A Decentralized Incentive Mechanism for Company-Wide Energy Consumption Reduction

by

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Abstract

This thesis proposes a decentralized reward-based incentive mechanism to address the problem of noncomplying subsidiaries when the parent company wish to meet its targeted energy consumption level. Besides its effectiveness in ensuring compliance, the proposed mechanism is advantageous as it is able to induce the optimal subsidiary behavior that maximizes the company profit given a carefully chosen reward allocation scheme. In addition, when the company is willing to trade part of its profit for an operationally simple mechanism, simple uniform allocation scheme is highly effective when the subsidiaries exhibit certain degree of symmetry.

The results above are drawn from our investigation on a more general model: Cournot competition under a joint constraint. For this model, we study the equilibrium behavior under free competition and compare the profit and total surplus achieved with the corresponding values when different levels of coordination are introduced in the market (i.e., the Monopoly market and the society-wide coordinated market). We establish tight upper bounds for the profit and total surplus loss due to lack of coordination as functions of various market characteristics (i.e., number of firms, intensity of competition and asymmetry between firms).

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Finally, I dedicate this thesis to my parents, whose unconditional love and support made everything possible.

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1 Introduction

1.1 Motivation

In response to increasing social pressure, more and more companies, especially large corporations, are committed to energy consumption reduction goals as part of their Corporate Social Responsibility (CSR) initiative. PepsiCo, for instance, has pledged to cut water consumption by 20 percent, electricity by 20 percent and fuel by 25 percent by 2015. P&G also announced plans to cut energy and water use by 20 percent by 2012 (see [4]).

In the long run, these consumption reduction targets encourage technology innovation, and ultimately increase energy efficiency. In the short run, however, since the output level is often directly linked to the energy consumption level, the limitations in energy consumption effectively place a cap on the output level. This is often then translated into a series of production quotas for subsidiaries or divisions within the parent company. This gives rise to the typical 'principal-agent' problem where divisional managers deliberately overlook their assigned quotas in order to achieve a higher divisional profit, especially when the burden of over-consumption is not borne by each individual division or subsidiary.

The problem is further complicated when divisions compete with each other. In this competitive setting, the demand of a particular division is adversely affected by the output of other divisions. This could happen when divisions sell the same or slightly differentiated products in the same market, (i.e. different brands of shampoo from P&G). In this setting, the divisional goals of individual managers may differ significantly from the optimal strategy for the company as a whole.

Since company-wide coordination is often operationally impossible or too costly, it is necessary for the parent company to design alternative mechanisms to induce the 'optimal' behavior of its divisional managers. That is, a total consumption level under the promised energy reduction goal, and a distribution among divisions that maximizes the total profit for the company.

One common approach to align the different incentives is to use financial compensation.

For example, since 2008, Intel has established an employee engagement mechanism where a portion of each employee's variable compensation is dependent upon the company achieving its environmental sustainability goals [1]. Similarly, Hilton Hotel successfully met its goal of 5 percent reduction in energy consumption by tying the hotel general managers' annual bonuses to the energy performance at each property [2].

However, despite its effectiveness in encouraging compliance, simple financial compensation does not guarantee the maximum profit for the company without the 'correct' quota allocation. In this thesis, we study one possible reward-based mechanism to coordinate the behavior of divisional managers. We show that with careful choice of parameters, this mechanism is able to achieve the maximum profit for the company under the committed energy/resources consumption reduction goal. In addition, it is possible to avoid the complicated quota allocation process by using a simple (uniform) reward scheme if the company is willing to sacrifice a certain degree of optimality for operational simplicity. We quantify this sacrifice as the lower bound of the ratio between the profit achieved under the uniform scheme and the maximum possible profit under company-wide coordination.

The results obtained for this proposed reward-based mechanism can also be interpreted in the context of a more general problem: Cournot competition under a joint constraint (We will further elaborate the connection between the two problems in Chapter 2). Cournot (quantity) competition is often seen in industries where the output level could not be easily adjusted in the short-run (i.e. the manufacturing industry). Under this setting, the decision on output quantity is made first and the price is determined subsequently by the market clearing price. As firms compete with each other, the selling price of a particular firm decreases when any of the firms in the market (itself or its competitors) increase their output quantities. Equilibrium is reached when no firm can increase its profit by unilaterally changing its output quantity. Joint constraints arise when a common resource is shared among the competing firms (i.e., when firms purchase raw material from the same supplier with limited capacity). Compared to problems with disjoint constraints, the equilibrium behavior is more complicated under joint constraints since every player's strategy must belong to the feasible strategy space determined by its competitors. As a result, under this model, firms influence each other through both the market price and the feasible strategy space.

A large part of this thesis is dedicated to investigate the effectiveness of the equilibrium strategies for the model described above. More specifically, we compare the profit and social surplus achieved under free competition with the values achieved when different levels of coordination are introduced in the market. The losses of profit and total surplus are quantified as functions of various market characteristics (i.e., number of firms, intensity of competition and asymmetry between firms). In addition, we will look into the equilibrium strategies themselves and explain the underlying rationale behind the discrepancies in price and quantity.

1.2 Literature Review

Issues of coordination and efficiency have been studied extensively in the supply chain management literature. Since vertical or horizontal integration is not always possible, various contracts have been designed with the aim of aligning different incentives and maximizing the total profit for the supply chain. We refer the reader to Cachon and Lariviere (2005) for a discussion on revenue-sharing contracts, and Cachon and Kok (2010) for a comparison between different types of contracts.

Besides designing mechanisms to coordinate the system, a large body of research focused on quantifying the loss of efficiency due to lack of coordination. The concept of 'Price of Anarchy' was first introduced in Koutsoupias and Papadimitriou (1999) as the ratio between the performance of the worst case Nash equilibrium and the performance of the coordinated solution. This concept has been used extensively to quantify the loss of efficiency in transportation networks (see Roughgarden and Tardos (2002), Roughgarden (2005) and Perakis (2007)).

For the loss of efficiency in uncoordinated one-tier supply chains, two types of models have been considered : Betrand (1883) competition where firms compete by setting prices, and Cournot (1838) competition where firms decide the quantities they are willing to sell and the selling price is then determined by the market clearing price. Loss of efficiency under Betrand competition has been studied, in for example, Farahat and Perakis (2007), Sun (2006) and Bernstein and Federgruen (2003). For Cournot competition models, Guo and Yang (2005) studied the loss of social surplus when firms face no production constraints. Kluberg and Perakis (2009) considered a model with general convex production constraints. They provided upper bounds for the loss of total surplus and profit which depend only on the number of competing firms and their market power.

The model studied in this thesis is different from the literature as it deals with Cournot competition under a joint constraint. While competition in the unconstrained (or uncoupled constrained) models lead to the unique Nash-Equilibrium (where no firm can improve its profit by unilaterally changing its strategy), competition under 'joint' or 'coupled' constraints belongs to the class of Generalized Nash-Equilibrium problems (GNEPs). The concept of Generalized Nash-Equilibrium (GNE) (often referred to as pseudo-game or abstract economy) was first formally introduced in Debreau (1952). It is a generalization of the Nash-Equilibrium concept where the choice of an action by one player affects both the pay-off and the domain of actions of other players. Under some convexity assumptions, GNEP can be reformulated as a 'constrained quasi-optimization problem' so that a point is a GNE if and only if it is a solution of the corresponding quasi-variational inequality (see Bensoussan, 1974). Unlike the class of Nash equilibrium problems, uniqueness of solution is rare for GNEPs. Rosen (1965) introduced the notion of the normalized Nash-equilibrium as a special case of the Generalized Nash-Equilibrium and provided conditions on its existence and uniqueness.

GNEPs have a wide application in modeling problems where a common resource is shared by players (see for example, Pang et al. (2007) for an application to power allocation problems in telecommunications, and Adida and Perakis (2009) for a model in dynamic pricing and inventory management). We refer the reader to Facchinei and Kanzow (2007) for a comprehensive survey on GNEPs.

The model considered in this thesis is closely related to the applications of GNE to the environmental economics problems (see for example, Haurie and Krawczyk (1997), Krawczyk (2000), Krawczyk and Uryasev (2000)). Breton et al. (2005) provided a game-theoretic interpretation of a joint implementation mechanism of Kyoto's protocol where the restrictions on the prescribed party emission reduction units (ERU) are modeled as joint constraints faced by all participating countries. Krawczyk (2005) considered a river basin pollution game where the regional government enforces compliance through Pigouvian taxes. This model is similar to ours in terms of the discussion on the one-to-one correspondence between a tax strategy and the equilibrium behavior. However, as most literature on GNEPs(see for example, Morgan and Scalzo (2004), and Pang and Fukushima (2005)) , the focus of this paper is on establishing conditions for existence and solution algorithms. The closest resemblance of our model is the environmental compliance problem studied in Tidball and Zaccour (2005). Three different scenarios are analyzed and compared in their paper, namely, the noncooperative scenario where each player optimizes under a separate environmental constraint; the cooperative scenario where coordination among players is introduced to optimize under a single constraint on the total emission level; and the umbrella scenario where players remain independent but face a joint constraint together. The utility functions considered are general concave functions. However, additional assumptions were imposed in their model to ensure the existence of an unique normalized Nash equilibrium, and the comparisons between different scenarios are based on the total emissions levels only. This is different from the Cournot competition model considered in this thesis as we allow the existence of multiple equilibria and compare between scenarios where different levels of coordination (i.e., the Monopoly market and the society-wide coordinated market) are introduced in the system. This thesis also considers other issues, including how the equilibrium prices and quantities compare to those in other settings (i.e., the Monopoly market and the society-wide coordinated market), as well as the corresponding profits and social surpluses.

1.3 Main Contributions and Thesis Outline

The contributions of this research project are two-fold:

Firstly, we propose a decentralized reward-based incentive mechanism that is applicable to companies with the goal of regulating the strategies of their subsidiaries under a pre-committed energy consumption target. We demonstrate the equivalence of the proposed scheme with a Cournot competition game under a joint constraint. To the best of our knowledge, this is among the first attempts to utilize the abstract concept of Generalized Nash-Equilibrium in a more practical setting;

Secondly, we quantified the loss of efficiency due to free competition in Oligopoly markets as a function of market characteristics (i.e., number of firms, intensity of competition and asymmetry between firms). The results obtained are new to the literature as the constraint considered in our model is a joint constraint. Since the equilibrium strategies are not unique in our model, we focused on two special equilibria: the worst Nash-Equilibrium and the Normalized Nash-Equilibrium. In addition, for a market consisting of symmetric (identical) firms, we characterized the worst and best Oligopoly equilibrium analytically at any fixed capacity level and provided insights on the equilibrium prices and quantities.

This thesis is structured as follows: In Chapter 2, we first introduce the notations and assumptions used throughout this thesis. We then present the mathematical formulation of a reward-based incentive mechanism and show its equivalence to the general model of Cournot competition under a joint constraint. In Chapter 3, we analyze the loss of profit and social surplus for both the Worst Nash-Equilibrium and the Normalized Nash-Equilibrium under the general model without any restrictions on the market characteristics. Chapter 4, 5 and 6 each presents a special case with various market characteristics: Chapter 4 considers a market consisting of many symmetric firms. That is, firms have the same price potential and are uniform in their abilities to influence their own selling prices as well as the prices of their competitors; we then relax this assumption to consider firms with only symmetric price potential and only symmetric price influence in Chapter 5 and Chapter 6, respectively. In Chapter 7, we discuss the implications of the results obtained from Chapter 3 to Chapter 5 on the proposed decentralized incentive-mechanism. Finally, Chapter 8 discusses the conclusions of this thesis.

2 Model Description

Consider a company with n subsidiaries. The parent company is committed to a maximum total energy consumption level, denoted by C. Each subsidiary sells a single differentiated product targeting basically the same market. Assume that the n subsidiaries are the only major players in the market and compete with each other through the quantities of the products they are selling. The output level at each subsidiary is decided simultaneously and independently by the subsidiary managers with the objective of maximizing the profit at the subsidiary level. We use the vector $\mathbf{d} = (d_1, .., d_n)$ to describe the subsidiary output levels, where d_i denotes the quantity produced by subsidiary i.

It is a common practice in the industry (see [3]) that an Environmental Conservation committee is established by the parent company to ensure the compliance with the committed energy reduction target. The strategy employed by this independently funded committee is to offer a financial reward to each subsidiary for a company-level total energy consumption below the target. For each unit of energy consumption below the target C, the committee offers a reward of μ to the subsidiaries. Each subsidiary receives a portion of the reward according to their weights in the allocation vector $\mathbf{w} = (w_1, ..., w_n)$ (Assume that $w_i > 0$ and $\sum w_i = 1$.). That is, the unit reward received by subsidiary i is μw_i. Since the Environmental Conservation committee is usually also responsible for monitoring other energy reduction activities within the company, it is of its interest to minimize the actual payout of the reward without compromising its effectiveness.

2.1 Notations and Assumptions

The Cournot competition model considered in this thesis adopts the same set of assumptions discussed in Kluberg and Perakis (2009), which is drawn from the traditional approach in the literature (see Vives (2001), Chapter 6 for Cournot competition models).

- The Cournot (quantity) competition between subsidiaries is modeled by an affine price-quantity relationship:

$$\mathbf{p}(\mathbf{d}) = B\bar{\mathbf{d}} - B\mathbf{d},$$

where $B = \mathbf{M}^{-1}$ for some symmetric, strictly diagonally dominant M-matrix[1] $\mathbf{M}$. That is,

$$M = \begin{bmatrix} M_{11} & -M_{12} & \dots & -M_{1n} \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ -M_{n1} & \dots & -M_{n(n-1)} & M_{nn} \end{bmatrix}$$

for some $M_{ii} > \sum_{j \neq i}^{n} M_{ij}$ for all $i = 1, 2, ..., n$, and $M_{ij} \geq 0$, for all $i, j = 1, 2, ..., n$.

In addition, we use Γ to denote the diagonal matrix containing the diagonal elements of B.

- We refer to the vector $\bar{\mathbf{p}} = B\bar{\mathbf{d}}$ as the price potential since it represents the maximum market prices of the products. The corresponding vector $\bar{\mathbf{d}}$ are referred to as the demand potential.
- Ignoring the production costs, the profits of the firms are the same as the revenues:

$$\Pi(\mathbf{d}) = \mathbf{d}^T\mathbf{p}(\mathbf{d}) = \mathbf{d}^T(\bar{\mathbf{p}} - B\mathbf{d})$$

- The consumer surplus is computed using a quadratic utility function of a representative consumer:

$$CS(\mathbf{d}) = \frac{1}{2}\mathbf{d}^T B\mathbf{d}$$

(see Kluberg and Perakis(2009) for the derivation of consumer surplus).

---

[1] A matrix $\mathbf{A}$ is called an M-Matrix if $\mathbf{A} \in \mathbf{Z}_n$ and $\mathbf{A}$ is positive stable (if every eigenvalue has positive real part), where $\mathbf{Z}_n = \{\mathbf{A} = [a_{ij}] \in \mathbf{M}_n(\Re) : a_{ij} \leq 0 \forall i \neq j, i, j = 1, 2, ..., n\}$ (refer to Johnson (1982) for properties of M-matrices).

- Total social surplus considers the welfare of both the firms and the consumers. It is the sum of firm profits and consumer surplus:

$$TS(\mathbf{d}) = \Pi(\mathbf{d}) + CS(\mathbf{d}) = \mathbf{d}^T(\bar{\mathbf{p}} - \frac{1}{2}B\mathbf{d})$$

- Market Power is defined as:

$$r_i = \frac{\sum_{j\neq i}^{n} \mathbf{M}_{ij}}{\mathbf{M}_{ii}} \in [0,1), \forall i = 1,2,...,n, \text{ and } r = \max_i r_i.$$

  r measures the intensity of competition since when $r \to 0$, M is a diagonal matrix, and the strategy of a particular firm have little influence over the strategy of other firms; when $r \to 1$, the total demand in the market remains constant regardless of the market prices, suggesting a highly competitive market where the customers lost by a particular firm following a price increase are captured entirely by its competitors. We refer the reader to Kluberg and Perakis(2009) for a discussion on the market power.

- We further assume that the energy consumption level is linked directly to the output quantity. As a result, the targeted energy consumption level is modeled as a constraint on the sum of the total output:

$$\mathbf{e}^T\mathbf{d} \leq C.$$

In addition, we formally present here the definitions of the Nash Equilibrium and the Generalized Nash Equilibrium:

Definition (Nash Equilibrium) A point is called a Nash Equilibrium if no player can improve his utility function by unilaterally changing its strategy.

Definition (Generalized Nash Equilibrium) A point is called a Generalized Nash Equilibrium if no player can improve his utility function by unilaterally changing its strategy within the feasible strategy space defined by the strategies of other players.

Note here that the Generalized Nash Equilibrium reduces to the standard Nash equilibrium when the feasible strategy space do not depend on the rival players' strategies.

2.2 Reward-Based Incentive Mechanism

In this decentralized setting where the subsidiaries and the Environmental Conservation committee all act according to their own interests, their behavior can be modeled through the following bi-level game: the Environmental Conservation committee is the leader of the game who anticipates each subsidiary's output quantity as a function of the reward allocation vector $\mathbf{w}$ and unit reward μ. Given a predetermined allocation scheme $\mathbf{w}$, the Environmental Conservation committee chooses the optimal unit reward μ^* in order to minimize the actual payments of the reward:

$$\begin{aligned} \mu^* = \text{argmin}_{\mu}\mu(C - \sum_{i=1}^{n} d_i^*(\mu)) \\ \text{s.t.} \sum_{i=1}^{n} d_i^*(\mu) \leq C \\ \mu \geq 0 \end{aligned}$$

For subsidiary i, the problem is to choose the optimal strategy d_i^* given the reward scheme $(\mathbf{w}, \mu)$ and the strategies of other divisions $\mathbf{d}_{-i}^*(\mu)^2$:

$$\begin{aligned} d_i^*(\mu) = \text{argmax}_{d_i \geq 0} & \underbrace{d_i(\bar{p}_i - \sum_{j \neq i}^{n} B_{ij} d_j^*(\mu) - B_{ii} d_i)}_{\text{sales}} \\ & + \underbrace{w_i \mu (c - \sum_{j \neq i}^{n} d_j^*(\mu) - d_i)^+}_{\text{compliance reward}} \end{aligned}$$

Note that there are infinitely many possible reward allocation vectors $\mathbf{w}$. For the rest of this thesis, we refer to the particular allocation where $w_1 = w_2 = ... = w_n = \frac{1}{n}$ as the uniform reward allocation scheme.

---

[2] We use $\mathbf{d}_{-i}$ to denote the vector resulted from removing the ith element from the vector $\mathbf{d}$.

2.3 Cournot competition under a joint constraint

In this section, we introduce a more general model: Cournot competition under a joint constraint.

In the absence of coordination, each firm chooses the optimal output level that maximizes its profit. In addition, the strategy employed by each firm must lie inside the feasible strategy space defined by the joint constraint on the total output:

$$\begin{aligned} d_i^{OP} &= \text{argmax}_{d_i} \Pi(d_i, \mathbf{d}_{-i}^{OP}) \\ \text{s.t. } & d_i + \sum_{j \neq i} d_j^{OP} \leq C \\ & d_i \geq 0 \end{aligned} \tag{1}$$

For the rest of this thesis, we refer to problem (1) as the Oligopoly problem, and use $S = \left\{\mathbf{d}^{OP}\right\}$ to denote the set of equilibria.

The KKT conditions of the above problem are:

$$\begin{cases} \bar{\mathbf{p}} - (B + \Gamma)\mathbf{d}^{OP} - \mu + \lambda = \mathbf{0} & \\ \mathbf{e}^T \mathbf{d}^{OP} \leq c & \\ \mu_i (c - \mathbf{e}^T \mathbf{d}) = 0 & \\ \lambda_i d_i^{OP} = 0 & \forall i = 1, 2, ..., n \\ \lambda_i \geq 0, \mu_i \geq 0, d_i^{OP} \geq 0 & \end{cases}$$

The solution to the Oligopoly problem is not unique since the multipliers μ_i (corresponding to the joint constraint) are not related to each other. However, if we specify the relationship between these multipliers using a strictly positive vector $\mathbf{w}$ such that

$$\mu_i = \frac{\mu_0}{w_i}, i = 1, ..., n$$

for some $\mu_0 \geq 0$, then there exists a unique equilibrium. In particular, the solution corresponding to the vector $\mathbf{w}$ with $w_1 = w_2 = ... = w_n$ is called the Normalized Nash-Equilibrium. A rigorous proof on the existence and uniqueness of the equilibrium corresponding to every specified positive vector $\mathbf{w}$ can be found in Rosen(1965) (see Theorems 3 and 4).

