

Essays in Dynamic Contracting

by

Suehyun Kwon

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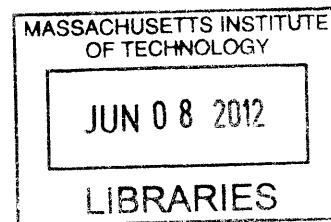
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Author ✓

Department of Economics

May 14, 2012

Certified by

Glenn Ellison

Gregory K. Palm (1970) Professor of Economics

Thesis Supervisor

Certified by

Muhamet Yildiz

Professor of Economics

Thesis Supervisor

Accepted by

Michael Greenstone

3M Professor of Environmental Economics

Chairman, Department Committee on Graduate Studies

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Abstract

This thesis examines three models of dynamic contracting. The first model is a model of dynamic moral hazard with partially persistent states, and the second model considers relational contracts when the states are partially persistent. The last model studies preference for delegation with learning.

In the first chapter, the costly unobservable action of the agent produces a good outcome with some probability, and the probability of the good outcome corresponds to the state. The states are unobservable and follow an irreducible Markov chain with positive persistence. The chapter finds that an informational rent arises in this environment. The second best contract resembles a tenure system: the agent is paid nothing during the probationary period, and once he is paid, the principal never takes his outside option again. The second best contract becomes stationary after the agent is tenured. For discount factors close to one, the principal can approximate his first best payoff with review contracts.

The second chapter studies relational contracts with partially persistent states, where the distribution of the state depends on the previous state. When the states are observable, the optimal contracts can be stationary, and the self-enforcement leads to the dynamic enforcement constraint as with i.i.d. states. The chapter then applies the results to study the implications for the markets where the principal and the agent can be matched with new partners.

The third chapter studies preference for delegation when there is a possibility of learning before taking an action. The optimal action depends on the unobservable state. After the principal chooses the manager, one of the agents may receive a private signal about the world. The agent decides whether to disclose the signal to the manager, and the manager chooses an action. In an equilibrium, the agents' communication strategies depend on the manager's prior. The principal prefers a manager with some difference in prior belief to a manager with the same prior.

Thesis Supervisor: Glenn Ellison

Title: Gregory K. Palm (1970) Professor of Economics

Thesis Supervisor: Muhamet Yildiz
Title: Professor of Economics

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Chapter 1

Dynamic Moral Hazard with Persistent States

1.1 Introduction

There is a large literature on repeated moral hazard with i.i.d. states or fully persistent states.¹ But there are circumstances that are better described by partially persistent states. Productivity of an economic sector varies from year to year, but high productivity is more likely to come after a productive year than after an unproductive year. Technological advances are also often clustered, and much advance can be made in a short amount of time. It is worthwhile to consider dynamic moral hazard with partially persistent states. In this chapter, I examine such a model; I note that the persistent states create an informational rent for the agent and explore properties of the second best contracts.

This chapter develops a model of principal-agent problem in which the underlying environment is partially persistent. The principal hires the agent over an infinite horizon, and each period, the agent can work or shirk. The costly unobservable action, work, produces a good outcome with some probability, and the probability

¹Dutta and Radner (1994), Fudenberg, Holmstrom and Milgrom (1990), Holmstrom and Milgrom (1987), Malcomson and Spinnewyn (1988), Radner (1981, 1985), Rey and Salanie (1990), Rogerson (1985), Rubinstein and Yaari (1983)

of the good outcome depends on the state. The states are unobservable, and they follow an irreducible Markov chain with positive persistence. The principal observes the outcome and pays the agent. The principal can commit to a long term contract. Both parties are risk-neutral, and the agent is subject to limited liability.

I start in Section 1.3 with an analysis of one and two period versions of the model. The two period model illustrates one of the basic insights: the deviations of the agent create information asymmetry, and the agent receives an informational rent if the principal wants him to work in both periods. When the states are persistent, the outcome has informational value in addition to its payoff consequence. If the principal believes that the agent worked this period, the principal updates his belief about the state after observing the outcome. The agent's deviation leads to both the unproductive outcome and the information asymmetry between the principal and the agent. Since the bad outcome lowers the prior of being in the good state, the principal assigns a strictly lower probability to the good state than the agent in the period after the agent deviates. Working in the following period after shirking, the agent can ensure himself a strictly positive amount of rent by a one-shot deviation. I characterize the informational rent as a function of the discount factor, the persistence of the states and the informativeness of the outcomes, and I give conditions under which the principal wants the agent to work in both periods.

Section 1.4 describes the informational rent and derives an upper bound on the informational rent. The agent's deviation creates information asymmetry between the principal and the agent, and in all periods following the deviation, the agent assigns weakly higher probabilities to the good state than the principal does. After any history, if the principal wants the agent to work, the principal has to provide as much rent as what the agent can get from deviating. However, I show that the principal can attain an upper bound on informational rents by offering a contract that is stationary from the second period. When a contract is stationary, the information asymmetry doesn't play a role in the continuation values of the agent, and the agent works in any given period if the expected payment is greater than the cost. The upper bound increases with the discount factor, the persistence of the states, and the

informativeness of the outcomes. If the discount factor, the persistence of the states, or the informativeness of the outcomes is small, the principal can approximate his first best payoff with a contract that is stationary from the second period.

In Section 1.5, I consider a special case in which the states correspond to the outcomes of working. First, I consider contracts that induce the agent to work in every period. The main result is that the upper bound on informational rents from Section 1.4 turns out to be the lower bound on the rent also. Hence, the cost-minimizing contract that induces the agent to work in every period is stationary from the second period. The deterministic mapping from the states to the outcomes turns out to be a necessary condition for a history-independent contract to be optimal.

Section 1.6 provides further results on the form of the second best contract in the case studied in Section 1.5. The main result is that the second best resembles a tenure system. The agent is paid nothing during the probationary period, and once the agent is paid, the principal never takes his outside option again. By backloading the payment, the principal can provide better incentives to the agent, and he can offer a continuation contract with a higher expected outcome. The principal doesn't benefit from backloading the payment only if he is already inducing the agent to work in every period, and therefore, once the agent is getting paid, he is tenured, and the principal never takes his outside option again. The second best contract also becomes stationary after the agent is tenured. Since the principal never takes his outside option again, he can offer the cost-minimizing contract as the continuation contract. When the state variable is whether or not working will produce a good outcome, the cost-minimizing contract is stationary from the second period, and the second best contract becomes stationary. The principal's information changes over time, but if the principal uses his information, the agent can deviate and create information asymmetry; the principal is better off by committing not to use his information.

The chapter also considers what happens in the limit as the discount factor goes to one. Here, I show that the principal can approximate his first-best payoff when the discount factor is close to one. The proof of the result is not based on computing the second best contract. Instead, I just note that the principal could employ a review

strategy with a sufficiently long block so that the probability of meeting the ergodic distribution by working in every period is close to one. Having a lump sum transfer and continuing with the contract when the agent meets the quota, the principal can ensure that the agent works in every period. When the discount factor is close to one, the principal's payoff from each block gets arbitrarily close to his first best payoff with a sufficiently high probability, and the principal can approximate his first best payoff.

The informational aspect of the rent in my chapter is related to the ratchet effect in dynamic adverse selection. Including Laffont and Tirole (1988), there have been numerous papers on ratchet effect. In these papers, the principal cannot commit to a long term contract, and there is much pooling in the first period. The principal can commit to a long term contract in my model, but if the principal were to use his information, the agent can deviate and create information asymmetry between the principal and the agent. The persistence of the states leads to the informational component of the agent's action, and the rent is informational.

The second best takes a particular form in my model: after a probationary period, the agent is tenured, and the continuation contract becomes stationary. The probationary period of the second best contract shares similarities with Chassang (2010). Fong and Li (2010) also find that the optimal contract has a probationary phase. In these papers, the principal cannot commit to a long term contract and the environment is i.i.d., whereas in my model, the principal can commit to a long term contract, and the states are partially persistent. The stationarity of the continuation contract after tenure is related to the literature on sticky wages. In Townsend (1982), long-term contracts and inefficient tie-ins can be optimal under private information. The stationary payments of the second best of my model shows that the stationarity of a long-term contract is not necessarily because of enforcement costs.

The first best approximation under little discounting is related to folk theorem results. Review strategies are first introduced by Radner (1981, 1985), and Fudenberg, Holmstrom and Milgrom (1990) show conditions under which the first best can be approximated with short term contracts. Other papers on the approximation of the

first best include Rubinstein (1979), Rubinstein and Yaari (1983), and Dutta and Radner (1994).

Another paper on repeated moral hazard in which the outcome carries information about both the agent's effort and the future profitability is DeMarzo and Sannikov (2011). Their model is in continuous time, and the firm's fundamental evolves over time according to a Brownian motion.

Lastly, there is also literature on dynamic adverse selection with persistent private information. With partially persistent types, the optimal contract is often history-contingent, but the principal achieves efficiency in the limit. Papers with Markovian types include Battaglini (2005) and Athey and Bagwell (2008). Battaglini (2005) considers consumers with Markovian types, and Athey and Bagwell (2008) study collusion with persistent private shocks. Escobar and Toikka (2010) show the folk theorem result with Markovian types and communication.

The rest of the chapter is organized as the following. Section 1.2 describes the model, and one and two period examples are described in Section 1.3. Section 1.4 discusses the informational rent. Section 1.5 discusses the special case when the states correspond to the outcomes of working, and I characterize the second best of the special case in Section 1.6. The first best approximation is considered in Section 1.7. Section 1.8 concludes.

1.2 Model

The principal hires the agent over an infinite horizon $t = 1, 2, \dots$. The common discount factor is $\delta < 1$, and the principal can commit to a long term contract.

Each period, the agent can work or shirk, and the outcome is either 1 or 0, which I call the good outcome and the bad outcome. Shirking costs nothing to the agent, but it produces the bad outcome with probability 1. Work costs $c > 0$ to the agent, but it produces a good outcome with some probability. The probability of the good outcome depends on the state. The agent's action is unobservable to the principal, and the principal only observes the outcome.

There are two states, the good state (state 1) and the bad state (state 2). Throughout the chapter, the subscript 1 refers to the good state, and the subscript 2 refers to the bad state. The probability of the good outcome is p_H in the good state, and it is p_L in the bad state. The probability is strictly higher in the good state than in the bad state, and $0 \leq p_L < p_H \leq 1$. The state is unobservable to both parties. The states follow an irreducible Markov chain with positive persistence. Specifically, let M be the Markov transition matrix for the state transition with entries

$$M_{ij} = \Pr(s_{t+1} = j \mid s_t = i),$$

where s_t is the state in period t . The next assumption states that the states are partially persistent.

Assumption 1.1 (Persistence). *The Markov matrix M for the state transition satisfies*

$$\det M > 0, \quad 0 < M_{ij} < 1, \forall i, j.$$

The positive persistence of the states is captured by the condition $\det M > 0$. The determinant of the Markov matrix is

$$\begin{aligned} \det M &= M_{11}M_{22} - M_{12}M_{21} \\ &= M_{11}(1 - M_{21}) - (1 - M_{11})M_{21} \\ &= M_{11} - M_{21}. \end{aligned}$$

Since M_{11} and M_{21} are the probabilities of the good state after the good state and the bad state, respectively, the determinant is the difference in the probabilities of being in the good state. When the states have positive persistence, the probability of being in the good state is strictly higher after being in the good state than after being in the bad state. Note also that I'm assuming that there is always a positive probability of transiting from state i to state j , for all $i, j = 1, 2$, which implies that the Markov chain is irreducible.

The principal can take an outside option in any period, but in the first best, the

principal wants the agent to work in every period. Let π^t be the principal's prior on the state at the beginning of period t . π^t is a vector of beliefs, and π_i^t is the probability that the state in period t will be state i . I assume that the initial prior π^1 satisfies $M_{21} \leq \pi_1^1 \leq M_{11}$. Then, for all $t \geq 1$, we have $M_{21} \leq \pi_1^t \leq M_{11}$. Let \underline{u} be the payoff to the principal from his outside option. I assume that the payoff to the agent is zero if the principal takes his outside option. The following assumption says that it is efficient to have the agent work for any given prior on the state. I also assume that taking the outside option is better than not inducing the agent to work, which implies that if the principal doesn't want the agent to work in a given period, the principal takes his outside option.

Assumption 1.2 (Efficiency). *The parameters M, c, p_H, p_L and \underline{u} are such that*

$$M_{21}p_H + M_{22}p_L - c > \underline{u} > 0.$$

Both the principal and the agent are risk-neutral, and the agent is subject to limited liability. There are three constraints to consider, IR, IC and limited liability. IR means that in period 1, the agent receives at least the payoff from his outside option in expectation by participating in the contract. Limited liability requires that the agent receives non-negative payments. I normalize the outside option of the agent to zero, and IR is implied by the IC constraint and limited liability. The IC constraint has to be satisfied at every node at which the principal wants the agent to work. I assume that if indifferent between working and shirking, the agent chooses to work.

Throughout the chapter, h_t refers to the outcome in period t . When the principal takes the outside option, I denote it by $h_t = -1$. The history h^t is the sequence of outcomes up to period t , and I denote it by $h^t = h_1 \cdots h_t$. The initial history is an empty set and is denoted by $h^0 = h_0 = \emptyset$. $h^t \sqsubset h^k$ refers to the history that h^t is followed by h^k .

In the first best, if the actions of the agent were observable, the principal can pay the cost if and only if the agent works. However, the states and the actions are unobservable, and the principal only observes the outcomes. A contract specifies the

principal's decisions to take his outside option and history-contingent payments $w(h^t)$ for all histories h^t .

1.3 One and Two Period Examples

This section describes the one and two period examples of the model. The main observation these models bring out is that an informational rent arises when the principal wants the agent to work in both periods. The informational rent doesn't exist in the one period example, but it arises in the two period example when the principal wants the agent to work in both periods. The rent is proportional to the discount factor and the persistence of the states. When the discount factor is low, it can be optimal to have the agent work in every period, and the optimal contract is independent of the history.

1.3.1 One Period

In this section, I consider the case when the principal hires the agent for one period and show that the principal can leave no rent to the agent.

Let $\pi = (\pi_1, \pi_2)$ be the common prior; π_1 is the probability of the good state. Since the principal only observes the outcome, and not the action of the agent, he offers payments $w(0)$ and $w(1)$ as a function of the outcome. The agent's IC constraint is given by

$$-c + \pi \begin{pmatrix} p_H \\ p_L \end{pmatrix} w(1) + (1 - \pi \begin{pmatrix} p_H \\ p_L \end{pmatrix}) w(0) \geq w(0).$$

The agent is subject to limited liability, and this imposes

$$w(0) \geq 0, \quad w(1) \geq 0.$$

The optimal contract for the principal is to provide

$$w(0) = 0, \quad w(1) = \frac{c}{\pi \begin{pmatrix} p_H \\ p_L \end{pmatrix}}.$$

The expected rent to the agent is

$$-c + \pi \binom{p_H}{p_L} w(1) + (1 - \pi \binom{p_H}{p_L}) w(0) = 0,$$

and the agent receives zero rent in the one period model.

1.3.2 Two Periods

In this section, I consider the two period example of the model. I note that the principal has to leave an informational rent to the agent if he wants the agent to work in both periods. The rent is proportional to the discount factor and the persistence of the states. For low discount factors, it can be optimal to leave the rent and have the agent work in both periods. The principal can provide a history-independent contract to have the agent work in both periods.

Proposition 1.1. *Suppose the principal wants the agent to work in both periods. The rent to the agent is bounded from below by*

$$\delta c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi^1 \binom{1-p_H}{1-p_L} \tilde{\pi}^2 \binom{p_H}{p_L}}$$

where

$$\tilde{\pi}^2 = \left(\frac{\pi_1^1 (1 - p_H)}{\pi^1 \binom{1-p_H}{1-p_L}}, \frac{\pi_2^1 (1 - p_L)}{\pi^1 \binom{1-p_H}{1-p_L}} \right) M$$

is the principal's prior in the beginning of period 2 after the bad outcome in period 1.

Proof. We have the following expressions for the priors in the beginning of period 2. After the good outcome, the prior becomes

$$\pi^2 = \left(\frac{\pi_1^1 p_H}{\pi^1 \binom{p_H}{p_L}}, \frac{\pi_2^1 p_L}{\pi^1 \binom{p_H}{p_L}} \right) M.$$

After the bad outcome, when the principal believes that the agent worked in the first

period, his prior in the following period is

$$\tilde{\pi}^2 = \left(\frac{\pi_1^1(1-p_H)}{\pi^1\left(\frac{1-p_H}{1-p_L}\right)}, \frac{\pi_2^1(1-p_L)}{\pi^1\left(\frac{1-p_H}{1-p_L}\right)} \right) M.$$

There are two IC constraints to consider in period 1. Denote by $w(0)$, $w(1)$, $w(00)$, $w(01)$, $w(10)$ and $w(11)$ the history-contingent payments. Also denote by R_1 and R_0 the rents to the agent in period 2 after the good outcome and the bad outcome, respectively. The rents are given by the following expressions:

$$\begin{aligned} R_1 &= -c + \pi^2 \binom{p_H}{p_L} w(11) + (1 - \pi^2 \binom{p_H}{p_L}) w(10), \\ R_0 &= -c + \tilde{\pi}^2 \binom{p_H}{p_L} w(01) + (1 - \tilde{\pi}^2 \binom{p_H}{p_L}) w(00). \end{aligned}$$

The one-shot deviation of the agent is to shirk in period 1 and work again in period 2. When the agent deviates and shirks in period 1, he doesn't update his posterior after observing the outcome. The agent's prior in the beginning of period 2 is $\pi^1 M$. The IC constraint for the one-shot deviation is given by

$$\begin{aligned} & -c + \pi^1 \binom{p_H}{p_L} (w(1) + \delta R_1) + (1 - \pi^1 \binom{p_H}{p_L}) (w(0) + \delta R_0) \\ & \geq w(0) + \delta (-c + \pi^1 M \binom{p_H}{p_L} w(01) + (1 - \pi^1 M \binom{p_H}{p_L}) w(00)). \end{aligned}$$

The second IC constraint is for which the agent deviates twice in a row and shirks in both periods. The IC constraint is given by

$$\begin{aligned} & -c + \pi^1 \binom{p_H}{p_L} (w(1) + \delta R_1) + (1 - \pi^1 \binom{p_H}{p_L}) (w(0) + \delta R_0) \\ & \geq w(0) + \delta w(00). \end{aligned}$$

On the other hand, the IC constraints for period 2 are the following: after the good outcome, the IC constraint is

$$R_1 \geq w(10),$$

and after the bad outcome, the IC constraint is

$$R_0 \geq w(00).$$

The second period IC constraint after the bad outcome is equivalent to

$$w(01) - w(00) \geq \frac{c}{\tilde{\pi}^2 \binom{p_H}{p_L}}.$$

From the positive persistence of the states, we know that

$$-c + \pi^1 M \binom{p_H}{p_L} w(01) + (1 - \pi^1 M \binom{p_H}{p_L}) w(00) \geq w(00)$$

whenever $R_0 \geq w(00)$ holds. Therefore, in period 1, it is sufficient to consider the one-shot deviation of the agent.

The limited liability implies $w(0) + \delta w(00) \geq 0$, and the first period IC constraint for the one-shot deviation becomes

$$\begin{aligned} & -c + \pi^1 \binom{p_H}{p_L} (w(1) + \delta R_1) + (1 - \pi^1 \binom{p_H}{p_L}) (w(0) + \delta R_0) \\ & \geq w(0) + \delta (-c + \pi^1 M \binom{p_H}{p_L} w(01) + (1 - \pi^1 M \binom{p_H}{p_L}) w(00)) \\ & = w(0) + \delta w(00) + \delta (-c + \pi^1 M \binom{p_H}{p_L} (w(01) - w(00))) \\ & \geq \delta c \left(-1 + \frac{\pi^1 M \binom{p_H}{p_L}}{\tilde{\pi}^2 \binom{p_H}{p_L}} \right) \\ & = \delta c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi^1 \binom{1-p_H}{1-p_L} \tilde{\pi}^2 \binom{p_H}{p_L}}. \end{aligned}$$

□

Proposition 1.1 shows that the IC constraint and the limited liability imply a minimum rent to the agent when the principal wants the agent to work in both periods. Note that the bound on the rent is proportional to the discount factor and the persistence of the states. If the states were i.i.d., i.e., $\det M = 0$ or $p_H = p_L$, the

principal can have the agent work in both periods and leave no rent.

Note also that the rent to the agent depends on the payments in the second period after the bad outcome. The payments in the second period after the good outcome doesn't matter for the minimum rent to the agent. When the agent deviates, the principal believes that the agent worked but the outcome is bad; the principal offers the payments in the second period as if the outcome of working was bad in the first period, and the agent's continuation value after deviation is determined by the payments in the second period after the bad outcome.

Proposition 1.1 shows that there exists a lower bound on the rent the principal has to leave in order to have the agent work in both periods. In Proposition 1.2, I show that the lower bound is tight, and I also show a possible form of the contract the principal can provide.

Proposition 1.2. *Suppose the principal wants the agent to work in both periods. The principal can achieve the minimum rent by offering the identical contract in period 2 independent of the outcome in period 1:*

$$\begin{aligned} w(0) &= w(00) = w(10) = 0, \\ w(1) &= \frac{c}{\pi^1(p_H)}, \\ w(11) &= w(01) = \frac{c}{\tilde{\pi}^2(p_H)}, \end{aligned}$$

where

$$\tilde{\pi}^2 = \left(\frac{\pi_1^1(1-p_H)}{\pi^1(1-p_L)}, \frac{\pi_2^1(1-p_L)}{\pi^1(1-p_H)} \right) M$$

is the principal's prior in the beginning of period 2 after the bad outcome in period 1.

The contract in Proposition 1.2 is one of the many contracts that the principal can provide to attain the lower bound on the rent. Since both the principal and the agent are risk-neutral, the principal can always delay the payment and pay the agent later. The above contract is nice because of the stationarity and the simplicity. The principal makes positive payments only for the good outcome, and in particular, the

payments in the second period are independent of the outcome in the first period. The principal is leaving the rent to the agent because the deviation of the agent leads to information asymmetry between the principal and the agent, and one way to take care of the information asymmetry is to provide identical payments in the second period, regardless of the outcome in the first period.

Proposition 1.1 and Proposition 1.2 assume that the principal wants the agent to work in both periods. I will next show what happens when the principal takes his outside option in some periods. Since the outside option is inefficient, the principal incurs a loss in outcome by taking his outside option. However, depending on the timing of the outside option, the principal can prevent the information asymmetry from the agent's deviation and therefore, reduce the rent to the agent.

If the principal takes his outside option in period 2 after the good outcome, the IC constraint in period 1 doesn't change. The principal has to leave just as much rent as he would if the agent were to work in period 2 after the good outcome. Since the outside option is inefficient, the principal won't take his outside option after the good outcome.

On the other hand, if the principal takes his outside option in period 2 after the bad outcome, the principal doesn't have to leave the rent to the agent. The IC constraint in period 1 becomes

$$-c + \pi^1 \binom{p_H}{p_L} (w(1) + \delta R_1) + (1 - \pi^1 \binom{p_H}{p_L}) (w(0) + \delta R_0) \geq w(0),$$

and $R_0 = 0$. By offering the following contract, the principal leaves no rent to the agent:

$$\begin{aligned} w(0) &= w(10) = 0, \\ w(1) &= \frac{c}{\pi^1 \binom{p_H}{p_L}}, \\ w(11) &= \frac{c}{\pi^2 \binom{p_H}{p_L}}, \end{aligned}$$

where

$$\pi^2 = \left(\frac{\pi_1^1 p_H}{\pi^1 \binom{p_H}{p_L}}, \frac{\pi_2^1 p_L}{\pi^1 \binom{p_H}{p_L}} \right) M$$

is the prior of the principal in the beginning of period 2 after the good outcome in period 1.

