# Numerical Properties of Pseudo-effective Divisors

by

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## Abstract

Suppose that $X$ is a smooth variety and $L$ is an effective divisor. One of the main goals of birational geometry is to understand the asymptotic behavior of the linear series $|mL|$ as $m$ increases. The two most important features of the asymptotic behavior - the Iitaka dimension and the Iitaka fibration - are subtle and difficult to work with. In this thesis we will construct approximations to these objects that depend only on the numerical class of $L$. The main interest in such results arises from the Abundance Conjecture which predicts that the Iitaka fibration for $K_X$ is determined by its numerical properties.

In the second chapter we study a numerical approximation to the Iitaka dimension of $L$. For a nef divisor $L$, this quantity is a classical invariant known as the numerical dimension. There have been several proposed extensions of the numerical dimension to pseudo-effective divisors in [Nak04] and [BDPP04]. We show that these proposed definitions coincide and agree with many other natural notions. Just as in the nef case, the numerical dimension $\nu(L)$ of a pseudo-effective divisor $L$ should measure the maximum dimension of a subvariety $W \subset X$ such that the "positive restriction" of $L$ is big along $W$.

In the third chapter, we analyze how the properties of the Iitaka fibration $\phi_L$ for $L$ are related to the numerical properties of $L$. Although the numerical dimension detects the existence of "virtual sections", it does not have a direct relationship with the Iitaka fibration. However, we do construct a rational map that only depends on the numerical class of $L$ and approximates the Iitaka fibration. This rational map is the maximal possible fibration for which a general fiber $F$ satisfies $\nu(L|_F) = 0$. Thus, this chapter recovers and extends the work of [Eck05] from an algebraic viewpoint. Finally, we use the pseudo-effective reduction map to study the Abundance Conjecture. [1]

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# Contents

# Chapter 1

# Introduction

Suppose that $X$ is a smooth variety and $L$ is an effective divisor. One of the main goals of birational geometry is to understand the asymptotic behavior of the linear series $|mL|$ as $m$ increases. The two most important features of the asymptotic behavior - the Iitaka dimension and the Iitaka fibration - are subtle and difficult to work with. In this thesis we will construct approximations to these objects that depend only on the numerical class of $L$. These approximations provide a foundation for understanding the discrepancy between asymptotic behavior and predictions of numerical invariants.

The main interest in such results arises from the Abundance Conjecture, one of the key components of the birational classification of varieties. The conjecture states that the Iitaka fibration of the canonical divisor $K_X$ is precisely determined by numerical properties. We will show later on that the general framework laid out here provides additional insights in the special case of the canonical divisor.

In the second chapter we study a numerical approximation to the Iitaka dimension of $L$ known as the numerical dimension $\nu(L)$. For nef divisors $L$, this is a classical notion that has been extensively studied. [Nak04] and [BDPP04] have proposed several extensions of the numerical dimension to pseudo-effective divisors. The main result of this chapter is that these proposed definitions coincide and agree with many other natural notions.

**Theorem 2.0.2.** *Let $X$ be a normal variety and let $L$ be a pseudo-effective divisor. In the following, $W$ will denote an intersection of very general very ample divisors, and $A$ some fixed sufficiently ample divisor. Then the following quantities coincide:*

*Volume conditions:*

1. $\max\left\{k \in \mathbb{Z}_{\geq 0} \,\middle|\, \limsup_{m\to\infty} \frac{h^0(X, \lfloor mL \rfloor + A)}{m^k} > 0\right\}$.
2. $\max\{\dim W \,|\, \lim_{\epsilon\to 0} \mathrm{vol}_{X|W}(L + \epsilon A) > 0\}$.
3. $\max\{\dim W \,|\, \inf_{\phi:Y\to X} \mathrm{vol}_{\widetilde{W}}(P_\sigma(\phi^* L)|_{\widetilde{W}}) > 0\}$ *where $\phi$ varies over all birational maps such that no exceptional center contains $W$ and $\widetilde{W}$ denotes the strict transform of $W$.*

*Positive product conditions:*

4. $\max\{k \in \mathbb{Z}_{\geq 0} | \langle L^k \rangle \neq 0\}$.

5. $\max\{\dim W | \langle L \rangle_{X|W}$ *is big* $\}$.

*Geometric condition:*

6. $$\min \left\{ \dim W \,\middle|\, \begin{array}{l} \phi^* L - \epsilon E \text{ is not pseudo-effective for any } \epsilon > 0 \\ \text{where } \phi : Bl_W(X) \to X \text{ and } \mathcal{O}_X(-E) = \phi^{-1}\mathcal{I}_W \cdot \mathcal{O}_{Bl_W X} \end{array} \right\}.$$

*By convention, if* $L$ *is big we interpret this expression as returning* $\dim(X)$.

*This common quantity is known as the numerical dimension of* $L$*, and is denoted* $\nu(L)$*. It only depends on the numerical class of* $L$.

The restricted volume $\mathrm{vol}_{X|W}$ is explained in Definition 2.3.3, and the restricted positive product $\langle - \rangle_{X|W}$ is described in Section 2.4.

Just as in the nef case, the numerical dimension of a pseudo-effective divisor measures the maximum dimension of a subvariety $W \subset X$ such that the "restriction" of $L$ is big along $W$. An important subtlety is that "restriction" no longer means $L|_W$. Since $L$ is no longer nef, the positivity of $L$ along $W$ is best measured by throwing away the contributions of the base locus of $L$.

The numerical dimension satisfies several important properties:

**Theorem 2.0.4.** *Let* $X$ *be a normal variety,* $L$ *a pseudo-effective divisor.*

1. *We have* $0 \leq \nu(L) \leq \dim(X)$ *and* $\kappa(X, L) \leq \nu(L)$.

2. $\nu(L) = \dim(X)$ *iff* $L$ *is big, and* $\nu(L) = 0$ *iff* $P_\sigma(L) \equiv 0$.

3. *When* $L$ *is nef,* $\nu(L)$ *coincides with the usual numerical dimension.*

4. *If* $L'$ *is pseudo-effective, then* $\nu(L + L') \geq \nu(L)$.

5. *If* $f : Y \to X$ *is any surjective morphism from a normal variety* $Y$ *then* $\nu(f^*L) = \nu(L)$.

6. *We have* $\nu(L) = \nu(P_\sigma(L))$.

7. *Suppose that* $f : X \to Z$ *has connected fibers, and* $F$ *is a very general fiber of* $f$*. Then* $\nu(L) \leq \nu(L|_F) + \dim(Z)$.

We also define a restricted numerical dimension $\nu_{X|V}(L)$ which measures the positivity of $L$ along a subvariety $V$. The restricted numerical dimension satisfies analogues of Theorems 2.0.2 and 2.0.4.

**Definition 2.0.6.** *Let* $X$ *be a normal variety,* $V$ *a subvariety not contained in* $\mathrm{Sing}(X)$*, and* $L$ *a pseudo-effective divisor such that* $V \not\subset \mathbf{B}_-(L)$*. We define the restricted numerical dimension* $\nu_{X|V}(L)$ *to be*

$$\nu_{X|V}(L) := \max\{\dim W | \lim_{\epsilon \to 0} \mathrm{vol}_{X|W}(L + \epsilon A) > 0\}$$

*where* $W$ *is an intersection of* $V$ *with very general very ample divisors and* $A$ *is some fixed ample divisor. The restricted numerical dimension only depends on the numerical class of* $L$.

In the third chapter, we construct a fibration that approximates the Iitaka fibration of $L$ but only depends on the numerical class of $L$. This fibration is known as the pseudo-effective reduction map and was previously defined by [Eck05].

**Theorem 3.0.8** ([Eck05], Proposition 1.5 and Definition 4.1). *Let $X$ be a smooth projective variety and $L$ a pseudo-effective $\mathbb{R}$-divisor on $X$. There is a birational model $\phi : Y \to X$ and a morphism $\pi : Y \to Z$ with connected fibers satisfying*

1. *For a general fiber $F$ of $\pi$, we have $\nu(L|_F) = 0$.*
2. *The pair $(Y, \pi)$ is the maximal quotient satisfying (1): if $\phi' : Y' \to X$ is a birational map and $\pi' : Y' \to Z'$ a morphism with connected fibers that satisfies (1), then there is a dominant rational map $\psi : Z' \dashrightarrow Z$ such that $\pi = \psi \circ \pi'$ as rational maps.*

*The pair $(Y, \pi)$ is determined up to birational equivalence and depends only on the numerical class of $L$. We call it the pseudo-effective reduction map associated to $L$.*

Since [Eck05] obtains this result from an analytic perspective, the main contribution of this paper is to recover and extend the work of [Eck05] using algebraic techniques. We will discuss the relationship with Eckl's work in Section 3.2.3. Our approach yields a more explicit description of the pseudo-effective reduction map.

**Theorem 3.0.9.** *Let $X$ be a smooth projective variety and $L$ a pseudo-effective $\mathbb{R}$-divisor on $X$. The pseudo-effective reduction map for $L$ is the generic quotient of $X$ by all movable curves $C$ satisfying $\nu_{X|C}(L) = 0$.*

The pseudo-effective reduction map has the following important properties:

**Theorem 3.0.10.** *Let $X$ be a smooth variety and let $L$ be a pseudo-effective divisor.*

1. *Suppose that $\phi : Y \to X$ is a birational map. The pseudo-effective reduction map for $\phi^* L$ is birationally equivalent to the pseudo-effective reduction map for $L$.*
2. *If $\kappa(X, L) \geq 0$ then the Iitaka fibration for $L$ factors birationally through the pseudo-effective reduction map.*
3. *The pseudo-effective reduction map for $P_\sigma(L)$ is birationally equivalent to the pseudo-effective reduction map for $L$.*
4. *If $\kappa(X, L) \geq 0$, there is a birational model $\phi : W \to X$ and a morphism $f : W \to Z$ birationally equivalent to the pseudo-effective reduction map such that there is a divisor $D$ on $Z$ and an effective divisor $E$ on $W$ with $\mu^* L \sim_{\mathbb{Q}} f^* D + E$ and the section rings $R(X, L) = R(Z, D)$ coincide.*

A pseudo-effective divisor $L$ is said to be abundant if $\kappa(X, L) = \nu(L)$. It is well-known that abundant nef divisors have many special geometric properties. It turns out that these properties generalize to pseudo-effective divisors in a natural way. In particular, $L$ is abundant iff the pseudo-effective reduction map for $L$ is birationally equivalent to the Iitaka fibration (as shown in [Eck05]). The importance of abundance derives from the following reformulation of the Abundance Conjecture:

**Conjecture 3.7.3.** *Let $(X, \Delta)$ be a klt pair. Then $K_X + \Delta$ is abundant.*

The pseudo-effective reduction map naturally leads to an inductive approach for the Abundance Conjecture. Using the work of [Amb04], we show the following result:

**Theorem 3.0.11.** *Let $(X, \Delta)$ be a klt pair. Assume that $K_X + \Delta$ is pseudo-effective, and let $f : X \dashrightarrow Z$ denote the pseudo-effective reduction map. If Conjecture 3.7.3 holds on $Z$, then $K_X + \Delta$ is abundant.*

In this approach to the Abundance Conjecture, the key question is whether the pseudo-effective reduction map for $K_X + \Delta$ maps to a variety of smaller dimension. By Theorem 3.0.9, this question can be answered by finding curves satisfying a numerical condition. Similar work has appeared in the recent preprint [Siu09].

# Chapter 2

# Comparing Numerical Dimensions

Suppose that $X$ is a smooth variety and $L$ an effective divisor. One of the main goals of birational geometry is to understand the asymptotic behavior of the linear series $|mL|$ as $m$ increases. When $L$ is a big divisor, many features of the asymptotic behavior are captured by invariants that can be calculated numerically. However, for an arbitrary pseudo-effective divisor $L$ the numerical invariants no longer predict the behavior of sections.

In this chapter we define the numerical dimension $\nu(L)$ of $L$. The numerical dimension is an approximation to the Iitaka dimension of $L$ that only depends on the numerical class of $L$. In a loose sense it measures the discrepancy between predictions of numerical invariants and the actual asymptotic behavior of sections. For a nef divisor $L$, the numerical dimension is a classical invariant.

**Definition 2.0.1.** Let $X$ be a normal projective variety of dimension $n$, $L$ a nef divisor. Fix an ample divisor $A$. The numerical dimension of $L$ is defined to be

$$\nu(L) := \max\{k \in \mathbb{Z}_{\geq 0} | L^k \cdot A^{n-k} \neq 0\}.$$

Thus, the numerical dimension of a nef divisor $L$ is the maximal dimension of a subvariety $W$ of $X$ such that $L|_W$ is big. [Nak04] and [BDPP04] propose several extensions of the numerical dimension to pseudo-effective divisors. Our main result is that the proposed definitions of numerical dimension coincide and agree with many other natural notions. (Since some of the definitions in the following theorem are quite involved, we will recall them later in the paper.)

**Theorem 2.0.2.** *Let $X$ be a normal variety and let $L$ be a pseudo-effective divisor. In the following, $W$ will denote an intersection of very general very ample divisors, and $A$ some fixed sufficiently ample divisor. Then the following quantities coincide:*

*Volume conditions:*

1. $\max\left\{k \in \mathbb{Z}_{\geq 0} \left| \limsup_{m\to\infty} \frac{h^0(X, \lfloor mL \rfloor + A)}{m^k} > 0 \right.\right\}$.

2. $\max\{\dim W | \lim_{\epsilon\to 0} \mathrm{vol}_{X|W}(L+\epsilon A) > 0\}$.

3. $\max\{\dim W \mid \inf_{\phi: Y \to X} \mathrm{vol}_{\widetilde{W}}(P_\sigma(\phi^* L)|_{\widetilde{W}}) > 0\}$ *where* $\phi$ *varies over all birational maps such that no exceptional center contains* $W$ *and* $\widetilde{W}$ *denotes the strict transform of* $W$.

*Positive product conditions:*

4. $\max\{k \in \mathbb{Z}_{\geq 0} \mid \langle L^k \rangle \neq 0\}$.

5. $\max\{\dim W \mid \langle L \rangle_{X|W}$ *is big* $\}$.

*Geometric condition:*

6. $$\min\left\{\dim W \;\middle|\; \begin{array}{l} \phi^* L - \epsilon E \text{ is not pseudo-effective for any } \epsilon > 0 \\ \text{where } \phi : Bl_W(X) \to X \text{ and } \mathcal{O}_X(-E) = \phi^{-1}\mathcal{I}_W \cdot \mathcal{O}_{Bl_W X} \end{array}\right\}.$$

   *By convention, if* $L$ *is big we interpret this expression as returning* $\dim(X)$.

*This common quantity is known as the numerical dimension of* $L$*, and is denoted* $\nu(L)$*. It only depends on the numerical class of* $L$.

The restricted volume $\mathrm{vol}_{X|W}$ is explained in Definition 2.3.3, and the restricted positive product $\langle - \rangle_{X|W}$ is described in Section 2.4.

**Remark 2.0.3.** The numerical dimension also admits a natural interpretation with respect to separation of jets, reduced volumes, and the other invariants considered in [ELM+09].

Just as in the nef case, the numerical dimension of a pseudo-effective divisor measures the maximum dimension of a subvariety $W \subset X$ such that the "restriction" of $L$ is big along $W$. An important subtlety is that "restriction" no longer means $L|_W$. Since $L$ is no longer nef, the positivity of $L$ along $W$ is best measured by throwing away contributions of the base locus of $L$.

There are two ways of making this intuition precise. One way to measure positivity is to use intersection products. The positive product of [BDPP04] gives a precise method of calculating intersections while discounting the contributions of the base locus of $L$. Another way of measuring positivity is to calculate the rate of growth of sections. Since the base locus of $L$ may increase the dimension of $H^0(W, mL|_W)$, it is better to work with the image of the restriction map $H^0(X, \mathcal{O}_X(mL)) \to H^0(W, \mathcal{O}_W(mL|_W))$. These subspaces more accurately reflect the positivity of $L$ along $W$.

The numerical dimension satisfies several important properties:

**Theorem 2.0.4.** *Let* $X$ *be a normal variety,* $L$ *a pseudo-effective divisor.*

1. *We have* $0 \leq \nu(L) \leq \dim(X)$ *and* $\kappa(X, L) \leq \nu(L)$.

2. $\nu(L) = \dim(X)$ *iff* $L$ *is big, and* $\nu(L) = 0$ *iff* $P_\sigma(L) \equiv 0$.

3. *When* $L$ *is nef,* $\nu(L)$ *coincides with the usual numerical dimension.*

4. *If $L'$ is pseudo-effective, then $\nu(L+L') \geq \nu(L)$.*

5. *If $f : Y \to X$ is any surjective morphism from a normal variety $Y$ then $\nu(f^*L) = \nu(L)$.*

6. *We have $\nu(L) = \nu(P_\sigma(L))$.*

7. *Suppose that $f : X \to Z$ has connected fibers, and $F$ is a very general fiber of $f$. Then $\nu(L) \leq \nu(L|_F) + \dim(Z)$.*

**Remark 2.0.5.** Since bigness is an open condition, one might expect that the numerical dimension is lower semicontinuous as a function on $\overline{NE}^1(X)$. This turns out to be false; see Example 2.7.6.

As we study different numerical invariants, we will come across two common themes. First, we will define numerical invariants for pseudo-effective divisors by taking the limit of asymptotic invariants of nearby big divisors. Since asymptotic invariants of big divisors can be calculated numerically, this approach ensures that our invariant is numerical in nature. It also allows us to avoid the difficulties of working with pseudo-effective divisors of negative Iitaka dimension. In some sense the numerical invariants will measure the existence of "virtual sections" of $L$.

The second theme is that it is useful to study not only numerical invariants on $X$ but also restricted numerical invariants along subvarieties $V$. In particular, we can define the notion of a restricted numerical dimension of $L$ along a subvariety $V$. Just as in the non-restricted case, the restricted numerical dimension should measure the maximal dimension of a very general subvariety $W \subset V$ such that the "positive restriction" of $L$ is big along $W$.

**Definition 2.0.6.** Let $X$ be a normal variety, $V$ a subvariety not contained in $\mathrm{Sing}(X)$, and $L$ a pseudo-effective divisor such that $V \not\subset \mathbf{B}_-(L)$. We define the restricted numerical dimension $\nu_{X|V}(L)$ to be

$$\nu_{X|V}(L) := \max\{\dim W \mid \lim_{\epsilon \to 0} \mathrm{vol}_{X|W}(L+\epsilon A) > 0\}$$

where $W$ is an intersection of $V$ with very general very ample divisors and $A$ is some fixed ample divisor. The restricted numerical dimension only depends on the numerical class of $L$.

The restricted numerical dimension satisfies (slightly weaker) analogues of Theorems 2.0.2 and 2.0.4. It does not occur in the classical analysis because when $L$ is a nef divisor we have $\nu_{L|V}(L) = \nu_V(L|_V)$. Nevertheless, it is useful to have the additional flexibility for a general pseudo-effective divisor. The restricted numerical dimension satisfies several compatibility relations:

**Proposition 2.0.7.** *Let $X$ be a normal variety, $V$ a normal subvariety not contained in $\mathrm{Sing}(X)$, and $L$ a pseudo-effective divisor such that $V \not\subset \mathbf{B}_-(L)$. Then*

1. *$\nu_{X|V}(L) \leq \nu(L)$.*

*2.* $\nu_{X|V}(L) \leq \nu(L|_V)$.

*3. If $H$ is a general very ample divisor and $\nu_{X|V}(L) < \dim V$, then $\nu_{X|V\cap H}(L) = \nu_{X|V}(L)$.*

## 2.1 Background

All schemes will lie over the base field $\mathbb{C}$. A variety will always be an irreducible reduced projective scheme. We will usually restrict our attention to normal varieties $X$.

Throughout the paper we will be working with $\mathbb{R}$-Cartier divisors. Since the behavior of $\mathbb{R}$-divisors can be subtle, a couple of remarks are in order. In the literature, often asymptotic invariants (such as the volume) are only defined for $\mathbb{Q}$-divisors. It is then checked that these invariants can be extended to continuous functions on the entire big cone. It is easy to see that this continuous extension coincides with the naive extension of the definition to big $\mathbb{R}$-divisors. Thus, there is usually no difficulty in considering asymptotic invariants of big divisors.

The behavior of $\mathbb{R}$-divisors on the pseudo-effective boundary is much more subtle. However, as mentioned in the introduction, we will define numerical invariants by taking a limit of invariants of nearby big divisors. Thus, we will avoid the potential difficulties of the boundary case. We refer to [Nak04] as a reference for properties of pseudo-effective $\mathbb{R}$-divisors.

### 2.1.1 $\mathbb{R}$-divisors

Suppose that $L = \sum_i a_i L_i$ is an $\mathbb{R}$-Cartier divisor (henceforth "divisor"). As usual, the support of $L$, denoted $\mathrm{Supp}(L)$, is defined to be the set of points $x$ for which a local equation of one of the $L_i$ is not a unit in $\mathcal{O}_{x,X}$. We define the round-down $\lfloor L \rfloor := \sum_i \lfloor a_i \rfloor L_i$. Then $\lfloor L \rfloor$ is an integral Cartier divisor, and we denote the difference $\{L\} := L - \lfloor L \rfloor$. We will use the notation $\mathfrak{b}(\lfloor L \rfloor)$ to denote the base ideal of the linear system $|\lfloor L \rfloor|$ and $\mathrm{Bs}(\lfloor L \rfloor)$ to denote the set-theoretic base locus.

We say that two $\mathbb{R}$-divisors $L, L'$ are linearly equivalent if $L - L'$ is a principal divisor. That is, we must have $\{L\} = \{L'\}$ and $\lfloor L \rfloor - \lfloor L' \rfloor = \mathrm{div}(f)$ for some meromorphic function $f$ on $X$. We will write $L \sim L'$ to denote linear equivalence. Then the complete linear system associated to $L$ is

$$|L| = |\lfloor L \rfloor| + \{L\}.$$

Suppose that $|L|$ is not empty. Then sections of $|\lfloor L \rfloor|$ define a rational map $\Phi_{|L|} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(\lfloor L \rfloor)))$. We denote the closure of the image by $W_L$. The Iitaka dimension $\kappa(X, L)$ is defined to be

$$\kappa(X, L) = \max\{\dim W_{mL} \mid |mL| \neq \varnothing\}$$

unless $|mL|$ is empty for every $m$, in which case we set $\kappa(X, L) = -\infty$. We can also construct an Iitaka fibration $\pi : X \dashrightarrow Z$ for $L$: it is characterized up to birational equivalence by the property that $\dim(Z) = \kappa(X, L)$ and $\kappa(F, L|_F) = 0$ for a very general fiber $F$ of $\pi$.

We will say that two $\mathbb{R}$-divisors $L, L'$ are $\mathbb{R}$-linearly equivalent (respectively $\mathbb{Q}$-linearly equivalent) if $L - L'$ is a $\mathbb{R}$-linear combination (resp. $\mathbb{Q}$-linear combination) of principal divisors. We write this relation as $L \sim_{\mathbb{R}} L'$ (resp. $L \sim_{\mathbb{Q}} L'$). We write $L \equiv L'$ to denote numerical equivalence, and use the notation $[L]$ to denote the class of $L$ in $N^1(X)$. $A^p(X)$ will denote the space of rational equivalence classes of codimension $p$ cycles and $N^p(X)$ will denote the space of numerical classes of codimension $p$ cycles.