2.4 Equivalence of the two models

Theorem 2.1 *Every equilibrium strategy in the bi-level game belongs to the set of Oligopoly equilibrium S. In particular, when the reward scheme $\mathbf{w}$ is uniform, the resulting equilibrium strategy is the unique Normalized Nash Equilibrium.*

Proof Given a reward-allocation scheme $\mathbf{w}$, the output quantity for each subsidiary is a function of the unit reward μ given by:

$$d_i^*(\mu) = \text{argmax}_{d_i \geq 0} \left\{ d_i(\bar{p}_i - \sum_{j \neq i}^{n} B_{ij} d_j^*(\mu) - B_{ii} d_i) + w_i \mu (c - \sum_{j \neq i}^{n} d_j^*(\mu) - d_i) \right\}$$

Writing down the KKT conditions for the above problem, we have that

$$\begin{cases} \bar{p}_i - \sum_{j \neq i}^{n} B_{ij} d_j^* - 2B_{ii} d_i^* - w_i \mu + \lambda_i = 0 & \\ \lambda_i d_i^* = 0 & \forall i = 1, 2, ..., n \\ \lambda_i \geq 0, d_i^* \geq 0 & \end{cases}$$

$$\Leftrightarrow \begin{cases} \bar{\mathbf{p}} - (B + \Gamma)\mathbf{d}^* - \mu \mathbf{w} + \lambda = \mathbf{0} \\ \lambda_i d_i^* = 0 \\ \lambda_i \geq 0, d_i^* \geq 0 \end{cases}$$

where $\mathbf{w}$ and λ are vectors of w_i and λ_i for $i = 1, 2, ..., n$, respectively.

$$\begin{aligned} \Rightarrow \mathbf{e}^T \mathbf{d}^* &= \mathbf{e}^T (B + \Gamma)^{-1} (\bar{\mathbf{p}} - \mu \mathbf{w} + \lambda) \\ &= \mathbf{e}^T (B + \Gamma)^{-1} (\bar{\mathbf{p}} + \lambda) - \mathbf{e}^T (B + \Gamma)^{-1} \mathbf{w} \mu \end{aligned}$$

Note here that $\mathbf{e}^T \mathbf{d}^*$ is a continuous and decreasing function of μ [3]. Hence given a fixed consumption target C, we can always ensure compliance by using a sufficiently large unit reward μ.

---

[3] This follows since $\mathbf{e}^T (B + \Gamma)^{-1} = \underbrace{\mathbf{e}^T B^{-1}}_{>0} \underbrace{B (B + \Gamma)^{-1}}_{>0} > \mathbf{0}$

Now consider the problem faced by the Environmental Conservation committee:

$$\begin{aligned}&\min_{\mu} F(\mu)\\ &\text{s.t. } \mathbf{e}^T\mathbf{d}^*(\mu) \leq C\\ &\mu \geq 0\end{aligned}$$

where $F(\mu) = \mu(C - \mathbf{e}^T\mathbf{d}^*(\mu))$. $F(\mu) \geq 0$ for every feasible μ.

- If $\mathbf{e}^T\mathbf{d}(0, \mathbf{w})) \leq C$, then $\mu = 0$ is the optimal solution since $F(0) = 0$;
- If $\mathbf{e}^T\mathbf{d}(0, \mathbf{w})) > C$, since $\mathbf{e}^T\mathbf{d}(\mu, \mathbf{w})$ is continuously decreasing in μ, there exists a strictly positive μ^* satisfying $\mathbf{e}^T\mathbf{d}(\mu^*, \mathbf{w}) = C$. μ^* is the optimal solution since $F(\mu^*) = \mu^*(C - \mathbf{e}^T\mathbf{d}^*(\mu)) = 0$.

Summarizing the above discussions, the solution for the bi-level game is characterized by

$$\left\{\begin{array}{c}\bar{\mathbf{p}} - (B+\Gamma)\mathbf{d}^* - \mu^*\mathbf{w} + \lambda = \mathbf{0}\\ \mu^*(C - \mathbf{e}^T\mathbf{d}^*) = 0\\ \mathbf{e}^T\mathbf{d}^* \leq C\\ \lambda \geq \mathbf{0}\\ \lambda_i d_i^* = 0, d_i^* \geq 0\\ \mu^* \geq 0\end{array}\right. \Leftrightarrow \left\{\begin{array}{c}\bar{\mathbf{p}} - (B+\Gamma)\mathbf{d}^* - \mu^*\mathbf{w} + \lambda = \mathbf{0}\\ \mu^* w_i(C - \mathbf{e}^T\mathbf{d}^*) = 0\\ \mathbf{e}^T\mathbf{d}^* \leq C\\ \lambda \geq \mathbf{0}\\ \lambda_i d_i^* = 0, d_i^* \geq 0\\ \mu^* \geq 0\end{array}\right.$$

since $w_i > 0$ for $i = 1, 2, ..., n$.

$\mu w_i = \tilde{\mu}_i$ satisfies the KKT conditions for problem (1). In addition, when $w_1 = w_2 = ... = w_n$, we have $\tilde{\mu}_1 = ... = \tilde{\mu}_n$, which by definition, is the unique Normalized Nash-Equilibrium. ∎

2.5 Company Objective

If company-wide coordination is possible, the optimal output level is the one that maximizes the sum of all divisional profits under the committed energy consumption level. In the general Cournot Competition model, this corresponds to the Monopoly game when the n firms are controlled by one single authority:

$$
\begin{aligned}
\mathbf{d}^{MP} = \operatorname{argmax}_{\mathbf{d}} \mathbf{d}^T(\bar{\mathbf{p}} - \mathbf{B}\mathbf{d}) \\
\text{s.t. } \mathbf{e}^T\mathbf{d} \leq C \\
\mathbf{d} \geq \mathbf{0}
\end{aligned} \tag{2}
$$

For the rest of this thesis, we refer to the above optimization problem as the Monopoly problem and use $\mathbf{d}^{MP}$ to denote the corresponding optimal solution.

Theorem 2.2 *When the constraint is active for the Monopoly problem ($\mathbf{e}^T\mathbf{d}^{MP} = C$), $\mathbf{d}^{MP}$ belongs to the set of equilibrium strategies of the Oligopoly problem S.*

Proof To prove this theorem, we first show that when the constraint is active for the Monopoly problem, it must be active for the Oligopoly problem as well.

This follows since in the unconstrained case, $\mathbf{d}^{OP} = (B+\Gamma)^{-1}B\bar{\mathbf{d}}$ and $\mathbf{d}^{MP} = \frac{1}{2}\bar{\mathbf{d}}$,

$$
\begin{aligned}
\mathbf{e}^T\mathbf{d}^{OP} &= \mathbf{e}^T(B+\Gamma)^{-1}B\bar{\mathbf{d}} \\
&= \frac{1}{2}\mathbf{e}^T(B+\Gamma)^{-1}(2B)\bar{\mathbf{d}} \\
&\geq \frac{1}{2}\mathbf{e}^T(B+\Gamma)^{-1}(B+\Gamma)\bar{\mathbf{d}} \\
&= \frac{1}{2}\mathbf{e}^T\bar{\mathbf{d}} \\
&= \mathbf{e}^T\mathbf{d}^{MP}
\end{aligned}
$$

For every solution $\mathbf{d}^{OP}$ that belongs to the set of equilibrium strategies of the Oligopoly problem, the following Quasi-Variational-Inequality holds (see Bensoussan (1974)):

$$
\left(-\bar{p}_i + (B+\Gamma)_i\mathbf{d}^{OP}\right)(d_i - d_i^{OP}) \geq 0 \ \forall d_i \in K(\mathbf{d}_{-i}^{OP}) \tag{3}
$$

where $K(\mathbf{d}_{-i}^{OP})$ is the feasible strategy space for player i given the strategy $\mathbf{d}_{-i}^{OP}$ of the other players. Since the constraint is active for every oligopoly equilibrium, the feasible strategy space is simply $K(\mathbf{d}_{-i}^{OP}) = \left\{d_i | d_i + \sum_{j\neq i}^{n} d_j^{OP} \leq C, d_i \geq 0\right\} = \left\{d_i | 0 \leq d_i \leq d_i^{OP}\right\}$.

Let $I = \left\{ i | d_i^{OP} > 0 \right\}$. From equation (3), we have

$$-\bar{p}_i + (B+\Gamma)_i \mathbf{d}^{OP} \leq 0, \ \forall i \in I.$$

On the other hand, $\mathbf{d}^{MP}$ satisfies the following KKT conditions for problem (2):

$$\begin{cases} -\bar{\mathbf{p}} + 2B\mathbf{d}^{MP} + \mu\mathbf{e} - \lambda = 0 \\ \mu(C - \mathbf{e}^T\mathbf{d}^{MP}) = 0 \\ \mathbf{e}^T\mathbf{d}^{MP} \leq C \\ \mu \geq 0 \\ \lambda_i d_i^{MP} = 0, \lambda_i \geq 0, \forall i = 1, 2, ..., n \end{cases}$$

For every i such that $d_i^{MP} > 0$, $-\bar{p}_i + 2B_i\mathbf{d}^{MP} + \mu = \lambda_i = 0$.

$$\begin{aligned} \Rightarrow -\bar{p}_i + (B+\Gamma)_i \mathbf{d}^{MP} &\leq -\bar{p}_i + 2B_i\mathbf{d}^{MP} \\ &= -\mu_i \\ &< 0 \\ \Rightarrow \mathbf{d}^{MP} &\in S \end{aligned}$$

∎

This implies that when the restriction on energy consumption decreases the maximum possible profit for the company (when the constraint is active for the Monopoly problem), with a carefully chosen allocation scheme **w**, the reward-based incentive mechanism is able to induce the optimal subsidiary strategy that minimizes this loss.

2.6 Social Objective

From the society's perspective, (that is, considering also the consumers' utility) the optimal production quantity solves the following problem:

$$\begin{aligned} \mathbf{d}^{SMAX} &= \text{argmax}_{\mathbf{d}} \mathbf{d}^T(\bar{\mathbf{p}} - \frac{1}{2}\mathbf{B}\mathbf{d}) \\ &\text{s.t. } \mathbf{e}^T\mathbf{d} \leq C \\ &\mathbf{d} \geq \mathbf{0} \end{aligned}$$

For the rest of this thesis, we refer to the above optimization problem as the 'SMAX' problem and use $\mathbf{d}^{SMAX}$ to denote the corresponding optimal solution.

2.7 Existence and Uniqueness of Solutions

In our model, the price-quantity relationship is modeled through a linear function and an M matrix so that the resulting expression for total social surplus and firm profit are concave functions of the output quantity $\mathbf{d}$. Hence both the SMAX problem and the Monopoly problem are maximization problems of a concave function over a simplex constraint. The existence and uniqueness of an optimal solution follow easily.

The existence of the set of equilibrium strategies for the Oligopoly problem and the uniqueness of the Normalized Nash-Equilibrium follows from Rosen (1965).

3 General Case

In this chapter, we present some results that are applicable for the general model without any assumptions on the market characteristics. To avoid confusion on notations, all results from Chapter 3 to Chapter 6 are presented in the context of a Cournot competition model under a joint constraint. We will discuss the implications of these results on the incentive mechanism in Chapter 7.

Theorem 3.1 *Without any constraint on the output quantity, competition among firms always results in an equilibrium strategy that is more beneficial to the consumers compared with the case when firms are allowed to collude:* $CS(OP) \geq CS(MP)$. *Equality is achieved when firms are independent.*

Proof Refer to Appendix A.1.

Theorem 3.2 *Without any constraint on the output quantity, the social surplus achieved under free competition is always larger than the social surplus achieved in the monopoly market:* $TS(OP) \geq TS(MP) = \frac{3}{4}TS(SMAX)$. *Equality is achieved when firms are independent.*

Proof Refer to Appendix A.2.

This suggests that in the absence of production constraints, free competition is always preferred from the society's perspective. It is also interesting to notice that regardless of the number of firms, the maximum loss in social surplus (25%) is always achieveable when firms have no influence over the price of their competitiors. In this case, the equilibrium strategy coincides with the Monopoly strategy so that the firms are able to extract the maxmium profit from consumers even in the decentralized setting.

The two theorems above are extensions of the results proven in Kluberg and Perakis (2009) to the totally unconstrained case as we relaxed the restriction of $\mathbf{d} \leq \bar{\mathbf{d}}$. It is also shown in their paper that these results still hold in the constrained case when every firm faces a single separate constraint. However, for the joint constraint considered in this thesis, these results are no longer true as there will be multiple equilibrium strategies for the Oligopoly problem when the constraint is active. In fact, as will be shown in the following chapters, even for under the Normalized Nash-Equilibrium which is unique, the corresponding social and consumer surplus could be less than the ones achieved when firms collude. In these cases, colluding induces a more efficient allocation of scarce resource among firms which benefit both the firms themselves and the society as a whole.

Theorem 3.3 *When the joint constraint is active for the Oligopoly problem and the price potentials are asymmetric across firms, there is no nontrivial constant upper bound on the percentage loss in total surplus and profit for the worst equilibrium solution.*

This result can be shown through the following example of duopoly competition:

- $N = 2$
- $\mathbf{B} = \begin{bmatrix} \beta_1 & \alpha \\ \alpha & \beta_2 \end{bmatrix}$, for some $\beta_1 \geq \beta_2 > \alpha \geq 0$,
- $\bar{\mathbf{d}} = \begin{bmatrix} \bar{d}_0 \\ \bar{d}_0 \end{bmatrix}$, for some $\bar{d}_0 > 0$.

$$\Rightarrow \bar{p}_1 = (\beta_1 + \alpha)\bar{d}_0 \geq (\beta_2 + \alpha)\bar{d}_0 = \bar{p}_2$$

when $C < min\left\{\frac{\bar{p}_1 - \bar{p}_2}{2(\beta-\alpha)}, \frac{\bar{p}_2}{2\beta}\right\}$, the optimal solution of the SMAX problem and the Monopoly problem is given by (see Appendix A for the derivation of solutions for the duopoly problem)

$$\mathbf{d}^{SMAX} = \mathbf{d}^{MP} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

On the other hand, consider one particular equilibrium for the Oligopoly problem: $\mathbf{d}^{worst} = \begin{bmatrix} 0 \\ c \end{bmatrix}$, we have

$$\frac{r_2}{r_1} = \frac{\beta_2}{\beta_1} \leq \frac{TS(\mathbf{d}^{worst})}{TS(SMAX)} = \frac{C(\bar{p}_2 - \frac{1}{2}\beta_2 C)}{C(\bar{p}_1 - \frac{1}{2}\beta_1 C)}\frac{\bar{p}_2}{\bar{p}_1} = \frac{\beta_2 + \alpha}{\beta_1 + \alpha} = \frac{r_2 + 1}{r_1 + 1}$$

$$\frac{r_2}{r_1} = \frac{\beta_2}{\beta_1} \leq \frac{\Pi(\mathbf{d}^{worst})}{\Pi(MP)} = \frac{C(\bar{p}_2 - \beta_2 C)}{C(\bar{p}_1 - \beta_1 C)} \leq \frac{\bar{p}_2}{\bar{p}_1} = \frac{\beta_2 + \alpha}{\beta_1 + \alpha} = \frac{r_2 + 1}{r_1 + 1}$$

Both the loss in total surplus ($1 - \frac{TS(\mathbf{d}^{worst})}{TS(SMAX)}$) and loss in profit ($1 - \frac{\Pi(\mathbf{d}^{worst})}{\Pi(MP)}$) goes up to 100% as $\frac{r_2}{r_1} \to 0$.

As shown in the example above, when the capacity constraint is extremely restrictive, both the Monopoly solution and the SMAX solution will allocate all available capacities to the most 'efficient' firms. However, the set of Oligopoly equilibrium solutions, in this extreme case, encompass all feasible solutions. The maximum loss occurs when all capacities are allocated to the least efficient firm.

Theorem 3.4 *When the constraint is active for the Oligopoly problem, the loss in total surplus for the Normalized Nash-Equilibrium is bounded by:*

$$TS(NNE) \geq \frac{5}{6} TS(MP)$$

$$TS(MP) \geq \frac{2}{3} TS(SMAX)$$

$$TS(NNE) \geq \frac{5}{9} TS(SMAX)$$

Proof The results follow from Kluberg and Perakis (2009) since the Variational Inequality (VI) formulation for the Normalized Nash-Equilibrium is the same as the VI for the disjoint constraints. ∎

Although these bounds are derived on general convex feasible sets and are not tight for the single constraint considered in our model, they do provide some insights on the distinct feature of using the uniform reward allocation. As suggested in the theorem, while the loss in total surplus for the worst Oligopoly equilibrium is unbounded, we are guaranteed a no more than 44% loss for the Normalized Nash-Equilibrium.

4 Many Symmetric Firms

In this chapter, we consider the situation where n symmetric firms compete in the market. By symmetry, we mean that firms have the same demand potential and are identical in their abilities to influence the market prices.

4.1 Assumptions and Closed-form Solutions

- The demand potentials are symmetric across firms: $\bar{\mathbf{d}} = \bar{d}_0\mathbf{e}$, for some $\bar{d}_0 > 0$;
- The assumption on symmetric price influence is reflected on the uniformity of matrix B: holding the quantities of other firms constant, when firm i increases its quantity by 1 unit, its own selling price decreases by B_{ii} and the price of firm j decrease by B_{ij} $(j \neq i)$. By assuming that firms are identical in their abilities to influence their own selling price, we have that $B_{11} = B_{22} = \ldots = B_{ii} = \beta$, for some $\beta > 0$. Similarly, the assumption that firms are identical in their abilities to influence the prices of their competitors implies $B_{ij} = \alpha$, for some $0 \leq \alpha < \beta$ for all $j \neq i$:

$$\mathbf{B} = \begin{bmatrix} \beta & \alpha & \dots & \alpha \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ \alpha & \dots & \alpha & \beta \end{bmatrix} = \mathbf{M}^{-1} = \begin{bmatrix} M & -m & \dots & -m \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ -m & \dots & -m & M \end{bmatrix}^{-1},$$

  for some $M > (n-1)m \geq 0$.

- $\bar{\mathbf{p}} = \mathbf{B}\bar{\mathbf{d}} = \bar{p}_0\mathbf{e}$, where $\bar{p}_0 = (\beta + (n-1)\alpha)\bar{d}_0$;
- Market power: $r = \frac{(n-1)m}{M} = \frac{(n-1)\alpha}{(n-2)\alpha+\beta}$

The closed-form solutions are summarized as follows: (refer to Appendix B for a proof)

$$\begin{aligned} d_i^{MP} &= \min\left\{\frac{C}{n}, \frac{\bar{p}_0}{2(\beta+(n-1)\alpha)}\right\} \\ d_i^{SMAX} &= \min\left\{\frac{C}{n}, \frac{\bar{p}_0}{\beta+(n-1)\alpha}\right\} \quad \forall i = 1, 2, \cdots, n. \\ d_i^{NNE} &= \min\left\{\frac{C}{n}, \frac{\bar{p}_0}{2\beta+(n-1)\alpha}\right\} \end{aligned}$$

$$\mathbf{d}^{OP} = \begin{cases} \frac{\bar{p}_0}{2\beta+(n-1)\alpha}\mathbf{e} & \text{, when } \frac{\bar{p}_0}{2\beta+(n-1)\alpha} \leq \frac{C}{n}; \\ \left\{\mathbf{d}|0 \leq d_i \leq \frac{\bar{p}_0-\alpha C}{2\beta-\alpha}, \sum d_i = C\right\} & \text{, otherwise.} \end{cases}$$

When the capacity is restrictive, the Monopoly solution, SMAX solution and Normalized Nash-equilibrium coincide: $\left\{\mathbf{d}|\ d_i = \frac{C}{n}, \forall i\right\}$. The set of Oligopoly Equilibria, on the other hand, encompass all possible allocations when $C \leq \frac{\bar{p}_0}{2\beta}$.

4.2 Loss of efficiency

Theorem 4.1 *For symmetric firms facing a single joint capacity constraint, the profit achieved under free competition compared with the maximum possible profit achieved when firms collude is characterized as follows:*

1. *When the capacity constraint is not active for both the Monopoly problem and the Oligopoly problem (i.e., $C \geq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}}$), then*

   $$\frac{\Pi(OP)}{\Pi(MP)} = \frac{4(1+\frac{1}{\frac{1}{r}-1+\frac{1}{n-1}})}{(2+\frac{1}{\frac{1}{r}-1+\frac{1}{n-1}})^2} \geq \frac{4n}{(n+1)^2};$$

   The inequality is tight under intense competition (i.e., $r \to 1$).

2. *When the capacity constraint is active for both problems (i.e., $C \leq \frac{\bar{p}_0}{2\alpha+\frac{2(\beta-\alpha)}{n}}$), the profit loss for the worst Oligopoly Equilibrium is no more than 50% percent:*

   $$\frac{\Pi(OP)}{\Pi(MP)} \geq \left\{ \begin{array}{ll} 1 - \frac{\frac{1}{k}-\frac{1}{n}}{\frac{1}{(n-1)(\frac{1}{r}-1)}\frac{1}{k}+\frac{2}{k}-\frac{1}{n}} > \frac{1}{2-\frac{k}{n}} & \textit{when } \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}} < C \leq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k+1}} \\ & k = 1, 2, ..., n-1 \\ 1 - \frac{1-\frac{1}{n}}{\frac{1}{(n-1)(\frac{1}{r}-1)}+2-\frac{1}{n}} > \frac{1}{2-\frac{1}{n}} & \textit{when } 0 < C \leq \frac{\bar{p}_0}{2\beta} \end{array} \right\} > \frac{1}{2}$$

   When $C \in \left(\frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}}, \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k+1}}\right]$, $k = 1, 2, ...n-1$, tightness of the bound is achieved when $C \to \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}}$;

   When $C \in \left(0, \frac{\bar{p}_0}{2\beta}\right]$, tightness of the bound is achieved when $C = \frac{\bar{p}_0}{2\beta}$.

   The maximum profit loss of 50% is achieved when there exist numerous independent firms (i.e., $r \to 0, n \to \infty$).

