state <v>, hence it cannot be a constant. Instead, η, the eddy viscosity, should be a time-dependent, nonrandom function which indicates a definite trend for <v>.

One of the greatest challenges in statistical mechanics is to find an explicit form for the eddy viscosity η. But first we should look for the analog between Brownian motion and turbulence. The two-dimensional incompressible, inviscid Navier-Stokes equation can be written in an orthogonal representation for the modal amplitude* ϕ_k as (see appendix A):

$$\frac{d}{dt}\phi_k = \sum_{p,q} a_{kpq}\phi_p\phi_q \qquad (2.1.4)$$

The above equation, though deterministic in form, will generate randomness similar to the phenomena of Brownian motion (see Orszag, 1974).

Our task here is to seek an alternative model equation which can simulate the statistics of Eq. (2.1.4) and preserve the integral constraints (e.g., energy and enstrophy), while also guaranteeing realizability (i.e., no negative value for energy). Other

* For example: Eq. (2.1.4) can be interpreted as a simplified form of two-dimensional vorticity equation in (5.2.1) where ϕ_k ... are the real and imaginary parts of Fourier components of eddy stream function which are linearly independent under the conditions of the incompressibility and reality.

considerations also include the feasiblility of using the equations in a computational model.

The easiest stochastic approximation for the above nonlinear system is to replace

$$\sum_{p,q} a_{kpq}\phi_p\phi_q$$

with a random forcing term, i.e.:

$$\frac{\partial}{\partial t}\phi_k = \sqrt{2}\sum_{pq} \psi_p{}^{I}\ \psi_q{}^{II} \qquad (2.1.5)$$

The stochastic processes $\psi_p{}^I$, $\psi_q{}^{II}$ satisfy, for any i, j:

(i) $\langle\psi_i{}^I\rangle = \langle\psi_i{}^{II}\rangle = 0$

(ii) $\langle\psi_i{}^I(t)\ \psi_j{}^I(t')\rangle = \langle\psi_i{}^{II}(t)\ \psi_j{}^{II}(t')\rangle = \langle\phi_i(t)\ \phi_j(t')\rangle\ \delta_{ij}$

(iii) $\psi_i{}^I, \psi_i{}^{II}$ and $\phi_i(0)$ are statistically unrelated.

The kronecker delta δ_{ij} appearing in the second property is to accommodate the homogeneous assumption of this chapter. A more detailed discussion between diagonal correlations and homogeneity will be given in section 3-1. The superscripts I, II represents different ensembles. The factor 2 is included to produce a proper energy equation. Eq. (2.1.5) can be interpreted as follows. Suppose the randomness has been introduced from the initial condition $\{\phi_k{}^0\}$, where {}

represents an ensemble. Then ψ_p^I, ψ_q^{II} were drawn separately from a pool of solutions. From Eq. (2.1.5), the equatio for a two-time second moment $R_k(t,t')$, defined as $R_k(t,t') = \langle\phi_k(t)\,\phi_k(t')\rangle$, can be derived as:

$$\begin{aligned}
\frac{d}{dt} R_k(t,t') &= \sqrt{2} \sum_{p,q} a_{kpq} \langle \psi_p^{\,I}(t) \psi_q^{\,II}(t) \phi_k(t') \rangle \\
&= \sqrt{2} \sum_{p,q} a_{kpq} \langle \psi_p^{\,I}(t) \psi_q^{\,II}(t) \int_0^{t'} \sqrt{2} \sum_{p,q} a_{kpq} \psi_p^{\,I}(s) \psi_q^{\,II}(s)\, ds \rangle \\
&= 2 \sum_{p,q} a_{kpq}^{\;2} \int_0^{t'} R_p(t,s) R_q(t,s)\, ds \quad . \qquad (2.1.6)
\end{aligned}$$

It can be seen that for zero-time separation $R_k(t',t')$, a measurement of energy, the right side of Eq.(2.1.6) is positive definite. Hence the energy will increase monotonically. This is not unlike when we stir a cup of tea randomly, with energy being constantly fed into the fluid in the cup.

It is necessary for a stochastic model to conserve energy, specifically in the energy equation. But for a simple stochastic model equation as we will introduce in this study, the energy is fluctuating in a single realization. This shortcoming is not serious, and will be tolerated. The generalized Langevin equation (Leith, 1971) for two-dimensional turbulence is:

$$\frac{\partial}{\partial t} \phi_k + \int_0^t \eta_k(t,s)\, \phi_k(s)\, ds = \sqrt{2} \sum_{p,q} a_{kpq}\, \psi_p^{\,I}\, \psi_q^{\,II} \qquad (2.1.7)$$

where η_k is called the eddy damping rate or eddy viscosity. The similarities between Eq. (2.1.7) for 2-D turbulence and Eq. (2.1.3) for Brownian motion are striking. Both have a random forcing term related to a "source" and a linear eddy damping term related to a "sink," though in Eq. (2.1.7) the eddy viscosity with a past time integration is written with an explicit memory. The above stochastic linear differential equation can easily be solved with the help of a temporal Green's function $G_k(t,t')$, which satisfies

$$\frac{\partial}{\partial t} G_k(t,t') + \int_0^t \eta_k(t,s)\, G_k(s,t')\, ds = \delta(t,t') \qquad (2.1.8)$$

$G_k(t', t') = 1$ and $G_k(t,t') = 0$ as $t < t'$. Hence,

$$\phi_k(t) = \int_0^t G_k(t,s) \Big(\sqrt{2} \sum_{p,q} a_{kpq}\, \psi_p{}^{I}(s)\, \psi_q{}^{II}(s) \Big) ds + \phi_k(0)\, G_k(t,0) \qquad (2.1.9)$$

The equation for the two-time second moment $R_k(t,t')$ is now:

$$\frac{\partial}{\partial t} R_k(t,t') + \int_0^t \eta_k(t,s)\, R_k(s,t')\, ds =$$

$$= 2 \sum_{p,q} a_{kpq}{}^2 \int_0^{t'} G_k(t',s)\, R_p(t,s)\, R_q(t,s)\, ds \qquad (2.1.10)$$

The nonrandom Green's function $G_k(t',s)$ is supposed to be statistically independent of the random forcing terms on the right side of Eq. (2.1.7). Hence, we can decouple:

$$\langle \psi_p^{\,I}(t)\; \psi_q^{\,II}(t)\; \psi_p^{\,I}(s)\; \psi_q^{\,II}(s)\; G_k(t',s)\rangle \;=\; R_p(t,s)\; R_q(t,s)\; G_k(t',s) \tag{2.1.11}$$

Once the eddy viscosity $\eta_k(t,s)$ has been determined, Eq. (2.1.10) and Eq. (2.1.11) form a closed system for $R_k(t,t')$ and $G_k(t,t')$. The choice which satisfies all the statistical properties of the original equation in Eq. (2.1.4) is the result of the DIA expansion (Kraichnan, 1970; Leith, 1971):

$$\eta_k(t,s) \;=\; -\,4 \sum_{p,q} a_{kpq}\; a_{pkq}\; G_p(t,s)\; R_q(t,s) \tag{2.1.12}$$

Various forms of η_k other than Eq. (2.1.12) also satisfy the energy conservation law. As Salmon (1978) has shown, η_k in Eq. (2.1.12) is determined uniquely only after the second law of thermodyanmics is imposed.

In the next section we will present a more formal derivation of DIA, which will illustrate the mathematical consistancy of related modern turbulence theories.

2.2 Direct Interaction Approximation

The rather cumbersome derivation of DIA (see, for example, Leslie, 1974) can be simplified greatly by using the concept of eddy viscosity, introduced in the previous section. Here we adopt a formulation of Orszag's (1974), by adding the explicit eddy viscosity term on both sides of the model Eq. (2.1.4):

$$\frac{\partial}{\partial t}\phi_k(t) + \int_0^t \eta_k(t,s)\,\phi_k(s)\,ds =$$

$$= \varepsilon\left(\sum_{p,q} a_{kpq}\,\phi_p(t)\,\phi_q(t) + \int_0^t \eta_k(t,s)\,\phi_k(s)\,ds\right). \qquad (2.2.1)$$

$\eta_k(t,s)$, as before, is assumed to be a nonrandom function. The perturbation parameter ε, which will be set to 1 later, is a measure of the nonlinear scrambling and the subsequent generation of non-Gaussianity, rather than the smallness of the terms in brackets. The DIA starts by expanding

$$\phi_k(t) = \phi_k^{(0)} + \varepsilon\,\phi_k^{(1)} + \varepsilon^2\,\phi_k^{(2)}\ldots.$$

and

$$\frac{\partial}{\partial t}\phi_k^{(0)}(t) + \int_0^t \eta_k(t,s)\,\phi_k^{(0)}(s) = 0 \qquad (2.2.2)$$

$$\frac{\partial}{\partial t}\phi_k^{(1)}(t) + \int_0^t \eta_k(\varepsilon,s)\,\phi_k^{(1)}(s)\,ds =$$

$$= \sum_{p,q} a_{kpq}\,\phi_p^{(0)}(t)\,\phi_q^{(0)}(t) + \int_0^t \eta_k(t,s)\,\phi_k^{(0)}(s)\,ds \qquad (2.2.3)$$

A crucial initial assumption is made that:

$\phi_k^{(0)}(0)$ belongs to a zero-mean Gaussian distribution

$$\phi_k^{(1)}(0) = \phi_k^{(2)}(0) = \dots\dots = 0 \ .$$

This is to simulate how the non-Gaussian terms (cumulants) grow. Since the zero-order Eq. (2.2.2) is a linear function, $\phi_k^{(0)}(t)$ will remain Gaussian for all t. The triple correlation terms on the right side of the two-time second moment equation:

$$\frac{\partial}{\partial t} \langle \phi_k(t) \ \phi_k(t') \rangle = \sum_{p,q} a_{kpq} \ \langle \phi_p(t) \ \phi_q(t) \ \phi_k(t') \rangle$$

can then be expanded:

$$\begin{aligned} \langle \phi_p(t) \ \phi_q(t) \ \phi_k(t') \rangle = & \ \langle \phi_p^{(0)}(t) \ \phi_q^{(0)}(t) \ \phi_k^{(0)}(t') \rangle \ + \\ & + \varepsilon \ (\langle \phi_p^{(1)}(t) \ \phi_q^{(0)}(t) \ \phi_k^{(0)}(t') \rangle \ + \\ & + \langle \phi_p^{(0)}(t) \ \phi_q^{(1)}(t) \ \phi_k^{(0)}(t') \rangle \ + \\ & + \langle \phi_p^{(0)}(t) \ \phi_q^{(0)}(t) \ \phi_k^{(1)}(t') \rangle) . \end{aligned}$$

The first term vanishes for Gaussian distributed elements. To calculate $\phi_p^{(1)}(t)$, $\phi_q^{(1)}$, $\phi_k^{(1)}, \dots\dots$, a Green's function related to the zero-order Eq. (2.2.2) is defined as:

$$\frac{\partial}{\partial t} G_k(t,t') + \int_{t'}^{t} \eta_k(t,s)\, G_k(s,t')\, ds = 0 \qquad \text{for} \qquad t > t'$$

$$G_k(t,t') = 1 \qquad t = t' \tag{2.2.4}$$

$$G_k(t,t') = 0 \qquad t<t'$$

Hence:

$$\phi_p^{(1)}(t) = \int_0^t G_p(t,s) \; [\sum_{i,j} a_{pij}\, \phi_i^{(0)}(s)\, \phi_j^{(0)}(s) + \int_0^s \eta_p(t,s')\, \phi_p^{(0)}(s')\, ds']\, ds$$

The triple correlations for example, $<\phi_p^{(1)}(t) \quad \phi_q^{(0)}(t) \; \phi_k^{(0)}(t')>$, can be evaluated by:

$$<\phi_p^{(1)}(t)\, \phi_q^{(0)}(t)\, \phi_k^{(0)}(t')> = \int_0^t G_p(t,s) \; [\sum_{i,j} a_{pij} <\phi_i^{(0)}(s)\, \phi_j^{(0)}(s)\, \phi_q^{(0)}(t)\, \phi_k^{(0)}(t')> + + \int_0^s \eta_p(s,t') <\phi_p^{(0)}(s')\, \phi_q^{(0)}(t)\, \phi_k^{(0)}(t')>\, ds'\,]\, ds.$$

Again, the Gaussian properties for the zero-order system have been used repeatedly. The triple correlation $<\phi_p^{(0)}(s') \quad \phi_q^{(0)}(t) \; \phi_k^{(0)}(t')>$ is thereby eliminated and the fourth-order correlation can be factored into:

$$\langle\phi_i^{(0)}(s)\ \phi_j^{(0)}(s)\ \phi_q^{(0)}(t)\ \phi_k^{(0)}(t')\rangle = R_i^{(0)}(s,s)\ R_q^{(0)}(t,t')\ \delta_{ij}\ \delta_{qk} + R_k^{(0)}(t',s)\ R_q^{(0)}(t,s)\ \delta_{ki}\ \delta_{qj} + R_k^{(0)}(t',s)\ R_q^{(0)}(t,s)\ \delta_{kj}\ \delta_{qi}$$

$$= 2\ R_k^{(0)}(t',s)\ R_q^{(0)}(t,s)$$

(As a reminder, $R_k^{(0)}(t',s) = \langle\phi_k^{(0)}(t')\ \phi_k^{(0)}(s)\rangle$ and $i \neq j$, $q \neq k$.)

The DIA is completed by taking the lowest order approximation of the ε-expansion:

$$\langle\phi_p(t)\ \phi_q(t)\ \phi_k(t')\rangle \approx \varepsilon\,[\langle\phi_p^{(1)}(t)\ \phi_q^{(0)}(t)\ \phi_k^{(0)}(t')\rangle + \langle\phi_p^{(0)}(t)\ \phi_q^{(1)}(t)\ \phi_k^{(0)}(t')\rangle + \langle\phi_p^{(0)}(t)\ \phi_q^{(0)}(t)\ \phi_k^{(1)}(t')\rangle]$$

$$= \varepsilon\,[4a_{pkq}\int_0^t G_p(t,s)\ R_q^{(0)}(t,s)\ R_k^{(0)}(t',s)\ ds + 2a_{kpq}\int_0^{t'} G_k(t',s)\ R_p^{(0)}(t,s)\ R_q^{(0)}(t,s)\ ds]$$

Now we set $\varepsilon = 1$ and take away the superscripts. That is, renormalize all the elements on the right-hand side. It is equivalent to replacing $R_k^{(0)}$.... by the summation of the infinite series $R_k(t,s) = R_k^{(0)}(t,s) + \varepsilon R_k^{(0)}(t,s) + ...$ The final equation for $R_k(t,t')$ is written as:

$$\frac{\partial}{\partial t} R_k(t,t') - 4 \sum_{p,q} a_{kpq}\, a_{pkq} \int_0^t G_p(t,s)\, R_q(t,s)\, R_k(t',s)\, ds$$
$$= 2 \sum_{p,q} a_{kpq}^{\;2} \int_0^{t'} G_k(t',s)\, R_p(t,s)\, R_q(t,s)\, ds \qquad (2.2.5)$$

Comparing Eq. (2.2.5) and Eq. (2.2.1), the natural choice for eddy viscosity is:

$$\eta_k(t,s) = -4 \sum_{p,q} a_{kpq}\, a_{pkq}\, G_p(t,s)\, R_q(t,s). \qquad (2.2.6)$$

Eq. (2.2.4), (2.2.5) and (2.2.6) form the closed system of DIA, which is also identical to the results in section 2.1 from heuristic arguments. The justification of DIA, especially the renormalization procedure, can be found in various critical reviews (e.g., Kraichnan, 1975a). The most persuasive evidence for the effectiveness of DIA may be the very good agreement between the results of DIA and numerical simulations for low Reynolds number flow (Herring and Kraichnan, 1972). The intimacy between DIA and the stochastic model provides a physical interpretation of the DIA. Since the DIA is the exact solution of certain form of the Langevin equation, so it suggests that a rational way to close the Navier-Stokes equation is to "abandon" the Navier-Stokes equation itself. Instead, we start with a stochastic model which can preserve all the necessary statistical properties of the Navier-Stokes equation and exhibit the most essential features of nonlinear interaction--that is, the random scrambling and irreversible relaxation--explicitly. The stochastic model should also be mathematically solvable. Hence in modern turbulence theories the closure problem seeks an exact solution for an approximate system, instead of an approximate solution for an exact system.

2.3 Eddy-Damped Markovian Model

The eddy-damped Markovian model (EDM) was motivated by the failure of the traditional cumulant-discard method. The same concept of linear relaxation of eddy viscosity is inserted into the model in a seemingly more direct way. The final results of EDM turn out to be just an abridged form of DIA. It is the connection between EDM and DIA which will inspire several simplification procedures relevant to this study (see chapter 4).

Write the single time, second moment $R_k = \langle \phi_k \; \phi_k \rangle$, third moment $T_{kpq} = \langle \phi_k \; \phi_p \; \phi_q \rangle$ and fourth moment $Q_{kpqR} = \langle \phi_k \; \phi_p \; \phi_q \; \phi_R \rangle$,... and so on. This time we put a dissipative viscosity term ν_k into the model equation:

$$\frac{\partial}{\partial t}\phi_k + \nu_k\phi_k = \sum_{p,q} a_{kpq}\phi_p\phi_q. \tag{2.3.1}$$

The equations for the second and third moment are:

$$\frac{\partial}{\partial t} R_k + 2\nu_k R_k = \sum_{p,q} a_{kpq} T_{kpq} \tag{2.3.2}$$

$$\frac{\partial}{\partial t} T_{kpq} + (\nu_k + \nu_p + \nu_q) T_{kpq} = \sum_{i,j} (a_{kij} Q_{pqij} + a_{pij} Q_{kqij} + a_{qij} Q_{kpij}). \tag{2.3.3}$$

The factorization of Q_{pqij} gives:

$$Q_{pqij} = R_i R_p \delta_{ij} \delta_{pq} + R_p R_q \delta_{ip} \delta_{jq} + R_p R_q \delta_{jp} \delta_{iq} + \text{cumulant}.$$

The earliest closure (Millionshtchikov, 1941) just neglects the cumulant term of fourth moments, which gives the second moment equation:

$$\frac{\partial}{\partial t} R_k + 2\nu_k R_k = \sum_{p,q} \int_0^t e^{-(\nu_k+\nu_p+\nu_q)(t-t')} [\, 8a_{kpq}\, a_{pkq} R_q(t')\, R_k(t') + 4a_{kpq}{}^2\, R_p(t')\, R_q(t')\,]\, dt'. \tag{2.3.4}$$

Unfortunately, the numerical experiments by Ogura (1962) showed that negative energy appeared in the model evolution, implying that

Eq. (2.3.4) was physically unrealizable. The reason for the defect in this quasinormal method is now well known; let us call the bracket term $N_k(t') = [8a_{kpq}\ a_{pkq}\ R_q(t')\ R_k(t') + 4a_{kpq}^2\ R_k(t')\ R(t')]$ a nonlinear scrambling term, it can be seen in Eq. (2.3.4) the factor

$$e^{-(\nu_k + \nu_p + \nu_q)(t-t')}$$

serves as a "memory" weighting function of $N^k(t')$. For high Reynolds-number flow, ν_k, ν_p, ν_q have only negligible values. Hence the effect of $N_k(t)$ will be weighted equally either for an ancient past or for a recent past. If initially $R_k(0)$ is concentrated in a very narrow band k_0, the nonlinear scrambling term $N_{k_0}(0)$ tends (compared to a later time) to strongly reduce R_{k_0}; that is, to redistribute energy to modes other than k_0. After this initial stage the R_k-spectrum becomes rather flat; whereas the impact of $N_{k_0}(t)$ should be weakening, but the initial strong tendency of $N_{k_0}(0)$ is still remembered in Eq. (2.3.4) even when t advances beyond the initial stage. This persistent effect is the reason R_k drops below zero. By neglecting the fourth cumulant, a brake mechanism for nonlinear scrambling has been unjustifiably thrown away.

Obviously, if there does exist a linear viscous type memory in real nonlinear flow which is described by Eq. (2.3.4), this memory should reflect the dynamic state resulting mainly from nonlinear interactions, instead of dissipative viscosity ν_k, ν_p.... The very plausible remedy of EDM is to insert a linear eddy viscosity term $(\mu_k + \mu_p + \mu_q)\, T_{kpq}$ on the left side of the third moment Eq. (2.3.3) as a compensation for the neglected fourth cumulant. This effectively gives

$$\frac{\partial}{\partial t} R_k + \nu_k R_k = \sum_{p,q} \int_0^t e^{-\int_{t'}^{t} [\nu_k+\nu_p+\nu_q + \mu_k(s) + \mu_p(s) + \mu_q(s)]\, ds} N_k(t')\, dt' \qquad (2.3.5)$$

A Markovianization procedure is adopted to simplify the above equation. That is accomplished by choosing only the current values of N_k and μ_k, μ_p, μ_q on the right side of Eq. (2.3.5). Define:

$$\theta_{kpq} = \int_0^t e^{-(\nu_k+\nu_p+ \nu_p + \mu_k+ \mu_p+ \mu_q)(t-t')}\, dt'. \qquad (2.3.6)$$

The EDM equation of R_k becomes:

$$\frac{\partial}{\partial t} R_k + 2\nu_k R_k = \sum_{p,q} \theta_{kpq}\, N_k. \qquad (2.3.7)$$

A self-consistant prescription for μ_k calls for (see Orszag, 1974):

$$\mu_k = \upsilon_k - 4 \sum_{p,q} \theta_{kpq} a_{kpq} a_{pkq} R_q(t) \qquad (2.3.8)$$

The importance of EDM is to reduce the whole closure scheme to the problem of prescribing a single element θ_{kpq}, or μ_k. θ_{kpq}, with a dimension of time, can be considered as the dephasing time for the triple moment T_{kpq}. The uncertainties in θ_{kpq} will not upset the energy conservation law as long as the subscripts k, p, q are interchangeable and θ_{kpq} is bounded in time.

Eqs. (2.3.6), (2.3.7) and (2.3.8) form the complete set of the EDM. For comparison, we regroup the DIA, EDM equations in terms of single time second moment $R_k(t,t) = R_k$

The DIA model

$$\frac{\partial}{\partial t} R_k + 2 \int_0^t \eta_k(t,s)\, R\,(t,s)\, ds = 4 \sum a_{kpq}{}^2 \int_0^t G(t,s)\, R_p(t,s)\, R_q(t,s)\, ds \qquad (2.3.9)$$

$$\frac{\partial}{\partial t} G_k(t,t') + \int_{t'}^t \eta_k(t,s)\, G_k(s,t')\, ds = 0 \qquad \text{for } t > t'$$

$$G(t',t') = 1 \qquad (2.3.10)$$

$$G_k(t,t') = 0 \qquad \text{for } t < t'$$

$$\eta_k = -4 \sum_{p,q} a_{kpq}\, a_{pkq}\, G_p(t,s)\, R_q(t,s) \tag{2.3.11}$$

The EDM model [neglecting ν_k, integrating θ_{kpq} in Eq. (2.3.6)]:

$$\frac{\partial}{\partial t} R_k + 2\mu_k R_k = 4 \sum_{p,q} a_{kpq}^{\;2}\, \theta_{kpq}\, R_p\, R_q \tag{2.3.12}$$

$$\mu_k = -4 \sum_{p,q} a_{kpq}\, a_{pkq}\, \theta_{kpq}\, R_q \tag{2.3.13}$$

$$\theta_{kpq} = \frac{1-e^{-(\mu_k+\mu_p+\mu_q)t}}{\mu_k+\mu_p+\mu_q}\,. \tag{2.3.14}$$

The link between the DIA and the EDM is seen by approximating:

$$R_k(t,s) = G_k(t,s)\, R_k(t,t) \tag{2.3.15}$$

and letting:

$$\theta_{kpq}(t) = \int_0^t G_k(t,s)\, G_p(t,s)\, G_q(t,s)\, ds \tag{2.3.16}$$

and

$$G_k(t,s) = e^{-\mu_k(t-s)}. \tag{2.3.17}$$

The relationship in (2.3.15) is exact in thermal equlibrium states according to the fluctuation-dissipation theory (Kraichnan, 1958; Leith, 1975). We will examine this critical theory further via direct numerical experiments in chapter 5. The application of Eqs. (2.3.15) and (2.3.16) results in identical formulations for the DIA and the EDM. Hence the EDM is just an abridged form of the DIA.