The $\mathbb{R}$-stable base locus $\mathbf{B}_{\mathbb{R}}(L)$ is defined to be

$$\mathbf{B}_{\mathbb{R}}(L) := \bigcap \{ \operatorname{Supp}(C) \mid C \geq 0 \text{ and } C \sim_{\mathbb{R}} L \}.$$

This is always a Zariski-closed subset of $X$; we do not associate any scheme structure to it. When $L$ is a $\mathbb{Q}$-divisor, this is the same as the usual stable base locus, but for a general $\mathbb{R}$-divisor it will be smaller than the corresponding intersection over all $\mathbb{Q}$-linearly equivalent divisors (see [BCHM10], Lemma 3.5.3).

### 2.1.2 Asymptotic Valuations and Multiplier Ideals

Asymptotic valuations of big $\mathbb{R}$-divisors are an important example of a numerical invariant. Throughout this section we will let $v$ denote a discrete valuation of the function field $K(X)$ of $X$. When such a valuation measures the the order of vanishing along a subvariety $Z$ of $X$, we will denote it by $v_Z$. For a Cartier divisor $D$, we define $v(D)$ to be $v(f)$ where $f$ is any local defining equation for $D$ on a open set intersecting the center of $v$. We then extend the valuation by $\mathbb{R}$-linearity to arbitrary $\mathbb{R}$-divisors. Similarly, for an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, we can define the valuation $v(\mathcal{I})$ by first passing to a principalization of $\mathcal{I}$ and then using the definition for divisors. On an open affine $U \subset X$ intersecting the center of $v$, this is equivalent to taking the valuation of an element $f \in \mathcal{I}(U)$ "general" in the sense that it is a general linear combination of the generators of $\mathcal{I}(U)$.

**Definition 2.1.1.** Let $X$ be a normal variety. Suppose that $L$ is a divisor with $\kappa(L) \geq 0$ and $v$ is a discrete valuation on $X$. We define the asymptotic order of vanishing of $L$ along $v$ as follows:

$$v(\|L\|) = \min_{\substack{m \in \mathbb{Z}_{>0} \\ D \in |mL|}} \frac{1}{m} v(D).$$

We will almost always restrict our attention to the case when $L$ is big. In this case, asymptotic valuations only depend on the numerical class of $L$.

**Theorem 2.1.2** ([ELM+09], Theorem A and Lemma 3.3). *Let $X$ be a normal variety. If $L$ is big, then*

$$v(\|L\|) = \min_{\substack{D \equiv L \\ D \geq 0}} v(D).$$

*Furthermore, $v(\|-\|)$ is continuous on the big cone.*

Asymptotic valuations share a close relationship with the asymptotic multiplier ideal. Multiplier ideals are most commonly used for vanishing theorems. Thus, for the sake of clarity we will only work with multiplier ideals when the underlying variety $X$ is smooth. We begin by recalling the definition of a multiplier ideal:

**Definition 2.1.3.** Suppose that $X$ is smooth and that $L$ is an effective $\mathbb{R}$-divisor. Let $\phi : Y \to X$ be a log resolution of the pair $(X, L)$. We define the multiplier ideal of $L$ to be the ideal sheaf

$$\mathcal{J}(X, L) := \phi_* \mathcal{O}_Y(K_{Y/X} - \lfloor \phi^* L \rfloor).$$

Since $\phi_* \mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X$, this sheaf is in fact an ideal sheaf. It turns out to be independent of the choice of resolution. In some sense the multiplier ideal sheaf measures the singularities of $L$.

The asymptotic multiplier ideal sheaf associated to $L$ measures the singularities of a general divisor $D$ with $D \sim_{\mathbb{R}} L$. We first recall the definition for $\mathbb{Q}$-divisors:

**Definition 2.1.4.** Let $X$ be smooth. Suppose that $L$ is a $\mathbb{Q}$-divisor with $\kappa(X, L) \geq 0$. Consider the set of ideals $\{\mathcal{J}(X, D)\}$ where $D \sim_{\mathbb{R}} L$ (or equivalently $D \sim_{\mathbb{Q}} L$). It is shown in [Laz04] that this is a directed set under the inclusion relation. Since the underlying rings are Noetherian, there is a unique maximal element. We define $\mathcal{J}(X, \|L\|)$ to be this maximal element.

**Remark 2.1.5.** Suppose that $L$ is a big $\mathbb{Q}$-divisor. It turns out that the asymptotic multiplier ideal sheaf $\mathcal{J}(X, \|L\|)$ is the unique ideal sheaf satisfying the valuation conditions

$$v_Z(\mathcal{J}(\|L\|)) = \lfloor v_Z(\|L\|) - \operatorname{codim}(Z) + 1 \rfloor$$

where $Z$ ranges over all subvarieties in $X$. In particular, the asymptotic multiplier ideal sheaf only depends on the numerical class of $L$.

We would like to extend the notion of asymptotic multiplier ideal to $\mathbb{R}$-divisors. When $L$ is big, we can define the asymptotic multiplier ideal sheaf by perturbing by an ample divisor $A$.

**Definition 2.1.6.** Let $X$ be smooth. Suppose that $L$ is a big $\mathbb{R}$-divisor. Consider the set of ideal sheaves $\{\mathcal{J}(\|L - A\|)\}$ as $A$ varies over all ample divisors such that $L - A$ is a big $\mathbb{Q}$-divisor. This forms a directed set under the inclusion relation, so as before this set has a unique maximal element. We define $\mathcal{J}(\|L\|)$ to be this ideal sheaf.

**Remark 2.1.7.** There is some effective divisor $D \sim_{\mathbb{R}} L$ such that $\mathcal{J}(\|L\|) = \mathcal{J}(D)$. To see this, first choose an ample divisor $A$ sufficiently small so that $\mathcal{J}(\|L\|) = \mathcal{J}(\|L - A\|)$ and $L - A$ is a $\mathbb{Q}$-divisor. We may then take $D = D' + A'$ where $D' \sim_{\mathbb{R}} L - A$ and $A' \sim_{\mathbb{R}} A$ are sufficiently general.

Note also that $\mathcal{J}(\|L\|)$ is the unique ideal satisfying the valuation criteria

$$v_Z(\mathcal{J}(\|L\|)) = \lfloor v_Z(\|L\|) - \operatorname{codim}(Z) + 1 \rfloor$$

where $Z$ ranges over all subvarieties in $X$. Therefore this sheaf is an invariant of the numerical class of $L$.

The following theorem gives a basic comparison between asymptotic multiplier ideals and base loci. It is the analogue for $\mathbb{R}$-divisors of [Laz04],Theorem 11.2.21.

**Theorem 2.1.8.** *Let $X$ be smooth and let $L$ be a big $\mathbb{R}$-divisor on $X$. Fix a very ample $\mathbb{Z}$-divisor $H$ on $X$ such that $H + \lfloor D \rfloor$ is ample for every divisor $D$ supported on* Supp$(L)$ *with coefficients in the set $[-3, 3]$. Suppose that $b$ is a sufficiently large positive integer so that $\lfloor bL \rfloor - (K_X + (n+1)H)$ is numerically equivalent to an effective $\mathbb{Z}$-divisor $G$. Then for every $m \geq b$ we have*

$$\mathcal{J}(\|mL\|) \otimes \mathcal{O}_X(-G) \subseteq \mathfrak{b}(|\lfloor mL \rfloor|).$$

*Proof.* The condition on $H$ guarantees that for $m \geq b$ we can write

$$\begin{aligned} \lfloor mL \rfloor - G &\equiv \lfloor mL \rfloor - \lfloor bL \rfloor + K_X + (n+1)H \\ &\equiv ((m-b)L + A) + K_X + nH \end{aligned}$$

for some ample $\mathbb{R}$-divisor $A$. By applying Nadel vanishing and Castelnuovo-Mumford regularity, we find that

$$\mathcal{O}_X(\lfloor mL \rfloor) \otimes (\mathcal{O}_X(-G) \otimes \mathcal{J}(\|(m-b)L\|))$$

is globally generated for $m \geq b$. Since $\mathcal{J}(\|mL\|) \subset \mathcal{J}(\|(m-b)L\|)$, this proves the theorem. ☐

### 2.1.3 Base Loci

We next analyze some variants of the $\mathbb{R}$-stable base locus $\mathbf{B}_{\mathbb{R}}(L)$ of an effective divisor $L$. The stable base locus can exhibit quite subtle behavior. For example, it is not an invariant of the numerical class of $L$. The situation improves greatly if we perturb by an ample divisor. This approach to invariants was first considered in [Nak00], and studied more systematically in [ELM+05].

**Definition 2.1.9.** The *augmented base locus* is

$$\mathbf{B}_+(L) := \bigcap_{A \text{ ample}} \mathbf{B}_{\mathbb{R}}(L - A).$$

Note that $\mathbf{B}_+(L) \supset \mathbf{B}_{\mathbb{R}}(L)$. [ELM+05] verifies that the augmented base locus is equal to $\mathbf{B}_{\mathbb{R}}(L - A)$ for a sufficiently small ample divisor $A$. Thus $\mathbf{B}_+(L)$ is a Zariski-closed subset of $X$ and it only depends on the numerical class of $L$. Note that we do not associate any scheme structure to the augmented stable base locus.

**Example 2.1.10.** An $\mathbb{R}$-divisor $L$ is big iff $\mathbf{B}_+(L)$ is not equal to $X$.

For the second variant, we add on a small ample divisor.

**Definition 2.1.11.** The *restricted base locus* is

$$\mathbf{B}_-(L) = \bigcup_{A \text{ ample}} \mathbf{B}_{\mathbb{R}}(L + A).$$

It is clear that $\mathbf{B}_-(L) \subset \mathbf{B}_{\mathbb{R}}(L)$ and that the restricted base locus only depends on the numerical class of $L$. We do not associate any scheme structure to the restricted base locus.

**Example 2.1.12.** An $\mathbb{R}$-divisor $L$ is nef iff $\mathbf{B}_-(L)$ is empty.

Unlike the augmented base locus, the restricted base locus is probably not a Zariski-closed subset (although no examples are known of such pathological behavior). However, it is a countable union of subvarieties due to the following theorem.

**Theorem 2.1.13** ([Nak04],V.1.3). *Let $X$ be a smooth variety and $L$ be a pseudo-effective divisor. There is an ample $\mathbb{Z}$-divisor $A$ such that*

$$\bigcup_m \mathrm{Bs}(\lceil mL \rceil + A) = \mathbf{B}_-(L).$$

The restricted base locus is closely related to asymptotic valuations.

**Theorem 2.1.14** ([ELM+09],Theorem B). *Suppose that $X$ is a smooth variety and $L$ is a big divisor. Let $\nu$ be a discrete valuation of the function field of $X$ with center $Z$ on $X$. Then $\nu(\|L\|) > 0$ iff $Z \subset \mathbf{B}_-(L)$.*

We immediately obtain two important corollaries.

**Corollary 2.1.15.** *Suppose that $X$ is smooth and $L$ is a big $\mathbb{R}$-divisor. Then as sets*

$$\bigcup_m V(\mathcal{J}(\|mL\|)) = \mathbf{B}_-(L).$$

**Corollary 2.1.16.** *Suppose that $\phi : Y \to X$ is a birational map of smooth projective varieties and $L$ is a pseudo-effective $\mathbb{R}$-divisor on $X$. Then we have an equality of sets*

$$\mathbf{B}_-(\phi^* L) = \phi^{-1}\mathbf{B}_-(L).$$

We will need a condition that describes when a subvariety $V$ sits in "general position" with respect to the perturbed base loci of a divisor $L$.

**Definition 2.1.17.** Suppose that $V \subset X$ is a subvariety. We define the $V$-pseudo-effective cone $\mathrm{Psef}_V(X)$ to be the subcone of $\overline{NE}^1(X)$ generated by classes of divisors $L$ with $V \not\subset \mathbf{B}_-(L)$. We define the $V$-big cone $\mathrm{Big}_V(X)$ to be the cone generated by classes of divisors $L$ with $V \not\subset \mathbf{B}_+(L)$.

One may verify that $\mathrm{Psef}_V(X)$ is closed and $\mathrm{Big}_V(X)$ is its interior. Note also that $L|_V$ is pseudo-effective whenever $[L] \in \mathrm{Psef}_V(X)$. Although the following interpretation of $V$-bigness is just a rephrasing of the definitions, the different perspective will sometimes be useful.

**Definition 2.1.18.** Suppose that $V \subset X$ is a subvariety. If $L$ is an effective divisor such that $\mathrm{Supp}(L) \not\supset V$ we say $L \geq_V 0$.

The relationship with the earlier criteria is given by a trivial lemma.

**Lemma 2.1.19.** *Suppose that $V \subset X$ is a subvariety. If $L$ is a $V$-big divisor, then $L \sim_{\mathbb{R}} L'$ for some $L' \geq_V 0$.*

### 2.1.4 V-Birational Models

Suppose that $X$ is a normal variety and $V$ is a subvariety. In order to study the geometry of a divisor $L$ along $V$ with respect to a birational morphism $\phi : Y \to X$, we need to be careful about how $V$ intersects the exceptional centers of $\phi$. We will restrict ourselves to the following situation:

**Definition 2.1.20.** Let $X$ be a normal variety and $V$ a subvariety not contained in $\mathrm{Sing}(X)$. Suppose that $\phi : \widetilde{X} \to X$ is a birational map from a normal variety $\widetilde{X}$ such that $V$ is not contained in any $\phi$-exceptional center. Let $\widetilde{V}$ denote the strict transform of $V$. We say that $(\widetilde{X}, \widetilde{V})$ or $\phi : \widetilde{X} \to X$ is a $V$-birational model for $(X, V)$. When both $\widetilde{X}$ and $\widetilde{V}$ are smooth, we say that $(\widetilde{X}, \widetilde{V})$ is a smooth $V$-birational model.

The $V$-big cone satisfies natural compatibility statements for $V$-birational models.

**Proposition 2.1.21.** *Let $X$ be a normal variety and $V$ a subvariety not contained in* $\mathrm{Sing}(X)$. *Suppose that $\phi : \widetilde{X} \to X$ is a $V$-birational model. If $L$ is a $V$-big divisor then $\phi^*L$ is a $\widetilde{V}$-big divisor. Similarly, if $L$ is a $\widetilde{V}$-big divisor then $\phi_*L$ is $V$-big.*

*Proof.* We start with the second statement. Let $H$ be an ample divisor on $X$. Then $L - \epsilon\phi^*H$ is still $V$-big for sufficiently small $\epsilon > 0$. By Lemma 2.1.19, there is a divisor $D \equiv L - \epsilon\phi^*H$ such that $D \geq_{\widetilde{V}} 0$. Since $V$ is not contained in any $\phi$-exceptional center, it is also true that $\phi_*D \geq_V 0$. But then $L \equiv \phi_*D + \epsilon H$ is $V$-big.

To show the first statement, as before write $L \equiv D + \epsilon H$ where $H$ is ample and $D \geq_V 0$. Since $\phi^*D$ is $\widetilde{V}$-pseudo-effective, it suffices to check that $\phi^*H$ is $\widetilde{V}$-big. Lemma 2.1.22 below shows that $\mathbf{B}_+(A)$ is contained in the union of the $\phi$-exceptional locus and $\mathrm{Sing}(\widetilde{X})$. By assumption this set does not contain $\widetilde{V}$. □

**Lemma 2.1.22.** *Let $\phi : Y \to X$ be a birational map of normal varieties. If $A$ is an ample divisor on $X$, then $\mathbf{B}_+(\phi^*A)$ is contained in the union of the $\phi$-exceptional locus and* $\mathrm{Sing}(Y)$.

*Proof.* Let $\psi : \widetilde{Y} \to Y$ be a desingularization of $Y$ that is an isomorphism away from $\mathrm{Sing}(Y)$. Then certainly $\psi^{-1}\mathbf{B}_+(\phi^{-1}A) \subset \mathbf{B}_+(\psi^*\phi^*A)$. We can apply Theorem 2.3.7 to $\widetilde{Y}$ to see that $\mathbf{B}_+(\psi^*\phi^*A)$ is precisely the $(\phi \circ \psi)$-exceptional locus. Since this set is the preimage of the $\phi$-exceptional locus and $\mathrm{Sing}(Y)$, the lemma follows. □

### 2.1.5 Movable Curves

**Definition 2.1.23.** Let $X$ be a projective variety. The cone of nef curves $\overline{NM}_1(X)$ is the subcone of $N_1(X)$ that is dual to $\overline{NE}^1(X)$.

**Definition 2.1.24.** Let $X$ be a variety and $C$ be an irreducible reduced curve on $X$. If $C$ is a member of a flat algebraic family of curves dominating $X$, I will say that $C$ is a *movable* curve.

Movable curves play a key role in birational geometry. The following theorem gives a concrete link between movable curves and positivity notions for divisors.

**Theorem 2.1.25** ([BDPP04], Theorem 1.5). *Let $X$ be a projective variety. The cone of nef curves $\overline{NM}_1(X)$ is the closure of the cone generated by classes of movable curves.*

## 2.2 Divisorial Zariski Decomposition

For a pseudo-effective divisor $L$ on a surface, there is a classical decomposition of $L$ into a "positive part" and a "negative part" due to Zariski and Fujita. There have been many attempts to generalize this decomposition in higher dimensions. One important option was developed independently by Nakayama ([Nak04]) and by Boucksom ([Bou04]). In this section we outline the basics of this theory and then prove a couple important facts about the decomposition.

Both references consider only smooth varieties $X$. Unless noted otherwise, the proofs cited below will work for normal varieties $X$ with no changes.

**Definition 2.2.1.** Let $X$ be a normal variety and $L$ be a pseudo-effective divisor. For any prime divisor $\Gamma$, we define

$$\sigma_\Gamma(L) = \lim_{\epsilon \downarrow 0} v_\Gamma(\|L + \epsilon A\|).$$

One may verify that this definition is independent of the choice of $A$.

**Remark 2.2.2.** Since $v_\Gamma(\| - \|)$ is continuous on the big cone, when $L$ is big

$$\sigma_\Gamma(L) = v_\Gamma(\|L\|).$$

Thus $\sigma_\Gamma$ is a natural extension of the notion of asymptotic order of vanishing to the entire pseudo-effective cone. Although $\sigma_\Gamma$ is continuous on the big cone, in general it is only upper-continuous along the pseudo-effective boundary.

The following basic result about $\sigma_\Gamma$ is the key to defining the divisorial Zariski decomposition.

**Theorem 2.2.3** ([Nak04],[Bou04]). *Let $X$ be a normal variety. For any pseudo-effective $L$, there are only finitely many prime divisors $\Gamma$ with $\sigma_\Gamma(L) > 0$.*

This allows us to make the following definition:

**Definition 2.2.4.** Let $X$ be a normal variety, $L$ a pseudo-effective divisor. We define:

$$N_\sigma(L) = \sum \sigma_\Gamma(L)\Gamma \qquad\qquad P_\sigma(L) = L - N_\sigma(L)$$

The decomposition $L = N_\sigma(L) + P_\sigma(L)$ is called the divisorial Zariski decomposition of $L$.

**Remark 2.2.5.** We will soon demonstrate that $\mathrm{Supp}(N_\sigma(L))$ is exactly the divisorial components of $\mathbf{B}_-(L)$. Since the restricted base locus represents the obstruction to $L$ being nef, we may think of the positive part $P_\sigma(L)$ as being "nef in codimension 1." In this sense, it is a codimension 1 analogue of a Zariski decomposition for surfaces.

**Example 2.2.6.** When $L$ is nef, $L = P_\sigma(L)$. In contrast, if $E$ is an exceptional divisor of a blow-up then $E = N_\sigma(E)$.

**Example 2.2.7.** If $X$ is a smooth surface, then the divisorial Zariski decomposition of $L$ coincides with the usual Zariski decomposition. In particular, the support of $N_\sigma(L)$ is a union of curves with negative self-intersection matrix.

The following proposition records the basic properties of the divisorial Zariski decomposition.

**Proposition 2.2.8** ([Nak04]). *Let $X$ be a normal variety, $L$ a pseudo-effective divisor.*

1. *The negative part $N_\sigma(L)$ depends only on the numerical class of $L$.*
2. *$N_\sigma(L) \geq 0$ and $\kappa(X, N_\sigma(L)) = 0$.*
3. *$\mathrm{Supp}(N_\sigma(L))$ is precisely the divisorial part of $\mathbf{B}_-(L)$.*
4. *$H^0(X, \mathcal{O}_X(\lfloor mP_\sigma(L)\rfloor)) \to H^0(X, \mathcal{O}_X(\lfloor mL \rfloor))$ is an isomorphism for every $m \geq 0$.*

*Proof.* All four properties are proved for smooth varieties $X$ in [Nak04]. Since vanishing theorems are used in some of the proofs, we will check the normal case explicitly. Property (1) is a consequence of Theorem 2.1.2, which shows that asymptotic valuations are numerical invariants on the big cone. To check Property (3), let $\phi : Y \to X$ be a desingularization of $X$. Fix an ample $A$ on $X$; we want to analyze

$$\phi^{-1}\mathbf{B}_-(L) = \bigcup_m \mathbf{B}_\mathbb{R}\left(\phi^* L + \frac{1}{m}\phi^* A\right)$$

Let $T$ denote the $\phi$-exceptional locus. For any point $y$ of $Y$ not contained in $T$, Lemma 2.1.22 shows that there is an expression $\phi^*A \sim_\mathbb{R} H + E$ where $H$ is ample, $E$ is effective, and $y$ is not contained in the support of $E$. Thus, $\phi^{-1}\mathbf{B}_-(L)$ and $\mathbf{B}_-(\phi^*L)$ coincide away from $T$. In particular, the two sets share the same divisorial components outside of $T$, finishing the proof of Property (3). Properties (2) and (4) then follow immediately from the smooth case. $\square$

### 2.2.1 Movable Cone of Divisors

Note that $N_\sigma(L) = 0$ iff $\mathbf{B}_-(L)$ has no divisorial components. This simple observation leads to a different perspective on the divisorial Zariski decomposition.

**Definition 2.2.9.** Let $X$ be a normal variety. The movable cone $\overline{Mov}^1(X) \subset N^1(X)$ is the closure of the cone generated by the classes of all effective divisors $L$ such that $\mathbf{B}_-(L)$ has no divisorial components.

The following proposition helps clarify the terminology:

**Proposition 2.2.10** ([Bou04], Proposition 2.3). *Given any element $\alpha$ in the interior of $\overline{Mov}^1(X)$, there is a birational map $\phi : Y \to X$ and an ample divisor $A$ on $Y$ such that $[\phi_* A] = \alpha$.*

The positive part $P_\sigma(L)$ of the divisorial Zariski decomposition can be understood as a "projection" of $L$ onto the movable cone.

**Proposition 2.2.11** ([Nak04], Proposition III.1.14). *Let $X$ be normal, $L$ pseudo-effective. If $D$ is an effective divisor such that $L - D$ is movable, then $N_\sigma(L) \leq D$.*

We will also need a version that accounts for a subvariety $V$.