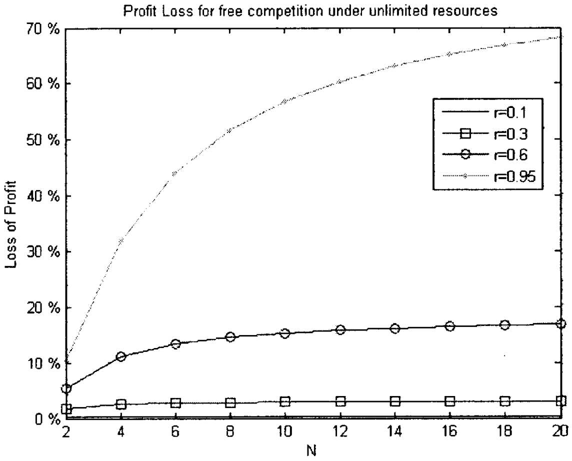

Figure 1: Profit loss for free competition under unlimited resource

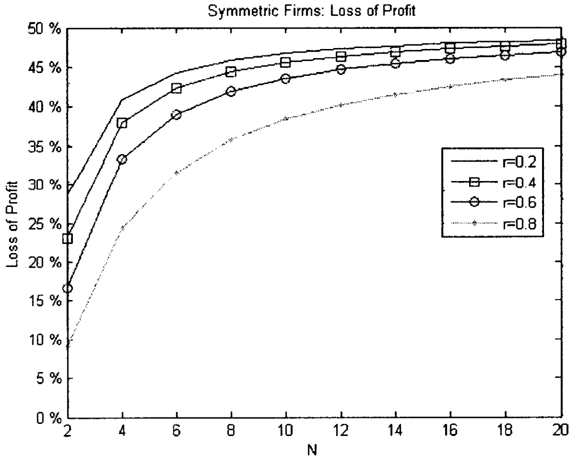

Figure 2: Maximum profit loss for free competition under extremely limited resource

3. *When the constraint is active for the Oligopoly problem but inactive for the Monopoly problem, the maximum loss of profit for the set of Oligopoly equilibria lies between the unconstrained bound in case 1 and the constrained bound in case 2.*

Figure 1 plots the profit loss ($= 1 - \frac{\Pi(OP)}{\Pi(MP)}$) for the Oligopoly equilibrium in the unconstrained case for $r = 0.1, 0.3, 0.6$ and 0.95, respectively. Figure 2 plots the profit loss for the worst Oligopoly equilibrium in the extremely constrained case (i.e., $C = \frac{\bar{p}_0}{2\beta}$) for $r = 0.2, 0.4, 0.6$ and 0.8, respectively (refer to the next section for observations and discussions of the figures).

Theorem 4.2 *For symmetric firms competing under one joint capacity constraint, the loss of total surplus for the worst Oligopoly Equilibrium compared with the SMAX solution is no more than 25%:*

1. *When the capacity constraint is not active for both the Oligopoly problem and the SMAX problem (i.e., $C \geq \frac{\bar{p}_0}{\alpha + \frac{\beta-\alpha}{n}}$), then*

$$\frac{TS(OP)}{TS(SMAX)} = \left(\frac{1}{r} + \frac{1}{n-1}\right) \frac{\frac{3}{r} + \frac{3}{n-1} - 2}{\left(\frac{2}{r} + \frac{2}{n-1} - 1\right)^2} \geq \frac{3}{4};$$

   Tightness of 25% is achieved when firms are independent (i.e., $r \to 0$);

2. *When the constraint is active for both problems (i.e., $C \leq \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{n}}$), then*

$$\frac{TS(OP)}{TS(SMAX)} \geq \left\{ \begin{array}{ll} 1 - \frac{\frac{1}{k} - \frac{1}{n}}{\frac{4}{k} + \frac{1}{(n-1)(\frac{1}{r}-1)}(\frac{2}{k}+1) - \frac{1}{n}}, & \text{when } \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k}} < c \leq \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k+1}} \\ & \text{for } k = 1, 2, \ldots n-1 \\ 1 - \frac{1}{3(1+\frac{1}{n-1})(1+\frac{1}{(n-1)(\frac{1}{r}-1)})+1}, & \text{when } c \leq \frac{\bar{p}_0}{2\beta} \end{array} \right\} > \frac{3}{4}$$

   Similar to the previous theorem, when $C \in \left(\frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k}}, \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k+1}}\right]$, $k = 1, 2, \ldots n-1$, tightness of the bound is achieved when $C \to \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k}}$. When $C \in \left(0, \frac{\bar{p}_0}{2\beta}\right]$, tightness of the bound is achieved when $C = \frac{\bar{p}_0}{2\beta}$.

   The maximum loss of 25% is achieved when the market consists of numerous independent firms (i.e., $r \to 0, n \to \infty$).

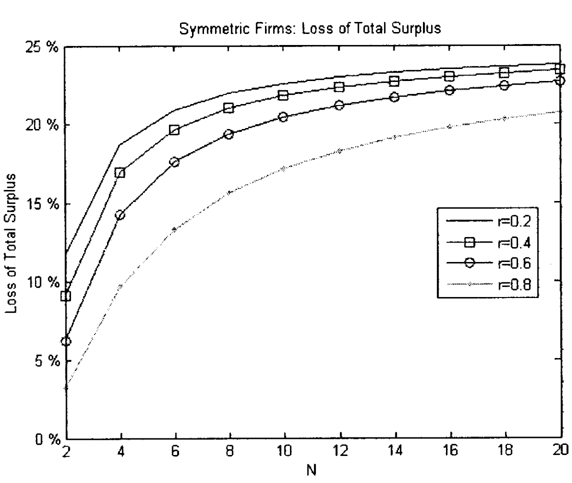

Figure 3: Maximum loss of total surplus for free competition under extremely limited resource

3. *The loss of total surplus when the constraint is active for the SMAX problem but not for the Oligopoly problem lies above the unconstrained bound in case 1.*

Figure 3 plots the maximum loss of total surplus ($= 1 - \frac{TS(OP)}{TS(SMAX)}$) in the extremely constrained case (i.e., $C = \frac{\bar{p}_0}{2\beta}$) for different market powers.

Under the assumption of symmetric firms, the worst Oligopoly equilibrium in terms of profit is also the worst in terms of total surplus. For a given capacity level $C \in \left(\frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}}, \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k+1}}\right]$, the worst equilibrium can be characterized analytically as :

$$d_i^w = \begin{cases} \frac{\bar{p}_0-\alpha c}{2\beta-\alpha} & \forall i = 1,2,...,k \\ c - k\frac{\bar{p}_0-\alpha c}{2\beta-\alpha} & i = k+1 \\ 0 & \forall i = k+2,...,n \end{cases}$$

(refer to Appendix B for a proof). Figure 4 compares the profit loss of the constructed worst equilibrium with randomly generated equilibria when $C < \frac{\bar{p}_0}{2\beta}$.

It is also interesting to notice that although the maximum profit (and total surplus) loss on each interval of C increases as k decreases ($C \in \left(\frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}}, \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k+1}}\right]$), monotonicity does

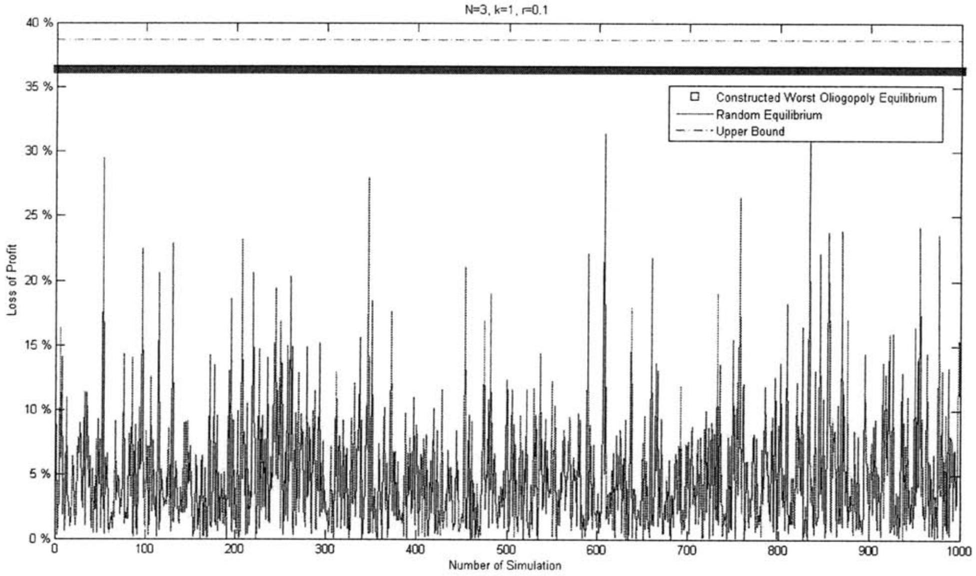

Figure 4: Comparison between Oligopoly Equilibrium

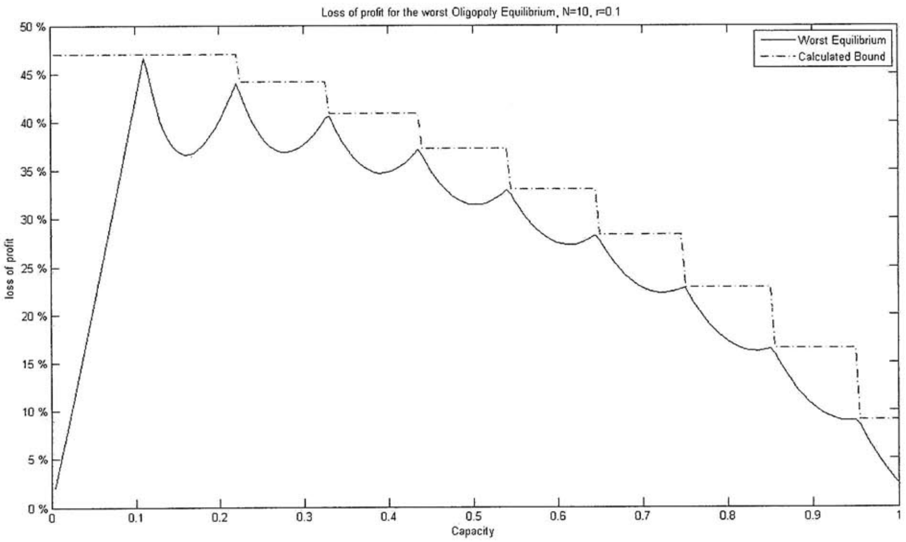

Figure 5: Profit loss for the worst equilibrium

not hold within each interval except when $C \in \left(0, \frac{\bar{p}_0}{2\beta}\right]$. Hence the loss of profit for the worst Oligopoly equilibrium is not a simple monotonic function of the capacity. Figure 5 demonstrates this observation by plotting the profit loss for the worst Oligopoly equilibrium as a function of capacity when $n = 10$ and $r = 0.1$.

Theorem 4.3 *When the capacity constraint is active for the SMAX, Monopoly and Oligopoly problem, then the Normalized Nash Equilibrium is the best Oligopoly equilibrium, that is, it achieves the maximum possible profit and total surplus:*

$$\frac{TS(NNE)}{TS(SMAX)} = 1 \text{ and } \frac{\Pi(NNE)}{\Pi(MP)} = 1$$

Proof The theorem follows since $\mathbf{d}^{SMAX} = \mathbf{d}^{MP} = \mathbf{d}^{NNE} = \begin{bmatrix} \frac{C}{n} \\ \vdots \\ \frac{C}{n} \end{bmatrix}$ when the constraint is active. ∎

4.3 Discussion

4.3.1 Quantity and price comparison

In the absence of constraints, free competition results in underproduction from a social perspective, and overproduction from a Monopoly perspective: $d_i^{SMAX} > d_i^{OP} \geq d_i^{MP}$. Since the market is symmetric, each firm is allocated the same amount of resources at equilibrium. As the constraint becomes restrictive, the equilibrium strategy for free competition is no longer unique. Among the set of equilibrium strategies, only the Normalized Nash-Equilibrium retains uniform allocation of resources.

When the constraint is active, the total quantity of product sold in the market is constant. Hence the profit of firms depends on the average market price:

$$\Pi(\mathbf{d}) = \sum_{i}^{n} d_i p_i(\mathbf{d}) = C\mu_p \ ,$$

where $\mu_p = \frac{\sum_i^n d_i p_i(\mathbf{d})}{\sum_{i=1}^n d_i}$ is the average market price.

Since firms are symmetric and the price is a decreasing function of the quantity, the average market price is maximized when each firm sells an equal portion of the total quantity: $\frac{C}{n}$. The average market price is minimized when all the products are sold through a single firm.

As a result, in terms of profit, the Normalized Nash-equilibrium is the best equilibrium which coincides with the Monopoly solution, and the worst equilibrium is the one with the least number of producing firms.

However, from the consumer's perspective, they would prefer to purchase from fewer firms with lower price. Hence the worst equilibrium in terms of profit is actually the best in terms of consumer surplus.

The SMAX problem is a trade off between the firms' preference for higher market price and the consumers' preference for lower market price. The optimal solution (which coincides with the Monopoly solution) indicates that the loss of company profit that resulted from lower market price is far more significant than the surplus gained by consumers.

4.3.2 Loss of profit and surplus for the worst Oligopoly equilibrium

With unlimited resources, the equilibrium strategy for free competition is unique and results in a maximum total surplus loss of 25% (when firms are independent from each other). However, there is no constant upper bound for the profit loss. As suggested by Theorem 4.1, the loss of profit magnifies as the number of firms increases and competition intensifies. In particular, as competition intensifies, the profit loss becomes more sensitive to the increase in n (see Figure 1). In case of fierce competition (i.e., r close to 1), the profit loss is 12% in a duopoly competition and increases rapidly to 67% when 10 firms compete in the market.

As resources become more scarce, free competition yields multiple equilibrium strategies. For a given capacity, the worst equilibrium in terms of both total surplus and profit is the one with the least number of producing firms. Compared with the optimal strategy (where every firm produces the same quantity), this solution is least efficient (25% loss in total surplus and 50% loss in profit) when the market consists of numerous independent firms (see Figure 2 and Figure 3 for the loss in profit and total surplus in the constrained case, respectively).

In order to understand how the efficiency of free competition is affected by the presence of a joint constraint, we compare the loss of profit and total surplus for the worst equilibrium in the constrained case with the unique equilibrium in the unconstrained case:

Figure 6 compares the percentage loss in profit. For fixed n, the blue line plots the profit loss of the worst Oligopoly equilibrium in the constrained case (when $k = 1$) as a function of the market power r while the green line represents the profit loss for the unique equilibrium in the unconstrained case. Given fixed r and n, the loss of profit for the worst equilibrium in the constrained case is generally greater than the unconstrained equilibrium. However, it is interesting to notice that this relationship is reversed when $r \to 1$, indicating that in an extremely competitive market, competition under limited resources is always more efficient (in terms of achieved profit) than competition under unlimited resources.

Figure 7 demonstrates a similar comparison for the loss in total surplus. While the loss of total surplus for the worst equilibrium in the constrained case is generally greater than the unconstrained equilibrium, when the number of firms is small, competition under limited

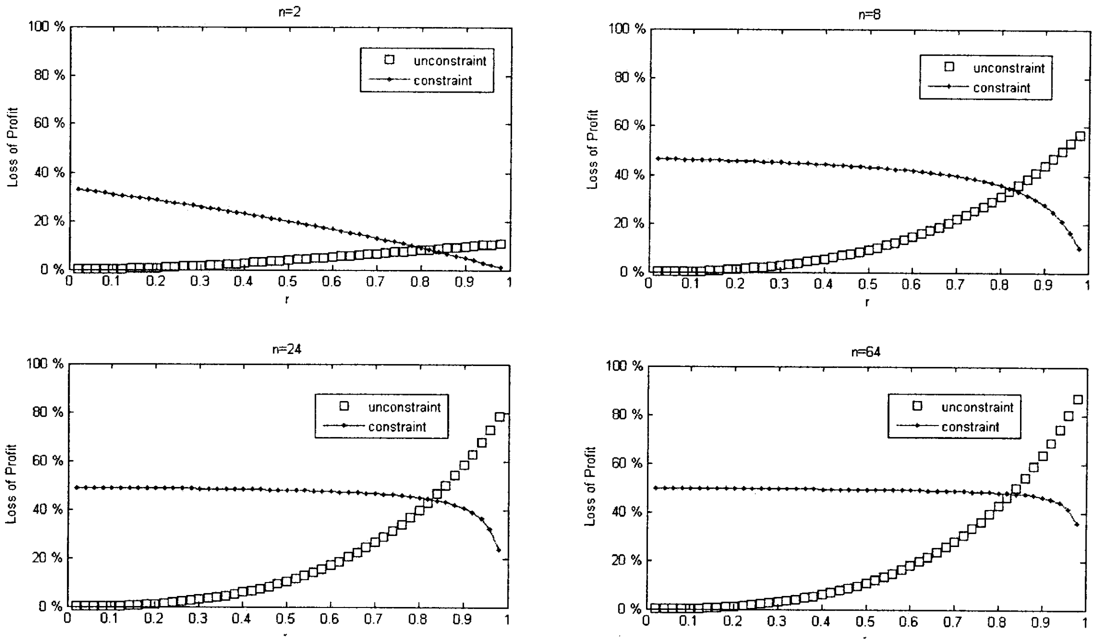

Figure 6: Effect of Joint Constraint:Loss of Profit

resources is always more efficient (in terms of total social surplus) regardless of the intensity of competition.

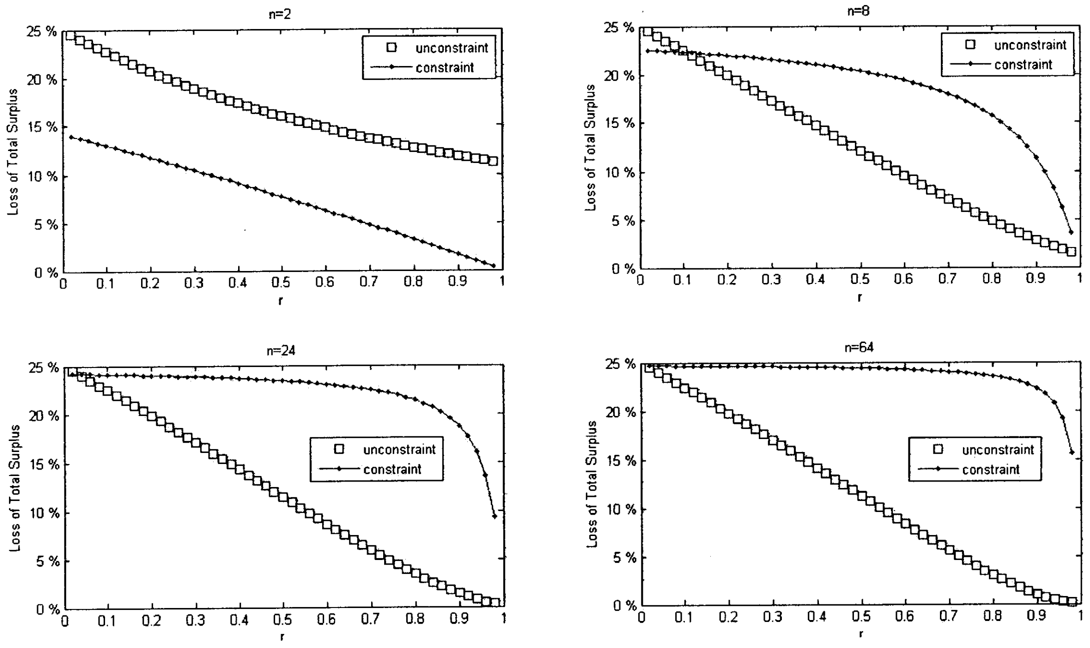

Figure 7: Effect of Joint Constraint:Loss of Total Surplus

5 Many Firms with Symmetric Price Potential

5.1 Assumptions and Closed-form Solutions

In this chapter, we relax the assumption of symmetric firms imposed in chapter 4 to consider firms with only symmetric price potentials: $\bar{\mathbf{p}} = B\bar{\mathbf{d}} = \mathbf{e}\bar{p}_0$, for some $\bar{p}_0 > 0$.

5.1.1 Closed-Form Solutions

The closed-form solutions for the SMAX, Monopoly and Normalized Nash Equilibrium are given by

$$\mathbf{d}^{SMAX} = min\left\{\bar{p}_0 B^{-1}\mathbf{e}, \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}} B^{-1}\mathbf{e}\right\}$$

$$\mathbf{d}^{MP} = min\left\{\frac{1}{2}\bar{p}_0 B^{-1}\mathbf{e}, \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}} B^{-1}\mathbf{e}\right\}$$

$$\mathbf{d}^{NNE} = min\left\{\bar{p}_0 (B+\Gamma)^{-1}\mathbf{e}, \frac{C}{\mathbf{e}^T (B+\Gamma)^{-1}\mathbf{e}} (B+\Gamma)^{-1}\mathbf{e}\right\}$$

(Refer to Appendix C.1 for a proof.)

5.2 Bounds for Loss of Efficiency

Theorem 5.1 *When the joint constraint is active and the price potentials are symmetric across firms, the loss of total surplus for the Normalized Nash-Equilibrium compared with the SMAX solutions is no more than 25%:*

$$\frac{TS(NNE)}{TS(SMAX)} \geq \max\left\{\frac{2}{3} + \frac{2}{3\,(2+r(n-1))}, \frac{3}{8}\frac{(2-r)^2}{\frac{3}{2}-r}\right\} \geq \frac{3}{4}.$$

The first bound dominates when n is small while the second bound dominates when n is large. Tightness of the bound is achieved when $r = 0$.

