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DIFFERENTIABILITY OF M-FUNCTIONALS OF LOCATION AND SCATTER BASED ON T LIKELIHOODS

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Abstract. The paper aims at finding widely and smoothly defined nonparametric location and scatter functionals. As a convenient vehicle, maximum likelihood estimation of the location vector μ and scatter matrix Σ of an elliptically symmetric t distribution on \mathbb{R}^d with degrees of freedom $\nu > 1$ extends to an M-functional defined on all probability distributions P in a weakly open, weakly dense domain U . Here U consists of P putting not too much mass in hyperplanes of dimension $< d$, as shown for empirical measures by Kent and Tyler (*Ann. Statist.* 1991). It is shown here that (μ, Σ) is analytic on U , for the bounded Lipschitz norm, or for $d = 1$, for the sup norm on distribution functions. For $k = 1, 2, \dots$, and other norms, depending on k and more directly adapted to t functionals, one has continuous differentiability of order k , allowing the delta-method to be applied to (μ, Σ) for any P in U , which can be arbitrarily heavy-tailed. These results imply asymptotic normality of the corresponding M -estimators (μ_n, Σ_n) . In dimension $d = 1$ only, the t_ν functional (μ, σ) extends to be defined and weakly continuous at all P .

1. INTRODUCTION

This paper is a longer version, with proofs, of the paper Dudley, Sidenko and Wang (2009). It aims at developing some nonparametric location and scatter functionals, defined and smooth on large (weakly dense and open) sets of distributions. The nonparametric view is much as in the work of Bickel and Lehmann (1975) (but not adopting, e.g., their monotonicity axiom) and to a somewhat lesser extent, that of Davies (1998). Although there are relations to robustness, that is not the main aim here: there is no focus on neighborhoods of model distributions with densities such as the normal. It happens that the parametric family of ellipsoidally symmetric t densities provides an avenue toward nonparametric

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location and scatter functionals, somewhat as maximum likelihood estimation of location for the double-exponential distribution in one dimension gives the median, generally viewed as a nonparametric functional.

Given observations X_1, \dots, X_n in \mathbb{R}^d let $P_n := \frac{1}{n} \sum_{j=1}^n \delta_{X_j}$. Given P_n , and the location-scatter family of elliptically symmetric t_ν distributions on \mathbb{R}^d with $\nu > 1$, maximum likelihood estimates of the location vector μ and scatter matrix Σ exist and are unique for “most” P_n . Namely, it suffices that $P_n(J) < (\nu + q)/(\nu + d)$ for each affine hyperplane J of dimension $q < d$, as shown by Kent and Tyler (1991). The estimates extend to M-functionals defined at all probability measures P on \mathbb{R}^d satisfying the same condition; that is shown for integer ν and in the sense of unique critical points by Dümbgen and Tyler (2005) and for any $\nu > 0$ and M-functionals in the sense of unique absolute minima in Theorem 3, in light of Theorem 6(a), for pure scatter and then in Theorem 6(e) for location and scatter with $\nu > 1$. A method of reducing location and scatter functionals in dimension d to pure scatter functionals in dimension $d + 1$ was shown to work for t distributions by Kent and Tyler (1991) and only for such distributions by Kent, Tyler and Vardi (1994), as will be recalled after Theorem 6.

So the t functionals are defined on a weakly open and weakly dense domain, whose complement is thus weakly nowhere dense. One of the main results of the present paper gives analyticity (defined in the Appendix) of the functionals on this domain, with respect to the bounded Lipschitz norm (Theorem 9(d)). An adaptation gives differentiability of any given finite order k with respect to norms, depending on k , chosen to give asymptotic normality of the t location and scatter functionals (Theorem 18) for arbitrarily heavy-tailed P (for such P , the central limit fails in the bounded Lipschitz norm). In turn, this yields delta-method conclusions (Theorem 20(b)), uniformly over suitable families of distributions (Proposition 22); these statements don’t include any norms, although their proofs do. It follows in Corollary 24 that continuous Fréchet differentiability of the t_ν location and scatter functionals of order k also holds with respect to affinely invariant norms defined via suprema over positivity sets of polynomials of degree at most $2k + 4$.

For the delta-method, one needs at least differentiability of first order. To get first derivatives with respect to probability measures P via an implicit function theorem we use second order derivatives with respect to matrices. Moreover, second order derivatives with respect to P (or in the classical case, with respect to an unknown parameter) can improve the accuracy of the delta-method and the speed of convergence of approximations. It turns out that derivatives of arbitrarily high order are obtainable with little additional difficulty.

For norms in which the central limit theorem for empirical measures holds for all probability measures, such as those just mentioned, bootstrap central limit

theorems also hold [Giné and Zinn (1990)], which then via the delta-method can give bootstrap confidence sets for the t location and scatter functionals.

In dimension $d = 1$, the domain on which differentiability is proved is the class of distributions having no atom of size $\nu/(\nu + 1)$ or larger. On this domain, analyticity is proved, in Theorem 9(e), with respect to the usual supremum norm for distribution functions. Only for $d = 1$, it turns out to be possible to extend the t_ν location and scatter (scale) functionals to be defined and weakly continuous at arbitrary distributions (Theorem 25).

For general $d \geq 1$ and $\nu = 1$ (multivariate Cauchy distributions), a case not covered by the present paper, Dümbgen (1998, §6) briefly treats location and scatter functionals and their asymptotic properties.

Weak continuity on a dense open set implies that for distributions in that set, estimators (functionals of empirical measures) eventually exist almost surely and converge to the functional of the distribution. Weak continuity, where it holds, also is a robustness property in itself and implies a strictly positive (not necessarily large) breakdown point. The t_ν functionals, as redescending M-functionals, downweight outliers. Among such M-functionals, only the t_ν functionals are known to be uniquely defined on a satisfactorily large domain. The t_ν estimators are \sqrt{n} -consistent estimators of t_ν functionals where each t_ν location functional, at any distribution in its domain and symmetric around a point, (by equivariance) equals the center of symmetry.

It seems that few other known location and scatter functionals exist and are unique and continuous, let alone differentiable, on a dense open domain. For example, the median is discontinuous on a dense set. Smoothly trimmed means and variances are defined and differentiable at all distributions in one dimension, e.g. Boos (1979) for means. In higher dimensions there are analogues of trimming, called peeling or depth weighting, e.g. the work of Zuo and Cui (2005). Location-scatter functionals differentiable on a dense domain apparently have not been found by depth weighting thus far (in dimension $d > 1$).

The t location and scatter functionals, on their domain, can be effectively computed via EM algorithms [cf. Kent, Tyler and Vardi (1994, §4); Arslan, Constable, and Kent (1995); Liu, Rubin and Wu (1998)].

2. DEFINITIONS AND PRELIMINARIES

In this paper the sample space will be a finite-dimensional Euclidean space \mathbb{R}^d with its usual topological and Borel structure. A *law* will mean a probability measure on \mathbb{R}^d . Let \mathcal{S}_d be the collection of all $d \times d$ symmetric real matrices, \mathcal{N}_d the subset of nonnegative definite symmetric matrices and $\mathcal{P}_d \subset \mathcal{N}_d$ the further subset of strictly positive definite symmetric matrices. The parameter spaces Θ considered will be \mathcal{P}_d , \mathcal{N}_d (pure scatter matrices), $\mathbb{R}^d \times \mathcal{P}_d$, or $\mathbb{R}^d \times \mathcal{N}_d$. For $(\mu, \Sigma) \in \mathbb{R}^d \times \mathcal{N}_d$, μ will be viewed as a location parameter and Σ as a

scatter parameter, extending the notions of mean vector and covariance matrix to arbitrarily heavy-tailed distributions. Matrices in \mathcal{N}_d but not in \mathcal{P}_d will only be considered in one dimension, in Section 9, where the scale parameter $\sigma \geq 0$ corresponds to $\sigma^2 \in \mathcal{N}_1$.

Notions of “location” and “scale” or multidimensional “scatter” functional will be defined in terms of equivariance, as follows.

Definitions. Let $Q \mapsto \mu(Q) \in \mathbb{R}^d$, resp. $\Sigma(Q) \in \mathcal{N}_d$, be a functional defined on a set \mathcal{D} of laws Q on \mathbb{R}^d . Then μ (resp. Σ) is called an *affinely equivariant location* (resp. *scatter*) *functional* iff for any nonsingular $d \times d$ matrix A and $v \in \mathbb{R}^d$, with $f(x) := Ax + v$, and any law $Q \in \mathcal{D}$, the image measure $P := Q \circ f^{-1} \in \mathcal{D}$ also, with $\mu(P) = A\mu(Q) + v$ or, respectively, $\Sigma(P) = A\Sigma(Q)A'$. For $d = 1$, $\sigma(\cdot)$ with $0 \leq \sigma < \infty$ will be called an *affinely equivariant scale functional* iff σ^2 satisfies the definition of affinely equivariant scatter functional. If we have affinely equivariant location and scatter functionals μ and Σ on the same domain \mathcal{D} then (μ, Σ) will be called an affinely equivariant location-scatter functional on \mathcal{D} .

To define M-functionals, suppose we have a function $(x, \theta) \mapsto \rho(x, \theta)$ defined for $x \in \mathbb{R}^d$ and $\theta \in \Theta$, Borel measurable in x and lower semicontinuous in θ , i.e. $\rho(x, \theta) \leq \liminf_{\phi \rightarrow \theta} \rho(x, \phi)$ for all θ . For a law Q , let $Q\rho(\phi) := \int \rho(x, \phi) dQ(x)$ if the integral is defined (not $\infty - \infty$), as it always will be if $Q = P_n$. An *M-estimate* of θ for a given n and P_n will be a $\hat{\theta}_n$ such that $P_n\rho(\theta)$ is minimized at $\theta = \hat{\theta}_n$, if it exists and is unique. A measurable function, not necessarily defined a.s., whose values are M-estimates is called an M-estimator.

For a law P on \mathbb{R}^d and a given $\rho(\cdot, \cdot)$, a $\theta_1 = \theta_1(P)$ is called the *M-functional* of P for ρ if and only if there exists a measurable function $a(x)$, called an *adjustment function*, such that for $h(x, \theta) = \rho(x, \theta) - a(x)$, $Ph(\theta)$ is defined and satisfies $-\infty < Ph(\theta) \leq +\infty$ for all $\theta \in \Theta$, and is minimized uniquely at $\theta = \theta_1(P)$, e.g. Huber (1967). As Huber showed, $\theta_1(P)$ doesn't depend on the choice of $a(\cdot)$, which can moreover be taken as $a(x) \equiv \rho(x, \theta_2)$ for a suitable θ_2 .

The following definition will be used for $d = 1$. Suppose we have a parameter space Θ , specifically \mathcal{P}_d or $\mathcal{P}_d \times \mathbb{R}^d$, which has a closure $\bar{\Theta}$, specifically \mathcal{N}_d or $\mathcal{N}_d \times \mathbb{R}^d$ respectively. The *boundary* of Θ is then $\bar{\Theta} \setminus \Theta$. The functions ρ and h are not necessarily defined for θ in the boundary, but M-functionals may have values anywhere in $\bar{\Theta}$ according to the following.

Definition. A $\theta_0 = \theta_0(P) \in \bar{\Theta}$ will be called the (extended) *M-functional* of P for ρ or h if and only if for every neighborhood U of θ_0 ,

$$(1) \quad -\infty \leq \liminf_{\phi \rightarrow \theta_0, \phi \in \Theta} Ph(\phi) < \inf_{\phi \in \Theta, \phi \notin U} Ph(\phi).$$

The above definition extends that of M-functional given by Huber (1967) in that if θ_0 is on the boundary of $\bar{\Theta}$ then $h(x, \theta_0)$ is not defined, $Ph(\theta_0)$ is defined only in a lim inf sense, and at θ_0 (but only there), the lim inf may be $-\infty$.

From the definition, an M-functional, if it exists, must be unique. If P is an empirical measure P_n , then the M-functional $\hat{\theta}_n := \theta_0(P_n)$, if it exists, is the maximum likelihood estimate of θ , in a lim sup sense if $\hat{\theta}_n$ is on the boundary. Clearly, an M-estimate $\hat{\theta}_n$ is the M-functional $\theta_1(P_n)$ if either exists.

For a differentiable function f , recall that a *critical point* of f is a point where the gradient of f is 0. For example, on \mathbb{R}^2 let $f(x, y) = x^2(1 + y)^3 + y^2$. Then f has a unique critical point $(0, 0)$, which is a strict relative minimum where the Hessian (matrix of second partial derivatives) is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, but not an absolute minimum since $f(1, y) \rightarrow -\infty$ as $y \rightarrow -\infty$. This example appeared in Durfee, Kronenfeld, Munson, Roy, and Westby (1993).

3. MULTIVARIATE SCATTER

This section will treat the pure scatter problem in \mathbb{R}^d , with parameter space $\Theta = \mathcal{P}_d$. The results here are extensions of those of Kent and Tyler (1991, Theorems 2.1 and 2.2), on unique maximum likelihood estimates for finite samples, to the case of M-functionals for general laws on \mathbb{R}^d .

For $A \in \mathcal{P}_d$ and a function ρ from $[0, \infty)$ into itself, consider the function

$$(2) \quad L(y, A) := \frac{1}{2} \log \det A + \rho(y' A^{-1} y), \quad y \in \mathbb{R}^d.$$

For adjustment, let

$$(3) \quad h(y, A) := L(y, A) - L(y, I)$$

where I is the identity matrix. Then

$$(4) \quad Qh(A) = \frac{1}{2} \log \det A + \int \rho(y' A^{-1} y) - \rho(y' y) dQ(y)$$

if the integral is defined.

As a referee suggested, one can differentiate functions of matrices in a coordinate free way, as follows. The d^2 -dimensional vector space of all $d \times d$ real matrices becomes a Hilbert space (Euclidean space) under the inner product $\langle A, B \rangle := \text{trace}(A'B)$. It's easy to verify that this is indeed an inner product and is invariant under orthogonal changes of coordinates in the underlying d -dimensional vector space. The corresponding norm $\|A\|_F := \langle A, A \rangle^{1/2}$ is called the Frobenius norm. Here $\|A\|_F^2$ is simply the sum of squares of all elements of A , and $\|\cdot\|_F$ is the specialization of the (Hilbert)-Schmidt norm for Hilbert-Schmidt operators on a general Hilbert space to the case of (all) linear operators on a

finite-dimensional Hilbert space. Let $\|\cdot\|$ be the usual matrix or operator norm, $\|A\| := \sup_{|x|=1} |Ax|$. Then

$$(5) \quad \|A\| \leq \|A\|_F \leq \sqrt{d}\|A\|,$$

with equality in the latter for $A = I$ and the former when $A = \text{diag}(1, 0, \dots, 0)$. In statements such as $\|A\| \rightarrow 0$ or expressions such as $O(\|A\|)$ the particular norm doesn't matter for fixed d .

The map $A \mapsto A^{-1}$ is C^∞ from \mathcal{P}_d onto itself. For fixed $A \in \mathcal{P}_d$ and as $\|\Delta\| \rightarrow 0$, we have

$$(6) \quad (A + \Delta)^{-1} = A^{-1} - A^{-1}\Delta A^{-1} + O(\|\Delta\|^2),$$

as is seen since $(A + \Delta)(A^{-1} - A^{-1}\Delta A^{-1}) = I + O(\|\Delta\|^2)$, then multiplying by $(A + \Delta)^{-1}$.

Differentiating $f(A)$ for $A \in \mathcal{S}_d$ is preferably done when possible in coordinate free form, or if in coordinates, when restricted to a subspace of matrices all diagonal in some fixed coordinates, or at least approaching such matrices. It turns out that all proofs in the paper can be and have been done in one of these ways.

We have the following, stated for $Q = Q_n$ an empirical measure in Kent and Tyler (1991, (1.3)). Here (7) is a redescending condition.

Proposition 1. *Let $\rho : [0, \infty) \rightarrow [0, \infty)$ be continuous and have a bounded continuous derivative on $[0, \infty)$, where*

$$\rho'(0) := \rho'(0+) := \lim_{x \downarrow 0} [\rho(x) - \rho(0)]/x.$$

Let $0 \leq u(x) := 2\rho'(x)$ for $x \geq 0$ and suppose that

$$(7) \quad \sup_{0 \leq x < \infty} xu(x) < \infty.$$

Then for any law Q on \mathbb{R}^d , Qh in (4) is a well-defined and C^1 function of $A \in \mathcal{P}_d$, which has a critical point at $A = B$ if and only if

$$(8) \quad B = \int u(y'B^{-1}y)yy'dQ(y).$$

Proof. By the hypotheses, the chain rule, and (6) we have for fixed $A \in \mathcal{P}_d$ as $\|\Delta\| \rightarrow 0$

$$\begin{aligned} \rho(y'(A + \Delta)^{-1}y) - \rho(y'A^{-1}y) &= \rho(y'[A^{-1} - A^{-1}\Delta A^{-1} + O(\|\Delta\|^2)]y) \\ &= -\rho'(y'A^{-1}y)y'A^{-1}\Delta A^{-1}y + o(\|\Delta\||y|). \end{aligned}$$

Since $y'A^{-1}\Delta A^{-1}y \equiv \text{trace}(A^{-1}yy'A^{-1}\Delta)$, it follows that the gradient ∇_A with respect to $A \in \mathcal{P}_d$ of $\rho(y'A^{-1}y)$ is given by

$$(9) \quad \nabla_A \rho(y'A^{-1}y) = -\frac{1}{2}u(y'A^{-1}y)A^{-1}yy'A^{-1}.$$

Given $A \in \mathcal{P}_d$ let $A_t := (1-t)I + tA \in \mathcal{P}_d$ for $0 \leq t \leq 1$. Then

$$\begin{aligned} \rho(y'A^{-1}y) - \rho(y'y) &= \int_0^1 \frac{d}{dt} \rho(y'A_t^{-1}y) dt \\ &= \int_0^1 \rho'(y'A_t^{-1}y) \text{trace} \left(A_t^{-1} y y' A_t^{-1} (A - I) \right) dt. \end{aligned}$$

For a fixed $A \in \mathcal{P}_d$, the A_t^{-1} are all in some compact subset of \mathcal{P}_d , so that their eigenvalues are bounded and bounded away from 0. From this and boundedness of $xu(x)$ for $x \geq 0$, it follows that $y \mapsto \rho(y'A^{-1}y) - \rho(y'y)$ is a bounded continuous function of y . We also have:

$$(10) \quad \text{For any compact } \mathcal{K} \subset \mathcal{P}_d, \quad \sup\{|h(y, A)| : y \in \mathbb{R}^d, A \in \mathcal{K}\} < \infty.$$

It follows that for an arbitrary law Q on \mathbb{R}^d , $Qh(A)$ in (4) is defined and finite. Also, $Qh(A)$ is continuous in A by dominated convergence and so lower semicontinuous.

For any $B \in \mathcal{S}_d$ let its ordered eigenvalues be $\lambda_1(B) \geq \lambda_2(B) \geq \dots \geq \lambda_d(B)$. We have for fixed $A \in \mathcal{P}_d$ as $\Delta \rightarrow 0$, $\Delta \in \mathcal{S}_d$, that

$$(11) \quad \log \det(A + \Delta) - \log \det A = \text{trace}(A^{-1}\Delta) - \|A^{-1/2}\Delta A^{-1/2}\|_F^2/2 + O(\|\Delta\|^3)$$

because

$$\begin{aligned} \log \det(A + \Delta) - \log \det A &= \log \det(A^{-1/2}(A + \Delta)A^{-1/2}) \\ &= \log \det(I + A^{-1/2}\Delta A^{-1/2}) = \sum_{i=1}^d \log(1 + \lambda_i(A^{-1/2}\Delta A^{-1/2})) \\ &= \sum_{i=1}^d \lambda_i(A^{-1/2}\Delta A^{-1/2}) - \lambda_i(A^{-1/2}\Delta A^{-1/2})^2/2 + O(\|\Delta\|^3) \end{aligned}$$

and (11) follows. By (9), and because the gradient there is bounded, derivatives can be interchanged with the integral, so we have

$$\begin{aligned} Qh(A + \Delta) &= Qh(A) + \frac{1}{2} \text{trace}(A^{-1}\Delta) - \int \rho'(y'A^{-1}y) y'A^{-1}\Delta A^{-1}y dQ(y) + o(\|\Delta\|) \\ &= Qh(A) + \frac{1}{2} \left\langle A^{-1} - \int u(y'A^{-1}y) A^{-1} y y' A^{-1} dQ(y), \Delta \right\rangle + o(\|\Delta\|). \end{aligned}$$

It follows that the gradient of the mapping $A \mapsto Qh(A)$ from \mathcal{P}_d into \mathbb{R} is

$$(12) \quad \nabla_A Qh(A) = \frac{1}{2} \left(A^{-1} - \int u(y'A^{-1}y) A^{-1} y y' A^{-1} dQ(y) \right) \in \mathcal{S}_d,$$

which, multiplying by A on the left and right, is zero if and only if

$$A = \int u(y'A^{-1}y) y y' dQ(y).$$

This proves the Proposition. \square

The following extends to any law Q the uniqueness part of Kent and Tyler (1991, Theorem 2.2).