**Proposition 2.2.12.** *Let $X$ be normal, $V$ a subvariety, and $L$ a $V$-pseudo-effective divisor. If $M$ is a movable divisor then $L \geq_V M$ iff $P_\sigma(L) \geq_V M$. If furthermore $L - M$ is $V$-big, then $P_\sigma(L) - M$ is also $V$-big.*

*Proof.* First suppose that $P_\sigma(L) \geq_V M$. Since $L$ is $V$-pseudo-effective, no component of $N_\sigma(L)$ contains $V$. Thus $L \geq_V M$. Conversely, suppose $L = M + E$ with $E \geq_V 0$. Since $M$ is movable, $N_\sigma(L) \leq E$. Thus $E - N_\sigma(L)$ is still effective and does not contain $V$ in its support, showing that $P_\sigma(L) \geq_V M$. The final statement follows from the first by applying Lemma 2.1.19 to $M - A$ for a small ample divisor $A$. □

### 2.2.2 Birational Properties

Suppose that $\phi : Y \to X$ is a birational map. If some $\phi$-exceptional divisors are centered in $\mathbf{B}_-(L)$, then $\mathbf{B}_-(\phi^* L)$ will have additional codimension 1 components. Thus, we obtain the following birational relation:

**Proposition 2.2.13** ([Nak04], III.5.16). *Let $X$ be a normal variety and $L$ a $V$-pseudo-effective divisor. Suppose $\phi : Y \to X$ is a birational map. Then $N_\sigma(\phi^* L) - \phi^* N_\sigma(L)$ is effective and $\phi$-exceptional.*

The divisorial Zariski decomposition should really be considered as an object on all birational models of $X$ simultaneously (in other words, as a b-divisor). We expect the decomposition on higher models to satisfy particularly nice properties. Sometimes we are in a very close analogue to the surface case.

**Definition 2.2.14.** Let $X$ be a normal variety and $L$ be a pseudo-effective divisor. We say that $L$ has a *Zariski decomposition* if there is a birational map $\phi : Y \to X$ from a smooth variety $Y$ such that $P_\sigma(\phi^* L)$ is nef. We say that $L$ has a rational Zariski decomposition if $N_\sigma(\phi^* L)$ and $P_\sigma(\phi^* L)$ are $\mathbb{Q}$-divisors.

An example due to Nakayama ([Nak04], Section IV.2) shows that Zariski decompositions do not always exist even for big divisors. Nevertheless, there is a sense in which the positive part $P_\sigma(\phi^* L)$ becomes "more nef" as we pass to higher models $\phi : Y \to X$. We make this intuition explicit in the following proposition. In fact, we will make the comparison in the context of a subvariety $V$.

**Proposition 2.2.15.** *Let $X$ be smooth, $V$ a subvariety, and $L$ a $V$-big divisor with $L \geq_V 0$. There is a fixed divisor $G$ so that for any sufficiently large $m$ there is a $V$-birational model $\phi_m : \widetilde{X}_m \to X$ centered in $\mathbf{B}_+(L)$ and a big and nef divisor $N_m$ on $\widetilde{X}_m$ with*

$$N_m \leq_{\widetilde{V}_m} P_\sigma(\phi_m^* L) \leq_{\widetilde{V}_m} N_m + \frac{1}{m}\phi_m^* G$$

*where $\widetilde{V}_m$ denotes the strict transform of $V$ under $\phi_m$.*

The proof is based on the comparison between multiplier ideals and base loci in Theorem 2.1.8. The condition $L \geq_V 0$ is a technical requirement due to the fact that we work with $\mathbb{Q}$-linear equivalence. We will often apply this proposition in the context of $\mathbb{R}$-linear equivalence so that Lemma 2.1.19 makes this condition redundant.

*Proof.* Fix a very ample $\mathbb{Z}$-divisor $H$ and an integer $b$ as in Theorem 2.1.8. Thus, for any $m \geq b$ we have

$$\mathcal{J}(\|mL\|) \otimes \mathcal{O}_X(-G) \subseteq \mathfrak{b}(|\lfloor mL \rfloor|).$$

Recall that $G$ can be chosen to be any effective $\mathbb{Z}$-divisor numerically equivalent to $\lfloor bL \rfloor - (K_X + (n+1)H)$. In particular, by choosing $b$ large enough, we may choose $G$ so that the base locus of $|G|$ is contained in $\mathbf{B}_+(L)$. Since this set does not contain $V$, we may ensure that $G \geq_V 0$.

Let $\phi_m : \widetilde{X}_m \to X$ be a resolution of the ideals $\mathfrak{b}(|\lfloor mL \rfloor|)$ and $\mathcal{J}(\|mL\|)$. Note that each $\phi_m$ is centered in $\mathbf{B}_+(L)$ and is thus $V$-birational. We write $\phi_m^{-1}\mathfrak{b}(|\lfloor mL \rfloor|) \cdot \mathcal{O}_{Y_m} = \mathcal{O}_{Y_m}(-E_m)$ and $\phi_m^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{Y_m} = \mathcal{O}_{Y_m}(-F_m)$. We also define the big and nef divisor $M_m := m\phi_m^* L - E_m - \phi_m^*\{mL\}$.

We know that $F_m + \phi_m^* G \geq E_m$ for all sufficiently large $m$. Replacing $G$ by $G + L$ allows us to take into account the fractional part of $mL$ so that

$$F_m + \phi_m^* G \geq E_m + \phi_m^*\{mL\}$$

for large $m$. Note that we still have that $G \geq_V 0$. Furthermore, since $L$ is $V$-big we know that $F_m \geq_{\widetilde{V}_m} 0$. Thus, the inequality in the equation above is a $\widetilde{V}_m$-inequality. Furthermore $N_\sigma(m\phi_m^* L) \geq_{\widetilde{V}_m} F_m$. In all, we get $P_\sigma(m\phi_m^* L) \leq_{\widetilde{V}_m} M_m + \phi_m^* G$. Dividing by $m$ and setting $N_m := M_m/m$, we get $P_\sigma(\phi_m^* L) \leq_{\widetilde{V}_m} N_m + \frac{1}{m}\phi_m^* G$. The inequality $N_m \leq_{\widetilde{V}_m} P_\sigma(\phi_m^* L)$ follows from Proposition 2.2.12 and the fact that $E_m + \phi_m^*\{mL\} \geq_{\widetilde{V}_m} 0$. □

## 2.3 Restricted Volume

Perhaps the most basic numerical invariant that describes the asymptotic behavior of sections is the volume $\mathrm{vol}(L)$ of a big divisor. In this section we will explore a variant known as the restricted volume. This notion originated in the work of Hacon-M$^{c}$Kernan and Takayama. [ELM$^{+}$09] systematically develops the theory of this invariant, and we will often refer to it.

**Definition 2.3.1.** Suppose that $X$ is a projective variety, $V$ is a subvariety of $X$, and $\mathcal{F}$ is a coherent sheaf. We define the space $H^0(X|V,\mathcal{F})$ to be the image of the restriction map of sections of $\mathcal{F}$ from $X$ to $V$. That is,

$$H^0(X|V,\mathcal{F}) := \mathrm{Im}(H^0(X,\mathcal{F}) \to H^0(V,\mathcal{F}|_V)).$$

**Example 2.3.2.** Let $X$ be a variety and $V$ be a subvariety of $X$. Suppose that $\mathcal{F}$ is a locally free sheaf and $L$ a $\mathbb{Z}$-divisor satisfying $L \geq_V 0$. We will show that the map $\mathcal{F} \to \mathcal{F}(L)$ induces an injection

$$H^0(X|V,\mathcal{F}) \to H^0(X|V,\mathcal{F}(L))$$

Since $\mathcal{I}_V$ is prime, the only associated prime to $\mathcal{O}_V$ is $\mathcal{I}_V$ itself. By assumption $V$ is not contained in $\mathrm{Supp}(L)$. Thus the map $\mathcal{O}_V \to \mathcal{O}_V(D)$ is injective. Tensoring by $\mathcal{F}$ is exact, so the map $\mathcal{F}|_V \to \mathcal{F}(D)|_V$ is also injective. Since the commuting diagram

$$\begin{array}{ccc} H^0(X,\mathcal{F}) & \to & H^0(X,\mathcal{F}(D) \\ \downarrow & & \downarrow \\ H^0(V,\mathcal{F}|_V) & \to & H^0(V,\mathcal{F}(D)|_V) \end{array}$$

has injective rows, any element in the kernel of $H^0(X,\mathcal{F}) \to H^0(X|V,\mathcal{F}(D))$ must also lie in the kernel of the restriction map to $V$.

We now focus on the case when $\mathcal{F} = \mathcal{O}_X(\lfloor L \rfloor)$ for an $\mathbb{R}$-divisor $L$. Just as the volume measures the rate of growth of the space of sections of $L$, the restricted volume measures the rate of growth of space of restricted sections of $L$ to $V$.

**Definition 2.3.3.** Suppose that $X$ is a projective variety, $V$ is a $d$-dimensional subvariety of $X$, and $L$ is a divisor. We define the restricted volume $\mathrm{vol}_{X|V}(L)$ to be

$$\mathrm{vol}_{X|V}(L) := \limsup_{m\to\infty} \frac{h^0(X|V,\mathcal{O}_X(\lfloor mL \rfloor))}{m^d/d!}.$$

**Example 2.3.4.** When $L$ is an ample $\mathbb{Z}$-divisor, the restricted volume can be calculated by taking intersections. Since $H^1(X,\mathcal{I}(V)\otimes\mathcal{O}_X(mL))$ vanishes for $m$ sufficiently large, we find $\mathrm{vol}_{X|V}(L) = \mathrm{vol}_V(L|_V) = L^d \cdot V$.

The restricted volume gives us a more precise tool for understanding the numerical properties of a divisor $L$. As with the other quantities we consider, it turns out to be a numerical and birational invariant.

**Proposition 2.3.5** ([ELM+09],Lemma 2.4). *Let $X$ be a normal variety, $V$ a subvariety not contained in* $\mathrm{Sing}(X)$*, and $L$ any divisor on $X$. Suppose that $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a $V$-birational model. Then* $\mathrm{vol}_{\widetilde{X}|\widetilde{V}}(\phi^* L) = \mathrm{vol}_{X|V}(L)$.

**Theorem 2.3.6** ([ELM+09], Theorem A). *Let $X$ be a normal variety, $V$ a subvariety not contained in* $\mathrm{Sing}(X)$*, and $L$ a $V$-big divisor. Then* $\mathrm{vol}_{X|V}(L) > 0$*, and this quantity only depends on the numerical class of $L$. Furthermore,* $\mathrm{vol}_{X|V}(L)$ *varies continuously on the $V$-big cone.*

In some cases, the limit as $L$ approaches the boundary of the $V$-big cone can also be understood. More precisely, the restricted volume identifies the maximal components of $\mathbf{B}_+(L)$.

**Theorem 2.3.7** ([ELM+09], Theorem C). *If $X$ is smooth and $L$ is a divisor on $X$, then $\mathbf{B}_+(L)$ is the union of all positive dimensional subvarieties $V$ such that* $\mathrm{vol}_{X|V}(L) = 0$.

In general, it does not seem possible to say much for $V \subset \mathbf{B}_+(L)$, even when $V$ is not contained in the stable base locus of $L$. [ELM+09] gives an example of a base point free $L$ and a curve $C$ such that $\mathrm{vol}_{X|C}$ is not continuous near $L$.

**Example 2.3.8** ([ELM+09], Example 5.10). Let $f : X \to \mathbb{P}^3$ be the blow-up along a line $l$. The exceptional divisor $E$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and we identify $f|_E$ with projection onto the first component. We let $L$ be the pullback of a hyperplane $H$ on $\mathbb{P}^3$. Note that $L$ is big and base point free but $\mathbf{B}_+(L) = E$.

Let $C$ be a curve of type $(2, 1)$ in $E$. By Theorem 2.3.5 $h^0(X|C, \mathcal{O}_X(mL)) = h^0(\mathbb{P}^3|l, \mathcal{O}_{\mathbb{P}^3}(mH)) = m+1$, so that the restricted volume is 1. However, consider the ample divisors $L_j := L - \frac{1}{j}E$. Since the restricted volume for an ample divisor is just an intersection number, $\mathrm{vol}_{X|C}(L_j) = L_j \cdot C = 2 + \frac{1}{j}$. Thus the restricted volume is not continuous near $L$.

Despite this difficulty [PT09] gives the following characterization of when $\mathrm{vol}_{X|V}(L) > 0$ for a general $V$:

**Theorem 2.3.9** ([PT09], Theorem 1.1). *Let $X$ be a smooth variety. Suppose $L$ is a $\mathbb{Q}$-divisor with $\kappa(X, L) \geq 0$ and suppose $V$ passes through a general point of $X$. Then* $\mathrm{vol}_{X|V}(L) > 0$ *iff the Iitaka fibration of $L$ is generically finite on $V$.*

## 2.4 The Restricted Positive Product

Fujita realized that one can study the asymptotic behavior of sections of a big divisor $L$ by analyzing the ample divisors sitting beneath $L$ on higher birational models. The positive product (developed in [Bou04] and [BDPP04]) is a construction that encapsulates this approach to asymptotic behavior. In this section, we'll consider the restricted positive product constructed in [BFJ09].

Conceptually speaking, the restricted positive product $\langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \rangle_{X|V}$ is a class on $V$ that represents the "positive intersection" of the $L_i$ along $V$. In contrast to the usual intersection product $L_1 \cdot L_2 \cdot \ldots \cdot L_k \cdot V$, the restricted positive product throws away the contributions of the base loci of the $L_i$. The result is a rational equivalence class of cycles that gives a more precise measure of the positivity of the $L_i$ along $V$. In order to avoid working with the singularities of $V$, we will define the restricted positive product as a collection of classes on smooth $V$-birational models that are compatible under pushforward.

### 2.4.1 Approximating Big Divisors by Ample Divisors

Any big divisor can be written $L \sim_{\mathbb{R}} A+E$ for an ample divisor $A$ and effective divisor $E$. In general, the ample part $A$ will not have a particularly close relationship to $L$. However, if one allows birational modifications, it turns out that one can find such an expression so that $A$ is a close approximation to $L$. We first make our terminology precise.

**Definition 2.4.1.** Let $X$ be a projective variety and $L$ a big divisor. A Fujita approximation of $L$ is a birational model $\phi : Y \to X$ and a decomposition $\phi^*L \sim_{\mathbb{R}} A + E$ where $A$ is an ample divisor and $E$ is an effective divisor. We will sometimes refer to $A$ as the Fujita approximation if the other data are implicit.

The basic realization of Fujita is that the volume of a big divisor $L$ can be approximated arbitrarily well by the volume of a Fujita approximation $A$. In fact, many other asymptotic constructions can also be calculated using Fujita approximations. In this section, our goal is to show that for smooth varieties the positive product can be computed using only Fujita approximations $\phi : Y \to X$ centered in $\mathbf{B}_-(L)$. The first step is the following proposition:

**Proposition 2.4.2.** *Let $X$ be a smooth variety, $V$ a subvariety of $X$, and $L$ a $V$-big divisor. Then there is a sequence of $V$-birational maps $\phi_m : \widetilde{X}_m \to X$ centered in $\mathbf{B}_-(L)$ and Fujita approximations $\phi_m^*L \sim_{\mathbb{R}} A_m + E_m$ with $E_m \geq_{\widetilde{V}} 0$ such that the limit $\lim_{m\to\infty} \phi_{m*}A_m$ exists and satisfies*

$$\lim_{m\to\infty} \phi_{m*}A_m \sim_{\mathbb{R}} P_\sigma(L).$$

This proposition is well-known if we allow centers in $\mathbf{B}_+(L)$ (for example, it is a consequence of Proposition 2.2.15). An easy rescaling argument allows us to pass to $\mathbf{B}_-(L)$.

*Proof.* Recall the result of Proposition 2.2.15. Assuming that $L \geq_V 0$, the proposition allows us to find $\phi_m$ centered in $\mathbf{B}_+(L)$ and big and nef divisors $N_m$ such that $N_m \leq_{\widetilde{V}} \phi_m^*L$. Furthermore, $[\phi_{m*}N_m]$ converges to $[P_\sigma(L)]$ in the natural metric on $N^1(X)$. As a consequence, we see that $\lim_{m\to\infty} \phi_{m*}N_m = P_\sigma(L)$.

In order to construct the $A_m$, we perturb the situation slightly. Since $L$ is $V$-big, we may write $L \sim_{\mathbb{R}} H + F$ for some ample divisor $H \geq_V 0$ and some $V$-big divisor $F$ with $F \geq_V 0$. Define $D_j := H + \frac{j}{j+1}F$. Since $D_j \equiv \frac{j}{j+1}L + \frac{1}{j+1}H$, there is a containment $\mathbf{B}_+(D_j) \subset \mathbf{B}_-(L)$.

Since $D_j \geq_V 0$, we may apply Proposition 2.2.15 to $D_j$ to find a sequence of $V$-birational maps $\phi_{m,j}$ centered in $\mathbf{B}_-(L)$ and big and nef divisors $N_{m,j} \leq_{\widetilde{V}_{m,j}} \phi^*_{m,j} D_j$ such that

$$\lim_{m \to \infty} \phi_{m,j*} N_{m,j} = P_\sigma(D_j).$$

Consider $\phi^*_{m,j} L \sim_{\mathbb{R}} \phi^*_{m,j} D_j + \frac{1}{j+1} \phi^*_{m,j} F$. Since $F$ is $V$-big, we may remove a small ample and add it to $N_{m,j}$ to write $\phi^*_{m,j} L \sim_{\mathbb{R}} A_{m,j} + E_{m,j}$ where

- $A_{m,j}$ is ample and $\lim_{m \to \infty} \phi_{m,j*} A_{m,j} = P_\sigma(D_j)$.
- $E_{m,j}$ is $\widetilde{V}_{m,j}$-big and so by Lemma 2.1.19 we may assume $E_{m,j} \geq_{\widetilde{V}_{m,j}} 0$.

Note that $P_\sigma(D_j)$ converges to $P_\sigma(H+F)$ as $j$ increases. Since $P_\sigma(H+F) \sim_{\mathbb{R}} P_\sigma(L)$, a subsequence of the $A_{m,j}$ will suffice. □

### 2.4.2 Definition and Basic Properties

The restricted positive product $\langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \rangle_{X|V}$ is a rational equivalence class of cycles on $V$ that represents the intersections of the "positive parts" of the $L_i$ with $V$. Since we do not make any assumptions on the singularities of $V$, it is best to think of the restricted positive product as a collection of classes associated to smooth $V$-birational models $(\widetilde{X}, \widetilde{V})$ that is compatible under pushforward.

Since we will work with cycles of arbitrary codimension, we will need the following notation:

**Definition 2.4.3.** Let $X$ be a smooth variety of dimension $n$. Suppose that $K$, $K'$ are two classes in $A^{n-k}(X)$. We write $K \succeq K'$ if $K - K'$ is contained in the closure of the cone generated by effective cycles of dimension $k$. We will use the same notation for numerical equivalence classes of cycles.

The following lemma from [BFJ09] lies at the heart of the restricted positive product.

**Lemma 2.4.4.** *Let $X$ be a smooth variety, $V$ a smooth subvariety, and $L_1, \ldots, L_k$ $V$-big divisors. Consider the classes*

$$\phi_*(N_1 \cdot N_2 \cdot \ldots \cdot N_k \cdot \widetilde{V}) \in A^k(V)$$

*where $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ varies over all smooth $V$-birational models, the $N_i$ are nef, and $E_i := \phi^* L_i - N_i$ is a $\mathbb{Q}$-divisor satisfying $E_i \geq_{\widetilde{V}} 0$. Then these classes form a directed set under the relation $\preceq$.*

**Corollary 2.4.5.** *The classes*

$$\phi_*(N_1 \cdot N_2 \cdot \ldots \cdot N_k \cdot \widetilde{V}) \in A^k(V)$$

*admit a unique maximum as $(\widetilde{X}, \widetilde{V})$ varies over all smooth $V$-birational models.*

*Proof.* Following [BFJ09], we first prove that the numerical classes

$$[\phi_*(N_1 \cdot N_2 \cdot \ldots \cdot N_k \cdot \widetilde{V})] \in N^k(V)$$

vary in a compact set. Suppose that we choose ample divisors $A_i$ on $X$ such that $A_i - L_i$ is ample and effective. Then for any $N_i$ as in the statement we have $N_i \leq \phi^* A_i$. By the push-pull formula every class satisfies

$$\phi_*(N_1 \cdot N_2 \cdot \ldots \cdot N_k \cdot \widetilde{V}) \preceq A_1 \cdot \ldots \cdot A_k \cdot V$$

The set of all numerical classes satisfying $0 \preceq [\alpha] \preceq [A_1 \cdot \ldots \cdot A_k \cdot V]$ is compact since $N^k(V)$ is finite-dimensional.

Next, we note that the set of rational equivalence classes $K$ satisfying $0 \preceq K$ is a proper cone. In particular, suppose that $\alpha_i$ is a directed set of rational equivalence classes under $\preceq$ whose numerical classes converge. Then the maximum of the $\alpha_i$ under $\preceq$ is uniquely defined as a rational equivalence class. □

The restricted positive product is defined as the maximum class occurring in the previous corollary.

**Definition 2.4.6.** Let $X$ be a smooth variety and $V$ be a smooth subvariety. Let $L_1, L_2, \ldots, L_k$ be $V$-big divisors. We define the cycle

$$\langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \rangle_{X|V} \in A^k(V)$$

to be the maximum over all admissible models $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ of

$$\phi_*(N_1 \cdot N_2 \cdot \ldots \cdot N_k \cdot \widetilde{V})$$

where the $N_i$ are nef and $E_i := \phi^* L_i - N_i$ is a $\mathbb{Q}$-divisor satisfying $E_i \geq_{\widetilde{V}} 0$. In the special case $X = V$, we write $\langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \rangle_X$.

If $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a smooth $V$-birational model, then

$$\phi_* \langle \phi^* L_1 \cdot \ldots \cdot \phi^* L_k \rangle_{\widetilde{X}|\widetilde{V}} = \langle L_1 \cdot \ldots \cdot L_k \rangle_{X|V}.$$

Thus, if $X$ is normal and $V$ is any subvariety not contained in $\mathrm{Sing}(X)$, we define the restricted positive product as the collection of classes $\langle \phi^* L_1 \cdot \ldots \cdot \phi^* L_k \rangle_{\widetilde{X}|\widetilde{V}}$ as $(\widetilde{X}, \widetilde{V})$ varies over smooth $V$-birational models.

**Remark 2.4.7.** As we consider all $V$-birational models $\phi : \widetilde{X} \to X$, the $N_i|_{\widetilde{V}}$ will be big and nef, but not every big and nef divisor $H_i \leq \phi^*(L_i|_V)$ can be obtained in this way. Thus we see that

$$\langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \rangle_{X|V} \preceq \langle L_1|_V \cdot L_2|_V \cdot \ldots \cdot L_k|_V \rangle_V$$

In general equality will not hold.

The restricted positive product satisfies a number of important properties.

**Proposition 2.4.8** ([BFJ09],Proposition 4.6). *The restricted positive product is symmetric, homogeneous of degree* 1*, super-additive in each variable, and continuous on the p-fold product of the $V$-big cone.*

Since the product is continuous, this allows us to define a limit as we approach the pseudo-effective cone.

**Definition 2.4.9.** Let $X$ be a normal variety, $V$ a subvariety not contained in $\mathrm{Sing}(X)$, and $L_1, L_2, \dots, L_k$ $V$-pseudo-effective divisors. Fix an ample divisor $A$ on $X$. We define the class

$$\langle L_1 \cdot L_2 \cdot \dots \cdot L_k \rangle_{X|V} = \lim_{\epsilon \to 0} \langle (L_1 + \epsilon A) \cdot (L_2 + \epsilon A) \cdot \dots \cdot (L_d + \epsilon A) \rangle_{X|V}.$$

**Remark 2.4.10.** It is easy to see that this definition is independent of the choice of $A$. In fact, suppose that $B_i$ is any sequence of $V$-big divisors converging to 0. Then by sandwiching the $B_i$ between suitably chosen ample divisors, we see that

$$\lim_{i \to \infty} \langle (L_1 + B_i) \cdot (L_2 + B_i) \cdot \dots \cdot (L_d + B_i) \rangle_{X|V} = \langle L_1 \cdot L_2 \cdot \dots \cdot L_k \rangle_{X|V}.$$

As a consequence, we see that even when the $L_i$ are only $V$-pseudo-effective the restricted positive product still forms a compatible system under pushforward so that our definition makes sense when $X$ is singular.