Figures 8, 9 and 10 compare the derived lower bound of the ratio $\frac{TS(NNE)}{TS(SMAX)}$ using simulation. Here the simulation result represents the smallest ratio $\frac{TS(NNE)}{TS(SMAX)}$ of 1000 randomly generated scenarios with fixed n and r (Refer to the next section for observations and discussions of the figures).

Theorem 5.2 *When the joint constraint is active and the price potentials are symmetric across firms, then the loss of company profit for the Normalized Nash-Equilibrium compared with the Monopoly solution is no more than 33.3%:*

$$\frac{\Pi(NNE)}{\Pi(MP)} \geq 2 - 2\delta + \frac{3}{4}\delta^2 \geq \frac{2}{3}$$

where $2 - r \leq \delta \leq 2$*. Tightness of the bound is achieved when* $r = 0$*.*

Figure 11 compares the derived lower bound of the ratio $\frac{\Pi(NNE)}{\Pi(MP)}$ using simulation.

Theorem 5.3 *Loss of efficiency for the worst Oligopoly equilibrium in the constrained case is characterized by:*

$$\frac{TS(OP)}{TS(SMAX)} \geq \frac{3}{4} + \frac{3}{4}\frac{1}{(4B_{MM}e^T\Gamma^{-1}e - 1)} > \frac{3}{4}$$

$$\frac{\Pi(OP)}{\Pi(MP)} \geq \frac{1}{2} + \frac{1}{2}\frac{1}{(2B_{MM}(e^T\Gamma^{-1}e) - 1)} > \frac{1}{2}$$

where $B_{MM} = \max\{B_{ii}\}$*. Tightness for both bounds is achieved when* $C \leq \frac{\bar{p}_0}{2B_{MM}}$*.*

5.3 Discussion

5.3.1 Quantity and price comparison

With the assumption of only symmetric price potentials, the SMAX solution still coincides with the Monopoly solution when the resource available is restrictive. However, the Normalized Nash Equilibrium is no longer the best equilibrium except for 2 special cases:

1. when firms are independent, so that $\mathbf{d}^{MP} = \mathbf{d}^{NNE}$;
2. when the price influence (matrix B) is uniform, so that the market is fully symmetric as considered in Chapter 4.

Similar with the symmetric market, while both the Monopoly and SMAX problem tries to maximize the average market price, the worst Oligopoly Equilibrium is the one that results in the minimum market price. With symmetric price potentials, the selling price is given by

$$p_i(\mathbf{d}) = \bar{p}_0 - \sum_{j=1}^{n} B_{ij} d_j$$

so that the smallest possible market price to sell quantity C is $p_{min} = \bar{p}_0 - B_{MM}C$, where $B_{MM} = \max\{B_{ii}\}$. In this case, all products are sold through firm M. Firm M is the least efficient firm in the market in the sense that if all firms are to operate in their own niche market, the market price of firm M's product is not able to match the price of other firms at a given output level due to possibly inferior quality or other factors that decrease customers' perceived value for the product.

5.3.2 Upper bounds for loss of efficiency

Although the maximum loss of total surplus is both 25% for the worst Oligopoly Equilbrium and the Normalized Nash-equilibrium. The worst case senarios are different:

In terms of capacity, the loss of both profit and total surplus for the Normalized Nash-Equilibrium is a nondecreasing function of the capacity C, that is, the maximum loss of total surplus happens in the totally unconstrained case and the maximum loss of profit occurs when the unconstrained Monopoly solution satisfies the capacity constraint exactly. However, the loss of efficiency is maximized for the worst Oligopoly equilibrium when capacity is extremely restrictive (i.e., $C \leq \frac{\bar{p}}{2B_{MM}}$).

In terms of market power, the loss of both total surplus and profit for the Normalized Nash-Equilibrium is maximized under intense competition (that is, when $r \to 1$). In fact, when $r \leq 0.1$, our bounds guarantee a no larger than 3.3% loss in total surplus and a 9.25% loss in profit (see Figure 8 to Figure 11). Although our bounds are only tight when $r = 0$, simulation results exhibit similar trend in the loss of efficiency as an increasing function of r. On the other hand, for the worst Oligopoy equilibrium, regardless of the number of players and the market power, maximum losses (25% for total surplus and 50% for profit) occur when all the rescources are allocated to the firm with significantly larger diagonal entry in matrix B compared with its compeitors (see the derived bound in Theorem 5.3).

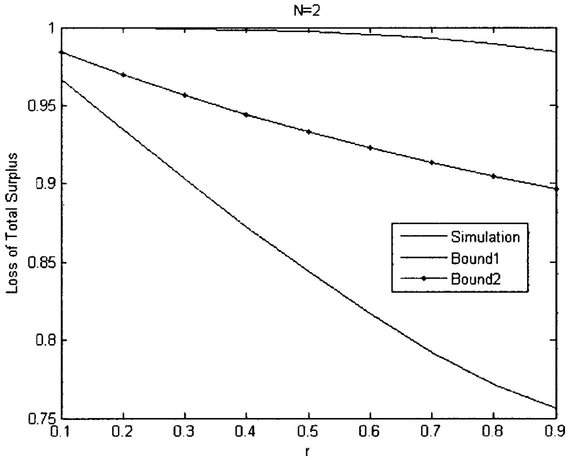

Figure 8: Loss of Total Surplus for the Normalized Nash-Equilibrium: N=2

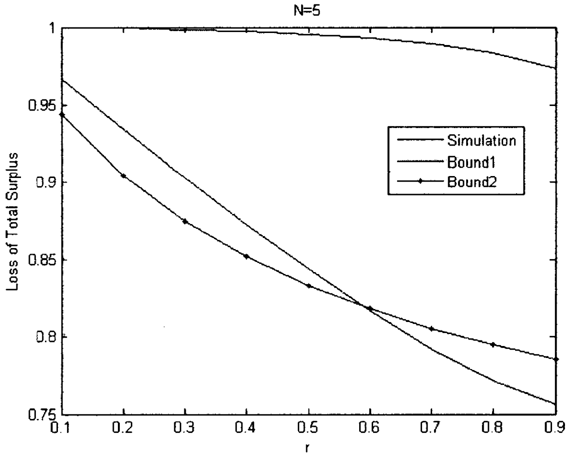

Figure 9: Loss of Total Surplus for the Normalized Nash-Equilibrium: N=5

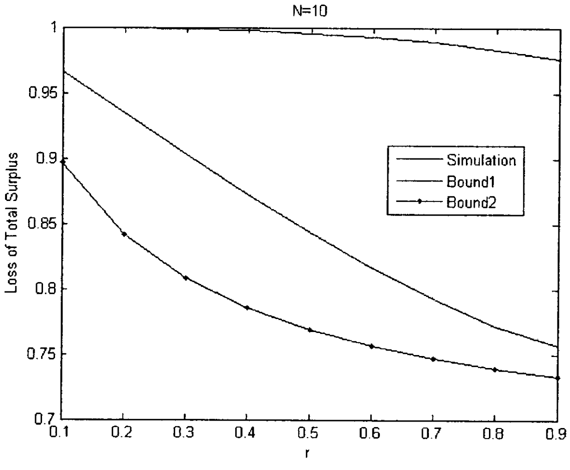

Figure 10: Loss of Total Surplus for the Normalized Nash-Equilibrium: N=10

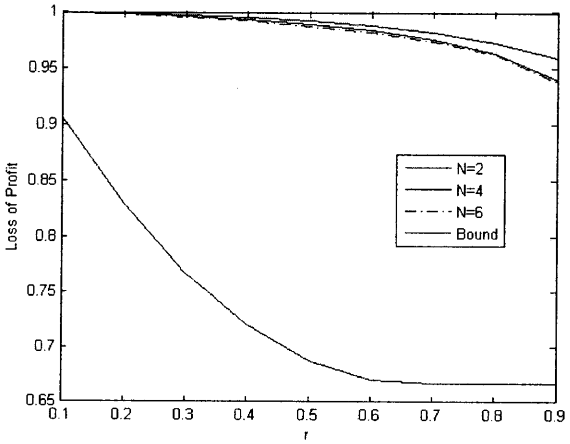

Figure 11: Loss of Profit Total Surplus for the Normalized Nash-Equilibrium

6 Many Firms with Symmetric Price Influence

6.1 Assumptions and Closed-form Solutions

This chapter considers the case where firms are symmetric in their ability to influence the market prices:

$$\mathbf{B} = \begin{bmatrix} \beta & \alpha & \dots & \alpha \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ \alpha & \dots & \alpha & \beta \end{bmatrix}$$

for $\beta > \alpha \geq 0$;

Unlike the closed-form solutions obtained in previous chapters, there is no guarantee in this case that the solution is strictly positive. Depending on the variance of the demand potential $\bar{\mathbf{d}}$ and the capacity C, the closed-from solutions for firms with symmetric price influences are given by:

Assume that $0 < \bar{d}_1 \leq \bar{d}_2 \leq \ldots \leq \bar{d}_n$.

Define $\mu_k = \frac{1}{k}\sum_{i=n-k+1}^{n} \bar{d}_i$ and $var(k) = \sum_{i=n-k+1}^{n}(\bar{d}_i - \mu_k)^2$, for $k = 1, 2, .., n$.

$$d_i^{SMAX} = \begin{cases} \bar{d}_i - \mu_{k_1} + \frac{C}{k_1} & \text{for } i = n - k_1 + 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where k_1 is the largest integer satisfying $\bar{d}_{n-k_1+1} - \mu_{k_1} + \frac{C}{k_1} \geq 0$.

$$d_i^{MP} = \begin{cases} \frac{\bar{d}_i - \mu_{k_2}}{2} + \frac{C}{k_2} & \text{for } i = n - k_2 + 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where k_2 is the largest integer satisfying $\frac{\bar{d}_{n-k_2+1} - \mu_{k_2}}{2} + \frac{C}{k_2} \geq 0$.

$$d_i^{NNE} = \begin{cases} \frac{\beta-\alpha}{2\beta-\alpha}(\bar{d}_i - \mu_{k_3}) + \frac{C}{k_3} & \text{for } i = n - k_3 + 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

where k_3 is the largest integer satisfying $\frac{\beta-\alpha}{2\beta-\alpha}(\bar{d}_{n-k_3+1} - \mu_{k_3}) + \frac{C}{k_3} \geq 0$.

(Refer to Appendix D for the derivation of these solutions.)

Notice here that $1 \leq k_1 \leq k_2 \leq k_3 \leq n$. As suggested by the closed-form solutions above, the Normalized Nash-Equilibrium always tries to include more firms in the market. This is consistent with the intuition that when the resource is scarce, more efficient firms will be given priority in the resource allocation process under both company-wide and society-wide coordination. The less efficient firms will be forced to quit the market more quickly than in the decentralized setting. In the case of large variation in the demand potential or restrictive capacity, $k_1 = k_2 = k_3 = 1$ so that all resources will be allocated to the firm with the highest demand potential.

For the rest of this Chapter, we will focus on the case when the capacity constraint is 'reasonably' restrictive. More specifically, we make the following assumptions on the capacity:

1. The unconstrained solution is infeasible for all three problems: $\frac{1}{2}\sum \bar{d}_i > C$.
2. No firm is forced to quit the market: $k_1 = k_2 = k_3 = n$,

$$\bar{d}_1 - \mu_n + \frac{C}{n} \geq 0 \Rightarrow C \geq n(\mu_n - \bar{d}_1).$$

As a result, the capacity constraint satisfies $\frac{1}{2}n\mu_n \geq C \geq n(\mu_n - \bar{d}_1)$.

6.2 Bounds on Loss of Efficiency

Theorem 6.1 *Under the assumptions described in the previous section, loss of total surplus for the Normalized Nash-Equilibrium compared with the SMAX solution is characterized by:*

$$\frac{TS(NNE)}{TS(SMAX)} \geq 1 - \frac{1-\theta}{(2-\theta)^2\left[1 + \frac{3n}{k(k+1)} + (\frac{3n(n-1)}{k(k+1)} - 1)\theta\right]} > \frac{3}{4},$$

where $\theta = \frac{\alpha}{\beta} = \frac{1}{\frac{n-1}{r} - (n-2)} \in [0,1)$, *and* $k = |I|$ *for* $I = \left\{i|\bar{d}_i < \mu_n\right\}$. $(0 \leq k \leq n-1)$

For each k, *tightness of the above bound is achieved when the demand potential satisfies*

$$\bar{d}_i = \begin{cases} \frac{C}{n} & \text{for } i = 1, 2, ...k, \\ \frac{2C}{n} & \text{for } i = k+1, ...n-1, \\ \frac{(k+2)C}{n} & \text{for } i = n. \end{cases}$$

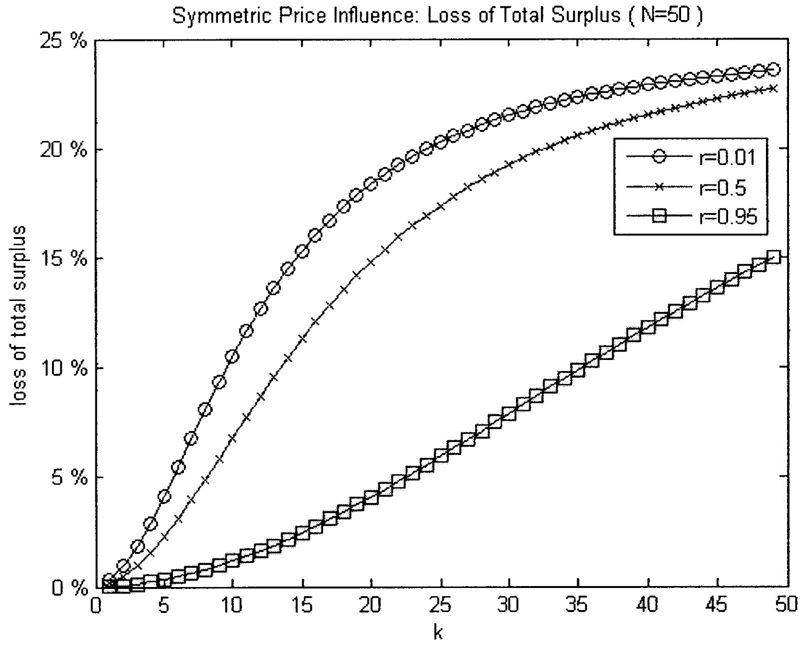

Figure 12: Loss of Total Surplus for Normalized Nash-Equilibrium

In addition, the maximum loss of 25% occurs when

1. *firms are independent ($r = 0$); and*

2. *demand potentials are highly non-uniform across firms ($k = n - 1$); and*

3. *the number of firms are extremely large ($n \to \infty$).*

In the theorem above, k is defined to be the size of the set $I = \left\{ i | \ \bar{d}_i < \mu_n \right\}$. It is a measure of variation of the demand potential since $k = 0$ implies that $\bar{d}_i \geq \mu_n$ for all i. Therefore, $\bar{d}_1 = \bar{d}_2 = ... = \bar{d}_n = \mu_n$. On the other hand, when $k = n - 1$, every demand potential is below the mean μ_n expect $\bar{d}_n$, suggesting a wider span of the demand potentials.

Figure 12 illustrates the loss of total surplus for the Normalized Nash Equilibrium as an increasing function of k when $n = 50$ for r=0.01, 0.5 and 0.95, respectively (Refer to the next section for observations and discussions of the figures).

Theorem 6.2 *The loss of profit for the Normalized Nash-Equilibrium compared with the Monopoly solution is given by:*

$$\begin{aligned}\frac{\Pi(NNE)}{\Pi(MP)} \geq = 1 - & \frac{\theta^2(1-\theta)}{(2-\theta)^2\left[1+\frac{4n}{k(k+1)}+(\frac{4n(n-1)}{k(k+1)}-1)\theta\right]} \\ \geq 1 - & \frac{\theta^2(1-\theta)}{(2-\theta)^2\left[1+\frac{4}{n-1}+3\theta\right]} (\textit{when } k = n-1) \\ > 1 - & 0.281 \ (\textit{when } n \to \infty, \theta^2 = \frac{1}{2}) \\ > 0.97, &\end{aligned}$$

where θ and k are the same as defined in Theorem 6.1. For each k, tightness of the bound is again achieved when the demand potential satisfies

$$\bar{d}_i = \begin{cases} \frac{C}{n} & \textit{for } i = 1, 2, \ldots k, \\ \frac{2C}{n} & \textit{for } i = k+1, \ldots n-1, \\ \frac{(k+2)C}{n} & \textit{for } i = n. \end{cases}$$

Figure 13 illustrates the loss of profit for the Normalized Nash Equilibrium as an increasing function of k when $n = 1000$ and $r = 0.9997$.

Unlike the loss of total surplus, the loss of profit is not a monotonic function of the market power. Figure 14 shows the loss of profit as a function of the market power when $k = N - 1$ for N ranging from 2 to 16.

In the independent case, the Normalized Nash-Equilibrium coincides with the Monopoly solution. In the case of extremely fierce competition, the percentage loss is also 0 since every feasible solution results in the same total surplus and profit:

$B = \frac{\alpha}{n}\mathbf{e}\mathbf{e}^T + (\beta - \alpha)\mathbf{I} \to \frac{\alpha}{n}\mathbf{e}\mathbf{e}^T$ as $r \to 1$. Therefore,

$$TS(\mathbf{d}) = \mathbf{d}^T B(\bar{\mathbf{d}} - \frac{1}{2}\mathbf{d}) \Rightarrow \frac{\alpha}{n}\mathbf{d}^T\mathbf{e}\mathbf{e}^T(\bar{\mathbf{d}} - \frac{1}{2}\mathbf{d}) = \frac{\alpha}{n}C(\mathbf{e}^T\bar{\mathbf{d}} - \frac{1}{2}C), \text{ and}$$

$$\Pi(\mathbf{d}) = \mathbf{d}^T B(\bar{\mathbf{d}} - \mathbf{d}) \Rightarrow \frac{\alpha}{n}\mathbf{d}^T\mathbf{e}\mathbf{e}^T(\bar{\mathbf{d}} - \mathbf{d}) = \frac{\alpha}{n}C(\mathbf{e}^T\bar{\mathbf{d}} - C),$$

for all $\mathbf{d}$ satisfying $\sum \mathbf{d}_i = C$.

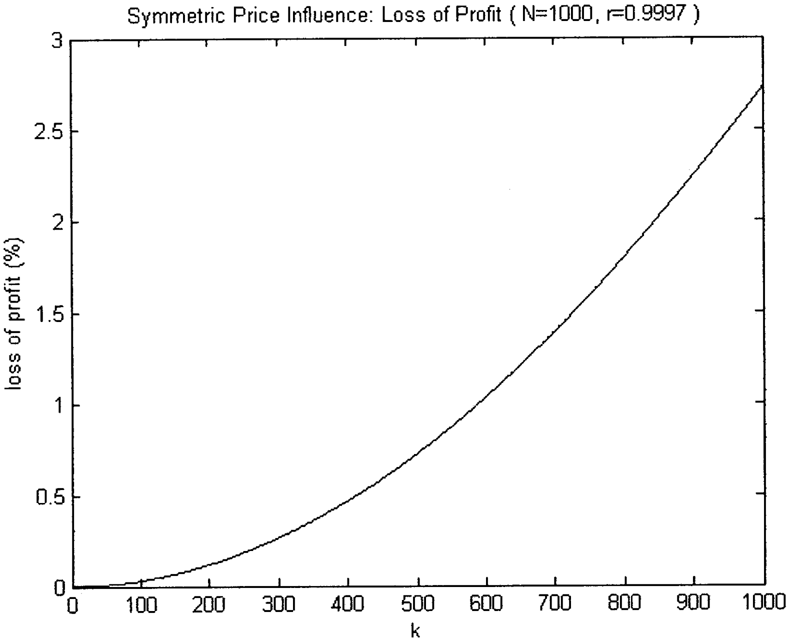

Figure 13: Loss of Profit for the Normalized Nash-Equilibrium

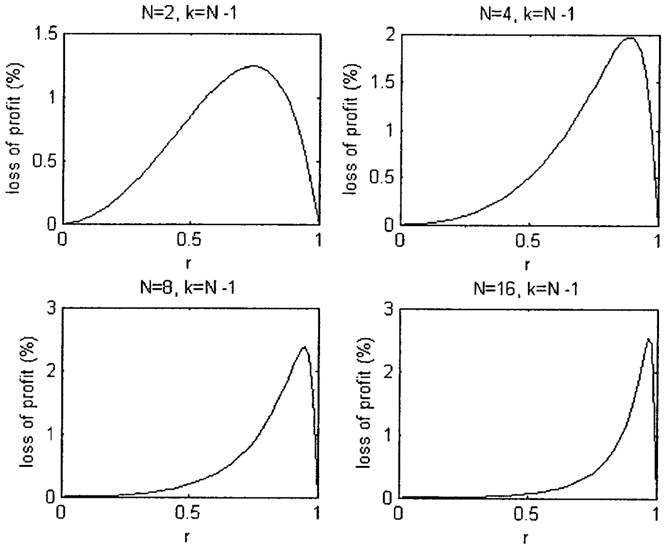

Figure 14: Loss of Profit for the Normalized Nash-Equilibrium

Theorem 6.3 *When firms collude, the resulting total surplus is no less than the total surplus achieved under the Normalized Nash-Equilibrium:*

$$1 \le \frac{TS(MP)}{TS(NNE)} \le 1 + \frac{1}{4}\frac{\theta(4-\theta)(1-\theta)}{3(1+(n-1)\theta)(2-\theta)^2\frac{n}{k(k+1)} + (3-\theta)(1-\theta)^2},$$

where θ and k are the same as defined in Theorem 6.1. For each k, tightness of the above bound is achieved when the demand potential satisfies

$$\bar{d}_i = \begin{cases} \frac{C}{n} & \textit{for } i = 1, 2, ...k \\ \frac{2C}{n} & \textit{for } i = k+1, ...n-1 \\ \frac{(k+2)C}{n} & \textit{for } i = n \end{cases}$$

The minimum ratio of 1 *is achieved when firms are independent or engage in intense competition ($r \to 0$).*

Figure 15 plots the upper bound of the ratio $\frac{TS(MP)}{TS(NNE)}$ as a function of k for $n = 50$, and r varies from 0.01 to 0.99.

