RATING SYSTEMS AND LEARNING

by

WARREN DOUGLAS SMITH

Submitted to the departments of MATHEMATICS and PHYSICS in partial fulfillment of requirements for a

DOUBLE B.S. IN MATH AND PHYSICS

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

January 1984

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Learning and Rating Systems Abstract Draft 2.0 Warren Smith

This thesis will concern rating systems: systems that produce quantitative measures of the ability of players in a league, based on pairwise comparison. By "quantitative", it is meant that win odds for a game between two players in the league may be estimated from their ratings. This subject may be of interest to gamblers, gaming leagues, psychologists, consumer groups, and industry. Two kinds of rating systems, 'static' and 'dynamic' systems will be considered; questions addressed will include definitions, rating system design, noise in rating systems, and distribution of ratings. Side topics will include a 'theory of learning' and various models of games. There will be some confirmation of theory by experiment. Includes a short bibliography.

This thesis will be one of very few (two?) mathematically well founded studies of rating systems, and is the first (as far as I know) to consider distributions of ratings and rating system noise.

Warren Douglas Smith Author **Richard Stanley Thesis Supervisor**

Rating systems and learning Table of Contents Warren Smith

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Introduction

The best way to introduce the reader to the idea of rating systems is to describe some actual rating systems. The most widely used and general rating system is the Elo system for rating chess players. Based on win/loss tournament results, the Elo system assigns "ratings" (numbers) to each player in the league. (A "league" is a set of players.) A player's rating is a quantitative measure of his ability in chess. Thus a rating of 1300 means the player is an average tournament player, a rating of 2200 is a master, and the world champion has a rating of 2750 or so. (The higher the rating, the better the player.) Elo ratings are a quantitative measure of ability because, by comparing two players' ratings, one may deduce an approximate value for the probability that one player will beat the other at chess. (E.g. in the Elo system, a rating difference of 100 points implies that the higher ranked player will win about 3 times for every time that he loses.)

The Elo system was designed by Arpad E. Elo and is described in [Elo].

Other rating systems in common use in the sports world are the:

"computer ranking system" of the Association of Tennis Professionals, which produces an ordered list of tennis players, e.g. it (1983) ranks Ivan Lendl 3rd in the world. I have not been able to get information about the algorithm used by the AOTP.

"New York Times Computer Football Rankings" appearing every friday in the sports section of the New York Times. This system rates the teams in the college top 20 and NFL. The system assigns the top team a rating of 1.000 and gives the other teams ratings between 0 and 1 in a way that "reflects strength... relative to the top team." Based on the description published with the rankings, the algorithm is sophisticated and highly football specific. (Unlike the Elo chess system, which is readily applicable to games other than chess.) The N.Y.T. system also has the disadvantage, relative to the Elo system, that there is no published way to generate game result odds based on ratings.

One could also imagine applying the ideas of rating systems to the design of quantitative psychological tests or to scholastic competitions.

In mathematical terms, what do rating systems do? The most general rating system would use as input data a chronologically ordered list of tournament results, where by "tournament result", I I mean a three-tuple

(player1, player2, game result),

where player1 and player2 are different players in the league,

and game result is a member of a finite or infinite
set of "results" of a game between players 1 and 2.

The rating system would produce as output a list of ratings (reals), one for each player in the league.

Further, each rating system would have associated with it a function which, given any two ratings as input, would output a probability distribution indicating approximate values for the probabilities that any given game result will happen in a game between two players with the given ratings.

The above picture will be called the "Generalized Game".

Problems to be Surmounted

Rating Triangles.

We will refer to situations in which A beats B, B beats C, and C beats A consistently as "rating triangles". It will be seen that it is theoretically possible to design rating systems which can predict and successfully handle rating triangles. However these systems are all impractical because they use far too much information and require the rating system to be too game specific for our taste. We will specifically eliminate these systems from consideration in this paper after a brief discussion in part II. There is then no longer any way to get around the rating triangle problem; the best we can do is to hope that whatever game we are rating does not suffer too severely from triangles. Experiment shows that this is the case for most human players of real life games; probably the reason for this is that the existence of triangles depends on the existence of specific player strengths and weaknesses. Human players (unlike imaginary players) tend to compensate for perceived weaknesses and to try for well rounded play - thus avoiding rating triangles.

Notation and Conventions

I will indicate references by [xxx pg y] where xxx is a cryptic string referring to a code in the bibliography. References to [Elo] will be by item number rather than page number. References to [HOMF] may be to either formula number or page number, depending on context.

Formula numbers will be in square brackets, e.g. [20]. Formulas will generally be counted by ten.

I will often use variable names that are more than one letter long. All variables will be separated by spaces, so there should be no trouble distinguishing "ab" and "a b".

Occasionally I will save typing labor by using square brackets to denote subscripting, e.g.

$$A_b$$

will be written

A[b].

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Defining a Rating System.

The essential components of a rating system are
1. machinery for assigning ratings to each player based on tournament results,
2. a function which can be used to deduce approximate probabilities for results of games between any two players who have ratings.

This section will be concerned with the second component.

Thus, let a game have n possible results (scores). We desire a function

$$[10] \qquad wp(r1,r2): \mathbb{R}^2 \text{ ----> } (\text{probability})^n ,$$

which, given the ratings of two players, returns the probabilities of each possible result of a match between them.

We will say that the function wp "defines" the rating system.

Is it possible for a rating system to be defined that will always yield exact win probabilities for conventional games such as chess or tennis? The answer is yes, although such a rating system would be quite impractical. The reason why it is possible to create such an "exact" rating system is because everything about each player may be encoded in their (real number) rating. To make matters more concrete, let us consider the game tree of chess. (Tennis may also be thought of as possessing a game tree; given the positions and momentums of the ball and players, (as nodes) the branches of the tree will consist of all the possible actions the players can take at that moment in time. This will be a very large game tree, but a finite one, assuming that we take a quantum mechanical view of nature. Similarly, every conventional game I can think of may be described by a game tree.) Now every branch of the tree may be labelled by the probability that a player would choose that branch in a game, assuming he started at the position at the branch's top node. Such a labelled tree completely describes a player. Further, Cantorian theory makes it clear that any such tree may be completely encoded as one very high precision real number. Suppose that each player was given his own labelled, encoded game tree as his rating! It is clear that a function wp(r1,r2) could be designed that would take two such encoded trees and return the probability that one tree won vs the other.

Clearly the rating system defined by the function described above will be "exact". Such systems are of some theoretical interest because they can easily handle rating triangles without any contradictions, etcetera.

However this kind of system is incredibly impractical! Even if one could find some way of determining the labelled game trees for some player (perhaps by taking him apart) the amount of information in each tree would be staggering.

We conclude that it is essential to limit the amount of information in our rating systems.

This is not as easy as one might expect. For example, a little thought will show that demanding that the function wp(r1,r2) be continuous will not eliminate tree encoder systems. However, demanding that wp(r1,r2) be differentiable in r1 and r2 will work, and agrees with intuitive notions about what properties a player's rating should have; e.g. a small change in one's rating should result in a small proportional change in one's game result probabilities.

Therefore from now on we will assume that wp(r1,r2) is differentiable.

Another condition that one would like wp(r1,r2) to obey is that, as r1+r2 remains constant and r1-r2 becomes greater, we would like the game probability vector wp(r1,r2) to reflect "better play" by the first player. For an abstract set of possible game results "S", it is unclear how to define this notion in general. Therefore, from now on we will only consider "binary games" with two possible results: (from the viewpoint of the first player)
"win" or "loss".
(See Good for a way to handle "ternary" games with a possible "draw" result. Elo, on the other hand, considers it `inordinately laborious' to consider draws in real rating systems [8.91 Elo].)

Then we may define a function

[20] wp(r1,r2): $\mathbb{R}^2$ ------> Probability,

which, given two ratings r1 and r2, gives an approximate probability that player #1 would beat player #2 in a match between them.

If the rating system is to be applied to a "binary symmetric game" ("BSG") (A BSG is a binary game in which one's winning chances are not affected by whether one is the first or second player. Example: "Modified chess", a game in which [players are assigned colors at random and play chess; in the event of draws, they continue playing more games until one of them wins] is a BSG.) then we may assume that

[30] wp(r1,r2) + wp(r2,r1) = 1

Finally, the rest of this paper will be almost exclusively concerned with "linear" BSG rating systems, for which wp(r1,r2) depends only on r1-r2 and not on r1+r2.

I know of no way to rigorously justify my (tacit) assumption that self consistent (or nearly self consistent) linear rating systems exist that will work well for most real BSG's, although some plausibility arguments may be made. (E.g. the "Timesharing" and "WM^n" BSG models to be discussed later in this paper have consistent linear rating systems; if they are assumed to be widely applicable models of games, then it follows that the linear rating system assumption is widely applicable.)

However, despite its arbitrariness, the restriction of our study to linear rating systems reaps great advantages in mathematical simplicity, from both the theoretical and from the applications standpoint. Some of these advantages follow from the fact that certain equations become linear, others follow from the fact that only linear rating systems do not have a "special" zero rating; the ratings are isotropic. (Although, as a matter of fact, we will reintroduce the concept of the zero rating in the section of this paper on rating distributions.)

For linear BSG's, we may define the rating system with the real valued, real domained function

[40] wp(ΔR)=(probability that player with rating r1 will beat player with rating r2, for ΔR=r1-r2).

and require this function to obey:

$$wp'(x)>0 \quad \forall x \in \mathbb{R}$$

$wp(-\infty)=0, \quad wp(+\infty)=1$

$wp(x) + wp(-x) = 1.$

In order to further restrict wp(x), we will need to create models of games and of ability and study them. In this section, I will tacitly assume that the rating systems we are discussing are BSG linear systems.