In general the restricted positive product is only upper semi-continuous along the $V$-pseudo-effective boundary.

**Remark 2.4.11.** [KM08] gives an alternative extension of the restricted positive product to the pseudo-effective boundary. The main theorem of [KM08] shows that for any two effective nef $\mathbb{Q}$-divisors $N_1, N_2$ there is some higher model $\phi : Y \to X$ so that the coefficient-wise maximum $\max(\phi^* N_1, \phi^* N_2)$ is also nef without any bigness hypotheses. Thus, we can define $\langle L_1 \cdot \dots \cdot L_d \rangle^*$ to be the maximum of pushforwards of all products $N_1 \cdot \dots \cdot N_d$ on higher models, where the $N_i \leq \phi^* L_i$ are effective nef $\mathbb{Q}$-divisors. When the $L_i$ are big this coincides with the positive product. It is unclear whether the two coincide when the $L_i$ are only effective.

For our purposes, the most important case of the restricted positive product is the following:

**Proposition 2.4.12** ([ELM+09], Proposition 2.11). *Let $X$ be a normal variety, $V$ a d-dimensional subvariety not contained in* $\mathrm{Sing}(X)$*, and $L$ a $V$-big divisor. Then* $\deg \langle L^d \rangle_{X|V} = \mathrm{vol}_{X|V}(L)$.

We will need a few more properties of the restricted positive product. The first is the observation that when the $L_i$ are nef, the restricted positive product reduces to the usual intersection product.

**Lemma 2.4.13.** *Let $X$ be a projective variety, $V$ a subvariety, and $L_1, \ldots, L_k$ $V$-pseudo-effective divisors. Suppose $N$ is a nef divisor. Then*

$$\langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \cdot N \rangle_{X|V} = \langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \rangle_{X|V} \cdot N.$$

*If $H$ is a very general element of a basepoint free linear system, then*

$$\langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \rangle_{X|V} \cdot H = \langle L_1 \cdot L_2 \cdot \ldots \cdot L_k \rangle_{X|V \cap H}.$$

*Proof.* The first property is shown in [BFJ09], Proposition 4.7. To show the second, consider a countable set of smooth $V$-birational models $\phi_m : \widetilde{X}_m \to X$ such that the restricted positive product can be computed using nef divisors on the $\widetilde{X}_m$. Choose $H$ sufficiently general so that it does not contain any $\phi_m$-exceptional center. Then the strict transform of $V \cap H$ is a cycle representing the class $\phi_m^* H \cdot \widetilde{V}$. Thus we can identify the classes

$$\begin{aligned} \phi_{m*}(N_1 \cdot N_2 \cdot \ldots \cdot N_k \cdot \widetilde{V} \cdot H) &= \phi_{m*}(N_1 \cdot N_2 \cdot \ldots \cdot N_k \cdot \phi_m^* H \cdot \widetilde{V}) \\ &= \phi_{m*}(N_1 \cdot N_2 \cdot \ldots \cdot N_k \cdot \widetilde{V \cap H}) \end{aligned}$$

□

The restricted positive product also has a natural compatibility with the divisorial Zariski decomposition.

**Proposition 2.4.14.** *Let $X$ be a normal variety, $V$ a subvariety not contained in* $\mathrm{Sing}(X)$, *and $L_1, \ldots, L_k$ $V$-pseudo-effective divisors. Then*

$$\langle L_1 \cdot \ldots \cdot L_k \rangle_{X|V} = \langle P_\sigma(L_1) \cdot \ldots \cdot P_\sigma(L_k) \rangle_{X|V}.$$

*Proof.* First suppose that the $L_i$ are $V$-big. Since any nef divisor is movable, Proposition 2.2.12 shows that for any of the $N_i$ as in Definition 2.4.6 we have $P_\sigma(\phi^* L_i) \geq_{\widetilde{V}} N_i$. We also know that $N_\sigma(\phi^* L_i) \geq_{\widetilde{V}} \phi^* N_\sigma(L_i)$ since $V$ is not contained in $\mathbf{B}_-(L_i)$. Combining the two inequalities yields

$$\phi^* P_\sigma(L_i) \geq_{\widetilde{V}} N_i.$$

Thus the two classes are computed by taking a maximum over the same sets, showing that they are equal.

Now suppose that the $L_i$ are only $V$-pseudo-effective. Fix an ample divisor $A$ on $X$. We will show that

$$\langle P_\sigma(L_1) \cdot \ldots \cdot P_\sigma(L_k) \rangle_{X|V} = \lim_{\epsilon \to 0} \langle P_\sigma(L_1 + \epsilon A) \cdot \ldots \cdot P_\sigma(L_k + \epsilon A) \rangle_{X|V}.$$

Combining this equality with the $V$-big case finishes the proof. Note that

$$P_\sigma(L + \epsilon A) - P_\sigma(L) = \epsilon A + (N_\sigma(L) - N_\sigma(L + \epsilon A))$$

is $V$-big. Thus, our claim follows from Remark 2.4.10, which shows that if $\{B_j\}$ is any sequence of $V$-big divisors converging to 0 then

$$\langle P_\sigma(L_1)\cdot\ldots\cdot P_\sigma(L_k)\rangle_{X|V} = \lim_{j\to\infty}\langle (P_\sigma(L_1)+B_j)\cdot\ldots\cdot(P_\sigma(L_d)+B_j)\rangle_{X|V}.$$

□

### 2.4.3 Alternate Definitions

Since the restricted positive product is continuous, we can alter the definitions slightly,

**Theorem 2.4.15** ([BFJ09], Proposition 2.13). *Suppose that $X$ is normal, $V$ a subvariety not contained in* $\mathrm{Sing}(X)$*, and $L_1, L_2, \ldots, L_k$ are $V$-big divisors. Then the following cycles in $A^k(V)$ coincide:*

1. $\langle L_1\cdot L_2\cdot\ldots\cdot L_k\rangle_{X|V}$.
2. *The maximum over all $V$-birational models $\phi:\widetilde{X}\to X$ of*

$$\phi_*(N_1\cdot N_2\cdot\ldots\cdot N_d\cdot\widetilde{V})$$

*where $N_i$ is nef and $\phi^*L_i - N_i$ is $V$-pseudo-effective.*

There is another characterization of the restricted positive product that emphasizes the relationship with the divisorial Zariski decomposition.

**Theorem 2.4.16.** *Let $X$ be a normal variety, $V$ a subvariety not contained in* $\mathrm{Sing}(X)$*, and $L_1,\ldots,L_k$ $V$-pseudo-effective divisors. Then*

$$\langle L_1\cdot\ldots\cdot L_k\rangle_{X|V} = \inf_\phi \phi_*\langle P_\sigma(\phi^*L_1)|_{\widetilde{V}}\cdot\ldots\cdot P_\sigma(\phi^*L_k)|_{\widetilde{V}}\rangle_{\widetilde{V}}$$

*where $\phi:(\widetilde{X},\widetilde{V})\to(X,V)$ ranges over all smooth $V$-birational models.*

**Remark 2.4.17.** Since $P_\sigma(\phi^*L)\le\phi^*P_\sigma(L)$, the classes on the right satisfy a filtering property. Therefore the infimum on the right hand side exists.

*Proof of Theorem 2.4.16:* It suffices to consider the case when $X$ and $V$ are smooth. Proposition 2.4.14 shows that the restricted positive product is invariant under passing to $P_\sigma(L_i)$. As remarked earlier, on any $V$-birational model $\phi:(\widetilde{X},\widetilde{V})\to(X,V)$

$$\langle P_\sigma(\phi^*L_1)\cdot\ldots\cdot P_\sigma(\phi^*L_d)\rangle_{\widetilde{X}|\widetilde{V}}\preceq\langle P_\sigma(\phi^*L_1)|_{\widetilde{V}}\cdot\ldots\cdot P_\sigma(\phi^*L_d)|_{\widetilde{V}}\rangle_{\widetilde{V}}$$

This shows the inequality $\preceq$. Conversely, recall that Proposition 2.2.15 shows that (up to $\mathbb{R}$-linear equivalence) there is a divisor $G$ such that for any $m$ there is a $V$-birational model $\phi:Y\to X$ with

$$N_{m,i}\le_{\widetilde{V}} P_\sigma(\phi_m^*L_i)\le_{\widetilde{V}} N_{m,i}+\frac{1}{m}\phi_m^*G$$

for some nef divisors $N_{m,i}$. By definition

$$\phi_*(N_{m,1} \cdot \ldots \cdot N_{m,d} \cdot \widetilde{V}) \preceq \langle L_1 \cdot \ldots \cdot L_d \rangle|_{X|V}.$$

Taking a limit as $m$ increases proves the theorem. □

## 2.5 Twisted Linear Series

It was observed by Iitaka that linear series of the form $|\lfloor mL + A \rfloor|$ play an important role in governing the numerical behavior of $L$. Due to the presence of the auxiliary divisor $A$, we call these "twisted" linear series. Twisted linear series are much more subtle than the classical case. There are a number of basic questions about such series that remain open.

We begin by studying the asymptotic behavior of $H^0(X, \mathcal{O}_X(\lfloor mL + A \rfloor))$ as $m$ grows. This behavior is governed by a "twisted Iitaka dimension." Following Nakayama, we denote this quantity by $\kappa_\sigma$. We will see later that $\kappa_\sigma$ gives one of the many equivalent notions of numerical dimension.

**Definition 2.5.1.** Let $X$ be a normal variety, $L$ a pseudo-effective $\mathbb{R}$-divisor, and $A$ any divisor. If $H^0(X, \mathcal{O}_X(\lfloor mL + A \rfloor))$ is non-zero for infinitely many values of $m$, we define

$$\kappa_\sigma(L; A) := \max \left\{ k \in \mathbb{Z}_{\geq 0} \,\middle|\, \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(\lfloor mL + A \rfloor))}{m^k} > 0 \right\}.$$

Otherwise we define $\kappa_\sigma(L; A) = -\infty$. The $\sigma$-dimension $\kappa_\sigma(X, L)$ is defined to be

$$\kappa_\sigma(L) := \max_A \{\kappa_\sigma(L; A)\}.$$

Since we are interested in perturbations by ample divisors, we will only consider the case when $A$ is a sufficiently ample $\mathbb{Z}$-divisor.

**Remark 2.5.2.** Note that as we increase $m$ the class of the divisor $\lceil mL \rceil - \lfloor mL \rfloor$ is bounded. Thus if we replace $\lfloor - \rfloor$ by $\lceil - \rceil$ in Definition 2.5.1, the result is unchanged as the difference can be absorbed by the divisor $A$.

The basic result about $\kappa_\sigma$ is the following:

**Proposition 2.5.3** ([Nak04], V.1.4). *Let $X$ be smooth. Fix a very ample divisor $H$, and suppose that $A$ is an ample $\mathbb{Z}$-divisor such that $A - K_X - (n+1)H$ is ample. Then a divisor $L$ is pseudo-effective iff $h^0(X, \mathcal{O}_X(\lceil mL \rceil + A)) > 0$ for every $m \geq 0$.*

In the next section we will prove that $\kappa_\sigma$ is a birational invariant. Thus, $\kappa_\sigma(L) \geq 0$ iff $L$ is pseudo-effective. Our main theorem shows that $\kappa_\sigma$ coincides with the numerical dimension, so we will see that all the properties of Theorem 2.0.4 hold for $\kappa_\sigma$.

By analogy to the non-twisted case, one wonders if there is a polynomial in $m$ of degree $\kappa_\sigma(L)$ that is an upper bound on the rate of growth of $H^0(X, \mathcal{O}_X(\lfloor mL \rfloor + A))$.

**Question 2.5.4.** Suppose that $L$ is pseudo-effective with $\kappa_\sigma(L) = k$. Is it true that for some sufficiently ample divisor $A$ there are positive constants $\alpha$ and $\beta$ such that

$$\alpha m^k \leq h^0(X, \mathcal{O}_X(\lfloor mL + A \rfloor)) \leq \beta m^k$$

for all sufficiently divisible $m$?

### 2.5.1 Restricted Twisted Linear Series

As usual, we will also be interested in a restricted notion that measures growth along a subvariety $V$.

**Definition 2.5.5.** Let $X$ be a normal variety, $V$ a subvariety not contained in $\mathrm{Sing}(X)$, $L$ a pseudo-effective $\mathbb{R}$-divisor, and $A$ any divisor. If $H^0(X|V, \mathcal{O}_X(\lfloor mL + A \rfloor))$ is non-zero for infinitely many values of $m$, we define

$$\kappa_\sigma(X|V, L; A) := \max\left\{ k \in \mathbb{Z}_{\geq 0} \,\middle|\, \limsup_{m\to\infty} \frac{h^0(X|V, \mathcal{O}_X(\lfloor mL + A \rfloor))}{m^k} > 0 \right\}.$$

Otherwise, we define $\kappa_\sigma(X|V, L; A) = -\infty$. The restricted $\sigma$-dimension $\kappa_\sigma(X|V, L)$ is defined to be

$$\kappa_\sigma(X|V, L) := \max_A \{\kappa_\sigma(X|V, L; A)\}.$$

Our first goal is to show that the restricted $\sigma$-dimension is a birational invariant. We will imitate the proof of [Nak04], V.2.7.

**Proposition 2.5.6.** *Let $X$ be a normal variety, $V$ a subvariety not contained in $\mathrm{Sing}(X)$, and $L$ a pseudo-effective divisor. Suppose that $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a $V$-birational model. Then*

$$\kappa_\sigma(X|V, L) = \kappa_\sigma(\widetilde{X}|\widetilde{V}, \phi^* L).$$

*Proof.* For any ample $\mathbb{Z}$-divisor $A$ on $X$, there is an ample $\mathbb{Z}$-divisor $H$ on $\widetilde{X}$ with $H - \phi^* A$ ample and effective. Thus we can naturally identify

$$H^0(X, \mathcal{O}_X(\lfloor mL \rfloor + A)) \subset H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\lfloor m\phi^* L \rfloor + H)).$$

Since there is an injection $H^0(V, \mathcal{O}_V((\lfloor mL \rfloor + A)|_V) \subset H^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}((\lfloor m\phi^* L \rfloor + H)|_{\widetilde{V}}))$, we see that $\kappa_\sigma(X|V, L) \leq \kappa_\sigma(\widetilde{X}|\widetilde{V}, \phi^* L)$.

Conversely, let $H$ be an ample divisor on $\widetilde{X}$. Choose an ample divisor $A$ on $X$ such that $A \geq_V \phi_* H$. Note that $\phi^* \lceil mL \rceil - \lfloor m\phi^* L \rfloor$ is effective and $\phi$-exceptional. Since every $\phi$-exceptional divisor $E$ satisfies $E \geq_{\widetilde{V}} 0$, we have $\lfloor m\phi^* L \rfloor + H \leq_{\widetilde{V}} \phi^*(\lceil mL \rceil + A)$. Thus

$$H^0(\widetilde{X}|\widetilde{V}, \mathcal{O}_{\widetilde{X}}(\lfloor m\phi^* L \rfloor + H) \subset H^0(\widetilde{X}|\widetilde{V}, \mathcal{O}_{\widetilde{X}}(\phi^*(\lceil mL \rceil + A)).$$

If a section of $\mathcal{O}_{\widetilde{X}}(\phi^*(\lceil mL \rceil + A))$ vanishes along $\widetilde{V}$, the corresponding section on $X$

vanishes along $V$. So

$$h^0(\widetilde{X}|\widetilde{V}, \mathcal{O}_{\widetilde{X}}(\lfloor m\phi^*L \rfloor + H)) \leq h^0(X|V, \mathcal{O}_X(\lceil mL \rceil + A))$$

and our conclusion follows from Remark 2.5.2. □

Conjecturally, restricted $\kappa_\sigma$ coincides with the restricted numerical dimension. Unfortunately, I have only been able to show that $\nu_{X|V}(L) \leq \kappa_\sigma(X|V, L)$. The following question seems to be closely related to the reverse inequality.

**Question 2.5.7.** Let $X$ be a smooth variety, $V$ a subvariety of $X$, and $L$ a $V$-pseudo-effective divisor. Is it true that

$$\kappa_\sigma(X|V, L) = \kappa_\sigma(X|V, P_\sigma(L))?$$

Despite this gap in our knowledge, one can show that restricted $\kappa_\sigma$ satisfies many properties similar to those of $\kappa_\sigma$.

### 2.5.2 Comparisons with Other Asymptotic Invariants

Our next goal is to relate restricted twisted linear series to our earlier constructions. We first record an easy consequence of Nadel vanishing.

**Lemma 2.5.8.** *Let $X$ be a smooth variety and $V$ be a smooth subvariety. Suppose that $L$ is a big $\mathbb{Z}$-divisor and $D \equiv L$ is an effective divisor such that $V$ is an isolated component of the support of $\mathcal{J}(D)$. Then the restriction maps $H^0(X, \mathcal{O}_X(K_X+L)) \to H^0(V, \mathcal{O}_V(K_X + L))$ are surjective.*

The following proposition gives a lower bound on the dimension of the space of restricted sections.

**Proposition 2.5.9.** *Let $X$ be a smooth variety, $V$ a smooth subvariety, and $L$ a $V$-big divisor satisfying $L \geq_V 0$. Then there is an ample divisor $A$ depending only on $X$ such that for any $V$-birational model $\phi : \widetilde{X} \to X$ and any big and nef divisor $N$ satisfying $0 \leq_{\widetilde{V}} N \leq_{\widetilde{V}} \phi^*L$ after passing to a higher $V$-birational model $\check{X}$ we have*

$$H^0(\check{V}, \mathcal{O}_{\check{V}}(\lceil N \rceil + \phi^*A)) \subset H^0(X|V, \mathcal{O}_X(\lceil L \rceil + A))$$

*Proof.* Since $N \geq_{\widetilde{V}} 0$, by passing to a higher $V$-birational model we may assume that $N$ has simple normal crossing support. We may also assume that $\mathrm{Supp}(N)$ has transverse intersection with $V$.

Let $f : Y \to X$ be the blow-up of $V$ with exceptional divisor $F$. Set $k$ to be the codimension of $V$ so that $K_{Y/X} = (k-1)F$. Choose an ample divisor $H$ on $Y$ such that $H + kF$ is $f$-numerically trivial. We then choose an ample $\mathbb{Z}$-divisor $A_1$ such that $A_1 - f_*(H+kF)$ is ample. We choose ample $A_2$ such that $A_2 - K_X$ is also ample.

Note that the map $\phi : \widetilde{X} \to X$ may have some exceptional centers contained in $V$. Let $E_1, \ldots, E_k$ be the exceptional divisors above such centers. Choose $a_i$ so that $v_{E_i}(\phi^*A_1 - a_iE_i) = 0$. Each $a_i$ is an integer no larger than $k$. We define $E = \sum a_iE_i$.

Note that $E \leq K_{\widetilde{X}/X}$ since every exceptional center contained in $V$ has codimension at least $k+1$.

The next step is a multiplier ideal calculation. Let $\psi : W \to \widetilde{X}$ be a log resolution of $\mathrm{Supp}(N) \cup \mathrm{Supp}(\phi^* A_1) \cup \mathrm{exc}(\phi)$. By passing to a higher model, we may assume that there is a map $g : W \to Y$ that is a composition of blow-ups along smooth centers. We let $F_W$ denote the strict transform of $F$. There is a divisor $A_W \equiv \psi^*\phi^* A_1$ such that $A_W \geq \psi^* E$ and $\lfloor A_W \rfloor = kg^* F$. Now, $\lfloor A_W \rfloor - \psi^* E$ only has positive coefficients over $\psi$-exceptional divisors. In particular, $K_{W/\widetilde{X}} - \lfloor A_W \rfloor + \psi^* E \geq -F_W$. Since the coefficients of $\psi^* \lceil N \rceil - \psi^* N$ over $\psi$-exceptional divisors are fractional, we find that there is some divisor $M \equiv \lceil N \rceil + \phi^* A_1 - E$ with $\mathcal{J}(M) = \mathcal{I}_{\widetilde{V}}$.

Applying Lemma 2.5.8 to $\lceil N \rceil + (\phi^* A_1 - E) + \phi^*(A_2 - K_X)$, we see that the restriction map

$$
\begin{aligned}
&H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + \lceil N \rceil + (\phi^* A_1 - E) + \phi^*(A_2 - K_X))) \\
&\qquad \to H^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(K_{\widetilde{X}} + \lceil N \rceil + (\phi^* A_1 - E) + \phi^*(A_2 - K_X)))
\end{aligned}
$$

is surjective. Since $K_{\widetilde{X}/X} - E$ is effective and $\phi$-exceptional, we have $K_{\widetilde{X}/X} - E \geq_{\widetilde{V}} 0$. Thus for any divisor $D$ on $X$ we have

$$
H^0(X|V, \mathcal{O}_X(D)) = H^0(\widetilde{X}|\widetilde{V}, \mathcal{O}_{\widetilde{X}}(D + K_{\widetilde{X}/X} - E).
$$

Similarly, note that $\lceil N \rceil \leq_{\widetilde{V}} \lceil \phi^* L \rceil \leq_{\widetilde{V}} \phi^* \lceil L \rceil$. Since $K_{\widetilde{X}} + \lceil N \rceil + \phi^* A_1 + \phi^*(A_2 - K_X) = \lceil N \rceil + \phi^*(A_1 + A_2) + (K_{\widetilde{X}/X} - E)$, we obtain

$$
\begin{aligned}
H^0(\widetilde{V}&, \mathcal{O}_{\widetilde{V}}(\lceil N \rceil + \phi^*(A_1 + A_2))) \\
&\subset H^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(\lceil N \rceil + \phi^*(A_1 + A_2) + K_{\widetilde{X}/X} - E)) \\
&= H^0(\widetilde{X}|\widetilde{V}, \mathcal{O}_{\widetilde{X}}(\lceil N \rceil + \phi^*(A_1 + A_2) + K_{\widetilde{X}/X} - E)) \\
&\subset H^0(\widetilde{X}|\widetilde{V}, \mathcal{O}_{\widetilde{X}}(\phi^* \lceil L \rceil + \phi^*(A_1 + A_2) + K_{\widetilde{X}/X} - E)) \\
&= H^0(X|V, \mathcal{O}_X(\lceil L \rceil + A_1 + A_2)).
\end{aligned}
$$

Setting $A = A_1 + A_2$ finishes the proof. □

## 2.6 Nakayama Constants

Suppose that $L$ is an ample divisor and $V$ a subvariety in $X$. Let $\phi : Y \to X$ be a resolution of the ideal $\mathcal{I}_V$, and define the divisor $E$ by the equation $\mathcal{O}_Y(-E) = \phi^{-1}\mathcal{I}_V \cdot \mathcal{O}_Y$. The Seshadri constant

$$
\varepsilon(L, V) := \max\{\, \tau \mid \phi^* L - \tau E \text{ is nef } \}
$$

measures "how ample" $L$ is along the subvariety $V$. Seshadri constants play an important role in understanding the positivity properties of ample divisors. There is a similar notion that can be defined for any $V$-big divisor $L$: the moving Seshadri constant is

$$\varepsilon(\|L\|, V) := \max_{\phi^* L \geq_{\widetilde{V}} A} \varepsilon(A, \widetilde{V})$$

where $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ varies over all $V$-birational models.