Proposition 2. *Under the hypotheses of Proposition 1 on ρ and $u(\cdot)$, if in addition $u(\cdot)$ is nonincreasing and $s \mapsto su(s)$ is strictly increasing on $[0, \infty)$, then for any law Q on \mathbb{R}^d , Qh has at most one critical point $A \in \mathcal{P}_d$.*

Proof. By Proposition 1, suppose that (8) holds for $B = A$ and $B = D$ for some $D \neq A$ in \mathcal{P}_d . By the substitution $y = A^{1/2}z$ we can assume that $A = I \neq D$.

Let t_1 be the largest eigenvalue of D . Suppose that $t_1 > 1$. Then for any $y \neq 0$, by the assumed properties of $u(\cdot)$, $u(y'D^{-1}y) \leq u(t_1^{-1}y'y) < t_1 u(y'y)$. It follows from (8) for D and I that for any $z \in \mathbb{R}^d$ with $z \neq 0$,

$$z'Dz = \int u(y'D^{-1}y)(z'y)^2 dQ(y) < t_1 \int u(y'y)(z'y)^2 dQ(y) = t_1 |z|^2,$$

where the last equation implies that Q is not concentrated in any $(d-1)$ -dimensional vector subspace $z'y = 0$ and so the preceding inequality is strict. Taking z as an eigenvector for the eigenvalue t_1 gives a contradiction.

If $t_d < 1$ for the smallest eigenvalue t_d of D we get a symmetrical contradiction. It follows that $D = I$, proving the Proposition. \square

We saw in the preceding proof that if there is a critical point, Q is not concentrated in any proper linear subspace. More precisely, a sufficient condition for existence of a minimum (unique by Proposition 2) will include the following assumption from Kent and Tyler (1991, (2.4)). For a given function $u(\cdot)$ as in Proposition 2, let $a_0 := a_0(u(\cdot)) := \sup_{s>0} su(s)$. Since $s \mapsto su(s)$ is increasing, we will have

$$(13) \quad su(s) \uparrow a_0 \quad \text{as} \quad s \uparrow +\infty.$$

Kent and Tyler (1991) gave the following conditions for empirical measures.

Definition. For a given number $a_0 := a(0) > 0$ let $\mathcal{U}_{d,a(0)}$ be the set of all probability measures Q on \mathbb{R}^d such that for every linear subspace H of dimension $q \leq d-1$, $Q(H) < 1 - (d-q)/a_0$, so that $Q(H^c) > (d-q)/a_0$.

If $Q \in \mathcal{U}_{d,a(0)}$, then $Q(\{0\}) < 1 - (d/a_0)$, which is impossible if $a_0 \leq d$. So we will need $a_0 > d$ and assume it, e.g. in the following theorem. In the t_ν case later we will have $a_0 = \nu + d > d$ for any $\nu > 0$. For $a(0) > d$, $\mathcal{U}_{d,a(0)}$ is weakly open and dense and contains all laws with densities. In part (b), Kent and Tyler (1991, Theorems 2.1 and 2.2) proved that there is a unique $B(Q_n)$ minimizing $Q_n h$ for an empirical $Q_n \in \mathcal{U}_{d,a(0)}$.

Theorem 3. *Let $u(\cdot) \geq 0$ be a bounded continuous function on $[0, \infty)$ satisfying (7), with $u(\cdot)$ nonincreasing and $s \mapsto su(s)$ strictly increasing. Then for $a(0) = a_0$ as in (13), if $a_0 > d$,*

(a) *If $Q \notin \mathcal{U}_{d,a(0)}$, then Qh has no critical points.*

(b) *If $Q \in \mathcal{U}_{d,a(0)}$, then Qh attains its minimum at a unique $B = B(Q) \in \mathcal{P}_d$ and has no other critical points.*

Proof. (a): Tyler (1988, (2.3)) showed that the condition $Q(H) \leq 1 - (d - q)/a_0$ for all linear subspaces H of dimension $q > 0$ is necessary for the existence of a critical point as in (8) for $Q = Q_n$. His proof shows necessity of the stronger condition $Q_n \in \mathcal{U}_{d,a(0)}$ when $su(s) < a_0$ for all $s < \infty$ (then the inequality Tyler [1988, (4.2)] is strict) and also applies when $q = 0$, so that $H = \{0\}$. The proof extends to general Q , using (7) for integrability.

(b): For any A in \mathcal{P}_d , let the eigenvalues of A^{-1} be $\tau_1 \leq \tau_2 \leq \dots \leq \tau_d$, where $\tau_j \equiv \tau_j(A)$ for each j . Let A be diagonalized. Then, varying A only among matrices diagonalized in the same coordinates, by (12),

$$(14) \quad \frac{\partial Qh(A)}{\partial \tau_j} = \frac{1}{2\tau_j} \left[\tau_j \int y_j^2 u \left(\sum_{i=1}^d \tau_i y_i^2 \right) dQ(y) - 1 \right].$$

Claim 1: For some $\delta_0 > 0$,

$$(15) \quad \inf\{Qh(A) : \tau_1(A) \leq \delta_0/2\} \geq (\log 2)/4 + \inf\{Qh(A) : \tau_1(A) \geq \delta_0\}.$$

To prove Claim 1, we have $xu(x) \downarrow 0$ as $x \downarrow 0$ since $u(\cdot)$ is right-continuous at 0, and so by dominated convergence using (7), there is a $\delta_0 > 0$, not depending on the choice of Euclidean coordinates, such that for any $t < \delta_0$, $\int t|y|^2 u(t|y|^2) dQ(y) < 1/2$. We can take $\delta_0 < 1$. Then, since $s \mapsto su(s)$ is increasing, it follows that for each $j = 1, \dots, d$, if $\tau_j < \delta_0$ then $\tau_j \int y_j^2 u(\tau_j y_j^2) dQ(y) < 1/2$ and so $\tau_j \int y_j^2 u(\sum_{i=1}^d \tau_i y_i^2) dQ(y) < 1/2$ since $u(\cdot)$ is nonincreasing. It follows by (14) that

$$(16) \quad \partial Qh(A)/\partial \tau_j < -1/(4\tau_j), \quad \tau_j < \delta_0.$$

If $\tau_1 < \delta_0/2$, let r be the largest index $j \leq d$ such that $\tau_j < \delta_0$. For any $0 < \zeta_1 \leq \dots \leq \zeta_d$ let $A(\zeta_1, \dots, \zeta_d)$ be the diagonal matrix with diagonal entries $1/\zeta_1, \dots, 1/\zeta_d$. Starting at τ_1, \dots, τ_d and letting ζ_j increase from τ_j up to δ_0 for $j = r, r-1, \dots, 1$ in that order, we get, specifically at the final step for ζ_1 ,

$$(17) \quad Qh(A(\tau_1, \dots, \tau_d)) - Qh(A(\delta_0, \dots, \delta_0, \tau_{r+1}, \dots, \tau_d)) \geq (\log 2)/4.$$

So (15) follows, for any small enough $\delta_0 > 0$, and Claim 1 is proved. At this stage we have not shown that either of the infima in (15) is finite.

Let $\mathcal{M}_0 := \{A \in \mathcal{P}_d : \tau_1(A) \geq \delta_0\}$. Then by iterating (17) for δ_0 divided by powers of 2, we find that for $k = 1, 2, \dots$, for any $A \in \mathcal{P}_d$ with $\tau_1(A) \leq \delta_0/2^k$,

there is an $A' \in \mathcal{M}_0$ with $\tau_j(A') = \tau_j(A)$ whenever $\tau_j(A) \geq \delta_0$ and

$$(18) \quad Qh(A) \geq Qh(A') + k(\log 2)/4.$$

Let $\delta_1 := \delta_0/2 < 1/2$. Then by (15),

$$(19) \quad \inf\{Qh(A) : \tau_1(A) < \delta_1\} \geq (\log 2)/4 + \inf\{Qh(A) : \tau_1(A) \geq \delta_1\}.$$

Next, *Claim 2* is that if $\{A_k\}$ is a sequence in \mathcal{P}_d , with $\tau_{j,k} := \tau_j(A_k)$ for each j and k , such that $\tau_{d,k} \rightarrow +\infty$, with $\tau_{1,k} \geq \delta_1$ for all k , then $Qh(A_k) \rightarrow +\infty$. If not, then taking subsequences, we can assume the following:

- (i) $\tau_{d,k} \uparrow +\infty$;
- (ii) For some $r = 1, \dots, d$, $\tau_{r,k} \rightarrow +\infty$, while for $j = 1, \dots, r-1$, $\tau_{j,k}$ is bounded;
- (iii) For each $j = r, \dots, d$, $1 \leq \tau_{j,k} \uparrow +\infty$;
- (iv) For each $k = 1, 2, \dots$, let $\{e_{j,k}\}_{j=1}^d$ be an orthonormal basis of eigenvectors of A_k in \mathbb{R}^d where $A_k e_{j,k} = \tau_{j,k} e_{j,k}$. As $k \rightarrow \infty$, for each $j = 1, \dots, d$, $e_{j,k}$ converges to some e_j .

Then $\{e_j\}_{j=1}^d$ is an orthonormal basis of \mathbb{R}^d . Let S_j be the linear span of e_1, \dots, e_j for $j = 1, \dots, d$, $S_0 := \{0\}$, $D_j := S_j \setminus S_{j-1}$ for $j = 1, \dots, d$ and $D_0 := \{0\}$. We have by (4) that $Qh(A_k) = \sum_{j=1}^d \zeta_{j,k}$ where for $j = 1, \dots, d$

$$(20) \quad \zeta_{j,k} := -\frac{1}{2} \log \tau_{j,k} + \int_{D_j} \rho(y' A_k^{-1} y) - \rho(y' y) dQ(y),$$

noting that on D_0 , the integrand is 0. So we need to show that $\sum_{j=1}^d \zeta_{j,k} \rightarrow +\infty$. If we add and subtract $\rho(\delta_1 y' y)$ in the integrand and note that $\rho(y' y) - \rho(\delta_1 y' y)$ is a fixed bounded and thus integrable function, by (10), letting

$$(21) \quad \gamma_{j,k} := -\frac{1}{2} \log \tau_{j,k} + \int_{D_j} \rho(y' A_k^{-1} y) - \rho(\delta_1 y' y) dQ(y),$$

we need to show that $\sum_{j=1}^d \gamma_{j,k} \rightarrow +\infty$. Since $\tau_{j,k} \geq \delta_1$ for all j and k and by (ii), $\gamma_{j,k}$ are bounded below for $j = 1, \dots, r-1$. Because $Q \in \mathcal{U}_{d,a(0)}$, there is an a with $d < a < a_0$ close enough to a_0 so that for $j = r, \dots, d$,

$$(22) \quad \alpha_j := 1 - \frac{d-j+1}{a} - Q(S_{j-1}) > 0,$$

noting that S_{j-1} is a linear subspace of dimension $j-1$ not depending on k . It will be shown that as $k \rightarrow \infty$,

$$(23) \quad T_m := -\frac{a\alpha_m}{2} \log \tau_{m,k} + \sum_{j=m}^d \gamma_{j,k} \rightarrow +\infty$$

for $m = r, \dots, d$, which for $m = r$ will imply Claim 2. The relation (23) will be proved by downward induction from $m = d$ to $m = r$.

For coordinates $y_j := e'_j y$, each $\varepsilon > 0$ and $j = r, \dots, d$, we have

$$(24) \quad \tau_{j,k}(e'_{j,k} y)^2 \geq (1 - \varepsilon) \tau_{j,k} y_j^2$$

for $k \geq k_{0,j}$ for some $k_{0,j}$. Choose ε with $0 < \varepsilon < 1 - \delta_1$. Let $k_0 := \max_{r \leq j \leq d} k_{0,j}$, so that for $k \geq k_0$, as will be assumed from here on, (24) will hold for all $j = r, \dots, d$. It follows then that since $\tau_{i,k} \geq \delta_1$ for all i ,

$$(25) \quad \rho(y' A_k^{-1} y) \geq \rho(\delta_1 y' y + (1 - \varepsilon - \delta_1) \tau_{j,k} y_j^2)$$

for $j = r, \dots, d$. For such j it follows that

$$\gamma_{j,k} \geq \gamma'_{j,k} := -\frac{1}{2} \log \tau_{j,k} + \int_{D_j} \rho(\delta_1 y' y + (1 - \varepsilon - \delta_1) \tau_{j,k} y_j^2) - \rho(\delta_1 y' y) dQ(y).$$

For $j = r, \dots, d$ and $\tau \geq \delta_1 > 0$ we have

$$\begin{aligned} 0 &\leq \tau \frac{\partial}{\partial \tau} [\rho(\delta_1 y' y + (1 - \varepsilon - \delta_1) \tau y_j^2) - \rho(\delta_1 y' y)] \\ &= \frac{\tau}{2} (1 - \varepsilon - \delta_1) y_j^2 u(\delta_1 y' y + (1 - \varepsilon - \delta_1) \tau y_j^2) \leq \frac{a_0}{2}, \end{aligned}$$

and the quantity bounded above by $a_0/2$ converges to $a_0/2$ as $\tau \rightarrow +\infty$ by (13) for all $y \in D_j$ since $y_j \neq 0$ there. Because the derivative is bounded, the differentiation can be interchanged with the integral, and we have

$$\frac{\partial \gamma'_{j,k}}{\partial \tau_{j,k}} = \frac{1}{2\tau_{j,k}} \left[\tau_{j,k} (1 - \varepsilon - \delta_1) \int_{D_j} y_j^2 u(\delta_1 y' y + (1 - \varepsilon - \delta_1) \tau_{j,k} y_j^2) dQ(y) - 1 \right]$$

where the quantity in square brackets converges to $a_0 Q(D_j) - 1$ as $k \rightarrow \infty$ and so

$$\partial \gamma'_{j,k} / \partial \tau_{j,k} \sim [a_0 Q(D_j) - 1] / (2\tau_{j,k}).$$

Choose a_1 with $a < a_1 < a_0$. It follows that for k large enough

$$(26) \quad \gamma_{j,k} \geq \frac{1}{2} [a_1 Q(D_j) - 1] \ln(\tau_{j,k}),$$

with equality if $Q(D_j) = 0$ and strict inequality otherwise.

Now beginning the inductive proof of (23) for $m = d$, we have $\alpha_d = 1 - a^{-1} - Q(S_{d-1}) = Q(D_d) - a^{-1}$, so $(1 + a\alpha_d)/2 = aQ(D_d)/2$, and $\gamma_{d,k} - (a\alpha_d/2) \log \tau_{d,k} \rightarrow +\infty$ by (26) for $j = d$.

For the induction step in (23) from $j+1$ to j for $j = d-1, \dots, r$ if $r < d$, it will suffice to show that

$$T_j - T_{j+1} = \gamma_{j,k} + \frac{a\alpha_{j+1}}{2} \log \tau_{j+1,k} - \frac{a\alpha_j}{2} \log \tau_{j,k}$$

is bounded below. Since $a > 0$, $\alpha_{j+1} > 0$ by (22), and $\tau_{j+1,k} \geq \tau_{j,k}$, it will be enough to show that

$$\Delta_{j,k} := \gamma_{j,k} + \frac{a}{2} (\alpha_{j+1} - \alpha_j) \log \tau_{j,k}$$

is bounded below. Inserting the definitions of α_j and α_{j+1} from (22) gives

$$\Delta_{j,k} = -\frac{a}{2}Q(D_j) \log \tau_{j,k} + \int_{D_j} \rho(y' A_k^{-1} y) - \rho(\delta_1 y' y) dQ(y).$$

This is identically 0 if $Q(D_j) = 0$. If $Q(D_j) > 0$, then $\Delta_{j,k} \rightarrow +\infty$ by (26) for j . The inductive proof of (23) and so of Claim 2 is complete.

By (18), (19), and Claim 2, we then have

$$(27) \quad Qh(A) \rightarrow +\infty \text{ if } \tau_1(A) \rightarrow 0 \text{ or } \tau_d(A) \rightarrow +\infty \text{ or both, } A \in \mathcal{P}_d.$$

The infimum of $Qh(A)$ equals the infimum over the set \mathcal{K} of A with $\tau_1(A) \geq \delta_1$ by (19) and $\tau_d(A) \leq M$ for some $M < \infty$ by Claim 2. Then \mathcal{K} is compact. Since Qh is continuous, in fact C^1 , it attains an absolute minimum over \mathcal{K} at some B in \mathcal{K} , where its value is finite and it has a critical point. By Claims 1 and 2 again, $Qh(B) < \inf_{A \notin \mathcal{K}} Qh(A)$. Thus Qh has a unique critical point B by Proposition 2, and Qh has its unique absolute minimum at B . So the theorem is proved. \square

4. LOCATION AND SCATTER t FUNCTIONALS

The main result of this section, Theorem 6, is an extension of results of Kent and Tyler (1991, Theorem 3.1), who found maximum likelihood estimates for finite samples, and Dümbgen and Tyler (2005) for M-functionals, defined as unique critical points, for integer ν , to the case of M-functionals in the sense of absolute minima and any $\nu > 0$.

Kent and Tyler (1991, §3) and Kent, Tyler and Vardi (1994) showed that location-scatter problems in \mathbb{R}^d can be treated by way of pure scatter problems in \mathbb{R}^{d+1} , specifically for functionals based on t log likelihoods. The two papers prove the following (clearly A is analytic as a function of Σ , μ and γ , and the inverse of an analytic function, if it exists and is C^1 , is analytic, e.g. Deimling [1985, Theorem 15.3 p. 151]):

Proposition 4. (i) For any $d = 1, 2, \dots$, there is a 1-1 correspondence between matrices $A \in \mathcal{P}_{d+1}$ and triples (Σ, μ, γ) where $\Sigma \in \mathcal{P}_d$, $\mu \in \mathbb{R}^d$, and $\gamma > 0$, given by $A = A(\Sigma, \mu, \gamma)$ where

$$(28) \quad A(\Sigma, \mu, \gamma) = \gamma \begin{bmatrix} \Sigma + \mu\mu' & \mu \\ \mu' & 1 \end{bmatrix}.$$

The correspondence is analytic in either direction.

(ii) For $A = A(\Sigma, \mu, \gamma)$, we have

$$(29) \quad A^{-1} = \gamma^{-1} \begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1}\mu \\ -\mu'\Sigma^{-1} & 1 + \mu'\Sigma^{-1}\mu \end{bmatrix}.$$

(iii) If (28) holds, then for any $y \in \mathbb{R}^d$ (a column vector),

$$(30) \quad (y', 1)A^{-1}(y', 1)' = \gamma^{-1} (1 + (y - \mu)'\Sigma^{-1}(y - \mu)).$$

For M-estimation of location and scatter in \mathbb{R}^d , we will have a function $\rho : [0, \infty) \mapsto [0, \infty)$ as in the previous section. The parameter space is now the set of pairs (μ, Σ) for $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathcal{P}_d$, and we have a multivariate ρ function (the two meanings of ρ should not cause confusion)

$$(31) \quad \rho(y, (\mu, \Sigma)) := \frac{1}{2} \log \det \Sigma + \rho((y - \mu)' \Sigma^{-1} (y - \mu)).$$

For any $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathcal{P}_d$ let $A_0 := A_0(\mu, \Sigma) := A(\Sigma, \mu, 1) \in \mathcal{P}_{d+1}$ by (28) with $\gamma = 1$, noting that $\det A_0 = \det \Sigma$. Now ρ can be adjusted, in light of (10) and (30), by defining

$$(32) \quad h(y, (\mu, \Sigma)) := \rho(y, (\mu, \Sigma)) - \rho(y, (0, I)).$$

Laws P on \mathbb{R}^d correspond to laws $Q := P \circ T_1^{-1}$ on \mathbb{R}^{d+1} concentrated in $\{y : y_{d+1} = 1\}$, where $T_1(y) := (y', 1)' \in \mathbb{R}^{d+1}$, $y \in \mathbb{R}^d$. We will need a hypothesis on P corresponding to $Q \in \mathcal{U}_{d+1, a(0)}$. Kent and Tyler (1991) gave these conditions for empirical measures.

Definition. For any $a_0 := a(0) > 0$ let $\mathcal{V}_{d, a(0)}$ be the set of all laws P on \mathbb{R}^d such that for every affine hyperplane J of dimension $q \leq d - 1$, $P(J) < 1 - (d - q)/a_0$, so that $P(J^c) > (d - q)/a_0$.

The next fact is rather straightforward to prove.