The first and simplest model of ability that we will consider will be the "Timesharing Model". The essential point of the timesharing model will be as follows. Consider two players "X" and "Y". Let X beat Y with probability PXY > 1/2. Then we will model the situation by saying that during a fraction 2(1-PXY) of the time X plays "on the same level" as Y. The rest of the time, X plays at an unbeatably higher level. ("At the same level" means wins 1/2 the time, loses the other 1/2.) Then let us consider three players "A", "B", and "C", listed in order of decreasing ability. Then (Using similar variable definitions)

$$[50] \quad PAC = 2\,PAB - 1 + 2\,(1-PAB)\,PBC \qquad \text{if } PAC, PBC, PAB > 1/2$$

Now let RA = A's rating, RB = B's rating, etc. If we let $x = RA-RB$, $y = RB-RC$, then

$$[60] \quad wp(x+y) = 2\,[wp(x) + wp(y) - wp(x)\,wp(y)] - 1 \qquad \text{for } x,y > 0$$

Any function wp(ΔR) which satisfies the above functional equation for all real $x,y>0$ will define a self-consistent rating system for timesharing players. And indeed the functional equation may be solved. We set $h(x)=1-wp(x)$ to get:

$$1-h(x+y) = 2\,(2 - h(x) - h(y) - (1-h(x))\,(1-h(y))) - 1$$

$$1-h(x+y) = 1 - 2\,h(x)\,h(y)$$

$$[70] \quad h(x+y) = 2\,h(x)\,h(y) \qquad \text{for } x,y > 0$$

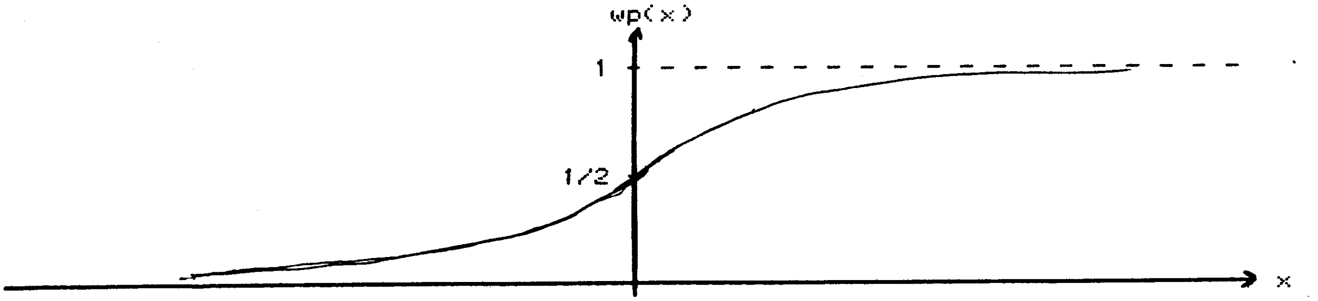
Recalling that $h(0)=1-wp(0)=1/2$ and that $h'(x)=-wp'(x)<0$ now forces the general solution:

$$h(x) = 2^{-(Q\,x + 1)} \qquad \text{for any } Q,x>0$$

And now symmetry gives us the general solution for all real x:

$$[80] \quad wp(x) = 1/2\,[\,1 - (2^{-|Q\,x|} - 1)\,\text{sign}(Q\,x)\,]$$

A Graph of wp(x) in [80]



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We will now consider a slightly more complicated BSG model, the "Wrong Move" ("WM") model. This model involves an imaginary game, the "WM game". The rules of WM are as follows:

> Players make moves simultaneously. (If one were to move first, the game would not be symmetric...) There are two kinds of moves a player can make: "wrong" moves and "right" moves. If both players make the same kind of move, the game continues. If one player makes a "wrong" move while his opponent makes a "right" move, the "right" mover wins the game, and the game is over.

We will assume that each player X makes "right" moves with probability PX, and that all moves are independent.

Then the probability that A beats B is (with x = RA-RB = A's rating - B's rating)

[90]

$$wp(x) = \sum_{n=0}^{\infty} [\, PA\ PB + (1-PA)\ (1-PB)\,]^{n} \quad PA\ (1-PB) \;=\; \frac{PA\ (1-PB)}{PA\ (1-PB) + PB\ (1-PA)}$$

if we similarly set y=RB-RC, then it is a matter of simple algebra to show that

$$[100] \qquad wp(x+y) = \frac{wp(x)\ wp(y)}{wp(x)\ wp(y) + (1-wp(x))\ (1-wp(y))}$$

where wp(y)=prob(B beats C) and wp(x+y)=prob(A beats C).

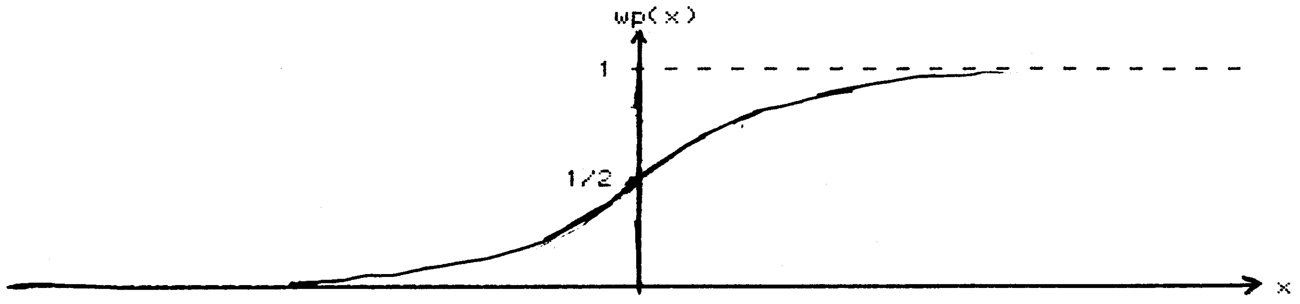
This functional equation may be solved as follows. If it has a solution, then this solution clearly must hold for y infinitesimal. The differential equation that results from taking y small (with boundary conditions wp(0)=1/2) may be solved to yield

$$[110] \qquad wp(x) = \frac{1}{1 + \exp(-Q\ x)} \qquad \text{for any positive } Q$$

This must be the solution if there is one; we will now show that this is, in fact, a solution by back substituting into the functional equation.

$$\frac{wp(x)\ wp(y)}{wp(x)\ wp(y)+(1-wp(x))(1-wp(y))} = \frac{1}{1 + \exp(-Q\ x)\ \exp(-Q\ y)} = wp(x+y)$$

Graph of wp(x) in [110]



Consider a player who makes right moves with probability 1/2. Let this player have rating "$R_{\frac{1}{2}}$ ". Then if a player whose rating is R (and who makes correct moves with probability p) plays the $R_{\frac{1}{2}}$ player, he will win with probability (by [90])

$$[130] \qquad \frac{1}{1 + \exp(-Q\,[R - R_{\frac{1}{2}}\,])} = \frac{1/2\ p}{1/2\ p + 1/2\ (1-p)} = p$$

This equation relates rating R to right move frequency p in WM.

Another useful quantity is the expected length of a WM game (in number of moves). This is

$$E(\text{length}) = \sum_{k=1}^{\infty} k\ [PA\ PB + (1-PA)\ (1-PB)]^{k-1}\ [PA\ (1-PB) + PB\ (1-PA)]$$

$$[140] \qquad = \frac{1}{PA\ (1-PB) + PB\ (1-PA)}$$

Finally, a handy equation based on [110] is

$$[145] \qquad wp'(x) = \frac{Q\ \exp(Q\ x)}{[1 + \exp(Q\ x)]^{2}}$$

The wrong move model might at first be thought to be a simplistic way to model mistakes in games. One can imagine a great number of models that are more "sophisticated" than the WM model, but which are based on the same fundamental idea. For example, one could imagine a model in which one needs to make two more mistakes than one's opponent, followed by a period of 5 consecutive correct moves by one's opponent, in order to lose.

However, the WM model is more widely applicable than it appears. It will soon become clear that a very wide class of these "more sophisticated" models have the same expression for wp(x) as the WM model!

> Why is the WM model so widely applicable? I believe that the following interpretation of the WM game may help answer this question.
>
> 1. Every real life game can be thought of as having a finite game tree. (as discussed earlier.)
> 2. Every move in this tree, with the aid of fairly simple arbitrary definitions, may be thought of as either "right" or "wrong".
> 3. The WM model (and also the WM^n model to be discussed) is basically equivalent to thinking of the game tree as a homogeneous entity - ignoring the complexity that arises from looking too closely at the tree, one envisions the tree as populated by uniform densities of right and wrong moves.

As an example of a wide class of game models, we will consider the "WM^n" model. The WM^n model will in itself be a countably infinite set of models. However, many other plausible WM type models, even though not members of the set of models embraced by WM^n, will be similar enough to some members of WM^n that they will be well approximated.

The WM^n model will involve the same imaginary game as the WM model, except that play will continue until one player has made n more mistakes than his opponent, for n some positive integer. (For n=1, the WM^n model reduces to the WM model.) The more mistaken player will then lose the game.

This model may be mathematically attacked by reducing it to a "restricted random walk" problem, which is a standard problem in basic probability. We look upon the WM^n game as a kind of chain of WM games, all of which have equal win/loss probabilities.

$$\begin{array}{c}\text{WIN for}\\\text{"leftward"}\end{array}\quad \overbrace{\text{WM}\quad \text{WM}\ \ldots\ \text{WM}}^{\text{n WM games}}\quad \text{START(WM)}\quad \overbrace{\text{WM}\ \ldots\ \text{WM}\quad \text{WM}}^{\text{n WM games}}\quad \begin{array}{c}\text{WIN for}\\\text{"rightward"}\end{array}$$

The rules of WM^n may be restated in terms of the "chain" diagram above as:
Two players, the "rightward" and the "leftward" player, participate in WM^n. They start play at the "START" node of the chain. At each node, they play a game of WM. Both players then walk over to the neighboring node in the direction of the WM winner's name, and play continues. The game is over when a "win" node is reached.

Now let R be the probability that the rightward player will make nonwrong moves, and let L be the probability that the leftward player will make nonwrong moves. Further, let

$$[150] \qquad p = \frac{R\,(1-L)}{R\,(1-L) + L\,(1-R)}$$

be the probability that the rightward player would beat the leftward at a game of WM^1, and let

$$[160] \qquad q = 1-p = \frac{L\,(1-R)}{R\,(1-L) + L\,(1-R)} \quad .$$

The problem is clearly a restricted random walk of the type treated in Hoel, Port, and Stone (eqn 21 pg. 222), and we may immediately write

$$[170] \qquad \text{Prob(right beats left)} = \frac{1-(q/p)^{n}}{1-(q/p)^{2\,n}} = \frac{p^{n}}{p^{n} + q^{n}}$$

or

$$[180] \qquad \text{PRL} = \text{Prob(right beats left)} = \frac{[R\,(1-L)]^{n}}{[R\,(1-L)]^{n} + [L\,(1-R)]^{n}}$$

It is now a matter of algebra to see that (Where I am introducing a third player with right move probability K and all symbols have usual meaning)

$$[190] \qquad \text{PRK} = \frac{\text{PRL PLK}}{\text{PRL PLK} + (1-\text{PRL})(1-\text{PLK})} = \frac{[L\,(1-K)\,R\,(1-L)]^{n}}{[L\,(1-K)\,R\,(1-L)]^{n} + [K\,(1-L)\,L\,(1-R)]^{n}}$$

(The lefthand equality is made clear by cancellation of the terms involving L in the numerator & denominator of the expression on the far right.)

Now if we set x=(R's rating-L's) and y=(L's rating-K's) we get

$$[200] \qquad wp(x+y) = \frac{wp(x)\ wp(y)}{wp(x)\ wp(y) + (1-wp(x))\ (1-wp(y))}$$

The same functional equation as for the WM model with obviously the same solution [110].

We may also relate rating to right move probabilities in a similar way as before by considering a player who makes right moves 1/2 the time, and who has rating $X_{\frac{1}{2}}$; using L=1/2 in [180] gives

$$[210] \qquad \frac{1}{1+\exp(-Q\ [X-X_{\frac{1}{2}}\ ])} = \frac{R^n}{R^n + (1-R)^n} = \frac{1}{1 + [(1-R)/R]^n}$$

This leads to this relation between rating X and right move probability R:

$$[220] \qquad \frac{1}{1 + \exp(\ -Q/n\ [X-X_{\frac{1}{2}}\ ])} = R$$

This is the same as [130] except that Q is divided by n.