It is interesting to show the Langevin equation corresponding to EDM:

$$\frac{\partial}{\partial t}\phi_k + \eta_k\phi_k = \sqrt{2}\,W(t)\sum_{p,q}(\theta_{kpq})^{1/2}a_{kpq}\,\psi_p^{\,I}\,\psi_q^{\,II} \qquad (2.3.18)$$

where W(t) is a white noise process, defined as:

$$\langle W(t)\,W(t')\rangle = \delta(t-t') \quad ,$$

which accounts essentially for the Markovianization in EDM. The greatest advantage of EDM-type models (including the more advanced test-field model, see Kraichnan, 1971a) is their simplicity. One can add more physical features into the model without introducing unnecessary mathematical intricacies. A good example is to incorporate periodic motion into the model Eq. (2.1.4),

$$\frac{\partial}{\partial t}\phi_k + i\omega_k\phi_k = \sum_{p,q} a_{kpq}\,\phi_p\phi_q$$

where ω_k is the frequency corresponding to certain restoring mechanisms. It can easily be derived that the dephasing time for the triple correlation $\langle\phi_k\phi_p\phi_q\rangle$ is:

$$\theta_{kpq} = \frac{\mu_k+\mu_p+\mu_q}{(\mu_k+\mu_p+\mu_q)^2+(\omega_k+\omega_p+\omega_q)^2} \qquad \text{as } t\to\infty\,.$$

For weak turbulence (μ_k, μ_p, $\mu_q \to 0$) the above relationship will approach the wave resonance condition (see Holloway and Hendershott, 1977):

$$\theta_{kpq} \to \delta(\omega_k + \omega_p + \omega_q)$$

Another example to show the versatility of EDM is to add a further simplification to the closure. The eddy damping rate μ_k can be prescribed by either

$$\mu_k = c\left(\int_0^k k'^2\, E(k')\, dk'\right)^{1/2} \qquad \text{(Pouquet, 1975)} \qquad (2.3.19)$$

to include nonlocal interactions between different modes, or by

$$\mu_k = c'(k^3E(k))^{1/2} \qquad \text{(Orszag, 1970; Leith, 1971)} \qquad (2.3.20)$$

where E(h) is the energy spectrum, and c,c' are two universal constants determined by numerical simulation.

Nevertheless, the direct interaction approximation still remains the most fundamental and systematic method to solve nonlinear problems (Kraichnan, 1959, has claimed that the DIA is a general method which can extend beyond the turbulence model). As in this study, we often found the most convenient way to treat an inhomogenous turbulence problem is to follow the DIA expansion strictly, then use a corresponding stochastic model for physical interpretation, and make the approximations from Eq. (2.3.15) to Eq. (2.3.17) whenever appropriate. Our goal, again, is not to evaluate different closure methods, but to obtain information suitable for practical applications from a turbulence model.

2.4 Modified Closure Scheme

Any review (even as brief as this one) on DIA cannot be complete without mentioning a major defect of DIA, which was discovered by Kraichnan (1964c). Consider in a given realization of the Navier-Stokes equation, an inserted uniform velocity U_0 acting on the Fourier decomposed mode ϕ_k

$$\phi_k(t) \rightarrow e^{-ik\, U_0 t}\phi_k(0)$$

by Galilean transformation. If U_0 is selected randomly from an ensemble which is Gaussian and isotropic, the total effect of Galilean transformaton on the two-time correlations [as well as the Green functions in DIA (see Rose and Sulem, 1978)] will be:

$$\langle\phi_k(t)\ \phi_k(t')\rangle \ = \ e^{-\,1/6\ k^2\langle U_0{}^2\rangle(t-t')^2}\langle\phi_k(t)\ \phi_k(t')\rangle$$

where k, U_0 are the scalar values of k, U. But for homogeneous turbulence as far as the single time correlations (hence energy specrum) are concerned, a Galilean transformation should not change the final solutions. DIA is unable to avoid the above exponential factor, and will fail to maintain the random Galilean invariant. The trouble is clearly caused by using an Eulerian coordinate. A small fluid parcel, following a Lagrangian trajectory, is being both advected by large scale motion and deformed internally by pressure forces at the same time. The advection of a small parcel by a large parcel has the same effect as a Galilean transformation. But in the Navier-Stokes equation of spectral form these two effects are generally grouped without distinction in the nonlinear interaction coefficients. This problem appears only when two-time correlations have to be calculated.

Kraichnan (1965; 1966) using a generalized velocity field $U(x,t|s)$, defined as the velocity of a fluid element at time s, which at time t has passed through a position x, was able to recover some Lagrangian features in the Navier-Stokes equation. Both the Lagrangian history direct interaction (LHDI) and abridged Lagrangian history direct interaction (ALHDI) models are able to preserve random Galilean invariants and to duplicate Kolmogorov spectra in the inertial range. Unfortunately, these models are much too complicated to have any practical usage. Other alternatives may be the introduction of some phenomenological-inspired "empirical" values into simple models like EDM. A good example is the closure scheme in Eq. (2.3.19). By adjusting the constant c, we can correct the overestimation of eddy damping rates in the Eulerian coordinate.

The test field model (TFM) designed by Kraichnan (1971) may represent the state-of-the-art closure technique. It is capable of distinguishing between advection and straining effects but still maintains the simplicity of an EDM. The only drawback of the TFM is a need for an adjustable constant determined empirically from experiments.

Despite the artifice, these simple closures work well in solving geophysical problems (e.g., Leith, 1971; Salmon, 1978). To conclude this chapter, our attitude toward selecting the available turbulence theories can best be described by a quote from an experienced closure modeler (Rick Salmon, 1981, private communication):

> "...Mercifully, however, many properties of the solutions to the closure equations are rather insensitive to what you use in θ_{kpq}.... We oceanographers and meteorologists are often interested in more qualitative and robust statistics than spectral shape.... Therefore I advocate using the simplest θ_{kpq} (closure) possible...."

Chapter 3: Inhomogeneous Turbulence Model

This chapter starts by describing the general concept of homogeneity and the common sources of inhomogeneity encountered in a turbulence model. A special type of inhomogeneity is chosen. In the second section, we extend the DIA from a homogeneous model in chapter 2 to an inhomogeneous model. The remaining part of the chapter is devoted to further simplifications of the inhomogeneous DIA.

In section 3.3, an expansion of off-diagonal elements in terms of diagonal elements, called the DIA diagonalization (DIAD) process*, is introduced. The DIAD can be interpreted as a degenerate case of the homogeneous model and related to a stochastic equation. In section 3.4, a practical example of shear flow is given. The application of the DIAD on a shear flow inspires us to impose a weakly inhomogeneous assumption (WIA), i.e., a scale separation between the inhomogeneous mean shear and turbulence. In section 3.5 an eddy diffusion equation is deduced under WIA. The calculation of the vorticity flux is thereby simplified.

The assumption of WIA demands a modification of the DIAD. In the final section, we describe an asymptotic expansion based on the

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* The general concept of the DIAD in this study was formed after several fruitful discussions with Dr. Herring during a visit to NCAR three years ago. In the later stage we "rediscovered" Kraichnan's (1964b) work that shed light on the DIAD but was formulated under different context and considerations. Part of Kraichnan's formalism was adopted in this chapter, though the interpretation of the DIAD in section 3.3 is entirely ours.

smallness of the inhomogeneous terms. Henceforth, the spectral transfer problem becomes completely independent of the spatial transfer problem. The convergent inhomogeneous expansion plus the DIAD is the center of this study.

3.1 Definitions of Homogeneity and Origins of Inhomogeneity

A physical system, which is invariant under the translation $y \to y + h$, where y is the independent spatial variable and h is an arbitrary constant, will be referred to as one-point homogeneous (along y).

In statistical turbulence literature, homogeneity is defined slightly differently (see, e.g., Monin and Yaglom, 1972). Here, a zero-mean field ϕ is homogeneous if the ensemble-averaged two-point correlation $R(y_1, y_2)$ depends only on the distance between y_1 and y_2, and not on the positions of y_1 and y_2, i.e., if:

$$R(y_1, y_2) = \langle\phi(y_1)\ \phi(y_2)\rangle = R(y_1 - y_2).$$

We will denote a system satisfying the above as two-point homogeneous.

A system $\{\phi\}$ which satisfies the condition of one-point homogeneity, will automatically satisfy the condition of two-point homogeneity. Since the origin of the y-coordinate can be shifted by an arbitrary value h:

$$R(y_1, y_2) = \langle\phi(y_1)\ \phi(y_2)\rangle = \langle\phi(y_1-h)\ \phi(y_2-h)\rangle = R(y_1-h, y_2-h).$$

By defining a pair of new variables $D = y_1 - y_2$ and $E = y_1 + y_2$, it follows that:

$$R(D,E) = R(D,E-2h) \quad ,$$

The above equation holds true for all h, hence R is independent of E and only depends on the distance between y_1 and y_2.

Nonlinear differential equations, whose independent variables do not appear explicitly (e.g., $\partial^3\phi/\partial y^3 + \phi(\partial\phi/\partial y) = 0$), are called autonomous equations. An autonomous equation satisfies the condition of one-point homogeneity, as well as two-point homogeneity.

Some systems can be nonautonomous and not one-point homogeneous but still be two-point homogeneous. A notable example is the system which describes constant shear flow (e.g., Hinze, 1975). Although a shift of origin along the transverse axis changes the governing equation, the effect of this shift is equivalent to that of adding a constant uniform flow in the longitudinal direction. Therefore, the correlations between any two points along the transverse axis remain unaffected.

From now on we will refer to the term "homogeneous" exclusively for two-point homogeneity.

For a well-posed physical problem, we need also to consider initial and boundary conditions. The presence of a physical boundary will certinly ruin the homogeneity (either one-point or two-point) in a system since every point in space will be "tagged" by its position relative to the boundary. This difficulty in homogeneous turbulence theories is circumvented by defining a domain which is confined in a box, with each side of the box obeying a cyclic boundary condition. The sides thus effectively extend to infinity. However, for each truncated Fourier transformation the flow domain relates to an elementary box with limited size. This so-called periodic boundary condition is sometimes also called the homogeneous boundary condition.

Consider a dynamic system which is functionally homogeneous and satisfies homogeneous boundary conditions. For each realization the initial condition is chosen from an ensemble that each spatial point has the same probability distribution. Assume that the system is ergodic. Therefore the statistics of this time-evolving system can only be homogeneous. This is the rationale for seeking a homogeneous turbulence solution for the N-S equaton.

A basic question is: when an initial condition is inhomogeneous, or when the system has only a slight inhomogeneous tendency, as in the radiative-dynamic system described in chapter 1, how will the inhomogeneity manifest itself in the final statistics (assuming an equilibrium state is attained)? In particular, will any inhomogeneity prevail in the formal case with the N-S equation? Such a question may be too hard to answer precisely. It is generally believed that the nonlinear interactions in a fully realized turbulent flow will "homogeneize" spatial differences within a finite time. But we also observe numerous natural phenomena which are clearly "dehomogeneization" processes (for example, the formation of the jet stream and hence the celebrated "negative viscosity" hypothesis by Starr, 1968). Perhaps a more answerable question is how stable is the homogeneity in a flow system? That is, if a very weak source of inhomogeneity is introduced into an apparently homogenous flow system, will this inhomogeneity grow?

Consider a flow equation in the general form:

$$\frac{\partial}{\partial t} \phi + (aL_y) \phi + N_y \phi \phi = 0 \qquad (3.1.1)$$

where N_y is the bilinear differential operator and y is the independent spatial variable. The linear differential operator (aL_y) represents $[a^n(\partial^n/\partial y^n) + a^{n-1}(\partial^{n-1}/\partial y^{n-1}) + \ldots]$ where a^n, $a^{n-1}, \ldots$ are coefficients. If a_n, a_{n-1}, ... are independent of y, the model equation (3.1.1) is a spatially autonomous equation. Hence the inhomogeneity can only enter the problem from either the initial conditions or the boundary conditions. This kind of question usually demands a complicated mathematical approach: either a time-dependent probability density function like Eq. (3.1.2) or a differential integral expression which fits the boundary condition (for example, the thermal convection DIA model by Kraichnan, 1964a).

Another source of inhomogeneity comes from the nonconstant coefficient $a(y)$. In our study, it can be identified as a time-independent inhomogeneous mean shear (which must also satisfy the periodic boundary condition). Rewrite Eq. (3.1.1) as:

$$\frac{\partial}{\partial t} \phi(t) + aL_y \phi(t) + a'(y) L'_y \phi(t) + N_y \phi(t) \phi(t) = 0 \qquad (3.1.2)$$

in order to distinguish the constant coefficient linear operator aL_y from the nonconstant coefficient linear operator $a'(y)\ L_y'$. In spectral form Eq. (3.1.2) can be written as

$$\frac{\partial}{\partial t}\phi_k(t) + L_k\phi_k(t) = \delta \sum_{i,j} a'_{kij}\ T_i\ \phi_j(t) + \sum_{p,q} a_{kpq}\phi_p(t)\ \phi_q(t) \tag{3.1.3}$$

where

$$(\phi \rightarrow \phi_L,\ aL_y\phi \rightarrow L_k\phi_k,\ a(y)'L_y\phi \rightarrow \sum_{i,j} a'_{kij}T_i\phi_j,\ N_y\phi\phi \rightarrow \sum_{p,q} a_{kpq}\phi_p\phi_q).$$

The inhomogeneous term

$$\sum_{i,j} a'_{kij}\ T_i\phi_j(t)$$

describes the interaction between the inhomogeneous mean shear $\sum T_i$ and the turbulent field. It is, in fact, a degenerate form of the nonlinear interaction term

$$\sum_{p,q} a_{kpq}\ \phi_p(t)\ \phi_q(t).$$

We assume that mean shear is confined to a few selected modes T_i and is time independent. In our study, we will concentrate on this type of problem, especially in the limit where $\delta \ll O(1)$, which will be referred to as the weakly inhomogeneous assumption. As we later show, it is possible to find a simplified expression for the off-diagonal correlations, or inhomogeneous dynamic fluxes. The natural tendency of flow systems to dissipate inhomogeneity can then be investigated within the context of negative (or positive) viscosity

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3.2 Inhomogeneous DIA Model

In this section, the DIA expansion is extended to an inhomogeneous model:

$$\frac{\partial}{\partial t} \phi_k(t) + \sum_j L_{kj} \phi_j(t) = \sum_{p,q} a_{kpq} \phi_p(t) \phi_q(t) \qquad (3.2.1)$$

where, in short form, the generalized linear operation is written as

$$L_{kj} = L_k - \sum_i a'_{kij} T_i$$

corresponding to terms in the equation (3.1.3). An inhomogeneous Green's function G_{km} is defined as follows: if an infinitesimal impulse $\delta f_m(s)$ is suddenly imposed on mode n at time s of a flow system described by Eq. (3.2.1), a series of these disturbances in the past results in a response $\delta\phi_k(t)$ of mode k at time t. Schematically,

$$\delta\phi_k(t) \leftarrow \boxed{\begin{array}{l}\text{Flow system}\\ \text{Eq. (3.2.1)}\end{array}} \leftarrow \sum_m \delta f_m(s)$$

Since the disturbance is small, the response of the flow system is linear. The Green's function $g_{km}(t,s)$ satisfies

$$\delta\phi_k(t) = \sum_m \int_0^t g_{km}(t,s)\,[\delta f_m(s)]\,ds \qquad (3.2.2)$$

Following Kraichnan's notation, g_{km} can be written as $g_{km}(t,s) = \delta\phi_k(t)/\delta f_m(s)$ and an essemble averaged Green's function can be defined as $G_{km}(t,s) = \langle g_{km}(t,s)\rangle$, satisfying $G_{km}(t,t) = 1$ and $G_{km}(t,s) = 0$ for $t < s$.

The above description of the Green's function $g_{km}(t,s)$ within an individual realization is consistant with our definition of $G_k(t,s)$ (or $G_{kk'}(t,s)$ written in our current notation) in chapter 2. Since most of the randomness of $g_{km}(t,s)$ generated by non-linear interactions will be filtered out when an ensemble average is taken, only the feature of irreversible relaxation (in linear eddy damping form) will appear in the equation for $G_{km}(t,s)$. The Langevin equation corresponding to Eq. (3.2.1) is:

$$\frac{\partial}{\partial t}\phi_k(t) + \sum_j L_{kj}\phi_j(t) + \sum_j \int_0^t \eta_{kj}(t,s)\,\phi_j(s)\,ds$$

$$= \sum_{p,q} \sqrt{2} a_{kpq} \psi_p^{I}(t) \psi_q^{II}(t) \quad (3.2.3)$$

where the inhomogeneous eddy damping term satisfies

$$\eta_{kj}(t,s) = - \sum_{p,q} \sum_{mn} 4a_{kpq} a_{mnj} G_{pm}(t,s) R_{qn}(t,s) \quad , \quad (3.2.4)$$

and the random forcing term on the right side of Eq. (3.2.3) is defined as in chapter 2, Eq. (2.1.5), except that now second moments of two different scales are permissible:

$$\langle \psi_i^{I}(t) \psi_j^{I}(t') \rangle = \langle \psi_i^{II}(t) \psi_j^{II}(t') \rangle = \langle \phi_i(t) \phi_j(t') \rangle .$$

The closed form for two-time two-scale second moment equation:

$$\frac{\partial}{\partial t} \langle \phi_k(t) \phi_L(t') \rangle + \sum_j L_{kj} \langle \phi_j(t) \phi_L(t') \rangle = \sum_{p,q} a_{kpq} \langle \phi_p(t) \phi_q(t) \phi_L(t') \rangle \quad (3.2.5)$$

is given by the following equations:

$$\frac{\partial}{\partial t} \langle \phi_k(t) \phi_L(t') \rangle + \sum_j L_{kj} \langle \phi_j(t) \phi_L(t') \rangle +$$

$$+ \sum_j \int_0^t \eta_{kj}(t,s) \langle \phi_j(s) \phi_L(t') \rangle \, ds$$

$$= \sum_{p,q} \sum_{jmn} 2a_{kpq} a_{jmn}$$

$$\int_0^{t'} G_{Lj}(t',s) \langle\phi_q(t)\ \phi_n(s)\rangle \langle\phi_p(t)\phi_m(s)\rangle\, ds \tag{3.2.6}$$

$$\frac{\partial}{\partial t} G_{kL}(t,t') + \sum_j L_{kj}G_{jL}(t,t') + \sum_j \int_{t'}^{t} \eta_{kj}(t,s)\ G_{jL}(s,t')\, ds = t \quad \text{for } t > t'. \tag{3.2.7}$$

A formal derivation of the above equations is given in appendix B. It is believed that the inhomogeneous DIA model is beyond our computational capability.* Hence we adopt the following simplification.

3.3 Diagonalization of DIA

The very heart of the DIA is the perturbation expansion of non-Gaussianity in cumulants and the subsequent renormalizaton. For our inhomogeneous turbulence problem, the same methodology can be extended to a DIA diagonalization process, where the role played by Gaussianity in the DIA is replaced by homogeneity, and that of cumulants by the off-diagonal correlations. Kraichnan (1946b) illustrated this in a thermal convection problem as a method to reduce the complexity of computations. In his model, inhomogeneity was induced from the boundaries and mean flow was time evolving. This is in contrast with our

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* Leslie (1973) estimated that it would take about 10 man-years to program an inhomogeneous DIA model at large Reynold's number.

assumptions. Hence we put the DIAD into a new context in our study. Here we rederive the DIA diagonalization in the following condensed and modified form.

The model equation (3.1.4) is repeated as:

$$\frac{\partial}{\partial t} \phi_k(t) + L_k \phi_k(t) = \varepsilon \left[\sum_{p,q} a_{kpq} \phi_p(t) \phi_q(t) + \sum_{i,j} a'_{kij} T_i \phi_j(t) \right] . \tag{3.3.1}$$

Both the nonlinear terms $\sum_{p,q} a_{kpq} \phi_p(t) \phi_q(t)$

and inhomogeneous terms $\sum_{ij} a'_{kij} T_i \phi_j(t)$

are labeled with the parameter ε. We expand:

$$\phi_k = \phi_k^{(0)}(t) + \varepsilon \, \phi_k^{(1)}(t) \ldots .$$

The zero-order system gives:

$$\frac{\partial}{\partial t} \phi_k^{(0)}(t) + L_k \phi_k^{(0)}(t) = 0. \tag{3.3.2}$$

The above equation is functionally homogeneous and linear. (In fact, the linear effect is dominated by an implicit eddy damping term.)

Given an initial condition that $\phi_k^{(0)}(0)$ is Gaussian distributed AND homogeneous, $\phi_k^{0}(t)$ will remain so according to the DIA expansion. The first-order solution:

$$\phi_k^{(1)}(t) = \int_0^t G_{kk}^{(0)}(t,s) \, [\sum_{p,q} a_{kpq}\phi_p^{(0)}(s)\phi_q^{(0)}(s) + \sum_{ij} a'_{kij}T_i\phi_j^{(0)}(s)]ds \tag{3.3.3}$$

and the lowest order nonvanishing off-diagonal correlation

$$<\phi_k(t) \; \phi_L(t')> \;\approx\; \varepsilon \; [<\phi_k^{(0)}(t) \; \phi_L^{(1)}(t')> + <\phi_k^{(1)}(t) \; \phi_L^{(0)}(t')>]$$

can be deduced from:

$$\begin{aligned} <\phi_k^{(1)}(t) \; \phi_L^{(0)}(t')> &= \int_0^t G_{kk}^{(0)}(t,t) \\ &\quad [\sum_{p,q} a_{kij} <\phi_p^{(0)}(s) \; \phi_q^{(0)}(s) \; \phi_L^{(0)}(t')> \\ &\quad + \sum_{ij} a'_{kij} \; T_i <\phi_j^{(0)}(s) \; \phi_L^{(0)}(t')> \delta_{jL}] \; ds \\ &= a'_{kmL} \; T_m \int_0^t G_{kk}^{(0)}(t,s) \; R_{LL}^{(0)}(t',s) \; ds. \end{aligned} \tag{3.3.4}$$

The simplifications come from repeatedly using the properties of homogeneity and Gaussianity in the zero-order system. The next step of the DIAD is to retain the lowest-order terms in ε and then set $\varepsilon = 1$. Finally replace the zero-order terms with the exact terms, making superscripts unnecessary.