The principal also leaves no rent to the agent if he takes his outside option in period 1. When the principal takes his outside option in period 1, period 2 is identical to the one period model with prior $\pi^1 M$, and the agent gets no rent.

If the principal mixes the continuation contracts, it has the same effect as taking the linear combination of the IC constraints, and it convexifies the set of payoffs.

Therefore, the agent gets a rent only if the principal wants him to work in period 1 and also in period 2 after the bad outcome in period 1 with a strictly positive probability.

However, if the discount factor is small for the given persistence of the states, it can be optimal to have the agent work in both periods and leave the rent. The amount of outcome the principal loses by taking his outside option in period 1 is

$$-c + \pi^1 \binom{p_H}{p_L} - \underline{u}.$$

If he takes his outside option in period 2 after the good or bad outcomes in period 1, it is

$$\delta \pi^1 \binom{p_H}{p_L} (-c + \pi^2 \binom{p_H}{p_L} - \underline{u})$$

and

$$\delta \pi^1 \binom{1-p_H}{1-p_L} (-c + \tilde{\pi}^2 \binom{p_H}{p_L} - \underline{u}),$$

respectively.

If the loss in outcome is greater than the rent to the agent, the principal will choose to have the agent work in both periods and leave the rent. This is the case if

$$\delta c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi^1 \binom{1-p_H}{1-p_L} \tilde{\pi}^2 \binom{p_H}{p_L}} < -c + \pi^1 \binom{p_H}{p_L} - \underline{u}$$

and

$$\delta c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi^1 \binom{1-p_H}{1-p_L} \tilde{\pi}^2 \binom{p_H}{p_L}} < \delta \pi^1 \binom{1-p_H}{1-p_L} (-c + \tilde{\pi}^2 \binom{p_H}{p_L}) - \underline{u}.$$

These calculations show that the principal leaves the rent only if he wants the agent to work in both periods. If the principal takes his outside option in the first period or in the second period after the bad outcome, the principal can leave no rent to the agent.

The above calculations also show that it can be optimal to have the agent work both periods and leave him with the rent. Since the outside option is inefficient, the principal incurs loss in outcome by taking the outside option, and if the loss in outcome is greater than the rent to the agent, the principal prefers to leave the rent and have the agent work in both periods.

Proposition 1.3. *Suppose the parameters satisfy the following two inequalities:*

$$\delta \leq \frac{-c + \pi^1 \binom{p_H}{p_L} - \underline{u}}{c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi^1 \binom{1-p_H}{1-p_L} \tilde{\pi}^2 \binom{p_H}{p_L}}},$$

$$c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi^1 \binom{1-p_H}{1-p_L} \tilde{\pi}^2 \binom{p_H}{p_L}} \leq \pi^1 \binom{1-p_H}{1-p_L} (-c + \tilde{\pi}^2 \binom{p_H}{p_L}) - \underline{u}.$$

It is optimal to induce work in both periods and leave the rent to the agent. The optimal contract is the contract given in Proposition 1.2. Otherwise, it is optimal to take the outside option in some periods.

There are a few things to note here that will hold in the general model with the infinite horizon.

Remark 1.1. *The principal has to leave the rent only if he wants the agent to work in both period 1 and period 2 after the bad outcome in period 1. Also, in the one period example, the principal doesn't have to leave the rent. This shows that the rent is an informational rent; the principal is leaving the rent because the agent's deviation leads to a different prior in the following period.*

Remark 1.2. *The minimum rent is proportional to the discount factor and the persistence of the states. When the environment is i.i.d., the principal can leave no rent*

and have the agent work in both periods.

Remark 1.3. *The composition of the continuation value in period 2 after the bad outcome matters for the IC constraint in period 1. After the good outcome in period 1, only the continuation value matters, and the composition of it in period 2 doesn't matter.*

Remark 1.4. *If the principal wants the agent to work in both periods, he can offer a history-independent contract. The principal's prior in the second period depends on the outcome, but the contract doesn't depend on the principal's information.*

1.4 Informational Rent in the Infinite Horizon Model

This section discusses the general model with an infinite horizon. As was the case with the two-period model, the informational rent arises in this environment. In this section, I provide upper and lower bounds on the informational rent. If the discount factor, the persistence of the states or the informativeness of the outcome is small, the principal can approximate his first-best payoff.

Before discussing the IC constraints of the agent, I will first define two notations. Let $P(h^t \sqcup h^k | h^t, \pi)$ be the conditional probability of history $h^t \sqcup h^k$ given h^t when the prior on the states in the period after h^t is π and the agent works in every period in which the principal doesn't take his outside option. Note that π is the prior on the states in the period following h^t . The principal updates his posterior given history h^t , and the posterior needs to be multiplied by the Markov matrix M as it transits to the following period.

The second notation is $V(h^t, \pi)$. A contract specifies history-contingent payments, $w(h^t)$ for all histories h^t . $V(h^t, \pi)$ is the agent's continuation value from working in every period when the continuation contract is conditional on the history h^t and π is the prior on the states in the period following h^t :

$$V(h^t, \pi) = \sum_{k=0}^{\infty} \delta^k \sum_{h^k} u(h^t \sqcup h^k),$$

where $u(h^t \sqcup h^k)$ is the expected payoff from working after $h^t \sqcup h^k$. Formally, $u(h^t \sqcup h^k)$ is given by

$$u(h^t \sqcup h^k) = q(h^t \sqcup h^k)(-P(h^t \sqcup h^k|h^t, \pi)c + P(h^t \sqcup h^k 1|h^t, \pi)w(h^t \sqcup h^k 1) + P(h^t \sqcup h^k 0|h^t, \pi)w(h^t \sqcup h^k 0)),$$

where the principal takes his outside option with probability $1 - q(h^t \sqcup h^k)$ after $h^t \sqcup h^k$.

Consider the IC constraints of the agent in period t given history h^{t-1} . Let π^t be the prior on the states. The agent can deviate for T periods before he starts working again, and there is an infinite sequence of IC constraints. The IC constraint for deviating for T periods is

$$V(h^{t-1}, \pi^t) \geq \sum_{k=1}^T \delta^{k-1} q(h^{t-1} \sqcup \tilde{h}^{k-1}) w(h^{t-1} \sqcup \tilde{h}^k) + \delta^T V(h^{t-1} \sqcup \tilde{h}^T, \pi^t M^T),$$

where $\tilde{h}^0 = \emptyset$ and $h^{t-1} \sqcup \tilde{h}^k, 1 \leq k \leq T$, are defined by

$$h_{t-1+k} = \begin{cases} 0 & \text{if the agent is induced to work but shirks,} \\ -1 & \text{if the principal takes his outside option.} \end{cases}$$

When the agent deviates, it has two effects. The first effect is the outcome consequence; by shirking in period t , the agent produces the bad outcome, and the continuation contract corresponds to the bad outcome in period t . The second effect is the information asymmetry between the principal and the agent. Since the states are unobservable, the principal and the agent have priors on the states. When the principal believes that the agent worked in a given period, the principal updates his prior after observing the outcome. However, if the agent deviates in period t , the agent doesn't update his prior after observing the bad outcome. In period $t + 1$, the principal and the agent have different priors on the state; when the principal believes that the agent worked in period t , the bad outcome lowers the prior on the good state, and the agent assigns strictly higher probability on the good state than the principal does.

In the IC constraints of the agent, the agent receives the payments for the bad outcomes while he deviates. After deviating for T periods, the continuation contract is the one for the history $h^{t-1} \sqcup \tilde{h}^T$. The second effect, the information asymmetry, is captured by the term $\pi^t M^T$. The updating of the priors and the Markov transition preserve the ordering of the priors, and once the agent deviates, in all future periods, he assigns a weakly higher probability to the good state than the principal does.

After each history h^{t-1} , there is a sequence of IC constraints for the agent. If the principal wants the agent to work, all the IC constraints have to be satisfied, and the maximum continuation value of all possible deviations is the rent the agent gets by working in period t .

Proposition 1.4. *After history h^{t-1} , the rent for the agent is bounded from below by*

$$\max_{T \geq 1} \left[\sum_{k=1}^T \delta^{k-1} q(h^{t-1} \sqcup \tilde{h}^{k-1}) w(h^{t-1} \sqcup \tilde{h}^k) + \delta^T V(h^{t-1} \sqcup \tilde{h}^T, \pi^t M^T) \right],$$

where π^t is the prior given history h^{t-1} , and the principal takes his outside option with probability $1 - q(h^t \sqcup h^k)$ after $h^t \sqcup h^k$. $\tilde{h}^0 = \emptyset$ and $h^{t-1} \sqcup \tilde{h}^k, 1 \leq k \leq T$, are defined by

$$h_{t-1+k} = \begin{cases} 0 & \text{if the agent is induced to work but shirks,} \\ -1 & \text{if the principal takes his outside option.} \end{cases}$$

Proposition 1.4 shows that there is a lower bound on the rent to the agent. On the other hand, the principal can attain an upper bound on the informational rent by offering the following contract:

$$w(h^t 1) = \frac{c}{\tilde{\pi}^{t+1} \binom{p_H}{p_L}}, w(h^t 0) = 0, \forall t \geq 0,$$

where $\tilde{\pi}^{t+1}$ is the prior on the states given history $h^t = \underbrace{0 \cdots 0}_t$.

When the contract is stationary, the payments don't depend on the history of the outcomes, and the information asymmetry between the principal and the agent has

no effect on the continuation contract. As long as the expected payment in the given period is above the cost, the agent is willing to work. Since $\tilde{\pi}^t$ assigns the lowest probability on the good state among the priors that can arise after any history of the outcomes of $t - 1$ periods, the above contract provides enough incentives for the agent to work in every period. The expected rent to the agent is bounded from above by

$$\begin{aligned}
& \sum_{t=1}^{\infty} \delta^{t-1} \left(-c + \pi^1 M^{t-1} \begin{pmatrix} p_H \\ p_L \end{pmatrix} \frac{c}{\pi^t \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \right) \\
&= \sum_{t=2}^{\infty} \delta^{t-1} \left(-c + \pi^1 M^{t-1} \begin{pmatrix} p_H \\ p_L \end{pmatrix} \frac{c}{\pi^t \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \right) \\
&\leq \sum_{t=1}^{\infty} \delta^t \left(-c + \pi^1 M^t \begin{pmatrix} p_H \\ p_L \end{pmatrix} \frac{c}{M_2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \right) \\
&= \frac{c}{M_2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \frac{\delta \det M}{1 - \delta \det M} (p_H - p_L) \left(\frac{\delta}{1 - \delta} M_{21} + \pi_1^1 \right).
\end{aligned}$$

Note that the upper bound is zero when the states are i.i.d.. The principal can induce working in every period and leave no rent to the agent.

When the discount factor is small for the given parameters, or when the persistence of the states or the informativeness of the outcomes is small, the principal can approximate his first best payoff by offering the stationary contract.

Proposition 1.5. *There exists an upper bound on the rent given by*

$$\frac{c}{M_2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \frac{\delta \det M}{1 - \delta \det M} (p_H - p_L) (\delta M_{21} + (1 - \delta) \pi_1^1).$$

Given $\epsilon > 0$, there exists $\bar{\delta}$ such that for $\delta < \bar{\delta}$, the principal can approximate his first best payoff by ϵ with a contract that is stationary from period 2. Conversely, for given δ , there exists \bar{D} and Δ_p such that if $\det M < \bar{D}$ or $p_H/p_L < \Delta_p$, the principal can approximate his first best payoff with a contract that is stationary from period 2.

1.5 States as Outcomes: Stationary Contracts as Cost-Minimizing Contracts

This section discusses the case in which the states correspond to the outcomes of working. The good state is in which work produces the good outcome, and work produces the bad outcome in the bad state. In this environment, the cost-minimizing contract that induces the agent to work in every period can be stationary, and the principal can offer a constant payment for the good outcome and minimize the rent to the agent. I obtain a tight lower bound on the rent to the agent when the principal wants the agent to work in every period, and the lower bound increases with the discount factor and the persistence of the states. The second best contracts are always fully history-contingent, and the stationary contracts are not optimal in general, but I'll show in Section 1.6 that the stationary contract is part of the second best contract.

When the states correspond to the outcomes of working, the mapping from the states to the outcomes is deterministic. If the states were observable, the principal would know the outcome of working from the state. Having a constant prior means that the states are distributed i.i.d. in every period. However, when the states follow a Markov matrix, the prior after the good outcome is $M_1 = (M_{11}, M_{12})$ and the prior after the bad outcome is $M_2 = (M_{21}, M_{22})$. This implies that work produces the good outcome with probability M_{11} after the good outcome, and it produces the good outcome with probability M_{21} after the bad outcome. Having different probabilities $M_{11} > M_{21}$ precisely captures the persistence of the outcomes of working. An alternative interpretation would be that the good state follows a good outcome in which work is productive with probability M_{11} ; the bad state follows a bad outcome, and work is productive with probability $M_{21} < M_{11}$.

Consider an agent who is in charge of making an innovation. The agent's effort will be productive if an innovation is available, and it will be unproductive if an innovation is unavailable. The good state then will be the state in which an innovation is available, and the bad state is when it is not. Since the states are unobservable, the principal and the agent will have priors on the states from the previous outcome

and whether the agent has worked or not, and the priors are the probabilities that they think work will produce a good outcome. Other examples can be modeled in a similar way.

When the states correspond to the outcomes of working, the cost-minimizing contract to have the agent work in every period can be completely stationary from the second period. The principal can offer a constant payment for the good outcome and minimize the rent to the agent.

Proposition 1.6. *Suppose $p_H = 1, p_L = 0$. If the principal wants the agent to work in every period, a cost-minimizing contract is to provide*

$$\begin{aligned} w(1) &= \frac{c}{\pi_1^1}, w(0) = 0, \\ w(h^t 1) &= \frac{c}{M_{21}}, w(h^t 0) = 0, \forall h^t, t \geq 1. \end{aligned}$$

Before proving Proposition 1.6, I will first characterize the lower bound on the rent to the agent when the principal wants him to work in every period. The proof of Proposition 1.6 follows by showing that the principal attains the lower bound with the stationary contract.

Proposition 1.7. *Suppose $p_H = 1, p_L = 0$. If the principal wants the agent to work in every period, the average rent to the agent is bounded from below by*

$$\frac{\delta \det M}{1 - \delta \det M} c \left(\delta + (1 - \delta) \frac{\pi_1^1}{M_{21}} \right).$$

Proof of Proposition 1.7. Consider the IC constraint for the one-shot deviation after history h^t . When the states correspond to the outcomes of working, the prior on the state is completely determined by the state in the previous period. In particular, the prior on the state after the good outcome is $M_1 = (M_{11}, M_{12})$, and the prior after the bad outcome believing that the agent worked in the previous period is $M_2 = (M_{21}, M_{22})$. Therefore, any subgame after the good outcome is identical, and any subgame after the bad outcome is also identical.

If the principal wants the agent to work in every period, he doesn't take his outside option after any history h^t . The IC constraint for the one-shot deviation is given by

$$\begin{aligned} -c + \pi^{t+1} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} &\geq w(h^t 0) + \delta(-c + \pi^{t+1} M \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix}) \\ \Leftrightarrow V_1 - V_2 &\geq \frac{c}{\pi_1^{t+1}} + \delta \det M(V'_1 - V'_2), \end{aligned}$$

where π^{t+1} is the principal's prior after h^t and V_1 is the sum of the present compensation and the continuation value after $h^t 1$. V_2, V'_1 and V'_2 are defined similarly for $h^t 0, h^t 01$ and $h^t 00$.

Let V_1^t, V_2^t be the sum of the present compensation and the continuation value after $h^t = \underbrace{0 \cdots 0}_{t-1} 1$ and $h^t = \underbrace{0 \cdots 0}_t$. We have the following set of IC constraints:

$$\begin{aligned} V_1^1 - V_2^1 &\geq \frac{c}{\pi_1^1} + \delta \det M(V_1^2 - V_2^2), \\ V_1^t - V_2^t &\geq \frac{c}{M_{21}} + \delta \det M(V_1^{t+1} - V_2^{t+1}), \forall t \geq 2. \end{aligned}$$

From Proposition 1.5, the IC constraint and the limited liability condition, we know that $V_1^t - V_2^t$ is bounded for all t under an optimal contract, and summing over the IC constraints, we get

$$\begin{aligned} V_1^t - V_2^t &\geq \frac{c}{M_{21}} \frac{1}{1 - \delta \det M}, \forall t \geq 2, \\ V_1^1 - V_2^1 &\geq \frac{c}{\pi_1^1} + \frac{c}{M_{21}} \frac{\delta \det M}{1 - \delta \det M}. \end{aligned}$$

Together with

$$\begin{aligned} V_2^t &= w(\underbrace{0 \cdots 0}_t) + \delta(-c + M_{21}V_1^{t+1} + M_{22}V_2^{t+1}) \\ &\geq \delta(-c + M_{21}(V_1^{t+1} - V_2^{t+1}) + V_2^{t+1}), \forall t \geq 1 \end{aligned}$$

we have

$$V_2^t \geq \delta \left(\frac{\delta \det M}{1 - \delta \det M} c + V_2^{t+1} \right)$$

and

$$V_2^1 \geq \frac{\delta}{1-\delta} \frac{\delta \det M}{1-\delta \det M} c.$$

The average rent to the agent is bounded from below by

$$\begin{aligned} & (1-\delta)(-c + \pi_1^1 V_1^1 + \pi_2^1 V_2^1) \\ &= (1-\delta)(-c + \pi_1^1(V_1^1 - V_2^1) + V_2^1) \\ &\geq (1-\delta)(-c + \pi_1^1(\frac{c}{\pi_1^1} + \frac{c}{M_{21}} \frac{\delta \det M}{1-\delta \det M})) + \frac{\delta}{1-\delta} \frac{\delta \det M}{1-\delta \det M} c) \\ &= \frac{\delta \det M}{1-\delta \det M} c(\delta + (1-\delta) \frac{\pi_1^1}{M_{21}}). \end{aligned}$$

□

From Proposition 1.7, we can show Proposition 1.6 as the following.

Proof of Proposition 1.6. When the principal offers a stationary contract, in any period following a history h^t , the agent chooses to work as long as

$$-c + \pi^{t+1} \left(\frac{w(h^t 1)}{w(h^t 0)} \right) \geq w(h^t 0).$$

The agent's IC constraints become myopic because there is no gain in creating information asymmetry. If the principal offers a constant payment for the good outcome, the payment is unaffected by the principal's prior, and the continuation value after a deviation is the same as the continuation value on the equilibrium path. Therefore, in deciding whether to work or shirk, the agent only cares about the payment in the current period, and as long as the expected payoff from working is greater than the payment for the bad outcome, the agent chooses to work.

The contract specified in Proposition 1.6 provides the following constant payments:

$$w(h^t 1) = \frac{c}{M_{21}}, w(h^t 0) = 0, \forall h^t, t \geq 1.$$

Since the agent's prior on the state satisfies $\pi_1^t \geq M_{21}$, the agent's IC constraints are

satisfied after any history $h^t, t \geq 1$. In period 1, the principal pays

$$w(1) = \frac{c}{\pi_1^1}, w(0) = 0,$$

and again, the agent chooses to work.

To show the optimality of the contract, we need to show that the rent to the agent is minimized with this contract. Under the specified contract, the agent gets rent if and only if the outcome in the previous period is good, and in each of those periods, he gets

$$-c + M_{11}w(h^t1) = \frac{\det Mc}{M_{21}}.$$

The probability of a good outcome in period t given the initial prior π^1 is

$$\pi^1 M^{t-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the average rent to the agent under the contract is

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^t \pi^1 M^{t-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\det Mc}{M_{21}} = \frac{\delta \det Mc}{1 - \delta \det M} \left(\delta + (1 - \delta) \frac{\pi_1^1}{M_{21}} \right),$$

which is the lower bound on the rent to the agent given in Proposition 1.7. Therefore, the principal can attain the lower bound on the rent with a stationary contract, and the cost-minimizing contract to have the agent work in every period can be stationary. \square

Proposition 1.6 shows that the cost-minimizing contract that induces the agent to work in every period can be made to be stationary when the state variable is whether working will produce a good outcome. I'll show in Section 1.6 that the stationary contract is a part of the second best, but the next two propositions show that stationary contracts are not optimal more generally: the assumption of a deterministic mapping from states to outcomes is needed, and even under the deterministic mapping, the principal prefers to take his outside option after an enough number of bad outcomes. First, Proposition 1.8 shows that in order for an optimal contract to be

history-independent, it is necessary that the mapping from the states to the outcomes is deterministic. I define the history-independent contracts as contracts with payments of the following form:

$$\begin{aligned} w(h^t 1) &= w_t(1), \forall h^t, \\ w(h^t 0) &= w_t(0), \forall h^t. \end{aligned}$$

Proposition 1.8. *The second best contract can be independent of the history only if the following condition holds:*

$$p_H = 1, p_L = 0.$$

The proof of Proposition 1.8 goes as the following. When a contract is history-independent, the agent's IC constraints become myopic since there is no gain from creating information asymmetry. As long as the expected payoff from working in the given period is greater than the payment for the bad outcome, the agent chooses to work. Then the principal can adjust the payments after some histories, and by front-loading the payments for which the IC constraints don't bind, the principal can keep the continuation value constant while reducing the deviation payoff of the agent. It allows the principal to lower the payments after some histories and reduce the rent. The only exception is when $p_H = 1, p_L = 0$ and the principal doesn't benefit from front-loading the payment.

The next proposition shows that in the special case with $p_H = 1, p_L = 0$, the principal wants to take his outside option after some histories. Together with Proposition 1.8, the proposition shows that the second best contracts in the general model will be fully history-contingent and the stationary contracts are not optimal.

Proposition 1.9. *Suppose $p_H = 1, p_L = 0$. There exists $t_0 > 0$ such that for any $\delta > 0$, a contract that involves taking the outside option after t_0 bad outcomes since period 1 gives a strictly higher payoff to the principal than the cost-minimizing contract that induces working in every period.*

The proof of Proposition 1.9 consists of two steps. The first step is to show that for $p_H = 1, p_L = 0$, it is sufficient to consider the one-shot deviations of the agent.

When the IC constraints for the one-shot deviations are satisfied, the agent doesn't deviate more than once at a time, and all IC constraints are satisfied.

The second step is to show that by taking the outside option after some history, the principal can lower the payments to the agent leading up to the specified history. If the principal takes his outside option after t_0 bad outcomes from period 1, the principal can lower the payments $w(\underbrace{0 \cdots 0}_k 1)$ for $0 \leq k < t_0$. When the reduction in the rent is greater than the loss in outcome from taking the outside option, the principal prefers to take the outside option after $h^{t_0} = \underbrace{0 \cdots 0}_{t_0}$.

The number of bad outcomes before the principal takes his outside option holds uniformly for all discount factors δ . Given the Markov matrix M , there exists t_0 such that for any discount factor δ , taking the outside option after t_0 bad outcomes strictly dominates inducing the agent to work in every period.

1.6 Second Best: Optimality of Tenure Contracts

This section characterizes the second best contracts in the same special case of the model studied in Section 1.5: I assume that the state variable is whether or not the project will succeed in period t . I show that in this case, the second best contracts take the form of a tenure system. The agent is paid nothing during the probationary period, and once the principal makes the positive payment, the agent is tenured, and the principal never takes his outside option again. There is no loss of generality in assuming that the principal makes positive payments only for the good outcome, and after two periods since the initial payments, the principal can offer a stationary contract. I also provide a recursive formulation to decide how long the probationary period lasts and what the initial payment is.