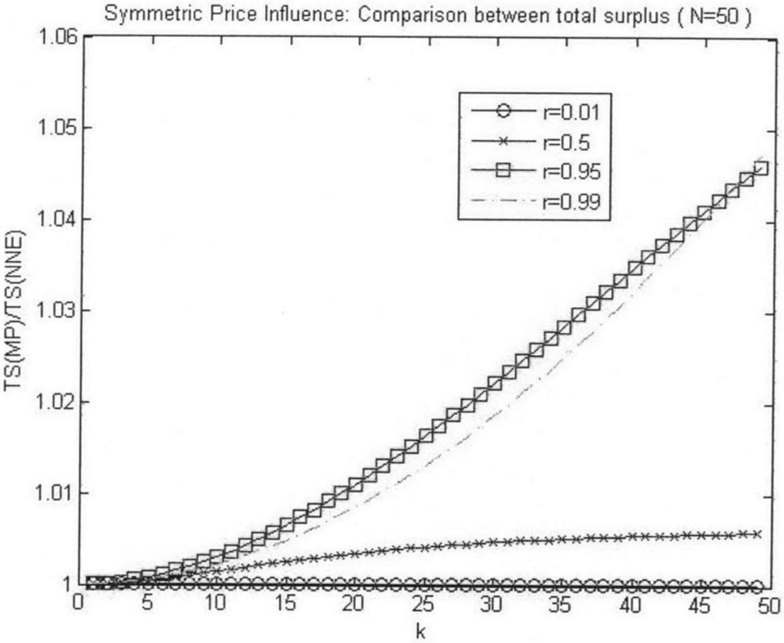

Figure 15: Comparison of total surplus between the Monopoly solution and the Normalized Nash-Equilibrium

6.3 Discussion

Compared with the case of symmetric firms discussed in Section 4, the presence of variation in the demand potential increases the losses of both total surplus and company profit for the Normalized Nash-Equilibrium. As suggested by the tight bounds established in Theorem 6.1 and Theorem 6.2, the greater the variation, the greater the loss (see Figure 13 and Figure 12). In a market with several independent firms, the Normalized Nash Equilibrium could lose up to 25% of total surplus compared with the SMAX solution. However, in terms of company profit, the Normalized Nash-equilibrium remains highly efficient as the loss never exceeds 3%.

Given a fixed level of variation in the demand potential, the loss of total surplus increases as the level of competition decreases and the number of firms increases. The trend with respect to the level of competition is different from the symmetric price potential case considered in Chapter 5, where the loss of total surplus for the Normalized Nash Equilibrium is maximized when the competition is intense.

The loss of profit as a function of r and n is more complicated: as competition intensifies, the Normalized Nash-Equilibrium deviates from the coordinated Monopoly solution and the loss of profit magnifies. After reaching its maximum at certain level of competition depending on the number of firms, the loss of profit decreases rapidly to zero again since every feasible solution results in the same profit under intense competition (see Figure 14).

It is also interesting to notice that while Theorem 3.2 suggested that in the absence of capacity constraints, the Oligopoly equilibrium is always preferred from a social perspective. In the constrained case however, even though the Monopoly solution no longer coincides with the SMAX solution (as is true in Chapters 4 and 5), it still guarantees a social surplus no smaller than the Normalized Nash-equilibrium (see Figure 15). In addition, this superiority in total surplus becomes more obvious as competition intensifies (except for the extreme case when $r \to 1$).

7 Implications for Reward Allocation Mechanism

In this chapter, we discuss the implications of the results presented in the previous chapters on the reward allocation mechanism.

- When the subsidiaries are symmetric, the uniform allocation scheme is the optimal scheme that achieves the maximum possible profit for the company under the committed energy consumption target. This result holds regardless of the number of subsidiaries and the intensity of the competition among them. Hence the uniform allocation scheme is highly preferable in this case given both its operational simplicity and profitability.

- When only a symmetric price potential of subsidiaries is guaranteed, the profit loss for the uniform scheme is still small if the level of competition is moderate. The bound derived in Chapter 5 guarantees no more than 10% loss of profit compared with the optimal allocation scheme when the market power $r \leq 0.1$ and a no more than 33.3% loss in the worst case senario. However, since the bound we derived is not tight as competition intensifies, the actual loss of profit for the uniform allocation scheme may be well below the theoretical bound. This is also suggested by the simulation results.

  Nevertheless, as the uniform scheme is no longer optimal in this situation, it is up to the management to decide whether the operational simplicity of the uniform reward allocation is significant enough to justify the loss in profit, especially when subsidiaries compete aggressively with each other.

- Finally, when only the price influences are symmetric, the uniform allocation scheme is highly desirable as it guarantees a no more than 3% loss of profit compared with the optimal scheme even in the case of intensive competition.

8 Conclusions

In this thesis, we quantified the loss of efficiency (in terms of total surplus and company profit) resulted from Cournot Competition under a joint constraint. Since the equilibrium strategies are not unique when the joint constraint is active, we focused on two particular equilibrium solutions: the worst Oligopoly equilibrium and the Normalized Nash-Equilibrium. In Chapter 3, we quantified the loss in total surplus in the general unconstrained case. For the constrained case, we demonstrated that no constant upper bounds of efficiency loss exist for the worst Oligopoly equilibrium solution without the assumption of symmetric price potential. For a market consisting of symmetric firms, we established tight upper bounds for the efficiency loss of the worst Oligopoly equilibrium as a function of the number of firms n and the market power r and derived an analytical expression for the worst Oligopoly equilibrium for all possible ranges of the capacity constraint C. We then relaxed our assumptions to consider firms with only symmetric potential and only symmetric price influence, and provided constant upper bounds for the loss of efficiency for both the Normalized Nash-Equilibrium and the worst Oligopoly equilibrium.

Another focus of this thesis is to propose a decentralized reward-based incentive mechanism for companies with many subsidiaries to meet their energy consumption targets. We demonstrated how this scheme fits in the general model described above. Besides its effectiveness in ensuring compliance, we proved the existence of the optimal reward allocation scheme that maximizes company profit and derived it explicitly when subsidiaries are symmetric. In addition, we discussed the effectiveness of the uniform reward allocation scheme based on the results derived for the general model.

Two possible extensions to the current model include:

1. Examine the loss of efficiency for the general case without any assumptions on market characteristics;

2. Extend the current model to incorporate multiple joint constraints.

A Calculations and Proofs for Chapter 3

A.1 Calculations

The KKT conditions of different problems are summarized as follows:

SMAX:

$$\begin{cases} \bar{\mathbf{p}} - B\mathbf{d}^{SMAX} - \mu\mathbf{e} + \lambda = \mathbf{0} & \\ \mathbf{e}^T\mathbf{d}^{SMAX} \le c & \\ \mu(c - \mathbf{e}^T\mathbf{d}) = 0 & \\ \lambda_i d_i^{SMAX} = 0 & \forall i = 1, 2, ..., n \\ \lambda_i \ge 0, \mu \ge 0, d_i^{SMAX} \ge 0 & \end{cases}$$

MP:

$$\begin{cases} \bar{\mathbf{p}} - 2B\mathbf{d}^{MP} - \mu\mathbf{e} + \lambda = \mathbf{0} & \\ \mathbf{e}^T\mathbf{d}^{MP} \le c & \\ \mu(c - \mathbf{e}^T\mathbf{d}) = 0 & \\ \lambda_i d_i^{MP} = 0 & \forall i = 1, 2, ..., n \\ \lambda_i \ge 0, \mu \ge 0, d_i^{MP} \ge 0 & \end{cases}$$

NNE:

$$\begin{cases} \bar{\mathbf{p}} - (B + \Gamma)\mathbf{d}^{NNE} - \mu\mathbf{e} + \lambda = \mathbf{0} & \\ \mathbf{e}^T\mathbf{d}^{NNE} \le c & \\ \mu(c - \mathbf{e}^T\mathbf{d}) = 0 & \\ \lambda_i d_i^{NNE} = 0 & \forall i = 1, 2, ..., n \\ \lambda_i \ge 0, \mu \ge 0, d_i^{NNE} \ge 0 & \end{cases}$$

The set of Oligopoly equilibria satisfy the following Cauchy-Variational Inequality:

$$\left(-\bar{p}_i + (B + \Gamma)_i \mathbf{d}^{OP}\right)(d_i - d_i^{OP}) \ge 0 \ \forall d_i \in K(\mathbf{d}_{-i}^{OP})$$

where $K(\mathbf{d}_{-i}^{OP}) = \left\{ d_i | d_i + \sum_{j \neq i}^{n} \le d_j^{OP} = C, d_i \ge 0 \right\}$.

A.2 Proofs

Theorem A.1 *In the absence of constraint, $CS(OP) \geq CS(MP)$. Equality is achieved when firms are indepdent.*

Proof In the unconstraint case, $\mathbf{d}^{SMAX} = \bar{\mathbf{d}}$, $\mathbf{d}^{MP} = \frac{1}{2}\bar{\mathbf{d}}$, and $\mathbf{d}^{OP} = (B+\Gamma)^{-1}B\bar{\mathbf{d}}$(follows directly from the KKT conditions).

$$\begin{aligned}
\Rightarrow B\mathbf{d}^{OP} - B\mathbf{d}^{MP} &= B((B+\Gamma)^{-1}B - \frac{1}{2}\mathbf{I})\bar{\mathbf{d}} \\
&= B(B+\Gamma)^{-1}(B - \frac{1}{2}(B+\Gamma))\bar{\mathbf{d}} \\
&= \frac{1}{2}\underbrace{B(B+\Gamma)^{-1}}_{\geq\mathbf{0}}\underbrace{(B-\Gamma)}_{\geq\mathbf{0}}\bar{\mathbf{d}} \\
&\geq \mathbf{0}
\end{aligned}$$

$\Rightarrow CS(OP) = \frac{1}{2}(\bar{\mathbf{d}}^{OP})^T B\bar{\mathbf{d}}^{OP} \geq \frac{1}{2}(\bar{\mathbf{d}}^{OP})^T B\bar{\mathbf{d}}^{MP} \geq \frac{1}{2}(\bar{\mathbf{d}}^{MP})^T B\bar{\mathbf{d}}^{MP} = CS(MP)$.
Equality holds when $B = \Gamma$ ($r = 0$). ▮

Theorem A.2 *In the absence of constraint, $TS(OP) \geq TS(MP) = \frac{3}{4}TS(SMAX)$. Equality is achieved when firms are indepdent.*

Proof

$$
\begin{aligned}
& TS(OP) - TS(MP) \\
& = \mathbf{d}_{OP}^T \left(B\bar{\mathbf{d}} - \frac{1}{2}B\mathbf{d}_{OP} \right) - \mathbf{d}_{MP}^T \left(B\bar{\mathbf{d}} - \frac{1}{2}B\mathbf{d}_{MP} \right) \\
& = (\bar{\mathbf{d}} - \frac{\mathbf{d}_{OP} + \mathbf{d}_{MP}}{2})^T B(\mathbf{d}_{OP} - \mathbf{d}_{MP}) \\
& (\text{Since } \mathbf{d}^{MP} = \frac{1}{2}\bar{\mathbf{d}}, \mathbf{d}^{OP} = (B+\Gamma)^{-1}B\bar{\mathbf{d}}) \\
& = \left(\bar{\mathbf{d}} - \frac{(B+\Gamma)^{-1}B\bar{\mathbf{d}} + \frac{1}{2}\bar{\mathbf{d}}}{2} \right)^T B \left((B+\Gamma)^{-1}B\bar{\mathbf{d}} - \frac{1}{2}\bar{\mathbf{d}} \right) \\
& = \bar{\mathbf{d}}^T \left(\frac{3}{4}\mathbf{I} - \frac{1}{2}B(B+\Gamma)^{-1} \right) B \left((B+\Gamma)^{-1}B - \frac{1}{2}\mathbf{I} \right) \bar{\mathbf{d}} \\
& = \frac{1}{2}\bar{\mathbf{d}}^T \left(\frac{3}{2}B - B(B+\Gamma)^{-1}B \right) \left((B+\Gamma)^{-1}B - \frac{1}{2}\mathbf{I} \right) \bar{\mathbf{d}} \\
& = \frac{1}{2}\bar{\mathbf{d}}^T \left(\frac{3}{2}B(B+\Gamma)^{-1}B - B(B+\Gamma)^{-1}B(B+\Gamma)^{-1}B \right. \\
& \left. -\frac{3}{4}B + \frac{1}{2}B(B+\Gamma)^{-1}B \right) \bar{\mathbf{d}} \\
& = \frac{1}{2}\bar{\mathbf{d}}^T \left(2B(B+\Gamma)^{-1}B - B(B+\Gamma)^{-1}B(B+\Gamma)^{-1}B - \frac{3}{4}B \right) \bar{\mathbf{d}} \\
& = \frac{1}{2}\bar{\mathbf{d}}^T(B(B+\Gamma)^{-1}B + B(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}B - \frac{3}{4}B)\bar{\mathbf{d}} \\
& = \frac{1}{2}\bar{\mathbf{d}}^T(B - B(B+\Gamma)^{-1}\Gamma + B(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}B - \frac{3}{4}B)\bar{\mathbf{d}} \\
& = \frac{1}{2}\bar{\mathbf{d}}^T(\frac{1}{4}B - B(B+\Gamma)^{-1}\Gamma + B(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}B)\bar{\mathbf{d}} \\
& (\text{since } \bar{\mathbf{d}}^T B\bar{\mathbf{d}} \geq \bar{\mathbf{d}}^T\Gamma\bar{\mathbf{d}}) \\
& \geq \frac{1}{2}\bar{\mathbf{d}}^T(\frac{1}{4}\Gamma - B(B+\Gamma)^{-1}\Gamma + B(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}B)\bar{\mathbf{d}}
\end{aligned}
$$

let $\bar{\mathbf{d}}^T\Gamma^{\frac{1}{2}} = x^T$, and $\bar{\mathbf{d}}^T B(B+\Gamma)^{-1}\Gamma^{\frac{1}{2}} = y^T$. We have

$$
\begin{aligned}
TS(OP) - TS(MP) & \geq \frac{1}{2}\bar{\mathbf{d}}^T(\frac{1}{4}\Gamma - B(B+\Gamma)^{-1}\Gamma + B(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}B)\bar{\mathbf{d}} \\
& = \frac{1}{2}(\frac{1}{4}x^Tx - x^Ty + y^Ty) \\
& = \frac{1}{2}(\frac{1}{2}x - y)^T(\frac{1}{2}x - y) \\
& \geq 0
\end{aligned}
$$

Equality holds when $B = \Gamma$ and $\frac{1}{2}x = y$. ∎

A.3 Calculations for the Duopoly Case

$$\mathbf{B} = \begin{bmatrix} \beta_1 & \alpha \\ \alpha & \beta_2 \end{bmatrix}, \beta_1 \geq \beta_2 > \alpha > 0$$

A.3.1 Monopoly Problem

- case 1: (constraint is not tight) $\mathbf{d}^{SO} = \frac{1}{2}\bar{\mathbf{d}} = \begin{bmatrix} \frac{\beta_2\bar{p}_1 - \alpha\bar{p}_2}{2(\beta_1\beta_2-\alpha^2)} \\ \frac{\beta_1\bar{p}_2 - \alpha\bar{p}_1}{2(\beta_1\beta_2-\alpha^2)} \end{bmatrix} > \mathbf{0}$ (by assumption)

  if $\mathbf{e}^T\bar{\mathbf{d}}^{SO} = \frac{(\beta_2-\alpha)\bar{p}_1+(\beta_1-\alpha)\bar{p}_2}{2(\beta_1\beta_2-\alpha^2)} \leq c$

- case 2: (constraint is tight) $\mathbf{d}^{SO} = \begin{bmatrix} \frac{2c(\beta_2-\alpha)-\bar{p}_2+\bar{p}_1}{2(\beta_1+\beta_2-2\alpha)} \\ \frac{2c(\beta_1-\alpha)-\bar{p}_1+\bar{p}_2}{2(\beta_1+\beta_2-2\alpha)} \end{bmatrix}$

  if

  $2c(\beta_1 - \alpha) - \bar{p}_1 + \bar{p}_2 \geq 0$ and $2c(\beta_2 - \alpha) - \bar{p}_2 + \bar{p}_1 \geq 0$

  and

  $\lambda = \bar{p}_1 - 2\alpha c - 2(\beta_1 - \alpha)d_1 = \frac{(\beta_2-\alpha)\bar{p}_1+(\beta_1-\alpha)\bar{p}_2-2c(\beta_1\beta_2-\alpha^2)}{\beta_1+\beta_2-2\alpha} \geq 0$

- case 3: $\mathbf{d}^{SO} = \begin{bmatrix} c \\ 0 \end{bmatrix}$

  if $\lambda = \bar{p}_1 - 2\beta_1 c \geq 0$ and $\mu_2 = \bar{p}_1 - \bar{p}_2 + 2c(\alpha - \beta_1) \geq 0$

- case 4: $\mathbf{d}^{SO} = \begin{bmatrix} 0 \\ c \end{bmatrix}$

  if $\lambda = \bar{p}_2 - 2\beta_2 c \geq 0$ and $\mu_1 = \bar{p}_2 - \bar{p}_1 + 2c(\alpha - \beta_2) \geq 0$

$\mathbf{d}^{SO}$ is unique as the above 4 cases are mutually exclusive.

A.3.2 Oligopoly Problem

- case 1:(constraint is not tight)

  $$\mathbf{d}^{NE} = \begin{bmatrix} \frac{2\beta_2\bar{p}_1-\alpha\bar{p}_2}{4\beta_1\beta_2-\alpha^2} \\ \frac{2\beta_1\bar{p}_2-\alpha\bar{p}_1}{4\beta_1\beta_2-\alpha^2} \end{bmatrix}$$

  if $\frac{(2\beta_2-\alpha)\bar{p}_1+(2\beta_1-\alpha)\bar{p}_2}{4\beta_1\beta_2-\alpha^2} \leq c$

- case 2:(constraint is tight) $\mathbf{d}^{NE}$ satisfies the following condition:

  1. $d_1 + d_2 = c$;
  2. $-\bar{p}_1 + 2\beta_1 d_1 + \alpha d_2 \leq 0$; if $d_1 > 0$

     $-\bar{p}_2 + 2\beta_2 d_2 + \alpha d_1 \leq 0$; if $d_2 > 0$
  3. $d_1 \geq 0$ and $d_2 \geq 0$

A.3.3 Normalized Nash Equilibrium

- case 1: $\mathbf{d}^{NNE} = \mathbf{d}^{NE}$when constraint is not tight;

- case 2: (constraint is tight) $\mathbf{d}^{SO} = \begin{bmatrix} \frac{c(2\beta_2-\alpha)-\bar{p}_2+\bar{p}_1}{2(\beta_1+\beta_2-\alpha)} \\ \frac{c(2\beta_1-\alpha)-\bar{p}_1+\bar{p}_2}{2(\beta_1+\beta_2-\alpha)} \end{bmatrix}$

  if

  $c(2\beta_1 - \alpha) - \bar{p}_1 + \bar{p}_2 \geq 0$ and $c(2\beta_2 - \alpha) - \bar{p}_2 + \bar{p}_1 \geq 0$

  and

  $\lambda = \bar{p}_1 - \alpha c - (2\beta_1 - \alpha)d_1 = \frac{(2\beta_2-\alpha)\bar{p}_1+(2\beta_1-\alpha)\bar{p}_2-c(4\beta_1\beta_2-\alpha^2)}{2(\beta_1+\beta_2-\alpha)} \geq 0$

- case 3: $\mathbf{d}^{SO} = \begin{bmatrix} c \\ 0 \end{bmatrix}$

  if $\lambda = \bar{p}_1 - 2\beta_1 c \geq 0$ and $\mu_2 = \bar{p}_1 - \bar{p}_2 + c(\alpha - 2\beta_1) \geq 0$

- case 4: $\mathbf{d}^{SO} = \begin{bmatrix} 0 \\ c \end{bmatrix}$

  if $\lambda = \bar{p}_2 - 2\beta_2 c \geq 0$ and $\mu_1 = \bar{p}_2 - \bar{p}_1 + c(\alpha - 2\beta_2) \geq 0$

$$\begin{aligned}
TS(d) =& d_1(\bar{p}_1 - \frac{1}{2}\beta_1 d_1 - \frac{1}{2}\alpha d_2) + d_2(\bar{p}_2 - \frac{1}{2}\beta_2 d_2 - \frac{1}{2}\alpha d_1) \\
=& d_1\bar{p}_1 + d_2\bar{p}_2 - \frac{1}{2}\left(\beta_1 d_1^2 - \alpha d_2 d_1 + \beta_2 d_2^2\right) \\
& \text{with } d_1 + d_2 = c \\
=& \bar{p}_2 c - \frac{1}{2}\beta_2 c^2 + d_1\left(\bar{p}_1 - \bar{p}_2 - \frac{1}{2}\beta_1 d_1 + \frac{1}{2}\alpha c - \frac{1}{2}\alpha d_1 + \beta_2 c - \frac{1}{2}\beta_2 d_1\right)
\end{aligned}$$

$$\Pi(d) = d_1\bar{p}_1 + d_2\bar{p}_2 - \left(\beta_1 d_1^2 - \alpha d_2 d_1 + \beta_2 d_2^2\right)$$

B Calculations and proofs for chapter 4

B.1 Calculations

Under the assumption of symmetric market, we have $\bar{\mathbf{p}} = \bar{p}_0\mathbf{e}$, and

$$\mathbf{B} = \begin{bmatrix} \beta & \alpha & \dots & \alpha \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ \alpha & \dots & \alpha & \beta \end{bmatrix}$$

.