Proposition 5. *For any law P on \mathbb{R}^d , $a > d + 1$, and $Q := P \circ T_1^{-1}$ on \mathbb{R}^{d+1} , we have $P \in \mathcal{V}_{d, a}$ if and only if $Q \in \mathcal{U}_{d+1, a}$.*

For laws $P \in \mathcal{V}_{d, a(0)}$ with $a(0) > d + 1$, one can prove that there exist $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathcal{P}_d$ at which $Ph(\mu, \Sigma)$ is minimized, as Kent and Tyler (1991) did for empirical measures, by applying part of the proof of Theorem 3 restricted to the closed set where $\gamma = A_{d+1, d+1} = 1$ in (30). But the proof of uniqueness (Proposition 2) doesn't apply in general under the constraint $A_{d+1, d+1} = 1$. For minimization under a constraint the notion of critical point changes, e.g. for a Lagrange multiplier λ one would seek critical points of $Qh(A) + \lambda(A_{d+1, d+1} - 1)$, so Propositions 1 and 2 no longer apply. Uniqueness will hold under an additional condition. A family of ρ functions that will satisfy the condition, as pointed out by Kent and Tyler [1991, (1.5), (1.6)], comes from elliptically symmetric multivariate t densities with ν degrees of freedom as follows: for $0 < \nu < \infty$ and $0 \leq s < \infty$ let

$$(33) \quad \rho_\nu(s) := \rho_{\nu, d}(s) := \frac{\nu + d}{2} \log \left(\frac{\nu + s}{\nu} \right).$$

For this ρ , u is $u_\nu(s) := u_{\nu, d}(s) := (\nu + d)/(\nu + s)$, which is decreasing, and $s \mapsto s u_{\nu, d}(s)$ is strictly increasing and bounded, so that (7) holds, with supremum and limit at $+\infty$ equal to $a_{0, \nu} := a_0(u_\nu(\cdot)) = \nu + d > d$ for any $\nu > 0$.

The following fact was shown in part by Kent and Tyler (1991) and further by Kent, Tyler and Vardi (1994), for empirical measures, with a short proof, and with equation (34) only implicit. The relation that ν degrees of freedom in dimension d correspond to $\nu' = \nu - 1$ in dimension $d + 1$, due to Kent, Tyler and Vardi (1994), is implemented more thoroughly in the following theorem and the proof in Dudley (2006). The extension from empirical to general laws follows from Theorem 3, specifically for part (a) of the next theorem since $a_0 = \nu + d > d$.

Theorem 6. For any $d = 1, 2, \dots$,

(a) For any $\nu > 0$ and $Q \in \mathcal{U}_{d, \nu+d}$, the map $A \mapsto Qh(A)$ defined by (4) for $\rho = \rho_{\nu, d}$ has a unique critical point $A(\nu) := A_\nu(Q)$ which is an absolute minimum;

In parts (b) through (f) let $\nu > 1$, let P be a law on \mathbb{R}^d , $Q = P \circ T_1^{-1}$ on \mathbb{R}^{d+1} , and $\nu' := \nu - 1$. Assume $P \in \mathcal{V}_{d, \nu+d}$ in parts (b) through (e). We have:

(b) $A(\nu')_{d+1, d+1} = \int u_{\nu', d+1}(z' A(\nu')^{-1} z) dQ(z) = 1$;

(c) For any $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathcal{P}_d$ let $A = A(\Sigma, \mu, 1) \in \mathcal{P}_{d+1}$ in (28). Then for any $y \in \mathbb{R}^d$ and $z := (y', 1)'$, we have

$$(34) \quad u_{\nu', d+1}(z' A^{-1} z) \equiv u_{\nu, d}((y - \mu)' \Sigma^{-1} (y - \mu)).$$

In particular, this holds for $A = A(\nu')$ and its corresponding $\mu = \mu_\nu \in \mathbb{R}^d$ and $\Sigma = \Sigma_\nu \in \mathcal{P}_d$.

(d)

$$(35) \quad \int u_{\nu, d}((y - \mu_\nu)' \Sigma_\nu^{-1} (y - \mu_\nu)) dP(y) = 1.$$

(e) For $h := h_\nu := h_{\nu, d}$ defined by (32) with $\rho = \rho_{\nu, d}$, (μ_ν, Σ_ν) is an M-functional for P .

(f) If, on the other hand, $P \notin \mathcal{V}_{d, \nu+d}$, then $(\mu, \Sigma) \mapsto Ph(\mu, \Sigma)$ for h as in part (e) has no critical points.

Kent, Tyler and Vardi (1994, Theorem 3.1) showed that if $u(s) \geq 0$, $u(0) < +\infty$, $u(\cdot)$ is continuous and nonincreasing for $s \geq 0$, and $su(s)$ is nondecreasing for $s \geq 0$, with $a_0 := \lim_{s \rightarrow +\infty} su(s) > d$, and if equation (35) holds with u in place of $u_{\nu, d}$ at each critical point (μ, Σ) of $Q_n h$ for any Q_n , then u must be of the form $u(s) = u_{\nu, d}(s) = (\nu + d)/(\nu + s)$ for some $\nu > 0$. Thus, the method of relating pure scatter functionals in \mathbb{R}^{d+1} to location-scatter functionals in \mathbb{R}^d given by Theorem 6 for t functionals defined by functions $u_{\nu, d}$ does not extend directly to other functions u . For $0 < \nu < 1$, we would get $\nu' < 0$, so the methods of Section 3 don't apply. In fact, (unique) t_ν location and scatter M-functionals may not exist, as Gabrielsen (1982) and Kent and Tyler (1991) noted. For example, if $d = 1$, $0 < \nu < 1$, and P is symmetric around 0 and nonatomic but concentrated near ± 1 , then for $-\infty < \mu < \infty$, there is a unique $\sigma_\nu(\mu) > 0$ where the minimum of $Ph_\nu(\mu, \sigma)$ with respect to σ is attained. Then $\sigma_\nu(0) \doteq 1$ and $(0, \sigma_\nu(0))$ is a saddle point of Ph_ν . Minima occur at some $\mu \neq 0, \sigma > 0$, and at (μ, σ) if and

only if at $(-\mu, \sigma)$. The Cauchy case $\nu = 1$ can be treated separately, see Kent, Tyler and Vardi (1994, §5) and references there.

When $d = 1$, $P \in \mathcal{V}_{1, \nu+1}$ requires that $P(\{x\}) < \nu/(1 + \nu)$ for each point x . Then Σ reduces to a number σ^2 with $\sigma > 0$. If $\nu > 1$ and $P \notin \mathcal{V}_{1, \nu+1}$, then for some unique x , $P(\{x\}) \geq \nu/(\nu + 1)$. One can extend (μ_ν, σ_ν) by setting $\mu_\nu(P) := x$ and $\sigma_\nu(P) := 0$, with (μ_ν, σ_ν) then being weakly continuous at all P , as will be shown in Section 9.

For $d > 1$ there is no weakly continuous extension to all P , because such an extension of μ_ν would give a weakly continuous affinely equivariant location functional defined for all laws, which is known to be impossible [Obenchain (1971)].

5. DIFFERENTIABILITY OF t FUNCTIONALS

One can metrize weak convergence by a norm. For a bounded function f from \mathbb{R}^d into a normed space, the sup norm is $\|f\|_{\text{sup}} := \sup_{x \in \mathbb{R}^d} \|f(x)\|$. Let V be a k -dimensional real vector space with a norm $\|\cdot\|$, where $1 \leq k < \infty$. Let $BL(\mathbb{R}^d, V)$ be the vector space of all functions f from \mathbb{R}^d into V such that the norm

$$\|f\|_{BL} := \|f\|_{\text{sup}} + \sup_{x \neq y} \|f(x) - f(y)\|/|x - y| < \infty,$$

i.e. bounded Lipschitz functions. The space $BL(\mathbb{R}^d, V)$ doesn't depend on $\|\cdot\|$, although $\|\cdot\|_{BL}$ does. Take any basis $\{v_j\}_{j=1}^k$ of V . Then $f(x) \equiv \sum_{j=1}^k f_j(x)v_j$ for some $f_j \in BL(\mathbb{R}^d) := BL(\mathbb{R}^d, \mathbb{R})$ where \mathbb{R} has its usual norm $|\cdot|$. Let $X := BL^*(\mathbb{R}^d)$ be the dual Banach space. For $\phi \in X$, let

$$\phi^* f := \sum_{j=1}^k \phi(f_j)v_j \in V.$$

Then because ϕ is linear, $\phi^* f$ doesn't depend on the choice of basis.

Let $\mathcal{P}(\mathbb{R}^d)$ be the set of all probability measures on the Borel sets of \mathbb{R}^d . Then each $Q \in \mathcal{P}(\mathbb{R}^d)$ defines a $\phi_Q \in BL^*(\mathbb{R}^d)$ via $\phi_Q(f) := \int f dQ$. For any $P, Q \in \mathcal{P}(\mathbb{R}^d)$ let $\beta(P, Q) := \|P - Q\|_{BL}^* := \|\phi_P - \phi_Q\|_{BL}^*$. Then β is a metric on $\mathcal{P}(\mathbb{R}^d)$ which metrizes the weak topology, e.g. Dudley (2002, Theorem 11.3.3).

Let U be an open set in a Euclidean space \mathbb{R}^d . For $k = 1, 2, \dots$, let $C_b^k(U)$ be the space of all real-valued functions f on U such that all partial derivatives $D^p f$, for $D^p := \partial^{[p]}/\partial x_1^{p_1} \cdots \partial x_d^{p_d}$ and $0 \leq [p] := p_1 + \cdots + p_d \leq k$, are continuous and bounded on U . Here $D^0 f \equiv f$. On $C_b^k(U)$ we have the norm

$$(36) \quad \|f\|_{k,U} := \sum_{0 \leq [p] \leq k} \|D^p f\|_{\text{sup}, U}, \quad \text{where } \|g\|_{\text{sup}, U} := \sup_{x \in U} |g(x)|.$$

Then $(C_b^k(U), \|\cdot\|_{k,U})$ is a Banach space. For $k = 1$ and U convex in \mathbb{R}^d it's easily seen that $C_b^1(U)$ is a subspace of $BL(U, \mathbb{R})$, with equal norm for $d = 1$.

Substituting $\rho_{\nu,d}$ from (33) into (2) gives for $y \in \mathbb{R}^d$ and $A \in \mathcal{P}_d$,

$$(37) \quad L_{\nu,d}(y, A) := \frac{1}{2} \log \det A + \frac{\nu + d}{2} \log [1 + \nu^{-1} y' A^{-1} y].$$

Then, reserving $h_\nu := h_{\nu,d}$ for the location-scatter case as in Theorem 6(e), we get in (3) for the pure scatter case

$$(38) \quad H_\nu(y, A) := H_{\nu,d}(y, A) := L_{\nu,d}(y, A) - L_{\nu,d}(y, I).$$

It follows from (11) and (37) that for $A \in \mathcal{P}_d$ and $C = A^{-1}$, gradients with respect to C are given by

$$(39) \quad G_{(\nu)}(y, A) := \nabla_C H_{\nu,d}(y, A) = \nabla_C L_{\nu,d}(y, A) = -\frac{A}{2} + \frac{(\nu + d)yy'}{2(\nu + y'Cy)} \in \mathcal{S}_d.$$

For $0 < \delta < 1$ and $d = 1, 2, \dots$, define an open subset of $\mathcal{P}_d \subset \mathcal{S}_d$ by

$$(40) \quad \mathcal{W}_\delta := \mathcal{W}_{\delta,d} := \{A \in \mathcal{P}_d : \max(\|A\|, \|A^{-1}\|) < 1/\delta\}.$$

For any $A \in \mathcal{P}_d$, $C = A^{-1}$, and $L_\nu := L_{\nu,d}$, let

$$I(C, Q, H) := QH_\nu(A) = \int L_\nu(y, A) - L_\nu(y, I) dQ(y),$$

$$J(C, Q, H) := \frac{1}{2} \log \det C + I(C, Q, H) = \frac{\nu + d}{2} \int \log \left[\frac{\nu + y'Cy}{\nu + y'y} \right] dQ(y).$$

Proposition 7. (a) *The function $C \mapsto I(C, Q, H)$ is an analytic function of C on the open subset \mathcal{P}_d of \mathcal{S}_d ;*

(b) *Its gradient is*

$$(41) \quad \nabla_C I(C, Q, H) \equiv \frac{1}{2} \left((\nu + d) \int \frac{yy'}{\nu + y'Cy} dQ(y) - A \right);$$

(c) *The functional $C \mapsto J(C, Q, H)$ has the Taylor expansion around any $C \in \mathcal{P}_d$*

$$(42) \quad J(C + \Delta, Q, H) - J(C, Q, H) = \frac{\nu + d}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int \frac{(y'\Delta y)^k}{(\nu + y'Cy)^k} dQ(y),$$

convergent for $\|\Delta\| < 1/\|A\|$;

(d) *For any $\delta \in (0, 1)$, $\nu \geq 1$ and $j = 1, 2, \dots$, the function $C \mapsto I(C, Q, H)$ is in $C_b^j(\mathcal{W}_{\delta,d})$.*

Proof. The term $\frac{1}{2} \log \det C$ doesn't depend on y and is clearly an analytic function of C , having derivatives of each order with respect to C bounded for $A \in \mathcal{W}_{\delta,d}$. For $\|\Delta\| < 1/\|A\|$, we can interchange the Taylor expansion of the logarithm with the integral and get part (c), (42). Then part (a) follows, and part (b) also from (39). For part (d), as in the Appendix, Proposition 29 and (94), the j th derivative $D^j f$ of a functional f defines a symmetric j -linear form $d^j f$, which in turn yields a j -homogeneous polynomial. Such polynomials appear

in Taylor series as in the one-variable case, (95). Thus from (42), the j th Taylor polynomial of $C \mapsto J(C, Q, H)$, times $j!$, is given by

$$(43) \quad d_C^j J(C, Q, H) = \frac{\nu + d}{2} (-1)^{j-1} (j-1)! \int \frac{(y' \Delta y)^j}{(\nu + y' C y)^j} dQ(y),$$

which clearly is bounded for $\|\Delta\| \leq 1$ when the eigenvalues of C are bounded away from 0, in other words $\|A\|$ is bounded above. Then the j th derivatives are also bounded by facts to be mentioned just after Proposition 29. \square

To treat t functionals of location and scatter in any dimension p we will need functionals of pure scatter in dimension $p+1$, so in the following lemma we only need dimension $d \geq 2$.

Usually, one might show that the Hessian is positive definite at a critical point in order to show it is a strict relative minimum. In our case we already know from Theorem 6(a) that we have a unique critical point which is a strict absolute minimum. The following lemma will be useful instead in showing differentiability of t functionals via implicit function theorems, in that it implies that the derivative of the gradient (the Hessian) is non-singular.

Lemma 8. *For each $\nu > 0$, $d = 2, 3, \dots$, and $Q \in \mathcal{U}_{d, \nu+d}$, at $A(\nu) = A_\nu(Q) \in \mathcal{P}_d$ given by Theorem 6(a), for $H_\nu = H_{\nu, d}$ defined by (38), the Hessian of QH_ν on \mathcal{S}_d with respect to $C = A^{-1}$ is positive definite.*

Proof. Each side of (42) equals

$$\frac{\nu + d}{2} \left[\int \frac{y' \Delta y}{\nu + y' C y} dQ(y) - \int \frac{(y' \Delta y)^2}{2(\nu + y' C y)^2} dQ(y) \right] + O(\|\Delta\|^3).$$

The second-order term in the Taylor expansion of $C \mapsto I(C, Q, H)$, e.g. (95) in the Appendix, using also (11) with C in place of A , is the quadratic form, for $\Delta \in \mathcal{S}_d$,

$$(44) \quad \Delta \mapsto \frac{1}{2} \left(\|A^{1/2} \Delta A^{1/2}\|_F^2 - (\nu + d) \int \frac{(y' \Delta y)^2}{(\nu + y' C y)^2} dQ(y) \right).$$

(Since differences of matrices in \mathcal{P}_d are in \mathcal{S}_d , it suffices to consider $\Delta \in \mathcal{S}_d$.) The Hessian bilinear form (2-linear mapping) $\mathcal{H}_{2,A}$ from $\mathcal{S}_d \times \mathcal{S}_d$ into \mathbb{R} defined by the second derivative at $C = A^{-1}$ of $C \mapsto I(C, Q, H)$, cf. (94), is positive definite if and only if the quadratic form (44) is positive definite. The Hessian also defines a linear map \mathcal{H}_A from \mathcal{S}_d into itself via the Frobenius inner product,

$$(45) \quad \langle \mathcal{H}_A(B), D \rangle = \text{trace}(\mathcal{H}_A(B)D) = \mathcal{H}_{2,A}(B, D)$$

for all $B, D \in \mathcal{S}_d$. Since $A \mapsto A^{-1}$ is C^∞ with C^∞ inverse from \mathcal{P}_d onto itself, it suffices to consider QH as a function of $C = A^{-1}$, in other words, to consider $I(C, Q, H)$. Then we need to show that (44) is positive definite in $\Delta \in \mathcal{S}_d$ at the unique $A = A_\nu(Q) \in \mathcal{P}_d$ such that $\nabla_A I(C, Q, H) = 0$ in (41), or equivalently

$\nabla_C I(C, Q, H) = 0$. By the substitution $z := A^{-1/2}y$, and consequently replacing Q by q with $dq(z) = dQ(y)$ and Δ by $A^{1/2}\Delta A^{1/2}$, we get $I = A_\nu(q)$. It suffices to prove the lemma for (I, q) in place of (A, Q) . We need to show that

$$(46) \quad \|\Delta\|_F^2 > (\nu + d) \int \frac{(z'\Delta z)^2}{(\nu + z'z)^2} dq(z)$$

for each $\Delta \neq 0$ in \mathcal{S}_d . By the Cauchy inequality $(z'\Delta z)^2 \leq (z'z)(z'\Delta^2 z)$, we have

$$\begin{aligned} (\nu + d) \int \frac{(z'\Delta z)^2}{(\nu + z'z)^2} dq(z) &\leq (\nu + d) \int \frac{(z'z)(z'\Delta^2 z)}{(\nu + z'z)^2} dq(z) \\ &\leq (\nu + d) \int \frac{(z'\Delta^2 z)}{\nu + z'z} dq(z) = \text{trace} \left(\Delta^2 (\nu + d) \int \frac{zz'}{\nu + z'z} dq(z) \right) \\ &= \text{trace}(\Delta^2) = \|\Delta\|_F^2, \end{aligned}$$

using (8) and (41) with $B = A = C = I$. Now, $z'z < \nu + z'z$ for all $z \neq 0$, and $z'\Delta^2 z = 0$ only for z with $\Delta z = 0$, a linear subspace of dimension at most $d - 1$. Thus $q(\Delta z = 0) < 1$, (46) follows and the Lemma is proved. \square

Example. For Q such that $A_\nu(Q) = I_d$, the $d \times d$ identity matrix, a large part of the mass of Q can escape to infinity, Q can approach the boundary of $\mathcal{U}_{d, \nu+d}$, and some eigenvalues of the Hessian can approach 0, as follows. Let e_j be the standard basis vectors of \mathbb{R}^d . For $c > 0$ and p such that $1/[2(\nu + d)] < p \leq 1/(2d)$, let

$$Q := (1 - 2dp)\delta_0 + p \sum_{j=1}^d \delta_{-ce_j} + \delta_{ce_j}.$$

To get $A_\nu(Q) = I_d$, by (8) and (41) we need $(\nu + d) \cdot 2pc^2 = \nu + c^2$, or $\nu = c^2[2p(\nu + d) - 1]$. There is a unique solution for $c > 0$ but as $p \downarrow 1/[2(\nu + d)]$, we have $c \uparrow +\infty$. Then, for each $q = 0, 1, \dots, d - 1$, for each q -dimensional subspace H where $d - q$ of the coordinates are 0, $Q(H) \uparrow 1 - \frac{d-q}{\nu+d}$, the critical value for which $Q \notin \mathcal{U}_{d, \nu+d}$. Also, an amount of probability for Q converging to $d/(\nu + d)$ is escaping to infinity. The Hessian, cf. (46), has d arbitrarily small eigenvalues $\nu/(\nu + c^2)$.

For the relatively open set $\mathcal{P}_d \subset \mathcal{S}_d$ and $G_{(\nu)}$ from (39), define the function $F := F_\nu$ from $X \times \mathcal{P}_d$ into \mathcal{S}_d by

$$(47) \quad F(\phi, A) := \phi^*(G_{(\nu)}(\cdot, A)).$$

Then F is well-defined because $G_{(\nu)}(\cdot, A)$ is a bounded and Lipschitz \mathcal{S}_d -valued function of x for each $A \in \mathcal{P}_d$; in fact, each entry is C^1 with bounded derivative, as is straightforward to check.