By means of [equation 24 page 222 Hoel, Port, Stone] we may write down the expected number of moves in a game of WM^n. It is

$$[230] \qquad E(\text{length of } WM^n) = \frac{n}{R-L}\ \frac{[R\ (1-L)]^n - [L\ (1-R)]^n}{[R\ (1-L)]^n + [L\ (1-R)]^n}$$

One major objection to the WM^n model is the behavior of the above formula for small n or R; namely, one gets small game lengths! In most real life games, no matter how badly one plays, the games will always last at least some constant length.
(Examples: It is rare for chess games to be shorter than 20 moves. Basketball, soccer, etc. games have fixed durations.)
In these games, the victor is a foregone conclusion long before the end of the game, so further play seems superfluous, but nevertheless it occurs. To model these games, I will introduce the game of "long WM^n", a game with the exact same rules as WM^n, except that if a the WM^n game was shorter than KL moves, then the players will play on until the length has become KL moves. (Although this "extra" part of the game will have no effect on the determination of victory.) Then

$$[240] \qquad \max[KL,\ E(WM^n\ \text{length})] < E(\text{length of long } WM^n) < KL + E(WM^n\ \text{length})$$

If both players of a WM game are very good, then the game tends to take a very long time. This might also be a valid objection to WM^n if it is to be applied to some games. (But it cannot be a valid objection to games like modified chess, which do take large amounts of time if both players are very good. For games with "overtime", or "extra innings", one could model the overtime as a WM^n game, which implies that modelling the entire game as a long WM^n game is acceptable.) I will not consider any WM^n fix to handle this objection, because I consider it to be an unimportant objection. Why? Because in real life, whenever one game will be insufficient to clearly demonstrate the superiority of one team over another, a match based on the "modified chess" principle or some other game lengthening principle is arranged. Besides, real life games have been purposely designed so that they are rarely too short.

It may seem pointless to introduce the long WM^n game, because the only way it differs from the WM game is in game length. The win probabilities, rating system, etc. are all exactly the same for the WM^n and the long WM^n game. However, there is a point to this maneuvering, and it will appear later in this paper when we will consider a theory of learning in games; this theory will involve the expected game length as a parameter.

It is also possible to consider a still more general variant of the WM model, which I call the "RL game" and is discussed in appendix 1. The RL game is general enough to include the WM^n model as a special case, and sophisticated enough to have rating triangles.

All of the linear BSG models discussed above have obeyed the "star property".
Star property:
[250] A differentiable function wp(x) obeys the star property iff

$$wp(x) + wp(-x) = 1, \quad wp(\infty) = 1, \quad wp'(x) > 0 \ \forall x, \quad \text{and}$$

$$wp(x) \sim \exp(Q\ x) \quad \text{for some positive } Q \text{ for } x \text{ large and negative.}$$

This leads me to propose the following [260]
Star Property Assumption:
Any win probability function wp(x) obeying the star property will define a linear BSG ranking system that will be satisfactory for application to most real world BSG's.

I will mostly use the WM form [110] for wp(x) in the future, however, because it is analytic and of simple form.

-----------------Part 4   Draft 2.0   Warren Smith-------------------

Static Rating Systems

At this point we have laid enough of a foundation to begin discussing actual rating systems. The first kind of rating system that we will consider will be the "static" rating system. The purpose of static rating systems will be to use the following input:

1. a win probability function that defines the rating system.

2. a set of tournament results "T", assumed independent. ("set of tournament results" means a subset of
   {(a,b): a,b different players} X (all possible results of one game) .)

to assign ratings to each player.

Let "R" represent an assignment of ratings to each player. Then we know Prob(T|R), the probability that given the match pairings of T between players rated with ratings R, the game results of T would happen, because our rating system is defined. (Strictly speaking, what I have called "Prob(T|R)" should be written "Prob(T[Results] | T[match pairings] and R)" .)

In fact, up to constants of proportionality (that reflect the many possible unknown time orderings T may have, and R normalization requirements; these constants being independent of R) we may immediately write down Prob(T|R):

$$[270] \qquad \mathrm{Prob}(T|R) \propto \prod_{(a,b,S)\in T} wp(R_a, R_b)_S$$

where wp is the function which defines the ranking system, i.e.

$$wp(x,y)_S$$

is the expected probability that, if a player with rating x played with one with rating y, the result of the game would be S, (as in [10])

and where R_x

is the rating of player x.

[270] follows immediately from the assumption that all games are independent, and from the law that the Prob(A $\cap$ B) = Prob(A) Prob(B) if A and B are independent events.

Now Prob(R|T) may be written by use of Bayes' law of probability:

$$[280] \qquad \mathrm{Prob}(R|T) \propto \mathrm{Prob}(T|R)\ \mathrm{Prob}(R)$$

Prob(R) may be written in terms of the probability density function for ratings x, pd(x):

$$[290] \qquad \mathrm{Prob}(R)\ d^{\#\ \text{players}}\vec{R} = \prod_{\substack{\text{all ratings}\\ x\in R}} pd(x)\ dx$$

The left hand side of this equation represents the probability that the (vector) rating assignment will fall inside a differential (# players) box.

It is NOT permissible to assume that all ratings are equally likely. In particular, if this is done with the wp(x)'s arising from the linear BSG models discussed in the last section, then any undefeated player's most likely and expected ratings will be infinite. Elo seems to have fallen into a trap something like this one. In [3.41 Elo], Elo proposes a method which, if taken literally, will assign infinite ratings to undefeated players.

Once the probability distribution of R|T is known, we may apply an arsenal of statistical methods to determine the "best" assignment of ratings R. Two favorite choices for the "best" assignment of ratings might be the expected rating assignment and the most likely rating assignment.

The "noise" in a static rating system may also be estimated by a variety of statistical methods. E.g. a measure of noise in a given rating might be the standard deviation in that rating with the others held constant.

I cannot think of any reasons why any of these measures should give more reliable results than any of the others when applied to static rating systems.

However, because of linearity, I will prefer the expected ratings for assignments and the standard deviation for a noise measure. The use of expectation ratings with a fixed form for the rating distribution has the additional advantage that, when used as an "initiation routine" for linear dynamic rating systems (which will be discussed later) rating inflation will automatically be eliminated.

We will now specialize our discussion of static rating systems to BSG linear systems. In this case we may specify tournament results as a matrix

[300]
$$T_{i,j} = T[i,j] = \text{number of times player \#i beat player \#j}$$

Then we may write:

[310]
$$\mathrm{Prob}(R|T) \propto \prod_{i,j} \left(wp(R[i]-R[j]) \right)^{T[i,j]} \prod_{k} pd(R[k])$$

Where wp(x) is now defined as in [40], the rating of player #i is R[i], and i,j, and k range over all the players. The expectation values for the R[i] may be written

[320]
$$E(R[i]) = \frac{\int \mathrm{Prob}(R|T)\, R[i]\, d^{\#players}\vec{R}}{\int \mathrm{Prob}(R|T)\, d^{\#players}\vec{R}}$$

(Note that it is not necessary to know the constant of proportionality in [270] & [280] to obtain any of the statistical quantities having to do with the ratings.)

The variances for the R[i] may also be written:

$$\mathrm{VAR}(R[i]) = \frac{\int \mathrm{Prob}(R|T)\,(R[i] - E(R[i])^{2}\, d^{\#players}\vec{R}}{\int \mathrm{Prob}(R|T)\, d^{\#players}\vec{R}} \qquad [330]$$

Given a definition for wp(x), a form for pd(x) and T, [310], [320], and [330] are all we need to create a rating assignment complete with error estimates.

But it will be impossible to evaluate the integral expressions of [320/30] until we derive analytic forms for the densities of ratings pd. Even then, as we will see when we do this, the integrals will have to be evaluated numerically.

Alternatively, instead of using analytic expressions for pd(x), one could use a histogram of an experimental distribution and use numerical integration to evaluate [320] and [330].

--------------------Part 5 Draft 2.0 Warren Smith--------------------

We want to derive analytic expressions for the distribution of ratings. The primary tool for this purpose will be the "fluid equation" of learning.

To derive this equation, we will use the following postulates: [340]
1. There is a differentiable probability density function for ratings x, pd(x). This function does not depend on time, i.e. the rating distribution is in static equilibrium.
2. New players ("beginners") are being introduced into the player pool at a constant rate. ("birth rate") All beginners have the same ability level, represented by the rating RB.
3. The birth rate is exactly balanced by the "death rate" D (fraction of league dying per unit time). Players die at a rate that is independent of their rating.

Now let L(x) be the average learning rate of a player with rating x; a player whose "learning rate" is L has a rating R that is increasing with time t with speed (dR/dt)=L. I will postulate that L(x)>0 for all x in an interval [RB,RB+ϵ), for some positive real ϵ. (I.e. the ability level of beginners increases with time.)

Then the "fluid equation"

[350] $$0 = \frac{d\ pd(x)}{dt} = - D\ pd(x) - \frac{d[L(x)\ pd(x)]}{dx} \qquad \text{for } x>RB,\ pd(x)>0$$

with boundary condition

$$L(RB)\ pd(RB) = D$$

holds. I am calling this the "fluid equation" because of the analogy to the mass conservation equation in one dimensional fluid mechanics. Here, we are conserving the number of players. "Flow rate" of players along the rating axis is L(x) pd(x); L(x) is analagous to a flow velocity.

In any case, the first term on the RHS of the fluid equation represents deaths. The second is due to buildup of rating "flow". The boundary condition represents births; the equation is only defined for x>RB.

If L(x) is known, this equation may be solved for pd(x). First, set

[360] $$G(x) = L(x)\ pd(x).$$

Now the fluid equation may be written

[370] $$- \frac{D}{L(x)}\, dx = \frac{dG}{G} \qquad \text{with boundary condition} \quad G(RB) = D.$$

This may be integrated...

[380] $$- \int_{RB}^{x} \frac{D}{L(x)}\, dx = \ln(G(x)/D)$$

...and solved for pd(x):

$$[390] \qquad pd(x) = \frac{D}{L(x)} \exp\left(- \int_{RB}^{x} \frac{D}{L(x)}\, dx \right) \quad \text{for } x>RB. \ (pd(x) = 0 \text{ for } x<RB.)$$

Similarly, if pd(x) is known, then L(x)/D may be deduced.
The fluid equation may be integrated directly to get

$$[400] \quad L(x)\, pd(x) - D = - \int_{RB}^{x} D\, pd(x)\, dx \qquad ;$$

this may be solved for L(x)/D to get

$$[410] \qquad \frac{L(x)}{D} = \frac{1}{pd(x)} \left(1 - \int_{RB}^{x} pd(x)\, dx \right) \qquad .$$

The fluid equation may also be used to deduce rating vs. time x(t) "development" curves for an "average" player. (Avg. P. always learns at the average rate for one with his rating, but never dies.) We write

$$[420] \qquad x(0) = RB, \qquad x'(t) = L(x)$$

so that

$$[430] \qquad \int_{RB}^{x} \frac{dR}{L(R)} = t \qquad .$$

This equation may be algebraically solved (sometimes) for x(t).