The closed-form solutions follow from the KKT conditions presented in Section A.1:

$$\begin{aligned}
d_i^{MP} =& \quad \min\left\{\frac{C}{n}, \frac{\bar{p}_0}{2(\beta+(n-1)\alpha)}\right\} \\
d_i^{SMAX} =& \quad \min\left\{\frac{C}{n}, \frac{\bar{p}_0}{\beta+(n-1)\alpha}\right\} \quad \forall i = 1, 2, \cdots, n. \\
d_i^{NNE} =& \quad \min\left\{\frac{C}{n}, \frac{\bar{p}_0}{2\beta+(n-1)\alpha}\right\}
\end{aligned}$$

B.2 Proofs

Theorem B.1 *When firms are symmetric,*

$$\frac{TS(NE)}{TS(SMAX)} \geq \begin{cases} \left(\frac{1}{r}+\frac{1}{n-1}\right)\frac{\frac{3}{r}+\frac{3}{n-1}-2}{\left(\frac{2}{r}+\frac{2}{n-1}-1\right)^2} \geq \frac{3}{4} & \textit{when } c > \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}} \\ 1-\frac{\frac{1}{k}-\frac{1}{n}}{\frac{4}{k}+\frac{1}{(n-1)(\frac{1}{r}-1)}(\frac{2}{k}+1)-\frac{1}{n}} > \frac{3}{4}, & \textit{when } \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}} < c \leq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k+1}} \\ & \textit{for } k = 1, 2, \ldots n-1 \\ 1-\frac{1}{3(1+\frac{1}{n-1})(1+\frac{1}{(n-1)(\frac{1}{r}-1)})+1} > \frac{3}{4}, & \textit{when } c \leq \frac{\bar{p}_0}{2\beta} \end{cases}$$

Proof For $\mathbf{d}$ with $d_1 = d_2 = \ldots. = d_n$,

$$\begin{aligned} TS(\mathbf{d}) &= nd_i(\bar{p}_0 - \frac{1}{2}n\alpha d_i - \frac{1}{2}(\beta-\alpha)d_i) \\ &= nd_i(\bar{p}_0 - \frac{1}{2}(\beta+(n-1)\alpha)d_i) \end{aligned}$$

When $c \geq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}}$(constraint is not restrictive for the OP problem),

$$\begin{aligned} \frac{TS(OP)}{TS(SMAX)} &\geq \frac{nd_i^{OP}(\bar{p}_0 - \frac{1}{2}(\beta+(n-1)\alpha)d_i^{OP})}{nd_i^{SMAX}(\bar{p}_0 - \frac{1}{2}(\beta+(n-1)\alpha)d_i^{SMAX})} \\ &= \frac{\beta+(n-1)\alpha}{2\beta+(n-1)\alpha}\frac{1-\frac{1}{2}\frac{\beta+(n-1)\alpha}{2\beta+(n-1)\alpha}}{\frac{1}{2}} \\ &\quad (\text{Since } \frac{1}{r}-1+\frac{1}{n-1} = \frac{\beta}{(n-1)\alpha},) \\ &= \frac{\frac{1}{r}-1+\frac{1}{n-1}+1}{2(\frac{1}{r}-1+\frac{1}{n-1})+1}\frac{1-\frac{1}{2}\frac{\frac{1}{r}-1+\frac{1}{n-1}+1}{2(\frac{1}{r}-1+\frac{1}{n-1})+1}}{\frac{1}{2}} \\ &= \frac{(3(\frac{1}{r}+\frac{1}{n-1})-2)(\frac{1}{r}+\frac{1}{n-1})}{\left[2(\frac{1}{r}+\frac{1}{n-1})-1\right]^2} \geq \frac{3}{4} \end{aligned}$$

The minimum of the ratio $\frac{3}{4}$ is achieved when $r = 0$.

When $0 < c < \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}}$(constraint is restrictive for the OP problem), we have a set of equilibrium strategies satisfying

$$\mathbf{d}^{OP} = \left\{\mathbf{d} | 0 \leq d_i \leq \frac{\bar{p}_0 - \alpha c}{2\beta-\alpha}, \sum d_i = c\right\}$$

$$\begin{aligned}TS(SMAX) &= \bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2}(\beta-\alpha)\sum(\frac{c}{n})^2\\ &= \bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2n}(\beta-\alpha)c^2.\end{aligned}$$

we have

$$\begin{aligned}TS(OP) =& \sum d_i^{NE}(\bar{p}_0 - \frac{1}{2}(\beta-\alpha)d_i^{NE} - \frac{1}{2}\alpha c)\\ =& \bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2}(\beta-\alpha)\sum d_i^{NE2}\\ \geq& \bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2}(\beta-\alpha)\sum d_i^{w2}\end{aligned}$$

Where $\mathbf{d}^w$ is the equilibrium with the largest Euclidean norm.

- Case 1: $c \leq \frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}$ ($\Leftrightarrow \bar{p}_0 \geq 2\beta c$)

$$d_i^w = \begin{cases} c, & i = 1;\\ 0, & \text{otherwise.}\end{cases}$$

$$\Rightarrow TS(NE) \geq \bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2}(\beta-\alpha)c^2$$

$$
\begin{aligned}
\frac{TS(OP)}{TS(SMAX)} \geq & \frac{\bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2}(\beta-\alpha)c^2}{\bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2n}(\beta-\alpha)c^2} \\
= & 1 - \frac{\frac{1}{2}(1-\frac{1}{n})(\beta-\alpha)c^2}{\bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2n}(\beta-\alpha)c^2} \\
& (\text{Since } \bar{p}_0 \geq 2\beta c) \\
\geq & 1 - \frac{\frac{1}{2}(1-\frac{1}{n})(\beta-\alpha)c^2}{(2\beta c)c - \frac{1}{2}\alpha c^2 - \frac{1}{2n}(\beta-\alpha)c^2} \\
= & 1 - \frac{\frac{1}{2}(1-\frac{1}{n})(\beta-\alpha)}{2\beta - \frac{1}{2}\alpha - \frac{1}{2n}(\beta-\alpha)} \\
= & 1 - (1-\frac{1}{n})\frac{1}{\frac{4\beta-\alpha}{\beta-\alpha} - \frac{1}{n}} \\
= & 1 - (1-\frac{1}{n})\frac{1}{\frac{3\alpha}{\beta-\alpha} + 4 - \frac{1}{n}} \\
& (\text{Since } \frac{\alpha}{\beta-\alpha} = \frac{1}{(n-1)(\frac{1}{r}-1)}) \\
= & 1 - (1-\frac{1}{n})\frac{1}{\frac{3}{(n-1)(\frac{1}{r}-1)} + 4 - \frac{1}{n}} \\
= & 1 - \frac{1}{3(1+\frac{1}{n-1})(1+\frac{1}{(n-1)(\frac{1}{r}-1)}) + 1} \\
> & 1 - \frac{1}{4+\frac{1}{n-1}} (\text{ when } r \to 1) \\
> & \frac{3}{4} (\text{ when } n \to \infty)
\end{aligned}
$$

- Case 2: $\frac{c}{n} \leq \frac{c}{k+1} \leq \frac{\bar{p}_0 - \alpha c}{2\beta-\alpha} < \frac{c}{k}$ for some $1 \leq k \leq n-1$. ($\Leftrightarrow \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k}} < c \leq \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k+1}}$)

  The worst case Oligopoly Equilibrium in terms of total surplus is given by:

$$
d_i^w = \begin{cases} \frac{\bar{p}_0-\alpha c}{2\beta-\alpha} & \forall i = 1, 2, ..., k \\ c - k\frac{\bar{p}_0-\alpha c}{2\beta-\alpha} & i = k+1 \\ 0 & \forall i = k+2, ..., n \end{cases}
$$

  (Because for any $\sum d_i = c$, if there exist index i and j such that $d_i > 0$, $d_j > 0$, and $d_i + d_j \leq \frac{\bar{p}_0-\alpha c}{2\beta-\alpha}$, we can always construct $\tilde{\mathbf{d}}$ such that $\sum d_i^2 < \sum \tilde{d}_i^2$ by letting

$$\tilde{d}_k = \begin{cases} d_i + d_j & k = i \\ 0 & k = j \\ d_k & \forall k \neq i, k \neq j \end{cases}$$

$$\begin{aligned} \sum d_i^2 - \sum \tilde{d}_i^2 =& d_i^2 + d_j^2 - \tilde{d}_i^2 - \tilde{d}_j^2 \\ =& d_i^2 + d_j^2 - (d_i + d_j)^2 \\ =& -2d_i d_j \\ <& 0 \end{aligned}$$

)

Hence

$$\begin{aligned} \frac{TS(OP)}{TS(SMAX)} \geq& \frac{\bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2}(\beta - \alpha)\sum (d_i^w)^2}{\bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2n}(\beta - \alpha)c^2} \\ =& \frac{\bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2}(\beta - \alpha)(k\left(\frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}\right)^2 + \left(c - k\frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}\right)^2)}{\bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2n}(\beta - \alpha)c^2} \\ =& 1 - \frac{1}{2}\frac{(k\left(\frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}\right)^2 + \left(c - k\frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}\right)^2 - \frac{c^2}{n})(\beta - \alpha)}{\bar{p}_0 c - \frac{1}{2}\alpha c^2 - \frac{1}{2n}(\beta - \alpha)c^2} \end{aligned}$$

Let $x = \frac{\bar{p}_0 - \alpha c}{c(2\beta - \alpha)}$, we have $\bar{p}_0 c = \left((2\beta - \alpha)x + \alpha\right)c^2$, $\frac{1}{k+1} \leq x < \frac{1}{k}$

$$\begin{aligned} \frac{TS(OP)}{TS(SMAX)} \geq& 1 - \frac{1}{2}\frac{(kx^2c^2 + (c - kcx)^2 - \frac{c^2}{n})(\beta - \alpha)}{\left((2\beta - \alpha)x + \alpha\right)c^2 - \frac{1}{2}\alpha c^2 - \frac{1}{2n}(\beta - \alpha)c^2} \\ =& 1 - \frac{1}{2}\frac{(kx^2 + (1 - kx)^2 - \frac{1}{n})(\beta - \alpha)}{(2\beta - \alpha)x + \frac{1}{2}\alpha - \frac{1}{2n}(\beta - \alpha)} \\ =& 1 - \frac{1}{2}\underbrace{\frac{kx^2 + (1 - kx)^2 - \frac{1}{n}}{\frac{2\beta - \alpha}{\beta - \alpha}x + \frac{1}{2}\frac{\alpha}{\beta - \alpha} - \frac{1}{2n}}}_{f(x)} \end{aligned}$$

Since

$$
\begin{aligned}
&f'(x) = \\
&\frac{(2kx-2k\,(1-kx))\,(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n})-\left(kx^2+(1-kx)^2-\frac{1}{n}\right)\frac{2\beta-\alpha}{\beta-\alpha}}{\left(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n}\right)^2} \\
&\frac{2k\,((1+k)x-1)\,(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n})-\left(kx^2+(1-kx)^2-\frac{1}{n}\right)\frac{2\beta-\alpha}{\beta-\alpha}}{\left(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
f'(\frac{1}{k}) &= \frac{2(\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k}+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n})-\left(\frac{1}{k}-\frac{1}{n}\right)\frac{2\beta-\alpha}{\beta-\alpha}}{\left(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n}\right)^2} \\
&= \frac{\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k}+\frac{\alpha}{\beta-\alpha}-\frac{1}{n}+\frac{1}{n}\frac{2\beta-\alpha}{\beta-\alpha}}{\left(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n}\right)^2} \\
&= \frac{\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k}+\frac{\alpha}{\beta-\alpha}-\frac{1}{n}+\frac{1}{n}+\frac{1}{n}\frac{\beta}{\beta-\alpha}}{\left(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n}\right)^2} \\
&= \frac{\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k}+\frac{\alpha}{\beta-\alpha}+\frac{1}{n}\frac{\beta}{\beta-\alpha}}{\left(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n}\right)^2} \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
f'(\frac{1}{k+1}) &= \frac{-\left(\frac{k}{(k+1)^2}+(1-\frac{k}{1+k})^2-\frac{1}{n}\right)\frac{2\beta-\alpha}{\beta-\alpha}}{\left(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n}\right)^2} \\
&= \frac{-\left(\frac{1}{1+k}-\frac{1}{n}\right)\frac{2\beta-\alpha}{\beta-\alpha}}{\left(\frac{2\beta-\alpha}{\beta-\alpha}x+\frac{1}{2}\frac{\alpha}{\beta-\alpha}-\frac{1}{2n}\right)^2} \\
&< 0
\end{aligned}
$$

This suggests that $f(x)$ is maximized at the boundary when $x = \frac{1}{k}$ or $\frac{1}{k+1}$. Comparing the two function values:

$$f(\frac{1}{k}) = \frac{(\frac{1}{k} - \frac{1}{n})}{\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k} + \frac{1}{2}\frac{\alpha}{\beta-\alpha} - \frac{1}{2n}}$$

$$f(\frac{1}{k+1}) = \frac{(\frac{1}{k+1} - \frac{1}{n})}{\underbrace{\frac{2\beta-\alpha}{\beta-\alpha}}_{A}\frac{1}{k+1} + \underbrace{\frac{1}{2}\frac{\alpha}{\beta-\alpha} - \frac{1}{2n}}_{B}}$$

$$\begin{aligned}
f(\frac{1}{k}) - f(\frac{1}{k+1}) &= \frac{(\frac{1}{k} - \frac{1}{n})}{A\frac{1}{k} + B} - \frac{(\frac{1}{k+1} - \frac{1}{n})}{A\frac{1}{k+1} + B} \\
&= \frac{(\frac{1}{k} - \frac{1}{n})(A\frac{1}{k+1} + B) - (\frac{1}{k+1} - \frac{1}{n})(A\frac{1}{k} + B)}{(A\frac{1}{k} + B)(A\frac{1}{k+1} + B)} \\
&= \frac{(\frac{1}{n}A + B)(\frac{1}{k} - \frac{1}{k+1})}{(A\frac{1}{k} + B)(A\frac{1}{k+1} + B)} \\
&> 0
\end{aligned}$$

Hence $f(x)$ is maximized at $x = \frac{1}{k}$. It is interesting to notice that f is not a monotonically decreasing function in x.

$$\begin{aligned}
\frac{TS(OP)}{TS(SMAX)} &= 1 - \frac{1}{2}f(x) \\
&\geq 1 - \frac{1}{2}f(\frac{1}{k}) \\
&= 1 - \frac{\frac{1}{k} - \frac{1}{n}}{\frac{2\beta-\alpha}{\beta-\alpha}\frac{2}{k} + \frac{\alpha}{\beta-\alpha} - \frac{1}{n}} \\
&= 1 - \frac{\frac{1}{k} - \frac{1}{n}}{\frac{4}{k} + \frac{\alpha}{\beta-\alpha}\frac{2}{k} + \frac{\alpha}{\beta-\alpha} - \frac{1}{n}} \\
&= 1 - \frac{\frac{1}{k} - \frac{1}{n}}{\frac{4}{k} + \frac{\alpha}{\beta-\alpha}(\frac{2}{k} + 1) - \frac{1}{n}} \\
&= 1 - \frac{\frac{1}{k} - \frac{1}{n}}{\frac{4}{k} + \frac{1}{(n-1)(\frac{1}{r}-1)}(\frac{2}{k} + 1) - \frac{1}{n}}
\end{aligned}$$

$$\frac{TS(OP)}{TS(SMAX)} \geq \begin{cases} \left(\frac{1}{r} + \frac{1}{n-1}\right) \frac{\frac{3}{r}+\frac{3}{n-1}-2}{\left(\frac{2}{r}+\frac{2}{n-1}-1\right)^2} \geq \frac{3}{4} & \text{when } c \geq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}} \\ 1 - \frac{\frac{1}{k}-\frac{1}{n}}{\frac{4}{k}+\frac{1}{(n-1)(\frac{1}{r}-1)}(\frac{2}{k}+1)-\frac{1}{n}} > \frac{3}{4}, & \text{when } \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}} < c \leq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k+1}} \\ & \text{for } k = 1, 2, \ldots n-1 \\ 1 - \frac{1}{3(1+\frac{1}{n-1})(1+\frac{1}{(n-1)(\frac{1}{r}-1)})+1} > \frac{3}{4}, & \text{when } c \leq \frac{\bar{p}_0}{2\beta} \end{cases} \quad \blacksquare$$

Theorem B.2 *When firms are symmetric,*

$$\frac{\Pi(OP)}{\Pi(MP)} \geq \begin{cases} \frac{4(1+\frac{1}{\frac{1}{r}-1+\frac{1}{n-1}})}{(2+\frac{1}{\frac{1}{r}-1+\frac{1}{n-1}})^2} \geq \frac{4n}{(n+1)^2} & \textit{when } c \geq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}} \\ 1 - \frac{\frac{1}{k}-\frac{1}{n}}{\frac{1}{(n-1)(\frac{1}{r}-1)}\frac{1}{k}+\frac{2}{k}-\frac{1}{n}} > \frac{1}{2}, & \textit{when } \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}} < c \leq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k+1}} \\ & \textit{for } k = 1, 2, \ldots n-1 \\ 1 - \frac{1-\frac{1}{n}}{\frac{1}{(n-1)(\frac{1}{r}-1)}+2-\frac{1}{n}} > \frac{1}{2}, & \textit{when } c \leq \frac{\bar{p}_0}{2\beta} \end{cases}$$

Proof When $c \geq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}}$ (constraint is not restrictive for the OP problem),

$$\begin{aligned} \frac{\Pi(OP)}{\Pi(MP)} &= \frac{nd_i^{OP}(\bar{p}_0 - (\beta + (n-1)\alpha)d_i^{OP})}{nd_i^{MP}(\bar{p}_0 - (\beta + (n-1)\alpha)d_i^{MP})} \\ &= \frac{2(\beta + (n-1)\alpha)}{2\beta + (n-1)\alpha} \frac{(\bar{p}_0 - (\beta + (n-1)\alpha)\frac{\bar{p}_0}{2\beta+(n-1)\alpha})}{\frac{1}{2}\bar{p}_0} \\ &= \frac{4(\beta + (n-1)\alpha)\beta}{(2\beta + (n-1)\alpha)^2} \\ &= \frac{4(1 + \frac{1}{\frac{1}{r}-1+\frac{1}{n-1}})}{(2 + \frac{1}{\frac{1}{r}-1+\frac{1}{n-1}})^2} \\ &> \frac{4n}{(n+1)^2} (\text{ when } r \to 1) \end{aligned}$$

When $0 < c < \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}}$ (constraint is restrictive for the OP problem),

$$\Pi(OP) = \bar{p}_0 c - \alpha c^2 - (\beta - \alpha) \sum d_i^2$$

- Case 1: $c \leq \frac{\bar{p}_0 - \alpha c}{2\beta-\alpha}$ ($\Leftrightarrow \bar{p}_0 \geq 2\beta c$)

$$d_i^w = \begin{cases} c, & i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \Rightarrow \Pi(OP) &\geq \Pi(\mathbf{d}^w) \\ &= \bar{p}_0 c - \alpha c^2 - (\beta - \alpha)c^2 \end{aligned}$$

$$\begin{aligned} \frac{\Pi(OP)}{\Pi(MP)} =& \frac{\bar{p}_0 c - \alpha c^2 - (\beta-\alpha)c^2}{\bar{p}_0 c - \alpha c^2 - (\beta-\alpha)\frac{c^2}{n}} \\ =& 1 - \frac{(\beta-\alpha)c^2(1-\frac{1}{n})}{\bar{p}_0 c - \alpha c^2 - (\beta-\alpha)\frac{c^2}{n}} \\ =& 1 - \frac{(\beta-\alpha)c^2(1-\frac{1}{n})}{2\beta c^2 - \alpha c^2 - (\beta-\alpha)\frac{c^2}{n}} \\ =& 1 - \frac{(\beta-\alpha)(1-\frac{1}{n})}{2\beta - \alpha - (\beta-\alpha)\frac{1}{n}} \\ =& 1 - \frac{1-\frac{1}{n}}{\frac{2\beta-\alpha}{\beta-\alpha} - \frac{1}{n}} \\ =& 1 - \frac{1-\frac{1}{n}}{\frac{\alpha}{\beta-\alpha} + 2 - \frac{1}{n}} \\ =& 1 - \frac{1-\frac{1}{n}}{\frac{1}{(n-1)(\frac{1}{r}-1)} + 2 - \frac{1}{n}} \end{aligned}$$

(since $\frac{\alpha}{\beta-\alpha} = \frac{1}{(n-1)(\frac{1}{r}-1)}$)