The results are:

$$R_{kL}(t,t') = a'_{kmL}\; T_m \int_0^t G_{kk}(t,s)\; R_{LL}(t',s)\; ds + a'_{Lmk}\; T_m \int_0^{t'} G_{LL}(t',s)\; R_{kk}(t,s)\; ds \qquad (k \neq L) \tag{3.3.5}$$

and

$$G_{kL}(t,t') = a'_{kmL}\; T_m \int_{t'}^{t} G_{kk}(t,s)\; G_{LL}(s,t')\; ds. \qquad (k \neq L,\; t > t') \tag{3.3.6}$$

The complete DIAD model for the flow system Eq. (3.3.1) consists of the following equations:

$$\frac{\partial}{\partial t} R_{kk}(t,t') + L_k R_{kk}(t,t') + \int_0^t \eta_{kk}(t,s) R_{kk}(s,t')\; ds + \int_0^t \eta'_{kk} R_{kk}(s,t')\; ds = \sum_{p,q} a_{kpq}{}^2 \int_0^{t'} G_{kk}(t',s)\; R_{pp}(t,s)\; R_{qq}(t,s)\; ds + \sum_{i,j} a'_{kij}{}^2 T_i{}^2 \int_0^t G_{kk}(t',s)\; R_{jj}(t,s)\; ds \tag{3.3.7}$$

$$\frac{\partial}{\partial t} G_{kk}(t,t') + L_k G_{kk}(t,t') + \int_{t'}^{t} \eta_{kk}(t,s)\, G_{kk}(s,t')\, ds$$

$$+ \int_{t'}^{t} \eta'_{kk}(t,s)\, G_{kk}(s,t')\, ds = 0 \qquad \text{for } t > t'$$

$$\text{and } G_{kk}(t',t') = 1 \quad \text{and} \quad G_{kk}(t,t') = 0 \quad \text{for } t < t' \tag{3.3.8}$$

where the eddy damping rates satisfy

$$\eta_{kk}(t,s) = -4 \sum_{p,q} a_{kpq}\, a_{pkq}\, G_{pp}(t,s)\, R_{qq}(t,s) \tag{3.3.9}$$

$$\eta_{kk}'(t,s) = - \sum_{i,j} a'_{kij}\, a'_{jki}\, G_{jj}(t,s)\, T_i^{\,2}. \tag{3.3.10}$$

We are not surprised to discover the analogies between the DIAD model and the homogeneous DIA model. As we stated before, Eq. (3.3.1) can be interpreteted as a degenerate case of homogeneous turbulence, whose elements are $\{T_i(\tau),\ T_m(\tau), \ldots,\ \phi_k(t),\ \phi_p(t), \ldots\}$. Some of the elements $\{T_i(\tau),\ T_m(\tau)\}$ have a period so long ($\tau \to \infty$) that they are essentially 'frozen" in space. As far as more rapidly fluctuating components $\{\phi_k(t) \ldots\}$ are concerned, $\{T_i(\tau) \ldots\}$ represent a time-independent inhomogeneity source. The results of DIAD are identical to those of the homogeneous DIA by assigning a_{kij}' to be a subset of a_{kpq}, and assuming a long-lasting phase relationship with the T_i mode, ($\langle T_i(t+\Delta t)\ T_i(t)\rangle = \langle T_i^2\rangle$ as $\Delta t \to \infty$). The Langevin equation for DIAD is henceforth found to be:

$$\frac{\partial}{\partial t}\phi_k(t) + L_k\phi_k(t) + \int_0^t \eta_{kk}(t,s)\phi_k(s)\,ds + \int_0^t \eta'_{kk}(t,s)\phi_k(s)\,ds$$

$$= \sum_{ij} a'_{kij}T_i\psi_j^{III}(t) + \sqrt{2}\sum_{p,q} a_{kpq}\psi_p^{I}(t)\psi_q^{II}(t) \quad . \tag{3.3.11}$$

All the random variables ψ_p^I, ψ_q^{II}, and ψ_j^{III} are selected from ensembles which are Gaussian and homogeneous, defined as in the homogeneous DIA model, i.e., Eq. (2.1.5). By introducing a third random group and an extra eddy damping rate $\eta_{kk}'(t,s)$, we have replaced the inhomogeneous term $\sum a'_{kij}T_i\phi_j$, in accordance with DIA treatment. It is interesting to see that the stochastic model in Eq. (3.3.1) contrives to preserve all the statistical properties (e.g., realizability and conservation of energy) for a system $\{T_i(\tau), T_m(\tau),\ldots, \phi_k(t), \phi_p(t),\ldots\}$ if time-evolving equations for $\{T_i(\tau), T_m(\tau)\ldots\}$ are included. But the freezing of the $\{T_0, T_m\ldots\}$ elements makes all the justifications in the DIA formalism impossible to prove. Hence the DIAD should be used with more caution than the DIA.

We recognize that Eq. (3.3.11) implies a return to isotropy and hence dictates a diffusion of inhomogeneity. This imposes a restriction when we apply the DIAD to an atmospheric problem where the diffusion phenomena only exist for a few selected cases.

3.4 Inhomogeneous Turbulent Shear Flow

Previously we have introduced turbulence theories in terms of a generalized spectral equation (see appendix A) whose terse form helped to enhance our mathematical perception. But since further simplification comes from practical considerations, it is necessary to reformulate the inhomogeneous turbulence theory is a less abstract framework. We also found it convenient to restrict the spectral model to a selected orthogonal basis, i.e., a Fourier expansion. Therefore in this section we use an example to illustrate the formulation of the inhomogeneous DIA and DIAD, and the need which leads to weakly inhomogeneous assumptions.

The simplest example of an inhomogeneous turbulence model may be that of a two-dimensional shear flow, with the mean shear prescribed and with cyclic boundary conditions. The model equations are derived from the system:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= \nu\nabla^2 u \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= \nu\nabla^2 v \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned} \tag{3.4.1}$$

The notation is conventional. Departures from the mean shear state are denoted by primes, i.e.,

$$u = \langle u(y) \rangle + u'$$

$$v = v'$$

$$p = p'.$$

The equations for the primed fields are:

$$\frac{\partial u'}{\partial t} + \langle u \rangle \frac{\partial u'}{\partial x} + v' \frac{\partial \langle u \rangle}{\partial y} + u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} + \frac{\partial p'}{\partial x} = \nu \nabla^2 u'$$

$$\frac{\partial v'}{\partial t} + \langle u \rangle \frac{\partial v'}{\partial x} + u' \frac{\partial v'}{\partial x} + v' \frac{\partial v'}{\partial y} + \frac{\partial p'}{\partial y} = \nu \nabla^2 u' \qquad (3.4.2)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0.$$

What is missing in Eq. (3.4.2) (or the subsequent tracer model in section 4.1) is the information about the maintenance of mean flow. Here we assume an external forcing source implicitly. But since there is no feedback between mean flow and eddies, the feedback between forcing and eddies is also lacking. This assumption of a fixed forcing source is contrary to the spirit of a climate model, in which the radiative and dynamic processes are mutually adjustable (see Stone 1972). Some additional assumptions will be needed for the climate modelers to take advantage of the weakly inhomogeneous turbulence theory.

Since the above system is an autonomous function of x, the flow is homogeneous in the x-direction. The pressure terms can be written in terms of velocity field by the condition of incompressibility. The simplified system is most easily expressed in terms of the vorticity

$$\xi' = -\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}$$

which satisfies:

$$\frac{\partial}{\partial t}\xi' + \langle u\rangle \frac{\partial}{\partial x}\xi' - v'\frac{\partial^2\langle u\rangle}{\partial y^2} + u'\frac{\partial}{\partial x}\xi' + v'\frac{\partial}{\partial y}\xi' = \nu\nabla^2\xi'. \tag{3.4.3}$$

A generalized mean wind field $\langle u\rangle$ can be assumed to consist of a homogeneous part $\langle u_H\rangle$, and an inhomogenous part $\langle u_I\rangle$,:

$$\langle u\rangle = \langle u_H\rangle + \langle u_I\rangle.$$

Each corresponds to a mean shear:

$$\langle S\rangle = \frac{d\langle u\rangle}{dy}, \quad \langle S_H\rangle = \frac{\partial\langle u_H\rangle}{\partial y} = \text{constant}, \quad \langle S_I\rangle = \frac{\partial\langle u_I\rangle}{\partial y}.$$

It can be seen that by introducing a homogenous shear $\langle S_H\rangle$ alone, the flow system Eq. (3.4.2) becomes nonautonomous with respect to y, but still maintains a two-point homogeneity in y.

We introduce the eddy stream function ϕ',

$$u' = -\frac{\partial\phi'}{\partial y}, \quad v' = \frac{\partial\phi'}{\partial x}, \quad \xi' = \nabla^2\phi'$$

ξ', ϕ' and $\langle S_I \rangle$ are then expanded in Fourier components:

$$\phi' = \sum_{\mathbf{k}} \phi_{\mathbf{k}}(t)\, e^{i\,\mathbf{k}\cdot\mathbf{x}} = -\sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}(t)}{k^2}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}$$

$$\langle S_I \rangle = \sum_{\Delta\,\mathbf{k}} \langle S_I \rangle_{\Delta\,\mathbf{k}}\, e^{i\Delta\,\mathbf{k}\cdot\mathbf{x}} = \sum_{\Delta\,\mathbf{k}} \langle S_I \rangle_{\Delta\,\mathbf{k}}\, e^{iM\Delta\,\mathbf{k}\cdot\mathbf{y}}$$

$$\mathbf{k} = (L_{\mathbf{k}}, m_{\mathbf{k}}), \quad \Delta\,\mathbf{k} = (0, m_{\Delta\,\mathbf{k}}).$$

The reason $\langle S_I \rangle$ is associated with $\Delta\,\mathbf{k}$ is to remind us of the possible scale separation between ϕ' and $\langle S_I \rangle$. A more detailed discussion about the notation usage can be found in Appendix C. The spectral equation is:

$$\left(\frac{\partial}{\partial t} + L_{\mathbf{k}}\right) \xi_{\mathbf{k}}(t) = \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} a'_{\mathbf{kij}} \langle S_I \rangle_{\mathbf{i}} \xi_{\mathbf{j}}(t) + \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} a_{\mathbf{kpq}} \xi_{\mathbf{p}}(t) \xi_{\mathbf{q}}(t) \quad (3.4.4)$$

where

$$L_{\mathbf{k}} = \nu k^2 - \langle S_H \rangle L_{\mathbf{k}} \frac{\partial}{\partial M_{\mathbf{k}}}$$

$$a'_{\mathbf{kij}} = L_{\mathbf{k}} \frac{M_{\mathbf{i}}}{j^2} - \frac{1}{m_{\mathbf{i}}}$$

$$a_{\mathbf{kpq}} = (C_{\mathbf{kp}} + C_{\mathbf{kq}})/2 \;, \quad C_{\mathbf{kp}} = (L_{\mathbf{k}} m_{\mathbf{p}} - L_{\mathbf{p}} m_{\mathbf{k}})/p^2.$$

The DIAD expansion in section 3.3 starts by setting:

$$\left(\frac{\partial}{\partial t} + L_{\mathbf{k}}\right)\xi_{\mathbf{k}}(t) = \varepsilon\left(\sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} a'_{\mathbf{kij}} \langle S\ \rangle_{\mathbf{i}}\,\xi_{\mathbf{j}}(t) + \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} a_{\mathbf{kpq}}\xi_{\mathbf{p}}(t)\xi_{\mathbf{q}}(t)\right)$$

and expanding

$$\xi_{\mathbf{k}}(t) = \xi_{\mathbf{k}}^{(0)}(t) + \varepsilon\,\xi_{\mathbf{k}}^{(1)}(t) + \ldots .$$

We obtain:

$$\xi_{\mathbf{k}}^{(1)}(t) = \int_0^t G_{\mathbf{k}}^{(0)}(t,s)\Big[\sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} a'_{\mathbf{kij}} \langle S_I\rangle_{\mathbf{i}}\,\xi_{\mathbf{j}}^{(0)}(s) + \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} a_{\mathbf{kpq}}\xi_{\mathbf{p}}^{(0)}(s)\xi_{\mathbf{q}}^{(0)}(s)\Big]\,ds.$$

An off-diagonal correlation

$$\langle\phi_{\mathbf{k}}^{(1)}(t)\ \phi_{\mathbf{k'}}^{(0)}(t')\rangle$$

becomes:

$$\langle\phi_{\mathbf{k}}^{(1)}(t)\ \phi_{\mathbf{k'}}^{(0)}(t')\rangle = a'_{\mathbf{k}\ \Delta\ \mathbf{k}-\mathbf{k'}} \langle S_I\rangle_{\Delta\ k}\,\frac{k'^2}{k^2}\int_0^t G_{\mathbf{k}}^{(0)}(t,s)\,\langle\phi_{-\mathbf{k'}}^{(0)}(s)\ \phi_{\mathbf{k'}}^{(0)}(t')\rangle\,ds \qquad (3.4.5)$$

where

$$\mathbf{k} + \mathbf{k}' = \Delta\,\mathbf{k} = (0,\ M_{\Delta\,\mathbf{k}}).$$

Since

$$\langle\phi_{\mathbf{k}}(t)\ \phi_{\mathbf{k}'}(t')\rangle \approx \varepsilon\,[\langle\phi_{\mathbf{k}}^{(1)}(t)\ \phi_{\mathbf{k}'}^{(0)}(t')\rangle + \langle\phi_{\mathbf{k}}^{(0)}(t)\ \phi_{\mathbf{k}'}^{(1)}(t')\rangle]$$

the inhomogeneous momentum flux*

$$M^{(1)}(Y) = -\sum_{\mathbf{k}}\sum_{\mathbf{k}'} m_{\mathbf{k}} L_{\mathbf{k}} \langle\phi_{\mathbf{k}}\phi_{\mathbf{k}'}\rangle\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}$$

can be found by letting $\varepsilon = 1$, $t = t'$ and removing the superscripts of the off-diagonal correlations in the DIAD expansion:

$$M^{(1)}(Y) = \sum_{k}\sum_{k'} - m_{\mathbf{k}} L_{\mathbf{k}} \langle S_I\rangle_{\Delta\,\mathbf{k}}\, e^{im_\Delta\,\mathbf{k}\cdot Y}[a'_{\mathbf{k}\ \Delta\,\mathbf{k}-\mathbf{k}'} \frac{k'^2}{\mathbf{k}^2}\int_0^t G_{\mathbf{k}}(t,s)$$

$$\langle\phi_{\mathbf{k}}(s)\phi_{\mathbf{k}}(t)\rangle ds + a'_{\mathbf{k}'\ \Delta\,\mathbf{k}-\mathbf{k}} \frac{\mathbf{k}^2}{k'^2}\int_0^t G_{\mathbf{k}'}(t,s)\ \langle\phi_{-\mathbf{k}}(s)\ \phi_{\mathbf{k}}(t)\rangle\ ds].$$

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* The relationship between fluxes and correlations can be found in appendix C.

The coefficient :

$$a'_{\mathbf{k}\ \Delta\ \mathbf{k}-\mathbf{k}'}\ \frac{k'^2}{k^2} = \frac{L_{\mathbf{k}}k'^2}{k^2}\ [\frac{m_{\Delta\ \mathbf{k}}}{k'^2} - \frac{1}{m_{\Delta\ \mathbf{k}}}] = \frac{2m_{\mathbf{k}}L_{\mathbf{k}}}{k^2} - \frac{L_{\mathbf{k}}}{m_{\Delta\ \mathbf{k}}}$$

since

$$L_{\mathbf{k}} + L_{\mathbf{k}'} = 0 \quad \text{and} \quad m_{\mathbf{k}} + m_{\mathbf{k}'} = m_{\Delta\ \mathbf{k}}.$$

The resulting inhomogeneous momentum flux is given by:

$$M^{(1)}(Y) = \sum_{\mathbf{k}} \sum_{\mathbf{k}'} - m_{\mathbf{k}}L_{\mathbf{k}} \langle S_I\rangle_{\Delta\ \mathbf{k}}\ e^{im_{\Delta\ \mathbf{k}}Y} \{[\frac{2m_{\mathbf{k}}L_{\mathbf{k}}}{k^2} - \frac{L_{\mathbf{k}}}{m_{\Delta\ \mathbf{k}}}] \int_0^t G_{\mathbf{k}}(t,s)$$

$$\langle\phi_{-\mathbf{k}'}(s)\ \phi_{\mathbf{k}'}(t)\rangle\ ds + [\frac{2m_{\mathbf{k}'}L_{\mathbf{k}'}}{k'^2} - \frac{L_{\mathbf{k}'}}{m_{\Delta\ \mathbf{k}}}]$$

$$\int_0^t G_{\mathbf{k}'}(t,s)\ \langle\phi_{-\mathbf{k}}(s)\phi_{\mathbf{k}}(t)\rangle\ ds\}. \qquad (3.4.6)$$

The above expression is simpler than the original expression

$$M^{(1)}(Y) = \sum_{\mathbf{k}} \sum_{\mathbf{k}'} L_{\mathbf{k}}M_{\mathbf{k}} \langle\phi_{\mathbf{k}}\ \phi_{\mathbf{k}'}\rangle\ e^{iM_{\Delta\ \mathbf{k}}Y}$$

because only diagonal elements appear on the right side of Eq. (3.4.6). There has been no restriction placed on the magnitude of the inhomogeneous term

$$\sum a'_{\mathbf{k}ij} \, T_i \, \phi_j$$

in DIAD, this will no longer be true if further simplification of Eq. (3.4.6) is needed. The next section will show how to reduce the expression Eq. (3.4.6).

3.5 Weakly Inhomogeneous Approximation

Consider the case when a scale separation between mean shear

$$\langle S_I \rangle = \sum_{\Delta \mathbf{k}} \langle S_I \rangle_{\Delta \mathbf{k}} \, e^{i \Delta \mathbf{k} \cdot \mathbf{x}}$$

and eddies

$$\phi' = \sum_{\mathbf{k}} \phi_{\mathbf{k}} \, e^{i \mathbf{k} \cdot \mathbf{x}}$$

exists. We can then make the approximation:

$$|\Delta \mathbf{k}| \ll |\mathbf{k}|, |\mathbf{k}'| \quad \text{or} \quad \mathbf{k} \approx \mathbf{k}' \qquad (3.5.1)$$

in the calculation of $M^{(1)}(Y_0)$ in Eq. (3.4.6). This approximation will be called the weakly inhomogeneous assumption (WIA).

It can be seen that:

$$M^{(1)}(Y) = \lim_{\mathbf{k}'\to-\mathbf{k}} \sum_{\mathbf{k}} \sum_{-\mathbf{k}+\Delta\mathbf{k}} [\langle S_I\rangle_{\Delta\mathbf{k}}\, e^{iM_{\Delta\mathbf{k}}Y}][\frac{-4m_{\mathbf{k}}^2 L_{\mathbf{k}}^2}{k^2}]\int_0^t G_{\mathbf{k}}(t,s)$$

$$\langle\phi_{-\mathbf{k}}(s)\ \phi_{\mathbf{k}}(t)\rangle\, ds$$

$$= \sum_{\Delta\mathbf{k}} [\langle S_I\rangle_{\Delta\mathbf{k}}\, e^{iM_{\Delta\mathbf{k}}Y}][-\sum_{\mathbf{k}} \frac{4m_{\mathbf{k}}^2 L_{\mathbf{k}}^2}{k^2}\int_0^t G_{\mathbf{k}}(t,s)$$

$$\langle\phi_{-\mathbf{k}}(s)\ \phi_{\mathbf{k}}(t)\rangle\, ds] \qquad (3.5.2)$$

where the reality condition holds for

$$G_{\mathbf{k}}(t,s) = G_{-\mathbf{k}}(t,s) \text{ and } \langle\phi_{-\mathbf{k}}(s)\ \phi_{\mathbf{k}}(t)\rangle = \langle\phi_{\mathbf{k}}(s)\ \phi_{-\mathbf{k}}(t)\rangle .$$

The decoupling of

$$\sum_{\Delta\mathbf{k}} \text{ and } \sum_{\mathbf{k}}$$

in the second line of the above equation is the amazing result of WIA. This allows (3.5.2) to be rewritten as

$$M^{(1)}(Y) = -D(t)\ \langle S_I(Y)\rangle \qquad (3.5.3)$$

where the eddy diffusivity D(t) is defined as:

$$D(t) = \sum_{\mathbf{k}} \frac{4m_{\mathbf{k}}^2 L_{\mathbf{k}}^2}{k^2} \int_0^t G_{\mathbf{k}}(t,s) \langle \phi_{-\mathbf{k}}(s)\, \phi_{\mathbf{k}}(t) \rangle \, ds \qquad (3.5.4)$$

and, as a reminder, the inhomogeneous mean shear

$$\langle S_I(Y) \rangle = \sum_{\Delta \mathbf{k}} \langle S_I \rangle_{\Delta \mathbf{k}} \, e^{i \Delta \mathbf{k} \cdot \mathbf{x}} = \sum_{m_{\Delta \mathbf{k}}} \langle S_I \rangle_{\Delta \mathbf{k}} \, e^{i M_{\Delta} \mathbf{k} \cdot Y}.$$

A more familiar form of Eq. (3.5.3) therefore emerges as:

$$Z(Y) = - D(t) \frac{d\langle \xi(Y) \rangle}{dY} \qquad (3.5.5)$$

where the vorticity flux $Z(Y) = - (dM^{(1)}/dY)$ (see appendix C) and mean vorticity gradient $(d\langle\xi\rangle/dY) = (d/dY)\,[- (d/dY)(\langle U_H\rangle + \langle U_I\rangle)] = - (d/dY)\,\langle S_I\rangle$. The diffusion form of Eq. (3.5.5) states that the vorticity flux generated by the inhomogeneous mean shear tends to reduce the inhomogeneity in the flow. It is therefore necessary to have an external inhomogeneous source to sustain $\langle S_I\rangle$. Under the assumptions of the DIAD and WIA, Eq. (3.5.5) indicates a homogeneization process in simple shear turbulence, a result well known in classical turbulence theory (see chapter 4).

3.6 Convergent Inhomogeneous Expansion

The weakly inhomogeneous assumption, the structure of off-diagonal correlations, and hence the inhomogeneous dynamic fluxes, all strongly suggest an asymptotic expansion based on the scale separation between the eddy and mean flow. The traditional two-scale expansion is indeed a valid tool to solve tracer problems, where a weak shear of the mean tracer field has introduced inhomogeneity, as shown in section 4.2. But the same method can hardly apply to shear flow problems. The difficulties come from using an Eulerian frame. More specifically, difficulties result when an inhomogeneity also appears in the advection term.

The vorticity equation for shear flow in Eq. (3.4.3) can be rewritten as

$$\frac{\partial}{\partial t}\xi' + \langle U_H\rangle \frac{\partial}{\partial x}\xi' + u'\frac{\partial}{\partial x}\xi' + v'\frac{\partial}{\partial y}\xi' - \nu\nabla^2\xi'$$
$$= -\langle U_I\rangle \frac{\partial}{\partial x}\xi' + v'\frac{\partial^2\langle U_I\rangle}{\partial y^2}$$

where we put the two inhomogeneous terms on the right-hand side of the equation. $\langle U_I\rangle(\partial/\partial x)\ \xi'$ will be called the inhomogeneous advection term and $v'\ (\partial^2\langle U_I\rangle/\partial y^2)$ as the straining term. We assign a mean stream function $\langle\phi_I\rangle$ and let $\langle U_I\rangle = -\ (\partial\langle\phi_I\rangle/\partial y)$. The weakly inhomogeneous assumption claims that $\langle\phi_I\rangle$ is slowly varying in y. We shall assign a new coordinate Y and a small parameter δ such that by multiple-scale expansion the original operator $\partial/\partial y$ can be written as $(\partial/\partial y) + \delta\ (\partial/\partial Y)$, and $\partial^2/\partial y^2$ as $(\partial^2/\partial y^2) + 2\delta\ (\partial/\partial y)(\partial/\partial Y) + \delta^2(\partial^2/\partial Y^2)$. The new variable system (x,y,Y,t) yields a zero-order homogeneous shear turbulence and a first-order inhomogeneous equation where the inhomogeneous advection term is the sole source of inhomogeneity. By the same analysis of DIAD, it is easily found that $M^{(1)}(Y) = 0$ at the level $O(\delta)$. The expansion rapidly becomes too complicated to solve before we reach the straining term which is at the third order $O(\delta^3)$.

A retreat to a Lagrangian coordinates, and hence classical theory may shed new light on solving this problem. As Rhines (1978) showed, with a weakly inhomogeneous assumption and a semi-Lagrangian frame, the diffusion-like equation for vorticity flux can be derived. Rhines' result essentially follows Taylor's eddy diffusion theory (see appendix D); hence it leads to the same difficulty in calculating Lagrangian correlations. Some empirical values wil be needed for calculating vorticity flux.

A more drastic assumption used by Yoshizawa (1979), is to eliminate the inhomogeneous advection terms from Eq. (3.4.1) all together in a two-scale DIA expansion. This may be a dubious task since $\langle U_I\rangle\ (\partial u'/\partial x)$, $\langle U_I\rangle\ (\partial u'/\partial x)$ and $v'\ (\partial\langle U_I\rangle/\partial y)$ in the momentum equation are all needed to maintain incompressibility, which in turn, preserve the form of the vorticity equation in Eq. (3.4.3). The Yoshizawa model also ignores the difference between a WIA expansion and a DIA expansion. This seems to lead to some inconsistancies in his theory.