To develop the reader's intuitive feel for this equation, I will list several interesting solutions of the fluid equation, along with the corresponding time development curves. In the list below I will define $y = x-RB$; and the solutions below will apply only for $y>0$.

$$[440] \quad \left[\begin{array}{l} L(x) = L_0 (L_1 y + 1)^{-P} \\ \quad (\text{for } P \geq -1) \\ \quad \Longleftrightarrow \\ \\ \text{where} \\ Z=\text{infemum}(\{y>0,\ 1(y)=0\}). \end{array} \right. \quad pd(x) = \begin{cases} \dfrac{D}{L_0} (L_1 y + 1)^{-\frac{D}{L_0 L_1} - 1} & \text{for } P = -1,\ y<Z \\[2ex] \dfrac{D}{L_0} (L_1 y + 1)^{P} \exp\left[- \dfrac{D}{L_0 L_1 (p+1)} [(L_1 y + 1)^{P+1} - 1] \right] & \text{for } P > -1,\ y<Z \end{cases}$$

$$\Longleftrightarrow \quad y(t) = \begin{cases} \frac{1}{L_1}\left(\exp\left[L_0 L_1 t\right] - 1\right) & \text{for } P = -1 \\ \frac{1}{L_1}\left(\left[L_0 L_1 (P+1) t + 1\right]^{1/(P+1)} - 1\right) & \text{for } P > -1,\ y(t)<Z \end{cases}$$

$$L(x) = L \exp(-Q y)$$

$$\Longleftrightarrow$$

$$pd(x) = \frac{D}{L} \exp\left[Q y - \frac{D}{L Q}(\exp(Q y) - 1)\right]$$

[450]

$$\Longleftrightarrow$$

$$y(t) = \ln(L Q t + 1)/Q$$

$$pd(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-((x-m)/\sigma)^2/2\right] = \frac{1}{\sigma} Z\left[\frac{x-m}{\sigma}\right]$$

$$\Longleftrightarrow$$

$$L(x) = D\sigma Q\left[\frac{x-m}{\sigma}\right]/Z\left[\frac{x-m}{\sigma}\right] \quad \text{where} \quad Q(u) = \int_u^\infty Z(v)\,dv$$

[460]

(see HOMF pg. 931-3 for information concerning Q,Z)

Note $L(x) \sim D\sigma^2/(x-m)$ for $x-m \gg \sigma$

$$L(x) \sim D\sigma^2\sqrt{2\pi}\exp\left[((x-m)/\sigma)^2/2\right] \quad \text{for } x-m \ll \sigma$$

$x(t)$ may not be solved for exactly. However for $t \gg D\sigma^2$,

$$x(t) \sim m + \frac{1}{\sigma}\sqrt{\frac{t}{D}} + O\left(\sqrt{\frac{1}{t}}\right)$$

This solution is valid only for RB = $-\infty$.

We will now develop the "heuristic model of learning", with the purpose of deriving an analytic form for L(x), and thus for pd(x).

The heuristic model will concern players of the long WM^n game discussed earlier. We will postulate that all players of the WM^n game have developed personal repertoires of heuristics that they use to determine whether a move will be correct. Each additional heuristic added to a player's repertoire gives him knowledge about the correctness of a fraction

P_{Heur}

of the moves he did not know anything about before. So if a player has h heuristics in his repertoire, he will make correct WM^n moves with probability

$$\frac{1}{1+\exp(-Q/n\,[R-R_{\frac{1}{2}}])} = a = 1 - Y\,(q_{Heur}^{\;h}) \qquad [470]$$

Where I am

> assuming that if one or more of a player's heuristics apply to some move decision, then he will move correctly with probability 1.
> assuming that if none of a player's heuristics apply to a move decision, he will move correctly anyway (i.e. through sheer luck) with probability 1-Y.
> using

$$p_{Heur} + q_{Heur} = 1. \qquad [480]$$

> calling the player's correct movement probability "a".
> calling his rating "R". The rating system is assumed to be defined by [110]; I am using [220] to relate ratings and correct movement probabilities in WM^n.

Further, I will assume that each player acts to increase his repertoire of heuristics ("learn") according to the following procedure. [490]

1. Propose a heuristic. All proposed heuristics will have a fixed probability Z of being valid heuristics.

2. Confirm or disprove the heuristic through experiments conducted while playing (rated games of) long WM^n. All players will be assumed to be playing 1 game of long WM^n per unit time interval, with all opponents in the league equally likely. It will be assumed that all hypotheses can be confirmed or disproved with a fixed number of experiments E. An "experiment" will mean any opportunity the player has to make a move decision in a game of WM in a situation in which none of his repertoire of accepted heuristics apply, but his tentative heuristic does.

3. If the tentative heuristic is valid, add it to the repertoire. If it is invalid, discard it.

4. Go back to step 1.

This procedure attempts to embody the scientific method.

We will now use the model above to derive an expression for dh/dt, relate R to h, and thus derive dR/dt = L(R).
First, we write down the expected number of experiments that our player can make in a game.

$$\overline{E}_{\text{1 game}} = \underset{\substack{\text{all}\\ \text{possible opponents}\\ \text{in league}}}{\text{Average}} \left[\text{Expected length of game with opponent} \times \text{probability that a given move can be an experiment.} \right] \qquad [500]$$

$$\simeq \text{Expected length of game with "typical" opponent} \times \text{probability that move can be an experiment.}$$

The "typical" opponent moves with correct move probability = b = a constant.

This "typical opponent" approximation will be good so long as pd(x) is sharply peaked, i.e. if we are considering large x, the condition that pd(x) falls off considerably faster than an exponential will be sufficient. Then using [240] and our "scientific method" in [500] gives for x large:

$$KL\ p_{Heur}\left((q_{Heur})^h\right) < \overline{E}_{1\ game} <$$

[510]

$$p_{Heur}\left((q_{Heur})^h\right)\left[KL + \frac{n}{b-1+[(q_{Heur})^h(Y q_{Heur} + p_{Heur}(1-Z))]}\right.$$

$$\left.\frac{[b(q_{Heur})^h(Y q_{Heur} + p_{Heur}(1-Z))]^n - [(1-b)(1-(q_{Heur})^h(Y q_{Heur} + p_{Heur}(1-Z)))]^n}{[b(q_{Heur})^h(Y q_{Heur} + p_{Heur}(1-Z))]^n + [(1-b)(1-(q_{Heur})^h(Y q_{Heur} + p_{Heur}(1-Z)))]^n}\right]$$

Now

[520] $$\frac{dh}{dt} = \frac{Z}{E}\,\overline{E}_{1\ game} \qquad \text{and} \qquad L(R) = \frac{dR}{dt} = \frac{dh}{dt}\frac{dR}{dh}$$

and using [470] gives

$$(q_{Heur})^h = \frac{1}{Y}\,\frac{1}{1 + \exp(+Q/n\,[R-R_{\frac{1}{2}}])} \qquad \text{and} \qquad \frac{dR}{dh} = \frac{-Q\ln(q_{Heur})}{1+\exp(-Q/n\,[R-R_{\frac{1}{2}}])}$$

[530]

If we now make the approximation

[540] $$\exp(Q/n\,[R-R_{\frac{1}{2}}]) >> 1$$

(Which is saying that high rated WM^n players will nearly always beat players who move correctly with probability 1/2. This approximation also has the interesting side effect that it gives a linear relationship between R [the player's rating] and h [the number of heuristics in his repertoire].)

and combine the last, four formulas, we get at last

$$L1\ \exp(-Q/n\,[R-R_{\frac{1}{2}}]) < L(R) < L2\ \exp(-Q/n\,[R-R_{\frac{1}{2}}])$$

[550] where

| This expression holds if $1-a << 1-b$, i.e our player is far higher rated than the "typical" player.

$$L2 = \frac{-p_{Heur}\ n\ Z\ Q\ \ln(q_{Heur})}{(1-b)\ Y\ E} + L1 > L1 = \frac{-p_{Heur}\ Z\ Q\ \ln(q_{Heur})\ KL}{E\ Y} > 0$$

This shows that (according to the heuristic model) high rated players learn at a rate which is a dying exponential function of their rating, and is indeed asymptotic to the upper bound of [550].

For players who are considerably lower rated than "typical" players, nearly all of their games will have KL moves, so that

$$[555] \qquad KL\ p_{Heur}\ (q_{Heur})^{h} \sim \overline{E}_{1\ game}$$

Using [470] on this leads to

$$[560] \qquad L(R) = \frac{L1}{1 + \exp(\ Q/n\ [R-R_{\frac{1}{2}}\ ])} \qquad \Bigg| \ \text{holds for } R \text{ much less than } R_{\text{Typical}}$$

where L1 is defined in [550] above.

There are a number of things wrong with our heuristic model ("HM") as far as modeling real games goes. HM assumes that all players, good or bad, have the same probability that they will propose valid heuristics, will take the same average amount of time to confirm them, and need to make the same number of mistakes in order to lose the game. In real life, good players may be able to recover from mistakes that would be fatal for a weak player, they may need to make less experiments on average to confirm a heuristic, etcetera. We will therefore adopt the "generalized heuristic model of learning" ("GHM"), as opposed to the regular heuristic model. In this model, we will use one set of values of E, Z, and n for the very strong players, and one set for the very weak players; i.e. we will consider E, n and Z to be functions of rating, rather than constants, and then will assume that these (unknown) functions are asymptotic to constants when ratings are either very large or very small. We still will be unable to predict L(R) for the medium rated players, but we will assume that it is some sort of smooth joining of the very strong and very weak player curves. Finally, the GHM will assume that even the very weak players will nearly always beat a probability 1/2 correct mover, so that we may use the approximation [540].

Then the conclusions of the generalized heuristic model (which applies when the rating system is defined by [110]) are (where I am using different definitions of the constants LW and LS than before, and the below holds only for Y=R-RB>0) are:

$$[570] \qquad L(R) = \begin{cases} LW\ \exp(\ -\ Q/nw\ Y) & \text{for } R \ll \text{"typical R", for some } LW,\ nw>0. \\ LS\ \exp(\ -\ Q/ns\ Y) & \text{for } R \gg \text{"typical R", for some } LS,\ ns>0. \\ \text{some kind of smooth join for R near "typical R".} & \end{cases}$$

From the standpoint of mathematics, the simplest kind of "smooth join" to consider is a corner, i.e.

$$[580] \qquad L(R) = \begin{cases} LW \ \exp(- Q/nw \ Y) & \text{for } Y \leq YC \\ LS \ \exp(- Q/ns \ Y) & \text{for } Y \geq YC \end{cases}$$

where $LS = LW \exp(Q \ YC \ [1/ns - 1/nw])$.

Applying the solution [450] of [390] results in the "GHMC" rating probability density:

$$[590] \qquad GHMC(R) = \begin{cases} pd1(Y) & \text{if } Y \leq YC \\ pd2(Y) & \text{if } Y \geq YC \end{cases}$$

where

$$pd1(Y) = \frac{D}{LW} \exp\left[- \frac{D}{LW} \frac{nw}{Q} \left[\exp(\frac{Q}{nw} Y) - 1 \right] + \frac{Q}{nw} Y \right]$$

$$pd2(Y) = \frac{D}{LS} \exp\left[- \frac{D}{LS} \frac{ns}{Q} \left[\exp(\frac{Q}{ns} Y) - \exp(\frac{Q}{ns} YC) \right] + \frac{Q}{ns} Y - K \right]$$

where $LS = LW \exp(Q \ YC \ [1/ns - 1/nw])$

and
$$K = \frac{D}{LW} \frac{nw}{Q} \left[\exp(\frac{Q}{nw} YC) - 1 \right]$$

and $Y = R-RB$, $YC = RC-RB$, $pd(R)$ defined only for $Y>0$.

This solution predicts that rating frequency drops off extremely fast (as a double exponential) for high rated players, but the low rating end of the distribution is a regular exponential tail.

------------------Part 5b Draft 2.0 Warren Smith------------------

Experimental Confirmation of the GHM

In this section, I will describe a fairly successful attempt to experimentally confirm the generalized heuristic model using available statistical data.