We will be interested in a related notion that can be defined for an arbitrary pseudo-effective divisor $L$.

**Definition 2.6.1.** Let $\mathcal{I}$ be an ideal sheaf on $X$ and let $L$ be a pseudo-effective divisor. Choose a resolution $\phi : Y \to X$ of $\mathcal{I}$ and define $E$ by setting $\mathcal{O}_Y(-E) = \phi^{-1}\mathcal{I} \cdot \mathcal{O}_Y$. We define the Nakayama constant

$$\varsigma(L, \mathcal{I}) := \max\{\, \tau \mid \phi^* L - \tau E \text{ is pseudo-effective } \}.$$

Of course, $\varsigma$ is independent of the choice of resolution. When $\mathcal{I}$ is the ideal sheaf of a subvariety $V$, we will also denote the Nakyama constant by $\varsigma(L, V)$.

Note that $\varsigma(L, V)$ can be positive even when $L$ is only pseudo-effective. This is in contrast to the movable Seshadri constant, which vanishes as we approach the pseudo-effective boundary. In a sense, the movable Seshadri constant measures the ampleness of $L$ along $V$, whereas the Nakayama constant measures the bigness of $L$ along $V$. Thus, the Nakayama constant shares a closer relationship with the other invariants we have considered.

There is a useful criterion which is closer in spirit to Nakayama's original formulation.

**Proposition 2.6.2.** *Let $X$ be a projective variety, $V$ a subvariety, and $L$ a pseudo-effective divisor. Then $\varsigma(L, V) > 0$ iff there is some sufficiently ample divisor $A$ on $X$ so that for any $q$,*

$$h^0(X, I_V^q \otimes \mathcal{O}_X(\lfloor mL + A \rfloor)) > 0$$

*for sufficiently large $m$. As $(q, m)$ varies over all pairs of positive integers satisfying the above criterion we have $\varsigma(L, V) = \limsup_{q \to \infty} \max_m \frac{q}{m}$.*

*Proof.* Let $\phi : Y \to X$ denote a smooth resolution of the ideal sheaf of $V$ and define $E$ by $\mathcal{O}_Y(-E) = \phi^{-1}\mathcal{I}_V \cdot \mathcal{O}_Y$. Note that $\phi^* L - \tau E$ is pseudo-effective for some $\tau > 0$ iff for any $b$, there is some $a$ so that $a\phi^* L - bE$ is pseudo-effective. By Proposition 2.5.3 (and Remark 2.5.2), this can be true iff for every $b$ there is some $a$ such that

$$h^0(Y, \mathcal{O}_Y(\lfloor c(a\phi^* L - bE) \rfloor + H)) > 0$$

for all $c$ and for some fixed ample divisor $H$. Choose ample $A \geq \phi_* H$. Replacing $\lfloor ac \rfloor$ by $m$ and $\lceil cb \rceil$ by $q$, we see that it is equivalent to require that for any $q$

$$h^0(X, I_V^q \otimes \mathcal{O}_X(\lfloor mL \rfloor + A)) > 0$$

for sufficiently large $m$. The last statement follows by comparing coefficients. $\square$

This allows us to show that the Nakayama constant behaves well with respect to $V$-birational models. We first need a lemma.

**Lemma 2.6.3.** *Let $X$ be a projective variety and $V$ be a subvariety of $X$. Suppose that $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a $V$-birational model. Then $\phi_* \mathcal{I}^q_{\widetilde{V}} = \mathcal{I}^q_V$ for all sufficiently large $q$.*

*Proof.* It is clear that the injection $\phi_* \mathcal{I}^q_{\widetilde{V}} \to \phi_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$ has image contained in $\mathcal{I}^q_V$. Thus, it suffices to show that the map $\phi_* \mathcal{I}^q_{\widetilde{V}} \to \mathcal{I}^q_V$ is surjective. Since $\widetilde{V}$ is a subvariety, we know that $\mathcal{I}^q_{\widetilde{V}} \supset \phi^{-1} \mathcal{I}^q_V \cdot \mathcal{O}_{\widetilde{X}}$. Thus, it suffices to show that the map

$$\phi_* \left( \phi^{-1} \mathcal{I}_V \cdot \mathcal{O}_{\widetilde{X}} \right)^q \to \mathcal{I}^q_V.$$

is surjective for sufficiently large $q$.

Note that if $\xi : \check{X} \to \widetilde{X}$ is a higher model, then

$$\xi_* \left( (\xi \circ \phi)^{-1} \mathcal{I}_V \cdot \mathcal{O}_{\check{X}} \right)^q \subset \left( \phi^{-1} \mathcal{I}_V \cdot \mathcal{O}_{\widetilde{X}} \right)^q.$$

So, it suffices to prove the statement for a sufficiently high model. In particular, we may assume that $\widetilde{X}$ admits a morphism $\psi : \widetilde{X} \to Bl_V X$. Let $E$ denote the Cartier divisor defined by the inverse image ideal sheaf on the blow-up of $V$. Then $\phi^{-1} \mathcal{I}_V \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-\psi^* E)$ and $\psi_*(\phi^{-1} \mathcal{I}_V \cdot \mathcal{O}_{\widetilde{X}}) = \mathcal{O}(-E)$. Thus, we have reduced to checking whether

$$f_* \mathcal{O}_{Bl_V X}(-qE) \supset \mathcal{I}^q_V.$$

where $f : Bl_V X \to X$ is the blow-up along $V$. However, since $-E$ is the relative twisting sheaf $\mathcal{O}(1)$ we know that

$$f^* f_* \mathcal{O}_{Bl_V X}(-qE) \to \mathcal{O}_{Bl_V X}(-qE)$$

is surjective for sufficiently large $q$, finishing the proof. □

**Proposition 2.6.4.** *Let $X$ be a normal variety, $V$ a subvariety, and $L$ a divisor. If $(\widetilde{X}, \widetilde{V})$ is a $V$-birational model for $(X, V)$, then $\varsigma(\phi^* L, \widetilde{V}) = \varsigma(L, V)$.*

*Proof.* We will use Proposition 2.6.2. First, note that for any ample $A$ on $X$ there is an ample $H$ on $\widetilde{X}$ with $H \geq_{\widetilde{V}} \phi^* A$. Furthermore we have $\mathcal{I}^q_{\widetilde{V}} \supset \phi^{-1} \mathcal{I}^q_V \cdot \mathcal{O}_{\widetilde{X}}$. Thus for any $(q, m)$ satisfying $h^0(X, \mathcal{I}^q_V \otimes \mathcal{O}_X(\lfloor mL \rfloor + A)) > 0$ we have $h^0(\widetilde{X}, \mathcal{I}^q_{\widetilde{V}} \otimes \mathcal{O}_{\widetilde{X}}(\lfloor m\phi^* L \rfloor + H)) > 0$. In particular $\varsigma(L, V) \leq \varsigma(\phi^* L, \widetilde{V})$.

Conversely, fix an ample $H$ on $\widetilde{X}$ and suppose that $(q, m)$ satisfies $h^0(\widetilde{X}, \mathcal{I}^q_{\widetilde{V}} \otimes \mathcal{O}_{\widetilde{X}}(\lfloor m\phi^* L \rfloor + H)) > 0$. For any ample $A \geq \phi_* H$ we have $\phi^* A \geq_{\widetilde{V}} \phi^* \phi_* H \geq_{\widetilde{V}} H$. By the push-pull formula, the previous lemma indicates that $h^0(X, \mathcal{I}^q_V \otimes \mathcal{O}_X(\lfloor mL \rfloor + A)) > 0$ when $q$ is sufficiently large. □

The following proposition indicates that the Nakayama constant satisfies the usual compatibility relations.

**Proposition 2.6.5.** *Let $X$ be a normal variety, $V$ a subvariety not contained in* $\mathrm{Sing}(X)$*, and $L$ a $V$-big divisor. Then*

$$\varsigma(L, V) = \varsigma(P_\sigma(L), V) = \max_{\phi^* L \geq A} \varsigma(A, \phi^{-1}\mathcal{I}_V \cdot \mathcal{O}_Y).$$

*where $\phi : Y \to X$ varies over all birational maps.*

*Proof.* We start with the equality on the left. It suffices to show the inequality $\leq$. Furthermore, by passing to a $V$-birational model we may assume that $X$ and $V$ are smooth. Let $\phi : Y \to X$ denote the blow-up of $V$ and let $E$ denote the exceptional divisor. Suppose that $\phi^* L - \tau E$ is pseudo-effective. Fix an ample $A$ on $Y$. For any $\epsilon > 0$, we find that $\phi^* L + \epsilon A \sim_{\mathbb{R}} \tau E + F$ for some effective $F$. Since $\mathrm{Supp}(E)$ is not contained in the restricted base locus, we know that $N_\sigma(\phi^* L + \epsilon A) \leq F$. Subtracting, we find that $P_\sigma(\phi^* L + \epsilon A) - \tau E$ is pseudo-effective. Taking a limit over $\epsilon$ and noting that $\phi^* P_\sigma(L) \geq P_\sigma(\phi^* L)$ completes the inequality.

We now turn to the equality on the right. Since $\varsigma$ can be computed on any model, the inequality $\geq$ is a result of the fact that any $A \leq \phi^* L$ satisfies $A \leq P_\sigma(\phi^* L) \leq \phi^* P_\sigma(L)$. Conversely, fix an ample divisor $H$ on $X$. Theorem 2.2.15 indicates that there are birational maps $\phi_m$ and big and nef divisors $N_m$ satisfying $N_m \leq P_\sigma(\phi_m^* L) \leq N_m + \frac{1}{m}\phi_m^* H$. Choosing ample divisors near the $N_m$, we see that the expression on the right hand side can be made arbitrarily close to $\varsigma(P_\sigma(L), V)$. □

Finally, we remark in passing that [Nak04] shows that $\varsigma(L, V)$ is controlled by what happens to a very general subvariety of dimension equal to $\dim(V)$.

**Proposition 2.6.6** ([Nak04], V.2.21). *Let $V$ be a $d$-dimensional subvariety of a smooth variety $X$ and let $L$ be a pseudo-effective divisor. If $\varsigma(L, V) = 0$, then for sufficiently general ample divisors $H_1, \ldots, H_{n-d}$ we have $\varsigma(L, H_1 \cap \cdots \cap H_{n-d}) = 0$.*

### 2.6.1 Restricted Nakayama Constants

Suppose that $V$ is a subvariety of $X$ and that $W$ is a subvariety of $V$. We will define the restricted Nakayama constant by considering the positivity of $\langle L \rangle_{X|V}$ along $W$.

**Definition 2.6.7.** Let $X$ be a normal variety, $V$ a subvariety not contained in $\mathrm{Sing}(X)$, and $W$ a subvariety of $V$ not contained in $\mathrm{Sing}(X)$ or $\mathrm{Sing}(V)$. Suppose that $L$ is a $V$-pseudo-effective divisor. We define the restricted Nakayama constant $\varsigma_{X|V}(L, W)$ as follows.

Let $\phi : \widetilde{X} \to X$ be a $W$-birational model such that $\widetilde{V}$ and $\widetilde{W}$ are both smooth. We can identify $\langle \phi^* L \rangle_{\widetilde{X}|\widetilde{V}}$ as a rational equivalence class of codimension 1 cycles on $\widetilde{V}$. Since $V$ is smooth, this corresponds to an $\mathbb{R}$-linear equivalence class of $\mathbb{R}$-Cartier divisors. Since $\varsigma$ only depends on numerical classes, it makes sense to define

$$\varsigma_{X|V}(L, W) := \varsigma_{\widetilde{V}}(\langle \phi^* L \rangle_{\widetilde{X}|\widetilde{V}}, \widetilde{W}).$$

This definition is independent of the choice of model since if $\psi : \check{X} \to \widetilde{X}$ is another smooth $W$-birational model then

$$\begin{aligned} \varsigma_{\widetilde{V}}(\langle \phi^* L \rangle_{\widetilde{X}|\widetilde{V}}, \widetilde{W}) &= \varsigma_{\check{V}}(\psi^* \langle \phi^* L \rangle_{\widetilde{X}|\widetilde{V}}, \check{W}) \\ &= \varsigma_{\check{V}}(P_\sigma(\psi^* \langle \phi^* L \rangle_{\widetilde{X}|\widetilde{V}}), \check{W}) \\ &= \varsigma_{\check{V}}(\langle \psi^* \phi^* L \rangle_{\check{X}|\check{V}}), \check{W}) \end{aligned}$$

Although it is not immediately clear that the restricted Nakayama constant reduces to our earlier definition when $V = X$, Proposition 2.6.4 shows the consistency of the two definitions since $\varsigma_{X|X}(L, V) = \varsigma_X(P_\sigma(L), V)$.

**Proposition 2.6.8.** *Let $X$ be smooth, $V$ a smooth subvariety of $X$, and $W$ a subvariety of $V$. Suppose that $L$ is a $V$-big divisor. Then*

$$\varsigma_{X|V}(L, W) = \max_{\phi^* L \geq_{\widetilde{V}} A} \varsigma_{\widetilde{V}}(A|_{\widetilde{V}}, \phi^{-1}\mathcal{I}_W \cdot \mathcal{O}_{\widetilde{X}})$$

*as $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ varies over all $V$-birational models.*

*Proof.* Since $V$ is smooth we know that $\varsigma_{X|V}(L, W) = \varsigma_V(\langle L \rangle_{X|V}, W)$. By definition the classes of $\phi_* A|_{\widetilde{V}}$ converge to $\langle L \rangle_{X|V}$. Thus we can conclude by applying Proposition 2.6.5. □

We would like to have a criterion for non-vanishing of the Nakayama constant in terms of sections. Unfortunately, I have been unable to prove an analogue of Proposition 2.6.2 for the restricted Nakayama constant.

**Question 2.6.9.** Let $X$ be a normal variety, $V$ a subvariety not contained in $\mathrm{Sing}(X)$, and $W$ a subvariety of $V$. Suppose $L$ is a $W$-pseudo-effective divisor. If there is some sufficiently ample divisor $A$ on $X$ so that for any $q$

$$h^0(X|V, I^q_{W/X} \otimes \mathcal{O}_X(\lfloor mL + A \rfloor)) > 0$$

for sufficiently large $m$, can we deduce that $\varsigma_{X|V}(L, W) > 0$?

This question is closely related to Question 2.5.7.

## 2.7 The Restricted Numerical Dimension

We now turn to the numerical dimension. Recall that for a nef divisor $L$, the numerical dimension measures the maximal dimension of a subvariety $W \subset X$ such that $L$ is big along $W$. It is precisely this intuition that we want to keep in extending our definition to a general pseudo-effective divisor. There have been several proposals of such a definition in [Nak04] and [BDPP04]. The common feature of all of these is that we should not consider the restriction $L|_W$ but a "positive restriction" of $L$. Our goal in this section is to show that these definitions coincide.

We will work with a restricted version throughout. The restricted numerical dimension of $L$ along $V$ measures the maximal dimension of a general subvariety $W \subset V$ such that the "positive restriction" of $L$ is big along $W$.

**Theorem 2.7.1.** *Let $X$ be a normal variety, $V$ a subvariety not contained in* $\mathrm{Sing}(X)$, *and $L$ a $V$-pseudo-effective divisor. In the following, $W$ will denote an intersection of very general very ample divisors with $V$, and $A$ some fixed sufficiently ample divisor. Then the following quantities coincide:*

1. $\max\{k \in \mathbb{Z}_{\geq 0} | \langle L^k \rangle_{X|V} \neq 0\}$.
2. $\max\{\dim W | \langle L^{\dim W} \rangle_{X|W} > 0\}$.
3. $\max\{\dim W | \lim_{\epsilon \to 0} \mathrm{vol}_{X|W}(L + \epsilon A) > 0\}$.
4. $\max\{\dim W | \liminf_{\phi} \mathrm{vol}_{\widetilde{W}}(P_\sigma(\phi^* L)|_{\widetilde{W}}) > 0\}$ *where* $\phi : (\widetilde{X}, \widetilde{W}) \to (X, W)$ *ranges over $W$-birational models.*
5. $\max\{\dim W | \langle L \rangle_{X|W}$ *is big* $\}$.
6. $\min\left\{\dim W | \varsigma_{X|V}(L, W) = 0\right\}$.
   *By convention, if $L$ is big we interpret this expression as returning* $\dim(V)$.

*This common quantity is known as the restricted numerical dimension of $L$, and is denoted $\nu_{X|V}(L)$. It only depends on the numerical class of $L$.*

*Furthermore, $\nu_{X|V}(L)$ is no greater than*

7. $\max\left\{k \in \mathbb{Z}_{\geq 0} \middle| \limsup_{m \to \infty} \frac{h^0(X|V, \lfloor mL \rfloor + A)}{m^k} > 0\right\}$.

*We have equality in the special case $X = V$. If we assume an affirmative answer to Question 2.6.9, then equality holds in general.*

(2) is the definition of numerical dimension in [BDPP04], while (6) and (7) correspond to $\kappa_\nu(L)$ and $\kappa_\sigma(L)$ in [Nak04] respectively.

We will prove Theorem 2.7.1 using a cycle of weak inequalities. Much of the work has been done already in previous propositions.

*Proof.* We first note that each of these quantities is invariant under passing to a smooth $V$-birational model. This is implicit in definitions of the characterizations (1), (2), (4), and (5). To see the invariance for (6), recall that by definition we first pass to a smooth $V$-birational model and take the strict transform of $W$. By Proposition 2.6.6, we may replace $W$ by an intersection of very general very ample divisors on our smooth model. The invariance is shown for (7) in Proposition 2.5.6. To show the invariance for (3), suppose that $A$ is an ample divisor on $X$. For any smooth $V$-birational model $\phi : \widetilde{X} \to X$, the pullback $\phi^* A$ is $\widetilde{V}$-big. By sandwiching $\phi^* A$ between suitable ample divisors on $\widetilde{X}$, we see that the limit defined in (3) is the same on $X$ and $\widetilde{X}$. Thus we will assume throughout that $X$ and $V$ are smooth.

(1) $\leq$ (2). Let $H_1, \ldots, H_{d-k}$ represent very general elements of a very ample linear system. Statement (1) is equivalent to the fact that $\langle L^k \rangle_{X|V} \cdot H_1 \cdot \ldots \cdot H_{d-k} > 0$. Proposition 2.4.13 allows us to conclude that $\langle L^k \rangle_{X|V \cap H_1 \cap \ldots \cap H_{d-k}} > 0$.

(2) $\leq$ (3). Proposition 2.4.12 shows that conditions set on $W$ in (2) and (3) are the same.

(3) $\leq$ (4). By Proposition 2.4.16 we know that

$$\mathrm{vol}_{X|W}(L + \epsilon A) = \inf_{\phi: \widetilde{X} \to X} \mathrm{vol}_{\widetilde{W}}(P_\sigma(\phi^*(L + \epsilon A))|_{\widetilde{W}})$$

where $\phi : (\widetilde{X}, \widetilde{W}) \to (X, W)$ varies over $V$-birational models. Since volume varies continuously on every model, taking the limit as $\epsilon$ goes to 0 proves that the two are equal.

(4) $\leq$ (5). By assumption there is a positive constant $\alpha$ that gives a lower bound for the volume of the divisor $P_\sigma(\phi^* L)|_{\widetilde{W}}$ where $\phi : (\widetilde{X}, \widetilde{W}) \to (X, W)$ is any $W$-birational model. Consider the pushforward of these divisors under $\phi_*$. The volume does not decrease under pushforward, so $\alpha$ is also a lower bound for the volume of the pushforwards on $W$. By Proposition 2.4.16 $\langle L \rangle_W$ is the limit of the classes of $\phi_* P_\sigma(L)|_{\widetilde{W}}$, so it must also be big.

(5) $\leq$ (6). We defer this inequality until later.

(6) $\leq$ (1). We need to consider the 0-case separately. Note that (1) is 0 precisely when $\langle L \rangle_{X|V}$ is numerically trivial. This means that (6) is also 0. Thus, we can prove that (6) $\leq$ (1) by considering the case where (6) is at least 2 and (1) is at least 1.

Set $k$ to be the value of (1), and suppose that $k$ is less than the value of (6). Let $W$ be a $k$-dimensional intersection of $V$ with general very ample divisors. Set $\tau = \varsigma_{X|W}(L, W) > 0$, and let $\phi : Y \to X$ be the blow-up of $W$ where $E$ is the exceptional divisor.

Fix a very ample divisor $H$ on $Y$. We will start by analyzing $\phi^* L + \epsilon H$ for small $\epsilon$. Suppose that $\psi_{i,\epsilon} : Y_{i,\epsilon} \to Y$ and $\psi_{i,\epsilon}(\phi^* L + \epsilon H) \geq_{\widetilde{V}_{i,\epsilon}} A_{i,\epsilon}$ is a sequence of $V$-birational Fujita approximations whose limit computes $\langle (\phi^* L + \epsilon H)^{d+1} \rangle_{Y|\widetilde{V}}$. By Proposition 2.6.8, we know that $A_{i,\epsilon}|_{\widetilde{V}_{i,\epsilon}} - \frac{\tau}{2} \psi_i^* E|_{\widetilde{V}_i}$ is pseudo-effective for $i$ sufficiently large. Thus

$$0 \leq \left( A_{i,\epsilon} - \frac{\tau}{2} \psi_{i,\epsilon}^* E \right) \cdot \widetilde{V_{i,\epsilon}} \cdot A_{i,\epsilon}^k \cdot \psi_{i,\epsilon}^* H^{d-k-1}.$$

By pushing forward and taking the limit over $i$, we find

$$0 \leq \langle (\phi^* L + \epsilon H)^{k+1} \rangle_{Y|\widetilde{V}} \cdot H^{d-k-1} - \frac{\tau}{2} \langle (\phi^* L + \epsilon H)^k \rangle_{Y|\widetilde{V}} \cdot E \cdot H^{d-k-1}.$$

This is true for all sufficiently small $\epsilon$, so

$$0 \leq \langle \phi^* L^{k+1} \rangle_{Y|\widetilde{V}} \cdot H^{d-k-1} - \frac{\tau}{2} \langle \phi^* L^k \rangle_{Y|\widetilde{V}} \cdot E \cdot H^{d-k-1}.$$

By assumption $\langle\phi^* L^k\rangle_{Y|\widetilde{V}} \neq 0$. By choosing sufficiently general elements $H_1, \ldots, H_{d-k-1} \in |H|$, we may ensure that $E \cap H_1 \cap \ldots \cap H_{d-k-1} \cap \widetilde{V}$ maps finitely onto $W$ via $\phi$. Since $W$ is sufficiently general, we have $\langle\phi^* L^k\rangle_{Y|\widetilde{V}} \cdot E \cdot H^{d-k-1} > 0$. Thus we obtain

$$0 < \langle\phi^* L^{k+1}\rangle_{X|V} \cdot H^{d-k-1}$$

contradicting the fact that $\langle L^{k+1}\rangle_{X|V} = 0$.

This completes all of the comparisons except (5) $\leq$ (6). We now turn to the quantity (7).