The first result of this section is that the second best contracts take the form of a tenure system. During the probationary period, the continuation value of the agent and the decision to take the outside option depends on the history of the outcomes, but the agent is paid nothing during this period. Once the principal makes a positive payment, the agent is tenured, and the principal never takes his outside option again.

Proposition 1.10. *Suppose $p_H = 1, p_L = 0$. Under the second best contract, once the principal makes a positive payment, the principal never takes his outside option again. For any history h^t such that $w(h^t) > 0$, the principal induces work after history $h^t \sqcup h^k, \forall h^k, k \geq 0$.*

The proof of Proposition 1.10 relies on the next proposition and the fact that the composition of the continuation value after the good outcome doesn't matter for the agent's IC constraints.

Proposition 1.11. *Suppose $p_H = 1, p_L = 0$. In characterizing the second best, there is no loss of generality in restricting attention to contracts under which the principal makes positive payments only for the good outcome.*

The proof of Proposition 1.11 is in the appendix.

Proof of Proposition 1.10. Given a contract, let R and L be the rent and the loss in outcome under the contract. L is defined to be

$$L = (1 - \delta)(Y_{FB} - Y),$$

where Y_{FB} is the expected discounted sum of the outcome in the first best, and Y is the expected discounted sum of the outcome under the given contract.

Consider the space of (R, L) for the initial prior π . I allow the principal to randomize continuation contracts, and the set of all feasible (R, L) is a convex set. In particular, there is a one to one mapping

$$f : [0, L_\pi] \rightarrow [0, \infty)$$

such that the set of feasible (R, L) is given by

$$X_\pi \equiv \{(R, L) | R \geq f(L), 0 \leq L \leq L_\pi\}$$

and

$$L_\pi \equiv (1 - \delta) \sum_{k=1}^{\infty} \delta^{k-1} (-c + \pi M^{k-1} \binom{1}{0} - \underline{u})$$

is the expected loss in outcome from taking the outside option forever.

From Proposition 1.7, we know that

$$f(0) = \frac{\delta \det M_c}{1 - \delta \det M} \left(\delta + (1 - \delta) \frac{\pi_1}{M_{21}} \right)$$

is the minimum rent to the agent under the cost-minimizing contract. From the fact that X_π is convex, we also know that $f(\cdot)$ is strictly decreasing in L .

Since both the principal and the agent are risk-neutral, the principal can always delay the payment. Suppose the principal makes a positive payment for h^t under the second best contract. From Proposition 1.11, we can assume that the principal makes the positive payment for a good outcome, and $h_t = 1$. After the good outcome, only the sum of the present compensation and the continuation value matters for the agent's IC constraint, and the principal can replace the continuation contract. Instead of paying $w(h^t)$ and continuing with $V(h^t, M_1)$, the principal can offer $\hat{w}(h^t) = 0$ and

$$\hat{V}(h^t, M_1) = \frac{1}{\delta} w(h^t) + V(h^t, M_1).$$

If $V(h^t, M_1) < f(0)$, we get

$$f^{-1}(\hat{V}(h^t, M_1)) < f^{-1}(V(h^t, M_1)).$$

The principal can replace the continuation contract with a contract with a lower L , and the principal's payoff strictly increases.

Therefore, if the principal makes a positive payment under the second best contract, he doesn't gain from delaying that payment, which means that the agent's continuation value $V(h^t, M_1)$ is at least as big as the minimum rent under the cost-minimizing contract; the principal doesn't lose anything in outcome under the continuation contract. The principal never takes his outside option after he makes a positive payment. \square

Proposition 1.10 and 1.11, together with Proposition 1.6, imply that the principal can restrict attention to the following form of contracts.

Proposition 1.12. *Suppose $p_H = 1, p_L = 0$. There is no loss of generality in restricting attention to the following contracts: suppose the principal makes a positive payment for the first time after $h^t = h^{t-1}$. In the following period, he offers*

$$w(h^t 1) = \frac{c}{M_{11}}, w(h^t 0) = 0,$$

and the next period and on, he offers

$$w(h^t \sqcup h^k 1) = \frac{c}{M_{21}}, w(h^t \sqcup h^k 0) = 0, \forall h^k, k \geq 1.$$

Note that the contract becomes completely stationary after two periods since the initial payment. In particular, we will observe that the principal makes the constant payments for the good outcomes on the equilibrium path. The states are changing, and the prior of the principal also changes over time, but it is optimal to commit not to use his information and offer a stationary contract. We know from Section 1.5 that the second best contracts are fully history-contingent, and it is never optimal to induce working in every period. However, the second best contracts turn out to be history-contingent only until the agent is tenured. Once the agent is tenured, it is optimal to induce working in every period, and the principal can offer a stationary contract regardless of his information.

Proposition 1.12 allows us to offer a stationary contract after two periods since the initial payment. Until the agent is tenured, the contract is history-contingent, and the timing of the initial payment and the outside options depends on the history of the outcomes. I will provide a recursive formulation to characterize the timing of the initial payment and the outside options, but before doing so, I'll prove one more proposition and a corollary on the dynamics of the continuation values.

I will call the periods before tenure probationary. The timing of the tenure is history-contingent, and any history before the tenure is granted is probationary. During the probationary period, the agent's continuation value strictly increases after the good outcome.

Proposition 1.13. *Suppose $p_H = 1, p_L = 0$. Given the history h^t , let R and R_1 be the agent's continuation values after h^t and $h^t 1$, respectively. For any h^t such that $h^t 1$ is in the probationary period, we have*

$$R < R_1.$$

Proof. From Assumption 2, the principal takes his outside option when the agent is not induced to work. Therefore, any outcome is on the equilibrium path from the principal's perspective, and the principal provides the payments specified in the contract. In particular, after each good outcome, the continuation value of the agent is exactly the amount the principal intends to provide under the contract.

During the probationary period, the agent is not paid anything. The agent's continuation value R can be written as

$$\begin{aligned} R &= -c + \pi_1 V_1 + \pi_2 V_2 \\ &< -c + V_1 \\ &= -c + \delta R_1 \\ &< R_1, \end{aligned}$$

where π is the prior on the states after h^t , and the first inequality follows from $V_1 > V_2$ under the optimal contract. \square

Corollary 1.1. *Suppose $p_H = 1, p_L = 0$. The second best contract never terminates after a good outcome.*

Proof. If the contract terminates, the agent's continuation value is zero. From $R_1 > R \geq 0$, the agent's continuation value is strictly positive after a good outcome, and the contract doesn't terminate. \square

The recursive formulation consists of two steps. The first step is to characterize the pairs of continuation values (V_1, V_2) with which the principal can induce working in the given period. The second step incorporates the loss in outcome under a given

contract. The principal chooses a contract that minimizes the sum of the rent to the agent and the loss in outcome.

Proposition 1.14. *Suppose $p_H = 1, p_L = 0$. The second best contract can be found from the following two sets: S is the largest self-generating set*

$S = \text{conv}(\{(\pi, V_1, V_2) | \exists T \geq 0, (\pi', V'_1, V'_2) \in S \text{ such that}$

$$(i) \pi' = M_2 M^T,$$

$$(ii) V_2 = \delta^{T+1}(-c + \pi' \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix}),$$

$$(iii) V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1}(\det M)^{T+1}(V'_1 - V'_2)\},$$

and

$X_\pi = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_{M_1}, (R_2, L_2) \in X_{M_2} \text{ such that}$

$$(i) \pi' = \pi M^T,$$

$$(ii) R = \delta^T(-c + \delta \pi' \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}),$$

$$(iii) L = (1 - \delta) \sum_{k=1}^T \delta^{k-1}(-c + \pi M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \underline{u}) + \delta^{T+1} \pi' \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

$$(iv) (\pi', \delta R_1, \delta R_2) \in S\})$$

is generated from X_1^0 and X_2^0 .

X_{M_1} and X_{M_2} are jointly determined as the limits of X_1^n, X_2^n :

$$X_1^0 = \{(R, 0) | R \geq R_1^* \equiv \frac{\delta \det M c}{1 - \delta \det M} (\delta + (1 - \delta) \frac{M_{11}}{M_{21}})\},$$

$$X_2^0 = \{(R, 0) | R \geq R_2^* \equiv \frac{\delta \det M c}{1 - \delta \det M}\},$$

where R_1^* and R_2^* are the rents to the agent under the cost-minimizing contracts for priors M_1 and M_2 , and

$X_1^{n+1} = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_1^n, (R_2, L_2) \in X_2^n \text{ such that}$

$$(i) \pi' = M_1 M^T,$$

$$(ii) R = \delta^T (-c + \delta \pi' \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}),$$

$$(iii) L = (1 - \delta) \sum_{k=1}^T \delta^{k-1} (-c + M_1 M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \underline{u}) + \delta^{T+1} \pi' \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

$$(iv) (\pi', \delta R_1, \delta R_2) \in S\},$$

$X_2^{n+1} = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_1^n, (R_2, L_2) \in X_2^n \text{ such that}$

$$(i) \pi' = M_2 M^T,$$

$$(ii) R = \delta^T (-c + \delta \pi' \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}),$$

$$(iii) L = (1 - \delta) \sum_{k=1}^T \delta^{k-1} (-c + M_2 M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \underline{u}) + \delta^{T+1} \pi' \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

$$(iv) (\pi', \delta R_1, \delta R_2) \in S\}.$$

The second best contract given the initial prior π is the one that minimizes $R + L$ such that $(R, L) \in X_\pi$.

Once we find $(R, L) \in X_\pi$ that minimizes $R + L$, the contract can be constructed as the following. Given $(R, L) \in X_\pi$, there exist $T \geq 0, (R_1, L_1) \in X_{M_1}, (R_2, L_2) \in X_{M_2}$ supporting (R, L) . The principal takes the outside option for T periods, and after the first period the agent is induced to work, the continuation contract is determined by (R_1, L_1) and (R_2, L_2) . If the outcome is good and $R_1 < R_1^*$, the contract continues with (R_1, L_1) . If the outcome is good and $R_1 \geq R_1^*$, the agent is tenured and is paid $\delta(R_1 - R_1^*)$ this period. The contract continues with $(R_1^*, 0)$, the contract specified in Proposition 1.12. If the outcome is bad and $0 < R_2 < R_2^*$, the contract continues with (R_2, L_2) . If the outcome is bad, but $R_2 = R_2^*$, the agent is tenured, and the principal pays

$$w(h^t 1) = \frac{c}{M_{21}}, w(h^t 0) = 0$$

from the following period. If the outcome is bad and $R_2 = 0$, the contract terminates.

1.7 First Best Approximation

In this section, I discuss the first best approximation of the general model. As the discount factor approaches one, the principal can get arbitrarily close to his first best payoff.

Proposition 1.15. *Given $\epsilon > 0$, there exists $\bar{\delta}$ such that for any $\delta > \bar{\delta}$, the principal's average per period payoff in the second best is within ϵ of his first best payoff.*

Consider the following review contract. The contract specifies a review block of T periods, a quota and a lump sum transfer. A quota is on the number of successful outcomes from the block. If the agent meets the quota, the principal pays the agent the discounted sum of the outcome subtracted by the lump sum transfer at the end of the review block, and the contract continues. If the agent fails to meet the quota, the principal pays the agent the discounted sum of the outcome, and the contract terminates.

First, consider the principal's payoff. Let s be the agent's strategy in the equilibrium, $p(s)$ be the minimum probability he meets the quota and X be the lump sum transfer. In general, the probability the agent meets the quota with a strategy depends on the prior on the state at the beginning of the review block, but we can take the minimum of the probabilities over the priors. Then the principal's average per period payoff is at least

$$\begin{aligned} & (1 - \delta)\delta^{T-1}p(s)X(1 + \delta^T p(s) + \delta^{2T} p(s)^2 + \dots) \\ &= \frac{\delta^{T-1}p(s)(1 - \delta)X}{1 - \delta^T p(s)} \\ &= \frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^T p(s)} \frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta}. \end{aligned}$$

Since the states exhibit positive persistence, the expected discounted sum of the outcome in the first best is the maximum when the pair starts with $\pi^1 = (1, 0)$. Let

\bar{y} be the average expected discounted sum of the outcome over the infinite horizon in the first best when $\pi^1 = (1, 0)$. When the following two inequalities hold,

$$\begin{aligned}\frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^T p(s)} &\geq 1 - \frac{\epsilon}{2}, \\ \frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta} &\geq (1 - \frac{\epsilon}{2})\bar{y} - c,\end{aligned}$$

the principal's payoff is at least $(1 - \epsilon)\bar{y} - c \geq \bar{y} - c - \epsilon$.

Roughly speaking, the first inequality says that the agent meets the quota with a high enough probability. The review block is sufficiently long to have $p(s)$ close to one. Then, there is also a lower bound on the discount factor so that δ^T is close to one. If the review block is too long for the given discount factor, the lump sum transfer the principal gets at the end of the review block is discounted too much for the principal's payoff to be close to his first best payoff. Therefore, the review block is sufficiently long to have a high probability of meeting the quota and yet not too long for the given discount factor so that the principal's payoff is not discounted too much.

The second inequality says that the lump sum transfer the principal gets on meeting the quota is close to his first best payoff. The expected outcome in the first best in any given period increases with the prior the pair puts on the good state, and together with the positive persistence, the expected discounted sum of the outcome is maximum when they start believing they are in the good state. If the lump sum transfer is above $(1 - \frac{\epsilon}{2})\bar{y} - c$, it is above $1 - \frac{\epsilon}{2}$ times the first best outcome for any initial prior, subtracted by the cost.

It remains to verify the agent's incentives that the agent will pass the quota with $p(s)$ close to one. Let $V(\pi)$ be the agent's continuation value when the review block starts with prior π . Since the agent can always choose to work in every period, letting s be the strategy of working in every period, we have

$$V(\pi) \geq (1 - \delta)(Y(\pi) - \frac{1 - \delta^T}{1 - \delta}c) + \delta^T p(s)(\mathbb{E}[V(\hat{\pi})] - \frac{1 - \delta}{\delta}X),$$

where $Y(\pi)$ is the expected discounted sum of the outcome from working in every period from a block with the initial prior π , and $\hat{\pi}$ is the prior in the beginning of the new block.

Let \underline{V} be the minimum of $V(\pi)$ over all priors π . Together with the fact that $Y(\pi)$ increases with π_1 , we get the following inequality:

$$\underline{V} \geq \frac{(1 - \delta)(Y((0, 1)) - \frac{1 - \delta^T}{1 - \delta}c) - \delta^T p(s) \frac{1 - \delta}{\delta} X}{1 - \delta^T p(s)}.$$

If $\underline{V} \geq \frac{1 - \delta}{\delta} X$, the agent always prefers to increase the probability of meeting the quota. Since the principal pays the agent the discounted sum of the outcome, subtracted by X on meeting the quota, the agent works in every period on the equilibrium path.

When the lump sum transfer is specified to

$$X \leq \delta(Y((0, 1)) - \frac{1 - \delta^T}{1 - \delta}c),$$

the inequality $\underline{V} \geq \frac{1 - \delta}{\delta} X$ is always satisfied. The last condition is to ensure that the discounted sum of the outcome on meeting the quota is weakly greater than X so that the principal can actually take away the lump sum transfer. A slightly stronger condition is

$$Q \geq X,$$

where Q is the number of good outcomes for the quota.

Therefore, when a review contract satisfies

$$\begin{aligned} \frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^T p(s)} &\geq 1 - \frac{\epsilon}{2}, \\ \frac{1 - \delta}{1 - \delta^T} X &\geq (1 - \frac{\epsilon}{2})\bar{y} - c, \\ X &\leq \delta(Y((0, 1)) - \frac{1 - \delta^T}{1 - \delta}c), \\ Q &\geq X, \end{aligned}$$

the agent chooses to work in every period, and the principal's payoff is within ϵ of his

first best payoff.

By the uniform weak law of large numbers, the principal can find $\bar{\delta}$ such that for any $\delta \geq \bar{\delta}$, there exist Q and T that satisfy the above conditions.

The above contract sets the quota on the number of good outcomes. Alternatively, we can set the quota directly on the discounted sum of the outcomes, which allows the principal to approximate his first best payoff in a more general environment. Generally speaking, we need the uniform weak law of large numbers for the discounted sum rather than the time average.

Proposition 1.16. *Let $x_0 \in \hat{X}$ be an initial condition and let $\{X_n\}_{n \geq 1}$ be an \mathbb{R}_+ -valued stochastic process satisfying the following condition: there exists μ such that for given $\epsilon > 0$, there exists δ_0, T_0 such that for every $k, (x_0, x_1, \dots, x_k), \delta \geq \delta_0, T \geq T_0$,*

$$\left| \frac{1-\delta}{1-\delta^T} \left(\sum_{t=1}^T \delta^{t-1} \mathbb{E}[X_{k+t}] \right) - \mu \right| < \epsilon$$

and

$$\Pr\left(\left| \frac{1-\delta}{1-\delta^T} \left(\sum_{t=1}^T \delta^{t-1} X_{k+t} \right) - \mu \right| > \epsilon \right) < \epsilon.$$

When the outcome of working follows a stochastic process $\{X_n\}$, for given $\epsilon > 0$, there exists $\bar{\delta}$ such that for $\delta \geq \bar{\delta}$, the principal's average per period payoff in the second best is within ϵ of his first best payoff.

The conditions in Proposition 1.16 say that the expected discounted sum of the outcomes converges uniformly and the uniform weak law of large numbers holds. A sufficient condition is that there are a finite number of states with an ergodic distribution and the outcome of working is bounded. First-order Markov chains that are irreducible and aperiodic have ergodic distributions, and we have the following proposition.

Proposition 1.17. *Suppose there are a finite number of states following an irreducible, aperiodic first-order Markov chain. The outcome of working is bounded. Given $\epsilon > 0$, there exists $\bar{\delta}$ such that for $\delta \geq \bar{\delta}$, the principal's average per period payoff in the second best is within ϵ of his first best payoff.*

1.8 Conclusion

I study a model of principal-agent problem in a persistent environment in this chapter and show that an informational rent arises when the states are partially persistent. When the states are partially persistent, the agent's effort has both a payoff consequence and informational value. If the principal believes that the agent worked this period, the principal infers about the state by observing the outcome, and the agent's deviation leads to a lower outcome and information asymmetry between the principal and the agent. Following a deviation, the agent assigns weakly higher probabilities to the good state than the principal does in all future periods, and the principal has to provide the maximum of all deviation payoffs the agent can get.

When the states correspond to the outcomes of working, the second best contract resembles a tenure system. In this environment, the principal makes positive payments only for the good outcome, and after the good outcome, only the sum of the present compensation and the continuation value matters for the agent's IC constraint. If the principal makes a positive payment after some history and takes his outside option with a positive probability under the continuation contract, the principal can backload the payment and replace the continuation contract with a contract with a higher expected payoff to the principal. The principal doesn't benefit from backloading the payment only if he is already inducing the agent to work in every period under the continuation contract. Therefore, if the agent is paid, the principal never takes his outside option again, and the agent is tenured.

After the agent is tenured, the principal offers the cost-minimizing contract, which is stationary from the second period. When the states correspond to the outcomes of working, we can express the deviation payoffs of the agent by his continuation values on the equilibrium path, without having to keep track of all future payments. If the principal uses his information to reduce the rent, the agent can deviate and get a positive rent from the one-shot deviation. One way to prevent the deviation is to always provide the payments as if the previous state was bad, and the stationary contract minimizes the rent to the agent.

For discount factors close to one, the principal can approximate his first best payoff. The contract combines the review contracts and the residual claimant argument. At the end of each review block, the agent is paid the discounted sum of the outcome from the block, subtracted by a lump sum transfer if he meets the quota. The principal and the agent continue with the contract only if the agent meets the quota, and the lump sum transfer is chosen so that the agent chooses to work in every period. The principal can use the law of large numbers to ensure that the agent meets the quota with a high probability by working in every period. The law of large numbers also allows the lump sum transfer to be close to the principal's payoff in the first best, and the principal's payoff gets arbitrarily close to his first best payoff as the discount factor goes to one.

In this chapter, I assumed that the principal can commit to a long term contract. When the states are persistent, the expected outcome in the future varies with the state in the given period. If the principal cannot commit to a long term contract, the persistence of the states puts further restrictions on the payments the principal can make, and the effect of the lack of commitment power will be magnified. It will be interesting to consider relational contracts in this environment and see what forms of commitment power or contracts can mitigate the effect of persistence.

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Chapter 2

Relational Contracts in a Persistent Environment

2.1 Introduction

Most literature assumes that in a repeated interaction, the states are independent and identically distributed over time. But the real-world interactions don't always take place in an i.i.d. environment. A shock to the cost of raw material is likely to persist for some time, and if it becomes costly to perform a task this year, a firm may not expect the cost of performing the task next year to be distributed in the same way as it would after a good year. The production technology this period can also depend on the past realization of the productivity. Anticipating the persistence of the states, the employees may not expect the same effectiveness of the compensation scheme every period, and the optimal compensation scheme may in fact depend on the state.

I study optimal relational contracts when the states are partially persistent and there is moral hazard. The principal and the agent trade every period over an infinite horizon, and both parties are risk-neutral with a common discount factor. Under a relational contract, the principal offers a compensation scheme each period, and the agent decides whether or not to accept it and how much effort to exert if he accepts the offer. The principal doesn't observe the agent's effort, which leads to

moral hazard, but he observes the outcome, which is a noisy signal of the agent's effort, and therefore can promise contingent payments on outcome.

At the beginning of each period, the payoff-relevant state is realized and becomes observable to both the principal and the agent. The states are persistent, and the distribution of the states next period given this period's state is known to both parties. I consider both exogenous states and endogenous states.

There is a large literature on relational contracts, including Levin (2003) and Baker, Gibbons, and Murphy (2002). Earlier literature on relational contracts focused on symmetric information case. (see for example, Shapiro and Stiglitz (1984), Bull (1987), MacLeod and Malcolmson (1989), Kreps (1990)) More recent papers consider environments with asymmetric information, and most of the literature assumes that the environment is either stationary or i.i.d. over time. My chapter is most closely related to Levin (2003), where he shows that for i.i.d. states, the principal can focus on maximizing the joint surplus and the optimal contracts can be stationary. The necessary and sufficient condition to implement an effort schedule with stationary contracts is that it satisfies the IC constraint and one other constraint. The optimal contract either implements the first best or is a step function. Other related literature is discussed at the end of this section.

Section 2.3 considers the results that hold for any type of persistence. As was the case with i.i.d. states, the distribution of the joint surplus between the principal and the agent can be separated from the problem of efficient-contracting, and in characterizing the Pareto-optimal contracts, it is sufficient to focus on the joint surplus from the relationship. The principal can always redistribute the surplus through the fixed wage in the initial period. When the states follow a first-order Markov chain, the realization of the state this period is a sufficient static for the distribution of the future states, and the principal can provide all incentives by the bonus payments at the end of this period. It is optimal to provide the same expected per period payoff to the agent in every state, and for each state, the principal can offer a contract that maximizes the joint surplus. In particular, the principal can offer a stationary contract every period. I define stationary contracts as contracts under which the compensation scheme is

identical in all periods; the wage and bonus payments are allowed to depend on the realization of the state and the outcome in the given period, but they don't depend on the history. Under a relational contract, there is a temptation to renege, and the self-enforcement leads to the dynamic enforcement constraint as in the i.i.d. case. The necessary and sufficient condition for an effort schedule to be implementable by a stationary contract is that it satisfies the IC constraint and the dynamic enforcement constraint. I also show that the optimal contract either implements the first best level of effort, or it takes the form of a step function.