From the derivation of DIAD shear turbulence in the last section, it can be seen that the net contribution of the inhomogeneous advection term to $M^{(1)}(Y)$, and hence spatial transfer, arises only from a small residual part of itself. (A large portion of the advection term contribution in $M^{(1)}(Y)$ are those terms associated with factor $L\,\mathbf{k}/m_{\Delta\,\mathbf{k}}$ in Eq. (3.4.6), which vanishes when we add the conjugate part.) The total effect of the inhomogeneous mean shear in Eulerian coordinates cannot be properly represented by the scaling of each individual term. Here we propose that adding a small parameter δ on the right side of the equation

$$\frac{\partial}{\partial t}\zeta' + \langle U_H\rangle \frac{\partial}{\partial x}\xi' + u'\frac{\partial \xi'}{\partial x} + v'\frac{\partial \xi'}{\partial y} - \nu\nabla^2\xi$$

$$= -\langle U_I\rangle \frac{\partial \zeta'}{\partial x} + v'\frac{\partial^2 \langle U_I\rangle}{\partial y^2} \qquad (3.6.1)$$

would be enough to ensure weak inhomogeneity in our model. The reason is simple: Since DIAD will be formulated on a Fourier basis which already represents the "exact" scaling, there is no need for further multiple-scale expansion. δ,($\delta \ll o(1)$), as a true asymptotic expansion parameter, should not be confused with the label parameter $\varepsilon(=1)$ in the DIA model. All that can be claimed in Eq. (3.6.1) is that the bracket part on the right-hand side of the equation is at least one order less in magnitude than those terms on the left-hand side, which can easily be achieved by assuming that the mean stream function $\langle\phi_I\rangle$ is slowly varying in y. In other words, it assumes that the mean inhomogeneous mean shear $\langle U_2\rangle \sim O(\delta)$.

Eq. (3.6.1) can be written back in the DIAD Langevin equation in generalized form (see Eq. (3.3.11):

$$\frac{\partial}{\partial t}\,\phi_k(t) + L_k\phi_k(t) + \int_0^t \eta_{kk}(t,s)\;\phi_k(s)\;ds + \delta^2\int_0^t \eta'_{kk}(t,s)\;\phi_k(s)\;ds$$

$$= \;\delta\;\sum_{ij} a'_{kij}T_i\psi_j{}^{III}(t) + \sqrt{2}\;\sum_{p,q} a_{kpq}\psi_p{}^{I}(t)\psi_q{}^{II}(t) \qquad (3.6.2)$$

An expansion

$$\phi_k(t) \;=\; \phi_k{}^{(0)} + \delta\;\phi_k{}^{(1)}$$

will yield the zero-order system:

$$\frac{\partial}{\partial t}\,\phi_k{}^{(0)}(t) + L_k\phi_k{}^{(0)}(t) + \int_0^t \eta_{kk}\phi_k{}^{(0)}(s)\;ds$$

$$= \;\sqrt{2}\;\sum_{p,q} a_{kpq}\psi_p^{I}\;(t)\psi_q^{II}\;(t)$$

and the first-order system

$$\frac{\partial}{\partial t}\,\phi_k{}^{(1)}(t) + L_k\phi_k{}^{(1)}(t) + \int_0^t \eta_{kk}\phi_k{}^{(1)}(s)\;ds$$

$$= \;\sum_{ij} a'_{kij}T_i\psi_j{}^{III}$$

where the eddy damping rate η_{kk} and random force forms $\psi_p{}^2$, $\psi_q{}^{II}$ are supposed to come only from homogeneous elements, i.e., the zero-order solutions. It can be seen that above expansion leads to a two-scale two-time correlation

$$\langle\phi_k{}^{(1)}(t)\ \phi_L{}^{(0)}(t')\rangle = a'_{kmL}\ T_m \int_0^t G_{kk}{}^{(0)}(t,s)\ R_{LL}{}^{(0)}(t's)\ ds$$

which is seemingly identical to the result from the DIAD in Eq. (3.3.4). But there is a significant difference in the definition of $G_k{}^{(0)}(t,s)$ here. Since δ is a true small parameter, the superscripts will remain intact and $G_{kk}{}^{(0)}(t,t')$ is defined as

$$\frac{\partial}{\partial t} G_{kk}{}^{(0)}(t,t') + L_k G_{kk}{}^{(0)}(t,t') + \int_{t'}^t \eta_{kk}(t,s)\ G_{kk}{}^{(0)}(s,t')\ ds = 0$$

for $t > t'$, where the diagonalized eddy damping term

$$\int_{t'}^t \eta'_{kk}(t,s)\ G_{kk}(s,t')\ ds$$

is considered to be a second-order effect.

A question may be raised that now the zero-order system, which describes a homogeneous turbulent flow, should be energetically self-consistant. According to stability analyses, a constant shear confined in two-dimensions cannot be unstable. Hence a homogeneous energy source for the eddies should be included in the model.

To be consistent with previous assumptions, the physical maintenance of the flow energy budget will not be considered here. We impose the problem as follows: An existing homogeneous turbulent flow is superimposed by a weakly inhomogeneous shear. Most activities of spectral transfer can be described by a homogeneous turbulence model, which is based on the Langevin equation. The nonlinear interaction can be classified by two features, one can be simulated by a Gaussian noise term and another by a linear eddy damping term. When inhomogeneity enters the problem in the form of a small, random perturbation, it is expected that this disturbance will be relaxed irreversibly toward the equilibrium state by the same homogeneous damping effect. Since the response of the flow system is supposed to be linear, the inhomogeneous solution (and hence the off-diagonal correlations) can be found with the help of a Green's function. By exerting the homogeneity property of zero-order systems, a very simplified form for the inhomogeneous dynamic flux is thus achieved.

Again, we should mention here that the relaxation of inhomogeneity can only be expected for the simple shear flow we present in this chapter. The method we developed so far might not be applicable when more complicated inhomogeneous turbulence problems arise.

Chapter 4: Eddy Diffusivity and Atmospheric Tracer Transfer

In previous chapters, we have discussed the motivations and techniques for constructing a stochastic model in place of a deterministic flow equation. As often occurs in modern turbulence theories, a rigorous proof is missing. Hence only results can justify the means. In this chapter, we seek an indirect justification. We will use the classical eddy diffusion theory (see appendix D) as a "standard of comparison." By no means do we consider that Taylor's theory provides a definite proof. Its deficiency is also well known (see for example Hinze, 1975). But by achieving good agreement between our results and Taylor's, the weakly inhomogeneous model will be shown to fulfill a role that bridges new and old turbulence theories.

The second half of this chapter is devoted to the calculation of an atmospheric tracer problem. Under the assumption that the only information available to us is the energy spectrum and mean tracer field in the atmosphere (as is often the case for meteorologists), the tracer flux can be calculated using our model without empirical aid, a conceptual improvement over the existing methods (for example, Kao, 1974). Though the model is crude, it shows how a geophysical problem can be clarified by modern turbulence theories.

4.1 Tracer Model

Taylor's eddy diffusion theory is described in appendix D. It is formulated under a physical situation which will be duplicated in this section.

The concentration density q of a passive tracer in a two-dimensional space (x,y) is governed by the conservation law:

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0. \qquad (4.1.1)$$

Consider that tracer field q is embedded in a turbulent field ϕ, which is an incompressible, zero-mean flow. As earlier, q can be divided into a mean part and an eddy part. The mean field is assumed to be constant in x direction and slowly varying in y direction

$$q = \langle q(y)\rangle + q'$$

and $\phi = \phi'$ is the eddy stream function. The equation for q' is

$$\frac{\partial q'}{\partial t} - \frac{\partial \phi'}{\partial y}\frac{\partial q'}{\partial x} + \frac{\partial \phi'}{\partial x}\frac{\partial q'}{\partial y} + \frac{\partial \phi'}{\partial x}\langle S_q\rangle = 0. \qquad (4.1.2)$$

In the above equation the mean tracer gradient $\langle S_q\rangle$ is defined as $\langle S_q\rangle = (\partial\langle q\rangle/\partial y)$ and can be prescribed in two parts:

$$\langle S_q \rangle = \langle S_{qH} \rangle + \delta \langle S_{qI}(y) \rangle$$

where the homogeneous part $\langle S_{qH} \rangle$ is a constant shear along y-axis and δ is a small parameter to measure weak inhomogeneity as described in section 3.6. A tracer dissipation term $\kappa^2 \nabla^2 q'$ can also be added on the right side of Eq (4.1.2).

A Fourier transform expression for Eq. (4.1.2) is

$$\frac{\partial}{\partial t} q_{\mathbf{k}} + L_{\mathbf{k}} q_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} b_{\mathbf{kpq}} \phi_{\mathbf{p}} q_{\mathbf{q}} + \delta \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} b'_{\mathbf{kij}} \langle S_{qI} \rangle_{\mathbf{i}} \, \phi_{\mathbf{j}} \qquad (4.1.3)$$

with the expansions

$$q' = \sum_{\mathbf{k}} q_{\mathbf{k}} e^{i\,\mathbf{k}\cdot\mathbf{x}}, \quad \phi' = \sum_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\,\mathbf{k}\cdot\mathbf{x}}, \quad \langle S_{qI} \rangle = \sum_{\Delta\,\mathbf{k}} \langle S_{qI} \rangle_{\Delta\,\mathbf{k}} e^{i\Delta\,\mathbf{k}\cdot\mathbf{x}}$$

$$\mathbf{k} = (L_{\mathbf{k}}, m_{\mathbf{k}}) \quad , \quad \Delta\,k = (0, m_{\Delta\,\mathbf{k}})$$

and definitions

$$L_k = \kappa\, k^2 + iL_{\mathbf{k}} \langle S_{qH} \rangle$$

$$b_{\mathbf{kpq}} = \mathbf{p} \times \mathbf{k}$$

$$b'_{\mathbf{kpq}} = -\, iL_{\mathbf{k}}.$$

To complete the above tracer model, a two-dimensional vorticity equation for ϕ',

$$\frac{\partial \nabla^2 \phi'}{\partial t} - \frac{\partial \phi'}{\partial y} \frac{\partial}{\partial x} \nabla^2 \phi' + \frac{\partial \phi'}{\partial x} \frac{\partial \nabla^2 \phi'}{\partial y} - \nu \nabla^2 \nabla^2 \phi' = 0 \qquad (4.1.4)$$

is written in spectral form:

$$\frac{\partial \phi_{\mathbf{k}}}{\partial t} + \nu k^2 \phi_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} a_{\mathbf{kpq}} \phi_{\mathbf{p}} \phi_{\mathbf{q}}$$

where ν is the dissipative viscosity and

$$a_{\mathbf{kpq}} = \frac{\mathbf{k} \mathbf{x} \mathbf{p}}{2k^2} (q^2 - p^2).$$

This has been defined slightly differently than before [e.g., Eq. (3.4.4)].

Since the inhomogeneity only enters the problem through $\langle S_{qI} \rangle$ and the turbulent field ϕ' is independent of tracer evolution, an expansion of δ will apply only to

$$q_{\mathbf{k}} = q_{\mathbf{k}}^{(0)} + \delta\, q_{\mathbf{k}}^{(1)}$$

so that

$$\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^{(0)}.$$

The zero-order system is thus

$$[\frac{\partial}{\partial t} + L_{\mathbf{k}}]\, q_{\mathbf{k}}^{(0)} = \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}} \phi_{\mathbf{p}}^{(0)} q_{\mathbf{q}}^{(0)}$$

$$[\frac{\partial}{\partial t} + \nu k^2]\, \phi_{\mathbf{k}}^{(0)} = \sum_{\mathbf{k=q=k}} a_{\mathbf{kpq}} \phi_{\mathbf{p}}^{(0)} \phi_{\mathbf{q}}^{(0)} . \qquad (4.1.5)$$

The above equations describe a homogeneous turbulent flow. Again, we assume an invisible homogeneous force to maintain the equilibrium state of

$$\phi_{\mathbf{k}}^{(0)} \text{ and } q_{\mathbf{k}}^{(0)} .$$

The DIA expansion of zero-order system yields (see appendix E):

$$[\frac{\partial}{\partial t} + L_{\mathbf{k}}] \langle q_{\mathbf{k}}^{(0)}(t) q_{-\mathbf{k}}^{(0)}(t')\rangle + \int_0^t \eta_{q\ \mathbf{k}}(t,s) \langle q_{-\mathbf{k}}^{(0)}(t') q_{\mathbf{k}}^{(0)}(s)\rangle\, ds$$

$$= \sum_{\mathbf{p+q=k}} b^2_{\mathbf{kpq}} \int_0^{t'} G_{-q\ k}^{(0)}(t',s) \langle \phi_{\mathbf{p}}^{(0)}(t) \phi_{-\mathbf{p}}^{(0)}(s)\rangle \langle q_{\mathbf{q}}^{(0)}(t) q_{-\mathbf{q}}^{(0)}(s)\rangle\, ds \qquad (4.1.6)$$

$$[\frac{\partial}{\partial t} + L_{\mathbf{k}}]\, G_{q\ \mathbf{k}}^{(0)}(t,t') + \int_{t'}^{t} \eta_{q\ \mathbf{k}}(t,s)\, G_{q\ \mathbf{k}}^{(0)}(s,t')\, ds = 0 \qquad \text{for } t>t' \qquad (4.1.7)$$

$$\eta_{q\ k} = - \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}}\, b_{\mathbf{q-pk}}\, G_{q\ \mathbf{q}}^{(0)}(t,s) \langle \phi_{\mathbf{p}}^{(0)}(t)\ \phi_{-\mathbf{p}}^{(0)}(s)\rangle \qquad (4.1.8).$$

Here we only list equations for the tracer field. It can be seen that Eq. (4.1.6) is the exact solution of the following stochastic equation:

$$\frac{\partial}{\partial t} + L_{\mathbf{k}} \quad q_{\mathbf{k}}^{(0)}(t) + \int_0^t \eta_{q\ k}(t,s)\ q_{\mathbf{k}}^{(0)}(s)\ ds = \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}} \psi_{\mathbf{p}}^{I}(t) \Theta_{\mathbf{q}}^{II}(t). \tag{4.1.9}$$

The random variables

$$\psi_{\mathbf{p}}^{I}(t) \quad , \quad \Theta_{\mathbf{q}}^{II}(t)$$

come from separate ensembles of Gaussian noise such that

$$\langle \psi_{\mathbf{p}}^{I}(t) \psi_{\mathbf{p'}}^{I}(t') \rangle = \langle \phi_{\mathbf{p}}^{(0)}(t)\ \phi_{\mathbf{p'}}^{(0)}(t') \rangle\ \delta_{\mathbf{p+p'}}$$

and

$$\langle \Theta_{\mathbf{q}}^{II}(t)\ \Theta_{\mathbf{q'}}^{II}(t') \rangle = \langle q_{\mathbf{q}}^{(0)}(t)\ q_{\mathbf{q'}}^{(0)}(t') \rangle\ \delta_{\mathbf{q+q'}}.$$

According to the weakly inhomogeneous model in section 3.6, the first-order solution

$$q_{\mathbf{k}}^{(1)}$$

corresponds to that of the stochastic equation

$$[\frac{\partial}{\partial t} + L_{\mathbf{k}}]\ q_{\mathbf{k}}^{(1)}(t) + \int_0^t \eta_{q\ \mathbf{k}}(t,s)\ q_{\mathbf{k}}^{(1)}(s)ds = \sum_{\mathbf{ij}} b'_{\mathbf{kij}} \langle S_{qI} \rangle_{\mathbf{i}} \psi_{\mathbf{j}}^{III}(t) \tag{4.1.10}$$

while the random variables

$$\psi_{\mathbf{j}}^{III}(t)$$

satisfy all the statistical properties of

$$\psi_{\mathbf{p}}^{I}(t)$$

but are chosen from a separate group. With the aid of the Green's function defined in Eq. (4.1.7)

$$q_{\mathbf{k}}^{(1)}$$

can be solved as:

$$q_{\mathbf{k}}^{(1)}(t) = \int_0^t G_{q\ \mathbf{k}}^{(0)}(t,s)\ \ [\sum_{\mathbf{ij}} b'_{\mathbf{kij}} \langle S_{qI} \rangle_{\mathbf{i}}\ \psi_{\mathbf{j}}^{III}(s)\ ds\,]\, ds.$$

The off-diagonal correlation is written as:

$$\langle q_{\mathbf{k}}^{(1)}(t)\ \phi_{\mathbf{k'}}^{(0)}(t')\rangle = b'_{k\ \Delta kk'}\ \langle S_{\mathbf{q}I}\rangle \int_0^t G_{\mathbf{q}\ k}^{(0)}(t,s)$$

$$\langle \phi_{\mathbf{k'}}^{(0)}(t')\ \phi_{-\mathbf{k}}^{(0)}(s)\rangle\ ds$$

where $\mathbf{k} + \mathbf{k'} = \Delta\,\mathbf{k}$. In the weakly inhomogeneous limit $\mathbf{k} + \mathbf{k'} \to 0$. The inhomogeneous tracer flux $Q^{(1)}(Y,t)$ (from appendix C) is found to be

$$Q^{(1)}(Y,t) = \text{Inhomogeneous } \langle v'q'\rangle$$

$$= -\sum_{\mathbf{k}} i\, L_{\mathbf{k}}\ \Big[\sum_{m_{\Delta\,\mathbf{k}}} \langle q_{\mathbf{k}}^{(1)}(t)\ \phi_{-\mathbf{k}+\Delta\,\mathbf{k}}^{(0)}(t)\rangle\ e^{im_{\Delta}\,\mathbf{k}Y}\Big]$$

$$= -\Big[\sum_{\mathbf{k}} L_{\mathbf{k}}^2 \int_0^t \langle G_{q\ k}^{(0)}(t,s)\rangle\langle \phi_{\mathbf{k}}^{(0)}(s)\ \phi_{-\mathbf{k}}^{(0)}(t)\rangle\ ds\Big]$$

$$\Big[\sum_{m\Delta\,\mathbf{k}} \langle S_{qI}\rangle_{\Delta\,\mathbf{k}} e^{im_{\Delta}\,\mathbf{k}Y}\Big]$$

$$= -D_q(t)\ \langle S_{qI}(y)\rangle \qquad (4.1.9)$$

with the tracer eddy diffusivity defined as:

$$D_q(t) = \sum_{\mathbf{k}} L_{\mathbf{k}}^2 \int_0^t \langle G_{q\ k}^{(0)}(t,s)\rangle\langle \phi_{\mathbf{k}}^{(0)}(s)\ \phi_{-\mathbf{k}}^{(0)}(t)\rangle\ ds. \qquad (4.1.10)$$

The time-dependence of $D_q(t)$ is a characteristic feature of stochastic diffusion equations. A comparison can be made between Eq. (4.1.9) and Taylor's result (see appendix D):

$$\langle v'q' \rangle = - D \frac{\partial \langle q \rangle}{\partial y} \qquad \text{(D.1)}$$

where

$$D_L = \lim_{t\to\infty} \langle v'^2 \rangle \int_0^t e^{-\tau/\tau_L} d\tau \qquad \text{(D.6)}$$

and the subscript L represents the Lagrangian frame and τ_L is defined as the Lagrangian integral time scale. Apply the connection from the DIA to EDM in section 2.3 (also see chapter 5) so that:

$$\langle \phi_{\mathbf{k}}^{(0)}(s)\ \phi_{-\mathbf{k}}^{(0)}(t) \rangle = G_{\mathbf{k}}^{(0)}(t,s)\ \langle \phi_{\mathbf{k}}^{(0)}(s)\ \phi_{-\mathbf{k}}^{(0)}(s) \rangle \quad t>s$$

$$G_{\mathbf{k}}^{(0)}(t,s) = e^{\mu_{\mathbf{k}}(t-s)}$$

$$G_{q\ \mathbf{k}}^{(0)}(t,s) = e^{\mu_{q\ \mathbf{k}}(t-s)}$$

and define

$$\tau_{\varepsilon k} = 1/(\mu_{\mathbf{k}} + \mu_{q\ \mathbf{k}})$$

$$\langle V'^2_{\varepsilon k} \rangle = L_{\mathbf{k}}^2\ \langle \phi_{\mathbf{k}}^{(0)}(t')\ \phi_{-\mathbf{k}}^{(0)}(t') \rangle \quad \text{independent of } t'.$$

By substituting τ = t-s, Eq. (4.1.0) becomes, at the limit $t\to\infty$,

$$D_q = \lim_{t\to\infty} \sum_{\mathbf{k}} \langle v'^2_{\varepsilon\,\mathbf{k}} \rangle \int_0^t e^{-\tau/\tau\varepsilon\,\mathbf{k}} d\tau. \qquad (4.1.10)$$

The remarkable similarity between Eq. (4.1.10) and Eq. (D.6) is partly an intentional result. From stationary turbulence the intensity of v'-variance is in both a Lagrangian or Eulerian frame, i.e., $\langle v'_{\varepsilon}{}^2 \rangle = \langle v'_L{}^2 \rangle$. Note that:

$$\mu_{\mathbf{k}},\ \mu_{q\ \mathbf{k}}$$

are sometimes called the decorrelation rates. It can be seen that

$$1/\mu_{\mathbf{k}} \text{ and } (1/\mu_{q\ \mathbf{k}}) ,$$

with a dimension of time, are measures of correlation time in

$$\langle \phi_{\mathbf{k}}{}^{(0)}\ \phi_{-\mathbf{k}}{}^{(0)} \rangle \text{ and } \langle q_{\mathbf{k}}{}^{(0)}\ q_{-\mathbf{k}}{}^{(0)} \rangle \quad .$$

The characteristic of being a passive tracer clearly indicates that $\mu_{q\ \mathbf{k}}$ is independent of the

$$\langle q_{\mathbf{k}}{}^{(0)}\ q_{-\mathbf{k}}{}^{(0)} \rangle$$

spectrum, as we can prove in Eq. (4.1.7) where

$$G_{q\,\mathbf{k}}^{(0)}(t,s)$$

is only a function of

$$\langle\phi_{\mathbf{q}}^{(0)}\ \phi_{\mathbf{q}}^{(0)}\rangle \quad \text{and} \quad \langle S_{qH}\rangle \quad .$$

The Eulerian integral time scale $\tau_{\varepsilon k}$ has a more complicated meaning than that of τ_L. It includes the linear (eddy and dissipative) damping effects on both tracer and dynamic turbulent fields.

We should caution that in Taylor's model there is no distinction between homogeneous shear and inhomogeneous shear. In the limit when the shear scale $(1/m_{\Delta\,\mathbf{k}})$ goes to infinity, both the mean shear and the tracer flux in Eq. (4.1.9) approach homogeneity. Hence Eq. (4.1.9) becomes an expression applicable to quasihomogeneous flow. But it is not clear that our eddy diffusion Eq. (4.1.9) can be extended automatically to calculate homogeneous tracer (or momentum) flux in a constant shear flow (suppose that the zero-order system is isotropic). We will not defend here the validity to compare the inhomogeneous eddy diffusivity in (4.1.9) with the total eddy diffusivity in Taylor's more genralized theory, as it will be shown in the following.

The same DIA-EDM connection can also simplify the vorticity eddy diffusion equation (3.5.4). We obtain

$$D = \lim_{t\to\infty} \sum_{\mathbf{k}} \frac{4L_{\mathbf{k}}^{2} m_{\mathbf{k}}^{2}}{k^2} \langle \phi_{\mathbf{k}}^{(0)} \phi_{-\mathbf{k}}^{(0)} \rangle \int_0^t e^{-2\mu_{\mathbf{k}}\tau} d\tau$$

$$= \sum_{\mathbf{k}} \frac{2L_{\mathbf{k}}^{2} m_{\mathbf{k}}^{2}}{\mu_{\mathbf{k}} k^2} \langle \phi_{\mathbf{k}}^{(0)} \phi_{-\mathbf{k}}^{(0)} \rangle. \tag{4.1.11}$$

Comparing D with a similar form for D_q:

$$D_q = \sum_{\mathbf{k}} \frac{L_{\mathbf{k}}^{2} \langle \phi_{\mathbf{k}}^{(0)} \phi_{-\mathbf{k}}^{(0)} \rangle}{(\mu_{\mathbf{k}} + \mu_{q\,\mathbf{k}})} \tag{4.1.12}$$

shows that the vorticity transfer is not a function of v'-variance alone. D includes an additional factor $m_{\mathbf{k}}^{2}/k^2$. This is because, while the vorticity field is convected by the stream field, the stream field itself is also being modified by the vorticity field. Hence it is expected that the transfer of a vorticity field is more "hesitant" than the transfer of a passive tracer field.

The turbulent tracer Prandtl number P_r, defined as $P_r = D/D_q$, is a measure of relative efficiency between vorticity and passive tracer transfers. Unless the value of a tracer (chemical) dissipation κ becomes too large, P_r should be a universal constant for all tracers, and depends primarily on the homogeneous energy spectrum. The calculation of

$\mu_{\mathbf{k}}$ and $\mu_{q\,\mathbf{k}}$

corresponding to a given stationary energy spectrum can only be done numerically (see the next section).