The only rating system that is:

1. In wide enough use to have compiled a large mass of statistical data

2. Is based on principles similar to my own ideas

is the Elo chess rating system, in use by the USCF (United States Chess Federation) and the FIDE (International chess body) as well as by other chess groups. The Elo system may be thought of as a BSG linear system defined by

$$[600] \qquad wp_{Elo}(x) = P\left(\frac{\sqrt{2\pi}\; x}{800\ elo}\right)$$

where P is the normal distribution function defined in [HOMF pg 931-3], and x is the rating difference measured in the units of "elo". (This formula may be deduced from [1.41,1.44, and 1.81 Elo].)

Elo system problems:
Elo's wp(x) does not obey the star property since it does not die as an exponential for large negative x, but rather like

$$[610] \qquad wp_{Elo}(x) \sim -\exp(-z^2/2)/z \qquad \text{where } z = \frac{\sqrt{2\pi}\; x}{800\ elo}$$

(The other criteria for the star property are met, however, i.e. the symmetry requirement and the positive derivative requirement; see [250].) Also, chess, the game that is being rated, is not a BSG, although Elo treats it as one.

These are not major flaws, however, because Elo's form for wp(x) is well approximated by functions which do obey the star property, and this approximation is good except for |x| large, an "unimportant" region. (In fact, Elo recommends approximating the curve of wp(x) by three straight lines in the majority of uses; this approximation is valid only for small |x|, which Elo considers to be the most "important" region. [Elo 1.81]) In any case, it is undeniable that the Elo system seems to work well in practice.

Our GHM model only applies if wp(x) is defined by [110], so we must find a form of [110] that is a good approximation to Elo's wp(x) for |x| small. Since both wp(x)'s have the same value (1/2) at x=0, we need only match derivatives at the origin:

$$[620] \qquad wp'_{Elo}(0) = \frac{1}{800\ elo} = wp'_{WM}(0) = \frac{Q}{4}$$

$$\Longrightarrow \qquad Q = 1/200\ elo^{-1}$$

With this choice of Q, [110] will be a good approximation (for |x| small) to Elo's wp(x).

It is now possible to compare statistical data collected over the years of operation of the Elo system and compare it with the predictions of the GHM. [Elo 7.33] gives tables of the 1978 distribution of chess ratings in the USCF for 34,403 chess players. Two least-squares curve fitting programs (one, which I wrote, running on my home Z80, the other, written by Mark Shattenburg [an MIT Aero grad student] running on a Data General Mini.) were used to fit various analytic forms for the rating probability density curve to a histogram of Elo's 1978 tables. Three analytic forms were so fit:

1. A normal distribution: fit parameters were the standard deviation, , and the mean m.
2. A "Maxwell Boltzmann" distribution of form

[630]

$$pd(x) = \begin{cases} \frac{4}{\sqrt{\pi}} \left(\frac{x-XZ}{XM}\right)^2 \frac{1}{XM} \exp\left[- \left(\frac{x-XZ}{XM}\right)^2 \right] , & \text{if } x>XZ \\ 0 , & \text{if } x<XZ \end{cases}$$

it is assumed that XM>0.
Fitting parameters were XM and XZ.

3. The GHMC distribution [590]. Fitting parameters were nw,ns,RB,RC, and LW; Q was obtained from [620].

The first two approximate forms for the distribution were suggested by Elo for empirical reasons. (Elo [9.1 and 7.35, 7.36].)

The results are tabulated in table VB.1 and depicted in graphs VB.1 and VB.2.

Further, based on Elo's tables and [410], it was possible to tabulate L(x) for the USCF player pool. (Table VB.2.) The logarithm of this function is depicted in graph VB.3.

Table VB.1

The 1978 USCF rating probability density: based on tables in [7.33 Elo].

x Rating Interval (Elo)	pd(x) Fraction of USCF in rating interval	N(x) Fit of normal density	MB(x) Fit of Maxwell Boltzmann density	GHMC(x) Fit of GHM "corner" dens.
150 ± 50	.0006	.0000	0	.0006
250 ± 50	.0008	.0001	0	.0010
350 ± 50	.0011	.0002	0	.0017
450 ± 50	.0027	.0008	0	.0028
550 ± 50	.0040	.0022	0	.0048
650 ± 50	.0077	.0055	0	.0081
750 ± 50	.0151	.0124	.0000	.0137
850 ± 50	.0260	.0249	.0107	.0238
950 ± 50	.0387	.0444	.0378	.0376
1050 ± 50	.0596	.0702	.0729	.0605
1150 ± 50	.0874	.0986	.1060	.0940

1250 ± 50	.1418	.1230	.1289	.1370
1350 ± 50	.1375	.1363	.1368	.1460
1450 ± 50	.1276	.1341	.1302	.1254
1550 ± 50	.1102	.1172	.1126	.1032
1650 ± 50	.0871	.0910	.0895	.0807
1750 ± 50	.0605	.0628	.0657	.0597
1850 ± 50	.0390	.0385	.0448	.0413
1950 ± 50	.0239	.0209	.0284	.0265
2050 ± 50	.0141	.0101	.0169	.0156
2150 ± 50	.0076	.0043	.0094	.0083
2250 ± 50	.0034	.0017	.0049	.0039
2350 ± 50	.0021	.0006	.0024	.0016
2450 ± 50	.0006	.0002	.0011	.0006
2550 ± 50	.0005	.0000	.0005	.0002
2650 ± 50	.0001	.0000	.0002	.0000
Chi squared:	0	8.01 e-4	13.5 e-4	2.20 e-4
|Correlation coeff|:	1	.9932	.9889	.9977
Fit parameters:	(none)	m=1386.5 elo σ=290.49 elo	XZ=745.18 elo XM=606.70 elo	RB=66.578 elo RC=1293.2 elo nw=.94180 ns=3.5943 LW/D=268.760 elo

---

The expected value of chi squared (the sum of the squares of the additive errors in each column) for the "correct" form of the probability density function is about

[635] chi squared(best expected) ≃ 2/34403 ≃ 0.6 e-4 .

This follows from assuming that the number of ratings in each rating interval are binomially distributed so that the sum of the variances of the number of players falling into each rating interval = 34,403 = total no. of players in USCF. Then rescaling (dividing by 34,403 squared) gives an expectation value of chi squared of 1/34,403. There is an additional chi squared from the error in approximating the average value of the density function over each rating interval by its midpoint value; this can be estimated graphically to be of approximately the same size as the noise contribution.

Analysis of results

The GHM model predicts an L(R) curve that, when plotted on log paper as is done in graph VB.3, should look like two straight lines (the asymptotics) joined by a "smooth join". It is clear from the graph that this is what is actually happening. The normal and MB alternatives (see [460]) predict quite different behavior.

The graphs VB.1 and VB.2 show that, of the three proposed alternatives, only the GHMC curve actually looks like the real thing. It is the only curve which captures the essential skewed character of the peak, and the only one (as it better seen from the tables) that fits the behavior at the tails of the distribution well. Further, the GHMC curve is the only one with a value of chi squared which approaches experimental error; chi[GHMC] is larger than chi[exp. error] by a factor of only 1.9 .

However the GHMC model suffers from the approximation of the smooth join in L(R) by a corner; thus the chi squared fit had to sacrifice accuracy in the tail of the distribution in order to gain it in the peak.

The GHMC model was, of course, not designed to fit peaks well, but rather to fit the tails of the distribution, which, in fact, it will do very well.

To sum up: the numerical evidence is sufficient to convince me that the GHMC model is good enough to model the distribution of Elo ratings in chess. The specific form of the GHMC distribution is good enough to be a useful analytic approximation to the Elo rating distribution, but is inaccurate enough that it can be excluded as the "correct" form for the probability density function with confidence of around 97% . The GHM model accurately describes the behavior of the tails of the experimental rating distribution where all competing forms fail. Finally, the GHMC model should be good enough to use in static rating system designs.

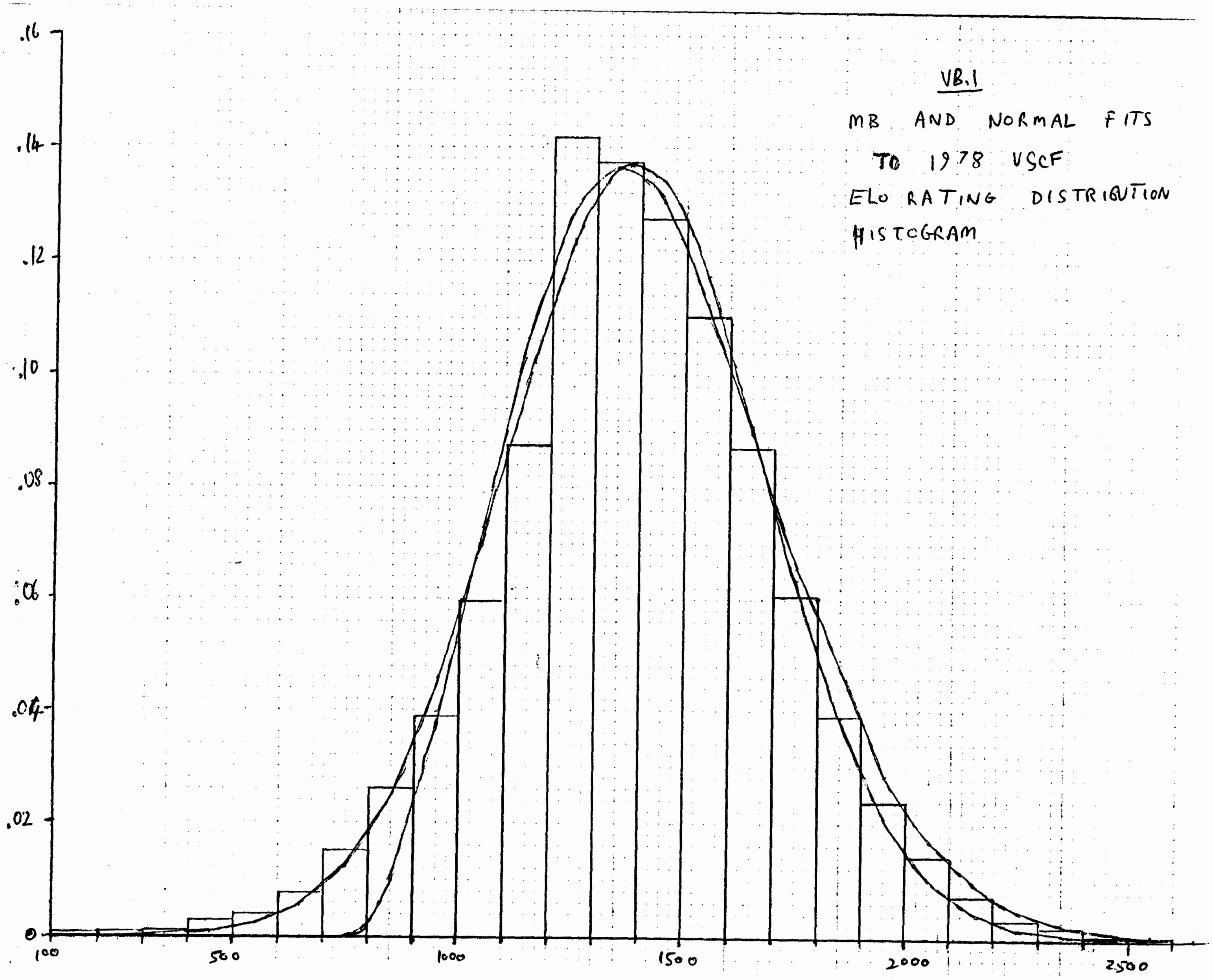
Table VB.2

L(R)/D curve for 1978 USCF rating distribution (34,403 players)

Based on [7.33 Elo].