In Section 2.4, I consider an alternative model in which the state is endogenous. From an applied perspective, there are often environments where the agent's effort affects the distribution of the state. Specifically, I consider the environment in which the productivity is the state variable. The distribution of productivity for the next period depends on the current productivity and the agent's effort, which implies that the agent's effort affects the distribution of states in all future periods. When the productivity is observable and the persistence is of first-order, however, most results in Section 2.3 generalize to this environment. The distribution of the joint surplus between the principal and the agent can be separated from the problem of efficient contracting, and the optimal contract can still be stationary. The productivity is a sufficient static for the distribution of future states, and the principal can still provide all incentives by the compensation scheme at the end of the period. The principal can offer a stationary contract that leaves the agent indifferent between accepting and rejecting the offer. I also show a version of dynamic enforcement constraint which is, together with the IC constraint, the necessary and sufficient condition to implement an effort schedule with such stationary contracts.

The next section discusses the implications for the markets for random matching in which the principal and the agent can be randomly, anonymously, and costlessly matched with a new partner. The nature of the state leads to starkly different implications for the market. The degree of cooperation varies with the nature of the state, and it also highlights the difference between the i.i.d. states and the persistent states. When the states are i.i.d., or if the states are persistent but common to all principal-

agent pairs, cooperation is impossible; the principal cannot induce any level of effort from the agent. On the other hand, if the state is persistent and agent-specific, there is no market, and the principal and the agent stay in the same relationship forever. If the state is persistent and relationship-specific, there will be a market, and the principal and the agent leave the current relationship if and only if the expected joint surplus falls below some threshold.

I also consider two mechanisms through which the persistence of the states affect relational contracts. When the states are persistent, the joint surplus in the first best can vary with the state, and incentive provision for given bonus cap can also vary with the state. I consider two mechanisms separately, holding the other constant. I find that in both cases, if the joint surplus in the first best increases with the state, or if the implementable level of effort for given bonus cap increases with the state, the difference in the joint surplus between the first best and the second best decreases with the state. The principal prefers relational contracts to full-commitment contracts if and only if the initial state is sufficiently high.

There are some papers on relational contracts with persistent states. Thomas and Worrall (2010) consider a two-sided incentive problem where the states and the efforts are observable and the players have limited liability. McAdmas (2011) considers joint-partnership games in which the states are persistent and both the states and efforts are observable. The players decide whether to stay in the relationship and how much effort to exert. The main difference from my model is that there is no asymmetric information in their models, and there is limited liability in Thomas and Worrall.

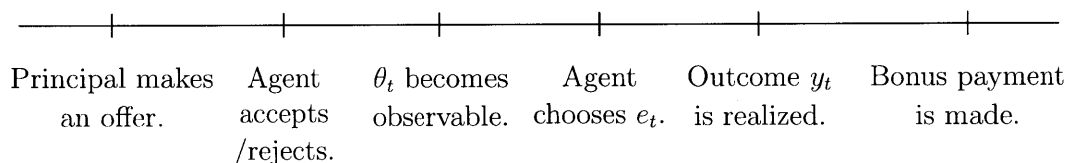
This chapter is also related to literature on partnership games with persistent states. Rotemberg and Saloner (1996) and Haltiwanger and Harrington (1991) study collusion in nonstationary markets. In Rotemberg and Saloner, the potential gain from deviating is higher in a higher state, and the future surplus is not affected by the state. In my first model of Section 2.6, the gain from deviating is constant across the states, and it is the future surplus that varies with the state; my model is closer to Haltiwanger and Harrington.

The market setting in my chapter is related to literature on repeated games with

rematching. Ghosh and Ray (1996), Kranton (1996) and Watson (1999) among others consider repeated interactions when the players can exit the relationship in any period. The stage game in these papers are similar to the prisoner's dilemma, and most of them don't have monetary transfers. The equilibrium strategy is often to start small, which contrasts with the stationary behavior in my model.

The rest of the chapter is organized as the following. Section 2.2 describes the model, and the general results are presented in Section 2.3. I consider an alternative model in Section 2.4 in which the state variable is endogenous, and Section 2.5 applies the results for the markets for random matching. Section 2.6 discusses the types of persistent states and their implications on relational contracts. Section 2.7 concludes.

Timing in Each Period



2.2 Model

The principal and the agent have the opportunity to trade over an infinite horizon, $t = 0, 1, 2, \dots$. Both the principal and the agent are risk-neutral, and the common discount factor is $\delta < 1$.

The principal has limited commitment power and can only employ relational contracts. At the beginning of period t , the principal offers a compensation scheme to the agent, which consists of a fixed salary w_t and a contingent payment b_t . Both the fixed salary and the contingent payment can be functions of the history, which I will define momentarily. The agent decides whether to accept the offer, and a payoff-relevant parameter θ_t is realized. Both the principal and the agent observe the state. Note that the principal offers the compensation scheme before the realization of the state; he offers a function of the state as fixed salary, and the bonus payment is a function of the performance outcome.

The state θ_t is drawn from the support $\Theta = [\underline{\theta}, \bar{\theta}]$. The distribution of the state θ_t depends only on the previous state θ_{t-1} . Denote the distribution of θ_t by $P(\theta_t|\theta_{t-1})$. The distribution of the state doesn't depend on the time index, and we have $P(\theta_t|\theta_{t-1}) = P(\theta_1|\theta_0)$ for all $t \geq 1$. In the initial period, the state θ_0 is distributed by $P_0(\theta_0)$. The distributions $P(\theta_t|\theta_{t-1})$ and $P_0(\theta_0)$ are known to both the principal and the agent.

Assumption 2.1. *The distribution of state θ_{t+1} when the previous state was θ_t is given by $P(\theta_{t+1}|\theta_t)$ and is identical for all $t \geq 0$.*

After the principal offers a compensation scheme, the agent decides whether or not to accept, $d_t \in \{0, 1\}$. If the agent accepts the compensation scheme, the agent chooses how much effort to exert, $e_t \in \mathcal{E} = [0, \bar{e}]$. The cost of effort, $c(e_t, \theta_t)$, increases with e with $c(e = 0, \theta) = 0$ for all θ and $c_{ee} > 0$. The agent's effort generates outcome y_t with the distribution $F(y|e, \theta)$ and the support $\mathcal{Y} = [\underline{y}, \bar{y}]$. The cdf of the outcome satisfies the Mirrlees-Rogerson constraints: $F(y|e, \theta)$ has the monotone likelihood ratio property, and $F(y|e = c^{-1}(x|\theta), \theta)$ is convex in x for any θ . The Mirrlees-Rogerson constraints ensure that the first order approach is valid; they are necessary for Proposition 2.4 and Section 2.6. The expected joint surplus can be written as a function of θ and e , $S(e, \theta) = \mathbb{E}[y|e, \theta] - c(e, \theta)$. Throughout the chapter, when capitalized, $S(e, \theta)$ denotes per-period joint surplus in state θ if the agent chooses effort e .

I allow the distribution of the outcome and the cost function to be dependent on the state. If neither of them depends on the state, we are back to i.i.d. environment, and in general, we can have one or the other to be state-dependent.

Each period, there are three pieces of payoff-relevant information: the cost-relevant parameter θ_t , the agent's effort e_t , and the outcome y_t . The agent observes all three parameters, but the principal observes only θ_t and y_t . The performance outcome is $\phi_t = \{\theta_t, y_t\}$, and the set of all performance outcome is denoted by Φ .

At the end of each period, the principal is obliged to pay the fixed salary w_t , but the contingent payment is only promised. Denote the total payment to the agent by

W_t , and $W_t = w_t + b_t$ if the contingent payment is made, and it is $W_t = w_t$ if not.

If the agent rejects the principal's offer, the parties receive their outside option for the period. The agent's outside option is \bar{u} , and the principal's outside option is $\bar{\pi}$. The joint surplus from the outside option is denoted by $\bar{s} = \bar{u} + \bar{\pi}$. I assume that for any state θ , the maximum joint surplus is strictly bigger than the outside option, but the outside option is weakly better than no effort. I also assume that the outside options $\bar{u}, \bar{\pi}$ are independent of the state and constant over time. In Section 2.5, I consider markets for random matching, and there will be endogenous outside options.

Assumption 2.2 (Efficiency). *For all $\theta \in \Theta$, $\max_e S(e, \theta) > \bar{s} \geq S(0, \theta)$.*

Given the distribution of the states, $P(\theta_{t+1}|\theta_t)$, we can define the distribution of $\theta_{t+\tau}$ given θ_t , $P(\theta_{t+\tau}|\theta_t)$. Let $p(\theta_{t+1}|\theta_t)$ be the pdf of θ_{t+1} , then we have

$$p(\theta_{t+\tau}|\theta_t) = \int \cdots \int p(\theta_{t+\tau}|\theta_{t+\tau-1}) \cdots p(\theta_{t+1}|\theta_t) d\theta_{t+\tau-1} d\theta_{t+1},$$

and $P(\theta_{t+\tau}|\theta_t)$ can be constructed from $p(\theta_{t+\tau}|\theta_t)$. The discounted payoffs to the parties from date t given θ_{t-1} are

$$u_t(\theta_{t-1}) = (1 - \delta) \mathbb{E} \left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} \{d_\tau (W_\tau - c(e_\tau, \theta_\tau)) + (1 - d_\tau) \bar{u}\} | \theta_{t-1} \right],$$

$$\pi_t(\theta_{t-1}) = (1 - \delta) \mathbb{E} \left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} \{d_\tau (y_\tau - W_\tau) + (1 - d_\tau) \bar{\pi}\} | \theta_{t-1} \right],$$

where the expectations are taken over $P(\theta_\tau|\theta_{t-1})$, $\tau \geq t$, and $F(\cdot|e, \theta)$. In period 0, the expectation is also taken over $P_0(\theta_0)$. At each period, the parties maximize their expected payoffs. I define the expected joint surplus from period t as

$$s_t(\theta_{t-1}) = u_t(\theta_{t-1}) + \pi_t(\theta_{t-1}).$$

Note that $s_t(\theta_{t-1})$ is the per period average expected joint surplus, as it is discounted by $1 - \delta$. When capitalized, $S(e, \theta)$ is the expected joint surplus from the given period for e, θ .

Let $h^t = (w_0, d_0, \phi_0, W_0, \dots, w_{t-1}, d_{t-1}, \phi_{t-1}, W_{t-1})$ be the history up to period t and \mathcal{H}^t be the set of possible period t histories. Given any period t and history h^t , a relational contract specifies the compensation the principal offers, whether or not the agent accepts it, and if the agent accepts the offer, the effort level. The compensation w_t, b_t are allowed to be functions of the history, and they are functions of the following form:

$$\begin{aligned} w_t &: \mathcal{H}^t \times \Theta \rightarrow \mathbb{R}, \\ b_t &: \mathcal{H}^t \times \Phi \rightarrow \mathbb{R}. \end{aligned}$$

A relational contract is self-enforcing if it forms a perfect public equilibrium of the repeated game.

2.3 Observable and Exogenous States

This section discusses the results that hold for any type of persistent states. The main results of this section generalize the characterization in Levin (2003) to persistent states. When the states are observable and exogenously given, the optimal contract can be stationary, and it is optimal to provide the same expected per period payoff in every state. The self-enforcement leads to the dynamic enforcement constraint as with i.i.d. states. An optimal contract either implements the first best level of effort or takes the form of a step function.

A relational contract forms a perfect public equilibrium of the repeated game, and there is multiplicity of equilibria. Instead of characterizing all relational contracts, I focus on efficient contracting and focus on the Pareto Frontier of the payoffs. The first result is to note that the problem of efficient contracting can be separated from the problem of distribution even if the states are persistent. The intuition is same as in Levin (2003). The principal can always adjust the fixed salary to redistribute the surplus.

Proposition 2.1. *Suppose there exists a relational contract with expected joint sur-*

plus $s > \bar{s}$. Any expected payoff pair (u, π) with $u \geq \bar{u}, \pi \geq \bar{\pi}, u + \pi = s$ can be implemented with a relational contract.

Proof. Consider the relational contract that provides s . The principal offers in the initial period $w(\theta_0), b(\phi_0)$, and if the agent accepts, he exerts effort $e(\theta_0)$. The continuation payoffs under the contract are denoted by $u(\phi_0)$ and $\pi(\phi_0)$, and the expected payoffs from the contract are u_0 and π_0 . Without loss of generality, we can assume that off the equilibrium path, the parties revert to the static equilibrium of $(\bar{u}, \bar{\pi})$. The first period payment W is a function of ϕ_0 .

The contract is self-enforcing if and only if the following conditions hold:

$$\begin{aligned}
(i) \quad & u_0 \geq \bar{u}, \pi_0 \geq \bar{\pi}, \\
(ii) \quad & e(\theta_0) \in \arg \max_e \mathbb{E}_y[(1 - \delta)W(\phi_0) + \delta u(\phi_0) | e, \theta_0] - c(e, \theta_0), \\
(iii) \quad & b(\phi_0) + \frac{\delta}{1 - \delta} u(\phi_0) \geq \frac{\delta}{1 - \delta} \bar{u}, \\
& -b(\phi_0) + \frac{\delta}{1 - \delta} \pi(\phi_0) \geq \frac{\delta}{1 - \delta} \bar{\pi},
\end{aligned}$$

and (iv) each continuation contract is self-enforcing.

Given any (u, π) such that $u \geq \bar{u}, \pi \geq \bar{\pi}, u + \pi = s$, the principal can offer the same $b(\phi_0)$ and continuation contracts and adjust $w(\theta_0)$ to

$$\hat{w}(\theta_0) \equiv w(\theta_0) + \frac{\pi - \pi_0}{1 - \delta}.$$

The conditions are satisfied with the new contract, and it provides (u, π) as the expected payoffs. \square

As long as the expected payoff is greater than the outside option, the parties are willing to initiate the contract. The principal can adjust the distribution of the joint surplus by the fixed salary of the initial period, and the resulting contract is still self-enforcing because the incentives are not affected. Given Proposition 2.1, we can restrict attention to optimal relational contracts that maximize the joint surplus from the contract.

The next result is that despite the persistence of the states, the maximum joint surplus can be achieved with stationary contracts. I define the stationarity of a contract as the following:

Definition 2.1. *A contract is stationary if $W_t = w(\theta_t) + b(\phi_t)$, $e_t = e(\theta_t)$ at every t on the equilibrium path for some $w : \Theta \rightarrow \mathbb{R}$, $b : \Phi \rightarrow \mathbb{R}$ and $e : \Theta \rightarrow \mathcal{E}$.*

Note that the contract is stationary on the equilibrium path. Without loss of generality, we can assume that off the equilibrium path, the parties revert to the static equilibrium of taking the outside option every period, $(\bar{u}, \bar{\pi})$. With a stationary contract, the principal offers the identical compensation scheme every period. The compensation scheme is independent of the history, and it only depends on the performance outcome of the given period. The fixed salary may depend on the state, but given the same state, the fixed salary is constant across the time.

Proposition 2.2. *The maximum joint surplus can be attained with a stationary contract. Furthermore, it can be achieved with a contract that provides the same expected payoff to the agent in every state.*

Proof. Suppose a contract that maximizes the joint surplus provides w_t, b_t and the agent chooses e_t . The first step is to construct an alternative contract \hat{w}_t, \hat{b}_t under which the agent chooses the same level of effort e_t and his expected payoff is constant in every state.

When the states are observable and exogenously given, the distribution of the states from period $t + 1$ only depends on θ_t , and the outcome y_t doesn't carry any information about the future states. The principal can adjust the contingent payment b_t and keep the expected payoff in each state constant. Specifically, consider the following contract. Let $u_t(h^t, \phi_t)$ be the continuation value of the agent under the given contract, and define \hat{w}_t, \hat{b}_t as the following:

$$\begin{aligned}\hat{b}_t(h^t, \phi_t) &\equiv b_t(h^t, \phi_t) + \frac{\delta}{1 - \delta}(u_t(h^t, \phi_t) - \bar{u}), \\ \hat{w}_t(h^t, \theta_t) &\equiv \bar{u} - \mathbb{E}_{y_t}[\hat{b}_t(h^t, \phi_t) | e_t(h^t, \theta_t)].\end{aligned}$$

From

$$\hat{b}_t(h^t, \phi_t) + \frac{\delta}{1-\delta}\bar{u} = b_t(h^t, \phi_t) + \frac{\delta}{1-\delta}u_t(h^t, \phi_t),$$

the agent chooses the same level of effort e_t under the new contract. The agent's expected payoff is \bar{u} for all t, h^t, θ_t .

The next step is to show that we can choose $\tilde{w} : \Theta \rightarrow \mathbb{R}, \tilde{b} : \Phi \rightarrow \mathbb{R}$ such that the principal offers \tilde{w}, \tilde{b} in every period. Consider \hat{w}_t and \hat{b}_t . The agent's expected payoff is constant over all t, h^t , and θ_t , which implies that the agent's IC constraint is determined by the within period compensation scheme. Specifically, the agent chooses e such that

$$e_t(h^t, \theta_t) \in \arg \max_e \mathbb{E}_y[\hat{b}_t(h^t, \phi_t)|e, \theta_t] - c(e, \theta_t).$$

When the agent's IC constraints are myopic, the principal can replace a compensation scheme for any given period with another compensation scheme without affecting the incentives. The principal can also treat each θ_t separately, because the state is observable before the agent chooses the effort. Specifically, let \tilde{b} be the compensation scheme that maximizes the expected per period joint surplus for state θ_t :

$$\tilde{b}(\theta_t, \cdot) \equiv \arg \max_{\hat{b}_t(h^t, \{\theta_t, \cdot\})} \mathbb{E}_y[y|e_t(h^t, \theta_t), \theta_t] - c(e_t(h^t, \theta_t), \theta_t).$$

If there's multiplicity of the compensation schemes, we can pick one without loss of generality.

Given $\tilde{b} : \Phi \rightarrow \mathbb{R}$, the agent chooses $e : \Theta \rightarrow \mathcal{E}$ such that

$$e(\theta_t) \in \arg \max_e \mathbb{E}_y[\tilde{b}(\phi)|e, \theta_t] - c(e, \theta_t).$$

Define \tilde{w} as

$$\tilde{w}(\theta_t) \equiv \bar{u} - \mathbb{E}_y[\tilde{b}(\phi)|e(\theta_t), \theta_t],$$

and we have a stationary contract that maximizes the expected joint surplus. By construction, it is self-enforcing, and it provides the same expected payoff to the agent in all t, h^t, θ_t . Let s^* be the minimum expected per period joint surplus over

the states under \tilde{b}, \tilde{w} :

$$s^* \equiv \min_{\theta} \{ \mathbb{E}_y [y | e(\theta), \theta] - c(e(\theta), \theta) \}.$$

The principal can adjust the fixed salary and can provide any u such that $\bar{u} \leq u \leq s^* - \bar{\pi}$ to the agent as the constant expected payoff. \square

From Propositions 2.1 and 2.2, we can focus on stationary contracts that maximize the joint surplus. I will next provide the necessary and sufficient condition for an effort schedule to be implementable by a stationary contract. When the states are observable and exogenously given, there is no information asymmetry about the distribution of future states. For the agent's IC constraints, only the sum of the present compensation and the continuation value matters, and in particular, the principal and the agent use the same probability distribution to evaluate the continuation values. Therefore, the principal can provide the incentives by the present compensation and provide the same expected payoff in all periods and all states. By doing so, the principal isolates the incentive provision to each period and the given state, and the principal can offer an identical compensation scheme in all periods for the given state. The maximum joint surplus can be attained with a stationary contract, and we can restrict attention to stationary contracts that maximize the joint surplus.

With relational contracts, neither the principal or the agent commits to the contingent payment, and there exists a temptation to renege on the promised payment. The contract is self-enforcing if the principal and the agent have no incentives to renege. Since we are interested in the maximum joint surplus, there is no loss of generality in assuming that a deviation leads to the static equilibrium behavior. If the principal offers an unexpected compensation scheme, the agent accepts the offer but exerts zero effort. Following a deviation, the parties receive their outside options $\bar{\pi}$ and \bar{u} .

Recall that when the states are persistent, the discounted payoffs at period t should be conditional on state θ_{t-1} :

$$u_t(\theta_{t-1}) = (1 - \delta)\mathbb{E}\left[\sum_{\tau=t} \delta^{\tau-t} \{d_\tau(W_\tau - c(e_\tau, \theta_\tau)) + (1 - d_\tau)\bar{u}\} | \theta_{t-1}\right],$$

$$\pi_t(\theta_{t-1}) = (1 - \delta)\mathbb{E}\left[\sum_{\tau=t} \delta^{\tau-t} \{d_\tau(y_\tau - W_\tau) + (1 - d_\tau)\bar{\pi}\} | \theta_{t-1}\right],$$

and the expected joint surplus from $t + 1$ is $s_{t+1}(\theta_t) = u_{t+1}(\theta_t) + \pi_{t+1}(\theta_t)$.

The principal makes the promised payment if and only if

$$\frac{\delta}{1 - \delta}(\pi_{t+1}(\theta_t) - \bar{\pi}) \geq \sup_y b(\theta_t, y), \forall \theta_t,$$

and for the agent to make the promised payment, we need

$$\frac{\delta}{1 - \delta}(u_{t+1}(\theta_t) - \bar{u}) \geq -\inf_y b(\theta_t, y), \forall \theta_t.$$

From Proposition 2.1, the principal can redistribute the surplus by adjusting the fixed wage, and the above inequalities can be combined in the dynamic enforcement constraint:

$$(DE) \quad \frac{\delta}{1 - \delta}(s_{t+1}(\theta_t) - \bar{s}) \geq \sup_y W(\theta_t, y) - \inf_y W(\theta_t, y).$$

The enforceable effort schedules are characterized by the agent's IC constraint and the dynamic enforcement constraint.

Proposition 2.3. *An effort schedule $e(\theta)$ with expected joint surplus $s(\theta)$ can be implemented with a stationary contract if and only if there exists a payment schedule $W : \Phi \rightarrow \mathbb{R}$ such that for all $\theta \in \Theta$,*

$$(IC) \quad e(\theta) \in \arg \max_e \mathbb{E}_y[W(\phi) | e, \theta] - c(e, \theta),$$

$$(DE) \quad \frac{\delta}{1 - \delta}(s(\theta) - \bar{s}) \geq \sup_y W(\theta, y) - \inf_y W(\theta, y).$$

Proof. (\Rightarrow) Suppose $e(\theta)$ is implementable. Let $u(\theta)$ and $\pi(\theta)$ be the continuation

value for the agent and the principal when the previous state was θ . The IC constraint has to be satisfied, and we also know that

$$\frac{\delta}{1-\delta}(\pi(\theta) - \bar{\pi}) \geq \sup_y b(\theta, y), \forall \theta, \quad (2.1)$$

$$\frac{\delta}{1-\delta}(u(\theta) - \bar{u}) \geq -\inf_y b(\theta, y), \forall \theta \quad (2.2)$$

have to hold. Adding the two inequalities, we have the dynamic enforcement constraint.