(5) $\leq$ (7). Choose an ample divisor $A$ as in Theorem 2.1.13. Thus for every positive integer $m$ there is a divisor $D_m \sim \lceil mL \rceil + A$ such that $D_m \geq_V 0$. For any $m$ there is a sequence of $W$-birational models $\phi_{j,m} : \widetilde{X}_{j,m} \to X$ centered in $\mathbf{B}_-(L)$ and big and nef divisors $N_{j,m}$ on $\widetilde{X}_{j,m}$ such that $N_{j,m} \leq_{\widetilde{W}_{j,m}} \phi_{j,m}^*(D_m)$ and $\phi_{j,m*}(N_{j,m}|_{\widetilde{W}_{j,m}})$ converge to $\langle D_m\rangle_{X|W} \sim_{\mathbb{R}} m\langle L + \frac{1}{m}A'\rangle_{X|W}$. By Proposition 2.5.9, we know there is some ample $A$ such that

$$H^0(\widetilde{W}_{j,m}, \mathcal{O}_{\widetilde{W}_{j,m}}(\lceil N_{j,m} \rceil + \phi^* A)) \subset H^0(X|W, \mathcal{O}_X(\lceil mL \rceil + A' + A))$$

Since $H^0(X|V, \mathcal{O}_X(\lceil mL \rceil + (A' + A)))$ is at least this large, it suffices to show that the spaces $H^0(\widetilde{W}_{j,m}, \mathcal{O}_{\widetilde{W}_{j,m}}(\lceil N_{j,m} \rceil + \phi^* A))$ grow at the necessary rate.

Choose a representative $B$ of $\langle L\rangle_{X|W}$ satisfying $B \geq 0$. Proposition 2.2.15 indicates that for any effective ample divisor $G$ there are models $\psi_m : W_m \to W$ and big and nef divisors $M_m$ on $W_m$ such that $M_m \leq mP_\sigma(\psi_m^* B) \leq M_m + \phi_m^* G$. Comparing against the $N_{j,m}$, we see that by replacing $A$ by $A + G$ we get the necessary rate of growth.

(7) $\leq$ (6). Nakayama proves this in the case $X = V$. If we assume a positive answer to Question 2.6.9, it should be possible to mimic his argument to prove it for general $V$.

Finally, we return to (5) $\leq$ (6). We have proved this in the case when $X = V$ and $L$ is a divisor on $X$. However, the inequality for general $V$ is precisely this situation for the variety $V$ and the divisor $\langle L\rangle_{X|V}$ on $V$. This finishes the equality of the first six characterizations, concluding the proof. □

The restricted numerical dimension is very natural from the viewpoint of birational geometry. It satisfies a number of important properties.

**Theorem 2.7.2.** *Let $X$ be a smooth variety, $V$ a subvariety of $X$, and $L$ a $V$-pseudo-effective divisor.*

1. *If $L'$ is also $V$-pseudo-effective, then $\nu_{X|V}(L + L') \geq \nu_{X|V}(L)$.*
2. *When $L$ is nef, $\nu_{X|V}(L) = \nu_V(L|_V)$.*
3. *We have $0 \leq \nu_{X|V}(L) \leq \dim(V)$.*

4. $\nu_{X|V}(L) = \dim(V)$ *iff* $\langle L \rangle_{X|V}$ *is big and* $\nu_{X|V}(L) = 0$ *iff* $\langle L \rangle_{X|V}$ *is numerically trivial.*

5. *If* $\phi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ *is a* $V$*-birational model then* $\nu_{\widetilde{X}|\widetilde{V}}(\phi^* L) = \nu_{X|V}(L)$.

6. *We have* $\nu_{X|V}(L) = \nu_{X|V}(P_\sigma(L))$.

*Proof.*

1. By definition $\langle L + L' \rangle_{X|W} \succeq \langle L \rangle_{X|W}$. So this inequality follows from characterization (5) of the restricted numerical dimension in Theorem 2.7.1.

2. The restricted volume of an ample divisor can be calculated as an intersection product, so the equality follows from characterization (3) in Theorem 2.7.1.

3. This set of inequalities is obvious.

4. This follows from characterizations (1) and (5) in Theorem 2.7.1.

5. We have checked this already in proving Theorem 2.7.1.

6. This follows from the fact that the restricted positive product is invariant under passing to $P_\sigma$ as demonstrated in Proposition 2.4.14.

□

**Remark 2.7.3.** It is important to note that we can have $\nu_{X|V}(L) = \dim(V)$ even when $L$ is not $V$-big. This is demonstrated in Example 2.3.8.

In the case when $X = V$, [Nak04] shows several additional properties.

**Proposition 2.7.4** ([Nak04],V.2.7). *Let* $X$ *be smooth,* $L$ *a pseudo-effective divisor.*

1. *We have* $\kappa(X, L) \leq \nu(L)$.

2. *Suppose that* $f : X \to Z$ *has connected fibers, and* $F$ *is a very general fiber. Then* $\nu(L) \leq \nu(L|_F) + \dim(Z)$.

3. *If* $f : Y \to X$ *is a surjective morphism then* $\nu(f^* L) = \nu(L)$.

We now recall the compatibility relations mentioned in the introduction. They all follow immediately from the definitions.

**Proposition 2.7.5.** *Let* $V$ *be a normal subvariety of* $X$ *and* $L$ *a* $V$*-pseudo-effective divisor. Then*

1. $\nu_{X|V}(L) \leq \nu(L)$.

2. $\nu_{X|V}(L) \leq \nu(L|_V)$.

3. *If* $H$ *is a very ample divisor and* $\nu_{X|V}(L) < \dim V$, *then* $\nu_{X|V \cap H}(L) = \nu_{X|V}(L)$.

It is interesting to note that $\nu$ is *not* lower semicontinuous as might be expected. This is a consequence of the fact that the restricted positive product is only upper semicontinuous on the boundary of the $V$-pseudo-effective cone.

**Example 2.7.6** ([BFJ09], Example 3.8)**.** Let $X$ be any smooth surface with infinitely many $-1$ curves. Take some compact slice of $\overline{NE}^1(X)$. We can choose a convergent sequence of distinct classes $\{\alpha_i\}$ on this compact slice such that each $\alpha_i$ lies on a ray generated by a $-1$ curve. Note that for any irreducible curve $C$ there is at most one $i$ for which $\alpha_i \cdot C < 0$. Thus $\beta := \lim_{i\to\infty} \alpha_i$ must be a nef class. Since $-1$ curves are contractible, we find that $\nu(\alpha_i) = 0$ for every $i$. However, a non-trivial nef class $\beta$ has $\nu(\beta) \geq 1$. Thus $\nu$ is not lower semicontinuous.

**Question 2.7.7.** What properties does $\nu$ satisfy along the $V$-pseudo-effective boundary?

# Chapter 3

# Pseudo-effective Reduction Maps

Suppose that $X$ is a smooth complex projective variety and $L$ an effective Cartier divisor on $X$. The Iitaka fibration $\phi_L$ associated to $L$ is a fundamental object of study in birational geometry. However, it is often very difficult to understand explicitly. Our goal in this chapter is to understand to what extent properties of $\phi_L$ are determined by the numerical class $[L] \in N^1(X)$. More precisely, we will construct a rational map, depending only on $[L]$, which approximates the Iitaka fibration associated to $L$. We are primarily interested in divisors $L$ whose numerical class is on the boundary of the pseudo-effective cone.

The main interest in such results arises from the Abundance Conjecture, one of the key components of the birational classification of varieties. Loosely speaking, the conjecture states that the Iitaka fibration of the canonical divisor $K_X$ is precisely determined by numerical properties. We will show later on that the general framework laid out here provides additional insights in the special case of the canonical divisor.

Since the Iitaka fibration $\phi_L$ for $L$ is characterized by the fact that $\kappa(F, L|_F) = 0$ for a very general fiber $F$, one might hope to construct a numerical approximation for $\phi_L$ by requiring that $L|_F$ is numerically trivial for a general fiber $F$. In [BCE$^+$02], the authors construct such a map when $L$ is a nef divisor. The main result is that there is a rational map $f : X \dashrightarrow Z$ with $L|_F \equiv 0$ that is maximal in the sense that any other such fibration factors birationally through it. The authors call this fibration the nef reduction map.

We would like to construct a similar map for an arbitrary pseudo-effective divisor $L$. It turns out that numerical triviality is no longer the right condition to impose on the fibers. Rather, we should require that the numerical dimension $\nu(L|_F)$ vanishes, which is a less restrictive condition when $L$ is not nef. Our goal is to demonstrate the following theorem of [Eck05]:

**Theorem 3.0.8** ([Eck05], Proposition 1.5 and Definition 4.1). *Let $X$ be a smooth projective variety and $L$ a pseudo-effective $\mathbb{R}$-divisor on $X$. There is a birational model $\phi : Y \to X$ and a morphism $\pi : Y \to Z$ with connected fibers satisfying*

1. *For a general fiber $F$ of $\pi$, we have $\nu(L|_F) = 0$.*

2. *The pair $(Y, \pi)$ is the maximal quotient satisfying (1): if $\phi' : Y' \to X$ is a birational map and $\pi' : Y' \to Z'$ a morphism with connected fibers that satisfies (1), then there is a dominant rational map $\psi : Z' \dashrightarrow Z$ such that $\pi = \psi \circ \pi'$ as rational maps.*

*The pair $(Y, \pi)$ is determined up to birational equivalence and depends only on the numerical class of $L$. We call it the pseudo-effective reduction map associated to $L$.*

Since [Eck05] obtains this result from an analytic perspective, the main contribution of this chapter is to recover and extend the work of [Eck05] using algebraic techniques. We will discuss the relationship with Eckl's work in Section 3.2.3. Our approach yields the following description of the pseudo-effective reduction map.

**Theorem 3.0.9.** *Let $X$ be a smooth projective variety and $L$ a pseudo-effective $\mathbb{R}$-divisor on $X$. The pseudo-effective reduction map for $L$ is the generic quotient of $X$ by all movable curves $C$ satisfying $\nu_{X|C}(L) = 0$.*

The pseudo-effective reduction map has the following important properties:

**Theorem 3.0.10.** *Let $X$ be a smooth variety and let $L$ be a pseudo-effective divisor.*

1. *Suppose that $\phi : Y \to X$ is a birational map. The pseudo-effective reduction map for $\phi^* L$ is birationally equivalent to the pseudo-effective reduction map for $L$.*

2. *If $\kappa(X, L) \geq 0$ then the Iitaka fibration for $L$ factors birationally through the pseudo-effective reduction map.*

3. *The pseudo-effective reduction map for $P_\sigma(L)$ is birationally equivalent to the pseudo-effective reduction map for $L$.*

4. *If $\kappa(X, L) \geq 0$, there is a birational model $\phi : W \to X$ and a morphism $f : W \to Z$ birationally equivalent to the pseudo-effective reduction map such that there is a divisor $D$ on $Z$ and an effective divisor $E$ on $W$ with $\mu^* L \sim_{\mathbb{Q}} f^* D + E$ and the section rings $R(X, L) = R(Z, D)$ coincide.*

A pseudo-effective divisor $L$ is said to be abundant if $\kappa(X, L) = \nu(L)$. It is well-known that abundant nef divisors have many special geometric properties. Using the pseudo-effective reduction map we will show that many of these properties generalize to the pseudo-effective case. The importance of abundance derives from the following reformulation of the Abundance Conjecture:

**Conjecture 3.7.3.** *Let $(X, \Delta)$ be a klt pair. Then $K_X + \Delta$ is abundant.*

The pseudo-effective reduction map naturally leads to an inductive approach to the Abundance Conjecture. Using the work of [Amb04], we show the following result:

**Theorem 3.0.11.** *Let $(X, \Delta)$ be a klt pair. Assume that $K_X + \Delta$ is pseudo-effective, and let $f : X \dashrightarrow Z$ denote the pseudo-effective reduction map. If Conjecture 3.7.3 holds on $Z$, then $K_X + \Delta$ is abundant.*

In this approach to the Abundance Conjecture, the key question is whether the pseudo-effective reduction map for $K_X + \Delta$ maps to a variety of smaller dimension. By Theorem 3.0.9, this question can be answered by finding curves satisfying a numerical condition. Similar work has appeared in the recent preprint [Siu09].

## 3.1 Background

In this section, we will review the basic tools needed in this chapter. We begin with a brief discussion of several basic notions in birational geometry. We then discuss the quotient theory of [Cam81] and [KMM92], which describes how to define a quotient by families of subvarieties.

We will continue to work with over the base field $\mathbb{C}$. All varieties are assumed to be projective unless otherwise qualified. Just as before, the term "divisor" means an $\mathbb{R}$-Cartier divisor unless otherwise qualified.

### 3.1.1 Birational Geometry

We will need the following standard terminology for birational maps.

**Definition 3.1.1.** Suppose that $\pi : X \dashrightarrow Z$ is a rational map and let $U \subset X$ be the locus on which $\pi$ is defined. Then we say that $\pi$ is *almost proper* if the general fiber of $\pi : U \to Z$ is proper. It is also common to say that $\pi$ is *almost holomorphic*, but we will not use this terminology.

**Definition 3.1.2.** Suppose that $\pi : X \dashrightarrow Z$ and $\pi' : X' \dashrightarrow Z'$ are two rational maps. We say that $\pi$ and $\pi'$ are *birationally equivalent* if there are birational maps $\phi : X \dashrightarrow X'$ and $\mu : Z \dashrightarrow Z'$ such that $\mu \circ \pi = \pi' \circ \phi$ on an open subset of $X$.

**Definition 3.1.3.** Suppose that $\pi : X \dashrightarrow Z$ is a rational map. We say that $\pi$ is a *birational contraction* if some resolution (and hence any resolution) of $\pi$ has connected fibers.

We also take this opportunity to recall a special case of Hironaka's flattening theorem.

**Theorem 3.1.4** ([Hir75]). *Let $f : Y \to X$ be a surjective morphism of (integral) projective varieties. There is a birational morphism $\psi : X' \to X$ satisfying the following: let $Y'$ denote the unique component of $X' \times_X Y$ that dominates $Y$ under the projection map. Then the natural morphism $\nu : Y' \to X'$ is flat.*

We may also require that the $X'$ constructed in the theorem is smooth.

### 3.1.2 Relations and Quotients

In this section we revisit the quotient theory developed by Campana in [Cam81] and Kollár, Miyaoka, and Mori in [KMM92] (and further developed in other papers). The presentation below can be found in [Kol96], Section IV.4.

Let $X$ be a variety. Suppose that we are given a family of subvarieties of $X$, that is, a family of algebraic varieties $s : U \to V$ admitting a morphism $w : U \to X$. We would like to construct a "quotient" $h : X \dashrightarrow Z$ by this family of subvarieties. At the very least, we should ask that a general element of our family be contracted to a point by $h$. In other words, any two general points connected by a chain of fibers of $s$ should be identified under $h$. In light of this requirement, we define an equivalence relation on $X$ by saying that two points are equivalent if they can be connected by such a chain. Surprisingly, it is possible to quotient out by this equivalence relation after setting only a few conditions on the morphisms $s, w$.

We now make the construction more precise. Let $(U, V, s, w)$ denote the family of subvarieties defined by the morphisms $s : U \to V$ and $w : U \to X$. We will allow $U$ and $V$ to be quasi-projective varieties, so that our family is not necessarily proper. Rather than dealing with just a single family, we allow finitely many families $(U_i, V_i, s_i, w_i)$ where $i$ ranges from 1 to $k$. Given such data, [Kol96] defines a constructible subset $\langle U_1, \ldots, U_k \rangle \subset X \times X$ such that two points $x_1, x_2$ can be connected by a chain of subvarieties iff $(x_1, x_2) \in \langle U_1, \ldots, U_k \rangle$.

There are two basic versions of the quotient theorem, and we will use both of them. The first version deals with open morphisms.

**Theorem 3.1.5** ([Kol96],IV.4.13). *Let $X$ be a normal variety and $(U_i, V_i, s_i, w_i)$ be finitely many families of quasi-projective subvarieties on $X$. Suppose that each $w_i$ is open and each $s_i$ is flat with irreducible fibers. Then there is an open subvariety $X^0 \subset X$ and a morphism $\pi : X^0 \to Z^0$ with connected fibers such that*

1. *$\langle U_1, \ldots, U_m \rangle$ restricts to an equivalence relation on $X^0$.*
2. *For every $z \in Z^0$ the closure of $\pi^{-1}(z)$ in $X$ coincides with the closure of a suitable $\langle U_1, \ldots, U_m \rangle$-equivalence class.*
3. *For every $z \in Z^0$ two general points of $\pi^{-1}(z)$ can be connected by a $(U_1, \ldots, U_m)$-chain of length at most $\dim X^0 - \dim Z^0$.*

This theorem shows that we can construct a quotient when we take into account only the "generic" connections. Using this theorem inductively, one obtains the (more common) version that deals with proper morphisms.

**Theorem 3.1.6** ([Kol96],IV.4.16). *Let $X$ be a normal variety and $(U_i, V_i, s_i, w_i)$ be finitely many families of subvarieties on $X$. Suppose that each $w_i$ is proper and that each $s_i$ is proper with connected fibers. Then there is an open subvariety $X^0 \subset X$, a proper morphism $h : X^0 \to Z$, and an open subset $Z^0 \subset Z$ such that*

1. *$\langle U_1, \ldots, U_m \rangle$ restricts to an equivalence relation on $X^0$.*
2. *For every $z \in Z^0$ the fiber $\pi^{-1}(z)$ coincides with a $\langle U_1, \ldots, U_m \rangle$-equivalence class.*
3. *For every $z \in Z^0$ two points of $\pi^{-1}(z)$ can be connected by a $(U_1, \ldots, U_m)$-chain of length at most $2^{\dim X - \dim Z} - 1$.*

There are also versions for infinite collections of families.

## 3.2 Previous Work

The concept of a reduction map associated to a pseudo-effective divisor has its origin in a preprint by H. Tsuji. Loosely speaking, given a variety $X$ and a pseudo-effective divisor $L$, we would like to describe the maximal quotient $\pi : X \dashrightarrow Z$ that is "trivial" with respect to $L$ in some sense. In the first two subsections, we will review [BCE+02] and [BDPP04], which quotient out by curves with $L \cdot C = 0$. In the third, we review [Eck05] which constructs a pseudo-effective reduction map using analytic techniques.

### 3.2.1 Nef Reduction

In [BCE+02] the authors construct a reduction map for nef divisors. They note that when $L$ is nef one can obtain a good theory by "quotienting" by all the curves $C$ with $L \cdot C = 0$.

**Theorem 3.2.1** ([BCE+02], Theorem 2.1). *Suppose that $X$ is normal and $L$ is a nef divisor. There exists an almost proper dominant map $\pi : X \dashrightarrow Z$ with connected fibers such that*

1. $L|_F \equiv 0$ *for every proper fiber $F$ of dimension* $\dim X - \dim Z$.
2. *The map $\pi$ is the maximal quotient satisfying (1): if $\pi' : X \dashrightarrow Z'$ is another almost proper dominant map satisfying (1), then there is a dominant rational map $\psi : Z' \dashrightarrow Z$ such that $\pi = \psi \circ \pi'$ as rational maps.*

*This map $f$ is unique up to birational equivalence.*

The map $\pi$ is constructed by applying Theorem 3.1.6 to all families of $L$-trivial curves. Property 2 follows automatically; the main difficulty is assigning a geometric meaning to the fibers of $\pi$. Theorem 3.1.6 guarantees that a general fiber $F$ admits a connecting family of $L$-trivial curves, but it is not immediately clear that this implies anything about the restriction of $L$ to $F$. The connection to the geometry is given by the following theorem.

**Theorem 3.2.2** ([BCE+02], Theorem 2.4). *Let $X$ be an irreducible projective variety and $L$ a nef divisor on $X$. If any two points of $X$ can be joined by a chain of irreducible curves with $L \cdot C = 0$, then $L \equiv 0$.*

### 3.2.2 Quotienting by Movable Curves

In [BDPP04] the authors consider how to generalize this work to arbitrary pseudo-effective divisors. Note that a rational map $\pi : X \dashrightarrow Z$ is characterized up to birational equivalence by the movable curves that it contracts. In other words, we do not lose anything by restricting our attention to movable curves, and it turns out that this constraint is well-suited for the study of a general pseudo-effective divisor $L$.

The authors of [BDPP04] define a reduction map by applying Theorem 3.1.6 to all families of movable curves $C$ with $L \cdot C = 0$. We obtain an almost proper rational map $\pi : X \dashrightarrow Z$, and again the main problem is to assign a geometric meaning to the fibers. The difficulty is that the fibers of this map do not have a nice interpretation. The following example shows that this map may contract more than the Iitaka fibration.

**Example 3.2.3.** ([BDPP04], Example 8.9) We construct an example of a smooth threefold $X$, a divisor $L$ on $X$ with $\kappa(X, L) = 1$, and a connecting family of movable curves $C$ such that $L \cdot C = 0$.

Let $X'$ be the smooth projective bundle

$$p : \mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbb{P}^1.$$

Each fiber of $p$ is isomorphic to $\mathbb{P}^2$. We define the divisor $F$ to be a fiber of $p$. We let $T'$ denote the section $\mathbb{P}(\mathcal{O}(-1))$. Note that $T'$ has normal bundle $N_{T'/X'} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Thus, we can construct the flop $\phi : X' \dashrightarrow X$ of $T'$. More explicitly, let $\pi : Y \to X'$ denote the blow up of $X'$ along $T'$, and let $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ denote the exceptional divisor. We can contract $E$ along either projection; one returns the map to $X'$, the other defines a new smooth variety $X$. We define the divisor $L$ on $X$ to be the strict transform of a fiber $F$ on $X'$. The pair $(X, L)$ will provide our example.

We first construct a connecting family of curves with $L \cdot C = 0$. Choose any line $C'$ in $F$ that avoids $T'$ and let $C$ denote the strict transform of this line. Since $\phi$ is an isomorphism on a neighborhood of $C$, we have $L \cdot C = F \cdot C' = 0$. We claim that deformations of $C$ define a connecting family of curves on $X$. Any two points in a strict transform of a fiber of $p$ can be connected by deformations of $C$. Furthermore, when we deform $C$ so that it intersects the flipped curve $T$, then $C$ breaks up into two components, one of which is $T$ itself. So, we can traverse between the strict transforms of two different fibers of $p$ via the flipped curve $T$, showing that any point can be connected to any other.

However, $\kappa(X, L) = \kappa(Y, \pi^*F + E) = 1$. Thus, the quotient of the family defined by $C$ contracts more than the Iitaka fibration.

Nevertheless, in some cases they are able to show that connection properties translate to geometric properties.

**Definition 3.2.4.** Let $X$ be smooth and $L$ be a pseudo-effective divisor. Suppose that $C_i$ are movable curves on $X$ with $L \cdot C_i = 0$. We say that the $C_i$ are *strongly connecting* if two general points of $X$ can be connected by deformations of the $C_i$ such that $L \cdot T = 0$ for every component $T$ of the connecting chain.

By generalizing the ideas of [BCE+02] the authors prove

**Theorem 3.2.5** ([BDPP04], Theorem 8.7). *Let $X$ be a smooth variety and $L$ be a pseudo-effective divisor. Suppose that $C_i$ is a strongly connecting collection of movable curves for $L$. Then $\nu(L) = 0$.*

### 3.2.3 Analytic Approach

Eckl has further developed the notion of a reduction map in a series of papers [Eck04a], [Eck04b], [Eck05]. In the first two, he works out a correct formulation of Tsuji's original ideas concerning multiplier ideals. In the third, he constructs a pseudo-effective reduction map. Since our goal is to give another construction of this map, we will briefly recall the ideas in this paper.

Suppose that $X$ is a compact Kähler manifold with Kähler class $\omega$. We first specify when a (not necessarily algebraic) foliation $\mathcal{F}$ is numerically trivial with respect to a pseudo-effective $(1,1)$-form $\alpha$.