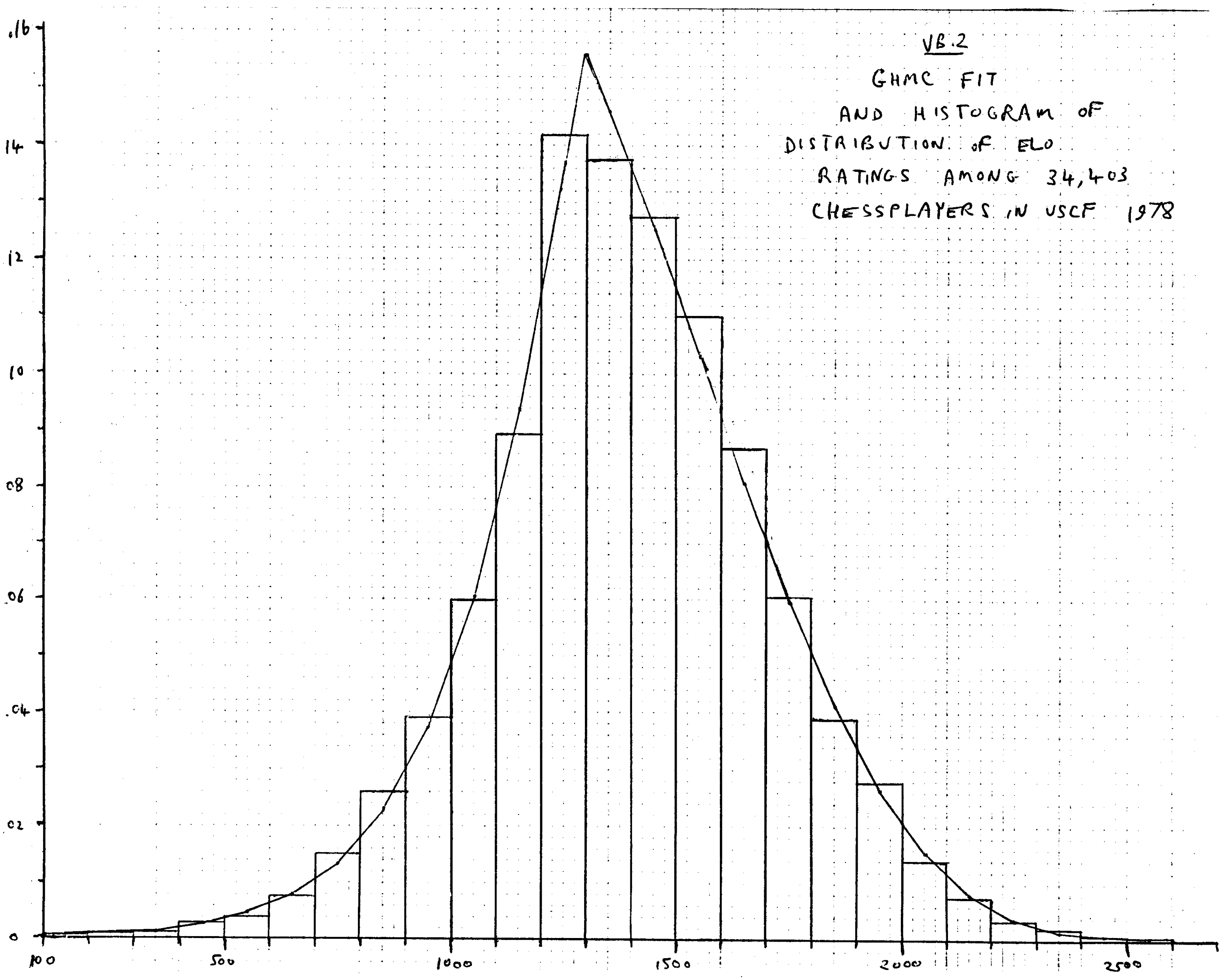
Rating R (in elo)	L(R)/D (in elo/player)	ln|L(R)/D|
150	1666 ± 120	7.418 ± .1
250	1249 ± 70	7.130 ± .06
350	907.3 ± 40	6.810 ± .05
450	368.9 ± 13	5.911 ± .03
550	248.2 ± 6	5.514 ± .02
650	128.1 ± 2	4.853 ± .01
750	64.59	4.168
850	36.72	3.603
950	23.83	3.171
1050	14.65	2.685
1150	9.150	2.214
1250	4.973	1.604
1350	3.967	1.378
1450	3.237	1.175
1550	2.669	0.982
1650	2.243	0.808
1750	2.011	0.699
1850	1.844	0.612
1950	1.692	0.527
2050	1.521	0.420
2150	1.368 ± .01	0.314
2250	1.441 ± .02	0.366 ± .01
2350	1.024 ± .05	0.025 ± .04
2450	1.333 ± .13	0.289 ± .1
2550	0.700 ± .4	-0.355 ± .4
2650	0.500 ± .3	-0.691 ± .7

The error bars are intended only as a rough guide. They are based on the number of significant figures given by Elo. Towards the middle of the table, the numbers are accurate to the first three digits given.

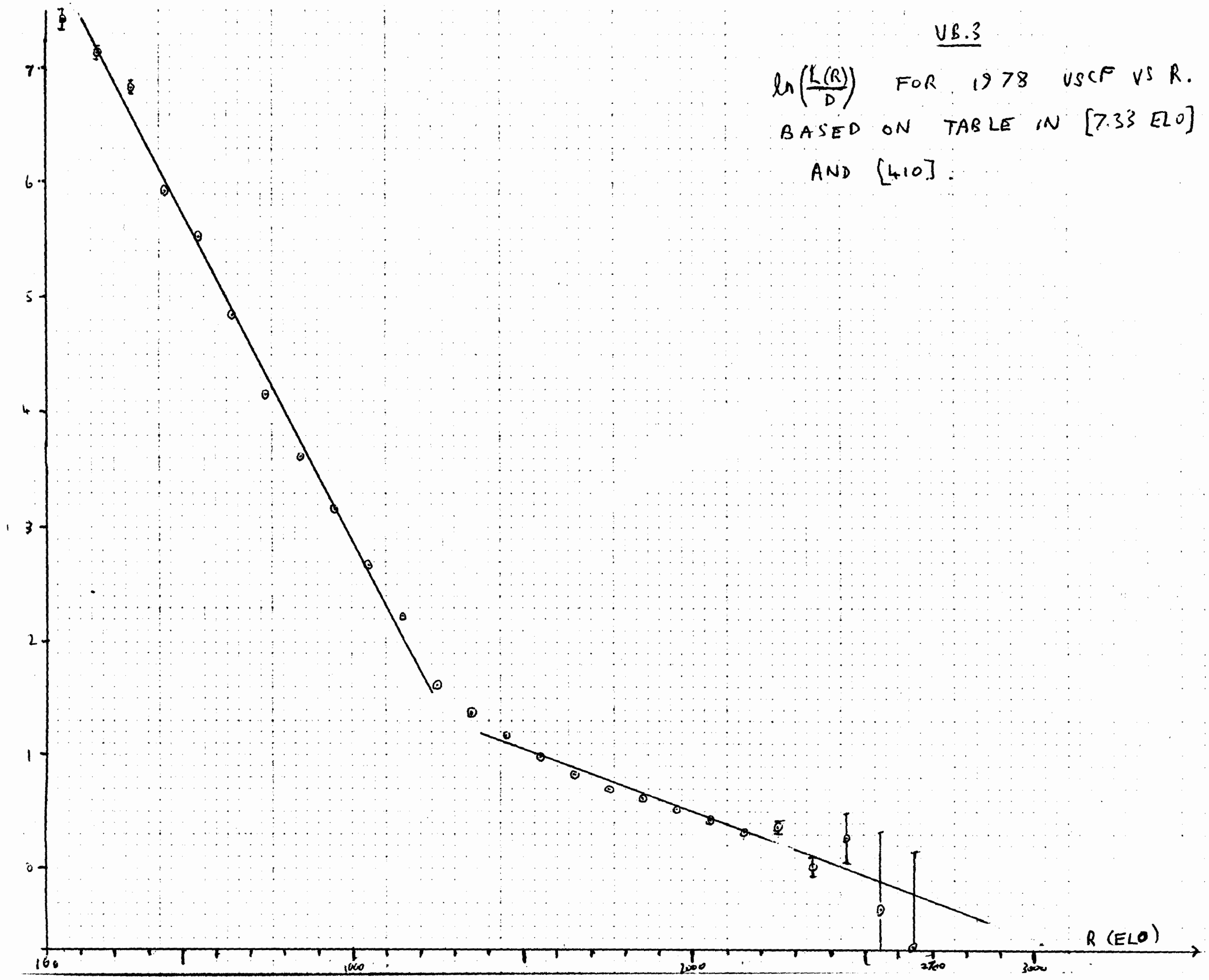


VB.1

MB AND NORMAL FITS TO 1978 USCF ELO RATING DISTRIBUTION HISTOGRAM



VB.2
GHMC FIT
AND HISTOGRAM OF
DISTRIBUTION OF ELO
RATINGS AMONG 34,403
CHESSPLAYERS IN USCF 1978



VB.3

$\ln\left(\frac{L(R)}{D}\right)$ FOR 1978 USCF VS R.
BASED ON TABLE IN [7.33 ELO]
AND [410].

Dynamic Ranking Systems

In a "dynamic" ranking system, every player in the league starts out with an initial ranking. After each game one plays, one's ranking is adjusted to reflect the result of the game.

Dynamic rating systems for generalized games are thus described by an iteration

[640] $(R1,R2) := \mathrm{dyn}(R1,R2,S)$,

where dyn( ) is a function with value being the new (adjusted) ratings R1 and R2 of two players 1 and 2 whose ratings were R1 and R2 respectively just before they played a game with result S.

We will now restrict ourselves to the BSG linear case as usual. BSG linear games may be described by the following (simpler) iteration:

[650] $R2 := R2 + \mathrm{dyn}(R2-R1)/2, \quad R1 := R1 - \mathrm{dyn}(R2-R1)/2$

which is the rating adjustment to be made after player 1 beats player 2. (This is the most general possible iteration which affects and depends only on R2-R1 and does not affect or depend on R2+R1)

Note that this iteration conserves the sum of the ratings of all the players in the league.

The first problem that we will ask ourselves concerning dynamic ranking systems will be:

Given that our ranking system is defined by wp(x) (obeying the star porperty) - what conditions need dyn(x) satisfy in order to ensure convergence of ratings to accurate ratings?
(A side problem will be: what do we mean by convergence?)

Our first necessary convergence condition will be what Good calls the "special principle of no incentives":

[660] $\mathrm{dyn}(x)\,\mathrm{wp}(x) = \mathrm{dyn}(-x)\,\mathrm{wp}(-x)$.

This states that one's expected rating increase after a game should be zero; i.e. correct ratings are stable. The reason Good calls this a "principle of no incentives" is that it denies players the capability to abuse the rating system. I.e. if this condition were not met for some x, then one could play players who were rated x above one's own rating and thus expect to gain (or lose) rating points in a way not justified by one's actual ability. I will call this the "weak stability condition" ("WSC").

The WSC may be solved generally for dyn(x) to give

[670] $\mathrm{dyn}(x) = h(|x|)\,\mathrm{wp}(-x)$

for some unknown function h(|x|)., which we would now like to restrict.