We can take a quick glance at the magnitude of P_r by assuming an isotropic

$$\langle \phi_{\mathbf{k}}^{(0)} \; \phi_{-\mathbf{k}}^{(0)} \rangle$$

spectrum and

$$\mu_{\mathbf{k}} \approx \mu_q \, \mathbf{k}$$

and transforming the two-dimensional wave vectors in D and D_I into a continuous, isotropic wave number space. This yields

$$P_r = \frac{D}{D_q} = \frac{\displaystyle\int_0^{2\pi} \cos^2\theta \, \sin^2\theta \, d\theta \; \frac{k^3 R_k(0)}{\mu_k} \, dk}{\displaystyle\int_0^{2\pi} \sin^2\theta \, d\theta \; \frac{k^3 R_k(0)}{2\mu_k} \, dk} = 1.$$

P_r is predicted as 1/2 in the Taylor's vorticity transfer theory and 1 in the Prandtl's momentum transfer theory (see Monin and Yaglom, 1972). The generally accepted observed value of P_r is approximately 0.7 (see Hinze, 1975). The reason that D and D_q stand at parity in our model is mainly because there are twice as many off-diagonal dynamic correlations

$$\langle\phi_{\mathbf{k}}\phi_{\mathbf{k'}}\rangle = \langle\phi_{\mathbf{k}}^{(0)}\phi_{\mathbf{k'}}^{(1)}\rangle + \langle\phi_{\mathbf{k}}^{(1)}\phi_{\mathbf{k'}}^{(0)}\rangle$$

as off-diagonal tracer correlations

$$\langle\phi_{\mathbf{k}}q_{\mathbf{k'}}\rangle = \langle\phi_{\mathbf{k}}^{(0)}q_{\mathbf{k'}}^{(1)}\rangle .$$

In the next section we will introduce an expansion method on the same tracer model which seems to appeal more directly to our physical intuition.

4.2 Two-Scale DIA Expansion

In section 3.6, we briefly mentioned the possibility of a two-scale DIA expansion method, which does not yield a diffusion formula for vorticity flux. But this technique is adaptable to tracer models. An advantage of the two-scale expansion is that the inhomogeneous mean shear stays in physical coordinates (see the implications in appendix C). Hence we can tag the inhomogeneity explicitly during the derivation, a potentially useful approach in dealing with a more complicated problem (for example, a two-level baroclinic model). Another advantage is that in two-scale DIA expansion the mean shear does not have to satisfy periodic boundary conditions. Hence we do not have to distinguish the difference between a homogeneous shear and an inhomogeneous shear.

The tracer Eq. (4.1.2) is repeated here as

$$\frac{\partial q'}{\partial t} - \frac{\partial \phi'}{\partial y}\frac{\partial q'}{\partial x} + \frac{\partial \phi'}{\partial x}\frac{\partial q'}{\partial y} + \frac{\partial \phi'}{\partial x}\langle S_{qI}(Y)\rangle = 0$$

where we neglect the homogeneous shear $\langle S_{qH}\rangle$. Since $\langle S_{qI}\rangle$ is slowly varying in y, we designate a new coordinate Y and a small parameter δ such that the original $\partial/\partial y$ can be written as $\partial/\partial y + \delta(\partial/\partial Y)$ in the new coordinate (x,y,Y,t). A Fourier transformation of Eq. (4.1.2) from (x,y,Y,t) coordinates to

$$(L_{\mathbf{k}}, m_{\mathbf{k}}, Y, t)$$

and an amplitude expansion

$$q_{\mathbf{k}} = q_{\mathbf{k}}^{(0)}(t) + \delta\, q_{\mathbf{k}}^{(1)}(Y,t)$$

$$\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^{(0)}(t)$$

yields as its solution

$$\frac{\partial}{\partial t} q_{\mathbf{k}}^{(0)} = \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}}\, \phi_{\mathbf{p}}^{(0)} q_{\mathbf{q}}^{(0)}$$

$$\frac{\partial}{\partial t} q_{\mathbf{k}}^{(1)} = \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}}\, \phi_{\mathbf{p}}^{(0)}\, q_{\mathbf{q}}^{(1)} + iL_{\mathbf{k}}\, \phi_{\mathbf{k}}^{(0)}\langle S_{qI}(Y)\rangle.$$

According to the DIA formalism, we define an ensemble-averaged Green's function

$$G_{\mathbf{k}}^{(0)}(t,s)$$

with respect to the linear response of the zero-order system; hence,

$$q_{\mathbf{k}}^{(1)}(t) = - iL_{\mathbf{k}} \langle S_{qI}(Y)\rangle \int_0^t G_{\mathbf{k}}^{(0)}(t,s)\ \phi_{\mathbf{k}}^{(0)}(s)\ ds$$

The half-transformed inhomogeneous correlation is given by

$$\langle q_{\mathbf{k}}^{(1)}(t,Y)\phi_{-\mathbf{k}}^{(0)}(t)\rangle = - iL_{\mathbf{k}}\langle S_{qI}(Y)\rangle \int_0^t G_{\mathbf{k}}^{(0)}(t,s)$$

$$\langle \phi_{\mathbf{k}}^{(0)}(s)\phi_{-\mathbf{k}}^{(0)}(t)\rangle\ ds.$$

The inhomogeneous (local) tracer flux

$$M^{(1)}(Y) = \sum_{\mathbf{k}} iL_{-\mathbf{k}}\langle q_{\mathbf{k}}^{(1)}\phi_{-\mathbf{k}}^{(0)}\rangle$$

is then found to be identical to that in Eq. (4.1.9).

The two-scale DIA expansion is only compatible with a weakly inhomogeneous model in a simple case like the tracer problem. It remains largely an unexplored method.

4.3 Decorrelation Rate

By use of the DIA and other analytic turbulence theories, eddy diffusivities can now be calculated whenever the homogeneous energy spectrum is known. An abridged form of the DIA Green's function is found by substituting

$$\langle G_{q\,\mathbf{k}}^{(0)}(t,t')\rangle \text{ for } e^{-\mu_{q\,\mathbf{k}}(t-t')}$$

and

$$\langle \phi_{\mathbf{k}}^{(0)}(t')\ \phi_{-\mathbf{k}}^{(0)}(t)\rangle \text{ for } e^{-\mu_{\mathbf{k}}(t-t')}R_{\mathbf{k}}^{(0)}$$

in Eqs. (4.1.7) and (4.1.8), where

$$R_{\mathbf{k}}^{(0)}$$

is the zero-time separation homogeneous spectrum

$$\langle \phi_{\mathbf{k}}^{(0)}\ \phi_{-\mathbf{k}}^{(0)}\rangle \quad .$$

We obtain

$$(-\mu_{q\,\mathbf{k}} + \kappa k^2)\, e^{-\mu_{q\,\mathbf{k}}(t-t')} \sum_{\mathbf{p}} \frac{b_{\mathbf{kpq}}\, b_{\mathbf{q-pk}}\, R_{\mathbf{p}}^{(0)}}{(\mu_{q\,\mathbf{p}} + \mu_{\mathbf{p}} + \mu_{q\,\mathbf{k}})}$$

$$\left(e^{-\mu_{q\,\mathbf{k}}(t-t')} - e^{-(\mu_{q\,k} + \mu_{\mathbf{k}})(t-t')}\right) = 0.$$

Let $t - t' = \tau$. By integrating the above equation from $\tau = 0 \to \infty$, we get:

$$\mu_{q\,\mathbf{k}} = \kappa k^2 + \sum_{\mathbf{p}} \frac{|\,\mathbf{kxp}\,|^2 R_{\mathbf{p}}(0)}{\mu_{q\,\mathbf{q}} + \mu_{\mathbf{p}}} \qquad (4.3.1)$$

$$\mu_{\mathbf{k}} = \nu K^2 + \sum_{\mathbf{p}} \frac{|\,\mathbf{kxp}\,|^2 \left(1 - \frac{p^2}{q^2}\right)\left(1 - \frac{p^2}{k^2}\right) R_{\mathbf{p}}^{(0)}}{\mu_{\mathbf{p}} + \mu_{\mathbf{q}}} . \qquad (4.3.2)$$

We will call Eqs. (4.3.1) and Eq. (4.3.2) as the abridged DIA (denoted as ADIA) expressions for

$$\mu_{q\,k} \text{ and } \mu_{\mathbf{k}} \quad .$$

The self-consistent eddy-damped Markovian model (SEDM), introduced in section 2.3 can be extended to the tracer model (see appendix E). The results for $\mu_{q\,\mathbf{k}}$ and $\mu_{\,\mathbf{k}}$ in the SEDM version are similar to those in ADIA. We obtain:

$$\mu_{q\,\mathbf{k}} = \kappa k^2 + \sum_{\mathbf{p}} \frac{|\,\mathbf{kxp}\,|^2 R_{\mathbf{p}}(0)}{\mu_{q\,k} + \mu_{\mathbf{p}} + \mu_{q\,\mathbf{q}}} \qquad (4.3.3)$$

and

$$\mu_{\mathbf{k}} = \nu K^2 + \sum_{\mathbf{p}} \frac{|\,\mathbf{kxp}\,|^2 (1 - \frac{p^2}{q^2})(1 - \frac{p^2}{k^2}) R_{\mathbf{p}}{}^{(0)}}{\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}}} . \qquad (4.3.4)$$

Both DIA and EDM (as well as the weakly inhomogeneous theory) are constructed within an Eulerian framework. The analogues of the expressions for eddy diffusivity between our model and Taylor's results seem to suggest a Lagrangian modification of $\mu_{\mathbf{k}}$ and $\mu_{q\,\mathbf{k}}$. This is not inconsistent with our previous discussion in section 2.4 where the DIA has to make a Lagrangian modification in order to maintain the Galilean invariant. For the sake of comparison, we list here two Lagrangian modified closure schemes:

$$\mu_{\mathbf{k}} = \nu k^2 + g^2 \Big(\sum_{\mathbf{p}} \frac{|\,\mathbf{kxp}\,|^2 R_{\mathbf{p}}(0)}{k^2 q^2 (\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}})}\Big) \qquad (4.3.5)$$

$$\mu_{\mathbf{k}} = \nu k^2 + \lambda \Big(\sum_{\mathbf{p}}^{|\,\mathbf{p}\,|<|\,\mathbf{k}\,|} p^2 R_{\mathbf{k}}{}^{(0)}\Big)^{1/2} \qquad (4.3.6)$$

Eq. (4.3.5) is the result from the test field model (TFM).

A detailed description of the derivation of Eq. (4.3.5) can be found in Holloway and Hendershott (1977). Eq. (4.3.6) is a discreet form of the phenomenological closure of Eq. (2.3.19). It measures the total effect of strain by large eddies on a smaller eddy with size $1/|\mathbf{k}|$. Both closures are semi-empirical, i.e., they have adjustable constants which are usually determined by direct numerical simulation. Here we adopt g = 0.6 from Leith and Kraichnan (1972) and λ = 0.3 from Pouquet et al. (1975).

In the next section all the above schemes will be tested with a prescribed large-scale atmospheric spectrum. The discrepancy between an Eulerian scheme and a Lagrangian scheme will also be investigated quantitatively.

4.4 Eddy Diffusivities by Large-Scale Atmospheric Eddies

For obtaining sample values of eddy diffusivity using the various closure schemes, we will consider an eddy spectrum described by

$$\langle\phi_{\mathbf{k}} \quad \phi_{-\mathbf{k}} \quad\rangle = \begin{cases} a|\mathbf{k}|^{-M-3} & \text{when } |\mathbf{k}| < k_R \\ b|\mathbf{k}|^{-N-3} & \text{when } |\mathbf{k}| > k_R . \end{cases} \tag{4.4.1}$$

The amplitudes of a,b are fixed by the total eddy kinetic energy:

$$E = \langle u'u'\rangle + \langle v'v'\rangle = \sum_{\mathbf{k}} E_k = \sum_{\mathbf{k}}^{k_{max}} k^2 \langle\phi_{\mathbf{k}}{}^{(0)}\phi_{-\mathbf{k}}{}^{(0)}\rangle ,$$

and the continuity of the spectrum: i.e., a k_R^{-M-3} = b k_R^{-N-3}. The dimensional wave vector $\mathbf{k}$ = (I/L,J/L), where I, J = 0,1,2,...25 and L = 4500 km corresponds to a midlatitudinal box. The value for total eddy kinetic energy is set at 450 m^2/sec, a typical value found close to the jet stream near 300 mb. The dividing wave number, k_R, represents the internal deformation scale in the atmosphere, which is generally recognized as the Rossby radius of deformation (see for example Salmon, 1978). The two segments of

$$\langle \phi_{\mathbf{k}}^{(0)} \phi_{-\mathbf{k}}^{(0)} \rangle$$

are an M-slope spectrum, called energy inertial range where inverse cascade of energy prevails in spectrum space, and an N-slope spectrum, called enstrophy inertial range where enstrophy cascade prevails and is downward to smaller eddies (see Kraichnan, 1975). We choose the values M = 0, N = 3 and $\mathbf{k}_R$ = (7/L,7/L) that roughly fits in the observed atmospheric energy spectrum by Wiin-Nielsen (1967). A comparison can be made only by transforming the two-dimensional energy spectrum $E_{\mathbf{k}}$ into a one-dimensional zonal wave number energy spectum. That is,

$$E_1(k_1) = \sum_{m_{\mathbf{k}}}^{|m_{\mathbf{k}}\,max|} E(\,k\,) \text{ for } k_1 = |L_{\mathbf{k}}|.$$

The prescribed $E_1(k)$ is plotted in Fig. 4.1.

The physical quantities to be computed and their respective formulae are listed in Table 4.1. The computations for most closure schemes are accomplished by interaction methods. The dissipative viscosity ν (or κ) has only small value which will not be included into the calculation. The results are summarized in Figs. 4.2 and 4.3. Fig. 4.2 is a plot of the decorrelation rates derived from different closure schemes vs wave number. Fig. 4.3 shows the percentage of eddy diffusivities contributed by eddies whose physical sizes are larger than that of the mode N. We conclude:

(1) The tracer eddy diffusivity D_q is found to be 2.4x-10^7 m^2/sec in ADIA and 2.8x10^7 m^2/sec in SEDM, which are on the same order as observed values*. Kao (1974) and Murgatroyed (1969) reported that D_q near the midlatitude jet stream area was observed to

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* It will be very difficult to classify a real geophysical flow as a homogeneous problem or an inhomogeneous problem. For example, the profile of the zonally-averaged temperature profile is symmetrical to the equator, hence it should be treated like a purely inhomogeneous problem within the domain from pole to pole. But if we consider the same mean temperature only in a hemisphere (or in certain regions), this problem will show a predominantly homogeneous features. Under such circumstances the suggestion in Section 4.1 which extends the concept of inhomogeneous eddy diffusivity to total eddy diffusivity will be very useful from a practical view.

As far as we know, there is no previous study to distinguish between a homogeneous eddy diffusivity and an inhomogeneous eddy diffusivity. Here we will refer the observed large-scale atmospheric eddy diffusivity based on the analysis of dispersion of particles from an instantaneous source. For example, in Kao's experiment, clusters of marked air particles (isobaric balloons) were released from a circle of radius 175 km. Following each particle's trajectory the relative dispersion between particles has been calculated and the ensemble-averaged integral time scale was derived from the particle-distance sphere. A total of 25920 particle pairs were analyzed. The eddy diffusivity was found from the formula in Appendix D.

be around 10^6 to 10^7 m^2/sec. There is no agreement on a defnite value of D_q in the literature.

(2) From Fig. 4.2, it can be seen that the Eulerian decorrelation rates are generally greater than the Lagrangian decorrelation rates (μ_k for TFM and PM), as one may expect. The overestimation of μ_k is especially crucial for smaller eddies since they are more vulnerable to the effect of advection. For the first five or six modes all three closures (ADIA, SEDM, TFM) agree very well on the values of μ_k.

(3) The μ_k-curve for phenomenological closure (PM), i.e., Eq. (2.3.19), is parallel to those of the other three closures, although it differs significantly in magnitude. Maybe a new empirical constant should be assigned to the energy inertial range. Nevertheless, the common agreement of shapes in μ_k curves indicates that all closure schemes preserve the features of phenomenological dephasing times.

(4) From Fig. 4.3 we can see that in Eulerian closure schemes 90% of the eddy diffusivities comes from the first six or seven modes: in the TFM, it is the first nine or ten modes. This makes the assumption of scale separation between mean and eddy fields a marginal one.

We will not deny that our experiment has been conducted under somewhat artificial conditions. It can easily be pointed out that in reality:

(1) Large-scale atmospheric motions act like a horizontal thermal convection model. IF the eddy diffusion formula does apply, it

should apply on both the quasigeostrophic potential vorticity flux and the heat flux (according to Wiin-Nielsen and Sela, 1971). In short, the baroclinicity should be included in the model.

(2) The large-scale atmospheric eddies are not fully turbulent, due to the presence of β-effect. At best they can only be described as being intermittent.

(3) The quasi-two-dimensionality of the atmosphere inherently contradicts the weakly inhomogeneous assumption, since two-dimensional turbulence theory predicts a reverse energy cascade to large-scale eddies. Hence the scale separation between the mean shear and eddies can hardly be expected to last.

It is possible to overcome the first two arguments for weakly inhomogeneous turbulence theories. Salmon (1978) has developed a two-layer baroclinic EDM model and Holloway and Hendershott (1977) succeeded in incoporating the β-effect into a TFM closure scheme. The difficulty associated with point three seems to be more formidable. Here we can only speculate that, if there is no weakly inhomogeneous assumption, there will be no simple expression (other than in spectral form) for the inhomogeneous dynamic fluxes.

Chapter 5: Comparison Between Results from a Directly Simulated Model and Turbulence Theories

Modern turbulence theories seek nonrandom statistics from a chaotic flow. We cannot find a better example than the ensemble-averaged Green's function $\langle g_{\alpha\beta}(t,t')\rangle$ defined in the DIA model. The study of predictability (Lorenz, 1969) tells us that any deviation in initial conditions will result in uncertainties in the final realization, especially for two-dimensional flow where aliasing errors in the small scales of motion will cascade and amplify to large-scale eddies. But the error growth follows a definite trend before it reaches the limit of predictability. This trend can be measured by $\langle g_{\alpha\beta}(t,t')\rangle$ and predicted to some extent from the fluctuation-dissipation theorem (FDT). It is the purpose of this chapter to calculate $\langle g_{\alpha\beta}(t,t')\rangle$ and verify the FDT by a direct simulation.

In section 5.1, we introduce the FDT and the next section is a description of the numerical model we will use. The remaining three sections are the reports of our experimental results on the energy spectrum, Green's functions and decorrelation rates when the thermal equilibrium state is reached. We would like to add a few more words about our motivation for writing this chapter. In the weakly inhomogeneous turbulence model, the zero-order system (ground state) is homogeneous. To calculate spatial transfer, we first have to deduce the homogeneous spatial-temporal spectrum and the Green's function. Therefore the investigation of various techniques to sort out homogeneous elements in this chapter will be considered as an essential step to complete our study.

5.1 Fluctuation-Dissipation Theory

Consider a physical system consisting of a number of dynamic variables $\{\phi_\alpha, \phi_\beta, \ldots\}$ and preserving certain statistical properties (e.g., integral constraints). The system with strong internal mixing among the variables is characterized by rapid fluctuations of $\{\phi_\alpha, \phi_\beta, \ldots\}$ in time evolution. For each pair of variables ϕ_α, ϕ_β there are two time scales of primary interest. The apparent one is a direct measurement of the time-lagged correlation $\langle\phi_\alpha(t)\, \phi_\beta(t')\rangle = R_{\alpha\beta}(\tau)$, $\tau = t - t' > 0$. The stationarity of the system assures us that only the time difference between and ϕ_α and ϕ_β contributes to $R_{\alpha\beta}$. The internal oscillating time scale

$$(T_{\alpha\beta})_{Int} = \int_0^\infty R_{\alpha\beta}(\tau)/R_{\alpha\beta}(0)\, d\tau \tag{5.1.1}$$

is thereby a description of the natural damping rate of $R_{\alpha\beta}$.

The other time scale is not so obvious. According to the definitions and notations in section 3.1, the ensemble-averaged Green's function is written as

$$G_{\alpha\beta}(t,t') = \langle \delta\phi_\alpha(t)/\delta f_\beta(t')\rangle \quad , \text{ or as } \quad G_{\alpha\beta}(\tau) \quad .$$

The linear relaxation time scale

$$(T_{\alpha\beta})_{rsp} = \int_0^\infty G_{\alpha\beta}(\tau)\, d\tau \qquad (5.1.2)$$

represents the response of the system to an infinitesimal perturbation. It measures the sensitivity of, say, a climate model, when an external forcing is suddenly applied. Under the condition of thermal equilibrium,* the fluctuation-dissipation theorem states that

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* For an ensemble of realizations of the physical system $\{\phi_\alpha, \phi_\beta, \ldots\}$, a probability density function p which satisfies the Liouville equation

$$\frac{\partial p}{\partial t} + \sum_\alpha \dot{\phi}_\alpha \frac{\partial p}{\partial \phi_\alpha} = 0$$

and conserves a single constant of motion (integral constraint, $E(\phi_d, \phi_p, \ldots)$, during the time evolution in phase space (see section 1.1), can be found to have a stationary solution

$$P_T = e^{-\alpha E}$$

where α is decided by initial conditions and is equivalent to temperature in a Maxwell-Boltzmann ensemble. The thermal equilibrium state is defined as the state when the ensemble evolution has reached the canonical distribution P_T. We call it thermal equilibrium state because P_T is stable under the random coupling between realizations in the ensemble provided the conservation law of E is not violated. For two-dimensional turbulence there are two integral constaints. The canonical distribution will be presented in section 5.3.

$$R_{\alpha\beta}(\tau) = \sum_r G_{\alpha r}(\tau)\, R_{r\beta}(0). \qquad (5.1.3)$$

The above equation can also be written as

$$R_{\alpha\beta}(\tau) = G_{\alpha\beta}(\tau)\, R_{\beta\beta}(0) \qquad (5.1.4)$$

since $R_{\alpha\beta}(0) = R_{\alpha\beta}(0)\,\delta_{\alpha\beta}$ in equilibrium state. The proof of the FDT can be found in Kraichnan (1958) and Leith (1975).

The implications of the FDT are that the physical system in thermal equilibrium state will be described by the same relaxation history after excitations either from natural oscillation or from an artificial source, providing that the outside "kick" (according to Bell, 1980) is small enough to induce only linear response. In the DIA model and the subsequent extensions of the inhomogeneous turbulence problem, the nonlinear term (internal excitations) is replaced and simulated by a Gaussian process (external excitations). In accordance with the FDT, the response of Green's function should store the same memory as if natural damping were taking its own course. The connection between the DIA and the EDM in Eq. (2.3.15) clearly illustrates this point.

It can be seen that in the thermal equilibrium state, there is only one time scale relevant. That is:

$$(T_{\alpha\beta})_{Int} = (T_{\alpha\beta})_{rsp} = \frac{1}{\mu_{\alpha\beta}}$$

where $\mu_{\alpha\beta}$ is the decorrelation rate for $R_{\alpha\beta}(\tau)$. The above relationship is especially useful for climate modelers. Since in climate problems the distinction between an external forcing and an internal excitation is often obscured. Neither can we perturb a real climate state artificially. As Leith (1975) advocated, the FDT should be extended to the study of climatic sensitivity.

The next sections are reports of numerical experiments, centered around the FDT. Our technique is not unlike Bell's experiment (1980). But in our model there are 1086 dynamic variables, compared to 20 variables in Bell's model. Therefore our experiments are a more certain verification of these turbulence theories.