For $d = 1$, and a finite signed Borel measure τ , let

$$(48) \quad \|\tau\|_{\mathcal{K}} := \sup_x |\tau((-\infty, x])|.$$

Let P and Q be two laws with distribution functions F_P and F_Q . Then $\|P - Q\|_{\mathcal{K}}$ is the usual sup (Kolmogorov) norm distance $\sup_x |(F_Q - F_P)(x)|$.

The next statement and its proof call on some basic notions and facts from infinite-dimensional calculus, which are reviewed in the Appendix.

Theorem 9. *Let $\nu > 0$ in parts (a) through (c), $\nu > 1$ in parts (d), (e).*

- (a) *The function $F = F_\nu$ is analytic from $X \times \mathcal{P}_d$ into \mathcal{S}_d where $X = BL^*(\mathbb{R}^d)$.*
- (b) *For any law $Q \in \mathcal{U}_{d, \nu+d}$, and the corresponding $\phi_Q \in X$, at $A_\nu(Q)$ given by Theorem 6(a), the partial derivative linear map $\partial_C F(\phi_Q, A)/\partial C := \nabla_C F(\phi_Q, A)$ from \mathcal{S}_d into \mathcal{S}_d is invertible.*
- (c) *Still for $Q \in \mathcal{U}_{d, \nu+d}$, the functional $Q \mapsto A_\nu(Q)$ is analytic for the BL^* norm.*
- (d) *For each $P \in \mathcal{V}_{d, \nu+d}$, the t_ν location-scatter functional $P \mapsto (\mu_\nu, \Sigma_\nu)(P)$ given by Theorems 3 and 6 is also analytic for the norm on X .*
- (e) *For $d = 1$, the t_ν location and scatter functionals μ_ν, σ_ν^2 are analytic on $\mathcal{V}_{1, \nu+1}$ with respect to the sup norm $\|\cdot\|_{\mathcal{K}}$.*

Proof. (a): The function $(\phi, f) \mapsto \phi(f)$ is a bounded bilinear operator, hence analytic, from $BL^*(\mathbb{R}^d) \times BL(\mathbb{R}^d)$ into \mathbb{R} , and the composition of analytic functions is analytic, so it will suffice to show that $A \mapsto G_{(\nu)}(\cdot, A)$ from (39) is analytic from the relatively open set $\mathcal{P}_d \subset \mathcal{S}_d$ into $BL(\mathbb{R}^d, \mathcal{S}_d)$. By easy reductions, it will suffice to show that $C \mapsto (y \mapsto yy' / (\nu + y'Cy))$ is analytic from \mathcal{P}_d into $BL(\mathbb{R}^d, \mathcal{S}_d)$. Fixing $C \equiv A^{-1}$ and considering $C + \Delta$ for sufficiently small $\Delta \in \mathcal{S}_d$, we get

$$(49) \quad \frac{yy'}{\nu + y'Cy + y'\Delta y} = yy' \sum_{j=0}^{\infty} \frac{(-y'\Delta y)^j}{(\nu + y'Cy)^{j+1}},$$

which we would like to show gives the desired Taylor expansion around C . For $j = 1, 2, \dots$ let $g_j(y) := (-y'\Delta y)^j (\nu + y'Cy)^{-j-1} \in \mathbb{R}$ and let f_j be the j th term of (49), $f_j(y) := g_j(y)yy' \in \mathcal{S}_d$. It's easily seen that for each j , f_j is a bounded Lipschitz function into \mathcal{S}_d . We have for all y , since $\nu + y'Cy \geq \nu + |y|^2/\|A\|$, that

$$(50) \quad |g_j(y)| \leq \|\Delta\|^j \|A\|^j / (\nu + |y|^2/\|A\|).$$

For the Frobenius norm $\|\cdot\|_F$ on \mathcal{S}_d , it follows that for all y

$$(51) \quad \|f_j(y)\|_F \leq \|\Delta\|^j \|A\|^{j+1}.$$

Thus for $\|\Delta\| < 1/\|A\|$, the series converges absolutely in the supremum norm. To consider Lipschitz seminorms, for any y and z in \mathbb{R}^d we have

$$\begin{aligned} & \|f_j(y) - f_j(z)\|_F^2 \\ &= \text{trace}[g_j(y)^2|y|^2yy' + g_j(z)^2|z|^2zz' - g_j(y)g_j(z)\{(y'z)yz' + (z'y)zy'\}] \\ &= g_j(y)^2|y|^4 + g_j(z)^2|z|^4 - 2g_j(y)g_j(z)(y'z)^2 \end{aligned}$$

and so, letting $G(y, z) := g_j(y)g_j(z)(y'z)^2 \in \mathbb{R}$ for any $y, z \in \mathbb{R}^d$, we have

$$(52) \quad \|f_j(y) - f_j(z)\|_F^2 = G(y, y) - 2G(y, z) + G(z, z).$$

To evaluate some gradients, we have $\nabla_y(y'B y) = 2B y$ for any $B \in \mathcal{S}_d$, and thus

$$\nabla_y g_j(y) = \frac{2(-y'\Delta y)^{j-1}}{(\nu + y'Cy)^{j+2}}[-j(\nu + y'Cy)\Delta y - (j+1)(-y'\Delta y)Cy].$$

It follows that for all y

$$|\nabla_y g_j(y)| \leq 2(j+1)\|\Delta\|^j\|A\|^{j-1/2}(\nu + 2\|C\||y|^2)(\nu + |y|^2/\|A\|)^{-5/2}$$

and so since $\|A\|\|C\| \geq 1$,

$$(53) \quad |\nabla_y g_j(y)| \leq (4j+4)\|\Delta\|^j\|A\|^{j+1/2}\|C\|(\nu + |y|^2/\|A\|)^{-3/2}.$$

Letting Δ_1 be the gradient with respect to the first of the two arguments we have

$$\Delta_1 G(y, z) = (y'z)^2 g_j(z) \Delta_y g_j(y) + 2g_j(y)g_j(z)(y'z)z.$$

For any $u \in \mathbb{R}^d$, having in mind $u = u_t = y + t(z - y)$ with $0 \leq t \leq 1$, we have

$$(54) \quad \begin{aligned} \Delta_1 G(u, z) - \Delta_1 G(u, y) &= [(u'z)^2 g_j(z) - (u'y)^2 g_j(y)] \nabla_u g_j(y) \\ &\quad + 2g_j(u)[g_j(z)(u'z)z - g_j(y)(u'y)y]. \end{aligned}$$

For the first factor in the first term on the right we will use

$$\nabla_v [(u'v)^2 g_j(v)] = 2g_j(v)(u'v)u + (u'v)^2 \nabla_v g_j(v).$$

From (50) and (53) it follows that for all u and v in \mathbb{R}^d

$$|\nabla_v [(u'v)^2 g_j(v)]| \leq \|\Delta\|^j\|A\|^j|u|^2|v| \left(\frac{2}{\nu + |v|^2/\|A\|} + \frac{(4j+4)\sqrt{\|A\|}\|C\||v|}{(\nu + |v|^2/\|A\|)^{3/2}} \right).$$

Now, for all v , $2|v|/(\nu + |v|^2/\|A\|) \leq \|A\|^{1/2}$ and $|v|^2/(\nu + |v|^2/\|A\|)^{3/2} \leq \|A\|$. It follows, integrating along the line (u, v) from $v = y$ to $v = z$ for each fixed u , that

$$|(u'z)^2 g_j(z) - (u'y)^2 g_j(y)| \leq |z - y| \|\Delta\|^j\|A\|^{j+3/2}|u|^2(4j+5)\|C\|.$$

By this and (53), since $|u|^2/(\nu + |u|^2/\|A\|)^{3/2} \leq \|A\|$, the first term on the right in (54) is bounded above by

$$(55) \quad (4j+5)^2 \|\Delta\|^{2j}\|A\|^{2j+3}\|C\|^2|y - z|.$$

For the second term on the right in (54), the second factor is $g_j(z)(u'z)z - g_j(y)(u'y)y$. The gradient of a vector-valued function is a matrix-valued function, in this case non-symmetric. We have

$$\nabla_v[g_j(v)(u'v)v] = (\nabla_v g_j(v))(u'v)v' + g_j(v)[uv' + (u'v)I].$$

It follows by (50) and (53) that for any v

$$\|\nabla_v[g_j(v)(u'v)v]\| \leq \|\Delta\|^j \|A\|^{j+1/2} |u| \{2 + (4j + 4)\|A\|\|C\|\}.$$

Multiplying by $2g_j(u)$, and integrating with respect to v along the line segment from $v = y$ to $v = z$, we get for the second term on the right in (54)

$$|2g_j(u)[g_j(z)(u'z)z - g_j(y)(u'y)y]| \leq \|\Delta\|^{2j} \|A\|^{2j+2} \|C\| (6j + 6) |z - y|.$$

Combining with (55) gives in (54)

$$\begin{aligned} & |\Delta_1 G(u, z) - \Delta_1 G(u, y)| \\ & \leq \|\Delta\|^{2j} \|A\|^{2j+2} \|C\| \{(4j + 5)^2 \|A\|\|C\| + (6j + 6)\} |z - y| \\ & \leq \|\Delta\|^{2j} \|A\|^{2j+3} \|C\|^2 (6j + 6)^2 |z - y|. \end{aligned}$$

Then integrating this bound with respect to u on the line from $u = y$ to $u = z$ we get

$$|G(z, z) - 2G(y, z) + G(y, y)| \leq \|\Delta\|^{2j} \|A\|^{2j+3} \|C\|^2 (6j + 6)^2 |y - z|^2$$

and so by (52) $\|f_j\|_L \leq \|\Delta\|^j \|A\|^{j+3/2} \|C\| (6j + 6)$. Since the right side of the latter inequality equals a factor linear in j , times $\|\Delta\|^j \|A\|^j$, times factors fixed for given A , not depending on j or Δ , we see that the series (49) converges not only in the supremum norm but also in $\|\cdot\|_L$ for $\|\Delta\| < 1/\|A\|$, finishing the proof of analyticity of $A \mapsto (y \mapsto yy' / (\nu + y'Cy))$ into $BL(\mathbb{R}^d, \mathcal{S}_d)$ and so part (a).

For (b), A_ν exists by Theorem 3 with $u = u_{\nu,d}$, so $a(0) = \nu + d > d$. The gradient of F with respect to A is the Hessian of QH_ν , which is positive definite at the critical point A_ν by Lemma 8 and so non-singular.

For (c), by parts (a) and (b), all the hypotheses of the Hildebrandt-Graves implicit function theorem in the analytic case, e.g. Theorem 30(c) in the Appendix, hold at each point $(\phi_Q, A_\nu(Q))$, giving the conclusions that: on some open neighborhood U of ϕ_Q in X , there is a function $\phi \mapsto A_\nu(\phi)$ such that $F(\phi, A_\nu(\phi)) = 0$ for all $\phi \in U$; the function A_ν is C^1 ; and, since F is analytic by part (a), so is A_ν on U . Existence of the implicit function in a BL^* neighborhood of ϕ_Q , and Theorem 3, imply that $\mathcal{U}_{d,\nu+d}$ is a relatively $\|\cdot\|_{BL}^*$ open set of probability measures, thus weakly open since β metrizes weak convergence. We know by Theorem 3, (33) and the form of $u_{\nu,d}$ that there is a unique solution $A_\nu(Q)$ for each $Q \in \mathcal{U}_{d,\nu+d}$. So the local functions on neighborhoods fit together to define one analytic function A_ν on $\mathcal{U}_{d,\nu+d}$, and part (c) is proved.

For part (d), we apply the previous parts with $d + 1$ and $\nu - 1$ in place of d and ν respectively. Theorem 6 shows that in the t_ν case with $\nu > 1$, $\mu = \mu_\nu$ and $\Sigma = \Sigma_\nu$ give uniquely defined M-functionals of location and scatter. Proposition

4 shows that the relation (28) with $\gamma \equiv 1$ gives an analytic homeomorphism with analytic inverse between A with $A_{d+1,d+1} = 1$ and (μ, Σ) , so (d) follows from (c) and the composition of analytic functions.

For part (e), consider the Taylor expansion (49) related to $G_{(\nu)}$, specialized to the case $d = 1$, recalling that we treat location-scatter in this case by way of pure scatter for $d = 2$, where for a law P on \mathbb{R} we take the law $P \circ T_1^{-1}$ on \mathbb{R}^2 concentrated in vectors $(x, 1)'$. The bilinear form $(f, \tau) \mapsto \int f d\tau$ is jointly continuous with respect to the total variation norm on f ,

$$\|f\|_{[1]} := \|f\|_{\text{sup}} + \sup_{-\infty < x_1 < \dots < x_k < +\infty, k=2,3,\dots} \sum_{j=2}^k |f(x_j) - f(x_{j-1})|,$$

and the sup (Kolmogorov) norm $\|\cdot\|_{\mathcal{K}}$ on finite signed measures (48). Thus it will suffice to show as for part (a) that the \mathcal{S}_2 -valued Taylor series (49) has entries converging in total variation norm for $\|\Delta\| < 1/\|A\|$.

An entry of the j th term $f_j((x, 1)')$ of (49) is a rational function $R(x) = U(x)/V(x)$ where V has degree $2j + 2$ and U has degree $2j + i$ for $i = 0, 1$, or 2 . We already know from (51) that $\|R\|_{\text{sup}} \leq \|\Delta\|^j \|A\|^{j+1}$. A zero of the derivative rational function $R'(x)$ is a zero of its numerator, which after reduction is a polynomial of degree at most $2j + 3$. Thus there are at most $2j + 3$ (real) zeroes. Between two adjacent zeroes of R' the total variation of R is at most $2\|R\|_{\text{sup}}$. Between $\pm\infty$ and the largest or smallest zero of R' , the same holds. Thus the total variation norm $\|R\|_{[1]} \leq (4j + 9)\|R\|_{\text{sup}}$. Since $\sum_{j=1}^{\infty} (4j + 9)\|\Delta\|^j \|A\|^{j+1} < \infty$ for $\|\Delta\| < 1/\|A\|$, the conclusion follows. \square

If a functional T is differentiable at P for a suitable norm, with a non-zero derivative, then one can look for asymptotic normality of $\sqrt{n}(T(P_n) - T(P))$ by way of some central limit theorem and the delta-method. For this purpose the dual-bounded-Lipschitz norm $\|\cdot\|_{BL}^*$, although it works for large classes of distributions, is still too strong for some heavy-tailed distributions. For $d = 1$, let P be a law concentrated in the positive integers with $\sum_{k=1}^{\infty} \sqrt{P(\{k\})} = +\infty$. Then a short calculation shows that as $n \rightarrow \infty$, $\sqrt{n} \sum_{k=1}^{\infty} |(P_n - P)(\{k\})| \rightarrow +\infty$ in probability. For any numbers a_k there is an $f \in BL(\mathbb{R})$ with usual metric such that $f(k)a_k = |a_k|$ for all k and $\|f\|_{BL} \leq 3$. Thus $\sqrt{n}\|P_n - P\|_{BL}^* \rightarrow +\infty$ in probability. Giné and Zinn (1986) proved equivalence of the related condition $\sum_{j=1}^{\infty} \Pr(j - 1 < |X| \leq j)^{1/2} < \infty$ for X with general distribution P on \mathbb{R} to the Donsker property [defined in Dudley (1999, §3.1)] of $\{f : \|f\|_{BL} \leq 1\}$. But norms more directly adapted to the functions needed will be defined in the following section.

6. SOME BANACH SPACES GENERATED BY RATIONAL FUNCTIONS

For the facts in this section, proofs are omitted if they are short and easy, or given briefly if they are longer. More details are given in Dudley, Sidenko, Wang and Yang (2007). Throughout this section let $0 < \delta < 1$, $d = 1, 2, \dots$ and $r = 1, 2, \dots$ be arbitrary unless further specified. Let \mathcal{MM}_r be the set of monic monomials g from \mathbb{R}^d into \mathbb{R} of degree r , in other words $g(x) = \prod_{i=1}^d x_i^{n_i}$ for some $n_i \in \mathbb{N}$ with $\sum_{i=1}^d n_i = r$. Let

$$\mathcal{F}_{\delta,r} := \mathcal{F}_{\delta,r,d} := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R}, f(x) \equiv g(x)/\prod_{s=1}^r (1 + x' C_s x), \right.$$

$$\left. \text{where } g \in \mathcal{MM}_{2r}, \text{ and for } s = 1, \dots, r, C_s \in \mathcal{W}_\delta \right\}.$$

For $1 \leq j \leq r$, let $\mathcal{F}_{\delta,r}^{(j)} := \mathcal{F}_{\delta,r,d}^{(j)}$ be the set of $f \in \mathcal{F}_{\delta,r}$ such that C_s has at most j different values (depending on f). Then $\mathcal{F}_{\delta,r} = \mathcal{F}_{\delta,r}^{(r)}$. Let $\mathcal{G}_{\delta,r}^{(j)} := \mathcal{G}_{\delta,r,d}^{(j)} := \bigcup_{v=1}^r \mathcal{F}_{\delta,v}^{(j)}$. We will be interested in $j = 1$ and 2 . Clearly $\mathcal{F}_{\delta,r}^{(1)} \subset \mathcal{F}_{\delta,r}^{(2)} \subset \dots \subset \mathcal{F}_{\delta,r}$ for each δ and r .

Let $h_C(x) := 1 + x' C x$ for $C \in \mathcal{P}_d$ and $x \in \mathbb{R}^d$. Then clearly $f \in \mathcal{F}_{\delta,r}^{(1)}$ if and only if for some $P \in \mathcal{MM}_{2r}$ and $C \in \mathcal{W}_\delta$, $f(x) \equiv f_{P,C,r}(x) := P(x)h_C(x)^{-r}$. The next two lemmas are straightforward:

Lemma 10. *For any $f \in \mathcal{G}_{\delta,r}^{(r)}$ we have $(\delta/d)^r \leq \|f\|_{\text{sup}} \leq \delta^{-r}$.*

Lemma 11. *Let $f = f_{P,C,r}$ and $g = f_{P,D,r}$ for some $P \in \mathcal{MM}_{2r}$ and $C, D \in \mathcal{P}_d$. Then*

$$(56) \quad (f - g)(x) \equiv \frac{x'(D - C)xP(x) \sum_{j=0}^{r-1} h_D(x)^{r-1-j} h_C(x)^j}{(h_C h_D)(x)^r}.$$

For $1 \leq k \leq l \leq d$ and $j = 0, 1, \dots, r - 1$, let

$$h_{C,D,k,l,r,j}(x) := x_k x_l P(x) h_C(x)^{j-r} h_D(x)^{-j-1}.$$

Then each $h_{C,D,k,l,r,j}$ is in $\mathcal{F}_{\delta,r+1}^{(2)}$ and

$$(57) \quad g - f \equiv - \sum_{1 \leq k \leq l \leq d} \sum_{j=0}^{r-1} (D_{kl} - C_{kl})(2 - \delta_{kl}) h_{C,D,k,l,r,j}.$$

For any $f : \mathbb{R}^d \rightarrow \mathbb{R}$, define

$$(58) \quad \|f\|_{\delta,r}^{*,j} := \|f\|_{\delta,r,d}^{*,j} := \inf \left\{ \sum_{s=1}^{\infty} |\lambda_s| : \exists g_s \in \mathcal{G}_{\delta,r}^{(j)}, s \geq 1, f \equiv \sum_{s=1}^{\infty} \lambda_s g_s \right\},$$

or $+\infty$ if no such λ_s, g_s with $\sum_s |\lambda_s| < \infty$ exist. Lemma 10 implies that for $\sum_s |\lambda_s| < \infty$ and $g_s \in \mathcal{G}_{\delta,r}^{(r)}$, $\sum_s \lambda_s g_s$ converges absolutely and uniformly on \mathbb{R}^d . Let $Y_{\delta,r}^j := Y_{\delta,r,d}^j := \{f : \mathbb{R}^d \rightarrow \mathbb{R}, \|f\|_{\delta,r}^{*,j} < \infty\}$. It's easy to see that each $Y_{\delta,r}^j$

is a real vector space of functions on \mathbb{R}^d and $\|\cdot\|_{\delta,r}^{*,j}$ is a seminorm on it. The next two lemmas and a proposition are rather straightforward to prove.