Another condition neccessary for convergence will be the "strong stability condition"

[675] $\mathrm{sign}(y-x) = \mathrm{sign}\Big[\,\mathrm{dyn}(x)\,\mathrm{wp}(y) - \mathrm{dyn}(-x)\,\mathrm{wp}(-y)\,\Big]$

which says that if one's ability is reflected by a rating y above one's opponent's, but the official rating difference is (erroneously) x, then the expected rating adjustment will be in the right direction, i.e. in the direction of y as seen from x. Note that the strong stability condition incorporates the weak one as the special case y=x, assuming sign(0)=0.

Using [670] in the strong stability condition yields

[680] $$\mathrm{sign}(y-x) = \mathrm{sign}(h(|x|))\ \mathrm{sign}\Big[\ wp(-x)\ wp(y) - wp(x)\ wp(-y)\ \Big] \quad ;$$

This may be simplified by the symmetry condition [250]

[690] $$\mathrm{sign}(y-x) = \mathrm{sign}(h(|x|))\ \mathrm{sign}\Big[\ wp(y) - wp(x)\ \Big]$$

And the monotonic increase property of wp(x) [250] then shows that the strong stability condition is equivalent to saying that

[700] $$h(|x|) > 0 \quad .$$

We might also require the "expected error decrease" condition:

[710] $$|y-x| > |y-x - wp(y)\ dyn(x) + wp(-y)\ dyn(-x)|$$

which states that the expected error in the official rating difference between two players will decrease after each game. Assuming that the strong stability condition holds, this is equivalent to saying that

[720] $$\frac{wp(y)\ dyn(x) - wp(-y)\ dyn(-x)}{y - x} < 2$$

This is the same as saying that

[730] $$\frac{wp(y)\ (1-wp(x)) - (1-wp(y))\ wp(x)}{y - x}\ h(|x|) < 2$$

or

[740] $$\frac{wp(y) - wp(x)}{y - x}\ h(|x|) < 2$$

Certainly a sufficient condition for the above to hold is that

[750] $$h(|x|) < 2/\max_{x}(wp'(x))$$

If wp(x) obeys the star property, then this condition may always be met, e.g. by the choice

[760] $$h(|x|) = h = \text{constant},$$

$$\text{for any positive } h < 2/\max(wp'(x)) \qquad ;$$

the star property assures that the upper bound above will exist.

It is a simple matter to see that if, after 1 game, expected error in rating difference between two players is decreased, then expected error in rating difference will also be decreased after n games for n any positive integer.

With this in mind, I will make the following definition: [770]
A dynamic BSG linear rating system will be said to "probabilistically converge" iff it meets the two stability conditions and the expected error decrease condition.

From now on, we will only be concerned with probabilistically convergent systems.

It is, as we have seen, a simple matter to design a probabilistically convergent system by choosing h(|x|). However, it is considerably harder to restrict our choice of h(|x|) further. I have not found any form for h(|x|) which I could argue to be better than a constant. I did try a brief experiment with using

$$[780] \qquad h(|x|) = 1/wp'(x)$$

which follows from applying ideas like those of the Newton-Raphson (non probabilistic) iteration to dynamic rating systems. This formula is not necessarally probabilistically convergent however! (And in fact it is not if wp(x) is given by [110], [80] or [600], or by any wp(x) obeying the star property, as is easy to see.)

Unless or until someone produces a good reason to choose some particular form for h(|x|), I will recommend utilizing [760] because of its simplicity.

Now the question arises: how reliable will the dynamic rating system we've just discussed be? How much "noise" will be inherent in the system?

I will answer this question for h(|x|) a constant. Some of my statements will be trivially generalizable to the h(|x|) non-constant case.

First, there will be noise inherent in the system due to players whose ratings have not yet converged from their initial guess rating (which was given to them when they were "born") to a rating representing their actual ability. If the standard deviation of all the ratings in the league is σ, then the expected "noise" (|additive error|) in a rating R due to this effect will be of order of magnitude

$$[790] \qquad \frac{\sigma^2}{2\,G\,h}$$

where G is the number of games one plays in a typical lifetime.

Second, there will be noise due to statistical fluctuations in the ratings of players whose ratings have converged; every time a player wins or loses a game, his rating will make a small discontinuous jump. It is to be expected that this noise will have size h or larger, but it is not obvious exactly how much larger.

For the purpose of estimating the size of this noise, we will set up the following model.

Let the difference of two player's ratings be have equilibrium value R0. Let the noise in this rating difference have value x, i.e. R-R0 = x. Then if the two players only play each other, x obeys the following iteration:

[800]
$$x := x + \begin{cases} h\,(1-f(x)) & \text{with probability} \quad p = f(0) \\ -h\,f(x) & \text{with probability} \quad q = 1-f(0) \end{cases}$$

where $f(x) = wp(x+R0)$.

Now let $D(x)$ be the probability density of x. (Presumably $D(x)$ is peaked near $x=0$.) Then $D(x)$ obeys the difference equation

[810]
$$D(x) = \frac{p\,D(y)}{1 - h\,f'(y)} + \frac{q\,D(z)}{1 - h\,f'(z)}$$

where $\quad x = y + h\,(1-f(y)) \quad$ and $\quad x = z - h\,f(z)$.

Now let us consider the above difference equation in the limit when x, y, z, and h are small. Then we may approximate $f(x)$ by its linear Maclaurin expansion

[820]
$$f(x) = f(0) + f'(0)\,x = p + s\,x\ .$$

Now we see that y and z are given by

[830]
$$y = \frac{x - h\,q}{1 - h\,s} \quad ; \quad z = \frac{x + h\,p}{1 - h\,s}$$

Then the difference equation [810] becomes (where I am calling $k = h\,s$ and I am using the identity $1/(1-k) = 1+k+k^2+\dots$):

[840]
$$(1-k)\,D(x) = p\,D\left[(x - h\,q)\,(1+k+k^2+\dots)\right] + q\,D\left[(x + h\,p)\,(1+k+k^2+\dots)\right]$$

Now we expand D in a taylor expansion about x:

[850]
$$\begin{aligned} 0 = {} & k\,D(x) + [p\,(k\,x - h\,q) + q\,(k\,x + h\,p)]\,[1+k+k^2+\dots]\,D'(x) + \\ & + [p^2\,(k\,x - h\,q)^2 + q^2\,(k\,x + h\,p)^2]\,[1+k+k^2+\dots]\,D''(x)/2 + \dots \end{aligned}$$

Taking advantage of the fact that $p+q=1$ gives

[860]
$$\begin{aligned} 0 = {} & k\,D(x) + k\,x/(1-k)\,D'(x) + \\ & \frac{(p^2 + q^2)(k\,x)^2 + 2\,(q-p)\,p\,q\,k\,x\,h + 2\,(p\,q\,h)^2}{(1-k)^2}\,\frac{D''(x)}{2} + \dots \end{aligned}$$

Now let us neglect all terms of order h and assume as we do so that x is of the same order of magnitude as the square root of h. (Then the nth derivative of $D(x)$ has the same order of magnitude as the $(-1-n)$st power of the square root of h.) (Also remember that $k = h\,s$.) We then divide the resulting equation by a common factor of h. Then:

$$[870] \quad O\left(x^2/h,\ 1\right) = s\,D(x) + s\,x\,D'(x) + h\,(p\,q)^2\,D''(x)$$

This differential equation has solution

$$[880] \quad D(x) = \frac{s}{\sqrt{2\pi}\,h}\ \frac{1}{(p\,q)^2}\ \exp\left[-\frac{s}{2\,h\,(p\,q)^2}\,x^2\right]$$

A normal distribution!
(Compare with [26.2.27 pg 933 HOMF].)

The standard deviation of this distribution is

$$[890] \quad \sqrt{h/wp'(R0)}\ \ p\ q \quad .$$

(Note the fact that x is about the same size as the square root of h justifies [admittedly circularly] the selection of negligible terms in [860]. This selection is also justifiable on the basis of numerical evidence which I have not presented here. [Monte Carlo experiments].)

Therefore a measure of the noise caused by statistical fluctuations of rating differences about their equilibrium value is given by

$$[900] \quad \sqrt{h/wp'(\sigma)}\ \ wp(\sigma)\ wp(-\sigma) \quad .$$

where σ is the standard deviation (or some other measure of the typical rating difference) of the rating distribution.

Finally, there is one additional source of dynamic rating system noise that we have still neglected. This is noise due to the presence of rating triangles. In our treament of statistical fluctuations we ignored this phenomenon by the device of considering two players who played only each other. However in a three player triangular universe, more noise is possible; in a triangle A>B>C>A we can imagine A's rating shooting up as he plays B and dropping as he plays C.

Consideration of the game tree for games like chess, etc. shows that there is no limit inherent in the game itself to the triangularity possible. (It is easy to construct three chess players who will beat one another in a triangular way with probability 1. In fact, in appendix 1, three players will be constructed with such behavior for the RL game model.)

It is therefore difficult or impossible to bound triangle noise, but one thing at least is clear: triangularity of players in a league should not depend on which rating system is used! We conclude that triangle noise is described by a typical noise amplitude

$$[910] \quad T(R)$$

Then total rating noise in a dynamic BSG linear system is given approximately by:

$$[920] \quad \sqrt{\left[T(R)\right]^2 + \left[wp(\sigma)\ wp(-\sigma)\ \sqrt{h/wp'(\sigma)}\right]^2 + \left[\sigma^2/(2\,h\,G)\right]^2}\ .$$

By setting the derivative of this with respect to h equal to zero, we may solve for an optimum value of h which minimizes the noise. This value is found to be approximately

[930] $$h_{opt} \simeq \left(\frac{\sqrt{wp'(\sigma)}\,\sigma^2}{wp(\sigma)\,wp(-\sigma)\,G} \right)^{2/3} .$$

For the Elo system, assuming that G = 500, and σ = 300 Elo gives non-triangle noise of around 30 elo and

[940] $$h_{opt\ Elo} \simeq 10 \text{ Elo}$$

- This noise estimate applies to a modified Elo system using [110] and [620] and using a dynamic ranking system; the real life Elo system is not a simple dynamic system and uses a form for wp(x) ([600]) which probably results in more noise than our estimate.

-----------------Part 7   Draft 2.0   Warren Smith-------------------

Notes on the Design of Real Rating Systems

We are now interested in applying our knowledge of rating systems to design a rating system for use in a real world application. I will not actually propose a universal system, since I believe that each system should be tailored to its own application. However, I believe that after reading through this thesis, the reader should have a sufficiently deep knowledge of rating systems to design his own.

A typical real world rating system would be neither a pure static nor a pure dynamic system. A pure static system would involve too much computational work. Pure dynamic systems are very desirable from the standpoint of simplicity and minimal computational labor, but a good method for initializing ratings into the system is required.

I would therefore recommend the use of a static rating system involving [110], [300], [310], [320], and [330] for the purpose of initializing new players into a dynamic system involving [110], [145], [650], [670], [760], [920], and [930]. Either an analytic approximation like [590] or else a "bootstrapped" experimental form for the rating distribution may be used in [310]. Note that [320] does not preclude the use of established ratings to help determine initial ratings of new players.