5.2 Two-Dimensional Turbulence Model

The two-dimensional, inviscid, barotropic vorticity equation (4.1.4)

$$\frac{\partial \nabla^2 \phi'}{\partial t} - \frac{\partial \phi'}{\partial y} \frac{\partial}{\partial x} \nabla^2 \phi' + \frac{\partial \phi'}{\partial x} \frac{\partial \nabla^2 \phi'}{\partial y} = 0$$

is written in dimensionless spectral form (see section 4.1):

$$\frac{\partial \phi_{\mathbf{k}}}{\partial t} = \sum_{\mathbf{p+q=k}} \frac{\mathbf{k x p}}{2k^2} (q^2 - p^2)\, \phi_{\mathbf{p}} \phi_{\mathbf{q}}. \tag{5.2.1}$$

We assume that ϕ' exists in a square periodic box $(2\pi,2\pi)$. The double Fourier series of ϕ' is truncated at k_{max} and set as 16. The time integration of Eq. (5.2.1) is a leap-frog scheme where the Jacobian term is calculated by pseudospectral method.* The total energy

$$E = \frac{1}{2} \sum_{\mathbf{k}}^{\sqrt{2}k_{max}} k^2 \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle$$

and total enstrophy

$$F = \frac{1}{2} \sum_{\mathbf{k}}^{\sqrt{2}k_{max}} k^4 \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle$$

are conserved in both Eq. (5.2.1) and the numerical integration. We define an unidirectional energy spectrum**

$$E(k) = \sum_{k-\Delta k<k<k+\Delta k} k^2 \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle$$

and enstrophy spectrum

$$F(k) = \sum_{k-\Delta k<k<k+\Delta k} k^4 \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle$$

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* The basic numerical code was kindly provided by Dr. Haidvogel at the Woods Hole Oceanographic Institution.

** In our results we have added a factor $2\pi k/N$ on $E(k)$, $F(k)$, where N is the number of discreet modes within the band $(k-\Delta k, k+\Delta k)$ in order to smooth the spectrum near low wave number regime.

where Δk is 1/2. The total energy and enstrophy

$$\tilde{E} = \sum_{k=1}^{k_{max}} E(k) \qquad \text{and} \qquad \tilde{F} = \sum_{k=1}^{k_{max}} F(k)$$

will approximately equal E and F as defined before. Since Eq. (5.2.1) is isotropic, E(k), F(k) represent isotropic spectra if the initial condition is also isotropic. We choose a random phase initial condition which will give $E = 2.09$ and $F = 84.53$ in the equilibrium state. Eq. (5.2.1) is integrated with a time step $\Delta t = 0.001$ until the time-averaged E(k) (averaged over about an estimated period t that is the correlation time scale for the lowest mode, i.e., $t \approx 1/\mu_1$, μ_1 as the decorrelation rate of mode 1) approach constant values. E(k) and F(k) are then compared with theoretical prediction.

5.3 Energy Spectrum in Thermal Equilibrium State

Kraichnan (1975b) has derived the canonical probability density function P_T for two-dimensional inviscid turbulence:

$$P_T = \exp(-\alpha E - \beta F)$$

which, in our case, P_T is approximately $\exp(-\alpha \tilde{E} - \beta \tilde{F})$ since the truncation is within a square box, instead of a circle. α, β are constants which can be derived as follows: the above P_T yields an isotropic energy spectrum

$$\tilde{E} = \sum_k E(k) = \sum_k^{k_{max}} \pi k(\alpha + \beta K^2)^{-1}$$

$$\tilde{F} = \sum_k F(k) = \sum_k^{k_{max}} \pi k^3(\alpha + \beta k^2)^{-1}.$$

Since E, F are known from initial condition, α, β can be obtained from iteration methods. We use a Newton-Raphson method to get α = 0.025, β = 1.61. Subsequently E(k) and F(k) can be calculated. We have plotted the thermal equilibrium state E(k) calculated both from Kraichnan's prediction and from directly simulated model in Fig. 5.1. The results appear to be in very good agreement. The slight discrepancy in lowest 1.2 modes is considered due to the relatively short sampling time for direct simulation.

5.4 Ensemble-Averaged Green's Function

After 2 or 3 time units, the two-dimensional turbulence model apparently evolves into the thermal equlibrium state. A particular mode $\alpha_{\mathbf{k}}$ (either a real part or a imaginary part of $\phi_{\mathbf{k}}$) is chosen. We add a sudden perturbation*

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* For two-dimensional model, we only integrate half the space of $\mathbf{k}$ since the realizability demands $\phi^*_{\mathbf{k}} = \phi_{-\mathbf{k}}$. A perturbation on $\phi_{\mathbf{k}}$ means a conjugated perturbation on $\phi_{-\mathbf{k}}$ at the same time.

$$\delta f_{\mathbf{k}}(t_0) = R(t_0)\,\varepsilon|\alpha_{\mathbf{k}}| \qquad (5.4.1)$$

into $\alpha_{\mathbf{k}}$ at time $t = t_0$, where $R(t_0)$ is a random number from -1 to 1 and ε is set as 0.1. The difference between the perturbed $\alpha'_{\mathbf{k}}$ and the unperturbed $\alpha_{\mathbf{k}}$ is then calculated by the two parallel evolutions:

$$\delta\alpha_{\mathbf{k}}(t) = \alpha'_{\mathbf{k}}(t) - \alpha_{\mathbf{k}}(t).$$

The Green's function is the normalized impulse response function

$$g_{\mathbf{k}}(t,t_0) = \delta\alpha_{\mathbf{k}}(t)/\delta f_{\mathbf{k}}(t_0).$$

We repeat the same procedure many times during the original evolution of $\alpha_{\mathbf{k}}$. The time intervals between perturbations are also randomly chosen. The ensemble-averaged Green's function is then defined as

$$G_{\mathbf{k}}(\tau) = \langle \delta\phi_{\mathbf{k}}(t)/\delta f_{\mathbf{k}}(t_0)\rangle = \frac{1}{M}\sum_{M} g_{\mathbf{k}}(\tau)$$

where $\tau = t - t_0$ and M is the number of times we perturbed $\alpha_{\mathbf{k}}$. The time-lagged convariance $R_{\mathbf{k}}(t-t_0) = \langle \alpha_{\mathbf{k}}(t)\ \alpha_{\mathbf{k}}(t_0)\rangle$ is also calculated from the results of direct simulation

$$R_{\mathbf{k}}(\tau) = T^{-1}\int_0^T \alpha_{\mathbf{k}}(t_0 + \tau)\ \alpha_{\mathbf{k}}(t_0)\ dt_0 \quad .$$

To save computations, we design a unidirectional linear regression element $Y_k(\tau)$,

$$Y_k(\tau) = \frac{1}{N} \sum_{k-\Delta k<k<k+\Delta k} R_{\mathbf{k}}(\tau)/R_{\mathbf{k}}(0)$$

where N is the number of discreet modes within the band k. We also write $G_{\mathbf{k}}(\tau)$ as $G_k(\tau)$. The results for k = 5, 10, 15 are plotted in Fig. 5.2, 5.3 and 5.4. It can be seen that the agreement between $Y_k(\tau)$ and $G_k(\tau)$ is excellent.

We also test the assumption of infinitesimal perturbation. The amplitude ε of forcing in Eq. (5.4.1) varies within a range from 0.001 to 10.0. The resulting Green's functions are presented in Fig. 5.5. It shows that even when forcing is about ten times as large as the magnitude of $\alpha_{\mathbf{k}}$, there is still no apparent discrepancy in results from the linear response assumption. This conclusion seems to suggest an application of the DIA to a flow system with only a few degrees of freedom.

It may be interesting to note that all the off-diagonal elements in our numerical model approximately vanish after we take an assemble average. The off-diagonal statistical Green's functions are three or four orders less in magnitude than the diagonal counterparts during the response time scale.

5.5 Time Scales in Turbulence

The decorrelation rates μ_k, as a measure of inverse time scale of k mode, can be calculated from the directly simulated model by

$$\mu_k = [\int_0^\infty Y_k(\tau)\, d\tau]^{-1}$$

The results are shown in Fig. 5.6. The linear relationship between μ_k and k is clearly an indication of phenomenological feature of Eulerian time scale. Since in the N-S equation we can relate

$$\mu_k \approx (U \cdot \nabla)_k \approx (E)^{1/2}\, k$$

where $U \cdot \nabla$ is a rough estimation of the advecting effect by total flow motion on a mode k.

For comparing observed μ_k with theoretical prediction, the abridged DIA equation (4.3.2) is used to calculate μ_k. The results are plotted as a dotted line in Fig. 5.6. The μ_k from the ADIA is larger than the observed μ_k, although (μ_k) ADIA is also a linear function of k. It seems that oscillations of $Y_k(\tau)$ near the end $\tau \rightarrow \infty$ could create uncertainties on the values of observed μ_k, especially near the smaller eddies. This nonrandom behavior is not accounted for in the abridged form when we assume

$$G_k(\tau) = e^{-\mu_k \tau} .$$

The unabridged DIA may be able to give a more faithful expression of observed μ_k.

Chapter 6: Concluding Remarks

We will close this thesis with a brief summary of the results. The implications of current studies will be discussed, especially those of geophysical interest. The shortcomings and limitations stemming from our various assumptions will be reviewed. Finally, this chapter will be concluded with some speculations for future improvements and extensions of our study.

6.1 Summary

We began this thesis with a review of modern turbulence theories. Recognition of the importance of the Langevin equation dominated our perceptions of nonlinear systems. The statistical theory of turbulence can be summed up by two basic concepts--a Gaussian process and a deterministic linear eddy-damping effect--which represent and replace the nonlinear terms in the Navier-Stokes equation. We have chosen the direct interaction approximation (DIA) as the foundation of this study. It can be found that the DIA, as an expansion about randomness, corresponds to the exact solutions of a certain Langevin-type stochastic model. We based most of our physical arguments on the stochastic model.

The limitation of a DIA inhomogeneous model lies not on a lack of technique to derive it, but on the complicated expressions of spectral form. Two steps were taken to simplify the problem: one was the DIA diagonalization (DIAD) process; the other was weakly inhomogeneous expansion. Both procedures were interpreted in the context of the relevant stochastic models. A modern rereading of the eddy diffusion conjectures then emerged. The positive-definite eddy diffusivities for inhomogeneous vorticity flux and tracer flux can thereby be deduced from the analytic turbulence theories.

To make a practical use of our theory, we have investigated the atmospheric tracer problem. The choice of a tracer model was motivated by meteorological considerations. We assumed that the atmospheric energy spectrum is known from observations. Therefore some abridged form of the turbulence theories can be employed to calculate the decorrelation rates. Derived eddy diffusivities were found to be compatible with observed values where an extension from inhomogeneous eddy diffusivity to total eddy diffusivity was made.

The last topic was a direct numerical simulation of two-dimensional turbulence. The intent of the experiment was to calculate the ensemble-averaged Green's function defined in the DIA from a directly simulated model. The precise agreement with the prediction of the fluctuation-dissipation theory not only justified some simplifications in our weakly inhomogeneous model, but also reassured us of the general validity of the modern turbulence theories.

6.2 Implications

In this study we have given an example demonstrating that analytic turbulence theory can be related to turbulence phenomenology. The particular problem we have considered here is the spatial transfer problem under the weakly inhomogeneous assumption. It appears that the DIA formalism can shed new light on the classical eddy diffusion theory, especially within the context of the Langevin-type stochastic model.

It is believed that the nonlinear terms in the N-S equation will cause a return to spatial symmetry (isotropy or homogeneity). The weakly inhomogeneous turbulence model also indicates a decrease of inhomogeneity in simple shear flow. This conclusion can be observed from the feedback tendency of the inhomogeneous dynamic flux, despite the fact that the mean shear was assumed to be time-independent in our model.

In the past the modern turbulence theories have been known to have difficulties in solving practical problems, since the more complicated problems often demand more advanced closure techniques, which rapidly increase the computations, not to mention the diminution of our physical intuition. On the other hand, the DIA formalism as "the only fully self-consistent analytic turbulence theory yet discovered" (according to Orszag, 1974), reveals so much insight into the fundamental features of nonlinear systems that no longer can phenomenological studies afford to ignore it. The implications of this study seems to indicate (to this writer) that more studies should be done to close the distance between the theoretical and phenomenological turbulence researchers. (A pioneering example has been given by Herring, 1977, who succeeded in connecting the results from a DIA model to empirical formalism for homogeneous shear turbulence. Also see Kraichnan, 1971b.)

6.3 Shortcomings and Limitations

The present study leaves two urgent questions unanswered. One concerns the use of Eulerian coordinates and the other concerns the weakly inhomogeneous assumption. Part of our limitations originate from the lack of feedback (i.e., time-independence) of mean shear.

The difficulty associated with an Eulerian frame is inherited from the DIA model (see section 2.4). While a Lagrangian modified closure (e.g., test field model) can be extended to inhomogeneous turbulence problems (Kraichnan, 1972), the lack of mutual adjustment between mean flow and eddies in our model prevents a direct adoption of such schemes. Nevertheless, the weakly inhomogeneous turbulence model has duplicated Taylor's eddy diffusion equation faithfully. A physically more tractable way (Herring, private communicaton) to treat inhomogeneous turbulence may be the formulation that starts from the two-point correlation equation (see Hinze, chapter 4, 1975), then expands the distance coordinates in spectral space and leaves the location coordinates in physical space. A multiple-scale expansion (see section 4.2) may then be applied on the location (inhomogeneous) coordinate under the weakly inhomogeneous assumption. It is expected that Taylor's eddy diffusion equation will reemerge from the above approach.

The assumption of weak inhomogeneity is indeed a necessary evil. Without this assumption, inhomogeneous turbulence theories can only resort to a system of equations including a full (diagonal and off-diagonal) spectrum. It may be more excusable if we consider the problem under the concept of homogeneity instability (see section 3.1). We will be satisfied to investigate whether the weak inhomogeneity grows or decays.

6.4 Future Studies

The foremost need for inhomogeneous turbulence studies is to build direct simulation models. It may be within our reach to simulate a shear flow like Eq. (3.4.4) or (4.1.3) at low Reynold's number regimes. After this is done, the weakly inhomogeneous turbulence theory can be verified under various situations. For example, we can test the tracer flux formula when the inhomogeneity is introduced by a mean wind, or by a mean tracer field, or by both. It is also interesting to investigate how periodic motions (e.g., those due to the β-effect) affect the spatial transfer.

Most elements in the second-moment closure model can also be measured from a directly simulated model. The comparison between the decorrelation rates in the EDM, or the ensemble-averaged Green's function in the DIA, and the statistics from a directly simulated model should be carried out to include forced and viscous turbulence flow, and baroclinic flow. The analytic turbulence theories are especially useful when we try to find the temporal correlations from the spatial correlations (see section 4.4). New experiments should comply with a more realistic atmospheric energy spectrum. At the end of this study we envision a climate model based on the weakly inhomogeneous theory. The resulting climate model may be able to describe both the global features (spectral transfer) and local features (spatial transfer) of the atmosphere. That is the eventual goal of this work.

Appendix A: The Navier-Stokes Equation in Generalized Spectral Form

Consider a fluid filling a region D. with boundary ∂D and satisfying the inviscid Navier-Stokes (N-S) equation:

$$\frac{\partial}{\partial t} \mathbf{U} + (\mathbf{U} \cdot \nabla) U = - \operatorname{grad} P$$

where the pressure term can be written in terms of velocity field by the incompressibility condition

$$\operatorname{div} U = 0 .$$

Assume a periodic condition at ∂D. Through Fourier transformation the N-S equation in wave number space becomes:

$$\frac{d}{dt} u_i(\mathbf{k} ,t) = - \frac{1}{2} \sqrt{-1}\, P_{ijk} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p} ,t)\, u_k(\mathbf{q} ,t)$$

with

$$P_{ijk} = k_j P_{ik}(\mathbf{k}) + k_k P_{ij}(\mathbf{k})$$

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2 , \quad k = | \mathbf{k} |$$

The summation convention is applied on the spatial direction coordinate (i,j,k). A finite-mode N-S equation is obtained by restricting $| \mathbf{k} | < k_{max}$, where k_{max} is a measure of the finest resolution in the discreet $\mathbf{k}$ -space mesh. The reality and the incompressibility condition demand, respectively

$$U_i(-\mathbf{k}) = \mathbf{u}_i^*(\mathbf{k})$$

$$k_i u_i(\mathbf{k},t) = 0$$

Hence for a pair of vectors $\mathbf{U}(\mathbf{k})$, $\mathbf{U}(-\mathbf{k})$, there are twelve real and imaginary parts, and only four of them are linear independent. A suitable way to choose an independent variable (Kraichnan, 1958) is to introduce an orthogonal space ($\mathbf{n_1},\mathbf{n_2},\mathbf{k}$) where $\mathbf{n_1},\mathbf{n_2}$ as the unit vectors in the plane normal to $\mathbf{k}$ and to project the real and imaginary parts of $u_i(\mathbf{k})$ on ($\mathbf{n_1},\mathbf{n_2}$) plane. The independent variables so chosen are then arranged in a one-dimensional sequence and each will be denoted by a new variable q_α, where α is assigned as a discreet integer. The N-S equation is thereby transformed to th generalized spectral equation:

$$\frac{\partial}{\partial t} q_\alpha = \sum_{\beta,r} a_{\alpha\beta r}\, q_\beta\, q_r$$

where the interaction coefficients preserve all nonlinear characteristic features of the N-S equation. A detailed discussion on the properties of $a_{\alpha\beta r}$ can be found in Leith (1971). In the case of two-dimensional incompressible turbulent flow, the velocity can be expressed in terms of a stream function. In spectral form, the real and imaginary parts of the stream function $\phi_{\mathbf{k}}$ are then chosen as the elements in $\{q_\alpha, q_\beta, \ldots\}$ and arranged in order. Since the reality condition requires $\phi^*_{\mathbf{k}} = \phi_{-\mathbf{k}}$, only half space of k has to be considered.

Appendix B: Inhomogeneous DIA Model

The two-time, two-scale correlation is defined as

$$R_{kL}(t,t') = \langle \phi_k(t)\ \phi_L(t') \rangle$$

and the Green's function

$$\langle G_{kL}(t,t') \rangle = \langle \delta\phi_k(t)/\delta f_L(t') \rangle$$

is described in section 3.2. From the inhomogeneous model Eq. (3.2.0), we obtain:

$$\frac{\partial}{\partial t} \langle R_{kL}(t,t') \rangle + \sum_j L_{kL} \langle R_{jL}(t,t') \rangle = \sum_{p,q} a_{kpq} \langle \phi_p(t) \phi_q(t) \phi_L(t') \rangle$$

and

$$\frac{\partial}{\partial t} G_{kL}(t,t') + \sum_j L_{kj} G_{jL}(t,t') = 2 \sum_{p,q} a_{kpq} \langle \phi\ (t)_p (\delta\phi_q(t)/\delta f_L(t')) \rangle$$

The DIA expansion can be constructed by the following algorithm designed by Kraichnan (1964a):

(1) Take an initial statistical ensemble such that the ϕ_k have a multivariate, normal distribution with zero mean at $t = 0$.

(2) Construct a zero-order equation for ϕ_k by setting all $a_{kpq} = 0$. Since this equation is linear, the zero-order amplitude have a multivariate normal distribution at all times.

Procedures (1) and (2) can be done by assigning a perturbation parameter ε to the terms associated with a_{kpq} and expand

$$\phi_k = \phi_k^{(0)} + \varepsilon\, \phi_k^{(0)} + \ldots$$

the zero-order system is

$$\frac{\partial}{\partial t} \phi_k^{(0)} + \sum_j L_{kj}\, \phi_j^{(0)} = 0$$

with a corresponding zero-order expansion of

$$\frac{\partial}{\partial t} g_{kL}^{(0)}(t,t') + \sum_j L_{kj} g_{jL}^{(0)}(t,t') = \delta f_L(t,t')$$

(3) Reintroduce the omitted (nonlinear) term as a perturbation. By interaction, expand the actual ϕ_k in powers of the zero-order amplitudes and the response matrix for the unperturbed equation. Make a similar expansion for the unaveraged actual response matrix.

The first order systems become:

$$\frac{\partial}{\partial t} \phi_k^{(1)} + \sum_j L_{kj}\, \phi_j^{(1)} = \sum_{p,q} a_{kpq}\, \phi_p^{(0)}\, \phi_q^{(0)}$$

$$\frac{\partial}{\partial t} g_{kL}^{(1)}(t,t') + \sum_j L_{kj}\, g_j^{(1)}(t,t') = 2\sum_{p,q} \phi_p^{(0)} g_{qL}^{(0)}(t,t')$$

which readily yield

$$\phi_k^{(1)}(t) = \sum_j \int_0^t g_{kj}^{(0)}(t,s)\,[\sum_{m,n}^{j} a_{jmn}\, \phi_m^{(0)}(s)\, \phi_n^{(0)}(s)]\, ds \tag{B.1}$$

$$g_{kL}^{(1)}(t) = \sum_j \int_{t'}^t g_{kj}^{(0)}(t,s)\,[\sum_{m,n}^{j} a_{jmn}\, \phi_m^{(0)}(s)\, g_{nL}^{(0)}(s,t')]\, ds \tag{B.2}$$

(4) Insert this expansion into

$S_{pqL} = \langle\phi_p(t)\ \phi_q(t)\ \phi_L(t')\rangle$ and $H_{pqL} = \langle\phi_p(t)(\delta\phi_q(t)/\delta f_L(t'))\rangle$.

Apply the rules for evaluating moments of normal distribution (see chapter 2). The formally exact power-series expansions of S_{pqL} and H_{pqL} are thereby obtained in terms of the covariance and response matrix of the zero-order amplitude.

For example,*

$$\langle \phi_p^{(0)}(t)\ \phi_q^{(0)}(t)\ \phi_L^{(1)}(t')\rangle = \sum_{jmn} 2a_{jmn} \int_0^t G_{Lj}^{(0)}(t',s)$$

$$R_{pm}^{(0)}(t,s)\ R_{qn}^{(0)}(t,s)\ ds$$

(5) Retain only the lowest-order terms in these expansions. Replace all the zero-order covariances and the response functions by actual covariances and the averaged response functions.