(\Leftarrow) Suppose $W(\phi)$ and $e(\theta)$ satisfy the IC constraint and the dynamic enforcement constraint. Define

$$b(\phi) = W(\phi) - \inf_{\phi} W(\phi),$$

$$w(\theta) = \bar{u} - \mathbb{E}_y[W(\phi)|e(\theta), \theta],$$

and consider the stationary contract with $w(\theta)$, $b(\phi)$ and $e(\theta)$. The parties revert to the static equilibrium if a deviation occurs. The agent receives \bar{u} as expected payoff in each state, and the principal receives $\pi(\theta) = s(\theta) - \bar{u}$ if the previous state was θ . By the dynamic enforcement constraint, $s(\theta) \geq \bar{s}$ and $\pi(\theta) \geq \bar{\pi}$ for all θ . From the IC constraint, the agent chooses $e(\theta)$ in each state θ , and it can be verified that Inequality (2.1) and (2.2) are satisfied. \square

Note that the continuation payoffs from period $t + 1$ matter in the dynamic enforcement constraint, but they don't enter the agent's IC constraint. Since the states are persistent, the continuation payoffs $u_{t+1}(\theta_t)$ and $\pi_{t+1}(\theta_t)$ depend on the state θ_t . But the principal also observes θ_t , and by Proposition 2.2, the principal can offer a stationary continuation contract and the constant continuation value, independent of the outcome y_t . Therefore, even though the agent's expected payoff from period t is $W(\phi_t) + \delta u_{t+1}(\theta_t)$, $u_{t+1}(\theta_t) = \bar{u}$, and it doesn't matter for the agent's IC constraint.

Lastly, from Proposition 2.3, we obtain the following characterization of optimal contracts. Due to risk-neutrality of both parties, the principal wants to use the

strongest incentives possible. If an optimal contract cannot induce the first best effort $e^{FB}(\theta_t)$ in state θ_t , the DE constraint binds, and the compensation scheme is a step function.

Proposition 2.4. *An optimal contract either (i) implements $e^{FB}(\theta_t)$ or (ii) takes the form of a step function at each θ_t . When $e(\theta_t) < e^{FB}(\theta_t)$, there exists $y(\theta_t)$ such that $W(\theta_t, y) = \bar{W}(\theta_t)$ for $y \geq y(\theta_t)$ and $W(\theta_t, y) = \underline{W}(\theta_t)$ for $y < y(\theta_t)$. $\bar{W}(\theta_t) = \underline{W}(\theta_t) + \frac{\delta}{1-\delta}(s_{t+1}(\theta_t) - \bar{s})$, and the likelihood ratio $f_e/f(y|e(\theta_t))$ changes the sign at $y(\theta_t)$.*

Proof. We know from Proposition 2.1 that we can focus on maximizing the joint surplus, and Proposition 2.2 implies that we can focus on stationary contracts. By the Mirrlees-Rogerson constraints, we can replace the agent's IC constraint with the first-order condition. The optimal stationary contract solves

$$\max_{e(\cdot), W(\cdot, \cdot)} \mathbb{E}_{\theta, y}[y - c|e(\theta), \theta]$$

subject to

$$\begin{aligned} \frac{d}{de} \{ \mathbb{E}_y[W(\theta, y) - c(e, \theta) | e = e(\theta), \theta] \} &= 0, \quad \forall \theta, \\ \frac{\delta}{1-\delta}(s(\theta) - \bar{s}) &\geq \sup_{\theta, y} W(\theta, y) - \inf_{\theta, y} W(\theta, y), \\ s(\theta) &= (1-\delta)\mathbb{E}[\sum_{t=0}^{\infty} \delta^t \{d_t(y_t - c(e_t, \theta_t)) + (1-d_t)\bar{s}\} | \theta]. \end{aligned}$$

From the Mirrlees-Rogerson constraints, the principal wants to maximize e when $e(\theta_t) < e^{FB}(\theta_t)$. We get

$$W(\theta_t, y) = \begin{cases} \bar{W}(\theta_t) & \text{if } y \geq y(\theta_t) \\ \underline{W}(\theta_t) & \text{if } y < y(\theta_t). \end{cases},$$

and f_e changes the sign at $y(\theta_t)$, and $\bar{W}(\theta_t) = \underline{W}(\theta_t) + \frac{\delta}{1-\delta}(s_{t+1}(\theta_t) - \bar{s})$. \square

The results in this section hold for any type of persistence. When the states are observable and exogenously given, there is no asymmetric information between the

principal and the agent regarding the distribution of future states. Together with risk-neutrality, the principal can provide all incentives by the bonus payments at the end of each period and offer the same continuation value in every state. The results in Levin (2003) extend to persistent states, and we have shown the following results. The problem of efficient contracting can be separated from the problem of distribution, and the joint-surplus can be maximized with stationary contracts. The necessary and sufficient condition to implement an effort schedule with stationary contracts is that it satisfies the IC constraint and the dynamic enforcement constraint. An optimal contract either implements the first best level of effort, or it is a step function in each state.

2.4 Endogenous States

This section considers an alternative model with endogenous states. The agent's effort and the productivity this period determine the distribution of the productivity next period, and the outcome is a function of the productivity. Since the agent's effort affects the distribution of the productivity, it is an endogenous state variable. However, when the productivity is observable to both the principal and the agent, most of the results in the previous section generalize to this model. The problem of efficient contracting can be separated from the distribution of joint surplus, and the maximum joint surplus can be attained with stationary contracts. An effort schedule is implementable with stationary contracts if and only if the IC constraint and the dynamic enforcement constraint are satisfied.

In practice, it is often natural to assume that the state variable is endogenous. Human capital is likely to be developed by the agent's effort over time, and the productivity is also often endogenous. If the outcome this period determines the productivity for the next period, the outcome itself is the state variable and is endogenous. Results in this section show that we can apply the similar analysis to relational contracts with endogenous states, as long as the state is observable to both the principal and the agent.

The productivity θ_t is drawn from $\Theta = [\underline{\theta}, \bar{\theta}]$. The distribution of θ_t depends on θ_{t-1} and e_{t-1} and is time-homogeneous. Denote the distribution by $P(\theta_t|\theta_{t-1}, e_{t-1})$. The distribution of θ_0 is given by $P_0(\cdot)$. Given θ_t , the principal gets the outcome $y_t = y(\theta_t)$ as a deterministic function of the productivity. A performance outcome is $\phi_t = (\theta_t, y_t, \theta_{t+1})$. Note that the outcome need not be deterministic. I assume it to be deterministic to simplify the analysis, but the same argument works if it is stochastic.

The timing of the model is the following. At the beginning of period t , the principal offers a contract to the agent, and the agent decides whether to accept it. The outcome is realized as a function of the productivity, which is known from previous period. The agent decides how much effort to exert, and the productivity for the next period is realized. The principal and the agent make the payments.

We have the following versions of Proposition 2.1-2.3. I omit the proofs since they are straightforward generalizations of the proofs of Proposition 2.1-2.3.

Proposition 2.5. *Suppose there exists a relational contract with expected joint surplus $s > \bar{s}$. Any expected payoff pair (u, π) with $u \geq \bar{u}, \pi \geq \bar{\pi}, u + \pi = s$ can be implemented with a relational contract.*

The proof of Proposition 2.5 is the same as the proof of Proposition 2.1 verbatim. The agent accepts the contract as long as the expected payoff is greater than his outside option, and the principal can always redistribute the surplus by the fixed wage.

Proposition 2.6. *The maximum joint surplus can be attained with a stationary contract. Furthermore, it is optimal to provide the same expected payoff to the agent in every state.*

The key to the proof of Proposition 2.6 is that θ' is a sufficient static about the outcome and states from the next period. Since the principal and the agent are risk-neutral and the productivity is observed before they make the payments, the principal can provide all incentives by the present compensation and provide a constant expected payoff to the agent in every state. Under an optimal contract, the

expected joint surplus for given state θ is constant, and the principal can choose the bonus payments to maximize the expected joint surplus.

Proposition 2.7. *An effort schedule $e(\theta)$ with expected joint surplus $s(\theta)$ can be implemented with a stationary contract, with a constant expected payoff to the agent, if and only if there exists a payment schedule $W : \Phi \rightarrow \mathbb{R}$ such that for all $\theta \in \Theta$,*

$$\begin{aligned} (IC) \quad & e(\theta) \in \arg \max_e \mathbb{E}_{\theta'}[W(\phi)|e, \theta] - c(e, \theta), \\ (DE) \quad & \frac{\delta}{1-\delta}(s(\theta') - \bar{s}) \geq W(\theta, y, \theta') - \inf_{\theta'} W(\theta, y, \theta'). \end{aligned}$$

In Proposition 2.7, the bonus cap now depends on the realization of the productivity for the next period. This is because the bonus payment is contingent on the productivity for the next period, which is the sufficient static for the expected joint surplus. The rest of the argument is the same as in the proof of Proposition 2.3.

When the states are observable to both the principal and the agent, there is no information asymmetry about the distribution of future states. Together with risk-neutrality, we obtain the stationarity theorem and the necessary and sufficient condition to implement an effort schedule. The difference from the exogenous states is that instead of having a uniform bonus cap for the given state, now the bonus cap depends on the realization of the productivity for the next period.

2.5 Market for Matching

This section considers a market for matching when there is a continuum of principal-agent pairs. In any given period, the principal and the agent have an option to exit the current relationship. If they exit, they will be randomly, anonymously and costlessly rematched with a new partner. The nature of the underlying state leads to different implications for the market. If the state is agent-specific, the principal-agent pairs remain in the current relationships regardless of the realization of the state or the past history, and there will be no market for matching. If the state is relationship-specific, there will be a market, and the principal leaves the relationship if and only

if the expected joint surplus falls below some threshold. If the state is a macro shock, common to all principal-agent pairs, then cooperation is impossible, and the principal cannot induce the agent to put in any effort. Cooperation is also impossible if the states are i.i.d..

The literature on relational contracts take the outside options as exogenous. The goal of this section is to consider the market and to endogenize the outside options. If a continuum of principal-agent pairs in the same contractual environment have options to be matched with new partners, the market forms endogenous outside options for the principal-agent pairs. The implications highlight the difference between the i.i.d. states and persistent states, and also the difference among the types of persistent states.

The timing of the game is the following. In each period, the principal offers a compensation scheme, and the agent decides whether or not to accept it. After the agent decides, the state is realized and becomes observable to both the principal and the agent. If the agent accepted, he decides how much effort to put in, and the outcome is realized. The principal and the agent make the contingent bonus payment and decide whether or not to stay in the relationship. If they both decide to stay, they move on to the next period. If one of them exits, both the principal and the agent will be matched with new partners and start in the next period. If the agent rejected the offer, both receive their outside options and decide whether to stay or exit.

With a market for matching, the outside options for the principal and the agent are endogenously determined in an equilibrium. However, given a continuum of principal-agent pairs, each pair takes the outside options as given, and we can apply the analysis from Section 2.3. I allow for exogenous outside options as well, but this doesn't affect the analysis, and we can restrict attention to endogenous outside options if desired.

As a benchmark, consider the i.i.d. states. Cooperation is impossible when there is frictionless market for matching.

Proposition 2.8. *Suppose the states are i.i.d., and the principal and the agent can be randomly, anonymously, and costlessly matched with a new partner. The principal*

cannot induce any level of effort from the agent.

Proof. After the outcome is realized, the principal makes the bonus payment if and only if

$$\frac{\delta}{1-\delta}(\pi - \bar{\pi}) \geq \sup_y b(\theta, y), \forall \theta,$$

and the agent makes the bonus payment if and only if

$$\frac{\delta}{1-\delta}(u - \bar{u}) \geq -\inf_y b(\theta, y), \forall \theta.$$

Together, we have

$$\frac{\delta}{1-\delta}(s - \bar{s}) \geq \sup_y b(\theta, y) - \inf_y b(\theta, y).$$

However, if they can be matched with a new partner and the states are i.i.d., $s = \bar{s}$, and the bonus payment has to be the same for all outcomes. The agent has no incentive to put in any effort. \square

2.5.1 Agent-Specific States

First, consider the case in which the state is the type of the agent. It can be interpreted as the productivity of the agent. When the agent is matched with a new principal, the distribution of the state is determined by his type in the last period, which is the last realization of the state in the agent's previous relationship. Then, there cannot be a market for matching, and all principal-agent pairs stay in their relationship forever.

Proposition 2.9. *Suppose when a principal and an agent is matched, the initial state is drawn from the distribution $P(\cdot|\theta)$ where θ is the last realization of the state of the agent. The principal and the agent never exit the current relationship, and there is no market for rematching.*

Proof. From Section 2.3, we can focus on the joint surplus from the relationship, and the principal and the agent remain in the current relationship if and only if $s(\theta) \geq \bar{s}$. Let $\hat{\Theta} = \{\theta | s(\theta) < \bar{s}\} \subset \Theta$ be the set of states after which the principal and the agent exit the relationship. Let F be the distribution of the states in the given period.

When the principal and the agent decide whether to stay in the relationship, the outside options must satisfy

$$\bar{s} = \int_{s \in \hat{\Theta}} s(\theta) dF,$$

which is a contradiction to the definition of $\hat{\Theta}$. Therefore, $\hat{\Theta}$ is degenerate and can only be \emptyset . Only the lowest type is indifferent between staying in and exiting the relationship. The principal and the agent never exit the relationship, and there is no market for rematching. \square

Proposition 2.9 shows that if the underlying state is the type of the agent, the market for matching turns into a market for lemons, and there will not be a market. Only the lowest type can exist in the market, and all principal-agent pairs stay in the current relationship.

2.5.2 Relationship-Specific States

Next, suppose that the state is specific to the pair of principal and agent. If they exit the current relationship, the initial state in a new relationship is drawn from a known distribution G and is i.i.d. across the new pairs of principals and agents. Then there is endogenous threshold for the joint surplus such that the principal and the agent exit the relationship if and only if the expected joint surplus falls below the threshold.

Proposition 2.10. *Suppose the initial state is i.i.d. across the new pairs of principals and agents and is drawn from a known distribution G . The principal and the agent exit the current relationship if and only if the expected joint surplus falls below some threshold. When the state is such that they will exit, the agent doesn't put in any effort.*

Proof. From the dynamic enforcement constraint in Section 2.3, the principal and the agent stay in the relationship if and only if $s(\theta) \geq \bar{s}$, where \bar{s} is the expected joint surplus from being matched with a new partner. If $s(\theta) < \bar{s}$, the bonus payment is the same for all outcomes, and the agent doesn't put in any effort. \square

When the state is specific to the principal-agent pair, they remain in the relationship if and only if the expected joint surplus is above the threshold. Since the optimal contract is stationary, the state in this period completely summarizes the expected joint surplus from the next period and on, and the exit behavior is determined by the realization of the state.

2.5.3 Macro Shocks

This section considers a macro shock. The state is common to all principal-agent pairs. In this case, the principal cannot induce the agent to put in any effort, and cooperation is impossible.

Proposition 2.11. *Suppose the state is common to all principal-agent pairs. The principal cannot induce the agent to put in any effort.*

Proof. The proof is the same as the i.i.d. case. If the state is common to all principal-agent pairs, $s = \bar{s}$ in the dynamic enforcement constraint, and the principal pays the same bonus for all payments. The agent has no incentive to put in any effort. \square

If the state is common to all principal-agent pairs, the expected joint surplus from the next period and on is the same whether they remain in the current relationship or are matched with new partners. Then, the principal and the agent have no incentive to pay the bonus payment, and without the bonus payments, cooperation is impossible.

2.6 Joint Surplus in the Second Best

I consider two types of persistence in this section. The first case is in which the joint surplus in the first best increases with the state. When the cost function is separable and strictly decreases with the state, incentive provision is identical in each state, and in particular, given a bonus cap, the principal can implement the same level of effort in every state. The second type of persistence I consider is when the incentive provision becomes easier in a higher state. The joint surplus in the first best is identical in all states. In both cases, the difference in joint surplus between the

first best and the second best strictly decreases with the state. The principal prefers relational contracts only if the initial state is sufficiently high.

2.6.1 Joint Surplus Varies with the State

In this section, I consider the case in which the joint surplus varies with the state and the incentive provision is constant across the states. Specifically, I assume the following.

Assumption 2.3. *The cost of effort is separable and strictly decreases with the state: there exist $c_1 : \mathcal{E} \rightarrow \mathbb{R}, c_2 : \Theta \rightarrow \mathbb{R}$ such that*

$$c(e, \theta) = c_1(e) + c_2(\theta), \forall e \in \mathcal{E}, \theta \in \Theta$$

and $c'_2 < 0$ for all $\theta \in \Theta$.

Assumption 2.4. *$F(\cdot|e, \theta)$ is independent of θ .*

Assumption 2.5. *$\theta_t > \theta'_t$ implies $P(\cdot|\theta_t)$ FOSD $P(\cdot|\theta'_t)$.*

I also define $\Delta W(\theta)$ as the minimum bonus cap necessary to be able to induce the first best level of effort in state θ . Given a state θ , $e^{FB}(\theta)$ can be a solution to

$$e(\theta) \in \arg \max_e \mathbb{E}_y[W(\phi)|e] - c(e, \theta),$$

$$\Delta W \geq \sup_y W(\theta, y) - \inf_y W(\theta, y)$$

if and only if $\Delta W \geq \Delta W(\theta)$.

As a benchmark, I first show the implications of Assumption 2.3 in the first best and in the case the principle has a within-period commitment power.

Proposition 2.12. *Suppose Assumption 2.3, 2.4 and 2.5 hold. The expected joint surplus in the first best strictly increases with the state, both in per period and in the future discounted joint surplus. The first best level of effort is constant across all states $\theta \in \Theta$. The minimum bonus cap to implement the first best level of effort,*

$\Delta W(\theta)$, is also constant across the state. If the principal can credibly promise $W(\phi)$, the principal implements one level of effort, $e^* = e^{FB}$ in all states.

Proof. The expected joint surplus in state θ is given by

$$\mathbb{E}_y[y|e] - c(e, \theta) = \mathbb{E}_y[y|e] - c_1(e) - c_2(\theta),$$

and the first best level of effort satisfies the first order condition,

$$\int_y y f_e(y|e) dy = c'_1(e).$$

Since the cost of effort is separable, the first order condition is independent of the state, and the first best level of effort is constant across the states. The cost strictly decreases with the state, and the expected per period joint surplus in the first best in state θ strictly increases with the state. By the persistence of states, the future discounted joint surplus also increases with the state.

From Proposition 2.4, an optimal contract either implements the first best level of effort or is a step function. When the dynamic enforcement constraint is binding, an optimal contract is a step function. From the first order condition

$$\int_y W(\theta_t, y) f_e(y|e) dy = c'_1(e),$$

$\Delta W(\theta)$ is constant across the states.

If the principal can commit to bonus payments, the only constraint is the agent's IC constraint. By the efficiency assumption, it is efficient to induce the first best level of effort than to take the outside option in all states θ , and the principal induces the first best level of effort in all θ . \square

Now consider relational contracts under Assumption 2.3. Define $s^{FB}(\theta)$ as the discounted future joint surplus when the previous state is θ . We know from Proposition 2.12 that $\Delta W(\theta)$ is constant over θ . Denote $\Delta W(\theta) = \Delta W^*$. If $s^{FB}(\theta) \geq \Delta W^*$, the principal can implement the first best level of effort in all states with relational

contracts, and the problem becomes trivial. I will make the following assumption:

Assumption 2.6. *The principal cannot induce the first best level of effort in the lowest state:*

$$s^{FB}(\underline{\theta}) < \Delta W^*.$$

Define $e(\theta|\Delta W)$ to be the solution to the optimization problem

$$\begin{aligned} \max_e \mathbb{E}_y[y - c|e, \theta] \quad \text{s.t. } e(\theta) \in \arg \max_e \mathbb{E}_y[W(\phi)|e] - c(e, \theta), \\ \Delta W \geq \sup_y W(\theta, y) - \inf_y W(\theta, y). \end{aligned}$$

If $\Delta W \leq \Delta W(\theta)$, the principal cannot implement the first best level of effort, and $e(\theta|\Delta W) < e^{FB}$. Since the principal can always mimic the payments with $\Delta W'$ if $\Delta W \geq \Delta W'$, the implementable level of effort weakly increases with the bonus cap, and we have $e(\theta|\Delta W) \geq e(\theta|\Delta W'), \forall \theta$.

Proposition 2.13. *The implementable level of effort $e(\theta|\Delta W)$ weakly increases with ΔW for all θ .*

Proof. The proof follows from the fact that the principal can always mimic the compensation scheme with $\Delta W'$ if $\Delta W \geq \Delta W'$. \square

Under relational contracts, the expected joint surplus from the following period limits the principal's ability to induce effort, and Assumption 2.3 states that the joint surplus in the first best strictly increases with the state. The implementable level of effort is lower in a worse state, and the difference in the expected joint surplus is reinforced by the implementable effort. Under Assumption 2.3, the joint surplus under relational contracts increases with the state, and the difference in the joint surplus between the first best and the second best decreases with the state.

Proposition 2.14. *Suppose Assumptions 2.3, 2.4, 2.5 and 2.6 hold. Let $s^{SB}(\theta)$ be the expected joint surplus under an optimal relation contract. $s^{SB}(\theta)$ strictly increases with θ , and $\frac{\partial s^{SB}}{\partial \theta} \geq \frac{\partial s^{FB}}{\partial \theta} > 0$. The difference in the joint surplus between the first best*

and the second best, $s^{FB}(\theta) - s^{SB}(\theta)$, weakly decreases with the state. The difference is strictly positive at $\underline{\theta}$, and it is weakly bigger than zero at all θ .

Proof. We know from Proposition 2.13 that the implementable level of effort, $e(\theta|\Delta W)$, weakly increases with ΔW . From Assumption 2.3, the expected joint surplus in the first best increases with the state, and Assumption 2.6 says that the expected joint surplus in the state $\underline{\theta}$ is less than the minimum bonus cap to induce the first best level of effort. Since the distribution of the states increases with the state in the sense of first-order stochastic dominance, the implementable level of effort under an optimal relational contract increases with the state, and the expected joint surplus in the second best also increases with the state.

Consider the difference in per period joint surplus between the first best and the second best.

$$\begin{aligned} & S(e^{FB}, \theta) - S(e(\theta|\Delta W), \theta) \\ &= (\mathbb{E}[y|e^{FB}] - c(e^{FB}, \theta)) - (\mathbb{E}[y|e(\theta|\Delta W)] - c(e(\theta|\Delta W), \theta)) \\ &= (\mathbb{E}[y|e^{FB}] - c_1(e^{FB})) - (\mathbb{E}[y|e(\theta|\Delta W)] - c_1(e(\theta|\Delta W))). \end{aligned}$$

Given ΔW , $e(\theta|\Delta W)$ is constant across the states, and we also know that

$$\mathbb{E}[y|e(\theta|\Delta W)] - c_1(e(\theta|\Delta W))$$

increases with ΔW . Therefore, the difference in the per period joint surplus,

$$S(e^{FB}, \theta) - S(e(\theta|\Delta W), \theta),$$

decreases with the state, and by the persistence of the states, the difference in the expected joint surplus also decreases with the state. From Assumption 2.6, the difference is strictly positive at $\underline{\theta}$. \square

When the per period joint surplus in the first best increases with the state, the persistence of the states enter the optimization problem through the bonus cap, and

the expected joint surplus under an optimal relational contract also increases with the state. The dynamic enforcement constraint magnifies the impact of persistent states, and the expected joint surplus varies more in the second best than in the first best.

2.6.2 Incentive Provision Varies with the State

In the first case, only the joint surplus varies with the state, and the incentive provision for given bonus cap was held constant across the states. Now, I am going to consider the alternative case in which the joint surplus in the first best is constant across the state but the incentive provision varies with the state.

I assume that the first best level of effort is constant across the states. This is without loss of generality for any interior solution e^{FB} . I also assume that for given bonus cap, the maximum per period joint surplus strictly increases with the state, and the principal cannot implement the first best level of effort in the worst state, even with the expected joint surplus in the first best.