For the purpose of numerically evaluating the multidimensional integrals in [320], I would recommend the use of product Laguerre rules for numerical integration, described and tabulated in [HOMF 25.4.45, and table 25.9]. It is possible to evaluate [320] in a way which does not require the direct numerical evaluation of integrals in a large number of dimensions, (a difficult numerical task) but instead only requires the ability to do one dimensional integrals. This iterational method involves:

- I0. Guess ratings for all players whose ratings we wish to determine except for player i's.
- I1. Set player i's rating to its expected value from the ONE dimensional integral [320] with all ratings except i's fixed.
- I2. Cycle i to a new player and loop back to step I1 until satisfied that the ratings have converged.

If this method is used, evaluation of [320] using the one dimensional Laguerre rules is recommended. Physical considerations assure fast convergence for this method, which (in addition to it's speed advantages) may have increased accuracy over a straightforward N-dimensional integration because of its "self correcting" feedback properties.

Dynamic rating systems must be updated after each game, which may represent an excessive computational labor. Thus the original Elo system was semidynamic: updated after each tournament rather than after each game, with the use of "average tournament ratings" instead of induvidual opponent ratings. I do not believe in this approach and think that remembering the time, date and opponents of each induvidual match (until the next rating update) is worth the trouble. The Elo system as used by the USCF includes several "kludges" that were required, probably to correct imperfections in the whole system. These include "bonus feedback points" that one gets when playing "underrated" opponents, and rating "floors" that prevent one's rating from dropping more than about 200 Elo below one's best ever rating. Also, higher rated players are rated using a different form for dyn(ΔR) which insures slower rating adjustments for higher rated players, but which does not conserve rating sum in matches between high and low rated players. My feeling is that all such kludges must be used with extreme caution.

Appendix: The RL Game

In this appendix, I will discuss the RL game model, which may be thought of as a generalization of the WM and WM^n game models, with similarities to many real life games. I invented the RL game with the idea that it might be a good way to investigate rating triangles and learning. In fact, I did gain some insights from this investigation, but some new ideas, like the fluid model, seemed to supersede the RL game ideas and exclude them from the main body of this thesis. Further, the RL game, although analyzable, turned out to be somewhat complicated to analyze, and investigations bogged down.

Despite my lack of success with the RL game model, I think it is interesting and yields some useful insights, and it may turn out to be a useful tool for further research, which is why I'm including it as an appendix.

The RL game:

The RL game may be thought of as a chain of WM games.

"L" Wins WM[-n] WM[1-n] ... WM[-1] WM[0] WM[1] ... WM[n-1] WM[n] "R" Wins

Both players (call the two players "R" and "L") start at the middle of the chain, where I've written "WM[0]". They then play a game of WM, with "R" making correct moves with probability R[0] and "L" with correct moves with probability L[0]. The winner of the WM[0] game then gets to move along the chain 1 place in his direction (Right for "R", left for "L") and the players play another WM game, with the difference that, if R and L are located on the diagram above in the spot labeled "WM[i]" (which I will call "position i") they play WM with correct move probabilities R[i] and L[-i] respectively. The first player to reach his "win" node wins the game and the game is over.

n is a nonnegative integer which characterizes the RL game. If n=0, the RL game reduces to the WM game. If R[i] = R[j] and L[i] = L[j] for all i,j {-n,1-n, ... -1,0,1, ... n-1,n} then the RL game reduces to the WM^n game. Let us call an RL game with some given n an RL[n] game. I will pay special attention to the RL[1] game in this appendix.

The RL game is similar to the game of American football. Each "down" in football corresponds to a single WM[i] game. In each down, the opposing teams struggle to move the ball in their direction across the football field, and if ever one of the teams falters, they lose ground at the next down. The dependence of R[i] and L[-i] on i in RL corresponds to the varying abilities of the football teams at the kinds of play required at different yard lines.

It is easy to see that RL is a binary symmetric game. In RL, there can be many different kinds of players. Each kind of player is totally defined by his labelled game tree (I.e. R[i] move probability vector) in the sense of the discussion on page 2.1, and it is the relative sophistication of the permitted game trees and corresponding player personalities that makes RL interesting; RL trees, as we will see, are sophisticated enough to allow for rating triangles and learning.

The first thing that we are interested in is the probability that any given RL player will beat another. We may consider the RL game as having 2n+3 possible position states, namely i ∈ {-n-1,-n,1-n,...n+1}. We then may set up a transition probability matrix:

[A20] $TP[i,j]$ = probability that game in position i will be in position j at the end of the $WM[i]$ subgame, for all $i,j \in \{-n-1,-n,1-n,\ldots n+1\}$.

Then if we define the vector W as

$$[A30] \quad W[i] = \frac{R[i]\,(1-L[-i])}{R[i]\,(1-L[-i]) + L[-i]\,(1-R[i])}$$

then by [90] we see that

$$[A40] \quad TP[i,j] = \begin{cases} 0 & \text{if } i=j \text{ or } j>i+1 \text{ or } j<i-1 \text{ or } (i=\pm(n+1) \text{ and } j/i) \\ 1 & \text{if } i=j=\pm(n+1) \\ 1-W[i] & \text{if } j=i-1 \\ W[i] & \text{if } j=i+1 \end{cases}$$

Now after an RL game has progressed for m WM subgames, the probability that the game will be in state j will be

$$[A50] \quad \mathrm{Prob}(\text{state } j) = TP[0,j]^{m}$$

In this equation, the right hand side is the $[0,j]$'th element of the (matrix TP raised to the m'th power).

Now let us define the following matrix and vectors:

$$[A60] \quad TPL[i,j] = \lim_{m\to\infty} TP[i,j]^{m} \quad ; \quad P[j] = TPL[0,j] \quad ; \quad S[i] = TPL[i,n+1]$$

The limit will exist for any "realistic" players R and L (i.e. a player R is realistic if $0<R[i]<1$ for all i $\{-n,1-n,\ldots n\}$) as is obvious from the background of the problem. Further, $P[j]$ will have the form

$$[A70] \quad P[j] = \begin{cases} \text{prob}(R \text{ beats } L) & \text{if } j=n+1 \\ 1-\text{prob}(R \text{ beats } L) & \text{if } j=-n-1 \\ 0 & \text{if } j \neq \pm(n+1) \end{cases}$$

and indeed

$$[A80] \quad \text{if } j \neq \pm(n+1)\text{, then} \quad TPL[i,j] = 0$$

$$\text{and} \quad TPL[i,n+1] + TPL[i,-n-1] = 1, \quad S[n+1] = 1, \quad S[-1-n] = 0$$

We may use [A60], [A70], and [A80] to set up a system of linear equations for the $S[i]$:

$$[A90] \quad S[i] = W[i]\,S[i+1] + (1-W[i])\,S[i-1] \quad \text{for} \quad i \in \{1-n,2-n,\ldots n-1\}$$

$$S[-n] = W[-n]\,S[1-n]$$

$$S[n] = (1-W[n])\,S[n-1] + W[n]$$

This system arises from the matrix equation TP TPL = TPL.

The system [A90] may now be solved for S[0]=Prob(R beats L), for example by Cramer's rule. To make our method clearer, we will apply it to the case n=1. In this case the matrix TP is:

$$\text{[A100]}\qquad TP = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-W[-1] & 0 & W[-1] & 0 & 0 \\ 0 & 1-W[0] & 0 & W[0] & 0 \\ 0 & 0 & 1-W[1] & 0 & W[1] \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix TPL is:

$$\text{[A110]}\qquad TPL = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-S[-1] & 0 & 0 & 0 & S[-1] \\ 1-S[0] & 0 & 0 & 0 & S[0] \\ 1-S[1] & 0 & 0 & 0 & S[1] \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now [A90], the system of equations corresponding to the matrix equation TP TPL = TPL, becomes:

$$\text{[A120]}\qquad \begin{aligned} S[-1] &= W[-1]\ S[0] \\ S[0] &= (1-W[0])\ S[-1] + W[0]\ S[1] \\ S[1] &= (1-W[1])\ S[0] + W[1] \end{aligned}$$

This may be rewritten in matrix form as

$$\text{[A130]}\qquad \begin{pmatrix} 0 \\ 0 \\ W[1] \end{pmatrix} = \begin{pmatrix} -1 & W[-1] & 0 \\ W[0]-1 & 1 & -W[0] \\ 0 & W[1]-1 & 1 \end{pmatrix} \begin{pmatrix} S[-1] \\ S[0] \\ S[1] \end{pmatrix}$$

Solving for S[0] by Cramer's rule now yields

$$\text{[A140]}\qquad S[0] = \frac{\text{DET} \begin{pmatrix} -1 & 0 & 0 \\ W[0]-1 & 0 & -W[0] \\ 0 & W[1] & 1 \end{pmatrix}}{\text{DET} \begin{pmatrix} -1 & W[-1] & 0 \\ W[0]-1 & 1 & -W[0] \\ 0 & W[1]-1 & 1 \end{pmatrix}}$$

Which may be written as:

[A150]
$$\text{Prob(R beats L)} = S[0] = \frac{W[0]\ W[1]}{1 - W[-1]\ (1-W[0]) - W[0]\ (1-W[1])}\ .$$

It is interesting to compile a short table of win probabilities in RL[1] using [A150]. Consider the following eight RL[1] players:

[A160]

Player X	X[-1]	X[0]	X[1]
Player A	.5	.5	.5
B	.5	.5	.9
C	.5	.9	.5
D	.5	.9	.9
E	.9	.5	.5
F	.9	.5	.9
G	.9	.9	.5
H	.9	.9	.9

Now let PXY denote the probability that player X will beat player Y at RL[1]. Then to five places of accuracy:

```
PAB=.35714 |
PAC=.1     |PBC=.16667 |                                  [A170]
PAD=.058140|PBD=.1     |PCD=.35714 |
PAE=.16667 |PBE=.5     |PCE=.64286 |PDE=.9     |
PAF=.1     |PBF=.35714 |PCF=.5     |PDF=.83333 |PEF=.16667 |
PAG=.021739|PBG=.1     |PCG=.16667 |PDG=.5     |PEG=.1     |PFG=.35714
PAH=.012195|PBH=.058140|PCH=.1     |PDH=.35714 |PEH=.021739|PFH=.1    PGH=.16667
```

The reader may make his own conclusions concerning the relative importance of offensive, defensive, and central strength in RL.

An interesting fact about RL is that rating triangles exist, and indeed three players A, B, and C may be found such that

Prob(A beats B), Prob(B beats C), and Prob(C beats A)

are all arbitrarally close to 1. As an example, consider these three RL[1] players:

[A180]

Player X	X[-1]	X[0]	X[1]
Player A	$1-e$	$1/2$	$1-e^3$
B	$1-e^3$	$1/2$	$1/2$
C	$1/2$	$1-e^2$	$1/2$

where $0 < e < 1$.

Then it may be shown that

$$\text{Prob(A beats B)} = 1/(1 + 2\,e) \qquad \text{Prob(B beats C)} = 1/(1 + 2\,e - 2\,e^3)$$

[A190]
$$\text{Prob(C beats A)} = (1 - e^2)/(1 + e - e^2 - e^4)$$

Thus for e small, these probabilities all approach 1; in fact, for $e < 0.1$, it may be shown that all of them are greater than

[A200] $$1 - 3 e.$$

This situation where arbitrarally severe triangles can exist is similar to the situation in real world games such as chess, where arbitrally severe triangles also exist, although only among imaginary non-human players.

A possible direction for future research might be to consider a population of RL players who learn as they play, and studying the result. I have not had success with this idea so far, however.

B

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