Lemma 12. *For any $j = 1, 2, \dots$,*

- (a) *If $f \in \mathcal{G}_{\delta,r}^{(j)}$ then $f \in Y_{\delta,r}^j$ and $\|f\|_{\delta,r}^{*,j} \leq 1$.*
- (b) *For any $g \in Y_{\delta,r}^j$, $\|g\|_{\text{sup}} \leq \|g\|_{\delta,r}^{*,j}/\delta^r < \infty$.*
- (c) *If $f \in \mathcal{G}_{\delta,r}^{(j)}$ then $\|f\|_{\delta,r}^{*,j} \geq (\delta^2/d)^r$.*
- (d) *$\|\cdot\|_{\delta,r}^{*,j}$ is a norm on $Y_{\delta,r}^j$.*
- (e) *$Y_{\delta,r}^j$ is complete for $\|\cdot\|_{\delta,r}^{*,j}$ and thus a Banach space.*

Lemma 13. *For any $j = 1, 2, \dots$, we have $Y_{\delta,r}^j \subset Y_{\delta,r+1}^j$. The inclusion linear map from $Y_{\delta,r}^j$ into $Y_{\delta,r+1}^j$ has norm at most 1.*

Proposition 14. *For any $P \in \mathcal{MM}_{2r}$, let $\psi(C, x) := f_{P,C,r}(x) = P(x)/h_C(x)^r$ from $\mathcal{W}_\delta \times \mathbb{R}^d$ into \mathbb{R} . Then:*

- (a) *For each fixed $C \in \mathcal{W}_\delta$, $\psi(C, \cdot) \in \mathcal{F}_{\delta,r}^{(1)}$.*
- (b) *For each x , $\psi(\cdot, x)$ has partial derivative $\nabla_C \psi(C, x) = -rP(x)xx'/h_C(x)^{r+1}$.*
- (c) *The map $C \mapsto \nabla_C \psi(C, \cdot) \in \mathcal{S}_d$ on \mathcal{W}_δ has entries Lipschitz into $Y_{\delta,r+2}^2$.*
- (d) *The map $C \mapsto \psi(C, \cdot)$ from \mathcal{W}_δ into $\mathcal{F}_{\delta,r}^{(1)} \subset Y_{\delta,r}^1$, viewed as a map into the larger space $Y_{\delta,r+2}^2$, is Fréchet C^1 .*

Theorem 15. *Let $r = 1, 2, \dots$, $d = 1, 2, \dots$, $0 < \delta < 1$, and $f \in Y_{\delta,r}^1$, so that for some a_s with $\sum_s |a_s| < \infty$ we have $f(x) \equiv \sum_s a_s P_s(x)/(1+x'C_s x)^{k_s}$ for $x \in \mathbb{R}^d$ where each $P_s \in \mathcal{MM}_{2k_s}$, $k_s = 1, \dots, r$, and $C_s \in \mathcal{W}_\delta$. Then f can be written as a sum of the same form in which the triples (P_s, C_s, k_s) are all distinct. In that case, the C_s , P_s , k_s and the coefficients a_s are uniquely determined by f .*

Proof. If $d = 1$, then $P_s(x) \equiv x^{2k_s}$ and $C_s \in (\delta, 1/\delta)$ for all s . We can assume the pairs (C_s, k_s) are all distinct. We need to show that if $f(x) = 0$ for all real x then all $a_s = 0$. Suppose not. Any f of the given form extends to a function of a complex variable z holomorphic except for possible singularities on the two line segments where $\Re z = 0$, $\sqrt{\delta} \leq |\Im z| \leq 1/\sqrt{\delta}$, and if $f \equiv 0$ on \mathbb{R} then $f \equiv 0$ also outside the two segments. For a given C_s take the largest k_s with $a_s \neq 0$. Then by dominated convergence for sums, $|a_s| = \lim_{t \downarrow 0} t^{k_s} |f(t + i/\sqrt{C_s})| = 0$, a contradiction (cf. Ross and Shapiro, 2002, Proposition 3.2.2).

Now for $d > 1$, consider lines $x = yu \in \mathbb{R}^d$ for $y \in \mathbb{R}$ and any $u \in \mathbb{R}^d$ with $|u| = 1$. We can assume the triples (P_s, C_s, k_s) are all distinct by summing terms where they are the same (there are just finitely many possibilities for P_s). There exist u (in fact almost all u with $|u| = 1$, in a surface measure or category sense) such that $P_s(u) \neq P_t(u)$ whenever $P_s \neq P_t$, and $u'C_s u \neq u'C_t u$ whenever $C_s \neq C_t$, since this is a countable set of conditions, holding except on a sparse set of u 's in the unit sphere. Fixing such a u , we then reduce to the case $d = 1$. \square

For any $P \in \mathcal{MM}_{2r}$ and any $C \neq D$ in \mathcal{W}_δ , let

$$f_{P,C,D,r}(x) := f_{P,C,D,r,d}(x) := \frac{P(x)}{(1+x'Cx)^r} - \frac{P(x)}{(1+x'Dx)^r}.$$

By Lemma 11, for C fixed and $D \rightarrow C$ we have $\|f_{P,C,D,r}\|_{\delta,r+1}^{*,2} \rightarrow 0$. The following shows this is not true if $r+1$ in the norm is replaced by r , even if the number of different C_s 's in the denominator is allowed to be as large as possible, namely r :

Proposition 16. *For any $r = 1, 2, \dots$, $d = 1, 2, \dots$, and $C \neq D$ in \mathcal{W}_δ , we have $\|f_{P,C,D,r}\|_{\delta,r}^{*,r} = 2$.*

The proof is similar to that of the preceding theorem.

Let $h_{C,\nu}(x) := \nu + x'Cx$, $r = 1, 2, \dots$, $P \in \mathcal{MM}_{2r}$, and

$$\psi_{(\nu)}(C, x) := \psi_{(\nu),r,P}(C, x) := P(x)/h_{C,\nu}(x)^r.$$

Then $\psi_{(\nu)}(C, x) \equiv \nu^{-r}\psi(C/\nu, x)$ and we get an alternate form of Proposition 14:

Proposition 17. *For any $d = 1, 2, \dots$, $r = 1, 2, \dots$, and $0 < \delta < 1$,*

(a) *For each $C \in \mathcal{W}_\delta$, $\nu^r\psi_{(\nu)}(C, \cdot) \in \mathcal{F}_{\delta/\nu,r,d}^{(1)}$.*

(b) *For each x , $\psi_{(\nu)}(\cdot, x)$ has the partial derivative*

$$\nabla_C \psi_{(\nu)}(C, x) = -rP(x)xx' / (\nu h_{C/\nu}(x))^{r+1} = -rP(x)xx' / h_{C,\nu}(x)^{r+1}.$$

(c) *The map $C \mapsto \nabla_C \psi_{(\nu)}(C, \cdot) \in \mathcal{S}_d$ on \mathcal{W}_δ has entries Lipschitz into $Y_{\delta/\nu,r+2}^2$.*

(d) *The map $C \mapsto \psi_{(\nu)}(C, \cdot)$ from \mathcal{W}_δ into $\mathcal{F}_{\delta/\nu,r}^{(1)}$, viewed as a map into $Y_{\delta/\nu,r+2}^2$, is Fréchet C^1 .*

Let $\mathbb{R} \oplus Y_{\delta,r}^j$ be the set of all functions $c+g$ on \mathbb{R}^d for any $c \in \mathbb{R}$ and $g \in Y_{\delta,r}^j$. Then c and g are uniquely determined since $g(0) = 0$. Let $\|c+g\|_{\delta,r,d}^{**,j} := |c| + \|g\|_{\delta,r,d}^{*,j}$.

7. FURTHER DIFFERENTIABILITY AND THE DELTA-METHOD

By (49), and (94), (95), and (96) in the Appendix, for any $0 < \delta < 1$, $C \in \mathcal{W}_\delta$, $\Delta \in \mathcal{S}_d$, and $k = 0, 1, 2, \dots$, the k th differential of $G_{(\nu)}$ from (39) with respect to C is given by

$$(59) \quad d_C^k G_{(\nu)}(y, A) \Delta^{\otimes k} = K_k(A) \Delta^{\otimes k} + g_k(y, A, \Delta)$$

with values in \mathcal{S}_d , where

$$g_k(y, A, \Delta) = \frac{\nu + d}{2} k! \frac{(-y'\Delta y)^k}{(\nu + y'Cy)^{k+1}} yy',$$

for some k -homogeneous polynomial $K_k(A)(\cdot)$ not depending on y . For $\Delta \in \mathcal{S}_d$, by the Cauchy inequality, $\sum_{i,j=1}^d |\Delta_{ij}| \leq \|\Delta\|_F d$, so each entry $g_k(\cdot, A, \Delta)_{ij} \in Y_{\delta/\nu,k+1,d}^1$ for $i, j = 1, \dots, d$, with

$$(60) \quad \|g_k(\cdot, A, \Delta)_{ij}\|_{\delta/\nu,k+1,d}^{*,1} \leq (\nu + d)k! (\|\Delta\|_F d/\nu)^k.$$

Thus $(d_C^k G_{(\nu)}(\cdot, A) \Delta^{\otimes k})_{ij} \in \mathbb{R} \oplus Y_{\delta/\nu, k+1, d}^1$. Let $X_{\delta, r, \nu}$ be the dual Banach space of $\mathbb{R} \oplus Y_{\delta/\nu, r, d}^2$, i.e. the set of all real-valued linear functionals ϕ on it for which the norm

$$\|\phi\|_{\delta, r, \nu} := \sup\{|\phi(f)| : \|f\|_{\delta/\nu, r, d}^{**, 2} \leq 1\} < \infty.$$

Let $X_{\delta, r, \nu}^0 := \{\phi \in X_{\delta, r, \nu} : \phi(c) = 0 \text{ for all } c \in \mathbb{R}\}$. For $\phi \in X_{\delta, r, \nu}^0$, by (58)

$$(61) \quad \begin{aligned} \|\phi\|_{\delta, r, \nu} &\equiv \|\phi\|_{\delta, r, \nu}^0 := \sup\{|\phi(0, g)| : \|g\|_{\delta/\nu, r, d}^{*, 2} \leq 1\} \\ &\leq \sup\{|\phi(0, g)| : g \in \mathcal{G}_{\delta/\nu, r}^{(2)}\} \leq \sup\{|\phi(0, g)| : g \in \mathcal{G}_{\delta/\nu, r}^{(r)}\}. \end{aligned}$$

For $A \in \mathcal{W}_{\delta, d}$ as defined in (40) and $\phi \in X_{\delta, r, \nu}$, define $F(\phi, A)$ again by (47), which makes sense since for any $r = 1, 2, \dots$, $G_{(\nu)}$ has entries in $Y_{\delta/\nu, 1, d}^1 \subset Y_{\delta/\nu, r, d}^2$. Proposition 16, closely related to Theorem 15, implies that in the following theorem $k + 2$ cannot be replaced by $k + 1$.

Theorem 18. *For any $d = 1, 2, \dots$, $k = 1, 2, \dots$, $0 < \nu < \infty$, and $Q \in \mathcal{U}_{d, \nu+d}$, there is a δ with $0 < \delta < 1$ such that the conclusions of Theorem 9 hold for $X = X_{\delta, k+2, \nu}$ in place of $BL^*(\mathbb{R}^d)$, $\mathcal{W}_{\delta, d}$ in place of \mathcal{P}_d , $\nu > 1$ in part (d), and analyticity replaced by C^k in parts (a), (c), and (d).*

Proof. To adapt the proof of (a), $A_\nu(Q)$ given by Theorem 6(a) exists and is in \mathcal{W}_δ for some $\delta \in (0, 1)$. Fix such a δ . For each $A \in \mathcal{W}_\delta$ and entry $f = G_{(\nu)}(\cdot, A)_{ij}$, we have $f = c + g \in \mathbb{R} \oplus Y_{\delta/\nu, 1, d}^1$, so $\phi(f)$ is defined for each $\phi \in X$. The map $C \mapsto G_{(\nu)}(\cdot, A)_{ij}$ is Fréchet C^1 from \mathcal{W}_δ into $\mathbb{R} \oplus Y_{\delta/\nu, 3, d}^2$ by Proposition 17(d), and since the term $-A$ in (39) not depending on y is analytic, thus C^∞ , with respect to $C = A^{-1}$. Now for $k \geq 2$ and $r = k - 1$ we consider $d_C^r G_{(\nu)}(\cdot, A) \Delta^{\otimes r}$ in (59) in place of $G_{(\nu)}(\cdot, A)$ and spaces $Y_{\delta/\nu, 2m-1+r, d}^m$ in place of $Y_{\delta/\nu, 2m-1, d}^m$ for $m = 1, 2$. Each additional differentiation with respect to C adds 1 to the power of $\nu + y'Cy$ in the denominator. Then the proof of (a), now proving C^k under the corresponding hypothesis, can proceed as before.

For (b), the Hessian is the same as before.

For (c), given $Q \in \mathcal{U}_{d, \nu+d}$ and $\delta > 0$ such that $A_\nu(Q) \in \mathcal{W}_{\delta, d}$, parts (a) and (b) give the hypotheses of the Hildebrandt-Graves implicit function theorem, C^k case, Theorem 30(b) in the Appendix. Also as before, there is a $\|\cdot\|_{\delta, k+2, \nu}$ neighborhood V of ϕ_Q on which the implicit function, say $A_{\nu, V}$, exists. By taking V small enough, we can get $A_{\nu, V}(\phi) \in \mathcal{W}_{\delta, d}$ for all $\phi \in V$. For any $Q' \in \mathcal{U}_{d, \nu+d}$ such that $\phi_{Q'} \in V$, we have uniqueness $A_{\nu, V}(\phi_{Q'}) = A_\nu(Q')$ by Theorem 3. Thus the C^k property of $A_{\nu, V}$ on V with respect to $\|\cdot\|_{\delta, k+2, \nu}$, given by the implicit function theorem, applies to $A_\nu(\cdot)$ on Q such that $\phi_Q \in V$, proving (c).

Part (d), again using earlier parts with $(d+1, \nu-1)$ in place of (d, ν) , and now with C^k , then follows as before. \square

Here are some definitions and a proposition to prepare for the next theorem. Recall that $O(d)$ is the group of all orthogonal transformations of \mathbb{R}^d onto itself

($d \times d$ orthogonal matrices). Then $O(d)$ is compact. Let χ_d be the Haar measure on the Borel sets of $O(d)$, invariant under the action of $O(d)$ on itself, normalized so that $\chi_d(O(d)) = 1$.

The *Grassmannian* $G(q, d)$ is the space of all q -dimensional vector subspaces of \mathbb{R}^d . Each $g \in O(d)$ defines a transformation of $G(q, d)$ onto itself. Fix $V \in G(q, d)$. For each Borel set $B \subset G(q, d)$, define a measure $\gamma_{d,q}(B) := \chi_d(\{g \in O(d) : gV \in B\})$. Then $\gamma_{d,q}$ is a probability measure on $G(q, d)$, invariant under the action of $O(d)$. The following may well be known, but we do not know a reference for it.

Proposition 19. *Let Q be any law on \mathbb{R}^d for $d \geq 2$. Then for each $q = 1, \dots, d-1$, $\gamma_{d,q}\{H \in G(q, d) : Q(H) = Q(\{0\})\} = 1$.*

Proof. Let $J(q) := J_Q(q) := \{H \in G(q, d) : Q(H) > Q(\{0\})\}$. For $q = 1$, the sets $H \setminus \{0\}$ for $H \in G(1, d)$ are disjoint, so $J(1)$ is countable and $\gamma_{d,1}(J(1)) = 0$.

We claim that if $1 \leq q < r < d$ and $K \in G(q, d)$, then $\gamma_{d,r}\{H \in G(r, d) : H \supset K\} = 0$. It suffices to prove this for $q = 1$. Let v be one of the two unit vectors $\pm v$ in K . Then for $g \in O(d)$, $K \subset gH$ if and only if $g^{-1}v \in H$. Now $g^{-1}v$ is uniformly distributed on the unit sphere and so is in H with probability 0 as claimed.

For $r = 1, \dots, d-1$ let $\mathcal{I}(r)$ be the set of all subspaces $H \in J(r)$ such that there is no $K \in J(q)$ with $1 \leq q < r$ and $K \subset H$. For any $H_1 \neq H_2$ in $\mathcal{I}(r)$ we have $H_1 \cap H_2 \in G(m, d)$ for some $m < r$ and $Q((H_1 \cap H_2) \setminus \{0\}) = 0$ by assumption. Thus the sets $H \setminus \{0\}$ for $H \in \mathcal{I}(r)$ are essentially disjoint for Q , with probability > 0 , so $\mathcal{I}(r)$ is countable for each r . It follows that for each $r = 1, \dots, d-1$,

$$\gamma_{d,r}(J(r)) = \sum_{q=1}^r \gamma_{d,r}\{H \in G(q, d) : H \supset K \text{ for some } K \in \mathcal{I}(r)\} = 0$$

by the claim and since each $\mathcal{I}(r)$ is countable. The Proposition is proved. \square

Here is a delta-method fact.

Theorem 20. (a) *For any $d = 2, 3, \dots$, $\nu > 0$, and $Q \in \mathcal{U}_{d,\nu+d}$ with empirical measures Q_n , we have $Q_n \in \mathcal{U}_{d,\nu+d}$ with probability $\rightarrow 1$ as $n \rightarrow \infty$ and $\sqrt{n}(A_\nu(Q_n) - A_\nu(Q))$ converges in distribution to a normal distribution $N(0, S)$ on \mathcal{S}_d . The covariance matrix S has full rank $d(d+1)/2$ if Q is not concentrated in any set where a non-zero second-degree polynomial vanishes, e.g. if Q has a density. For general $Q \in \mathcal{U}_{d,\nu+d}$, if $d = 1$ the rank is exactly 1, and for $d \geq 2$, the smallest possible rank of S is $d-1$.*

(b) *For any $d = 1, 2, \dots$, $1 < \nu < \infty$ and $P \in \mathcal{V}_{d,\nu+d}$ with empirical measures P_n , we have $P_n \in \mathcal{V}_{d,\nu+d}$ with probability $\rightarrow 1$ as $n \rightarrow \infty$ and the functionals μ_ν and Σ_ν are such that as $n \rightarrow \infty$,*

$$\sqrt{n} [(\mu_\nu, \Sigma_\nu)(P_n) - (\mu_\nu, \Sigma_\nu)(P)]$$

converges in distribution to some normal distribution with mean 0 on $\mathbb{R}^d \times \mathbb{R}^{d^2}$, whose marginal on \mathbb{R}^{d^2} is concentrated on \mathcal{S}_d . The covariance of the asymptotic normal distribution for $\mu_\nu(P_n)$ has full rank d . The rank of the covariance for $\Sigma_\nu(P_n)$ has the same behavior as the rank of S in part (a).

Proof. Let $k = 1$ or larger. Choose $0 < \delta < 1$ such that $A_\nu = A_\nu(Q) \in \mathcal{W}_\delta$. For (a), let $\Gamma_{\delta,\nu}^{k+2,d} := \mathcal{G}_{\delta/\nu,k+2,d}^{(k+2)}$. To control differences $P_n - P$ on classes $\Gamma_{\delta,\nu}^{k+2,d}$ we have the following.

By Lemma 10, for any $k = 1, 2, \dots$, $\Gamma_{\delta,\nu}^{k+2,d}$ is a uniformly bounded class of functions. It is a class of rational functions of the y_j and C_{kl} in which the polynomials in the numerators and denominators have degrees $\leq m := 2k + 4$. If $A(y)$ and $B(y)$ are any polynomials in y of degrees at most m , with $B(y) > 0$ for all y (as is the case here), then for any real c , the set $\{y : A(y)/B(y) > c\} = \{y : (A - cB)(y) > 0\}$, where $A - cB$ is also a polynomial of degree at most m .

Let $\mathcal{E}(r, d)$ be the collection of all sets $\{x \in \mathbb{R}^d : p(x) > 0\}$ for all polynomials p (in d variables) of degree at most r . Then for each r and d , $\mathcal{E}(r, d)$ is a VC (Vapnik-Chervonenkis) class of sets, e.g. Dudley (1999, Theorem 4.2.1). So $\Gamma_{\delta,\nu}^{k+2,d}$ is a VC major class of functions for $\mathcal{E}(2k + 4, d)$, and a VC hull class (defined in Dudley [1999, pp. 159-160]). It is uniformly bounded and has sufficient measurability properties by continuity in the parameter $A \in \mathcal{P}_d$ [Dudley (1999, Theorem 5.3.8)]. It follows that $\Gamma_{\delta,\nu}^{k+2,d}$ is a universal Donsker class [Dudley (1999, Corollary 6.3.16, Theorem 10.1.6)], in other words, for any $\delta > 0$ and $r = 1, 2, \dots$ and any law Q , $\sqrt{n} \int f d(Q_n - Q)$ is asymptotically normal (converges to a Gaussian process G_Q indexed by f) uniformly for $f \in \Gamma_{\delta,\nu}^{k+2,d}$. In particular we have the bounded Donsker property, i.e. $\sqrt{n} \|Q_n - Q\|_{\delta,k+2,\nu}$ is bounded in probability, where we now identify ϕ_Q with Q and likewise for Q_n . We also have that $\Gamma_{\delta,\nu}^{k+2,d}$ is a uniform Glivenko-Cantelli class by Dudley, Giné and Zinn (1991, Theorem 6), so that $\|Q_n - Q\|_{\delta,k+2,\nu} \rightarrow 0$ almost surely as $n \rightarrow \infty$. Thus almost surely for n large enough, $Q_n \in V$ for the neighborhood V of Q defined in the proof of Theorem 18, so $Q_n \in \mathcal{U}_{d,\nu+d}$ and $A_\nu(Q_n)$ is defined.