- Case 2: $\frac{c}{n} \leq \frac{c}{k+1} \leq \frac{\bar{p}_0 - \alpha c}{2\beta-\alpha} < \frac{c}{k}$ for some $1 \leq k \leq n-1$.

$$(\Leftrightarrow \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k}} < c \leq \frac{\bar{p}_0}{\alpha + \frac{2\beta-\alpha}{k+1}})$$

The worst case equilibrium in terms of profit is given by:

$$d_i^w = \begin{cases} \frac{\bar{p}_0 - \alpha c}{2\beta - \alpha} & \forall i = 1, 2, ..., k \\ c - k\frac{\bar{p}_0-\alpha c}{2\beta-\alpha} & i = k+1 \\ 0 & \forall i = k+2, ..., n \end{cases}$$

$$
\begin{aligned}
\frac{\Pi(OP)}{\Pi(MP)} \geq & \frac{\bar{p}_0 c - \alpha c^2 - (\beta - \alpha)\sum (d_i^w)^2}{\bar{p}_0 c - \alpha c^2 - \frac{1}{n}(\beta - \alpha)c^2} \\
= & \frac{\bar{p}_0 c - \alpha c^2 - (\beta - \alpha)(k\left(\frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}\right)^2 + \left(c - k\frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}\right)^2)}{\bar{p}_0 c - \alpha c^2 - \frac{1}{n}(\beta - \alpha)c^2} \\
= & 1 - \frac{(k\left(\frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}\right)^2 + \left(c - k\frac{\bar{p}_0 - \alpha c}{2\beta - \alpha}\right)^2 - \frac{c^2}{n})(\beta - \alpha)}{\bar{p}_0 c - \alpha c^2 - \frac{1}{n}(\beta - \alpha)c^2}
\end{aligned}
$$

Let $x = \frac{\bar{p}_0 - \alpha c}{c(2\beta - \alpha)}$, $\bar{p}_0 c = \left((2\beta - \alpha)x + \alpha\right)c^2$, $\frac{1}{k+1} \leq x \leq \frac{1}{k}$

$$
\begin{aligned}
\frac{\Pi(NE)}{\Pi(MP)} \geq & 1 - \frac{(kx^2c^2 + (c - kcx)^2 - \frac{c^2}{n})(\beta - \alpha)}{\left((2\beta - \alpha)x + \alpha\right)c^2 - \alpha c^2 - \frac{1}{n}(\beta - \alpha)c^2} \\
= & 1 - \frac{(kx^2 + (1 - kx)^2 - \frac{1}{n})(\beta - \alpha)}{(2\beta - \alpha)x - \frac{1}{n}(\beta - \alpha)} \\
= & 1 - \underbrace{\frac{kx^2 + (1 - kx)^2 - \frac{1}{n}}{\frac{2\beta - \alpha}{\beta - \alpha}x - \frac{1}{n}}}_{f(x)}
\end{aligned}
$$

$$
\begin{aligned}
f'(\frac{1}{k}) = & \frac{2(\frac{2\beta - \alpha}{\beta - \alpha}\frac{1}{k} - \frac{1}{n}) - (\frac{1}{k} - \frac{1}{n})\frac{2\beta - \alpha}{\beta - \alpha}}{(\frac{2\beta - \alpha}{\beta - \alpha}\frac{1}{k} - \frac{1}{n})^2} \\
= & \frac{\frac{2\beta - \alpha}{\beta - \alpha}\frac{1}{k} - \frac{2}{n} + \frac{2}{n} + \frac{1}{n}\frac{\alpha}{\beta - \alpha}}{(\frac{2\beta - \alpha}{\beta - \alpha}\frac{1}{k} - \frac{1}{n})^2} \\
= & \frac{\frac{2\beta - \alpha}{\beta - \alpha}\frac{1}{k} + \frac{1}{n}\frac{\alpha}{\beta - \alpha}}{(\frac{2\beta - \alpha}{\beta - \alpha}\frac{1}{k} - \frac{1}{n})^2} \\
& > 0
\end{aligned}
$$

$$
\begin{aligned}
f'(\frac{1}{k+1}) =& \frac{-(\frac{k}{(k+1)^2} + (1 - \frac{k}{k+1})^2 - \frac{1}{n})\frac{2\beta-\alpha}{\beta-\alpha}}{(\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k+1} - \frac{1}{n})^2} \\
=& \frac{-(\frac{1}{k+1} - \frac{1}{n})\frac{2\beta-\alpha}{\beta-\alpha}}{(\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k+1} - \frac{1}{n})^2} \\
<& 0
\end{aligned}
$$

This suggests that $f(x)$ is maximized at the boundary when $x = \frac{1}{k}$ or $\frac{1}{k+1}$. Comparing the two function values:

$$
f(\frac{1}{k}) = \frac{(\frac{1}{k} - \frac{1}{n})}{\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k} - \frac{1}{n}}
$$

$$
f(\frac{1}{k+1}) = \frac{(\frac{1}{k+1} - \frac{1}{n})}{\underbrace{\frac{2\beta-\alpha}{\beta-\alpha}}_{A}\frac{1}{k+1} - \underbrace{\frac{1}{n}}_{B}}
$$

$$
\begin{aligned}
f(\frac{1}{k}) - f(\frac{1}{k+1}) &= \frac{(\frac{1}{k} - \frac{1}{n})}{A\frac{1}{k} - B} - \frac{(\frac{1}{k+1} - \frac{1}{n})}{A\frac{1}{k+1} - B} \\
&= \frac{(\frac{1}{k} - \frac{1}{n})(A\frac{1}{k+1} - B) - (\frac{1}{k+1} - \frac{1}{n})(A\frac{1}{k} - B)}{(A\frac{1}{k} - B)(A\frac{1}{k+1} - B)} \\
&= \frac{(\frac{1}{k} - \frac{1}{n})(A\frac{1}{k+1} - B) - (\frac{1}{k+1} - \frac{1}{n})(A\frac{1}{k} - B)}{(A\frac{1}{k} - B)(A\frac{1}{k+1} - B)} \\
&= \frac{(\frac{1}{n}A - B)(\frac{1}{k} - \frac{1}{k+1})}{(A\frac{1}{k} - B)(A\frac{1}{k+1} - B)} \\
&> 0
\end{aligned}
$$

Hence $f(x)$ is maximized at $x = \frac{1}{k}$.

$$
\begin{aligned}
\frac{\Pi(OP)}{\Pi(MP)} =& 1 - f(x) \\
\geq& 1 - f(\frac{1}{k}) \\
=& 1 - \frac{\frac{1}{k} - \frac{1}{n}}{\frac{2\beta-\alpha}{\beta-\alpha}\frac{1}{k} - \frac{1}{n}} \\
=& 1 - \frac{\frac{1}{k} - \frac{1}{n}}{\frac{\alpha}{\beta-\alpha}\frac{1}{k} + \frac{2}{k} - \frac{1}{n}} \\
&(\text{ Since } \frac{\alpha}{\beta-\alpha} = \frac{1}{(n-1)(\frac{1}{r}-1)}) \\
=& 1 - \frac{\frac{1}{k} - \frac{1}{n}}{\frac{1}{(n-1)(\frac{1}{r}-1)}\frac{1}{k} + \frac{2}{k} - \frac{1}{n}} \\
>& \frac{1}{2 - \frac{k}{n}} (\text{ when } r \to 0)
\end{aligned}
$$

$$
\frac{\Pi(OP)}{\Pi(MP)} \geq \begin{cases}
\frac{4(1+\frac{1}{\frac{1}{r}-1+\frac{1}{n-1}})}{(2+\frac{1}{\frac{1}{r}-1+\frac{1}{n-1}})^2} \geq \frac{4n}{(n+1)^2} & \text{when } c \geq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{n}} \\
1 - \frac{\frac{1}{k}-\frac{1}{n}}{\frac{1}{(n-1)(\frac{1}{r}-1)}\frac{1}{k}+\frac{2}{k}-\frac{1}{n}} > \frac{1}{2}, & \text{when } \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k}} < c \leq \frac{\bar{p}_0}{\alpha+\frac{2\beta-\alpha}{k+1}} \\
& \text{for } k = 1, 2, ...n-1 \\
1 - \frac{1-\frac{1}{n}}{\frac{1}{(n-1)(\frac{1}{r}-1)}+2-\frac{1}{n}} > \frac{1}{2}, & \text{when } c \leq \frac{\bar{p}_0}{2\beta}
\end{cases} \qquad \blacksquare
$$

C Calculations and Proofs for Chapter 5

C.1 Closed-form solutions

The closed-form solutions when $\bar{\mathbf{p}} = \bar{p}_0\mathbf{e}$ follows from the KKT conditions presented in Section A.1 : ($\bar{\mathbf{d}} = B^{-1}\bar{\mathbf{p}} = \bar{p}_0 B^{-1}\mathbf{e}$)

$$
\mathbf{d}^{SMAX} = min\left\{\bar{p}_0 B^{-1}\mathbf{e}, \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}} B^{-1}\mathbf{e}\right\}
$$

$$
\mathbf{d}^{MP} = min\left\{\frac{1}{2}\bar{p}_0 B^{-1}\mathbf{e}, \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}} B^{-1}\mathbf{e}\right\}
$$

$$
\mathbf{d}^{NNE} = min\left\{\bar{p}_0 (B+\Gamma)^{-1}\mathbf{e}, \frac{C}{\mathbf{e}^T (B+\Gamma)^{-1}\mathbf{e}} (B+\Gamma)^{-1}\mathbf{e}\right\}
$$

C.2 Bounds for loss of efficiency

Lemma C.1 $(1-r)\Gamma^{-1}e \leq B^{-1}e \leq \Gamma^{-1}e$

Proof we have

$$\begin{cases} B_{ii}M_{ii} - \sum_{k\neq i} B_{ik}M_{ki} = 1 & \forall i \\ B_{ij}M_{jj} - \sum_{k\neq j} B_{ik}M_{kj} = 0 & \forall j \neq i \end{cases}$$

$$\Rightarrow B_{ii}M_{ii} \geq 1$$

By the definition of the market power r, we have

$$\begin{aligned} && rM_{ii} &\geq \textstyle\sum_{j\neq i} M_{ij} \forall i \\ \Rightarrow && -rM_{ii} &\leq -\textstyle\sum_{j\neq i} M_{ij} \\ \Rightarrow && (1-r)M_{ii} &\leq M_{ii} - \textstyle\sum_{j\neq i} M_{ij} \\ \Rightarrow && 1-r \leq (1-r)M_{ii}B_{ii} &\leq \left(M_{ii} - \textstyle\sum_{j\neq i} M_{ij}\right) B_{ii} \\ \Rightarrow && (1-r)\tfrac{1}{B_{ii}} &\leq M_{ii} - \textstyle\sum_{j\neq i} M_{ij} \\ \Rightarrow && (1-r)\Gamma^{-1}\mathbf{e} &\leq B^{-1}\mathbf{e} \end{aligned}$$

Theorem C.2 *When firms have symmetric price potentials,*

$$\frac{TS(NNE)}{TS(SMAX)} \geq \max\left\{\frac{2}{3} + \frac{2}{3\,(2 + r(n-1))}, \frac{3}{8}\frac{\delta^2}{\delta - \frac{1}{2}}\right\} \geq \frac{3}{4}$$

where $\delta = 2 - r$. *Tightness is achieved when* $r = 0$.

Proof Let

$$\mathbf{e}^T B^{-1}\Gamma^{\frac{1}{2}} = \mathbf{x}^T$$

$$\mathbf{e}^T (B + \Gamma)^{-1}\Gamma^{\frac{1}{2}} = \mathbf{y}^T$$

$$\mathbf{e}^T \Gamma^{-\frac{1}{2}} = \mathbf{z}^T$$

$$\begin{aligned} \Rightarrow (\mathbf{x} - \mathbf{y})^T\mathbf{z} =& \mathbf{e}^T \left(B^{-1} - (B+\Gamma)^{-1}\right) \Gamma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\mathbf{e} \\ =& \mathbf{e}^T (B+\Gamma)^{-1} \left((B+\Gamma)B^{-1} - \mathbf{I}\right) \mathbf{e} \\ =& \mathbf{e}^T (B+\Gamma)^{-1}\Gamma B^{-1}\mathbf{e} \\ =& \mathbf{x}^T\mathbf{y} \end{aligned}$$

$$\frac{TS(NNE)}{TS(SMAX)} = \frac{\bar{p}_0 C - \frac{C^2}{2}\frac{\mathbf{e}^T(B+\Gamma)^{-1}B(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2}}{\bar{p}_0 C - \frac{1}{2}\frac{C^2}{\mathbf{e}^T B^{-1}\mathbf{e}}}$$

$$= 1 - \frac{C}{2}\frac{\frac{\mathbf{e}^T(B+\Gamma)^{-1}B(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\bar{p}_0 - \frac{1}{2}\frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}}$$

$$(\text{Since } \bar{p}_0 \geq \frac{C}{\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e}})$$

$$\geq 1 - \frac{C}{2}\frac{\frac{\mathbf{e}^T(B+\Gamma)^{-1}B(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\frac{C}{\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e}} - \frac{1}{2}\frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}}$$

$$= 1 - \frac{1}{2}\frac{\frac{\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e} - \mathbf{e}^T(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\frac{1}{\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e}} - \frac{1}{2}\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}$$

$$= 1 - \frac{1}{2}\frac{\frac{1}{\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e}} - \frac{\mathbf{e}^T(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\frac{1}{\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e}} - \frac{1}{2}\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{\frac{\mathbf{e}^T(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} + \frac{1}{2}\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\frac{1}{\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e}} - \frac{1}{2}\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{\frac{\mathbf{y}^T\mathbf{y}}{\mathbf{y}^T\mathbf{z}} + \frac{1}{2}\frac{\mathbf{y}^T\mathbf{z}}{\mathbf{x}^T\mathbf{z}}}{1 - \frac{1}{2}\frac{\mathbf{y}^T\mathbf{z}}{\mathbf{x}^T\mathbf{z}}}$$

$$\geq \frac{1}{2} + \frac{1}{2}\frac{\frac{\mathbf{x}^T\mathbf{y} - \frac{1}{4}\mathbf{x}^T\mathbf{x}}{\mathbf{y}^T\mathbf{z}} + \frac{1}{2}\frac{\mathbf{y}^T\mathbf{z}}{\mathbf{x}^T\mathbf{z}}}{1 - \frac{1}{2}\frac{\mathbf{y}^T\mathbf{z}}{\mathbf{x}^T\mathbf{z}}}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{\frac{\mathbf{x}^T\mathbf{z} - \mathbf{y}^T\mathbf{z} - \frac{1}{4}\mathbf{x}^T\mathbf{z}}{\mathbf{y}^T\mathbf{z}} + \frac{1}{2}\frac{\mathbf{y}^T\mathbf{z}}{\mathbf{x}^T\mathbf{z}}}{1 - \frac{1}{2}\frac{\mathbf{y}^T\mathbf{z}}{\mathbf{x}^T\mathbf{z}}}$$

$$(\text{Let } \tilde{\delta} = \frac{\mathbf{x}^T\mathbf{z}}{\mathbf{y}^T\mathbf{z}})$$

$$= \frac{1}{2} + \frac{1}{2}\frac{(1 - \frac{1}{4})\tilde{\delta} - 1 + \frac{1}{2}\frac{1}{\tilde{\delta}}}{1 - \frac{1}{2}\frac{1}{\tilde{\delta}}}$$

$$= \frac{3}{8}\underbrace{\frac{\tilde{\delta}^2}{\tilde{\delta} - \frac{1}{2}}}_{f(\tilde{\delta})}$$

By Lemma 4.1, we have$(1-r)\mathbf{z} \leq \mathbf{x} \leq \mathbf{z}$.

$$\begin{aligned} \Rightarrow \tilde{\delta} &= \frac{\mathbf{x}^T\mathbf{z}}{\mathbf{y}^T\mathbf{z}} \\ &= \frac{\mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{z}}{\mathbf{y}^T\mathbf{z}} \\ &= 1 + \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{z}} \\ &\geq 1 + (1-r)\frac{\mathbf{y}^T\mathbf{z}}{\mathbf{y}^T\mathbf{z}} \\ &\geq 2 - r \end{aligned}$$

$f'(\tilde{\delta}) = \frac{2\tilde{\delta}(\tilde{\delta}-\frac{1}{2})-\tilde{\delta}^2}{(\tilde{\delta}-\frac{1}{2})^2} = \frac{\tilde{\delta}(\tilde{\delta}-1)}{(\tilde{\delta}-\frac{1}{2})^2} \geq 0$ since $\tilde{\delta} \in [1,2]$

Hence $\frac{TS(NNE)}{TS(SMAX)} \geq \frac{3}{8}\frac{(2-r)^2}{\frac{3}{2}-r} \geq \frac{3}{4}$. The second bound $\frac{2}{3} + \frac{2}{3(2+r(n-1))}$ follows from $\|d\|_B^2 \leq (1+r(n-1))\,\|d\|_\Gamma^2$.(See Kluberg and Perakis (2009)) ▮

Theorem C.3

$$\frac{\Pi(NNE)}{\Pi(MP)} \geq 2 - 2\delta + \frac{3}{4}\delta^2 \geq \frac{2}{3}$$

where $2 - r \leq \delta \leq 2$.

Proof

$$
\begin{aligned}
\frac{\Pi(NNE)}{\Pi(MP)} &= \frac{\bar{p}_0 C - C^2 \frac{\mathbf{e}^T(B+\Gamma)^{-1}B(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2}}{\bar{p}_0 C - \frac{C^2}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= 1 - C\frac{\frac{\mathbf{e}^T(B+\Gamma)^{-1}B(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\bar{p}_0 - \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&\text{(Since } \bar{p}_0 \geq \frac{2C}{\mathbf{e}^T B^{-1}\mathbf{e}}) \\
&\geq 1 - \frac{\frac{\mathbf{e}^T(B+\Gamma)^{-1}B(B+\Gamma)^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= 2 - \frac{\mathbf{e}^T(B+\Gamma)^{-1}B(B+\Gamma)^{-1}\mathbf{e}\ \mathbf{e}^T B^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} \\
&= 2 + \frac{\left(-\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e} + \mathbf{e}^T(B+\Gamma)^{-1}\Gamma(B+\Gamma)^{-1}\mathbf{e}\right)\mathbf{e}^T B^{-1}\mathbf{e}}{(\mathbf{e}^T(B+\Gamma)^{-1}\mathbf{e})^2} \\
&= 2 + \frac{\left(-\mathbf{y}^T\mathbf{z} + \mathbf{y}^T\mathbf{y}\right)\mathbf{x}^T\mathbf{z}}{(\mathbf{y}^T\mathbf{z})^2} \\
&\geq 2 + \frac{\left(-\mathbf{y}^T\mathbf{z} + \mathbf{x}^T\mathbf{y} - \frac{1}{4}\mathbf{x}^T\mathbf{x}\right)\mathbf{x}^T\mathbf{z}}{(\mathbf{y}^T\mathbf{z})^2} \\
&\geq 2 + \frac{\left(-\mathbf{y}^T\mathbf{z} + \mathbf{x}^T\mathbf{y} - \frac{1}{4}\mathbf{x}^T\mathbf{z}\right)\mathbf{x}^T\mathbf{z}}{(\mathbf{y}^T\mathbf{z})^2} \\
&= 2 + \frac{\left(-\mathbf{y}^T\mathbf{z} + (\mathbf{x}^T\mathbf{z} - \mathbf{y}^T\mathbf{z}) - \frac{1}{4}\mathbf{x}^T\mathbf{z}\right)\mathbf{x}^T\mathbf{z}}{(\mathbf{y}^T\mathbf{z})^2} \\
&= 2 + \frac{\left(-2\mathbf{y}^T\mathbf{z} + \frac{3}{4}\mathbf{x}^T\mathbf{z}\right)\mathbf{x}^T\mathbf{z}}{(\mathbf{y}^T\mathbf{z})^2} \\
&= 2 - 2\delta + \frac{3}{4}\delta^2
\end{aligned}
$$

where $2 - r \leq \delta \leq 2$. So $\frac{\Pi(NNE)}{\Pi(MP)}$ is minimized when $\delta = \frac{4}{3}$ with a minimum of $\frac{2}{3}$. ∎

Theorem C.4 $\frac{TS(OP)}{TS(SMAX)} \geq \frac{3}{4}$

Proof let $i^M = \text{argmax}_i B_{ii}$, define $\mathbf{d}^w$ as follows:

$$
d_k^w = \begin{cases} C & \text{for } k = i^M \\ 0 & \text{otherwise} \end{cases}
$$

$TS(OP) \geq TS(\mathbf{d}^w) = C(\bar{p}_0 - \frac{1}{2}B_{i^M i^M}C)$

$$\begin{aligned}
\frac{TS(OP)}{TS(SMAX)} &\geq \frac{C(\bar{p}_0 - \frac{1}{2}B_{i^M i^M}C)}{C(\bar{p}_0 - \frac{1}{2}\frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}})} \\
&= \frac{\bar{p}_0 - \frac{1}{2}B_{i^M i^M}C}{\bar{p}_0 - \frac{1}{2}\frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= 1 - \frac{C}{2}\frac{B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\bar{p}_0 - \frac{1}{2}\frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&\geq 1 - \frac{C}{2}\frac{B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{2B_{i^M i^M}C - \frac{1}{2}\frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= 1 - \frac{1}{2}\frac{B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{2B_{i^M i^M} - \frac{1}{2}\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= 1 - \frac{1}{4} + \frac{3}{4}\frac{\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{4B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= \frac{3}{4} + \frac{3}{4}\frac{1}{4B_{i^M i^M}\mathbf{e}^T B^{-1}\mathbf{e} - 1} \\
&\geq \frac{3}{4} + \frac{3}{4}\frac{1}{4B_{i^M i^M}\mathbf{e}^T \Gamma^{-1}\mathbf{e} - 1} \\
&\geq \frac{3}{4}
\end{aligned}$$

∎

Theorem C.5 $\frac{\Pi(OP)}{\Pi(MP)} \geq \frac{1}{2}$

Proof let $i^M = \text{argmax}_i B_{ii}$, define $\mathbf{d}^w$ as follows:

$$d_k^w = \begin{cases} C & \text{for } k = i^M \\ 0 & \text{otherwise} \end{cases}$$

$\Pi(OP) \geq \Pi(\mathbf{d}^w) = C(\bar{p}_0 - B_{i^M i^M}C)$

$$
\begin{aligned}
\frac{\Pi(OP)}{\Pi(MP)} &\geq \frac{C(\bar{p}_0 - B_{i^M i^M} C)}{C(\bar{p}_0 - \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}})} \\
&= \frac{\bar{p}_0 - B_{i^M i^M} C}{\bar{p}_0 - \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= 1 - C \frac{B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{\bar{p}_0 - \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&\geq 1 - C \frac{B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{2B_{i^M i^M} C - \frac{C}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= 1 - \frac{B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{2B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= 1 - \frac{1}{2} + \frac{1}{2} \frac{\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{2B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= \frac{1}{2} + \frac{1}{2} \frac{\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}}{2B_{i^M i^M} - \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}}} \\
&= \frac{1}{2} + \frac{1}{2} \frac{1}{2B_{i^M i^M} (\mathbf{e}^T B^{-1}\mathbf{e}) - 1} \\
&\geq \frac{1}{2} + \frac{1}{2} \frac{1}{2B_{i^M i^M} (\mathbf{e}^T \Gamma^{-1}\mathbf{e}) - 1} \\
&\geq \frac{1}{2}
\end{aligned}
$$

▮

D Calculations and Proofs for Chapter 6

D.1 Closed-form Solutions

Under the assumption of symmetric price influence,

$$
\mathbf{B} = \begin{bmatrix} \beta & \alpha & \dots & \alpha \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ \alpha & \dots & \alpha & \beta \end{bmatrix}
$$

for some $0 \leq \alpha < \beta$. Consider the KKT conditions for the SMAX problem when the constraint is restrictive:

$$\begin{cases} -B\bar{\mathbf{d}} + B\mathbf{d}^{SMAX} + \mu\mathbf{e} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \\ \sum_{i=1}^{n} d_i^{SMAX} = C \end{cases}$$

$$\Leftrightarrow \begin{cases} (\beta - \alpha)d_i^{SMAX} + \alpha C + \mu = \lambda_i + (\beta - \alpha)\bar{d}_i + \alpha \sum_{i=1}^{n} \bar{d}_i \\ \sum_{i=1}^{n} d_i^{SMAX} = C \end{cases}$$

Let $P = \{i | d_i^{SMAX} > 0\}$ and $P^C = \{i | d_i^{SMAX} = 0\}$.