**Definition 3.2.6** ([Eck05], Definition 0.6). Let $X$ be a compact Kähler manifold with Kähler form $\omega$ and let $\alpha$ be a pseudo-effective $(1,1)$-class. A foliation $\mathcal{F}$ is called numerically trivial with respect to $\alpha$ if for all $1 \leq p \leq n-1$ and for all test forms $u \in \mathcal{D}^{(n-p,n-p)}(X - \mathrm{Sing}\mathcal{F})$,

$$\lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_X |(T_{ac} + \epsilon\omega)^p \wedge u| = 0$$

where $T$ varies over all currents with analytic singularities representing $\alpha$ with $T \geq -\epsilon\omega$ and where $T_{ac}$ is the absolutely continuous part of $T$ in the Lebesgue decomposition.

Here, the condition $T \geq -\epsilon\omega$ acts as a "perturbation" to ensure that $T$ captures the properties of nearby big divisors. Eckl shows that $\alpha$ admits a maximal numerically trivial foliation $\mathcal{F}$. The main result of [Eck05] is that the codimension of the leaves of $\mathcal{F}$ is bounded by $\nu(L)$, and furthermore if $\kappa(X, L) = \nu(L)$, then this foliation coincides with the Iitaka fibration.

Similarly, there is a maximal fibration $f : X \dashrightarrow Z$ with fibers in $\mathcal{F}$. Eckl calls this the pseudo-effective reduction map and compares it to the notion of reduction map in [BDPP04].

Some of the arguments in [Eck05] are similar to the work in this chapter. In particular, [Eck05] constructs a generic quotient in the context of foliations. All of the arguments presented here were developed independently.

There are advantages to both the analytic and algebraic approach. [Eck05] is more geometric, yielding insights into the numerical dimension for compact Kähler manifolds. The algebraic approach yields a more explicit understanding of the pseudo-effective reduction map. It is also more suggestive of an approach to the problem in characteristic $p$.

## 3.3 Generic Quotients

The pseudo-effective reduction map will be defined by taking a quotient of $X$ by a collection of families of curves. The geometry of the connections defined by these families can be very subtle, as indicated by Example 3.2.3. In this section we will show that the situation improves if we isolate the generic behavior of the families.

Suppose that $X$ is a variety, and $(U_i, V_i, s_i, w_i)$ is a collection of surjective proper families of subvarieties such that the fibers of $s_i$ are generically irreducible. The generic quotient can be described very succinctly: it is the map $\pi : X \dashrightarrow Z$ found by applying Theorem 3.1.5 to the open sets of the $U_i$ on which $w_i$ is flat and $s_i$ is flat with irreducible fibers. In order to understand the properties of this map, we will give a different description in Construction 3.3.5 using the quotient theorem for proper families.

In order to study the generic quotient, we will need to understand how quotients change under birational transformations. Suppose that $\phi : Y \to X$ is a birational map. We first note that that points on $X$ are connected through the families $U_i$ iff there are points above them on $Y$ that are connected through the preimages of the $U_i$.

**Lemma 3.3.1.** *Let $X$ be a normal variety, $(U_i, V_i, s_i, w_i)$ a finite collection of proper families of subvarieties. Suppose that $\phi : Y \to X$ is a birational map. Then the quotient map associated to the $U_i$ is birationally equivalent to the quotient map associated to the families $(U_i \times_X Y, V_i, s_i \circ p_1, p_2)$ on $Y$.*

*Proof.* Let $\pi : X \dashrightarrow Z$ be the almost proper quotient map defined by the $U_i$, and consider the composite $\pi \circ \phi : Y \dashrightarrow Z$. This is also almost proper, and it obviously satisfies the criteria of Theorem 3.1.6. □

In contrast, if we focus on the strict transforms of the subvarieties defined by $U_i$, we expect the connections to change. Suppose $(U, V, s, w)$ is a proper family of subvarieties of $X$ and that $\phi : Y \to X$ is a birational morphism. If we are to have any hope of defining a strict transform family on $Y$, we should require that $w(U)$ is not contained in the center of any $\phi$-exceptional divisor. Assuming this, we define the strict transform family $f : U' \to U$ to be the unique component of $U \times_X Y$ that dominates $w(U)$. We have a natural map $w' : U' \to Y$ and projection $s' = s \circ f : U' \to V$, and denote the new family by $(U', V, s', w')$.

**Example 3.3.2.** Pick a point $x \in \mathbb{P}^2$. We denote the family of lines through $x$ by $U$. More explicitly, we define $U := Bl_x\mathbb{P}^2$, with $s : U \to \mathbb{P}^1$ the map with fibers the lines through $x$, and $w : U \to \mathbb{P}^2$ the map exhibiting the fibers of the first projection as the lines through $x$. The quotient map associated to $U$ takes $\mathbb{P}^2$ to a point.

However, suppose we blow-up $\mathbb{P}^2$ at $x$ to obtain a new surface $S$. Then $U \times_{\mathbb{P}^2} S$ has two components; one is isomorphic to $U$ and the other is a surface intersecting $U$ along the $(-1)$-curve representing tangent directions to $x$. The first component is the strict transform family $U'$, and the map $w' : U' \to S$ is an isomorphism. Thus, the quotient by $U'$ yields the natural map to $\mathbb{P}^1$.

The connections defined by strict transform families can be very subtle. However, there is one case for which there will be no major changes.

**Lemma 3.3.3.** *Suppose that $X$ is a normal variety, $(U, V, s, w)$ a surjective proper family on $X$ such that $w : U \to X$ is flat. Suppose $\phi : Y \to X$ is a birational morphism. Then the the strict transform family $U'$ on $Y$ is equal to $U \times_X Y$.*

*Proof.* Since flatness is preserved under base change, we have that $p_2 : U \times_X Y \to Y$ is a flat morphism. In particular, this means that $p_2$ is flat (and hence open) when restricted to each irreducible component of $U \times_X Y$. Since only one component of $U \times_X Y$ dominates $Y$, this implies that $U \times_X Y$ is the same as the strict transform family. □

Lemma 3.3.1 then yields the following compatibility result.

**Corollary 3.3.4.** *Suppose that $X$ is a normal variety, $(U_i, V_i, s_i, w_i)$ a surjective proper family on $X$ such that $w_i : U_i \to X$ is flat. Suppose $\phi : Y \to X$ is a birational morphism. Then the quotient map defined by the strict transform family $(U_i', V_i, s_i', w_i')$ on $Y$ is birationally equivalent to the quotient map defined by $(U_i, V_i, s_i, w_i)$.*

In constructing the generic quotient, we will only deal with families for which $w : U \to X$ is proper and surjective. This assumption means that we can define the strict transform family for any birational map $\phi : Y \to X$. More importantly, it means that every family contributes to the generic quotient, so that the construction has some nice properties. In view of the conditions of Theorem 3.2.5, we also require that each $s_i : U_i \to V_i$ have generically irreducible fibers. This requirement is justified by Example 3.4.2.

**Construction 3.3.5.** Let $X$ be a normal variety, $(U_i, V_i, s_i, w_i)$ a finite collection of surjective proper families of subvarieties such that $s_i$ has generically irreducible fibers. We construct a rational map $\pi : X \dashrightarrow Z$ which we call the *generic quotient* associated to the $U_i$. Although we will make a number of choices during the construction, the resulting rational map will be unique up to birational equivalence.

Lemma 3.3.3 guarantees that if the morphism $w_i : U_i \to X$ is flat, then the strict transform family of $U_i$ coincides with the total transform. Thus, we may apply Hironaka's flattening theorem (Theorem 3.1.4) to each of the morphisms $w_i : U_i \to X$ in turn to obtain a model $\psi : X' \to X$ such that every strict transform family $(U_i', V_i, s_i', w_i')$ has that $w_i'$ is flat. Let $\pi' : X' \dashrightarrow Z$ be the almost proper rational map defined by these strict transform families. We define the generic quotient to be the composition

$$\pi = \pi' \circ \psi^{-1} : X \dashrightarrow Z.$$

We first verify that the generic quotient is independent of the choices we made during the construction. Suppose that a different sequence of flattenings yielded an almost proper map $\widehat{\pi} : \widehat{X} \dashrightarrow \widehat{Z}$. Corollary 3.3.4 shows that the quotients do not change upon passing to a higher model. Thus, by passing to a common model of $X'$ and $\widehat{X}$ we see the construction is unique up to birational equivalence.

We also check that the generic quotient captures the generic behavior of the connecting families. By resolving the map, we may assume that $\pi' : X' \to Z$ is a morphism. Note that by Corollary 3.3.4, $\pi'$ is still the quotient of $X'$ by the strict transform families $U_i'$. Consider any open set $V_i^0 \subset V_i$ above which $s_i'$ is flat with irreducible fibers. Let $U_i'^0$ denote the preimage of $V_i^0$ under $s_i'$. Since $w'$ is flat, $w'|_{U_i'^0}$

is open. We denote these open families of subvarieties by $(U_i'^0, V_i^0, s_i^+, w_i^+)$ to distinguish it from the proper families. We can apply Theorem 3.1.5 to construct a rational map $\tau : X' \dashrightarrow W$. We show that this rational map is birationally equivalent to $\pi'$.

Since the closure in $X$ of a general fiber of $\tau$ is an equivalence class defined by fibers of the $s_i^+$, it is contained in an equivalence class defined by the $s_i'$. Thus $\tau$ factors birationally through $\pi'$. Conversely, let $\widehat{\tau} : \widehat{X} \to W$ be a resolution of $\tau$ and let $\widehat{w} : \widehat{U} \to \widehat{X}$ be a resolution of the family. Since the fibers of $s_i^+$ are proper, the strict transform of a fiber of $s_i^+$ is contracted by $\widehat{\tau} \circ \widehat{w}$. Thus, by the Rigidity Lemma the same is true for every fiber in the family $\widehat{U} \to V$. This shows that $\tau$ and $\pi'$ are birationally equivalent. As an important consequence, we see that two general points of $X'$ can be connected through the $U_i'$ iff they can be connected using only irreducible fibers.

**Remark 3.3.6.** Here we have defined a generic quotient for a finite set of families. We can also define it for an infinite set of families using the following well-known argument. Suppose we are given an infinite set $\{U_\alpha\}_{\alpha \in A}$ of proper surjective families of subvarieties. For a given finite subset $I \subset A$, we let $\pi_I : X \dashrightarrow Z_I$ denote the generic quotient defined by the corresponding $U_i$. We then define the generic quotient of the $U_\alpha$ to be any quotient $\pi_I$ such that $Z_I$ has the smallest possible dimension. We need to check that this is independent of the choice of subset $I$. Suppose that $\{U_I\}$ and $\{U_J\}$ are two finite subfamilies meeting the criteria. Consider the generic quotient $\pi_{I \cup J} : X \dashrightarrow Z_{I \cup J}$. Of course there are rational maps $Z_I \dashrightarrow Z_{I \cup J}$ and $Z_J \dashrightarrow Z_{I \cup J}$, and by minimality of dimension these maps are generically finite. Since the quotient is always a birational contraction, $\pi_I$ and $\pi_J$ are birationally equivalent.

We also record the following consequence of Lemma 3.3.3:

**Corollary 3.3.7.** *Let $X$ be normal, $(U_i, V_i, s_i, w_i)$ a collection of proper surjective families. Suppose that $\phi : Y \to X$ is birational. The generic quotient associated to $Y$ and the strict transform families $U_i'$ is birationally equivalent to the generic quotient associated to $X$ and the families $U_i$.*

Finally, the following proposition will be useful later.

**Proposition 3.3.8.** *Let $X$ be normal, $(U_i, V_i, s_i, w_i)$ a collection of proper surjective families. Let $\pi : X \dashrightarrow Z$ denote the generic quotient. Suppose that $\phi : Y \to X$ is any birational map, and $\pi' : Y \dashrightarrow Z'$ the quotient obtained by applying Theorem 3.1.6 to the strict transform families. Then $\pi'$ factors birationally through $\pi$.*

*Proof.* Let $(U_i', V_i, s_i', w_i')$ denote the strict transform families on $Y$. As in Construction 3.3.5, let $\psi : X' \to X$ be a birational map resolving the generic quotient. We may assume by blowing up further that $X'$ admits a morphism $f : X' \to Y$. By Proposition 3.3.1, the proper quotient associated to the families $(U_i' \times_Y X', V_i, s_i' \circ p_{1,i}, p_{2,i})$ is birationally equivalent to the quotient on $Y$. However, these families consist of the strict transform families, along with some other components we have not tossed out. Thus the corresponding map contracts at least as much as the generic quotient. □

## 3.4 $L$-trivial Reduction Map

We now have one way of constructing a reduction map associated to a pseudo-effective divisor $L$: take the generic quotient associated to all surjective families of curves $C$ with $L \cdot C = 0$. In this section we will study the properties of this map. The main result of the section is

**Theorem 3.4.1.** *Let $X$ be a smooth projective variety, and $L$ a pseudo-effective line bundle on $X$. Then there is some birational model $\phi : Y \to X$, and an almost proper map $\pi : Y \dashrightarrow Z$ with connected fibers satisfying*

1. *For a very general point $y \in Y$ and curve $C \subset Y$ through $y$ with $\dim(\pi(C)) > 0$, we have $\phi^* L \cdot C > 0$.*
2. *For a general fiber $F$ of $\pi$, we have $\nu(L|_F) = 0$.*
3. *The pair $(Y, \pi)$ is the minimal quotient satisfying (1): if $\widehat{\phi} : \widehat{Y} \to X$ is a birational map and $\widehat{\pi} : \widehat{Y} \dashrightarrow \widehat{Z}$ an almost proper map with connected fibers that satisfies (1), then $\widehat{\pi}$ factors birationally through $\pi$.*

*The pair $(Y, \pi)$ is determined up to birational equivalence. We call it the $L$-*trivial reduction map*. It only depends on the numerical class of $L$.*

*Proof (of Theorem 3.4.1):* To construct $\pi : Y \dashrightarrow Z$, take the generic quotient associated to all families of movable curves $C$ with $L \cdot C = 0$, in the sense of Remark 3.3.6. Clearly this map has Property (1). In Construction 3.3.5, we showed that the generic fiber of $\pi$ is connected by irreducible members of our families. Thus by Theorem 3.2.5 this map also has Property (2). Property (3) follows from Proposition 3.3.8, which says that the generic quotient is the minimal quotient as we look over all strict transform families. □

**Example 3.4.2.** It is necessary in Theorem 3.4.1 that the general member of our families be irreducible. For example, consider a Hirzebruch surface $\mathbb{F}_e$ such that the distinguished section $C_0$ has large negative self-intersection $e$. Let $F$ be a fiber of the projection $\mathbb{F}_e \to \mathbb{P}^1$, and consider the divisor $F + C_0$. This divisor has Iitaka dimension 1. However, it has vanishing intersection with the connecting family of reducible curves $(e-1)F + C_0$.

**Example 3.4.3.** It is also necessary to work with dominant families. Consider again Example 3.2.3, keeping the notation there. Let $\mathcal{C}$ denote the family of curves on $X$ defined by deforming $C$. Then $\mathcal{C} \times_X Y$ has two components: the strict transform family and an additional component lying above the exceptional divisor. By Proposition 3.3.1, the quotient of $Y$ by both of these families gives the map to a point. Thus if we allow non-movable curves in our quotient, we will run into the same difficulty as before.

We begin by checking the basic properties of the $L$-trivial reduction map.

**Proposition 3.4.4.** *Let $X$ be smooth and $L$ pseudo-effective. Let $\pi : X \dashrightarrow Z$ denote the $L$-trivial reduction map.*

1. *If $\kappa(X, L) \geq 0$ then the Iitaka fibration for $L$ factors birationally through $\pi$.*
2. *If $\phi : X' \to X$ is a birational map, then the $\phi^*L$-trivial reduction map is birationally equivalent to the $L$-trivial reduction map.*
3. *If $L$ is nef, then the $L$-trivial reduction map is birationally equivalent to the nef reduction map.*

*Proof.* Let $\phi : Y \to X$ be a resolution of the $L$-trivial reduction map. Note that $0 \leq \kappa(F, \phi^*L|_F) \leq \nu(F, \phi^*L|_F) = 0$ for a very general fiber $F$ of the $L$-trivial reduction map. Thus the Iitaka fibration factors birationally through it, showing Property (1).

Property (2) is a consequence of the birational invariance of the generic quotient as shown in Corollary 3.3.7.

To show Property (3), recall that the nef reduction map is constructed by quotienting out by all $L$-trivial curves. Thus, the nef reduction map factors birationally through $\pi$. Conversely, recall that $L$ is numerically trivial along a general fiber $F$ of the nef reduction map. Define a curve $C$ by intersecting $F$ with general very ample divisors. The first property of Theorem 3.4.1 applied to $C$ shows that the $L$-trivial reduction map factors birationally through the nef reduction map. □

**Proposition 3.4.5.** *Let $X$ be smooth, $L$ pseudo-effective. Let $\pi : X \dashrightarrow Z$ denote the $L$-trivial reduction map. We have*

$$\kappa(X, L) \leq \nu(X, L) \leq \dim(Z).$$

*Proof.* We just need to check the last inequality. Recall that $\nu$ is invariant under birational pull-back. Thus, we first make a birational transformation $\phi : Y \to X$ so that the $L$-trivial reduction map $\pi : Y \to Z$ is a morphism. By Proposition 2.0.4, if $\pi : Y \to Z$ is a fiber space and $F$ a general fiber then $\nu(L) \leq \nu(L|_F) + \dim Z$, finishing the proof. □

**Example 3.4.6.** Example 3.2.3 gives a good illustration of the $L$-trivial reduction map. We will keep the notation established there. Recall that we have a smooth threefold $X$, a line bundle $L$ with $\kappa(X, L) = 1$, and a connecting family of movable curves $C$ such that $L \cdot C = 0$. In particular, the connections defined by the family of curves do not reflect the geometry of $L$. However, if we pass from $X$ to the birational model $Y$ then the strict transform family of the curves is no longer connecting. In fact, two points are connected by the strict transform family iff they lie in a fiber of the map $g : Y \to \mathbb{P}^1$.

Let us confirm that this is actually the $L$-trivial reduction map. Since $g$ is the Iitaka fibration for $L$, it factors birationally through the $L$-trivial reduction map. Conversely, since two general points in a fiber of $g$ can be connected by a deformation of the strict transform of $C$, Property (3) of Theorem 3.4.1 shows that the $L$-trivial reduction map is precisely $g$.

As discussed in the introduction, we would like to find the "maximal" fibration such that $\nu(L|_F) = 0$ for a general fiber $F$. A classical example due to Zariski shows that the $L$-reduction map does not satisfy this property. We will see later that the pseudo-effective reduction map corrects this deficiency.

**Example 3.4.7.** We recall Zariski's example of a surface $S$ that carries a curve with negative self-intersection which can not be contracted in the category of projective varieties. Start by fixing a smooth cubic curve $C$ in $\mathbb{P}^2$. Since $C$ is a cubic curve, we can identify it with $\mathbb{C}/\Lambda$ for some lattice $\Lambda$. Furthermore, we can do this in such a way that for any line $H \subset \mathbb{P}^2$ the sum of the points of $H|_C$ lie on the lattice. Pick 10 points on $\mathbb{C}$ sufficiently general so that there is no non-trivial integral linear combination of them that lies on the lattice. We denote their images in $C$ by $p_1, \ldots, p_{10}$.

Let $S$ denote the blow up of $\mathbb{P}^2$ along these 10 points, and let $C'$ denote the strict transform of $C$. Since $C'$ has self intersection $-1$, we have that $C' = N_\sigma(C')$. We now show that there is no irreducible curve $T \subset S$ such that $T \cdot C' = 0$. Suppose there were such a curve. Clearly $T$ can not be exceptional, so the pushforward $f_*T$ must be a divisor on $\mathbb{P}^2$. We set $d$ to be the degree. Since $f_*T$ and $C$ can only intersect along the 10 points chosen above, we see that $dH|_C \sim f_*T|_C \sim \sum a_i p_i$ for some integer coefficients $a_i$. This contradicts the generality of the $p_i$.

Thus, the $C'$-trivial reduction map on $S$ is the identity map. However, the map from $S$ to a point satisfies the defining Property (2) of the $L$-trivial map, showing that the $L$-trivial reduction map is not maximal with respect to this property.

## 3.5 The Restricted Numerical Dimension for Curves

In order to improve upon the $L$-trivial reduction map, we must take the generic quotient of a different set of curves. The key property of the $L$-trivial reduction map is that $\nu(L|_F) = 0$ for a general fiber $F$. Since we want to focus on this numerical property on the fibers, the most natural restriction on our curves is $\nu_{X|C}(L) = 0$.

The main theorem in this section is the following:

**Theorem 3.5.1.** *Let $X$ be smooth and let $L$ be a pseudo-effective divisor. Suppose that $C$ is a movable curve very general in its family. The following are equivalent:*

1. $\nu_{X|C}(L) = 0$.
2. *There is a birational map $\phi : Y \to X$ centered in $\mathbf{B}_-(L)$ such that $P_\sigma(\phi^* L) \cdot C' = 0$ where $C'$ denotes the strict transform of $C$.*

Since numerical dimension is invariant under passing to $P_\sigma$, it's clear that (2) implies (1). For the other direction, we will need a lemma showing that we can separate movable curves from the restricted base locus.

**Lemma 3.5.2.** *Let $X$ be smooth and $L$ be a pseudo-effective divisor. Suppose that $C$ is a movable curve that is very general in its family. Then there is a birational map $\psi : Y \to X$, centered in $\mathbf{B}_-(L)$, such that the strict transform $C'$ of $C$ does not intersect $\mathbf{B}_-(P_\sigma(\phi^* L))$.*

*Proof.* Let $w : \mathcal{C} \to X$ denote a generically finite family found by deforming $C$. Recall that by Theorem 2.2.3 $\mathbf{B}_-(L)$ has only has finitely many divisorial components. The same is true for $w^{-1}(\mathbf{B}_-(L))$: any divisorial component is either the preimage of a divisorial component of $\mathbf{B}_-(L)$ or a $w$-exceptional divisor. Thus there is a birational map $\psi : Y \to X$ centered in $\mathbf{B}_-(L)$ that extracts the image of any divisorial component of $w^{-1}(\mathbf{B}_-(L))$.

Let $\widetilde{C}$ denote the strict transform of $C$, and $\widetilde{w} : \widetilde{\mathcal{C}} \to Y$ the strict transform family. Since $C$ is very general, the map $\widetilde{\mathcal{C}} \to \mathcal{C}$ is an isomorphism on a neighborhood of $\widetilde{C}$. Furthermore, $C$ only intersects divisorial components of $w^{-1}(\mathbf{B}_-(L))$. Thus $\widetilde{C}$ is disjoint from $\mathbf{B}_-(P_\sigma(\psi^*L))$. □

We can now finish the proof of Theorem 3.5.1.

*Proof of Theorem 3.5.1:* It only remains to prove the implication (1) ⇒ (2). Recall that $\nu_{X|C}(L) = 0$ iff

$$\inf_{\phi} P_\sigma(\phi^*L) \cdot \widetilde{C} = 0$$

where $\phi$ varies over all $L$-admissible birational maps and $\widetilde{C}$ denotes the strict transform of $C$.

Lemma 3.5.2 shows that there is a birational model $\phi : Y \to X$ centered in $\mathbf{B}_-(L)$ such that $\widetilde{C}$ is disjoint from $\mathbf{B}_-(P_\sigma(\phi^*L))$. Thus, the infimum described above is in fact equal to $P_\sigma(\phi^*L) \cdot \widetilde{C}$, proving the theorem. □

## 3.6 Pseudo-effective Reduction Map

We will define the pseudo-effective reduction map for a pseudo-effective divisor $L$ by taking the generic quotient with respect to all families of movable curves whose very general member satisfies $\nu_{X|C}(L) = 0$.