Assumption 2.7. *The first best level of effort is constant in all states. The per period joint surplus in the first best is constant across the states: $S(e^{FB}, \theta) = S^*$ for all θ .*

Assumption 2.8. *For given bonus cap ΔW , if the principal cannot induce the first best level of effort, the maximum per period joint surplus strictly increases with the state. i.e., $S(e(\theta|\Delta W), \theta)$ strictly increases with θ for all $e(\theta|\Delta W) < e^{FB}$.*

Assumption 2.9. *The principal cannot implement the first best level of effort in the lowest state, and $e(\underline{\theta}|s^{FB}) < e^{FB}$.*

Under the second set of assumptions, the expected joint surplus in the second best strictly increases with the state, and the difference in the expected joint surplus between the first best and the second best decreases with the state. We have the following proposition which is an analogue of Proposition 2.14.

Proposition 2.15. *Suppose Assumptions 2.5, 2.7, 2.8 and 2.9 hold. There exists $\theta^* \in \Theta$ such that $s^{SB}(\theta)$ strictly increases with θ for $\theta < \theta^*$, and $s^{SB}(\theta) = s^{FB}$ for $\theta \geq \theta^*$. The difference in the joint surplus between the first best and the second best, $s^{FB} - s^{SB}(\theta)$, decreases with the state. The difference is strictly positive at $\underline{\theta}$, and it is weakly bigger than zero at all θ .*

Proof. By Assumptions 2.8, 2.9 and the persistence of the states, the per period joint surplus in the second best weakly increases with θ , and it increases strictly for all θ such that $e(\theta|s^{SB}(\theta)) < e^{FB}$. Therefore, the expected joint surplus in the second best also increases with the state. Since the first best joint surplus is constant across the states, the difference between the first best and the second best decreases with the state. □

I have considered two types of persistent states. In both environments, the difference in the expected joint surplus between the first best and the second best decreases with the state. If the two factors, the level of joint surplus in the first best and the difficulty of incentive provision, move in the same direction, the effect will be magnified. If they move in the opposite directions, the difference in the joint surplus will be determined by which effect dominates.

2.6.3 Benefits from Relational Contracts

Suppose there exists a positive benefit from relational contracts. I define full-commitment contracts as contracts under which the principal specifies the compensation scheme as functions of history and commit to both the fixed wage and the bonus payments. In my model, the only constraint under full-commitment contracts is the agent's IC constraints, and the principal can implement the first best under full-commitment contracts.

There could be gains from relational contracts as it is often impractical to write complete contracts. Performance measures can be hard to describe, and often, the best performance measure is a subjective measurement. When there is positive benefit $x > 0$ from relational contracts, the principal prefers the relational contracts over full-

commitment contracts if and only if the benefit is bigger than the difference in the expected joint surplus.

Proposition 2.16. *Suppose Assumptions 2.3 and 2.6 hold. Let $x > 0$ be the benefit from relational contracts. The principal prefers relational contracts if and only if the prior on the states is sufficiently high:*

$$\int_{\theta_0} s^{SB}(\theta_0) dP_0(\theta_0) + x \geq \int_{\theta_0} s^{FB}(\theta_0) dP_0(\theta_0).$$

Proof. The principal can implement the first best with full-commitment contracts. Given prior P_0 on the state, the difference in the expected joint surplus between the full-commitment contract and the optimal relational contract is given by

$$\int_{\theta_0} (s^{FB}(\theta_0) - s^{SB}(\theta_0)) dP_0(\theta_0) - x.$$

□

Proposition 2.17. *Suppose Assumption 2.7, 2.8 and 2.9 hold. Let $x > 0$ be the benefit from relational contracts. The principal prefers relational contracts if and only if the prior on the states is sufficiently high:*

$$\int_{\theta_0} s^{SB}(\theta_0) dP_0(\theta_0) + x \geq s^{FB}.$$

Proof. The principal can implement the first best with full-commitment contracts, and the joint surplus in the first best is constant. □

2.7 Conclusion

I study relational contracts in a persistent environment in this chapter. I find that many of the general properties of the optimal relational contracts in i.i.d. states carry over to persistent states, if there is no asymmetric information about the state. The benchmark is when the states are observable and exogenously given. When the states

follow a first-order Markov chain, the state in any given period is a sufficient statistic for the distribution of future states. In particular, the outcome doesn't have any information about the distribution of future states, and the principal can provide the incentives by the bonus payments. It is optimal to provide the same expected per period payoff in every state.

If the continuation contract for a given state in some period provides the maximum joint surplus for the given state, the principal can provide the same continuation contract in every period for the given state. Since the agent gets the same expected payoff in all states, the agent's IC constraints are still satisfied when the principal replaces the continuation contract, and the optimal contract can be stationary. The principal can also redistribute the surplus through the fixed wage, and we get the dynamic enforcement constraint as with i.i.d. states. An effort schedule can be implemented with stationary contracts if and only if it satisfies the IC constraint and the dynamic enforcement constraint. As was the case with i.i.d. states, the principal can either implement the first best effort, or the optimal contract takes the form of a step function.

The properties of the optimal contracts carry over to endogenous states if there is no asymmetric information about the state. The maximum joint surplus can be attained with a stationary contract when the productivity is the state variable. When the productivity is observed before the principal makes the payment, there is no information asymmetry. The agent's effort affects the distribution of future states, but given the productivity for the next period, the distribution of future states is known both to the principal and the agent. The principal can adjust the present compensation and provide the incentives by bonus payments, while keeping the expected payoff constant. Then, the incentive provision in each state becomes myopic, and the principal can offer a stationary contract and maximize the joint surplus. A version of dynamic enforcement constraint, together with the IC constraint, is the necessary and sufficient condition to implement an effort schedule with such stationary contracts.

I show that the nature of the state has starkly different implications for the market when the principal and the agent can be randomly, anonymously, and costlessly

matched with new partners. Cooperation is impossible if the states are i.i.d., regardless of the nature of the state. If the states are persistent, we get varying degree of cooperation depending on the type of the state. If it's agent-specific, the principal and the agent stay in the relationship forever, and there is no market. If it's relationship-specific, they exit the current relationship if and only if the expected joint surplus falls below some threshold. With macro shocks, cooperation is impossible, and the principal cannot induce any level of effort.

Persistent states can affect the relational contracts through two mechanisms. The persistence of the states imply that if the joint surplus depends on the state, the bonus cap also varies with the state, and the implementable level of effort depends on the state, even if the incentive provision for the given bonus cap is identical in each state. On the other hand, the incentive provision for the given bonus cap can also change with the state. If the joint surplus in the first best increases with the state, or if the implementable level of effort for given bonus cap increases with the state, the difference in the joint surplus between the first best and the second best decreases with the state. The principal prefers the relational contracts to full-commitment contracts only if the initial state is sufficiently high.

I consider two types of persistent environment in this chapter. If the states are observable and exogenously given, or if the outcome is the state variable, the optimal contract can be stationary. However, if the states are unobservable, or if the agent's effort affects the distribution of the states, where the outcome is only a noisy signal of the state, there can be information asymmetry between the principal and the agent about the future states. The belief about the agent's effort matters for the future, and the relational contract will likely have to take into account the private information. It will be interesting to study relational contracts when the information about the future states is no longer symmetric.

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Chapter 3

Future Learning and Preference for Delegation

3.1 Introduction

When a task is assigned to a manager, the manager has payoff-relevant information about the task and chooses the optimal action given his information. The manager's information may stay the same throughout his term, but his information about the task may change between the time of the assignment and the performance of the task. The manager can learn about the optimal action at his own cost, and he can also learn from someone else about the optimal action. If he learns from someone else, the informant doesn't necessarily have to provide the information to the manager. The communication will occur only if the informant is willing to share the information. When and what the informant communicates with the manager depends on the identities of the manager and the informant. Anticipating the arrival of new information, the principal's choice of a manager must take into account this endogenous communication.

This chapter studies preference for delegation when the manager can learn about the state before taking the action. There is an unknown state of the world which determines the optimal action to take. The players have the same vNM utility from the action, but they differ in their beliefs about the state of the world. After the

principal chooses the manager, one of the agents may receive a private signal about the state. The agent decides whether or not to disclose the signal to the manager, and the manager updates his posterior belief. The manager cannot commit to an action ex ante and chooses the optimal action given his posterior.

I start in Section 3.3 with an analysis of the equilibria with binary signals. There are two states of the world, and there are also two signals, which increases the likelihood of each state. Given a prior on the state, a player has an expected utility as a function of his belief and the manager's action. The optimal action is determined by the posterior of the player. When an agent has a signal, he can either report it or withhold it from the manager. Reporting and withholding the signal lead to two different actions of the manager, and the agent compares his expected utility given his own posterior belief. Roughly speaking, the agent with a signal reports it if and only if it brings the posterior of the manager closer to his own posterior than not reporting it. The communication strategies are given by cutoff strategies. When the prior of the agent is close to the prior of the manager, he reports both signals when he has one, but if his prior is farther away from that of the manager, he reports only the favorable signal.

When the principal chooses the manager, he compares the equilibria of the subgames for the manager's prior beliefs. In general, there is multiplicity of equilibria for the given prior belief of the manager. However, there exist the smallest and the largest PBE of the subgame, and the extremal equilibria are monotone increasing with the manager's belief. Specifically, the agents' strategies are given by the cutoff points on the space of priors. In the extremal equilibria, the cutoffs are monotone increasing with the manager's belief.

The next result, which is the main result of the chapter, considers the principal's preference over the managers. The principal doesn't necessarily prefer the manager with the same prior, even though such a manager will take the optimal action from the principal's point of view in every subgame. When there is endogenous communication, the amount of communication depends on the manager's prior belief. There is always a first order gain from increase in communication, while the loss from the action choice

is of second order. The principal prefers a manager who will bring in more gain from communication. The change in the prior belief of the manager has two effects on the expected utility. Each effect is through the change in the amount of communication of each signal. When the manager's prior changes, it changes the measure of agents who will report the signals. The change in the measure, together with the probability of getting a signal and the loss from an unreported signal, determines the effect on the expected utility for each signal.

Whether the principal prefers a more moderate manager or a more extreme manager depends on the functional forms and the parameters. I show that there exist two distributions of the priors of the agents such that the equilibrium behavior of the subgame is identical for both distributions, but the principal prefers a more moderate manager for one distribution and a more extreme manager for another distribution.

Section 3.4 extends the results to a continuum of signals. In the last period, the manager's action is uniquely determined by his posterior belief. The agents' communication strategies are given by cutoff strategies. The intuition is the same as in the case with binary signals. An agent with a signal compares the expected utility from reporting and not reporting the signal. The agent will report the signal if and only if it leads to a more favorable action of the manager, and given the supermodularity of the expected utility, the strategies are characterized by cutoff points in the space of priors. There exist also the smallest and the largest PBE of the subgame. In the first period, the principal prefers the manager with the highest expected utility.

The next section considers the implication for voters' preferences when the voters vote sincerely. If the leader has a chance of learning from one of the voters after he is elected, the voters' preference over candidates takes into account that the leader's identity, or the prior, leads to endogenous communication. The main implication is that the voters no longer prefer the candidate with the same prior. The revealed preference doesn't hold any more, and a voter always prefers a candidate who will provide a first order gain from increase in communication.

There is a large literature on delegation. In Aghion and Tirole (1997), delegation

may lead to a suboptimal action from the principal's point of view ex post, but it increases the agent's incentives to acquire information. Che and Kartik (2009), on the other hand, shows that having a conflict of interest motivates the agent to look for information, both to persuade the principal and to avoid prejudice. In the setup of Che and Kartik (2009), delegation is suboptimal because it demotivates the agent. The principal in my model prefers delegation because of the gain from communication. A manager does not look for information himself, but his type, or the prior on the states, leads to endogenous communication, and the principal prefers to increase the communication.

Other papers on information acquisition and appointment of an advisor or a juror include Dur and Swank (2005) and Gerardi and Yariv (2008). Both of them consider binary decisions, and Dur and Swank find that the decision maker's preference for an adviser depends on the decision maker's type. In Gerardi and Yariv, the optimal juror is extremely biased in the opposite direction of the decision maker. My model has a continuum of actions, and the manager doesn't look for information by himself, which leads to different incentives for delegation.

The chapter is also related to literature in strategic communication. Since the signal is a hard-evidence, my chapter is closer to Grossman (1981) or Milgrom (1981) than Crawford and Sobel (1982). In deciding whether to report the signal, the agent weighs the expected utility from each choice, and he communicates the signal only if it leads to a more favorable action of the manager. However, unlike in Grossman (1981) or Milgrom (1981), a continuum of signals don't lead to unraveling in my model. In an equilibrium, for each signal, there is a strictly positive mass of agents who report the signal and a strictly positive mass of agents who don't.

Lastly, the information structure of my model is related to Banerjee and Somanathan (2001). Their model has one signal that increases the likelihood of one state, whereas my model has binary signals or a continuum of signals. The similarities are that the communication strategies are cutoff strategies and the optimal action is determined by the posterior belief. The voting environment in Section 3.5 is related to their environment, but Banerjee and Somanathan don't consider the

preference over the leader's prior.

The rest of the chapter is organized as the following. I present the model in Section 3.2, and the equilibria with binary signals are characterized in Section 3.3. Section 3.4 extends the model to a continuum of signals, and Section 3.5 discusses voters' preferences over candidates. Section 3.6 concludes.

3.2 Model

The principal has a task to delegate to a manager. The task provides a common payoff to the principal, the manager and the agents, and the payoff is determined by the action of the manager and the state of the world. After the principal chooses a manager, one of the agents may receive a signal that is informative about the state of the world. The agent with the signal decides whether or not to disclose the signal to the manager. The signal is a hard-evidence. The manager cannot commit to an action *ex ante*, and he chooses the action given his posterior belief.

There are two states of the world, θ_1 and θ_2 . The prior on the state is indexed by $p \in (0, 1)$, the belief that the true state is θ_1 . The *ex ante* distribution of the agents' beliefs is given by $G(\cdot)$, which is atomless and has a positive density everywhere.

The payoff from action x in state θ_i is given by $U_i(x)$ for $i = 1, 2$. U_i is concave, and U_1 increases with the action, while U_2 decreases with the action. Specifically, I assume

$$U_i'' < 0, U_2' < 0 < U_1', \forall x \in (0, 1),$$

$$U_1'(1) = U_2'(0) = 0.$$

The action is chosen from $[0, 1]$. If the players know that they are in state θ_1 , they want action $x = 1$, and if they know that they are in state θ_2 , they want action $x = 0$. In general, the optimal action depends on the prior on the state.

After the principal chooses a manager, one of the agents may receive a signal. I assume that the probability of getting a signal is identical for all agents, and at most

one agent receives a signal. There are two signals, s_1 and s_2 . The probability of getting a signal is given by

$$\Pr(s_1|\theta_1) = \mu_1, \Pr(s_1|\theta_2) = \mu_2, \Pr(s_2|\theta_1) = \mu'_1, \Pr(s_2|\theta_2) = \mu'_2.$$

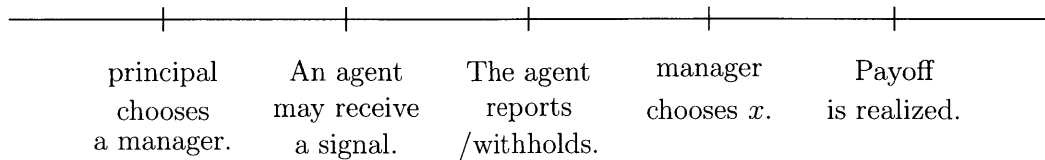
I assume that

$$\begin{aligned} \mu_1 &> \mu_2, \mu'_1 < \mu'_2, \\ \mu_1 + \mu'_1 &\leq 1, \mu_2 + \mu'_2 \leq 1, \\ \mu_1\mu'_2 - \mu_2\mu'_1 &< \min(\mu_1 - \mu_2, \mu'_2 - \mu'_1). \end{aligned}$$

The first two conditions mean that s_1 increases the likelihood of state θ_1 and s_2 increases the likelihood of state θ_2 . The next two conditions say that there is at most one signal, and the last condition is to ensure that the posterior beliefs are well-behaved.

When an agent has a signal, he decides whether or not to disclose the signal to the manager. The signal is non-falsifiable, and only the agent with the signal has a choice in this stage. After the communication stage, the manager updates his belief and chooses the action given his posterior. The manager cannot commit to an action ex ante. The payoff is realized for everyone. Throughout the game, the structure of the signal and the distribution of agents' beliefs are common knowledge. The timing of the game is given in the following graph.

Timing of the Game



3.3 Characterization of Equilibria

This section presents the results with binary signals. I show the existence of a pure strategy perfect Bayesian equilibrium, and I characterize the equilibria. The manager's action in the last period is uniquely determined by his posterior, and the agents' strategies are given by cutoff strategies. The extremal equilibria are monotone increasing with the manager's prior, and the principal prefers a manager with a different prior with probability one. Whether the principal prefers a more moderate manager or a more extreme manager depends on the distribution of the agents' prior, the probabilities of getting a signal and the functional form of the utility function.

I will solve for the equilibrium backwards. First, consider the players' expected utility in the last period. Let $V(p, x)$ be the expected utility from action x when the posterior belief is p . We have

$$V(p, x) = pU_1(x) + (1 - p)U_2(x).$$

From the concavity of U_1 and U_2 , we get the following proposition.

Proposition 3.1. *The manager's strategy is uniquely determined by his posterior belief. It strictly increases with his posterior.*

Proof. From

$$\frac{\partial^2}{\partial x^2} V(p, x) = pU_1''(x) + (1 - p)U_2''(x) < 0,$$

we know that $\partial V/\partial x$ strictly decreases with x . We know from $U_1'(1) = 0$, $U_2'(0) = 0$ that there exists a unique solution $x(p)$ that maximizes $V(p, x)$. The same conditions also guarantee that $x(p)$ is an interior solution. The first order condition can be written as

$$-\frac{U_1'(x)}{U_2'(x)} = \frac{1 - p}{p},$$

and the left hand side decreases with x , and the right hand side decreases with p . Therefore, $x(p)$ strictly increases with p . \square

Next, consider the agents' reporting strategies. Given the manager's prior \tilde{p} and

the agents' strategies, the manager can have one of the three posterior beliefs. Let $\pi_R^1(\tilde{p})$, $\pi_R^2(\tilde{p})$ be the posterior beliefs when an agent reports signal s_1 and s_2 , respectively. $\pi_N(\tilde{p})$ is the posterior belief when no signal is reported, which depends on the equilibrium strategies of the agents. Let $i_1(\tilde{p})$ and $i_2(\tilde{p})$ be the mass of agents who report signal s_1 and s_2 in an equilibrium, respectively. $\pi_N(\tilde{p})$ can be written as

$$\begin{aligned}\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p})) &= \frac{\tilde{p}(1 - \mu_1 i_1(\tilde{p}) - \mu'_1 i_2(\tilde{p}))}{\tilde{p}(1 - \mu_1 i_1(\tilde{p}) - \mu'_1 i_2(\tilde{p})) + (1 - \tilde{p})(1 - \mu_2 i_1(\tilde{p}) - \mu'_2 i_2(\tilde{p}))} \\ &= \frac{1}{1 + \frac{(1-\tilde{p})(1-\mu_2 i_1(\tilde{p})-\mu'_2 i_2(\tilde{p}))}{\tilde{p}(1-\mu_1 i_1(\tilde{p})-\mu'_1 i_2(\tilde{p}))}}.\end{aligned}$$

The posterior beliefs when a signal is reported are given by the following expressions:

$$\begin{aligned}\pi_R^1(p) &= \frac{p\mu_1}{p\mu_1 + (1-p)\mu_2}, \\ \pi_R^2(p) &= \frac{p\mu'_1}{p\mu'_1 + (1-p)\mu'_2}.\end{aligned}$$

Lemma 3.1. *The posterior beliefs satisfy the following inequality in any equilibrium:*

$$\pi_R^2(p) < \pi_N(p, i_1(\tilde{p}), i_2(\tilde{p})) < \pi_R^1(p).$$

The posterior beliefs strictly increase with p .

Proof. From the regularity condition, we have

$$\frac{\mu_2}{\mu_1} < \frac{1 - \mu_2 i_1(\tilde{p}) - \mu'_2 i_2(\tilde{p})}{1 - \mu_1 i_1(\tilde{p}) - \mu'_1 i_2(\tilde{p})} < \frac{\mu'_2}{\mu'_1}$$

for all $i_1(\tilde{p}), i_2(\tilde{p}) \in [0, 1]$. The fact that the posterior beliefs increase with p can be seen by dividing both the numerator and the denominator by the numerator. \square

Since the manager's action is uniquely determined by his posterior belief, the agent compares two expected utilities when he has a signal. If he has signal s_1 , and if he reports it, the manager's posterior becomes $\pi_R^1(\tilde{p})$, and the manager chooses $x(\pi_R^1(\tilde{p}))$. The agent's expected utility is given by $V(\pi_R^1(p), x(\pi_R^1(\tilde{p})))$, where p is the agent's prior. On the other hand, if he withholds his signal, the manager's posterior becomes

$\pi_N(\tilde{p})$, and he chooses $x(\pi_N(\tilde{p}))$. The agent's expected utility is $V(\pi_R^1(p), x(\pi_N(\tilde{p})))$. In the equilibrium, the agent reports s_1 if and only if

$$V(\pi_R^1(p), x(\pi_R^1(\tilde{p}))) \geq V(\pi_R^1(p), x(\pi_N(\tilde{p}))).$$

Similarly, the agent reports s_2 if and only if

$$V(\pi_R^2(p), x(\pi_R^2(\tilde{p}))) \geq V(\pi_R^2(p), x(\pi_N(\tilde{p}))).$$

The following lemma shows that the expected utility $V(p, x)$ is supermodular in (p, x) , and the agent's strategies are given by cutoff strategies.

Lemma 3.2. *The expected utility $V(p, x)$ is supermodular in (p, x) , and the agents' strategies are cutoff strategies.*

Proof. Consider the derivative of $V(p, x)$ with respect to p and x :

$$\frac{\partial^2}{\partial p \partial x} V(p, x) = U'_1(x) - U'_2(x) > 0, \quad \forall x,$$

which follows from $U'_1 > 0 > U'_2$.

When $V(p, x)$ is supermodular in (p, x) , the difference $V(p, x_1) - V(p, x_2)$ is monotone in p . If $x_1 > x_2$, there exists p' such that $V(p, x_1) \geq V(p, x_2)$ if and only if $p \geq p'$. Conversely, if $x_1 < x_2$, there exists p'' such that $V(p, x_1) \geq V(p, x_2)$ if and only if $p \leq p''$.

Together with the fact that $\pi_R^1(p)$ and $\pi_R^2(p)$ are strictly increasing with p , we know that the agents' strategies are given by cutoff strategies. In particular, from $\pi_R^1(\tilde{p}) > \pi_N(\tilde{p}) > \pi_R^2(\tilde{p})$, we know that there exists $p_1(\tilde{p})$ such that s_1 is reported if and only if $p \geq p_1(\tilde{p})$. s_2 is reported if and only if $p \leq p_2(\tilde{p})$ for some $p_2(\tilde{p})$. \square

Lemma 3.2 shows that the agents' strategies are cutoff strategies. Since the agents are indifferent between reporting a signal and withholding it if their prior belief is at

the cutoff, we get two indifference conditions.

$$\begin{aligned} V(\pi_R^1(p_1(\tilde{p})), x(\pi_R^1(\tilde{p}))) &= V(\pi_R^1(p_1(\tilde{p})), x(\pi_N(\tilde{p}))), \\ V(\pi_R^2(p_2(\tilde{p})), x(\pi_R^2(\tilde{p}))) &= V(\pi_R^2(p_2(\tilde{p})), x(\pi_N(\tilde{p}))). \end{aligned}$$

From

$$\pi_R^1(\tilde{p}) > \pi_N(\tilde{p}) > \pi_R^2(\tilde{p}),$$

we get

$$\pi_R^1(\tilde{p}) > \pi_R^1(p_1(\tilde{p})) > \pi_N(\tilde{p}) > \pi_R^2(p_2(\tilde{p})) > \pi_R^2(\tilde{p}),$$

and in particular, we have

$$p_1(\tilde{p}) < \tilde{p} < p_2(\tilde{p}).$$

There exists a neighborhood $[p_1(\tilde{p}), p_2(\tilde{p})]$ around the manager's prior in which the agents report both signals. If the agent's prior is to the left of the neighborhood, he reports only s_2 , and if his prior is to the right of the neighborhood, he reports only s_1 . Together, $p_1(\tilde{p})$ and $p_2(\tilde{p})$ determine the amount of communication in an equilibrium.