By Theorem 18(c) for $k = 1$ and (61), we have

$$(62) \quad A_\nu(Q_n) - A_\nu(Q) = (DA_\nu)(Q_n - Q) + o(\|Q_n - Q\|_{\delta,3,\nu})$$

as $n \rightarrow \infty$. The remainder term is $o_p(1/\sqrt{n})$ by the bounded Donsker property mentioned above.

To make DA_ν more explicit, one can use partial derivatives of F as follows. For any $\zeta \in X$ and $A_\nu := A_\nu(Q)$, we have $F(\phi_Q + \zeta, A_\nu) - F(\phi_Q, A_\nu) = F(\zeta, A_\nu)$, so the partial derivative of F with respect to ϕ at (ϕ_Q, A_ν) is the linear operator $D_\phi F : \zeta \mapsto \zeta(G_{(\nu)}(\cdot, A_\nu))$ from X into \mathcal{S}_d , which is continuous since each entry of $G_{(\nu)}(\cdot, A_\nu)$ is in $\Gamma_{\delta,\nu}^{k+2,d}$. The partial derivative of $F(\phi, A)$ with respect to C ,

at $A = A_\nu$, is given as mentioned previously by the Hessian (44), shown to be positive definite in Lemma 8.

Recall the Hessian linear map $\mathcal{H} := \mathcal{H}_A$ from \mathcal{S}_d to itself defined by (45). By a classical formula for derivatives of inverse functions, e.g. Deimling (1985, p. 150), $DA_\nu(\zeta) = -\mathcal{H}^{-1}D_\phi F(\phi_Q, A_\nu)(\zeta)$, from which

$$(63) \quad DA_\nu(Q_n - Q) = -\mathcal{H}^{-1} \left\{ \int G_{(\nu)}(y, A_\nu) d(Q_n - Q)(y) \right\}.$$

Multiplying by \sqrt{n} , the resulting expression is asymptotically normal by a finite-dimensional central limit theorem.

The rank of the covariance is preserved by the nonsingular \mathcal{H}^{-1} . The rank is the largest size of a subset S of the set $\{(i, j) : 1 \leq i \leq j \leq d\}$ for which the functions f_{ij} with $f_{ij}(y) := y_i y_j / (\nu + y' C y)$ for $(i, j) \in S$ are linearly independent with respect to Q modulo constant functions, i.e. there do not exist constants a_{ij} , $(i, j) \in S$, not all 0, and a constant c such that $\sum_{(i,j) \in S} a_{ij} f_{ij} = c$ almost surely for Q . By a linear change of variables we can assume that $A = I = C$.

For $d = 1$, f_{11} cannot be a constant a.s. since $Q \in \mathcal{U}_{1, \nu+1}$ is not concentrated in two points, so the rank (of the covariance) is exactly 1.

For any d , a linear dependence relation $\sum_{(i,j)} a_{ij} f_{ij} = c$ with a_{ij} not all 0 is equivalent to a quadratic polynomial equation $\sum_{(i,j)} a_{ij} y_i y_j = c(\nu + y' y)$ holding a.s. Q . If no such equation holds, e.g. Q has a density, then the rank has its maximum possible value $d(d+1)/2$.

For any $d \geq 2$, let e_j , $j = 1, \dots, d$, be the standard unit vectors in \mathbb{R}^d . Let

$$Q := \frac{1}{2d} \sum_{j=1}^d \left(\delta_{-\sqrt{d}e_j} + \delta_{\sqrt{d}e_j} \right).$$

Then for each i, j , $(\nu + d) \int y_i y_j dQ(y) / (\nu + |y|^2) = \delta_{ij}$, so $A = I = C$ as desired. Clearly $f_{ij} = 0$ Q -a.s. for $i \neq j$. One can check that $Q \in \mathcal{U}_{d, \nu+d}$ for any $d \geq 2$ and $\nu > 0$.

We have $\sum_{i=1}^d f_{ii} = |y|^2 / (\nu + |y|^2) = d / (\nu + d)$ almost surely with respect to Q , so the rank is at most $d - 1$. Conversely consider $g(y) := \sum_{i=1}^{d-1} a_i f_{ii}(y)$ where some $a_i \neq 0$. Then $g(y) = 0$ for $y = \pm \sqrt{d} e_d$ and $g(y) = a_i d / (\nu + d) \neq 0$ for $y = \pm \sqrt{d} e_i$, each occurring with Q -probability > 0 , so g is not constant a.s. Q , the $d - 1$ functions are not linearly dependent mod constants, and the rank is exactly $d - 1$ in this case.

Now for $d \geq 2$ and any $q \in \mathcal{U}_{d, \nu+d}$, still with $A = C = I$, by Proposition 19 and a rotation of coordinates we can assume that $Q(y_1 = 0) = Q(\{0\})$. We claim that then the functions f_{1j} for $j = 2, \dots, d$ are linearly independent mod constants with respect to Q . Suppose that for some real a_2, \dots, a_d not all 0 and constant c , $y_1 z(y) / (\nu + |y|^2) = c$ a.s. Q where $z(y) := \sum_{j=2}^d a_j y_j$. Since

$\int y_1 y_j dQ(y)/(\nu + |y|^2) = 0$ for $j \geq 2$ we must have $c = 0$ and so

$$1 = Q(y_1 z(y) = 0) = Q(z(y) = 0) + Q(y_1 = 0 \neq z(y))$$

but the latter probability is 0 by choice of y_1 . Thus $Q(z(y) = 0) = 1$ but $\{z(y) = 0\}$ is a $(d-1)$ -dimensional vector subspace, contradicting $Q \in \mathcal{U}_{d,\nu+d}$. Thus the rank is always at least $d-1$ for $d \geq 2$, which is sharp by the example.

Now $\sqrt{n}(A_\nu(Q_n) - A_\nu(Q))$ has the same asymptotic normal distribution as \sqrt{n} times the expression in (63) since the other term in (62) yields $\sqrt{n}o_p(1/\sqrt{n}) = o_p(1)$. So part (a) is proved.

For (b), we take $Q := P \circ T_1^{-1} \in \mathcal{U}_{d+1,\nu+d}$ and apply part (a) to it with d, ν replaced by $d+1, \nu' = \nu - 1$. We can write $Q_n = P_n \circ T_1^{-1}$. As in part (a), we will have almost surely $P_n \in \mathcal{V}_{d,\nu+d}$ for n large enough. From the resulting $A_{\nu'}$, we get μ_ν and Σ_ν for P and P_n via Proposition 4(a) with $\gamma = 1$. Then $(\mu_\nu)_j = (A_{\nu'})_{j,d+1}$ for $j = 1, \dots, d$, both for P, Q and for P_n, Q_n . We also have for $i, j = 1, \dots, d$,

$$(64) \quad (\Sigma_\nu(P))_{ij} = (A_{\nu'}(Q))_{ij} - (A_{\nu'}(Q))_{i,d+1}(A_{\nu'}(Q))_{j,d+1},$$

and likewise for P_n and Q_n . This transformation of matrices, although nonlinear, is smooth enough to preserve asymptotic normality (the finite-dimensional delta-method), where the following will show how uniformity in the asymptotics is preserved:

Lemma 21. *If random vectors $\{U_{in}\}_{i=1}^d$ for $n = 1, 2, \dots$ and a constant vector $\{U_i\}_{i=1}^d$ are such that as $n \rightarrow \infty$, $\sqrt{n}\{U_{in} - U_i\}_{i=1}^d$ converges in distribution to a normal distribution with mean 0 on \mathbb{R}^d , then so does*

$$(65) \quad \sqrt{n}(\{U_{in} - U_i\}_{i=1}^d, \{U_{in}U_{jn} - U_iU_j\}_{1 \leq i < j \leq d})$$

on $\mathbb{R}^{d(d+3)/2}$. For a family of $\{U_{in}\}$ and $\{U_i\}$ such that U_i are uniformly bounded and the convergence to normality of $\sqrt{n}\{U_{in} - U_i\}_{i=1}^d$ holds uniformly over the family, it does also for (65).

Proof. For one product term, we have

$$U_{in}U_{jn} - U_iU_j = (U_{in} - U_i)U_j + U_i(U_{jn} - U_j) + (U_{in} - U_i)(U_{jn} - U_j)$$

where the last term is $O_p(1/n)$ and so negligible and the other terms are jointly asymptotically normal. The uniformity holds for the first two terms since the U_i are uniformly bounded. Each factor in the last term is uniformly $O_p(1/\sqrt{n})$, so their product is uniformly $O_p(1/n)$. \square

Returning to the proof of Theorem 20(b), Lemma 21 for $U_{in} := A_{\nu'}(Q_n)_{i,d+1}$ and constants $U_i := A_{\nu'}(Q)_{i,d+1}$ gives asymptotic normality of $\sqrt{n}[\Sigma_\nu(P_n) - \Sigma_\nu(P)]_{ij}$ using (64).

Via an affine transformation of \mathbb{R}^d , we can assume that $\mu_\nu(P) = 0$ and $\Sigma_\nu(P) = I_d$. Then for $Q = P \circ T^{-1}$ we get $A_{\nu'}(Q) = I_{d+1}$. If for some a_1, \dots, a_d not all 0 we have $\sum_{j=1}^d a_j y_j y_{d+1}/(\nu + |y|^2) = c$ a.s. (Q) for a constant c , we must have

$c = 0$ and thus $\sum_{j=1}^d a_j y_j y_{d+1} = \sum_{j=1}^d a_j y_j = 0$ a.s. for Q , where the latter equation also holds a.s. (P), contradicting $P \in \mathcal{V}_{d,\nu+d}$. Thus the asymptotic normal distribution for $\mu_\nu(P_n)$ has full rank d . The rank of the covariance of the asymptotic normal distribution for $\Sigma_\nu(P_n)$ behaves as in part (a) by the same proof. Part (b) of the theorem is proved. \square

Now, here is a statement on uniformity as P and Q vary. Recall \mathcal{W}_δ as defined in (40).

Proposition 22. *For any $\delta > 0$ and $M < \infty$, the rate of convergence to normality in Theorem 20(a) is uniform over the set $\mathcal{Q} := \mathcal{Q}(\delta, M, \nu)$ of all $Q \in \mathcal{U}_{d,\nu+d}$ such that $A_\nu(Q) \in \mathcal{W}_\delta$ and*

$$(66) \quad Q(\{y : |y| > M\}) \leq (1 - \delta)/(\nu + d),$$

or in part (b), over all $P \in \mathcal{V}_{d,\nu+d}$ such that $\Sigma_\nu(P) \in \mathcal{W}_\delta$ and (66) holds for P in place of Q .

Remark. The example after Lemma 8 shows that $A = A_\nu(Q)$ itself does not control Q well enough to keep it away from the boundary of $\mathcal{U}_{d,\nu+d}$ or give an upper bound on the norm of \mathcal{H}_A^{-1} , which is needed for uniformity in the limit theorem. For a class \mathcal{Q} of laws to have the uniform asymptotic normality of A_ν , uniform tightness is not necessary, but a special case (66) of uniform tightness is assumed.

Proof. A transformation as in the proof of Lemma 8 gives a law q with $A_\nu(q) = I_d$ such that (66) holds with Q replaced by q and M by $K := M/\sqrt{\delta}$, noting that $\tau_1 \leq 1/\delta$ where τ_1 is the largest eigenvalue of $A_\nu(Q)^{-1}$.

In the proof of Theorem 20, it was shown that for any $\delta > 0$ and $k = 1, 2, \dots$, $\Gamma_{\delta,\nu}^{k+2,d}$ is a uniformly bounded VC major class of functions with sufficient measurability properties for empirical process limit theorems. To show that $\Gamma_{\delta,\nu}^{k+2,d}$ is a uniform Donsker class in the sense defined and characterized by Giné and Zinn (1991), one can apply a convex hull property proved by Bousquet, Koltchinskii and Panchenko (2002).

Take any $\Delta \in \mathcal{S}_d$ with $\|\Delta\|_F = 1$. In the following, probabilities and expectations are with respect to q . Let $X := (z'\Delta^2 z)/(\nu + z'z)$. Then $0 \leq X < 1$ for all z and by (8) with $Q = q$ and $B = I$, $EX = \text{trace}(\Delta^2)/(\nu + d) = 1/(\nu + d)$. Thus

$$\frac{1}{\nu + d} \leq \frac{\delta}{2(\nu + d)} + \Pr\left(X > \frac{\delta}{2(\nu + d)}\right),$$

so $\Pr(X > \delta/[2(\nu+d)]) \geq (1 - \frac{\delta}{2})/(\nu+d)$. Let $V := \{X > \delta/[2(\nu+d)], |z| \leq K\}$. Then by (66) for q and K we have $\Pr(V) \geq \delta/[2(\nu + d)] > 0$. Let $S := z'z/(\nu + z'z)$, $Y := X1_V$ and $Z := X1_{V^c}$. Then

$$E(XS) = E((Y + Z)S) \leq EZ + E(YK^2/(\nu + K^2)).$$

We have $E(Y\nu/(\nu + K^2)) \geq \alpha/(\nu + d)$ where $\alpha := \delta^2\nu/[4(\nu + d)(\nu + K^2)]$. Thus

$$(\nu + d)E(XS) = (\nu + d) \int \frac{(z'z)(z'\Delta^2z)}{(\nu + z'z)^2} dq(z) \leq 1 - \alpha.$$

This implies, by the proof of Lemma 8, that the eigenvalues of the Hessian \mathcal{H}_I for qH at I are all at least α and those of the Hessian \mathcal{H}_A for QH at A are at least $\alpha' := \delta^2\alpha$. Here α' depends on δ , M , ν , and d , but not otherwise on $Q \in \mathcal{Q}$. Bounds in the proof of Theorem 20 hold uniformly: specifically, in (63), $\|\mathcal{H}^{-1}\| \leq 4/(\delta^2\alpha)$ and the entries $G_{(\nu)}(\cdot, A_\nu)_{ij} \in \Gamma_{\delta,\nu}^{k+2,d}$, a uniform Donsker class. The remainder term $\sqrt{n}o(\|Q_n - Q\|_{\delta,k+2,\nu})$ in (62) is $o_p(1)$ uniformly over \mathcal{Q} by (61) since each $\Gamma_{\delta,\nu}^{k+2,d}$ is a uniform Donsker class. It follows that asymptotic normality of $\sqrt{n}(DA_\nu)(Q_n - Q)$ holds uniformly for $Q \in \mathcal{Q}$.

It remains to show that $\Pr(Q_n \in \mathcal{U}_{d,\nu+d})$, the probability that $A_\nu(Q_n)$ is defined, converges to 1 as $n \rightarrow \infty$ at a rate uniform over $Q \in \mathcal{Q}$. The class of all vector subspaces of \mathbb{R}^d is a VC class of sets with suitable measurability, so it is a uniform Glivenko-Cantelli class by Dudley, Giné and Zinn (1991, Theorem 6). For $q = 0, 1, \dots, d-1$, let $J(q)$ be the class of all q -dimensional vector subspaces of \mathbb{R}^d . We need to show that for each q ,

$$(67) \quad \sup_{Q \in \mathcal{Q}, H \in J(q)} Q(H) < 1 - \frac{d-q}{\nu+d}.$$

We can restrict to Q with $A_\nu(Q) = I_d$ without changing the suprema of Q of subspaces, replacing again M by $K := M/\sqrt{\delta}$. Then we can fix $H \in J(q)$ and let Q vary. Let $|z|_q^2 := z_{q+1}^2 + \dots + z_d^2$. By choice of coordinates we can take $H = \{z : |z|_q^2 = 0\}$. For each $Q \in \mathcal{Q}$, since $A_\nu(Q)$ is defined, we have $Q(H^c) > (d-q)/(\nu+d) \geq 1/(\nu+d)$. We also have by (66) $Q(|z| > M) \leq (1-\delta)/(\nu+d)$, so $Q(H^c \cap \{|z| \leq M\}) \geq \delta/(\nu+d)$. Now

$$\begin{aligned} \frac{d-q}{\nu+d} &= \int \frac{|z|_q^2 dq}{\nu + z'z} \leq \frac{\delta}{\nu+d} \cdot \frac{M^2}{\nu+M^2} + Q(H^c) - \frac{\delta}{\nu+d} \\ &= Q(H^c) - \frac{\delta\nu}{(\nu+d)(\nu+M^2)}. \end{aligned}$$

It follows that, replacing M by K to allow for the transformation,

$$Q(H) \leq 1 - \frac{d-q}{\nu+d} - \frac{\delta\nu}{(\nu+d)(\nu+K^2)},$$

which implies (67) and so finishes the proof of part (a).

As part of the proof of part (b), the next fact will show that the special-case tightness hypothesis (66) itself implies a bound on $\|A_\nu(Q)\|$ (although not, of course, on $\|A_\nu(Q)^{-1}\|$). A bound exists since A_ν has a breakdown point of $1/(\nu+d)$ with regard to mass going to infinity [Tyler (1986, §3); Dümbgen

and Tyler (2005, Theorem 5 and its proof)]. The next lemma provides specific constants which may not be sharp.

Lemma 23. *If $Q \in \mathcal{U}_{d,\nu+d}$, (66) implies $\|A_\nu(Q)\| \leq M^2(\nu + d - \delta)/(\delta\nu)$.*

Proof. $A_\nu(Q) \in \mathcal{P}_d$ exists by Theorem 6(a). Take coordinates in which $A := A_\nu(Q)$ is diagonalized with eigenvalues $1/\tau_i$, $i = 1, \dots, d$. We then have by (8) and $u_\nu(s) = (\nu + d)/(\nu + s)$ (just after (33)) that

$$\frac{1}{\tau_i} = (\nu + d) \int \frac{x_i^2 dQ(x)}{\nu + \sum_{j=1}^d \tau_j x_j^2}$$

for $i = 1, \dots, d$. The integral over $\{|x| > M\}$ is at most $(1 - \delta)/[(\nu + d)\tau_i]$ by (66). For $|x| \leq M$ we have

$$\frac{x_i^2}{\nu + \sum_{j=1}^d \tau_j x_j^2} \leq \frac{x_i^2}{\nu + \tau_i x_i^2} \leq \frac{M^2}{\nu + \tau_i M^2}.$$

Thus $\delta/\tau_i \leq (\nu + d)M^2/(\nu + \tau_i M^2)$, $\tau_i \geq \delta\nu/[M^2(\nu + d - \delta)]$ for all i , and the lemma follows. \square

Now to prove Proposition 22 part (b), i.e. as it relates to Theorem 20(b), let \mathcal{P} be the class of laws satisfying the hypotheses. For $P \in \mathcal{P}$, let $Q := P \circ T_1^{-1}$ as usual. Then (66) holds for Q with $M + 1$ in place of M . By Proposition 5, since $\nu > 1$ in part (b), $Q \in \mathcal{U}_{d+1,\nu+d}$. By Lemma 23, $\|A_{\nu'}(Q)\|$ are bounded uniformly for $P \in \mathcal{P}$ (recall $\nu' \equiv \nu - 1 > 0$). Next, $\det \Sigma_\nu(P) = \det A_{\nu'}(Q)$ by (28) with $\gamma = 1$, which holds by Theorem 6(b). This determinant is bounded below by $\|\Sigma_\nu^{-1}(P)\|^{-d} \geq \delta^d$, so the smallest eigenvalue of $A_{\nu'}(Q)$ is bounded below by $\delta^d \|A_{\nu'}(Q)\|^{-d}$, and $\|A_{\nu'}^{-1}(Q)\| \leq \|A_{\nu'}(Q)\|^d / \delta^d$, which is bounded uniformly for $P \in \mathcal{P}$.

Thus all the hypotheses of part (a) hold for $d + 1, \nu - 1$ in place of d, ν , and some $\delta' > 0$ in place of δ , depending on Q and P only insofar as the hypotheses of part (b) hold, so part (a) gives uniform asymptotic normality of $\sqrt{n}(A_{\nu'}(Q_n) - A_{\nu'}(Q))$ over all $P \in \mathcal{P}$. Taking the last column, that directly gives uniform asymptotic normality of $\sqrt{n}(\mu_\nu(P_n) - \mu_\nu(P))$. For $\sqrt{n}(\Sigma_\nu(P_n) - \Sigma_\nu(P))$ one can apply the delta-method for products, Lemma 21, which works uniformly for $|\mu_\nu(P)|$ bounded, as they are, so Proposition 22 is proved. \square

8. NORMS BASED ON CLASSES OF SETS

Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a vector space V such that for some $K < \infty$, $\|x\|_2 \leq K\|x\|_1$ for all $x \in V$. Let $U \subset V$ be open for $\|\cdot\|_2$ and so also for $\|\cdot\|_1$. Let $v \in U$ and suppose a functional T from U into some other normed space is Fréchet differentiable at v for $\|\cdot\|_2$. Then the same holds for $\|\cdot\|_1$ since the identity from V to V is a bounded linear operator from $(V, \|\cdot\|_1)$ to $(V, \|\cdot\|_2)$

and so equals its own Fréchet derivative everywhere on V , and we can apply a chain rule, e.g. Dieudonné [1960, (8.12.10)].