For every $i \in P$, $d_i^{SMAX} > 0$ and $\lambda_i = 0$,

$$\Rightarrow \alpha C + \mu < (\beta - \alpha)d_i^{SMAX} + \alpha C + \mu = \lambda_i + (\beta - \alpha)\bar{d}_i + \alpha \sum_{i=1}^{n} \bar{d}_i = (\beta - \alpha)\bar{d}_i + \alpha \sum_{i=1}^{n} \bar{d}_i$$

For every $j \in P^C$, $d_i^{SMAX} = 0$ and $\lambda_i \geq 0$,

$$\Rightarrow \alpha C + \mu = \lambda_j + (\beta - \alpha)\bar{d}_j + \alpha \sum_{i=1}^{n} \bar{d}_i > (\beta - \alpha)\bar{d}_j + \alpha \sum_{i=1}^{n} \bar{d}_i$$

Hence $\bar{d}_i > \bar{d}_j$ for every $i \in P$ and $j \in P^C$.

Without loss of generality, assume that $\bar{d}_1 \leq \bar{d}_2 \leq ... \leq \bar{d}_n$, then $P = \{i | d_i^{SMAX} > 0\} = \{n - k_1 + 1, n - k_1 + 2, ..., n\}$ for some $1 \leq k_1 \leq n$.

Now consider $i \in P$,

$$(\beta - \alpha)d_i^{SMAX} + \alpha C + \mu = (\beta - \alpha)\bar{d}_i + \alpha \sum_{i=1}^{n} \bar{d}_i$$

Summing up all the k_1 equations,

$$\begin{aligned} \Rightarrow \quad (\beta - \alpha)\textstyle\sum_{i\in P} d_i^{SMAX} + k_1(\alpha C + \mu) &= (\beta - \alpha)\textstyle\sum_{i\in P} \bar{d}_i + k_1\alpha \textstyle\sum_{i=1}^{n} \bar{d}_i \\ \Rightarrow \quad (\beta - \alpha)C + k_1(\alpha C + \mu) &= (\beta - \alpha)\textstyle\sum_{i\in P} \bar{d}_i + k_1\alpha \textstyle\sum_{i=1}^{n} \bar{d}_i \\ \Rightarrow \quad (\beta - \alpha)C + k_1(\alpha C + \mu) &= (\beta - \alpha)k_1\mu_{k_1} + k_1\alpha n\mu_n \\ \Rightarrow \quad \alpha C + \mu &= (\beta - \alpha)\mu_{k_1} + \alpha n\mu_n - \tfrac{(\beta-\alpha)C}{k_1} \end{aligned}$$

where $\mu_k = \frac{1}{k}\sum_{i=n-k+1}^{n} \bar{d}_i$.

$$\begin{aligned} d_i^{SMAX} &= \frac{1}{\beta-\alpha}\left((\beta-\alpha)\bar{d}_i + \alpha\sum_{i=1}^{n}\bar{d}_i - (\alpha C + \mu)\right) \\ &= \frac{1}{\beta-\alpha}\left[(\beta-\alpha)\bar{d}_i + \alpha\sum_{i=1}^{n}\bar{d}_i - ((\beta-\alpha)\mu_k + \alpha n\mu_n - \frac{(\beta-\alpha)C}{k_1})\right] \\ &= \bar{d}_i - \mu_{k_1} + \frac{C}{k_1} \end{aligned}$$

There exists k_1 to guarantee that $0 < d_{n-k+1}^{SMAX} \leq d_{n-k+2}^{SMAX} \leq \ldots \leq d_n^{SMAX}$ since starting from $k = n$, $f(k) = \bar{d}_{n-k+1} - \mu_k + \frac{C}{k}$ is an decreasing function of k with $f(1) = C > 0$. In addition, k_1 is the largest integer satisfying $\bar{d}_{n-k+1} - \mu_k + \frac{C}{k} > 0$ since if we keep on decreasing k, the KKT conditions for d_j^{SMAX}, $j \in P^C$ will be violated.

Summarizing the discussions above, the optimal solution of the SMAX problem is given by:

$$d_i^{SMAX} = \begin{cases} \bar{d}_i - \mu_{k_1} + \frac{C}{k_1} & \text{for } i = n - k_1 + 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

k_1 is the largest integer satisfying $\bar{d}_{n-k+1} - \mu_k + \frac{C}{k} > 0$.

The closed-form solutions for the Monopoly problem and the Normalized Nash-Equilibrium can be derived similarly:

$$d_i^{MP} = \begin{cases} \frac{\bar{d}_i - \mu_{k_2}}{2} + \frac{C}{k_2} & \text{for } i = n - k_2 + 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

k_2 is the largest integer satisfying $\frac{\bar{d}_{n-k_2+1} - \mu_{k_2}}{2} + \frac{C}{k_2} \geq 0$.

$$d_i^{NNE} = \begin{cases} \frac{\beta-\alpha}{2\beta-\alpha}(\bar{d}_i - \mu_{k_3}) + \frac{C}{k_3} & \text{for } i = n - k_3 + 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

k_3 is the largest integer satisfying $\frac{\beta-\alpha}{2\beta-\alpha}(\bar{d}_{n-k_3+1} - \mu_{k_3}) + \frac{C}{k_3} \geq 0$.

In addition, under our assumption of $k_1 = k_2 = k_3 = n$, the solutions are given by

$$\begin{aligned} d_i^{SMAX} &= \bar{d}_i - \mu_n + \tfrac{C}{n} \\ d_i^{NNE} &= \tfrac{\beta-\alpha}{2\beta-\alpha}(\bar{d}_i - \mu_n) + \tfrac{C}{n} \quad \forall i = 1, 2, \ldots, n \\ d_i^{MP} &= \tfrac{1}{2}(\bar{d}_i - \mu_n) + \tfrac{C}{n} \end{aligned}$$

$$
\begin{aligned}
TS(SMAX) &= (\beta + (n-1)\alpha)C\mu_n - \frac{\beta + (n-1)\alpha}{2n}C^2 + \frac{\beta - \alpha}{2}var(n) \\
TS(MP) &= (\beta + (n-1)\alpha)C\mu_n - \frac{\beta + (n-1)\alpha}{2n}C^2 + \frac{3(\beta - \alpha)}{8}var(n) \\
TS(NNE) &= (\beta + (n-1)\alpha)C\mu_n - \frac{\beta + (n-1)\alpha}{2n}C^2 + \frac{(3\beta - \alpha)(\beta - \alpha)^2}{2(2\beta - \alpha)^2}var(n) \\
\Pi(SMAX) &= (\beta + (n-1)\alpha)C\mu_n - \frac{\beta + (n-1)\alpha}{n}C^2 \\
\Pi(MP) &= (\beta + (n-1)\alpha)C\mu_n - \frac{\beta + (n-1)\alpha}{n}C^2 + \frac{(\beta - \alpha)}{4}var(n) \\
\Pi(NNE) &= (\beta + (n-1)\alpha)C\mu_n - \frac{\beta + (n-1)\alpha}{n}C^2 + \frac{\beta(\beta - \alpha)^2}{(2\beta - \alpha)^2}var(n)
\end{aligned}
$$

D.2 Bounds for Loss of Efficiency

Theorem D.1 *For firms with symmetric price influences, the loss of total surplus for the Normalized Nash-Equilibrium compared with the SMAX solution is characterized by:*

$$\frac{TS(NNE)}{TS(SMAX)} \geq 1 - \frac{1-\theta}{(2-\theta)^2\left[1 + \frac{3n}{k(k+1)} + (\frac{3n(n-1)}{k(k+1)} - 1)\theta\right]} > \frac{3}{4}$$

where $\theta = \frac{\alpha}{\beta} = \frac{1}{\frac{n-1}{r} - (n-2)} \in [0,1)$, *and* $k = |I|$ *for* $I = \left\{i | \bar{d}_i < \mu_n\right\}$. $(0 \leq k \leq n-1)$

Proof

$$
\begin{aligned}
\frac{TS(NNE)}{TS(SMAX)} &= \frac{(\beta + (n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{(3\beta-\alpha)(\beta-\alpha)^2}{2(2\beta-\alpha)^2}var(n)}{(\beta + (n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{\beta-\alpha}{2}var(n)} \\
&= 1 - \frac{var(n)(\beta - \alpha)}{2} \frac{1 - \frac{(3\beta-\alpha)(\beta-\alpha)}{(2\beta-\alpha)^2}}{(\beta + (n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{\beta-\alpha}{2}var(n)} \\
&= 1 - \frac{\beta^2(\beta - \alpha)var(n)}{2(2\beta - \alpha)^2} \frac{1}{(\beta + (n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{\beta-\alpha}{2}var(n)} \\
&= 1 - \frac{\beta^2(\beta - \alpha)var(n)}{2(2\beta - \alpha)^2} \frac{1}{(\beta + (n-1)\alpha)C\left(\mu_n - \frac{1}{2n}C\right) + \frac{\beta-\alpha}{2}var(n)} \\
&= 1 - \frac{\beta^2(\beta - \alpha)}{2(2\beta - \alpha)^2} \frac{1}{(\beta + (n-1)\alpha)\underbrace{\frac{C\left(\mu_n - \frac{1}{2n}C\right)}{var(n)}}_{X} + \frac{\beta-\alpha}{2}}
\end{aligned}
$$

Under the assumption that $\frac{1}{2}n\mu_n \geq C \geq n(\mu_n - \bar{d}_i)$ for $i = 1, 2, ..., n$ (discussed in Chapter 6),

$$\sum_{i=1}^{n} \bar{d}_i = n\mu_n \Rightarrow \sum_{i \notin I} \underbrace{(\bar{d}_i - \mu_n)}_{\geq 0} = \sum_{i \in I} \underbrace{(\mu_n - \bar{d}_i)}_{>0} \leq \frac{k}{n}C$$

$$\Rightarrow \sum_{i \notin I} (\bar{d}_i - \mu_n)^2 \leq (\sum_{i \notin I} \bar{d}_i - \mu_n)^2 \leq (\frac{k}{n}C)^2$$

where $I = \{i | \bar{d}_i < \mu_n\}$ and $k = |I|$.

$$\begin{aligned} \Rightarrow var(n) &= \sum_{i=1}^{n} (\bar{d}_i - \mu_n)^2 \\ &= \sum_{i \notin I}^{n} (\bar{d}_i - \mu_n)^2 + \sum_{i \in I}^{n} (\mu_n - \bar{d}_i)^2 \\ &\leq (\frac{k}{n}C)^2 + k(\frac{C}{n})^2 \\ &= \frac{k(k+1)}{n^2}C^2 \end{aligned}$$

This implies that

$$\begin{aligned} X &= \frac{C\left(\mu_n - \frac{1}{2n}C\right)}{var(n)} \\ &\geq \frac{C\left(\mu_n - \frac{1}{2n}C\right)}{\frac{k(k+1)}{n^2}C^2} \\ &= \frac{\left(\frac{\mu_n}{C} - \frac{1}{2n}\right)}{\frac{k(k+1)}{n^2}} \\ &\geq \frac{\left(\frac{2}{n} - \frac{1}{2n}\right)}{\frac{k(k+1)}{n^2}} \\ &= \frac{3}{2}\frac{n}{k(k+1)} \end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{TS(NNE)}{TS(SMAX)} &= 1 - \frac{\beta^2(\beta-\alpha)}{2(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)X + \frac{\beta-\alpha}{2}} \\
&\geq 1 - \frac{\beta^2(\beta-\alpha)}{2(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)\frac{3}{2}\frac{n}{k(k+1)} + \frac{\beta-\alpha}{2}} \\
&= 1 - \frac{\beta^2(\beta-\alpha)}{(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)\frac{3n}{k(k+1)} + \beta-\alpha} \\
&= 1 - \frac{1-\theta}{(2-\theta)^2\left[1+\frac{3n}{k(k+1)} + (\frac{3n(n-1)}{k(k-1)} - 1)\theta\right]}
\end{aligned}$$

where $\theta = \frac{\alpha}{\beta}$. ▮

Theorem D.2 *For firms with symmetric price influences, the total surplus of the Monopoly solution is always above the total surplus of the Normalized Nash-Equilibrium:*

$$1 \leq \frac{TS(MP)}{TS(NNE)} \leq 1 + \frac{1}{4}\frac{\theta(4-\theta)(1-\theta)}{3(1+(n-1)\theta)(2-\theta)^2\frac{n}{k(k+1)} + (3-\theta)(1-\theta)^2}.$$

Proof

$$TS(MP) = (\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{3(\beta-\alpha)}{8}var(n)$$

$$TS(NNE) = (\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{(3\beta-\alpha)(\beta-\alpha)^2}{2(2\beta-\alpha)^2}var(n)$$

$$\begin{aligned}
\Rightarrow \frac{TS(MP)}{TS(NNE)} &= \frac{(\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{3(\beta-\alpha)}{8}var(n)}{(\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{(3\beta-\alpha)(\beta-\alpha)^2}{2(2\beta-\alpha)^2}var(n)} \\
&= 1 + \frac{(\beta-\alpha)var(n)}{8}\frac{3 - \frac{4(3\beta-\alpha)(\beta-\alpha)}{(2\beta-\alpha)^2}}{(\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{(3\beta-\alpha)(\beta-\alpha)^2}{2(2\beta-\alpha)^2}var(n)} \\
&= 1 + \frac{\alpha(4\beta-\alpha)(\beta-\alpha)var(n)}{8(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{2n}C^2 + \frac{(3\beta-\alpha)(\beta-\alpha)^2}{2(2\beta-\alpha)^2}var(n)} \\
&= 1 + \frac{\alpha(4\beta-\alpha)(\beta-\alpha)}{8(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)\underbrace{\frac{C\mu_n - \frac{1}{2n}C^2}{var(n)}}_{X} + \frac{(3\beta-\alpha)(\beta-\alpha)^2}{2(2\beta-\alpha)^2}}
\end{aligned}$$

Since $X \geq \frac{3}{2}\frac{n}{k(k+1)}$ (shown in the previous theorem),

$$\begin{aligned}
\Rightarrow \frac{TS(MP)}{TS(NNE)} &\leq 1 + \frac{\alpha(4\beta-\alpha)(\beta-\alpha)}{8(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)\frac{3}{2}\frac{n}{k(k+1)} + \frac{(3\beta-\alpha)(\beta-\alpha)^2}{2(2\beta-\alpha)^2}} \\
&= 1 + \frac{\theta(4-\theta)(1-\theta)}{8(2-\theta)^2}\frac{1}{(1+(n-1)\theta)\frac{3}{2}\frac{n}{k(k+1)} + \frac{(3-\theta)(1-\theta)^2}{2(2-\theta)^2}} \\
&= 1 + \frac{1}{4}\frac{\theta(4-\theta)(1-\theta)}{3(1+(n-1)\theta)(2-\theta)^2\frac{n}{k(k+1)} + (3-\theta)(1-\theta)^2}
\end{aligned}$$

∎

Theorem D.3 *Loss of profit for the Normalized Nash-Equilibrium compared with the Monopoly solution is given by:*

$$\frac{\Pi(NNE)}{\Pi(MP)} \geq 1 - \frac{\theta^2(1-\theta)}{(2-\theta)^2\left[1 + \frac{4n}{k(k+1)} + (\frac{8n(n-1)}{k(k+1)} - 1)\theta\right]} > 0.98$$

where $\theta = \frac{\alpha}{\beta} = \frac{1}{\frac{n-1}{r}-(n-2)} \in [0,1)$, *and* $k = |I|$ *for* $I = \left\{i|\bar{d}_i < \mu_n\right\}$.

Proof With $\frac{1}{2}n\mu_n \geq C \geq n(\mu_n - \bar{d}_i)$ for $i = 1, 2, ..., n$,

$$\Pi(MP) = (\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{n}C^2 + \frac{(\beta-\alpha)}{4}var(n)$$

$$\Pi(NNE) = (\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{n}C^2 + \frac{\beta(\beta-\alpha)^2}{(2\beta-\alpha)^2}var(n)$$

$$\begin{aligned}
\Rightarrow \frac{\Pi(NNE)}{\Pi(MP)} &= \frac{(\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{n}C^2 + \frac{\beta(\beta-\alpha)^2}{(2\beta-\alpha)^2}var(n)}{(\beta+(n-1)\alpha)C\mu_n - \frac{\beta+(n-1)\alpha}{n}C^2 + \frac{(\beta-\alpha)}{4}var(n)} \\
&= 1 - \frac{(\beta-\alpha)\alpha^2}{4(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)\underbrace{\frac{C\mu_n - nC^2}{var(n)}}_{Y} + \frac{(\beta-\alpha)}{4}}
\end{aligned}$$

Similar with the previous theorem,

$$\begin{aligned} Y &= \frac{C\left(\mu_n - \frac{1}{n}C\right)}{var(n)} \\ &\geq \frac{C\left(\mu_n - \frac{1}{n}C\right)}{\frac{k(k+1)}{n^2}C^2} \\ &= \frac{\left(\frac{\mu_n}{C} - \frac{1}{n}\right)}{\frac{k(k+1)}{n^2}} \\ &\geq \frac{\left(\frac{2}{n} - \frac{1}{n}\right)}{\frac{k(k+1)}{n^2}} \\ &= \frac{n}{k(k+1)} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\Pi(NNE)}{\Pi(MP)} &= 1 - \frac{(\beta-\alpha)\alpha^2}{4(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)Y + \frac{(\beta-\alpha)}{4}} \\ &\geq 1 - \frac{(\beta-\alpha)\alpha^2}{4(2\beta-\alpha)^2}\frac{1}{(\beta+(n-1)\alpha)\frac{n}{k(k+1)} + \frac{(\beta-\alpha)}{4}} \\ &= 1 - \frac{\alpha^2(\beta-\alpha)}{(2\beta-\alpha)^2\left[(\beta+(n-1)\alpha)\frac{4n}{k(k+1)} + (\beta-\alpha)\right]} \\ &= 1 - \frac{\theta^2(1-\theta)}{(2-\theta)^2\left[(1+(n-1)\theta)\frac{4n}{k(k+1)} + (1-\theta)\right]} \\ &= 1 - \frac{\theta^2(1-\theta)}{(2-\theta)^2\left[1+\frac{4n}{k(k+1)} + \left(\frac{4n(n-1)}{k(k+1)} - 1\right)\theta\right]} \end{aligned}$$

where $\theta = \frac{\alpha}{\beta} = \frac{1}{\frac{n-1}{r} - (n-2)} \in [0,1)$. ∎

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