I have shown that the manager's strategy is uniquely determined by his posterior belief, and the agents' strategies are given by cutoff strategies. The agents are indifferent between reporting and withholding a signal on a set of measure zero. The next proposition establishes the existence of a pure strategy perfect Bayesian equilibrium.

Proposition 3.2. *There exists a pure strategy perfect Bayesian equilibrium. The agents' strategies are cutoff strategies, and the equilibrium strategies are characterized by two cutoffs, $p_1(\tilde{p}), p_2(\tilde{p}) \in (0, 1)$ where \tilde{p} is the manager's prior. Signal s_1 is reported if and only if $p \geq p_1(\tilde{p})$, and s_2 is reported if and only if $p \leq p_2(\tilde{p})$. We also have $p_1(\tilde{p}) < \tilde{p} < p_2(\tilde{p})$.*

Proof. The second part of the proposition is given in Lemma 3.2 and the following discussion. I'll now show the existence of an equilibrium.

Consider the following two equalities: Given \tilde{p} , p , there exist q_1 and q_2 such that

$$\begin{aligned} V(\pi_R^1(q_1), x(\pi_R^1(\tilde{p}))) &= V(\pi_R^1(q_1), x(p)), \\ V(\pi_R^2(q_2), x(\pi_R^2(\tilde{p}))) &= V(\pi_R^2(q_2), x(p)). \end{aligned}$$

Define $\hat{p}(p, \tilde{p})$ as

$$\hat{p}(p, \tilde{p}) = \pi_N(\tilde{p}, 1 - G(q_1(p)), G(q_2(p))).$$

As we vary p , we can think of \hat{p} as a mapping $\hat{p}(\cdot, \tilde{p}) : [0, 1] \rightarrow [0, 1]$. The fixed point of the mapping is the posterior $\pi_N(\tilde{p})$ in an equilibrium.

From $\pi_R^2(\tilde{p}) < \hat{p}(0, \tilde{p}), \hat{p}(1, \tilde{p}) < \pi_R^1(\tilde{p})$ and the fact that $V, \pi_R^1, \pi_R^2, \pi_N$ are continuous, there exists a fixed point such that $\hat{p}(p, \tilde{p}) = p$. Furthermore, the fixed point is in $(\pi_R^2(\tilde{p}), \pi_R^1(\tilde{p}))$. Each fixed point corresponds to an equilibrium, and the agents' strategies are given by $p_1(\tilde{p}) = q_1(p)$, $p_2(\tilde{p}) = q_2(p)$. \square

In general, there is multiplicity of equilibria. To see the reason for this, consider the mapping $\hat{p}(\cdot, \tilde{p})$ on $(\pi_R^2(\tilde{p}), \pi_R^1(\tilde{p}))$. Since $V(p, x)$ is supermodular, $q_1(p)$ and $q_2(p)$ increase with p . $\pi_N(\tilde{p}, 1 - G(q_1(p)), G(q_2(p)))$ increases with both $q_1(p)$ and $q_2(p)$, and the mapping $\hat{p}(p, \tilde{p})$ strictly increases with p . Therefore, $\hat{p}(\cdot, \tilde{p})$ can intersect with $y = x$ multiple times, which leads to multiplicity of equilibria.

When there is multiplicity of equilibria, the principal's preference over the managers in the first period depends on the equilibria of the subgames for each prior \tilde{p} . The next proposition shows that the extremal equilibria of the subgame are monotone with respect to the prior of the manager.

Proposition 3.3. *There exist the smallest and the largest PBE of the subgame when the manager's prior is \tilde{p} . Let $p_1^*(\tilde{p}), p_2^*(\tilde{p})$ be the cutoffs of the smallest PBE. $p_1^*(\tilde{p}), p_2^*(\tilde{p})$ are monotone increasing with \tilde{p} . Similarly, let $p_1^{**}(\tilde{p}), p_2^{**}(\tilde{p})$ be the cutoffs of the largest PBE. $p_1^{**}(\tilde{p}), p_2^{**}(\tilde{p})$ are monotone increasing with \tilde{p} .*

Proof. We know from Proposition 3.2 that the agents' strategies are given by the cutoffs in the space of priors. Given two cutoffs, $p_1(\tilde{p})$ and $p_2(\tilde{p})$, the posterior of the

manager with no reported signal is uniquely determined. Conversely, given $\pi_N(\tilde{p})$, the cutoffs are determined by the indifference conditions.

The proof of Proposition 3.2 provides that $q_1(p)$ and $q_2(p)$ increase with p . When there is multiplicity of equilibria, there exists the smallest and the largest $\pi_N(\tilde{p})$. The cutoffs corresponding to the smallest and the largest posteriors are the smallest and the largest, and there exist the smallest and the largest equilibria.

Consider $\hat{p}(p, \tilde{p})$ defined in the proof of Proposition 3.2, now also as a function of \tilde{p} . It can be easily verified that when \tilde{p} increases, $q_1(p)$ and $q_2(p)$ increase for each p , and $\hat{p}(p, \tilde{p})$ increases with \tilde{p} . The smallest equilibrium corresponds to the smallest fixed point of the mapping $\hat{p}(p, \tilde{p}) = p$, and if $\hat{p}(p, \tilde{p})$ increases for all p , the fixed point p also has to increase. Therefore, $q_1(p)$ and $q_2(p)$ also increase when \tilde{p} increases.

Similarly, the largest equilibrium corresponds to the largest fixed point $\hat{p}(p, \tilde{p}) = p$. If $\hat{p}(p, \tilde{p})$ increases for all p , the largest fixed point also increases, and the cutoffs $q_1(p)$ and $q_2(p)$ also increase. \square

In the first period, the principal has beliefs about the equilibrium of the subgame for each manager with prior \tilde{p} . Let \hat{p} be the prior of the principal, and $W(\hat{p}, \tilde{p})$ be the principal's expected utility from choosing the manager with prior \tilde{p} . $W(\hat{p}, \tilde{p})$ can be written as the following:

$$\begin{aligned} W(\hat{p}, \tilde{p}) = & i_1(\tilde{p})(\hat{p}\mu_1 + (1 - \hat{p})\mu_2)V(\pi_R^1(\hat{p}), x(\pi_R^1(\tilde{p}))) \\ & + i_2(\tilde{p})(\hat{p}\mu'_1 + (1 - \hat{p})\mu'_2)V(\pi_R^2(\hat{p}), x(\pi_R^2(\tilde{p}))) \\ & + (1 - i_1(\tilde{p})(\hat{p}\mu_1 + (1 - \hat{p})\mu_2) - i_2(\tilde{p})(\hat{p}\mu'_1 + (1 - \hat{p})\mu'_2)) \\ & \times V(\pi_N(\hat{p}, i_1(\tilde{p}), i_2(\tilde{p})), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p})))) \end{aligned}$$

where $i_1(\tilde{p})$ and $i_2(\tilde{p})$ are the measures of the agents who report signal s_1 and s_2 , respectively.

Since $V(p, x)$ is concave and maximized at an interior point, we know that

$$\left. \frac{\partial W}{\partial \pi_R^1(\tilde{p})} \right|_{\tilde{p}=\hat{p}} = \left. \frac{\partial W}{\partial \pi_R^2(\tilde{p})} \right|_{\tilde{p}=\hat{p}} = \left. \frac{\partial W}{\partial \pi_N(\tilde{p})} \right|_{\tilde{p}=\hat{p}} = 0.$$

The first order derivative of $W(\hat{p}, \tilde{p})$ can be written as

$$\frac{\partial W}{\partial \tilde{p}} \Big|_{\tilde{p}=\hat{p}} = \frac{\partial i_1(\tilde{p})}{\partial \tilde{p}} \frac{\partial W}{\partial i_1(\tilde{p})} \Big|_{\tilde{p}=\hat{p}} + \frac{\partial i_2(\tilde{p})}{\partial \tilde{p}} \frac{\partial W}{\partial i_2(\tilde{p})} \Big|_{\tilde{p}=\hat{p}}.$$

The third term in $W(\hat{p}, \tilde{p})$ is the expected utility when there is no reported signal, and it can be rewritten as

$$\begin{aligned} & \hat{p}(1 - \mu_1 i_1(\tilde{p}) - \mu'_1 i_2(\tilde{p})) U_1(x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p})))) \\ & + (1 - \hat{p})(1 - \mu_2 i_1(\tilde{p}) - \mu'_2 i_2(\tilde{p})) U_2(x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p})))) \\ = & V(\hat{p}, x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p})))) \\ & - i_1(\tilde{p})(\hat{p}\mu_1 + (1 - \hat{p})\mu_2) V(\pi_R^1(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p})))) \\ & - i_2(\tilde{p})(\hat{p}\mu'_1 + (1 - \hat{p})\mu'_2) V(\pi_R^2(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p}))))). \end{aligned}$$

Therefore, the partial derivatives with respect to the amount of communications are

$$\begin{aligned} \frac{\partial i_1(\tilde{p})}{\partial \tilde{p}} \frac{\partial W}{\partial i_1(\tilde{p})} &= \frac{\partial i_1(\tilde{p})}{\partial \tilde{p}} (\hat{p}\mu_1 + (1 - \hat{p})\mu_2) \\ &\quad \times (V(\pi_R^1(\hat{p}), x(\pi_R^1(\tilde{p}))) - V(\pi_R^1(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p}))))), \\ \frac{\partial i_2(\tilde{p})}{\partial \tilde{p}} \frac{\partial W}{\partial i_2(\tilde{p})} &= \frac{\partial i_2(\tilde{p})}{\partial \tilde{p}} (\hat{p}\mu'_1 + (1 - \hat{p})\mu'_2) \\ &\quad \times (V(\pi_R^2(\hat{p}), x(\pi_R^2(\tilde{p}))) - V(\pi_R^2(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p}))))). \end{aligned}$$

At $\tilde{p} = \hat{p}$, the difference in V is positive for both terms:

$$\begin{aligned} & V(\pi_R^1(\hat{p}), x(\pi_R^1(\tilde{p}))) - V(\pi_R^1(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p})))) > 0, \\ & V(\pi_R^2(\hat{p}), x(\pi_R^2(\tilde{p}))) - V(\pi_R^2(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p})))) > 0. \end{aligned}$$

The first-order effect on the measure of agents reporting signals depends on the equilibrium selection of the subgames. If the principal believes that they will be in the smallest equilibrium of the subgame for each \tilde{p} , then we have

$$\frac{\partial i_1(\tilde{p})}{\partial \tilde{p}} < 0 < \frac{\partial i_2(\tilde{p})}{\partial \tilde{p}}.$$

We get the same signs of the first-order effects if the principal believes that they will be in the largest equilibrium for each \tilde{p} .

Unless the first order derivative of $W(\hat{p}, \tilde{p})$ is zero at $\tilde{p} = \hat{p}$, the principal prefers a manager with some prior $\tilde{p} \neq \hat{p}$ to the manager with the same prior.

Proposition 3.4. *Let \hat{p} be the prior of the principal. The principal prefers a manager with $\tilde{p} \neq \hat{p}$ on a set of measure 1.*

Proof. Let $W(\hat{p}, \tilde{p})$ be the expected utility from choosing a manager with prior \tilde{p} , where \hat{p} is the prior of the principal. The prior of the manager has second order effects on the expected utility through the manager's action. There are first-order effects through communication, and the effect is given by

$$\left. \frac{\partial W}{\partial \tilde{p}} \right|_{\tilde{p}=\hat{p}} = \left. \frac{\partial i_1(\tilde{p})}{\partial \tilde{p}} \frac{\partial W}{\partial i_1(\tilde{p})} \right|_{\tilde{p}=\hat{p}} + \left. \frac{\partial i_2(\tilde{p})}{\partial \tilde{p}} \frac{\partial W}{\partial i_2(\tilde{p})} \right|_{\tilde{p}=\hat{p}},$$

where

$$\begin{aligned} \frac{\partial i_1(\tilde{p})}{\partial \tilde{p}} \frac{\partial W}{\partial i_1(\tilde{p})} &= \frac{\partial i_1(\tilde{p})}{\partial \tilde{p}} (\hat{p}\mu_1 + (1 - \hat{p})\mu_2) \\ &\quad \times (V(\pi_R^1(\hat{p}), x(\pi_R^1(\tilde{p}))) - V(\pi_R^1(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p}))))), \\ \frac{\partial i_2(\tilde{p})}{\partial \tilde{p}} \frac{\partial W}{\partial i_2(\tilde{p})} &= \frac{\partial i_2(\tilde{p})}{\partial \tilde{p}} (\hat{p}\mu'_1 + (1 - \hat{p})\mu'_2) \\ &\quad \times (V(\pi_R^2(\hat{p}), x(\pi_R^2(\tilde{p}))) - V(\pi_R^2(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p}))))). \end{aligned}$$

In general, $\partial W/\partial \tilde{p}$ is not zero at $\tilde{p} = \hat{p}$, and the principal prefers a manager with a different prior. \square

Proposition 3.4 shows that the principal prefers a manager with some different prior to the manager with the same prior. Whether the principal prefers a more moderate manager or a more extreme manager depends on the utility functions and the parameters. The next proposition shows that the preference of the principal can

go in both directions, when the equilibrium behavior of the subgame is held fixed.

Proposition 3.5. *There exist distributions of the agents' prior, $G_1(\cdot)$, $G_2(\cdot)$, such that the equilibrium strategies of the subgame after the manager is chosen are identical for both distributions, but the principal prefers a more moderate manager with $G_1(\cdot)$, and he prefers a more extreme manager with $G_2(\cdot)$.*

Proof. Suppose the distribution of the priors is G_1 . Let $p_1(\tilde{p})$, $p_2(\tilde{p})$ be the cutoffs of the agents' strategies in an equilibrium of the subgame with prior \tilde{p} . From the above discussion, we have $i_1(\tilde{p}) = 1 - G_1(p_1(\tilde{p}))$, $i_2(\tilde{p}) = G_1(p_2(\tilde{p}))$, and

$$\begin{aligned}\frac{\partial i_1(\tilde{p})}{\partial \tilde{p}} &= -g_1(p_1(\tilde{p})) \frac{\partial p_1}{\partial \tilde{p}}, \\ \frac{\partial i_2(\tilde{p})}{\partial \tilde{p}} &= g_1(p_2(\tilde{p})) \frac{\partial p_2}{\partial \tilde{p}}.\end{aligned}$$

We can always find another distribution G_2 such that $G_1(p_1(\tilde{p})) = G_2(p_1(\tilde{p}))$, $G_1(p_2(\tilde{p})) = G_2(p_2(\tilde{p}))$. Under distribution G_2 , $p_1(\tilde{p})$, $p_2(\tilde{p})$ form an equilibrium of the subgame. Since $\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p}))$ doesn't change, $V(\pi_R^1(\hat{p}), x(\pi_R^1(\tilde{p}))) - V(\pi_R^1(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p}))))$ and $V(\pi_R^2(\hat{p}), x(\pi_R^2(\tilde{p}))) - V(\pi_R^2(\hat{p}), x(\pi_N(\tilde{p}, i_1(\tilde{p}), i_2(\tilde{p}))))$ also don't change. We also know that $\hat{p}\mu_1 + (1-\hat{p})\mu_2$, $\hat{p}\mu'_1 + (1-\hat{p})\mu'_2$ don't change, and from the indifference conditions, $\partial p_1/\partial \tilde{p}$, $\partial p_2/\partial \tilde{p}$ also don't change. Given these values, we can find G_2 under which the sign of $\partial W/\partial \tilde{p}$ changes. Then, the equilibrium strategies of the subgame are identical under the two distributions, but under one distribution, the principal prefers a more moderate manager, but under another distribution, he prefers a more extreme manager. \square

In a perfect Bayesian equilibrium, the manager's action is uniquely determined by his posterior belief in the last period. The agents' strategies are given by cutoff strategies, and the agents mix on a set of measure zero. I've also shown the existence of a pure strategy perfect Bayesian equilibrium, and the extremal equilibria are monotone increasing with the manager's prior. The principal prefers a manager with a different prior to the one with the same prior, and whether he prefers a more moderate or a more extreme manager depends on underlying parameters and functional forms.

3.4 Continuum of Signals

This section discusses the environment where there is a continuum of signals. In a perfect Bayesian equilibrium, the manager's action is uniquely determined by his posterior, and the agents' strategies are given by cutoff strategies. Unlike Grossman (1981) or Milgrom (1981), the continuum of signals don't lead to unraveling, and the probability of each signal being reported lies strictly between 0 and 1. There exist the smallest and the largest PBE of the subgame, and the principal chooses a manager with the highest expected utility.

Let $S = [\underline{s}, \bar{s}]$ be the set of signals. The probability of getting a signal s is given by $\Pr(s|\theta_1) = f_1(s)$ and $\Pr(s|\theta_2) = f_2(s)$. The densities are strictly positive everywhere and atomless. There is at most one signal, and we have $\int f_1(s)ds \leq 1$, $\int f_2(s)ds \leq 1$. I also assume that the likelihood of state θ_1 increases with s : $f_1(s)/f_2(s)$ increases with s .

I assume the following for f_1 and f_2 for all s :

$$\frac{f_1(\underline{s})}{f_2(\underline{s})} < \frac{1 - \int_{\underline{s}}^s f_1(t)dt}{1 - \int_{\underline{s}}^s f_2(t)dt} < \frac{f_1(\bar{s})}{f_2(\bar{s})},$$

$$\frac{f_1(\underline{s})}{f_2(\underline{s})} < \frac{1 - \int_s^{\bar{s}} f_1(t)dt}{1 - \int_s^{\bar{s}} f_2(t)dt} < \frac{f_1(\bar{s})}{f_2(\bar{s})}.$$

The first result of this section is that the manager's action is uniquely determined by his posterior and increases with it.

Proposition 3.6. *In a PBE, the manager's action is uniquely determined by his posterior. $x(p)$ strictly increases with the manager's posterior p .*

Proof. From

$$\frac{\partial^2}{\partial x^2} V(p, x) = pU_1''(x) + (1-p)U_2''(x) < 0,$$

the first order condition strictly decreases with x . We know from $U_1'(1) = 0$, $U_2'(0) = 0$ that there exists a unique solution $x(p)$ that maximizes $V(p, x)$. The same conditions also guarantee that $x(p)$ is an interior solution. The first order condition can be

written as

$$-\frac{U'_1(x)}{U'_2(x)} = \frac{1-p}{p},$$

and the left hand side decreases with x , and the right hand side decreases with p . Therefore, $x(p)$ strictly increases with p . \square

In the last period, it doesn't matter whether there were binary signals or a continuum of signals. The manager updates his posterior depending on whether a signal is reported, and if so, which signal is reported. Given the posterior of the manager, the optimal action is uniquely determined by the concavity of the utility functions.

The next proposition characterizes the reporting strategies of the agents. As was the case with binary signals, the agent's strategy for a given signal is characterized by a cutoff point in the space of priors. Denote by $\pi_R(\tilde{p}, s)$ the manager's posterior when signal s is reported and his prior is \tilde{p} .

Proposition 3.7. *In a PBE, the agents' strategy given a manager's prior \tilde{p} is characterized by $r : S \rightarrow [0, 1]$ and $s_0 \in S$ such that (i) for $s < s_0$, the agent reports a signal if and only if $p \leq r(s)$, (ii) for $s > s_0$, the agent reports a signal if and only if $p \geq r(s)$, (iii) signal s_0 leads to the same posterior of the manager as no reported signal: $\pi_N(\tilde{p}, r) = \pi_R(\tilde{p}, s_0)$.*

Proof. Suppose $i(s)$ is the measure of agents who report signal s in an equilibrium. Given $i(s)$ for $s \in S$, there exists $\pi_N(\tilde{p})$, the posterior of the manager when no agent reports a signal.

Now, consider the agent with signal s . If he reports the signal, the manager chooses $x(\pi_R(\tilde{p}, s))$. If the agent withholds the signal, the manager chooses $x(\pi_N(\tilde{p}))$. Given his own posterior $\pi_R(p, s)$, the agent reports the signal if and only if

$$\Delta(p, s) \equiv V(\pi_R(p, s), x(\pi_R(\tilde{p}, s))) - V(\pi_R(p, s), x(\pi_N(\tilde{p}))) \geq 0.$$

$x(p)$ increases with p , and we have $x(\pi_R(\tilde{p}, s)) > x(\pi_N(\tilde{p}))$ if and only if $\pi_R(\tilde{p}, s) > \pi_N(\tilde{p})$. Since $V(p, x)$ is supermodular, $\Delta(p, s)$ increases with p when $\pi_R(\tilde{p}, s) > \pi_N(\tilde{p})$, and there exists $r(s)$ such that the agent reports signal s if and only if $p \geq r(s)$.

If $\pi_R(\tilde{p}, s) < \pi_N(\tilde{p})$, $\Delta(p, s)$ decreases with p . Therefore, there exists $r(s)$ such that the agent reports s if and only if $p \leq r(s)$. If $\pi_R(\tilde{p}, s) = \pi_N(\tilde{p})$, the agent is indifferent between reporting and withholding the signal.

An equilibrium is characterized by $r(s)$ and $s_0 \in S$ such that $\pi_N(\tilde{p}, r(s)) = \pi_R(\tilde{p}, s_0)$ and the agent reports $s < s_0$ if and only if $p \leq r(s)$. The agent reports $s > s_0$ if and only if $p \geq r(s)$. \square

Proposition 3.7 shows that the agents' strategies are given by the cutoffs in the space of priors. When the agent reports a signal depends on the relative size of $\pi_R(\tilde{p}, s)$ and $\pi_N(\tilde{p}, r)$, where the latter is an endogenous object. However, the following corollary shows that $\pi_R(r(s), s)$ strictly increases with s and coincides with $\pi_R(\tilde{p}, s)$ at $s = s_0$.

Corollary 3.1. *In an equilibrium, the posterior $\pi_R(r(s), s)$ strictly increases with s and is between $\pi_N(\tilde{p}, r)$ and $\pi_R(\tilde{p}, s)$. They all coincide at $s = s_0$. Specifically, we have*

$$\begin{aligned} \pi_R(\tilde{p}, s) &< \pi_R(r(s), s) < \pi_N(\tilde{p}, r) \text{ if } s < s_0, \\ \pi_R(\tilde{p}, s) &= \pi_R(r(s), s) = \pi_N(\tilde{p}, r) \text{ if } s = s_0, \\ \pi_N(\tilde{p}, r) &< \pi_R(r(s), s) < \pi_R(\tilde{p}, s) \text{ otherwise.} \end{aligned}$$

Proof. From the proof of Proposition 3.7, we have

$$\Delta(p, s) \equiv V(\pi_R(p, s), x(\pi_R(\tilde{p}, s))) - V(\pi_R(p, s), x(\pi_N(\tilde{p}))).$$

The agent is indifferent between reporting and withholding the signal at the cutoff, and $\Delta(r(s), s) = 0$.