If \mathcal{F} is a class of bounded real-valued functions on a set χ , measurable for a σ -algebra \mathcal{A} of subsets of χ , and ϕ is a finite signed measure on \mathcal{A} , (e.g. $P_n - P$) let $\|\phi\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\int f d\phi|$. For $\mathcal{C} \subset \mathcal{A}$ let $\|\phi\|_{\mathcal{C}} := \|\phi\|_{\mathcal{G}}$ where $\mathcal{G} := \{1_C : C \in \mathcal{C}\}$.

Let \mathcal{F} be a VC major class of functions for \mathcal{E} (defined in Dudley [1999, pp. 159-160]), where $\mathcal{E} \subset \mathcal{A}$ and suppose for some $M < \infty$, $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and $x \in \chi$. Then for any finite signed measure ϕ on \mathcal{A} having total mass $\phi(\chi) = 0$ (e.g., $\phi = P - Q$ for any two laws P and Q), we have

$$(68) \quad \|\phi\|_{\mathcal{F}} \leq 2M\|\phi\|_{\mathcal{E}},$$

by the rescaling $f \mapsto (f + M)/(2M)$ to get functions with values in $[0, 1]$ and then a convex hull representation [Dudley (1987, Theorem 2.1(a)) or (1999, Theorem 4.7.1(b))]; additive constants make no difference since $\phi(\chi) = 0$.

As noted in the proof of Theorem 20, each $\Gamma_{\delta, \nu}^{k+2, d}$ is a uniformly bounded VC major class for the VC class $\mathcal{E}(2k+4, d)$ of sets (positivity sets of polynomials of degree $\leq 2k+4$). So by (61) and (68), for some $M < \infty$ depending on r, δ, ν , and d , we have

$$(69) \quad \|\phi\|_{\delta, k+2, \nu} \leq 2M\|\phi\|_{\mathcal{E}(2k+4, d)}$$

for all finite signed measures ϕ on \mathbb{R}^d with $\phi(\mathbb{R}^d) = 0$. We have by the preceding discussion:

Corollary 24. *For each $d = 1, 2, \dots$, and $\nu > 1$, the Fréchet C^k differentiability property of the t_ν location and scatter functionals at each P in $\mathcal{V}_{d, \nu+d}$, as shown in Theorem 18 with respect to $\|\cdot\|_{\delta, k+2, \nu}$, also holds with respect to $\|\cdot\|_{\mathcal{E}(2k+4, d)}$.*

Each class $\mathcal{E}(r, d)$ for $r = 1, 2, \dots$ is invariant under all non-singular affine transformations of \mathbb{R}^d , and hence so is the norm $\|\cdot\|_{\mathcal{E}(r, d)}$. Davies (1993, pp. 1851-1852) defines norms $\|\cdot\|_{\mathcal{L}}$ based on suitable VC classes \mathcal{L} of subsets of \mathbb{R}^d and points out Donsker and affine invariance properties. The norms $\|\cdot\|_{\delta, r, \nu}$ are not affinely invariant.

On the other hand, note that M in (69) depends on δ , and there is no corresponding inequality in the opposite direction. Thus, Fréchet differentiability is strictly stronger for $\|\cdot\|_{\delta, k+2, \nu}$ than it is for $\|\cdot\|_{\mathcal{E}(2k+4, d)}$.

9. THE ONE-DIMENSIONAL CASE

In dimension $d = 1$, the scatter matrix Σ reduces to a number σ^2 . The ρ and h functions in this case become, for $\theta := (\mu, \sigma)$ with $\sigma > 0$, by (31) and (32),

$$(70) \quad \rho_\nu(x, \theta) := \log \sigma + \frac{\nu + 1}{2} \log \left(1 + \frac{(x - \mu)^2}{\nu \sigma^2} \right),$$

$$(71) \quad h_\nu(x, \theta) := \log \sigma + \frac{\nu + 1}{2} \log \left(\frac{1 + [(x - \mu)^2 / (\nu \sigma^2)]}{1 + x^2 / \nu} \right).$$

The function h_ν is bounded uniformly in x and for $|\mu|$ bounded and σ bounded away from 0 and ∞ . Thus it is integrable for any probability distribution P on \mathbb{R} . Let $Ph_\nu(\theta) := \int h_\nu(x, \theta) dP(x)$. In the next theorem, extended M-functionals are defined by (1) with $\theta := (\mu, \sigma) \in \Theta = \mathbb{R} \times (0, \infty)$ and $\bar{\Theta} = \mathbb{R} \times [0, \infty)$.

Theorem 25. *Let $d = 1$ and $1 < \nu < \infty$. Then:*

(a) *For any law Q on \mathbb{R} satisfying*

$$(72) \quad \max_t Q(\{t\}) < \nu / (\nu + 1),$$

the M-functional $(\mu, \sigma) = (\mu_\nu, \sigma_\nu)(Q)$ exists with $\sigma_\nu(Q) > 0$ and is the unique critical point with $\partial Q h_\nu / \partial \mu = \partial Q h_\nu / \partial \sigma = 0$. On the set of laws satisfying (72), (μ_ν, σ_ν) is analytic with respect to the dual-bounded-Lipschitz norm and thus weakly continuous.

(b) *For any law Q on \mathbb{R} , the extended M-functional $\theta_0(Q) := (\mu_\nu, \sigma_\nu)(Q) \in \bar{\Theta}$ exists for h_ν from (71).*

(c) *If $Q(\{s\}) \geq \nu / (\nu + 1)$ for some (unique) s , then $\mu_\nu(Q) = s$ and $\sigma_\nu(Q) = 0$.*

(d) *The map $Q \mapsto \theta_0(Q)$ is weakly continuous at every law Q . For X_1, X_2, \dots i.i.d. (Q) and empirical measures $Q_n := n^{-1} \sum_{j=1}^n \delta_{X_j}$, we thus have maximum likelihood estimates $\hat{\theta}_n = \theta_0(Q_n)$ existing for all n and converging to $\theta_0(Q)$ almost surely.*

Remark. The theorem doesn't extend to $0 < \nu \leq 1$. For some Q , points s in part (c) are not unique. For example if $\nu = 1$ (the Cauchy case) and $Q = \frac{1}{2}(\delta_{-1} + \delta_1)$, the likelihood is maximized on the semicircle $\mu^2 + \sigma^2 = 1$, as Copas (1975) noted.

Proof. Part (a) holds by the case of general dimension, Theorem 9(d), since $\sigma^2 \mapsto \sigma$ is analytic for $\sigma > 0$. The other parts are special to $d = 1$.

Let $D := (x - \mu)^2 + \nu \sigma^2$. Let $\nu \geq 1$ be fixed for the present and let $\rho = \rho_\nu$ and $h = h_\nu$. It's immediate from (70) and (71) that for any $\theta = (\mu, \sigma)$ with $0 < \sigma < \infty$ and any $x \in \mathbb{R}$,

$$(73) \quad \frac{\partial h(x, \theta)}{\partial \mu} = \frac{\partial \rho(x, \theta)}{\partial \mu} = \frac{(\nu + 1)(\mu - x)}{D},$$

$$(74) \quad \frac{\partial h(x, \theta)}{\partial \sigma} = \frac{\partial \rho(x, \theta)}{\partial \sigma} = \frac{1}{\sigma} \left[1 - \frac{(\nu + 1)(x - \mu)^2}{D} \right].$$

It's easily seen that for any $K > 0$ and all real y ,

$$(75) \quad |y| / (K + y^2) \leq 1 / (2\sqrt{K}).$$

It follows directly that for any x and μ , any $\sigma > 0$ and any $\nu \geq 1$, both partial derivatives (73) and (74) each have absolute values $\leq \nu / \sigma$, so for any $\delta > 0$, they

are bounded uniformly for $\sigma \geq \delta$. For $\theta = (0, 1)$ we have $h(x, \theta) \equiv 0$. Thus for any μ and $0 < \sigma < \infty$,

$$(76) \quad |h(x, \theta)| \leq \nu(|\log \sigma| + |\mu|/\sigma),$$

so h is bounded uniformly for μ bounded and $\delta \leq \sigma \leq 1/\delta$.

From (74) we see that $\partial Qh(\theta)/\partial\sigma = 0$ if and only if

$$(77) \quad F(\mu, \sigma) := \int \frac{(x - \mu)^2}{\nu\sigma^2 + (x - \mu)^2} dQ(x) = \frac{1}{\nu + 1}.$$

As σ decreases from $+\infty$ down to 0, the integrand increases from 0 up to $1_{x \neq \mu}$, strictly for $x \neq \mu$. Thus the integral increases from 0 up to $Q(\{\mu\}^c)$, strictly unless $Q(\{\mu\}) = 1$. So (77) for a fixed μ has a solution $\sigma := \sigma(\mu) > 0$ (depending on ν and Q) if and only if $Q(\{\mu\}^c) > 1/(\nu + 1)$, and the solution is unique. Then, moreover, $\partial Qh(\theta)/\partial\sigma$ will be < 0 for $0 < \sigma < \sigma(\mu)$ and > 0 for $\sigma > \sigma(\mu)$, so that $Qh(\mu, \sigma)$ has its unique minimum for the given μ at $\sigma = \sigma(\mu)$.

If $Q(\{\mu\}) \geq \nu/(\nu + 1)$, then $\sigma(\mu)$ is set equal to 0 (e.g. Copas [1975]), which is natural since for the given μ , $Qh(\mu, \sigma)$ has its smallest values as $\sigma \downarrow 0$.

Taking second partial derivatives we get

$$(78) \quad \partial^2 h / \partial \mu^2 = (\nu + 1)[\nu\sigma^2 - (x - \mu)^2]D^{-2},$$

$$(79) \quad \partial^2 h / \partial \sigma \partial \mu = 2(\nu + 1)\nu\sigma(x - \mu)D^{-2},$$

$$(80) \quad \frac{\partial^2 h}{\partial \sigma^2} = \frac{1}{\sigma^2} \left[(\nu + 1) \frac{(x - \mu)^2}{D} - 1 \right] + 2(\nu + 1)\nu \frac{(x - \mu)^2}{D^2}.$$

It's easily seen that these second partials are also bounded uniformly for $\sigma \geq \delta$ for any $\delta > 0$.

The following shows that $\sigma(\cdot)$ is C^1 and strictly positive except possibly at one large atom. (Here C^1 suffices for present purposes; it could be improved to analyticity, as in the proof of Theorem 9(c).)

Lemma 26. *On the set $U := U_{\nu, Q}$ of μ for which $Q(\{\mu\}) < \nu/(\nu + 1)$, namely the whole line if (72) holds or the complement of a point if it fails, the function $\mu \mapsto \sigma(\mu) > 0$ is C^1 , as is the function $\mu \mapsto Qh(\mu, \sigma(\mu))$.*

Proof. For each $\mu \in U$, we have $\sigma(\mu) > 0$, where $\sigma(\mu)$ is defined after (77) as the unique solution of $F(\mu, \sigma) = 1/(\nu + 1)$ for each $\mu \in U$. By (79), (80), and dominated convergence, F is C^1 . We have

$$\partial F(\mu, \sigma) / \partial \sigma = -2\nu\sigma \int (x - \mu)^2 D^{-2} dQ(x) < 0$$

for all $\mu \in U$ and all $\sigma > 0$. It follows from the implicit function theorem (e.g. Rudin (1976, Theorem 9.28)) that $\sigma(\cdot)$ is a C^1 function on U . Also, the function $(\mu, \sigma) \mapsto Qh(\mu, \sigma)$ is C^1 for $\sigma > 0$ by (73) and (74) and their integrated versions. Thus $\mu \mapsto Qh(\mu, \sigma(\mu))$ is C^1 on U , proving the lemma. \square

The following fact for laws concentrated in two points will be helpful, also in the Remark showing that σ_ν is non-Lipschitz at the end of this section.

Lemma 27. *Let $\nu > 1$ and $Q = q\delta_a + p\delta_b$ where $a < b$ and $0 \leq p = 1 - q \leq 1$. (a) If $1/(\nu + 1) < p < \nu/(\nu + 1)$, then Qh_ν has a unique critical point (μ_p, σ_p) , with $\sigma_p > 0$, at which the Hessian of Qh is strictly positive definite. Explicitly,*

$$(81) \quad \mu_p = \frac{\nu p - q}{\nu - 1}, \quad \sigma_p^2 = \frac{(\nu + 1)q\mu_p - \mu_p^2}{\nu} = \frac{\nu^2 pq - \nu(p^2 + q^2) + pq}{(\nu - 1)^2}.$$

(b) If $p \leq 1/(\nu + 1)$ or $p \geq \nu/(\nu + 1)$ then an M-functional $(\mu, \sigma) = (\mu_\nu(Q), \sigma_\nu(Q))$ exists with $\sigma_\nu(Q) = 0$ and $\mu_\nu(Q) = a$ or b respectively.

Proof. By an affine transformation we can assume that $a = 0$ and $b = 1$. For part (a), the equation $\partial Qh/\partial\mu = 0$ (73) times $1 - \mu$, the equations $\partial Qh/\partial\sigma = 0$ (74), (77), and straightforward calculations give unique solutions (81) for a critical point. Then $0 < \mu_p < 1$ by the hypotheses on p . For each $\nu > 1$, $\partial\sigma_p^2/\partial p = 0$ only at $p = 1/2$ where $\sigma_{1/2}^2 = 1/4$, a maximum. Also, $\sigma_p^2 \downarrow 0$ strictly as $p \downarrow 1/(\nu + 1)$ or $p \uparrow \nu/(\nu + 1)$. Thus $\sigma_p > 0$ for $1/(\nu + 1) < p < \nu/(\nu + 1)$ as assumed, and (μ_p, σ_p) is the unique critical point of Qh .

By Theorem 6 and Lemma 8, the Hessian of Qh as a function of $A \in \mathcal{P}_2$ at $A = A_{\nu-1}(Q \circ T_1^{-1})$ is positive definite. This remains true restricted to the subset where $\gamma = A_{22} = 1$ in Proposition 4(i), so that $A = \begin{pmatrix} \sigma_\mu^2 + \mu^2 & \mu \\ \mu & 1 \end{pmatrix}$, since, in suitable coordinates, a principal minor of a positive definite matrix is positive definite. It follows that the Hessian of Qh with respect to (μ, σ) at (μ_p, σ_p) is positive definite. So part (a) of Lemma 27 is proved.

Now for part (b), we can assume by symmetry that $p \leq 1/(\nu + 1)$ and want to prove $\mu_\nu = \sigma_\nu = 0$ are the M-functionals of Q . For all $\mu \neq 0$, by Lemma 26, $\sigma(\mu) > 0$ is defined such that $Qh(\mu, \sigma)$ is minimized for the given μ at $\sigma = \sigma_\mu := \sigma(\mu)$. (The notations σ_μ and σ_p are different.) Let $(Qh)(\mu) := (Qh)(\mu, \sigma(\mu))$ for $\mu \neq 0$, a C^1 function of μ by Lemma 26. To show that $d(Qh)(\mu)/d\mu$ has the same sign as μ for $\mu \neq 0$ is equivalent by (73) and since $\partial Qh(\mu, \sigma)/\partial\sigma|_{\sigma=\sigma(\mu)} = 0$, to showing that for $\mu \neq 0$,

$$(82) \quad \frac{(1-p)\mu^2}{\nu\sigma_\mu^2 + \mu^2} + \frac{p\mu(\mu-1)}{\nu\sigma_\mu^2 + (\mu-1)^2} > 0.$$

By (77) we have for $\mu \neq 0$

$$(83) \quad \frac{(1-p)\mu^2}{\nu\sigma_\mu^2 + \mu^2} + \frac{p(1-\mu)^2}{\nu\sigma_\mu^2 + (1-\mu)^2} = \frac{1}{\nu+1}.$$

Combining, we want to show that $(\nu + 1)p(1 - \mu) < \nu\sigma_\mu^2 + (1 - \mu)^2$ for $0 < p \leq 1/(\nu + 1)$. We need only consider $0 < \mu < 1$. If (82) fails, then for some such p and μ , $(\nu + 1)p(1 - \mu) - (1 - \mu)^2 \geq \nu\sigma_\mu^2$. Substituting in (83) gives, where the

denominators are necessarily positive,

$$\frac{(1-p)\mu^2}{(\nu+1)p(1-\mu)-1+2\mu} + \frac{1-\mu}{\nu+1} \leq \frac{1}{\nu+1},$$

so

$$\frac{(1-p)\mu}{[(\nu+1)p-1](1-\mu)+\mu} \leq \frac{1}{\nu+1},$$

but $(\nu+1)p-1 \leq 0$ implies the left side is at least $1-p \geq \nu/(\nu+1) > 1/(\nu+1)$ since $\nu > 1$, a contradiction. So (82) is proved. This implies that for any $\varepsilon > 0$,

$$(84) \quad \inf\{Qh(\mu) : 0 < |\mu| < \varepsilon\} < \inf\{Qh(\mu) : |\mu| \geq \varepsilon\}.$$

Next, if there is a sequence $\mu_j \rightarrow 0$ such that $\sigma(\mu_j) \geq \delta$ for some $\delta > 0$, then (83) gives a contradiction for j large enough. So $\sigma(\mu) \rightarrow 0$ as $\mu \rightarrow 0$. This implies that for any $\gamma > 0$

$$\inf\{Qh(\mu, \sigma) : |\mu| < \gamma, \sigma < \gamma\} < \inf\{Qh(\mu) : |\mu| \leq \gamma, \sigma \geq \gamma\},$$

because by (84), the inf is smallest for $|\mu|$ smallest, and then $\sigma(\mu)$ becomes $< \gamma$, so Qh for a given μ and $\sigma \geq \gamma$ is larger than at $\sigma(\mu)$. Also, by (74), $Qh(0, \sigma)$ is strictly decreasing as $\sigma \downarrow 0$. So part (b) of Lemma 27 is proved. \square

Next, let's consider a general Q such that (72) fails. The next fact, with part (a), implies parts (b) and (c) of Theorem 25.

Lemma 28. *Let $\nu > 1$ and let Q be a law on \mathbb{R} such that for some u , $Q(\{u\}) \geq \nu/(\nu+1)$. Then the (extended) M -functional of Q for ρ_ν or h_ν exists with $\mu_\nu(Q) = u$ and $\sigma_\nu(Q) = 0$.*

Proof. Since $\nu > 1$, u is uniquely determined. By a translation we can assume that $u = 0$. Then on the set $U := \{\mu \neq 0\}$, by Lemma 26, $\mu \mapsto \sigma_\mu > 0$ is a C^1 function, giving the infimum of $Qh(\mu, \sigma)$ for each $\mu \neq 0$. It will be shown that

$$(85) \quad \mu d(Qh)(\mu, \sigma_\mu)/d\mu > 0 \quad \text{for all } \mu \neq 0.$$

This is immediate if $Q = \delta_0$ from (73), so we can assume for $\beta := Q(\{0\})$ that $\nu/(\nu+1) \leq \beta < 1$. By (77) and Lemma 26, we have for each $\mu \neq 0$ that $\sigma_\mu > 0$ and

$$(86) \quad \frac{\beta\mu^2}{\nu\sigma_\mu^2 + \mu^2} + \int_{x \neq 0} \frac{(\mu-x)^2 dQ(x)}{\nu\sigma_\mu^2 + (\mu-x)^2} = \frac{1}{\nu+1}.$$

To prove (85), we need to show by (73) that for $\mu \neq 0$

$$(87) \quad \frac{\beta\mu^2}{\nu\sigma_\mu^2 + \mu^2} + \mu \int_{x \neq 0} \frac{(\mu-x)dQ(x)}{\nu\sigma_\mu^2 + (\mu-x)^2} > 0.$$

Combining (87) with (86), we need to show that for $\mu \neq 0$,

$$(88) \quad \int_{x \neq 0} \frac{x(x-\mu)dQ(x)}{\nu\sigma_\mu^2 + (\mu-x)^2} < \frac{1}{\nu+1}.$$

By (86), for $\mu \neq 0$,

$$(89) \quad \int_{x \neq 0} \frac{(\mu - x)^2 dQ(x)}{\nu \sigma_\mu^2 + (\mu - x)^2} = \frac{1}{\nu + 1} - \frac{\beta \mu^2}{\nu \sigma_\mu^2 + \mu^2}.$$

Now (88) will follow from (89) and the Cauchy-Schwarz inequality if

$$\int_{x \neq 0} \frac{x^2 dQ(x)}{\nu \sigma_\mu^2 + (\mu - x)^2} < \frac{\nu \sigma_\mu^2 + \mu^2}{(\nu + 1)[\nu \sigma_\mu^2 + \mu^2 \{1 - (\nu + 1)\beta\}]}.$$

By (86) again, $(\nu + 1)\beta \mu^2 < \nu \sigma_\mu^2 + \mu^2$ unless Q is concentrated at the two points $0, \mu$. That case is treated by Lemma 27(b), so we can neglect it here. Then the denominator of the last expression displayed is positive. Since $(\nu + 1)\beta \geq 1$ and $Q(x \neq 0) \leq 1/(\nu + 1)$, it will suffice to show that for all real x , and as always, $\mu \neq 0$,

$$\frac{x^2}{\nu \sigma_\mu^2 + (\mu - x)^2} \leq \frac{\mu^2 + \nu \sigma_\mu^2}{\nu \sigma_\mu^2}.$$

The fraction on the left goes to 1 as $x \rightarrow \pm\infty$, and there the inequality holds. At $x = 0$, a minimum of that fraction, the inequality also holds. Setting the derivative of the fraction equal to 0 gives one other root, where $x = \mu + (\nu \sigma_\mu^2 / \mu)$ and where the inequality holds (with equality just for this one value of x). Thus (88) and (85) are proved.