The results are:

$$\frac{\partial}{\partial t} R_{kL}(t,t') + \sum_j L_{kL}\ R_{jL}(t,t') = 4 \sum_{p,q} \sum_{m,n} a_{kpq}\ a_{mnj}$$

$$\int_0^t G_{pm}(t,s)\ R_{qn}(t,s)\ R_{jL}(s,t')\ ds$$

$$+ 2 \sum_{p,q} \sum_{j,m,n} a_{kpq}\ a_{jmn} \int_0^{t'} G_{Lj}(t',s)\ R_{qn}(t,s)\ R_{pm}(t,s)\ ds$$

---

* The factorization of a fourth moment

$$\langle \phi_p^{(0)}(t)\ \phi_q^{(0)}(t)\ \phi_m^{(0)}(s)\ \phi_n^{(0)}(s)\rangle$$

into $R_{pm}^{(0)}$. $R_{qn}^{(0)}$ should be done according to the diagonality along the homogeneous directions, which does not appear explicitly (see appendix C) in our model.

$$\frac{\partial}{\partial t} G_{kL}(t,t') + \sum_j L_{kj} G_{jL}(t,t') = \sum_{p,q} \sum_{jmn} a_{kpq}\, a_{mnj} \int_{t'}^{t} G_{pm}(t,s)\, R_{qn}(ts,)\, G_{jL}(s,t')\, ds = 0$$

for $t > t'$

which gives inhomogeneous DIA equations in Eq. (3.2.6) and Eq. (3.2.7)

Appendix C: Fluxes and Correlations

The contributions of eddies always appear in the form of eddy fluxes in climate models and as correlations in statistical turbulence models. A simple relation exists between dynamic fluxes and statistical correlations expanded in Fourier components. To simplify the notation and draw analogies with geophysical problems we consider a two-dimensional, incompressible flow which is homogeneous in the x (east-west) direction and inhomogeneous in the y (north-south) direction. The eddy field is described by

$$\text{eastward velocity} \quad u(\mathbf{x}) = \sum_{\mathbf{k}} -i\, m_{\mathbf{k}}\, \phi_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}$$

$$\text{northward velocity} \quad v(\mathbf{x}) = \sum_{\mathbf{k}} i L_{\mathbf{k}}\, \phi_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}$$

$$\text{vertical component of vorticity} \quad \xi(\mathbf{x}) = \sum_{\mathbf{k}} -k^2\, \phi_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}$$

$$\text{tracer concentration} \quad q(\mathbf{x}) = \sum_{\mathbf{k}} q_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}$$

where $\mathbf{x} = (x,y)$ is the spatial variable, $\mathbf{k} = (L_{\mathbf{k}}, m_{\mathbf{k}})$ is the wave number vector, k is the scale magnitude of $\mathbf{k}$, and $\phi_{\mathbf{k}}, q_{\mathbf{k}}$ are the corresponding Fourier components of eddy stream function and tracer concentration. Proper truncation of $\mathbf{k}$ and the reality condition

$$(\phi_{\mathbf{k}} = \phi^*_{-\mathbf{k}},\ q_{\mathbf{k}} = q^*_{-\mathbf{k}})$$

are assumed. The inclusion of tracer concentration sets the stage for tracer transport problems in chapter 4. Some important northward fluxes are:

$$\text{momentum flux} \quad M(\mathbf{x}) = \langle U(\mathbf{x})\, V(\mathbf{x})\rangle$$

$$\text{vorticity flux} \quad Z(\mathbf{x}) = \langle \xi(\mathbf{x})\, V(\mathbf{x})\rangle$$

$$\text{tracer flux} \quad Q(\mathbf{x}) = \langle q(\mathbf{x})\, V(\mathbf{x})\rangle$$

In most physical problems, the divergences of the above fluxes are more relevant. Therefore, we further divide the fluxes into a nondivergent (homogeneous) part and a divergent (inhomogeneous) part. At a fixed point x, the momentum flux can be written as:

$$\begin{aligned} M(\mathbf{x}) &= \langle [\sum_{\mathbf{k}} -i\, m_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\,\mathbf{k}\cdot\mathbf{x}}][\sum_{\mathbf{k'}} i\, L_{\mathbf{k'}} \phi_{\mathbf{k'}} e^{i\,\mathbf{k'}\cdot\mathbf{x}}]\rangle \\ &= -\sum_{\mathbf{k}} m_{\mathbf{k}} L_{\mathbf{k}} \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}}\rangle + \sum_{\mathbf{k}} \sum_{\mathbf{k'}} m_{\mathbf{k}} L_{\mathbf{k'}} \langle \phi_{\mathbf{k}} \phi_{\mathbf{k'}}\rangle\, e^{i\,(\mathbf{k}+\mathbf{k'})\cdot\mathbf{x}} \\ &\qquad (\mathbf{k}+\mathbf{k'} \neq 0) \end{aligned} \tag{C.1}$$

The first term on the right-hand side of the above equation consisting only of the antidiagonal correlations is independent of position **x** . This term will be denoted as the homogeneous momentum flux $M^{(0)}$:

$$M^{(0)}(\mathbf{x}) = \sum -m_{\mathbf{k}} L_{\mathbf{k}} \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle$$

It can be seen that $M^{(0)}$, though by definition a local flux, is identical everywhere. Therefore, the spatial (local) representation is indistinguishable from the spectral (global) representation. It can also be proved (e.g., Leslie, 1973) that the disappearance of the off-diagonal correlations, hence the second term on the right-hand side of Eq. (C.1), is a necessary and sufficient condition for turbulence to be homogeneous.

Since in this study the inhomogeneity only appears in y-direction, the inhomogeneous momentum flux can be written as

$$M^{(1)}(\mathbf{x}) = \sum_{\mathbf{k}} \sum_{\mathbf{k'}} - m_{\mathbf{k}} L_{\mathbf{k}} \langle \phi_{\mathbf{k}} \phi_{\mathbf{k'}} \rangle e^{i(\mathbf{k}+\mathbf{k'})\mathbf{x}} \delta(L_{\mathbf{k}} + L_{\mathbf{k'}}) \qquad \mathbf{k}+\mathbf{k'} \neq 0$$

$$= \sum_{\mathbf{k}} \sum_{m_{\Delta \mathbf{k}}} - m_{\mathbf{k}} L_{\mathbf{k}} \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}+\Delta \mathbf{k}} \rangle e^{i m_{\Delta \mathbf{k}} \cdot Y} \qquad (C.2)$$

where δ is the Kronecker delta. The gap wave number vector $\Delta \mathbf{k} = \mathbf{k}+\mathbf{k'} = (0, m_{\Delta \mathbf{k}})$, which shows a departure from diagonality, is a measure of the divergent (inhomogeneous) scale of fluxes ($\partial/\partial Y \approx m_{\Delta \mathbf{k}}$). It is very convenient to write the two-scale, off-diagonal correlation as $R(\mathbf{k}, \Delta \mathbf{k})$ instead of $R(\mathbf{k}, \mathbf{k'})$. We can assign a spatial variable Y to $\Delta \mathbf{k}$, and a y to $\mathbf{k}$, accordingly. O(Y) is now a scale for divergent flux and O(y) is a scale for eddy.

$M^{(0)}$ and $M^{(1)}$ can be written in similar forms if we define the functions

$$R_{\mathbf{k}}^{(1)}(Y) = \sum_{m_{\Delta \kappa}} \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}+\Delta \mathbf{k}} \rangle \, e^{i m_{\Delta} \mathbf{k} \cdot Y} \tag{C.3}$$

and

$$R_{\mathbf{k}}^{(0)} = \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle \quad ,$$

so that

$$M^{(1)}(Y) = - \sum_{\mathbf{k}} L_{\mathbf{k}} m_{\mathbf{k}} R_{\mathbf{k}}^{(1)}(Y) \quad ,$$

$$M^{(0)}(Y) = - \sum_{\mathbf{k}} L_{\mathbf{k}} m_{\mathbf{k}} R_{\mathbf{k}}^{(0)} \quad .$$

$R_{\mathbf{k}}^{(0)}(Y)$ can be considered as a half-transformed function corresponding to a two-scale correlation R($\mathbf{k}$, $\Delta \mathbf{k}$) where the inhomogeneous scale Y remains in physical space. The separability between the inhomogeneous (mean shear) scale Y and the eddy structural scale y, which we refer to as the weakly inhomogenneous assumption, holds the key to any simplification of inhomogeneous turbulence model.

Under this assumption, there are many cases (see section 3.6, chapter 4) where the summations of $\mathbf{k}$ and $\Delta\,\mathbf{k}$ can be decoupled such that Y will be recovered intact from a full spectral (including off-diagonal correlations) model to the physical space--an essential requirement for practical purposes.

The vorticity flux is:

$$Z(\,\mathbf{x}\,) \;=\; \sum_{\mathbf{k}} -\,i\,L_{\mathbf{k}} k^2 \,\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle + \sum_{\mathbf{k}} \sum_{\mathbf{k}'} -\,i\,L_{\mathbf{k}} k'^2 \,\langle \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \rangle \, e^{i\,(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}$$

The first term on the right-hand side vanishes since it is antisymmetrical with respect to the k axis. The divergent part can be simplified into:

$$Z^{(1)}(\,\mathbf{x}\,) \;=\; \sum_{\mathbf{k}} \sum_{\mathbf{k}'} -\,iL_{\mathbf{k}} \frac{(k'^2-k^2)}{2} \langle \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \rangle \, e^{i\,(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}$$

$$= \sum_{\mathbf{k}} \sum_{m_{\Delta\,\mathbf{k}}} \frac{[-iL_{\mathbf{k}}\; -2m_{\mathbf{k}} m_{\Delta\,\mathbf{k}} + m_{\Delta\,\mathbf{k}}^2]}{2} \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}\,+\Delta\,\mathbf{k}} \rangle \, e^{i m_{\Delta\,\mathbf{k}} Y}$$

$$= \sum_{\mathbf{k}} \sum_{m_{\Delta\,\mathbf{k}}} i\; m_{\Delta\,\mathbf{k}} L_{\mathbf{k}}\; m_{\mathbf{k}} \,\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}\,+\Delta\,\mathbf{k}} \rangle \, e^{i m_{\Delta\,\mathbf{k}} Y} \qquad (C.4)$$

The familiar relation between the vorticity flux and the momentum flux is thereby emerged

$$Z = -\frac{dM}{dY} = \sum_{\mathbf{k}} L_{\mathbf{k}} m_{\mathbf{k}} \frac{dR_{\mathbf{k}}^{(1)}}{dY} \qquad \text{(C.5)}$$

The tracer flux Q is:

$$Q = R_e \sum_{\mathbf{k}} i\, L_{\mathbf{k}} W_{\mathbf{k}}^{(0)} + R_e \sum_{\mathbf{k}} i\, L_{\mathbf{k}} W_{\mathbf{k}}^{(1)}$$

where the homogeneous tracer flux

$$W_{\mathbf{k}}^{(0)} = \sum_{\mathbf{k}} \langle \phi_{\mathbf{k}} q_{-\mathbf{k}} \rangle$$

and the inhomogeneous tracer flux

$$W_{\mathbf{k}}^{(1)} = \sum_{m,\Delta \mathbf{k}} \langle \phi_{\mathbf{k}} q_{-\mathbf{k} + \Delta \mathbf{k}} \rangle \, e^{im\Delta \mathbf{k} Y}$$

We will sum up the relationship between fluxes and correlations in the following table:

	Homogeneous	Inhomogeneous
Momentum flux	$M^{(0)} = -\sum_{\mathbf{k}} L_{\mathbf{k}} m_{\mathbf{k}} R_{\mathbf{k}}^{(0)}$	$M^{(1)} = -\sum_{\mathbf{k}} L_{\mathbf{k}} m_{\mathbf{k}} R_{\mathbf{k}}^{(1)}$
Vorticity flux	$Z^{(0)} = 0$	$Z^{(1)} = -\sum_{\mathbf{k}} L_{\mathbf{k}} m_{\mathbf{k}} \frac{dR_{\mathbf{k}}^{(1)}}{dY}$
Tracer flux	$Q^{(0)} = \sum_{\mathbf{k}} i\ L_{\mathbf{k}} W_{\mathbf{k}}^{(0)}$	$Q^{(1)} = \sum_{\mathbf{k}} i\ L_{\mathbf{k}} W_{\mathbf{k}}^{(1)}$

Appendix D: Taylor's Turbulent Diffusion Theory

The classic treatment of turbulent diffusion by G.I. Taylor (1921), based on the kinetic theory of gas, is still used with regard to practical engineering problems. Here we merely restate the essentials from the standard text (Hinze, 1975):

Assume that q is a transferable quantity in a two-dimensional space (x,y), and that the mean field $\langle q \rangle$ is constant in the x direction, varying only in the y direction. If the diffusion law is valid, the tracer flux $\langle q'v' \rangle$ along the y direction should be proportional to the mean field gradient $d\langle q \rangle/dy$:

$$\langle q'v' \rangle = - D_q \frac{d\langle q \rangle}{dy} \qquad \text{(D.1)}$$

where D_q is the supposed proportionality constant, commonly known as the eddy diffusivity. Note that D_q should not be considered as a constant parameter of fluid field.

From the theory of gases for the random motion of molecules, Taylor (1921) found that

$$D_q = \lim_{t \to \infty} \frac{\langle y^2(t) \rangle}{2t} \qquad \text{(D.2)}$$

$\langle y^2(t)\rangle$ was defined in the following way: a fluid particle, marked with the physical property $q(y(t_0),t_0)$, released at t_0 would conserve that physical property within a "free path" from $y(t_0)$ to $y(t_0+t)$.* Let $v'(t_0+t)$ be the Lagrangian velocity of a marked particle at any instance t. $y(t_0+t)$ is defined as Lagrangian displacement:

$$y(t_0+t) = \int_0^t v'(t_0+t')\, dt'$$

The variance of all "random walks" will yield the following result:

$$\langle y^2(t_0+t)\rangle = \int_0^t dt' \int_0^t dt'' \langle v'(t_0+t')\ v'(t_0+t'')\rangle$$

$$= 2\int_0^t dt' \int_0^{t'} dt'' \langle v'(t_0+t')\ v'(t_0+t'')\rangle$$

Introducing the Lagrangian auto-correlation coefficient $R_L(\tau)$ gives:

$$R_L(\tau) = \frac{\langle v'(t_0)\ v'(t_0+\tau)\rangle}{\langle v'^2(t_0)\rangle} \qquad (D.3)$$

---

* This is equivalent to the weakly inhomogeneous assumption as stated in chapter 3. A scale separation between the eddies and the mean flow is assumed implicitly.

For a stationary process the origin of time t_0 can be neglected.

Put t"-t'=t. Since

$$\int_0^t\int_0^{t'} R_L(\tau)dt'dt = \left| t'\int_0^{t'} d\tau\, R_L(\tau)\right|_0^t - \int_0^t dt't'R_L(t')$$

$$= t\int_0^t d\tau\, R_L(\tau) - \int_0^t d\tau\tau\, R_L(\tau)$$

Eq. (D.2) an be written as

$$\langle y^2(t)\rangle = 2\langle v'^2\rangle\int_0^t (t-\tau)\, R_L(\tau)\, d\tau$$

Note that for zero time separation, $R_L(0) = 1$. When τ becomes large, the particle will have "lost" memory totally. Hence $R_L(\tau) \to 0$ when $\tau \to \infty$.

For the longer period of time $t \to \infty$,

$$\langle y^2(t)\rangle = 2\langle v'^2\rangle\int_0^t t\, R_L(\tau)\, d\tau$$

In his original paper, Taylor treated $R_L(\tau)$ as an exponentially decaying function. He also mentioned that an oscillating damping form of $R_L(\tau)$ had been observed by L.T. Richardson, which he promptly attributed to some regularities in eddy motions. Since then many observations have confirmed Taylor's hypothesis (see Hinze, 1975).

Accordingly, a Lagrangian integral time scale τ_L can be defined as

$$\tau_L = \lim_{t \to \infty} \int_0^t R_L(\tau)\, d\tau \quad . \tag{D.4}$$

τ_L is the measure of the average persistence time of a particle in preserving its identity along the y direction. One possible form of R_L, consistent with Eq. (D.4), is

$$R\ (\tau) = e^{-\tau/\tau_L} \quad . \tag{D.5}$$

The final form of D_q in Taylor's theory gives

$$D_L = \lim_{t \to \infty} \langle v'^2 \rangle \int_0^t e^{-\tau/\tau_L}\, d\tau \quad . \tag{D.6}$$

The greatest contribution of Taylor's theory perhaps is to simplify the whole turbulent diffusion problem into a single empirical parameter $R_L(\tau)$ (or τ_L). We have observed the same kind of simplicity in the eddy-damped model (EDM) where the decorrelation rate $\mu_{\mathbf{k}}$ is the key function.

Appendix E: The DIA Tracer Model and the EDM Tracer Model

To solve the homogeneous tracer problem in Eq. (4.1.) the DIA expansion starts by constructing the corresponding Green's function as follows:

$$[\frac{\partial}{\partial t} + L_{\mathbf{k}}]\, g_{q\,\mathbf{k}}^{(0)}(t,t') = \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}} \phi_{\mathbf{p}}^{(0)} g_{q\,\mathbf{q}}(t,t') + \delta_{q\,k}(t-t')$$

$$g_{q\,\mathbf{k}}^{(0)}(t',t') = 1 \quad \text{and} \quad g_{q\,k}^{(0)}(t,t') = 0 \quad \text{when } t<t'$$

and assigning an expansion parameter ε on

$$[\frac{\partial}{\partial t} + L_{\mathbf{k}}]\, q_{\mathbf{k}}^{(0)} = \varepsilon \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}} \phi_{\mathbf{p}}^{(0)} q_{\mathbf{q}}^{(0)}$$

$$[\frac{\partial}{\partial t} + L_{\mathbf{k}}]\, g_{q\,k}^{(0)} = \varepsilon \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}} \phi_{\mathbf{p}}^{(0)} g_{q\,\mathbf{q}}^{(0)}$$

such that, we can expand

$$q_{\mathbf{k}}^{(0)} = q_{\mathbf{k}}^{\substack{(0)\\(0)}} + \varepsilon\, q_{\mathbf{k}}^{\substack{(1)\\(0)}} + \ldots$$

$$\phi_{\mathbf{k}}^{(0)} = \phi_{\mathbf{k}}^{\substack{(0)\\(0)}} + \varepsilon\, \phi_{\mathbf{k}}^{\substack{(1)\\(0)}} + \ldots$$

$$g_{\mathbf{k}}^{(0)} = g_{\mathbf{k}}^{\substack{(0)\\(0)}} + \varepsilon\, g_{\mathbf{k}}^{\substack{(1)\\(0)}} + \ldots$$

It is assumed that

$$\phi_{\mathbf{k}}{}^{(1)}_{(0)} = 0$$

since a disturbance of the passive tracer field will not affect the turbulence field $\phi_{\mathbf{k}}{}^{(0)}$. For simplicity, hereafter we will neglect the superscripts on the second deck, which represent the weakly inhomogenous (δ) expansion.

The zero-order system gives:

$$\left[\frac{\partial}{\partial t} + L_{\mathbf{k}}\right] q_{\mathbf{k}}{}^{(0)} = 0$$

$$\left[\frac{\partial}{\partial t} + L_{\mathbf{k}}\right] g_{q\,\mathbf{k}}{}^{(0)} = 0$$

The first-order system is solved by using the Green's function,

$$q_{\mathbf{k}}{}^{(1)}(t) = \int_{0}^{t} g_{q\,\mathbf{k}}{}^{(0)}(t,s) \left[\sum_{\mathbf{p+q=k}} b'_{\mathbf{kpq}}\phi_{\mathbf{p}}{}^{(0)}(s)\; q_{q\,\mathbf{k}}{}^{(0)}(s)\right] ds$$

$$g_{q\,\mathbf{k}}{}^{(1)}(t,t') = \int_{t'}^{t} g_{q\,\mathbf{k}}{}^{(0)}(t,s) \left[\sum_{\mathbf{p+q=k}} b'_{\mathbf{kpq}}\phi_{\mathbf{p}}{}^{(0)}(s)\; g_{q\,\mathbf{q}}{}^{(0)}(s,t')\right] ds$$

The triple correlation term

$$\langle\phi_{\mathbf{p}}(t)q_{\mathbf{q}}(t)q_{-\mathbf{k}}(t')\rangle \approx \varepsilon\left[\langle\phi_{\mathbf{p}}{}^{(0)}(t)q_{\mathbf{q}}{}^{(1)}(t)q_{-\mathbf{k}}{}^{(0)}(t')\rangle + \right.$$

$$+ \langle \phi_{\mathbf{p}}^{(0)}(t) q_{\mathbf{q}}^{(0)}(t) q_{-\mathbf{k}}^{(1)}(t') \rangle]$$

$$= \; b_{\mathbf{q}\text{-}\mathbf{p}\mathbf{k}} \int_0^t \langle g_{q\,\mathbf{q}}^{(0)} \; (t,s) \rangle \, \langle \phi_{\mathbf{p}}^{(0)}(t) \phi_{-\mathbf{p}}^{(0)}(s) \rangle$$

$$\langle q_{\mathbf{q}}^{(1)}(t) q_{-\mathbf{k}}^{(0)}(t') \rangle \; ds$$

$$+ \; b_{-\mathbf{k}\text{-}\mathbf{p}\text{-}\mathbf{q}} \int_0^{t'} \langle g_{q\; -\mathbf{k}}^{(0)}(t',s) \rangle \, \langle \phi_{\mathbf{p}}^{(0)}(t) \phi_{-\mathbf{p}}^{(0)}(s) \rangle$$

$$\langle q_{\mathbf{q}}^{(0)}(t) q_{-\mathbf{k}}^{(0)}(s) \rangle \; ds$$

and the double correlation

$$\langle \phi_{\mathbf{p}}^{(0)} \; (t) \; g_{q\; \mathbf{q}}^{(0)} \; (t-t') \rangle \approx \varepsilon \, \langle \phi_{\mathbf{p}}^{(0)} \; (t) g_{q\; \mathbf{q}}^{(1)}(t,t') \rangle$$

$$= \; b_{\mathbf{q}\text{-}\mathbf{p}\mathbf{k}} \int_{t'}^t \langle g_{q\,\mathbf{q}}^{(0)} \; (t,s) \rangle \, \langle \phi_{\mathbf{p}}^{(0)}(t) \phi_{-\mathbf{p}}^{(0)}(s) \rangle$$

$$\langle g_{q\; \mathbf{k}}^{(0)}(s,t') \rangle \; ds$$

can give the DIA tracer model in Eq. (4.1.6) to Eq. (4.1.8) after the superscripts have been removed. The derivation of a DIA model of 2-D homogeneous turbulence in Eq. (4.1.4) is almost identical to the above formalism. Hence we will not repeat here.

The EDM tracer model is to replace (write the linear equation in explicit dissipative viscosity terms) the triple correlation equation

$$[\frac{\partial}{\partial t} + \nu\, p^2 + \kappa\, k^2 + \kappa\, q^2] \langle \phi_{\mathbf{p}} q_{\mathbf{q}} q_{-\mathbf{k}} \rangle = \sum_{\mathbf{m+n=k}} b_{\mathbf{-kmn}} \langle \phi_{\mathbf{m}} q_{\mathbf{n}} \phi_{\mathbf{p}} q_{\mathbf{q}} \rangle$$

$$+ \sum_{\mathbf{L+H=q}} b_{\mathbf{qLH}} \langle \phi_{\mathbf{L}} q_{\mathbf{H}} \phi_{\mathbf{p}} q_{\mathbf{-k}} \rangle$$

$$+ \sum_{\mathbf{I+j=p}} a_{\mathbf{pij}} \langle \phi_{\mathbf{i}} q_{\mathbf{j}} \phi_{\mathbf{q}} q_{\mathbf{-k}} \rangle$$

with an eddy-damped randomized equation

$$(\frac{\partial}{\partial t} + \nu p^2 + \kappa k^2 + \kappa q^2 + \mu_{\mathbf{p}} + \mu_{q\ \mathbf{k}} + \mu_{q\ \mathbf{q}}) \langle \phi_{\mathbf{p}} q_{\mathbf{q}} q_{\mathbf{-k}} \rangle = b_{\mathbf{-k-p-q}} \langle \phi_{\mathbf{p}} \phi_{\mathbf{-p}} \rangle \langle q_{\mathbf{q}} q_{\mathbf{-q}} \rangle$$

$$+ b_{\mathbf{q-pk}} \langle \phi_{\mathbf{p}} \phi_{\mathbf{-p}} \rangle \langle q_{\mathbf{-k}} q_{\mathbf{-k}} \rangle$$

where we assume the cross correlation $\langle \phi_{\mathbf{p}}\, q_{\mathbf{-p}} \rangle = 0$. The same technique in section 2.3 is applied to Markovianize the second moment equation

$$[\frac{\partial}{\partial t} + \kappa k^2 \quad \langle q_{\mathbf{k}} q_{\mathbf{-k}} \rangle] = \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}}$$

$$\int_0^t e^{- \int_{t'}^{t} [\nu p^2 + \kappa k^2 + \kappa q^2 + \mu_{\mathbf{p}} + \mu_{q\ \mathbf{k}} + \mu_{q\ \mathbf{q}}) ds}$$

$$[b_{\mathbf{kpq}} \langle \phi_{\mathbf{p}}(t') \phi_{\mathbf{-p}}(t') \rangle \langle q_{\mathbf{q}}(t') q_{\mathbf{qq}}(t') \rangle +$$

$$+ b_{\mathbf{q-pk}} \langle \phi_{\mathbf{p}}(t') \phi_{\mathbf{-p}}(t') \rangle \langle q_{\mathbf{k}}(t') q_{\mathbf{-k}}(t') \rangle]\, dt'$$

A self consistent choice of $\mu_{q\,k}$ is:

$$\mu_{q\,k} = \kappa k^2 - \sum_{\mathbf{p+q=k}} b_{\mathbf{kpq}} b_{\mathbf{q-pk}} \left[\frac{1-e^{-(\mu_{q\,k}+\mu_{q\,q}+\mu_{\mathbf{p}})t}}{\mu_k+\mu_{q\,\mathbf{p}}+\mu_{q\,\mathbf{q}}} \right] \langle \phi_{\mathbf{p}} \phi_{\mathbf{-p}} \rangle$$

The exponential term in the bracket is vanished as $t \to 0$. The expression of Eq. (4.3.3) is then obtained.