*Proof of Theorem 3.0.8:* To construct $\pi : Y \to Z$, simply take the generic quotient associated to all families of movable curves such that $\nu_{X|C}(L) = 0$ for a very general member $C$. This map obviously satisfies Property (2).

In order to prove Property (1), we will need to compare against the $L$-trivial reduction map. First, identify a finite set of families of curves $\{\mathcal{C}_i\}$ that define the generic quotient. By Theorem 3.5.1, for any particular family $\mathcal{C}_i$ there is a model $\psi_i : X_i \to X$ such that $P_\sigma(\psi_i^*L) \cdot C = 0$ for the strict transform of a member of $\mathcal{C}_i$. Of course, this is also true for any higher model, so repeating the process we find one model $\psi : \widetilde{X} \to X$ so that $P_\sigma(\psi^*L)$ vanishes on every strict transform family. Then the pseudo-effective reduction map is birationally equivalent to the $P_\sigma(\psi^*L)$-trivial reduction map. This implies Property (1). □

**Corollary 3.6.1.** *Let $X$ be smooth and $L$ be pseudo-effective. There is some birational model $\phi : X' \to X$ such that the pseudo-effective reduction map for $L$ is birationally equivalent to the $P_\sigma(\phi^*L)$-trivial reduction map.*

Just as with the $L$-trivial reduction map, we can compare the pseudo-effective reduction map against the Iitaka fibration:

**Corollary 3.6.2.** *Let $X$ be smooth and $L$ be a divisor with $\kappa(X, L) \geq 0$. Then the Iitaka fibration factors birationally through the pseudo-effective reduction map for $L$.*

### 3.6.1 Properties of the Pseudo-effective Reduction Map

**Proposition 3.6.3.** *Let $X$ be smooth and $L$ be pseudo-effective.*

1. *If $\phi : X' \to X$ is a birational map, then the pseudo-effective reduction map for $L$ is birationally equivalent to the pseudo-effective reduction map for $\phi^*L$.*
2. *If $L$ is nef, then the pseudo-effective reduction map for $L$ is birationally equivalent to the nef reduction map.*
3. *The pseudo-effective reduction map for $L$ is birationally equivalent to the pseudo-effective reduction map for $P_\sigma(L)$.*

In fact, by Corollary 3.6.1 these properties can be reduced to the corresponding properties for the $L$-trivial reduction map.

*Proof.* To show (1), we apply Corollary 3.6.1 to see that both are birationally equivalent to the $P_\sigma(f^*L)$-trivial reduction map for some birational model $f : Y \to X$.

(2) and (3) follow from Corollary 3.6.1 and Proposition 3.4.4. □

**Example 3.6.4.** Suppose that $L$ has a Zariski decomposition $\phi : Y \to X$ with $\phi^*L = P+N$. Then the pseudo-effective reduction map for $L$ is birationally equivalent to the nef reduction map for $P$.

When the pseudo-effective reduction map for $L$ is non-trivial there are important consequences for the geometry of $L$.

**Theorem 3.6.5** ([Nak04], V.2.26). *Let $f : X \to Z$ be a morphism of smooth varieties and $L$ a pseudo-effective divisor on $X$ such that $\kappa(F, L|_F) = \nu(F, L|_F) = 0$ for a general fiber $F$ of $f$. Then there exists a morphism $g : W \to T$ birationally equivalent to $f$ and a divisor $D$ on $T$ such that $\mu^*L \sim_{\mathbb{Q}} g^*D+E$ where $E \leq N_\sigma(\mu^*L)$ is supported on the components dominating $T$.*

Thus, when $\kappa(X, L) \geq 0$ all the interesting geometry of $L$ can be detected on the base of the pseudo-effective reduction map.

We define the pseudo-effective dimension $p(X, L)$ of $L$ to be the dimension of the image of the pseudo-effective reduction map for $L$. Note that $p(X, L)$ is a birational invariant of $L$.

**Proposition 3.6.6.** *Let $X$ be smooth, $L$ pseudo-effective. We have*

$$\kappa(X, L) \leq \nu(X, L) \leq p(X, L).$$

*Proof.* Since each quantity is invariant under birational maps, this follows from the corresponding inequalities for the $L$-trivial reduction map in Proposition 3.4.5. □

### 3.6.2 Abundant Divisors

Abundant nef divisors exhibit a number of nice geometric properties. In this section we will show that abundant pseudo-effective divisors satisfy similar properties. The main result is the following list of equivalent conditions for abundance. The equivalence of (1), (2), and (3) was shown in [Eck05], and the equivalence of (2), (3), and (4) is proved in [Nak04]. Our contribution is to reprove the first set of equivalences from an algebraic perspective.

**Theorem 3.6.7.** *Let $X$ be a smooth variety, $L$ a divisor with $\kappa(L) \geq 0$. The following are equivalent:*

1. $\kappa(L) = \nu(L)$.
2. $\kappa(L) = p(L)$.
3. *Let $f : X' \to Z'$ be the Iitaka fibration for $L$ with morphism $\mu : X' \to X$. Then*
$$\nu(\mu^* L|_F) = 0$$
*for a very general fiber $F$ of $f$.*
4. *There is a birational map $\mu : W \to X$ and a contraction morphism $g : W \to T$ such that $P_\sigma(\mu^* L) \sim_{\mathbb{Q}} g^* B$ for some big divisor $B$ on $T$.*

We will first need a relative version of Theorem 2.1.25.

**Definition 3.6.8.** Let $X$ be normal and $f : X \to Z$ a morphism. We say that a movable curve $C$ on $X$ is *$f$-vertical* if $f(C)$ is a point.

**Theorem 3.6.9.** *Let $X$ be normal, $f : X \to Z$ a morphism. Let $\mathcal{K}$ denote the subcone $\mathcal{K} \subset \overline{NM}_1(X)$ consisting of curve classes that have vanishing intersection with the pullback of an ample divisor $H$ on $Z$. Then $\mathcal{K}$ is the closure of the cone generated by classes of $f$-vertical movable curves.*

*Proof.* Suppose the theorem fails. Fix an ample divisor $A$ on $X$. There is a divisor $D$ (not pseudo-effective) such that

- $D \cdot \alpha < 0$ for some $\alpha \in \mathcal{K}$.
- For some $\epsilon > 0$, $D \cdot C > \epsilon A \cdot C$ for every vertical movable $C$.

If $F$ is a very general fiber of $f$, then every curve $C \subset F$ that is movable in $F$ is also movable in $X$. So Theorem 2.1.25 implies that the restriction of $D$ to a very general fiber is big. This implies that $D$ is $f$-big, so that $D + mf^*H$ is pseudo-effective for some large $m$. But $(D + mf^*H) \cdot \alpha < 0$, a contradiction. □

We now turn to the proof of Theorem 3.6.7.

*Proof of Theorem 3.6.7:*

(1) $\Rightarrow$ (2): All the quantities are invariant under birational maps. So we may assume that the Iitaka fibration for $L$ is actually a morphism $f : X \to Z$. We would like to show that $\nu(F, L|_F) = 0$ for a general fiber $F$ of $f$. By the maximality of the pseudo-effective reduction map this implies $p(X, L) \leq \kappa(X, L)$, finishing the implication.

Recall that the numerical dimension $\nu(L)$ is defined to be the minimal dimension of a subvariety $V$ such that $\varsigma(L, V) = 0$. Choose such a subvariety $V$. Note that $V$ dominates $Z$ under $f$: if it did not, there would be some very ample $H$ on $Z$ such that $f^*H$ contains $V$. But since $f^*H \leq mL$ for some $m$, we would have $\varsigma(L, V) \geq \varsigma(f^*H, V) \geq 0$, a contradiction. In particular, $V \cap F$ is 0-dimensional for a general fiber $F$ of $f$.

Let $\phi : Y \to X$ be a birational map extracting $V$ as a divisor (which we denote by $E$), and let $g$ denote the composition $f \circ \phi : Y \to Z$. Since $V$ dominates $Z$ under $f$, the restriction of $E$ to a general fiber of $g$ has codimension 1. Fix an ample divisor $A$ on $Y$. By definition there is some $b$ such that $xL - yE + A$ is not pseudo-effective for any $x, y > b$. We will fix a specific $y > b$.

Choose any compact slice of $\overline{NM}_1(Y)$, and let $Q$ denote the closed region of this compact slice on which $-yE + A$ is non-positive. If $L$ is positive on $Q$, then (by compactness) $xL - yE + A$ is positive on all of $\overline{NM}_1(Y)$ for $x$ sufficiently large, a contradiction. Thus, there must be at least one curve class $\alpha \in \overline{NM}_1(X)$ with $(-yE + A) \cdot \alpha \leq 0$ and $L \cdot \alpha = 0$. This means that $E \cdot \alpha > 0$. Also, since $L \geq \epsilon f^*H$ for ample $H$ on $Z$ and sufficiently small $\epsilon$, we have $f^*H \cdot \alpha = 0$. Thus, $\alpha$ lies on the face of $\overline{NM}_1(Y)$ cut out by $g^*H$. By Lemma 3.6.9, we can find a sum of $g$-vertical movable curves $C := \sum a_i C_i$ that lies arbitrarily close to $\alpha$. In particular we may ensure $E \cdot C > 0$.

Choose a general fiber $G$ of $g$, and let $F = \phi(G)$ denote the corresponding fiber of $f$. By deforming the movable curves $C_i$, we may assume that they all lie in $G$. Thus we may identify $C$ as a curve class in $\overline{NM}_1(G)$. Of course $L|_G \cdot C = 0$ and $E|_G \cdot \alpha > 0$, showing that $\varsigma(L|_F, V \cap F) = 0$. Since $V \cap F$ has dimension 0, this shows that $\nu(L|_F) = 0$, finishing the implication.

(2) $\Rightarrow$ (3): Since all the quantities are birationally invariant, we may pass to higher models so that the Iitaka fibration $f : X' \to Z'$ and the pseudo-effective reduction map $\pi : X' \to Z$ are both morphisms. Since the Iitaka fibration factors birationally through the pseudo-effective reduction map, after possibly going to yet higher models, we find that there is a birational morphism $g : Z \to Z'$ such that $f = g \circ \pi$. But of course a general fiber of $f$ is also a general fiber of $\pi$, proving (3).

(3) $\Rightarrow$ (4): This is a consequence of Theorem 3.6.5.

(4) $\Rightarrow$ (1): It is clear that $P_\sigma(\mu^*L)$ is abundant. By Theorem 2.7.2, this is equivalent to the abundance of $L$. $\square$

Condition (4) of Theorem 3.6.7 implies that abundant divisors satisfy some of the same properties as big divisors. We will collect some of these here.

**Corollary 3.6.10.** *Let $X$ be a smooth variety and $L$ be an abundant divisor. Suppose that $D$ is $\mathbb{Q}$-linearly equivalent to an effective divisor and satisfies $D \equiv L$. Then $D$ is also abundant.*

*Proof.* It suffices to show that $\kappa(X, D) = \kappa(X, L)$. Recall that there is a birational map $\mu : X' \to X$ and an algebraic fiber space $f : X' \to Z$ such that $P_\sigma(\mu^* L) \sim_{\mathbb{Q}} f^* B$ for some big divisor $B$ on $Z$.

Note that $P_\sigma(\mu^* D)|_F \equiv 0$ for a general fiber $F$ of $f$. Since $D$ is $\mathbb{Q}$-linearly equivalent to an effective divisor, $P_\sigma(\mu^* D)|_F \sim_{\mathbb{Q}} 0$. That is, $\kappa(F, P_\sigma(\mu^* D)|_F) = 0$, so Theorem 3.6.5 implies that $P_\sigma(\mu^* D) \sim_{\mathbb{Q}} f^* D$ for some divisor $D$ on $Z$. Since clearly $D \equiv B$, we obtain abundance of $D$. ☐

**Proposition 3.6.11.** *Let $X$ be smooth, $L$ abundant. Fix an ample divisor $A$. Then any discrete valuation $v$ satisfies*

$$v(\|L\|) = \lim_{m \to \infty} v\left(\left\|L + \frac{1}{m}A\right\|\right).$$

*In particular the asymptotic valuation of an abundant divisor is a numerical invariant.*

*Proof.* The space of all discrete valuations is compact and the divisorial valuations are dense in this space (see [Vaq00]). Thus we may assume without loss of generality that $v$ measures the order of vanishing along a subvariety $T$. We first analyze the case when there is a flat morphism $f : X \to Z$ with connected fibers such that $L \sim_{\mathbb{Q}} f^* B$ for a big divisor $B$ on $Z$. We set $T' = f^{-1} f(T)$. Suppose that $T'_i$ are the irreducible components of $T'$. We define the function $w_{T'} : K(X) \to \mathbb{R}$ to be $w_{T'} = \min v_{T'_i}$, so that $w_{T'}$ measures the minimum order of vanishing along any component of $T'$.

Since the fibers of $f$ are equidimensional, by intersecting general hyperplane sections we may find a smooth subvariety $Z'$ mapping finitely onto $Z$ under $f$. Since $L$ is pulled-back from $Z$, we have

$$v_T(\|L\|) = w_{T'}(\|L\|) = w_{T' \cap Z'}(\|L|_{Z'}\|)$$

where $w_{T' \cap Z'}$ measures the minimum order of vanishing along any component of $T' \cap Z'$. In general we have inequalities

$$v_T\left(\left\|L + \frac{1}{m}A\right\|\right) \geq w_{T'}\left(\left\|L + \frac{1}{m}A\right\|\right) \geq w_{T' \cap Z'}\left(\left\|\left(L + \frac{1}{m}A\right)\Big|_{Z'}\right\|\right)$$

Since $f$ is finite on $Z'$, there is some $H$ sufficiently ample on $Z$ so that $(f^* H - A)|_{Z'}$

is ample. In all, we have

$$\begin{aligned} v_T(\|L\|) &\geq w_{T'}\left(\left\|L + \frac{1}{m}A\right\|\right) \\ &\geq w_{T'\cap Z'}\left(\left\|\left(L + \frac{1}{m}A\right)\Big|_{Z'}\right\|\right) \\ &\geq w_{T'\cap Z'}\left(\left\|\left(L + \frac{1}{m}f^*H\right)\Big|_{Z'}\right\|\right). \end{aligned}$$

Thus it suffices to show that the last term converges to $w_{T'\cap Z'}(\|L|_{Z'}\|)$ as $m$ increases. Let $Z'_i$ denote some component of $T' \cap Z'$. Since $B$ is big, we can write $B \sim_{\mathbb{Q}} D + \delta H$ for some small $\delta$. Then by the triangle inequality for valuations applied to $(1+\frac{1}{m})L \sim_{\mathbb{Q}} L + \frac{\delta}{m} f^*H + \frac{1}{m} f^*D$ we have

$$\frac{m+1}{m} v_{Z'_i}(\|L|_{Z'}\|) \leq v_{Z'_i}\left(\left\|\left(L + \frac{\delta}{m}f^*H\right)\Big|_{Z'}\right\|\right) + \frac{1}{m} v_{Z'_i}(f^*D|_{Z'}).$$

Taking the limit as $m$ goes to $\infty$ shows the desired equality for $v_{Z'_i}$, implying we also have equality for $w_{T'\cap Z'}$.

We now turn to the general case. Since valuations can be computed on any model, we may assume there is a flat morphism $f : X \to Z$ with connected fibers such that $P_\sigma(L) \sim_{\mathbb{Q}} f^*B$ for a big divisor $B$ on $Z$. The earlier calculation shows that $v(\|P_\sigma(L)\|) = \lim_{m\to\infty} v(\|P_\sigma(L) + \frac{1}{m}A\|)$. Since $P_\sigma(L + \frac{1}{m}A) - P_\sigma(L)$ is big, it is also true that

$$v(\|P_\sigma(L)\|) = \lim_{m\to\infty} v\left(\left\|P_\sigma\left(L + \frac{1}{m}A\right)\right\|\right).$$

[Nak04] III.1.8 shows that for a big divisor $L$ we have $v(\|B\|) = v(\|P_\sigma(B)\|) + v(N_\sigma(B))$. Using the triangle inequality, we see

$$\begin{aligned} v(\|L\|) &\leq v(\|P_\sigma(L)\|) + v(N_\sigma(L)) \\ &\leq \lim_{m\to\infty} v\left(\left\|P_\sigma\left(L + \frac{1}{m}A\right)\right\|\right) + v\left(N_\sigma\left(L + \frac{1}{m}A\right)\right) \\ &\leq \lim_{m\to\infty} v\left(\left\|L + \frac{1}{m}A\right\|\right) \\ &\leq v(\|L\|) \end{aligned}$$

finishing the proof. □

## 3.7 Relationship to the Minimal Model Program

The minimal model program gives a conjectural description of the Iitaka fibration for the canonical bundle $K_X$ in geometric terms. The main conjectures of the minimal model program are the following:

**Conjecture 3.7.1** (Existence of Minimal Models). Let $(X, \Delta)$ be a klt pair such that $K_X + \Delta$ is pseudo-effective. Then there is some birational contraction $\phi : X \dashrightarrow X'$ such that $(X', \phi_*\Delta)$ is klt, $K_{X'} + \phi_*\Delta$ is nef, and $R(X', K_{X'} + \phi_*\Delta) = R(X, K_X + \Delta)$.

**Conjecture 3.7.2** (Abundance Conjecture). Let $(X, \Delta)$ be a klt pair such that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semiample, i.e. there is a morphism $f : X \to Z$ so that $K_X + \Delta \sim_{\mathbb{R}} f^*A$ for some ample divisor $A$ on $Z$.

The essence of the Abundance Conjecture is that the asymptotic behavior of sections of $K_X + \Delta$ should be predicted by its numerical behavior. This idea was made precise in [Kaw85], which proves the Abundance Conjecture in the special case when $K_X + \Delta$ is nef and abundant (see also [Fuj09]). The following slightly stronger formulation occurs in [Nak04] and [BDPP04]:

**Conjecture 3.7.3.** Let $(X, \Delta)$ be a klt pair. Then $K_X + \Delta$ is abundant.

Note that the abundance of $K_X + \Delta$ would follow from Conjectures 3.7.1 and 3.7.2. In fact, these two conjectures imply the following description of the Iitaka fibration.

**Proposition 3.7.4.** *Let $(X, \Delta)$ be a klt pair, with $X$ smooth and $K_X + \Delta$ pseudo-effective. Assume Conjectures 3.7.1 and 3.7.2. Then the Iitaka fibration for $(K_X + \Delta)$, the $(K_X + \Delta)$-trivial reduction map, and the pseudo-effective reduction map for $(K_X + \Delta)$ are all birationally equivalent.*

*Proof.* Since the conjectures imply that $K_X + \Delta$ is abundant, Theorem 3.6.7 shows that the Iitaka fibration is birationally equivalent to the pseudo-effective reduction map for $K_X + \Delta$. Furthermore, since the map $\phi : X \dashrightarrow X'$ to the minimal model is a birational contraction, $\phi^{-1}$ is an isomorphism on a neighborhood of a general curve $C$ contained in a general fiber of $f$. In particular, $(K_X + \Delta) \cdot C = 0$ for such a curve. Thus the $(K_X + \Delta)$-trivial reduction map also yields the Iitaka fibration. □

Since the pseudo-effective reduction map satisfies nicer properties than the Iitaka fibration, it can be used to study Conjecture 3.7.3. The key question is to understand when the pseudo-effective reduction map is non-trivial. Thus, we make the following conjecture:

**Conjecture 3.7.5.** Let $(X, \Delta)$ be a klt pair. Suppose that $\nu_{X|C}(K_X + \Delta) > 0$ for every movable curve $C$ on $X$. Then $K_X + \Delta$ is big.

Our main application of the pseudo-effective reduction map is the following theorem:

**Theorem 3.7.6.** *Conjecture 3.7.5 holds for all $X$ of dimension at most $n$ iff Conjecture 3.7.3 and Conjecture 3.7.1 hold up to dimension $n$.*

This result is an extension of [Amb04], which proves a similar theorem in the case when $K_X + \Delta$ is nef. The main point is that the results of [BCHM10] allow us to contract components of $N_\sigma(K_X + \Delta)$, as was first noticed in [Dru09]. If we assume that $\kappa(K_X + \Delta) > 0$, we can use the recent work of [Lai09] instead. A similar result appears in the recent preprint [Siu09].

*Proof.* Since the reverse implication follows from Proposition 3.7.4, it suffices to show the forward implication. Let $(X, \Delta)$ be a klt pair. It is enough to consider the case when $K_X + \Delta$ is pseudo-effective but not big. Thus by assumption there is a movable curve on $X$ with $\nu_{X|C}(K_X + \Delta) = 0$.

Consider the (non-trivial) pseudo-effective reduction map for $K_X + \Delta$. By Lemma 3.6.5, after passing to a new model $\phi : X' \to X$ we may assume the pseudo-effective reduction map is a morphism $f : X' \to Z$ and also that $P_\sigma(\phi^*(K_X + \Delta)) \equiv f^*\Xi$ for some divisor $\Xi$ on $Z$. Now, we can write

$$K_{X'} + \Delta' + \sum a_i E_i \equiv \phi^*(K_X + \Delta) + \sum b_j F_j$$

where the $a_i$ and $b_i$ are positive, $\Delta'$ is the strict transform of $\Delta$, and the $E_i$ and $F_j$ are $\phi$-exceptional and share no components in common. Since the $F_j$ are exceptional, we have

$$P_\sigma(\phi^*(K_X + \Delta) + \sum b_j F_j) = P_\sigma(\phi^*(K_X + \Delta)).$$

(See for example [Nak04] III.5.14.) After replacing $(X, \Delta)$ by $(X', \Delta' + \sum a_i E_i)$, it suffices to show abundance under the additional assumption that there is a morphism $f : X \to Z$ such that $P_\sigma(K_X + \Delta)$ is $f$-numerically trivial.

The next step is to use the arguments of [Dru09] to contract $N_\sigma$. Recall that $N_\sigma(K_X+\Delta)$ has only finitely many components and each is contained in the restricted base locus of $K_X+\Delta$. Thus, there is some small ample $A$ such that each component is in the restricted base locus of $K_X+\Delta+A$. By [BCHM10], we can run the $K_X+\Delta+A$-minimal model program over $Z$. As a result, we obtain a model $\psi : X \dashrightarrow \widehat{X}$ and $g : \widehat{X} \to Z$ such that $K_{\widehat{X}} + \widehat{\Delta} + A$ is $g$-nef. In particular, we must contract every component of $N_\sigma(K_X + \Delta)$ that is not $f$-numerically trivial. Thus $K_{\widehat{X}} + \widehat{\Delta}$ is $g$-numerically trivial.

Finally, we apply the arguments of [Amb04] to the resulting variety.

**Theorem 3.7.7.** *Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair, and $f : X \to Y$ a $(K_X + \Delta)$-numerically trivial fibration. After passing to birational models $f' : X' \to Y'$ (where $\mu$ denotes the birational map $X' \to X$), we have*

$$f'^*(P_\sigma(K_{Y'} + \Delta')) = P_\sigma(\mu^*(K_X + \Delta))$$

*for some $\Delta'$ on $Y'$ such that $(Y', \Delta')$ is klt.*

Note that abundance follows immediately by induction, proving Conjecture 3.7.3. In particular, we have weak non-vanishing in dimension up to $n$, so the main result of [Bir09] implies Conjecture 3.7.1. □

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