The proof that $\mu_\nu(Q) = \sigma_\nu(Q) = 0$ is now completed as in the end of the proof of Lemma 27(b), where now if $\mu_j \rightarrow 0$ and $\sigma(\mu_j) \geq \delta > 0$, (86) is contradicted for j large enough. So Lemma 28 is proved. \square

It remains to prove part (d) of Theorem 25. To show the weak continuity of μ_ν and σ_ν at a law Q with $Q(\{t\}) \geq \nu/(\nu + 1)$ for some unique t , we can and do assume that $t = 0$. We want to show that if a sequence $P_k \rightarrow Q$ weakly, then $\mu_k := \mu_\nu(P_k) \rightarrow 0$ and $\sigma_k := \sigma_\nu(P_k) \rightarrow 0$. Taking subsequences, we can assume that $\mu_k \rightarrow \mu_0$ and $\sigma_k \rightarrow \sigma_0$ where $-\infty \leq \mu_0 \leq +\infty$ and $0 \leq \sigma_0 \leq +\infty$.

If $\sigma_k = 0$ for all k then we have $P_k(\{t_k\}) \geq \nu/(\nu + 1)$ for some t_k . By weak convergence, we must have $t_k \rightarrow 0$, and $\mu_k = t_k$ by Lemma 28, so the conclusion holds. Thus we can assume from here on that $\sigma_k > 0$ for all $k \geq 1$, taking another subsequence. For $k = 0, 1, 2, \dots$, let

$$I_k(x) := \frac{(\mu_k - x)^2}{\nu \sigma_k^2 + (\mu_k - x)^2},$$

with $I_0(x) := 1$ if $\sigma_0 = 0$. Then $0 \leq I_k(x) \leq 1$ for all x and k , a domination condition which is used below without further mention. For $k \geq 1$, since $\sigma_k > 0$, we have by (77) and Lemma 28 that

$$(90) \quad \int I_k dP_k = 1/(\nu + 1).$$

If $\sigma_0 = +\infty$ and μ_0 is finite, then as $k \rightarrow \infty$, $I_k \rightarrow 0$ uniformly on compact sets. Since P_k are uniformly tight, it follows that $\int I_k dP_k \rightarrow 0$, contradicting (90). If $\mu_0 = \pm\infty$ and σ_0 is finite, then $I_k \rightarrow 1$ uniformly on compact sets, so $\int I_k dP_k \rightarrow 1$, again contradicting (90).

So we have two remaining situations, μ_0 and σ_0 both finite or both infinite. First suppose both are finite. If $\sigma_0 > 0$ then as $k \rightarrow \infty$, $I_k(x) \rightarrow I_0(x)$ uniformly on compact sets. From this, the weak convergence and (90) it follows that $\int I_0(x) dQ(x) = 1/(\nu + 1)$, so $\sigma_0 = \sigma(\mu_0)$ for Q . For $k = 1, 2, \dots$ let

$$J_k(x) := \frac{\mu_k - x}{\nu\sigma_k^2 + (\mu_k - x)^2} \rightarrow J_0(x) := \frac{\mu_0 - x}{\nu\sigma_0^2 + (\mu_0 - x)^2}$$

uniformly on compact sets. Then $|J_k(x)| \leq 1/(2\sqrt{\nu}\sigma_k)$ for all x by (75), so J_k are uniformly bounded for k large enough or for $k = 0$. By Lemma 28, $\sigma_k > 0$ implies that each P_k satisfies (72). Then by Theorem 25(a) as already proved, (μ_k, σ_k) is a critical point for P_k , and so by (73) $\int J_k dP_k = 0$ for all $k \geq 1$. Then by weak convergence, $\int J_0 dQ = 0$. Thus (μ_0, σ_0) would be a critical point for Q . This implies by (85) that $\mu_0 = 0$, but that contradicts $\int I_0(x) dQ(x) = 1/(\nu + 1)$. So μ_0 finite and $\sigma_0 > 0$ are not compatible.

If μ_0 is finite and non-zero and $\sigma_0 = 0$ then we have $I_k(x) \rightarrow 1$ except possibly for $x = \mu_0$, and the convergence is uniform on compact subsets of $\{\mu_0\}^c$. Thus

$$\liminf_{k \rightarrow \infty} \int I_k dP_k \geq Q(\{\mu_0\}^c) \geq 1 - \frac{1}{\nu + 1} = \frac{\nu}{\nu + 1} > \frac{1}{\nu + 1},$$

again contradicting (90).

So the proof is complete except if $\mu_0 = \pm\infty$ and $\sigma_0 = +\infty$. Then by symmetry we can assume that $\mu_0 = +\infty$.

If $\sigma_k = o(\mu_k)$ as $k \rightarrow \infty$ then $I_k \rightarrow 1$, or if $\mu_k = o(\sigma_k)$ as $k \rightarrow \infty$ then $I_k \rightarrow 0$, in either case uniformly on compact sets and so contradicting (90). So, taking another subsequence, we can assume that as $k \rightarrow \infty$, $\mu_k/\sigma_k \rightarrow c$ for some c with $0 < c < \infty$. Then uniformly on bounded intervals, $I_k \rightarrow c^2/(\nu + c^2)$ as $k \rightarrow \infty$, an increasing function of c , so (90) implies that $c = 1$.

Since P_k are uniformly tight, take a constant $M < \infty$, with $M > 1$, large enough so that $P_k(|x| > M) \leq 1/(2(\nu + 1))$ for all k . On $[-M, M]$, the quantity $j_k(x) := j(x, \mu, \sigma, \nu)$ in parentheses in (71) whose logarithm is taken, for $\mu = \mu_k$ and $\sigma = \sigma_k$, satisfies asymptotically

$$j_k(x) \sim \frac{\nu + 1}{\nu + x^2} \geq \frac{\nu + 1}{\nu + M^2} \geq \frac{1}{M^2}.$$

Thus up to an additive constant going to 0 as $k \rightarrow \infty$,

$$(91) \quad \frac{\nu + 1}{2} \int_{-M}^M \log j_k(x) dP_k(x) \geq \left[\frac{\nu + 1}{2} - \frac{1}{4} \right] (-2 \log M) = - \left(\nu + \frac{1}{2} \right) \log M.$$

Now if k is large enough, $\sigma_k > 1$ and $6\nu\sigma_k^2 > 3\mu_k^2 + 2\nu$. Then

$$1 + \frac{(x - \mu_k)^2}{\nu\sigma_k^2} \geq \frac{1}{3\sigma_k^2} \left(1 + \frac{x^2}{\nu}\right)$$

for all x , by a short calculation. Thus $j_k(x) \geq 1/(3\sigma_k^2)$ and

$$\frac{\nu + 1}{2} \int_{|x| > M} \log j_k(x) dP_k(x) \geq \frac{1}{4}(-2 \log \sigma_k - \log 3).$$

Combining this with (91) and by (71) it follows for a constant α that as $k \rightarrow \infty$, $P_k h(\mu_k, \sigma_k) \geq (\log \sigma_k)/2 - \alpha \rightarrow +\infty$. But since $P_k h(0, 1) \equiv 0$, this contradicts the assumption that (μ_k, σ_k) give the M-functional of P_k and so completes the proof of continuity of (μ_ν, σ_ν) for weak convergence. Since $Q_n \rightarrow Q$ weakly a.s. for the empirical measures Q_n of Q (by the Glivenko-Cantelli and Helly-Bray theorems), part (d) and Theorem 9 are proved. \square

Remark. For $\nu > 1$, although (μ_ν, σ_ν) is defined and weakly continuous at all laws, it is not Lipschitz at some boundary points (for any norm): in Lemma 27, let $Q_\varepsilon := q_\varepsilon \delta_0 + p_\varepsilon \delta_1$ where $p := p_\varepsilon := (\nu - \varepsilon)/(\nu + 1)$ and $q := q_\varepsilon := (1 + \varepsilon)/(\nu + 1)$, $\varepsilon > 0$. In (81) we find that $\sigma_{p_\varepsilon}^2 = \varepsilon/(\nu - 1) + O(\varepsilon^2)$ as $\varepsilon \downarrow 0$. Let $\|\cdot\|$ be any norm defined on finite signed measures on \mathbb{R} , of which $\|\cdot\|_{BL}^*$ is just one example. Then

$$(92) \quad \|Q_\varepsilon - Q_0\| = \varepsilon \|\delta_1 - \delta_0\| / (\nu + 1),$$

$$(93) \quad |\sigma_\nu(Q_\varepsilon) - \sigma_\nu(Q_0)| = \sigma_\nu(Q_\varepsilon) \sim \sqrt{\varepsilon/(\nu - 1)}$$

as $\varepsilon \downarrow 0$. Thus $Q \mapsto \sigma_\nu(Q)$ is not Lipschitz and hence not Fréchet differentiable at Q_0 with respect to the norm $\|\cdot\|$, whatever it may be. Also, σ_ν^2 is not differentiable at Q_0 since $d\sigma_\nu^2(Q_\varepsilon)/d\varepsilon$ has left limit 0 and right limit $1/(\nu - 1) > 0$ at $\varepsilon = 0$.

10. APPENDIX

Derivatives in Banach spaces. Fréchet differentiability is often defined by statisticians, e.g. Huber (1981, §2.5), for functionals defined on the convex set of probability measures. As long as the definition is for a norm, this usually seems to cause no problems. But, in this paper, we need to apply implicit function theorems which require that a function(al) be defined on an open set in a Banach space. Thus we need the set U in the following usual mathematicians' definition of Fréchet differentiability to be open. No set of probability measures is open in any Banach space of signed measures.

Let X and Y be Banach spaces over the real numbers. Let $B(X, Y)$ be the space of bounded, i.e. continuous, linear operators A from X into Y , with the norm $\|A\| := \sup\{\|Ax\| : \|x\| = 1\}$. Let U be an open subset of X , $x \in U$,

and f a function from U into Y . Then f is said to be *Fréchet differentiable* at x iff there is an $A \in B(X, Y)$ such that

$$f(u) = f(x) + A(u - x) + o(\|u - x\|)$$

as $u \rightarrow x$. If so let $(Df)(x) := A$. Then f is said to be C^1 on U if it is Fréchet differentiable at each $x \in U$ and $x \mapsto Df(x)$ is continuous from U into $B(X, Y)$. Iterating the definition, the second derivative $D^2f(x) = D(Df)(x)$, if it exists for a given x , is in $B(X, B(X, Y))$, and the k th derivative $D^k f(x)$ will be in $B(X, B(X, \dots, B(X, Y)))$ with k B 's. Then f is called C^k on U if its k th derivative exists and is continuous on U . If f is C^k on U for all $k = 1, 2, \dots$, it is called C^∞ on U . In some cases, higher order derivatives will be seen to simplify or to reduce to more familiar notions.

Suppose X is a finite-dimensional space \mathbb{R}^d . Let e_1, \dots, e_d be the standard basis vectors of \mathbb{R}^d . If $x \in U$, an open set in \mathbb{R}^d , and $f : U \rightarrow Y$, partial derivatives are defined by $\partial f(x)/\partial x_j := \lim_{t \rightarrow 0} [f(x + te_j) - f(x)]/t$, the usual definition except that the functions are Y -valued. Just as for real-valued functions, f is C^1 from U into Y if and only if each $\partial f/\partial x_j$ for $j = 1, \dots, d$ exists and is continuous from U into Y , e.g. by Dieudonné [1960, (8.9.1)] and induction on d . Any linear map A from \mathbb{R}^d into Y is automatically continuous and is given by $A(x) \equiv \sum_{j=1}^d x_j A_j$ for some $A_j \in Y$, so we can identify A with $\{A_j\}_{j=1}^d \in Y^d$. Then if $Df(x)$ exists, each $\partial f(x)/\partial x_j$ exists and $Df(x) = \{\partial f(x)/\partial x_j\}_{j=1}^d$.

Again as for real-valued functions, we can define higher-order partial derivatives if they exist. Then, f is C^k from $U \subset \mathbb{R}^d$ into Y if and only if each partial derivative $D^p f(x) := \partial^{[p]} f / \partial x_1^{p_1} \dots \partial x_d^{p_d}$, with $p := (p_1, \dots, p_d)$ and $[p] := p_1 + \dots + p_d \leq k$, exists and is continuous from U into Y , e.g. by Dieudonné [1960, (8.9.1), (8.12.8)] and induction.

If $Y = \mathbb{R}^m$ is also finite-dimensional, we have $f(u) \equiv \{f_i(u)\}_{i=1}^m$ for some $f_i : U \rightarrow \mathbb{R}$, $i = 1, \dots, m$, and $\partial f(x)/\partial x_j = \{\partial f_i(x)/\partial x_j\}_{i=1}^m$ for each $j = 1, \dots, d$, if either the partial derivative on the left, or each one on the right, exists: Dieudonné [1960, (8.12.6)].

Let X and Y be real vector spaces. For $k \geq 1$, a mapping $T : (x_1, \dots, x_k) \mapsto T(x_1, \dots, x_k)$ from X^k into Y is called k -linear iff for each $j = 1, \dots, k$, T is linear in x_j if x_i for $i \neq j$ are fixed. T is called *symmetric* iff for each $\pi \in S_k$, the set of all permutations of $\{1, \dots, k\}$, we have $T(x_{\pi(1)}, \dots, x_{\pi(k)}) \equiv T(x_1, \dots, x_k)$. Any k -linear mapping T has a *symmetrization* T_s , which is symmetric, with

$$T_s(x_1, \dots, x_k) := \frac{1}{k!} \sum_{\pi \in S_k} T(x_{\pi(1)}, \dots, x_{\pi(k)}).$$

A function g from X into Y is called a k -homogeneous polynomial iff for some k -linear $T : X^k \rightarrow Y$, we have $g(x) \equiv g_T(x) := T(x, x, \dots, x)$ for all $x \in X$. Since $g_{T_s} \equiv g_T$ one can assume that T is symmetric. For the following, one can obtain T from g by the ‘‘polarization identity,’’ e.g. Chae (1985), Theorem 4.6.

Proposition 29. *For any two real vector spaces X and Y and $k = 1, 2, \dots$, there is a 1-1 correspondence between symmetric k -linear mappings T from X^k into Y and k -homogeneous polynomials $g = g_T$ from X into Y .*

Now suppose $(X, \|\cdot\|)$ and $(Y, |\cdot|)$ are normed vector spaces. It is known and not hard to show that a k -linear mapping T from X^k into Y is jointly continuous if and only if

$$\|T\| := \sup\{|T(x_1, \dots, x_k)| : \|x_1\| = \dots = \|x_k\| = 1\} < \infty,$$

and that a k -homogeneous polynomial g from X into Y is continuous if and only if $\|g\| := \sup\{|g(x)| : \|x\| = 1\} < \infty$. In general, for a symmetric k -linear T with $\|T\| < \infty$ we have $\|g_T\| \leq \|T\| \leq k^k \|g_T\|/k!$, e.g. Chae (1985), Theorem 4.13. The bounds are sharp in general Banach spaces [Kopeć and Musielak (1956)] but if X is a Hilbert space we have $\|g_T\| \equiv \|T\|$ [Bochnak and Siciak (1971)].

If f is a C^k function from an open set $U \subset X$ into Y then at each $x \in U$, $D^k f(x)$ defines a k -linear mapping $d^k f(x)$ from X^k into Y ,

$$(94) \quad d^k f(x)(x_1, \dots, x_k) := (\dots((D^k f)(x)(x_1))(x_2) \dots (x_k)).$$

Then $d^k f(x)$ is symmetric, e.g. Chae (1985), Theorem 7.9. The corresponding k -homogeneous polynomial $u \mapsto g_{d^k f(x)}(u)$ will be written as $u \mapsto d^k f(x)u^{\otimes k}$.

Also, f will be called *analytic* from U into Y iff it is C^∞ and for each $x \in U$ there exist an $r > 0$ and k -homogeneous polynomials V_k from X into Y for each $k \geq 1$ such that for any $v \in X$ with $\|v - x\| < r$, we have $v \in U$ and

$$(95) \quad f(v) = f(x) + \sum_{k=1}^{\infty} V_k(v - x).$$

It is known that then necessarily for each $k \geq 1$ and $u \in X$

$$(96) \quad V_k(u) = d^k f(x)u^{\otimes k}/k!.$$

For any Banach space X let $(X', \|\cdot\|')$ be the dual Banach space $B(X, \mathbb{R})$. The product $X' \times X$ with coordinatewise operations is a vector space and a Banach space with the norm $\|(\phi, x)\| := \|\phi\|' + \|x\|$. The mapping $\gamma : (\phi, x) \mapsto \phi(x)$ is C^∞ from $X' \times X$ into \mathbb{R} (it is analytic and a 2-homogeneous polynomial): for $\psi, \phi \in X'$ and $x, y \in X$ we have

$$\gamma(\psi, y) = \psi(y) = \phi(x) + (\psi - \phi)(x) + \phi(y - x) + (\psi - \phi)(y - x).$$

As $(\psi, y) \rightarrow (\phi, x)$, clearly $(\psi - \phi)(x)$ and $\phi(y - x)$ give first derivative terms and $(\psi - \phi)(y - x)$ a second derivative term. We have that $D\gamma$ is continuous (linear) and $D^2\gamma$ has a fixed value $(\eta, u) \mapsto ((\zeta, v) \mapsto \eta(v) + \zeta(u))$ in $B(X' \times X, B(X' \times X, \mathbb{R}))$, so $D^3\gamma \equiv 0$.

If U is an open subset of a Banach space Y and f is a C^k function from U into X , then

$$(97) \quad (\phi, u) \mapsto \phi(f(u))$$

is C^k on $X' \times U$ by a chain rule, e.g. Dieudonné [1960, (8.12.10)].

For a point x in a normed space $(X, \|\cdot\|)$ denote the open ball of radius r around x by $B_r(x) := \{y \in X : \|y - x\| < r\}$. The Hildebrandt-Graves implicit function theorem and related facts, essentially as stated by Deimling (1985, Theorem 15.1 p. 148, Corollary 15.1 p. 150, and Theorem 15.3 p. 151) are as follows:

Theorem 30. *Let X, Y, Z be real Banach spaces, $U \subset X$ and $V \subset Y$ neighborhoods of x_0 and y_0 respectively. Let $F : U \times V \rightarrow Z$ be jointly continuous, and continuously differentiable with respect to $y \in V$. Let F_2 be the (partial Fréchet) derivative of F with respect to $y \in V$, so that for each $x \in U$ and $y \in V$, $F_2(x, y)(\cdot)$ is a bounded linear operator from Y into Z . Suppose that $F(x_0, y_0) = 0$ and that $F_2(x_0, y_0)(\cdot)$ is onto Z and has a bounded inverse, i.e. it is a topological isomorphism of Y onto Z . Then there exist $r > 0$, $\delta > 0$ with $B_r(x_0) \subset U$ and $B_\delta(y_0) \subset V$ such that there is exactly one map T from $B_r(x_0)$ into $B_\delta(y_0)$ with $F(x, T(x)) = 0$ for all $x \in B_r(x_0)$, and:*

(a) T is continuous.

(b) If for some $m \geq 1$, $F \in C^m(U \times V)$, then for some ρ with $0 < \rho < r$, T is C^m on $B_\rho(x_0)$.

(c) If F is analytic on $U \times V$ then for some τ with $0 < \tau < r$, T is analytic on $B_\tau(x_0)$.

The two Banach spaces Y and Z are topologically isomorphic if they are finite-dimensional and of the same dimension, e.g. both are \mathbb{R}^d or both are \mathcal{S}_d as in the present paper. Then we need that the linear transformation $F_2(x_0, y_0)(\cdot)$, or the associated matrix of partial derivatives in coordinates, is non-singular.

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