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TABLE 4.1. Formula for Decorrelation Rate

$\mu_{\mathbf{k}}$: Dynamic Decorrelation Rate

The abridged direct interaction approximation (ADIA)

$$\mu_{\mathbf{k}} = \nu K^2 + \sum_{\mathbf{p}} \frac{|\mathbf{kxp}|^2 \left(1 - \frac{p^2}{q^2}\right)\left(1 - \frac{p^2}{k^2}\right) R_{\mathbf{p}}(0)}{\mu_{\mathbf{p}} + \mu_{\mathbf{q}}} .$$

The self-consistant eddy-damped Markovian model (SEDM)

$$\mu_{\mathbf{k}} = \nu K^2 + \sum_{\mathbf{p}} \frac{|\mathbf{kxp}|^2 \left(1 - \frac{p^2}{q^2}\right)\left(1 - \frac{p^2}{k^2}\right) R_{\mathbf{p}}(0)}{\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}}} .$$

The test field model (TFM)

$$\mu_{\mathbf{k}} = \nu k^2 + g^2 \sum_{\mathbf{p}} \frac{|\mathbf{kxp}|^2 R_{\mathbf{p}}(0)}{k^2 q^2 (\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}})}$$

The phenomenological model (FM)

$$\mu_{\mathbf{k}} = \nu k^2 + \lambda \left(\sum_{|\mathbf{p}| \leq |\mathbf{k}|} p^2 R_{\mathbf{k}}(0) \right)^{1/2}$$

$\mu_{\mathbf{k}}$: Tracer Decorelation Rate

ADIA $$\mu_{q\ \mathbf{k}} = \kappa k^2 + \sum_{\mathbf{p}} \frac{|\ \mathbf{kxp}\ |^2 R_{\mathbf{p}}(0)}{\mu_{q\ \mathbf{q}} + \mu_{\mathbf{p}}}$$

SEDM $$\mu_{q\ \mathbf{k}} = \kappa k^2 + \sum_{\mathbf{p}} \frac{|\ \mathbf{kxp}\ |^2 R_{\mathbf{p}}(0)}{\mu_{q\ k} + \mu_{\mathbf{p}} + \mu_{q\ \mathbf{q}}}$$

Tracer Eddy Diffusivity

$$D_q = \sum_{\mathbf{k}} \frac{L_{\mathbf{k}}^2 \langle \phi_{\mathbf{k}}^{(0)} \phi_{-\mathbf{k}}^{(0)} \rangle}{(\mu_{\mathbf{k}} + \mu_{q\ \mathbf{k}})}$$

Vorticity Eddy Diffusivity

$$D = \sum_{\mathbf{k}} \frac{2L_k^2 m_{\mathbf{k}}^2}{\mu_{\mathbf{k}} k^2} \langle \phi_{\mathbf{k}}^{(0)} \phi_{-\mathbf{k}}^{(0)} \rangle.$$

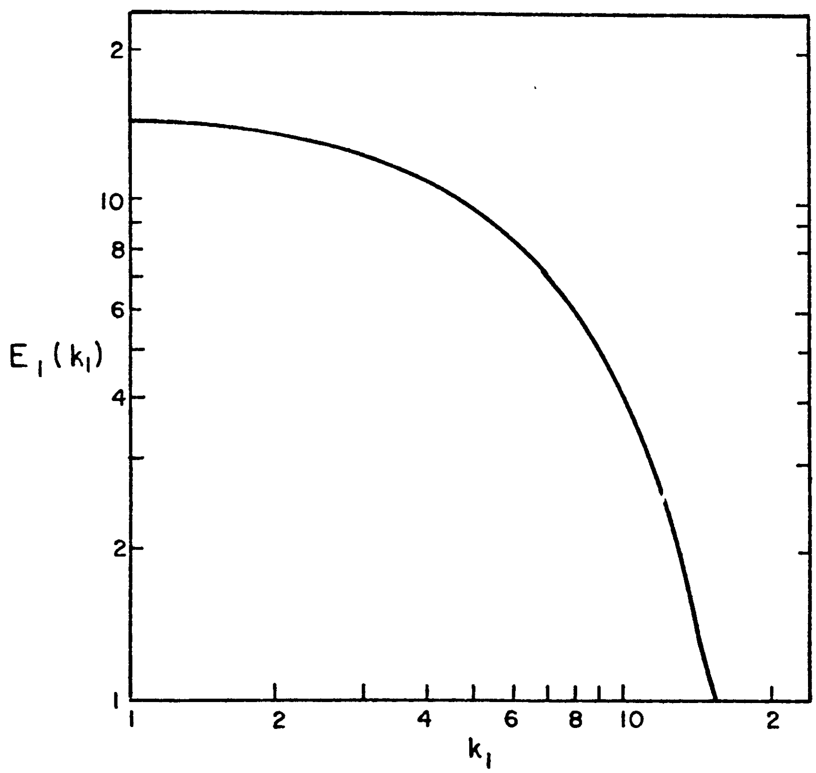

Fig. 4.1 Prescribed one-dimensional energy spectrum

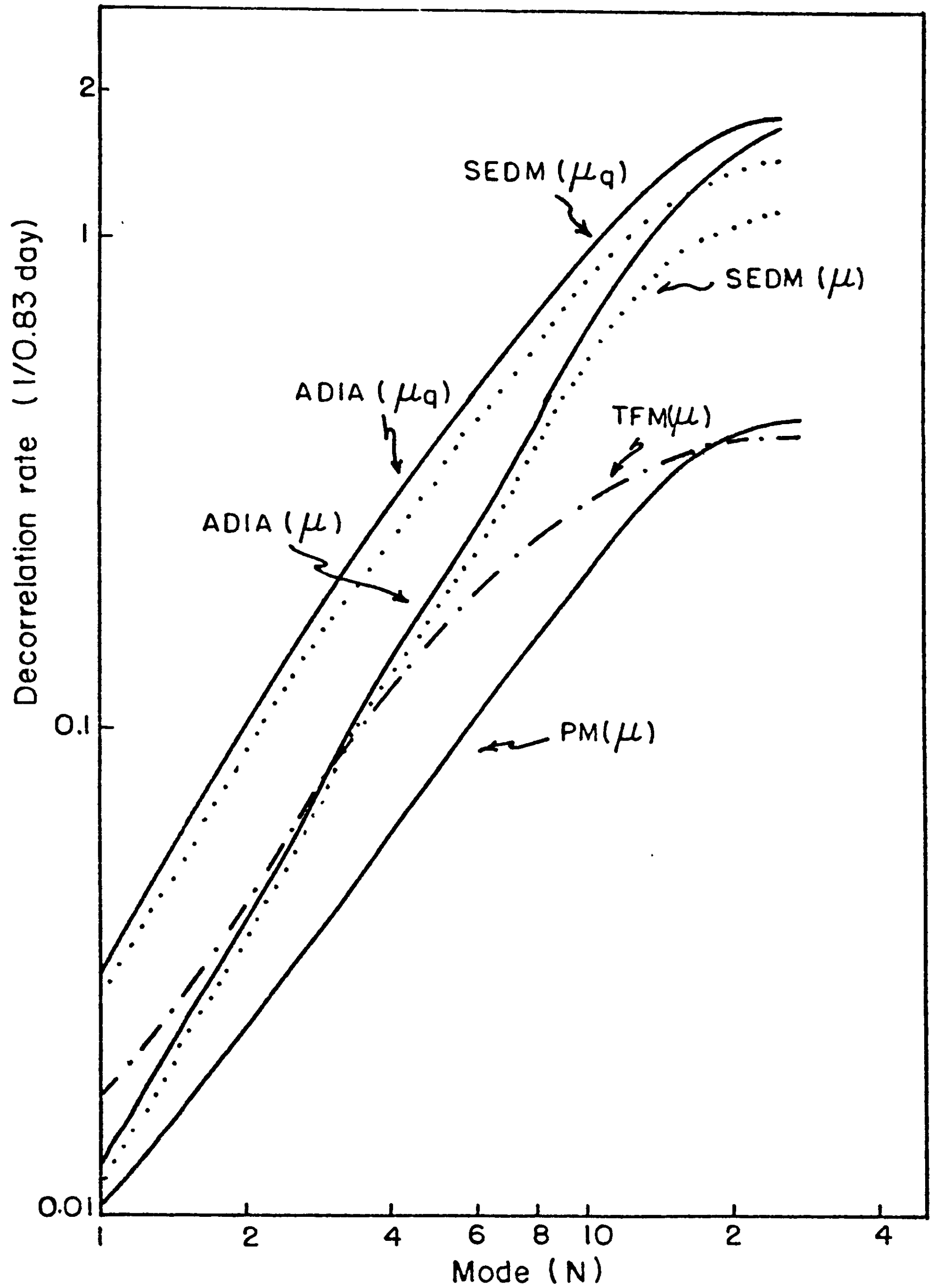

Fig. 4.2 Decorrelation rate vs. Mode (N)
Mode (N) corresponding to wave number
K = (0,N/L), N=1,2,3.

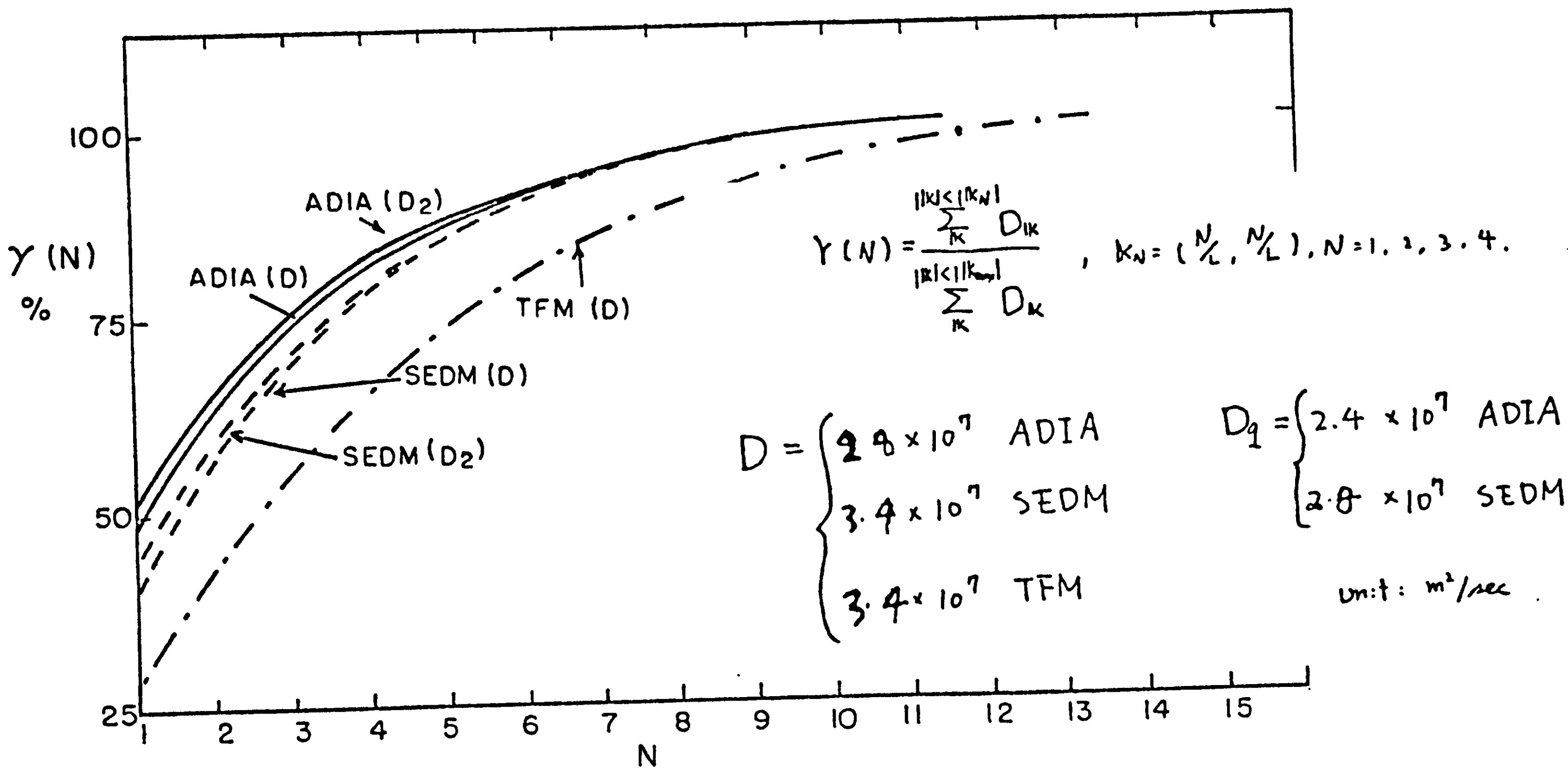

Fig. 4.3 The percentage Y(N) of eddy diffusive coeffecient under mode N.

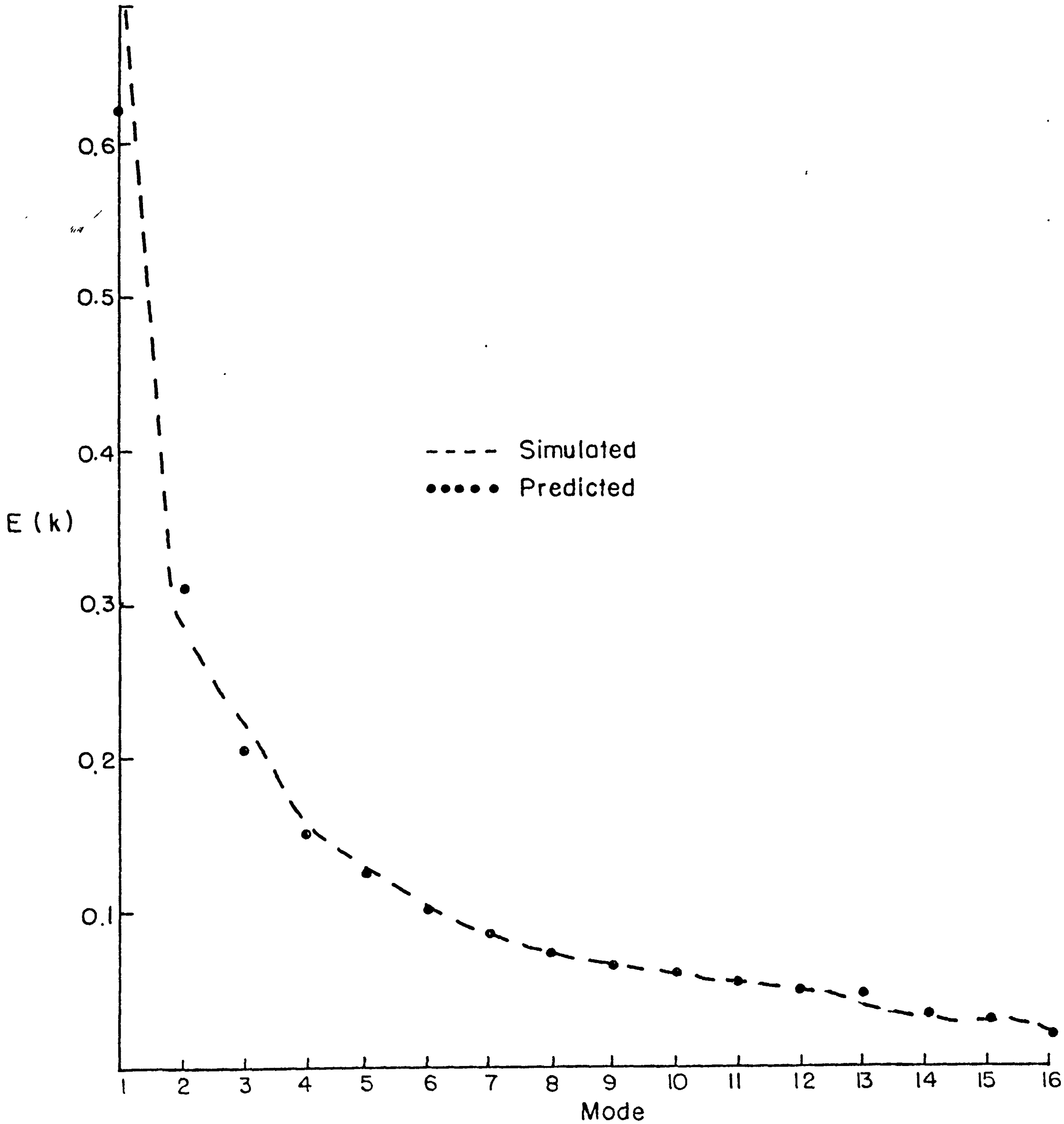

Fig. 5.1 Energy spectrum in thermal equilibrium state

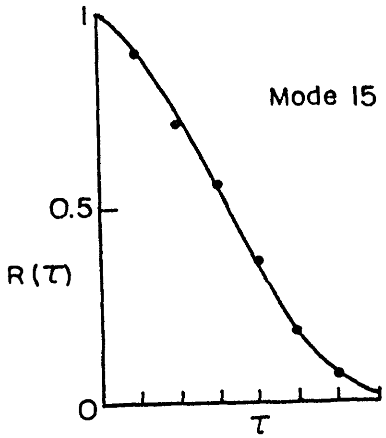

Fig. 5.2

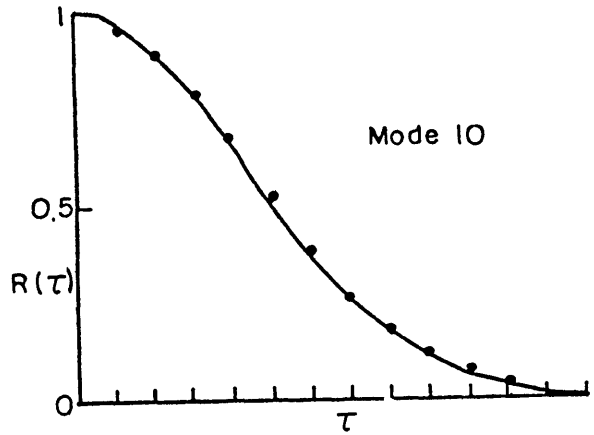

Fig. 5.3

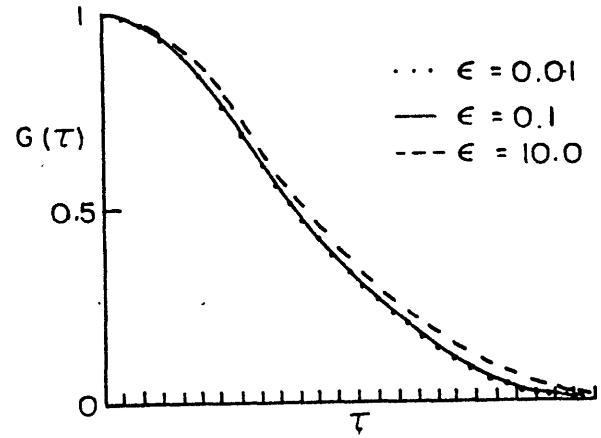

Fig. 5.5 The averaged Green's function corresponding to different Magnitudes of perturbation

....Green's function

ig. 5.4

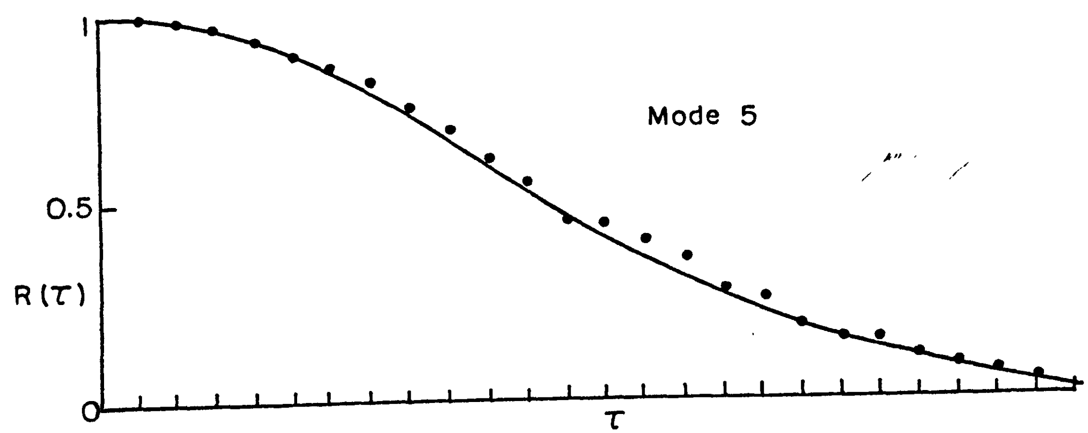

Fig. 5.2, 5.3, 5.4. The average Green's function (...) vs. the autocorrelation (solid line) for mode 15, 10, 5.

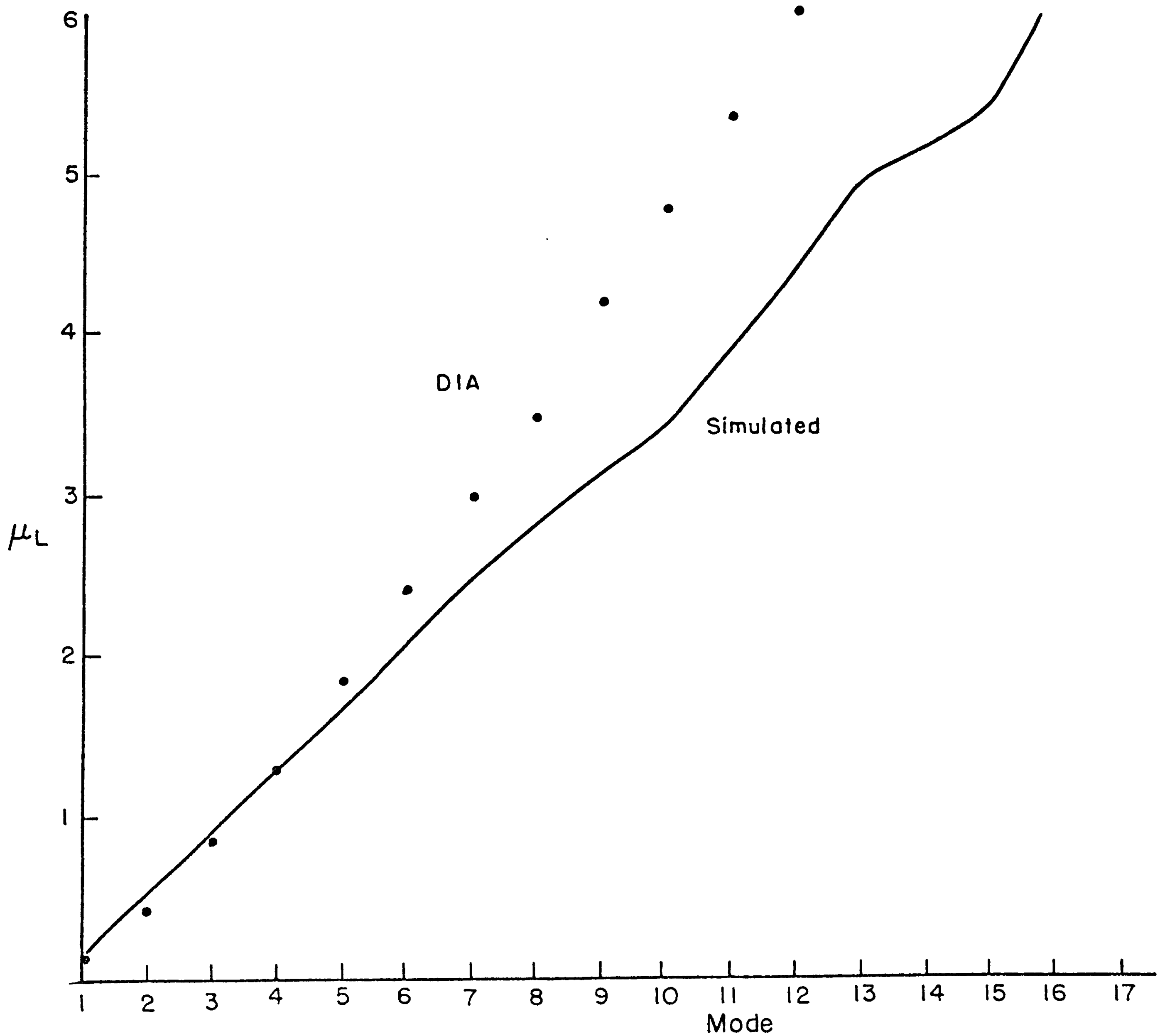

Fig. 56 Decorrelation rate (simulated vs. DIA)