In an equilibrium, $\pi_N(\tilde{p})$ is fixed, while $\pi_R(\tilde{p}, s)$ increases with s . From Proposition 3.7, we have

$$\pi_R(\tilde{p}, s) > \pi_N(\tilde{p}) \iff s > s_0,$$

and $\pi_N(\tilde{p}) < \pi_R(r(s), s) < \pi_R(\tilde{p}, s)$. For $p' < \pi_R(\tilde{p}, s)$, $s > s_0$, $V(p', x(\pi_R(\tilde{p}, s))) -$

$V(p', x(\pi_N(\tilde{p})))$ decreases with s . Together with the fact that $V(p', x(\pi_R(\tilde{p}, s))) - V(p', x(\pi_N(\tilde{p})))$ increases with p' when $s > s_0$, we know that $\pi_R(r(s), s)$ increases with s . By a similar argument, $\pi_R(r(s), s)$ increases with s when $s < s_0$. \square

The above corollary provides upper and lower bounds on $\pi_N(\tilde{p}, r)$. Together with the regularity conditions, we have $\pi_R(\tilde{p}, \underline{s}) < \pi_N(\tilde{p}, r) < \pi_R(\tilde{p}, \bar{s})$ in any equilibrium. By the fixed point theorem, we get the existence of a pure strategy PBE.

Proposition 3.8. *There exists a pure strategy perfect Bayesian equilibrium.*

Proof. We know from Proposition 3.7 and Corollary 3.1 that a perfect Bayesian equilibrium is characterized by $r : S \rightarrow [0, 1]$ and $s_0 \in S$ such that

if $s < s_0$, the agent reports s if and only if $p \leq r(s)$,

$$\pi_R(\tilde{p}, s) < \pi_R(r(s), s) < \pi_N(\tilde{p}, r),$$

if $s = s_0$, the agent is indifferent between reporting and withholding s_0 ,

$$\pi_R(\tilde{p}, s) = \pi_R(r(s), s) = \pi_N(\tilde{p}, r),$$

otherwise, the agent reports s if and only if $p \geq r(s)$,

$$\pi_N(\tilde{p}, r) < \pi_R(r(s), s) < \pi_R(\tilde{p}, s),$$

and the agents may mix between reporting and withholding a signal if and only if $p = r(s)$ or $s = s_0$.

Consider the following mapping $\hat{r} : [\pi_R(\tilde{p}, \underline{s}), \pi_R(\tilde{p}, \bar{s})] \times S \rightarrow [0, 1]$ given by

$$V(\hat{r}(p, s), x(\pi_R(\tilde{p}, s))) = V(\hat{r}(p, s), x(p)).$$

The agent with prior $\hat{r}(p, s)$ is indifferent between reporting and withholding signal s if the manager chooses $x(p)$ when no signal is reported. Since $V(p, x)$ has a unique interior maximand, \hat{r} is well-defined. Also define $\hat{s}(p)$ as $\pi_R(\tilde{p}, \hat{s}(p)) = p$.

Define $\hat{p} : [\pi_R(\tilde{p}, \underline{s}), \pi_R(\tilde{p}, \bar{s})] \rightarrow [0, 1]$ as the following:

$$\hat{p}(p) = \frac{\tilde{p}H_1(p)}{\tilde{p}H_1(p) + (1 - \tilde{p})H_2(p)},$$

where

$$H_1(p) = 1 - \int_{\underline{s}}^{\hat{s}(p)} G(\hat{r}(p, s)) f_1(s) ds - \int_{\hat{s}(p)}^{\bar{s}} (1 - G(\hat{r}(p, s))) f_1(s) ds,$$

$$H_2(p) = 1 - \int_{\underline{s}}^{\hat{s}(p)} G(\hat{r}(p, s)) f_2(s) ds - \int_{\hat{s}(p)}^{\bar{s}} (1 - G(\hat{r}(p, s))) f_2(s) ds.$$

The fixed point $\hat{p}(p) = p$ is the posterior of the manager when no signal is reported. \hat{r} provides the cutoffs for the agents' reporting strategies. By the continuity of the functions and the posteriors, \hat{p} is a continuous mapping. From the regularity conditions, $\hat{p}(p) \in (\pi_R(\tilde{p}, \underline{s}), \pi_R(\tilde{p}, \bar{s}))$, and by the fixed point theorem, there exists a fixed point of the mapping \hat{p} . \square

Proposition 3.8 shows that unlike in Grossman (1981) or Milgrom (1981), unraveling doesn't occur with a continuum of signals. In an equilibrium, the posterior of the manager when no signal is reported lies strictly between $\pi_R(\tilde{p}, \underline{s})$ and $\pi_R(\tilde{p}, \bar{s})$, and the mass of agents reporting signal s is in $(0, 1)$ for all $s \in S$.

The following proposition shows that there exist the smallest and the largest equilibria for the given prior of the manager.

Proposition 3.9. *Given the prior of the manager \tilde{p} , there exist the smallest and the largest equilibria of the subgame.*

Proof. The proof runs parallel to the proof of Proposition 3.3. Consider the mappings \hat{r} and \hat{p} from the proof of Proposition 3.8. We can extend the definition of the mappings so that \tilde{p} is a parameter. Since $V(p, x)$ is supermodular, $\hat{r}(p, s)$ increases with p for all $s \in S$. $\hat{r}(p, s)$ corresponding to the smallest and the largest fixed point of $\hat{p}(p) = p$ are the smallest and the largest equilibria of the subgame when the manager's prior is \tilde{p} . \square

Given an equilibrium of the subgame, the principal's expected utility can be written as a function of the prior of the manager. In the first period, the principal chooses the manager with prior \tilde{p} with the highest expected utility.

Proposition 3.10. *Given the principal's belief about the equilibrium of the subgame for each prior of the manager, the principal chooses the manager with the highest expected utility.*

3.5 Voting

This section discusses the implication for voters' preferences over candidates. If the leader has a chance to learn from the voters after he is elected, and if the voters vote sincerely, the voters don't prefer the candidate with the same prior the most. A voter strictly prefers a candidate who brings in gain from communication to a candidate with the same prior.

Specifically, consider the following environment. There is a continuum of voters with prior on the state θ_1, θ_2 . The distribution of the priors is given by $G(\cdot)$. There are two signals, s_1 and s_2 , with the following probabilities:

$$\Pr(s_1|\theta_1) = \mu_1, \Pr(s_1|\theta_2) = \mu_2, \Pr(s_2|\theta_1) = \mu'_1, \Pr(s_2|\theta_2) = \mu'_2$$

such that

$$\begin{aligned} \mu_1 &> \mu_2, \mu'_1 < \mu'_2, \\ \mu_1 + \mu'_1 &\leq 1, \mu_2 + \mu'_2 \leq 1, \\ \mu_1\mu'_2 - \mu_2\mu'_1 &< \min(\mu_1 - \mu_2, \mu'_2 - \mu'_1). \end{aligned}$$

After the leader is elected, one of the voters may receive a private signal and decides whether or not to report it to the leader. After the communication stage, the leader chooses a policy which provides an identical payoff to all voters and the leader himself. The leader cannot commit to a policy ex ante.

Assumption 3.1. *Voters vote sincerely.*

Proposition 3.11. *Under Assumption 3.1, a voter doesn't prefer the candidate with the same prior the most. On a set of measure 1, for given $p \in (0, 1)$, there exists*

a prior $\tilde{p} \neq p$ such that a voter with prior p prefers a candidate with prior \tilde{p} to a candidate with prior p .

Proof. The proof follows directly from Proposition 3.4. □

3.6 Conclusion

In this chapter, I study preference for delegation when the manager can learn before taking an action. There is an unobservable payoff-relevant state, and the players have prior beliefs about the state. After the principal chooses the manager, one of the agents may receive a private signal about the state. The agent with the signal decides whether or not to report the signal to the manager. The signal is hard-evidence. After the communication stage, the manager updates his posterior belief about the state and chooses an action. The manager cannot commit to an action *ex ante* and chooses the optimal action given his posterior belief.

In an equilibrium, the agents' strategies are given by cutoff strategies. This is the case with both binary signals and a continuum of signals. In an equilibrium, the manager's action is determined by his posterior belief, and the agent with a signal compares his expected utility from two actions the manager will choose, the one with a signal and the other one with no reported signal. Since the expected utility for given posterior and action is supermodular, the agent's strategy is characterized by a cutoff point in the space of priors.

The cutoffs of the agents' strategies depend on the prior of the manager, and the amount of communication in an equilibrium is endogenous. Anticipating the endogenous communication, the principal takes into account the communication when he chooses the manager. The manager with the same prior as the principal always chooses the optimal action from the principal's perspective. In any subgame, their ideal actions coincide. However, the effect on the expected utility through communication is of first-order, while the loss from the action choice is of second-order. The principal prefers a manager who bring in more gain from communication to a manager with the same prior. Whether he prefers a more moderate manager or a more

extreme manager depends on the functional forms and the parameters.

The implication for voting is that a voter no longer prefers a candidate with the same prior the most. If the voters vote sincerely, and if the leader has a chance to learn from the voters after he is elected, the voter takes into account that the identity of the leader affects the amount of information revealed in the equilibrium. The leader with a different prior will choose a sub-optimal policy from the voter's perspective, but the leader brings out more information and the ex ante expected utility is higher with some difference of priors.

When the principal prefers a manager with some difference in prior, he strictly prefers delegation over taking the action by himself. Even if he could be the one to communicate with the agents and take an action, the principal knows that someone else would bring out more information than he himself, and he prefers to delegate the decision making. The intuition generalizes to models when there is endogenous learning. If the agents can search for a signal at some cost, the level of effort in an equilibrium depends on both the manager's prior and the agent's prior. The level of effort is optimal from the agent's perspective, but the expected utility in an equilibrium is not equalized across the priors of the manager, and with probability one, the principal wants someone with a different prior to be the manager.

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Appendix A

Proofs

A.1 Proofs for Chapter 1

Proof of Proposition 1.2. There are three IC constraints to consider: there are two IC constraints in the second period after the good outcome or the bad outcome in the first period, and there is the IC constraint for the one-shot deviation in the first period. We know from the proof of Proposition 1.1 that the positive persistence implies that the IC constraint for the one-shot deviation is sufficient for the IC constraint for the double deviation.

The IC constraints are

$$\begin{aligned} R_1 &= -c + \pi^2 \binom{p_H}{p_L} w(11) + (1 - \pi^2 \binom{p_H}{p_L}) w(10) \\ &\geq w(10), \\ R_0 &= -c + \tilde{\pi}^2 \binom{p_H}{p_L} w(01) + (1 - \tilde{\pi}^2 \binom{p_H}{p_L}) w(00) \\ &\geq w(00), \\ -c + \pi^1 \binom{p_H}{p_L} (w(1) + \delta R_1) + (1 - \pi^1 \binom{p_H}{p_L}) (w(0) + \delta R_0) \\ &\geq w(0) + \delta (-c + \pi^1 M \binom{p_H}{p_L} w(01) + (1 - \pi^1 M \binom{p_H}{p_L}) w(00)), \end{aligned}$$

where π^2 and $\tilde{\pi}^2$ are the priors in the second period after the good and the bad

outcomes in the first period.

One can verify that the IC constraints are satisfied under the given contract, and the given contract yields the rent specified in Proposition 1.1. \square

Proof of Proposition 1.3. From Proposition 1.1, if the principal wants the agent to work in both periods, the minimum rent is

$$\delta c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi_1^1 \binom{1-p_H}{1-p_L} \tilde{\pi}^2 \binom{p_H}{p_L}}.$$

If the principal takes his outside option after the good outcome in the first period but wants the agent to work both in the first period and the second period after the bad outcome, he has to leave the same amount of rent as inducing the agent to work in both periods. Since the outside option is inefficient, the principal never wants to take his outside option only after the good outcome in the first period.

If the principal takes his outside option after the bad outcome in the first period, the principal doesn't have to leave any rent to the agent. The IC constraint in the first period becomes

$$-c + \pi^1 \binom{p_H}{p_L} (w(1) + \delta R_1) + (1 - \pi^1 \binom{p_H}{p_L}) w(0) \geq w(0),$$

and the principal can offer

$$\begin{aligned} w(0) &= w(10) = 0, \\ w(1) &= \frac{c}{\pi^1 \binom{p_H}{p_L}}, \\ w(11) &= \frac{c}{\pi^2 \binom{p_H}{p_L}}, \end{aligned}$$

where π^2 is the principal's prior in the second period after the good outcome in the first period. Since the principal is already leaving no rent, the principal prefers to have the agent work in the first period and the second period after the good outcome.

The loss in outcome in this case is

$$\delta\pi^1 \begin{pmatrix} 1 - p_H \\ 1 - p_L \end{pmatrix} (-c + \tilde{\pi}^2 \begin{pmatrix} p_H \\ p_L \end{pmatrix} - \underline{u}).$$

If the principal takes his outside option in the first period, the second period problem becomes the same as the one period model, and the principal can induce working in the second period without leaving any rent. The loss in outcome is

$$-c + \pi^1 \begin{pmatrix} p_H \\ p_L \end{pmatrix} - \underline{u}.$$

If the principal mixes the continuation contract, it has the same effect as taking the linear combination of the IC constraints, and it convexifies the set of payoffs.

Therefore, the principal's problem is to choose the contract that minimizes the sum of the rent and the loss in outcome, and he prefers to leave the rent to the agent if the loss from taking the outside option is greater than the rent. This happens when

$$\begin{aligned} \delta c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi^1 \begin{pmatrix} 1 - p_H \\ 1 - p_L \end{pmatrix} \tilde{\pi}^2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} &\leq -c + \pi^1 \begin{pmatrix} p_H \\ p_L \end{pmatrix} - \underline{u}, \\ \delta c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi^1 \begin{pmatrix} 1 - p_H \\ 1 - p_L \end{pmatrix} \tilde{\pi}^2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} &\leq \delta\pi^1 \begin{pmatrix} 1 - p_H \\ 1 - p_L \end{pmatrix} (-c + \tilde{\pi}^2 \begin{pmatrix} p_H \\ p_L \end{pmatrix} - \underline{u}). \end{aligned}$$

Rearranging the inequalities, we get the conditions given in the proposition.

If one of the inequalities doesn't hold, the expected loss in outcome from taking the outside option in some period is smaller than the rent to the agent, and the principal prefers to take his outside option in that period. \square

Proof of Proposition 1.4. After history h^{t-1} , the IC constraint for deviating for T periods is

$$V(h^{t-1}, \pi^t) \geq \sum_{k=1}^T \delta^{k-1} q(h^{t-1} \sqcup \tilde{h}^{k-1}) w(h^{t-1} \sqcup \tilde{h}^k) + \delta^T V(h^{t-1} \sqcup \tilde{h}^T, \pi^t M^T),$$

where the principal takes his outside option with probability $1 - q(h^t \sqcup h^k)$ after $h^t \sqcup h^k$.

$\tilde{h}^0 = \emptyset$ and $h^{t-1} \sqcup \tilde{h}^k, 1 \leq k \leq T$, are defined by

$$h_{t-1+k} = \begin{cases} 0 & \text{if the agent is induced to work but shirks,} \\ -1 & \text{if the principal takes his outside option.} \end{cases}$$

There is a sequence of IC constraints for $T \geq 1$, and the maximum of the arguments on the right hand side of the IC constraints is the minimum rent to the agent. Therefore, the rent to the agent is bounded from below by

$$\max_{T \geq 1} \left[\sum_{k=1}^T \delta^{k-1} q(h^{t-1} \sqcup \tilde{h}^{k-1}) w(h^{t-1} \sqcup \tilde{h}^k) + \delta^T V(h^{t-1} \sqcup \tilde{h}^T, \pi^t M^T) \right].$$

□

Proof of Proposition 1.5. Let π^t be the principal's prior given history $h^{t-1} = \underbrace{0 \cdots 0}_{t-1}$.

If the principal offers

$$w(h^t 1) = \frac{c}{\pi^{t+1} \binom{p_H}{p_L}}, w(h^t 0) = 0, \forall t \geq 0,$$

the agent is induced to work in every period. Since the continuation value of the agent doesn't depend on the history, the agent's IC constraint becomes myopic, and the probability the agent assigns on the good state is the lowest when all outcomes have been bad. Therefore, the agent chooses to work in every period with the above contract. The rent to the agent is given by

$$\sum_{t=1}^{\infty} \delta^{t-1} (-c + \pi^1 M^{t-1} \binom{p_H}{p_L} \frac{c}{\pi^t \binom{p_H}{p_L}}),$$

and there's no loss in outcome.

The above contract gives an upper bound on the difference in the principal's payoff between the first best and the second best. Using

$$\pi^t \binom{p_H}{p_L} \geq M_2 \binom{p_H}{p_L},$$

we have

$$\begin{aligned}
& \sum_{t=1}^{\infty} \delta^{t-1} \left(-c + \pi^1 M^{t-1} \begin{pmatrix} p_H \\ p_L \end{pmatrix} \frac{c}{\pi^t \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \right) \\
&= \sum_{t=2}^{\infty} \delta^{t-1} \left(-c + \pi^1 M^{t-1} \begin{pmatrix} p_H \\ p_L \end{pmatrix} \frac{c}{\pi^t \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \right) \\
&\leq \sum_{t=1}^{\infty} \delta^t \left(-c + \pi^1 M^t \begin{pmatrix} p_H \\ p_L \end{pmatrix} \frac{c}{M_2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \right) \\
&= \frac{c}{M_2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \frac{\delta \det M}{1 - \delta \det M} (p_H - p_L) \left(\frac{\delta}{1 - \delta} M_{21} + \pi_1^1 \right).
\end{aligned}$$

The difference in the average per period payoff of the principal between the first best and the second best is at most

$$\begin{aligned}
& (1 - \delta) \frac{c}{M_2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \frac{\delta \det M}{1 - \delta \det M} (p_H - p_L) \left(\frac{\delta}{1 - \delta} M_{21} + \pi_1^1 \right) \\
&= \frac{c}{M_2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}} \frac{\delta \det M}{1 - \delta \det M} (p_H - p_L) (\delta M_{21} + (1 - \delta) \pi_1^1).
\end{aligned}$$

Therefore, given $\epsilon > 0$, there exists $\bar{\delta}$ such that for $\delta < \bar{\delta}$, the principal can approximate his first best payoff by ϵ . Conversely, for given δ , there exists \bar{D} and Δ_p such that if $\det M < \bar{D}$ or $p_H/p_L < \Delta_p$, the principal can approximate his first best payoff by ϵ .

In addition, the principal can offer

$$\begin{aligned}
w(1) &= \frac{c}{\pi^1 \begin{pmatrix} p_H \\ p_L \end{pmatrix}}, w(0) = 0, \\
w(h^t 1) &= \frac{c}{M_2 \begin{pmatrix} p_H \\ p_L \end{pmatrix}}, w(h^t 0) = 0, \forall t \geq 1,
\end{aligned}$$

and he can approximate his first best payoff with a contract that is stationary from the second period. \square

Proof of Proposition 1.8. Consider the IC constraints of the agent. By deviating in period t given history h^{t-1} , the agent effectively replaces the continuation contract for $h^{t-1}1$ with the continuation contract for $h^{t-1}0$. When the payments are independent

of the history, the continuation contract from period $t + 1$ is identical whether $h_t = 0$ or $h_t = 1$, and the IC constraint becomes

$$-c + \pi^t \binom{p_H}{p_L} w(h^{t-1}1) + (1 - \pi^t \binom{p_H}{p_L}) w(h^{t-1}0) \geq w(h^{t-1}0).$$

In particular, the principal doesn't take his outside option in any period, and the agent is induced to work in every period.

Since the payments $w_{t-1}(1)$ and $w_{t-1}(0)$ satisfy the agent's IC constraint at all information sets in period t , it is necessary that

$$w_{t-1}(1) \geq w_{t-1}(0) + \frac{c}{\pi^t \binom{p_H}{p_L}}$$

for all π^t given t . In particular, let $\tilde{\pi}^t$ be the prior when all outcomes have been bad since period 1, and we get

$$\begin{aligned} w_{t-1}(1) &\geq w_{t-1}(0) + \frac{c}{\tilde{\pi}^t \binom{p_H}{p_L}}, \\ w_{t-1}(1) &> w_{t-1}(0) + \frac{c}{\pi^t \binom{p_H}{p_L}}, \end{aligned}$$

for all $\pi^t \neq \tilde{\pi}^t$ that can arise as a prior after some history. Since in each period t , there are only two levels of payments, the principal wants to provide the positive payment only after the good outcome, and we have $w_t(0) = 0$ for all t .

On the other hand, consider the IC constraints in period 1. From Section 4, the IC constraints in period 1 are given by

$$\begin{aligned} V(\emptyset, \pi^1) &\geq \sum_{k=1}^T \delta^{k-1} q(\underbrace{0 \cdots 0}_{k-1}) w(\underbrace{0 \cdots 0}_k) + \delta^T V(\underbrace{0 \cdots 0}_T, \pi^1 M^T) \\ &= \delta^T V(\underbrace{0 \cdots 0}_T, \pi^1 M^T), T \geq 1. \end{aligned}$$

We know from Proposition 1.5 that there exists a uniform upper bound on $V(\underbrace{0 \cdots 0}_T, \pi)$ under an optimal contract. $w(1) > 0$ implies that there exists $T > 0$ such that the

IC constraint for deviating T times in a row binds. Lowering $\delta^T V(\underbrace{0 \cdots 0}_T, \pi^1 M^T)$, the deviation payoff under the wrong continuation contract, relaxes the IC constraint, and it will allow the principal to lower $w(1)$, increasing his payoff.

Consider updating priors π and $\hat{\pi}$ after the good outcome. Without loss of generality, suppose $\pi_1 > \hat{\pi}_1$. After the good outcome, the priors become

$$\pi' = \left(\frac{\pi_1 p_H}{\pi \binom{p_H}{p_L}}, \frac{\pi_2 p_L}{\pi \binom{p_H}{p_L}} \right)$$

and

$$\hat{\pi}' = \left(\frac{\hat{\pi}_1 p_H}{\hat{\pi} \binom{p_H}{p_L}}, \frac{\hat{\pi}_2 p_L}{\hat{\pi} \binom{p_H}{p_L}} \right).$$

$\pi'_1 \geq \hat{\pi}'_1$, and the equality holds if and only if $p_L = 0$.

Consider history $h^t = \underbrace{0 \cdots 0}_T 1$. In evaluating $V(\underbrace{0 \cdots 0}_T, \pi^1 M^T)$, the agent assigns $\pi^1 M^T \binom{p_H}{p_L}$ as the probability on h^t and updates his prior after observing $h_t = 1$. On the equilibrium path, he has the same prior in the period after h^t if and only if $p_L = 0$. Suppose $p_L > 0$ and consider adjusting $w(h^t)$ and $w(h^t 1)$.

Let π and $\hat{\pi}$ be the priors of the agent in the period following h^t when he has deviated in the first T periods and on the equilibrium path, respectively. If the principal lowers $w(h^t 1)$ by Δ and raises $w(h^t)$ by $\delta \hat{\pi} \binom{p_H}{p_L} \Delta$, the agent's continuation value on the equilibrium path doesn't change. However, the deviation payoff, $V(\underbrace{0 \cdots 0}_T, \pi^1 M^T)$, changes by

$$\begin{aligned} & \delta P(\underbrace{0 \cdots 0}_T 1 | \underbrace{0 \cdots 0}_T, \pi^1 M^T) (\hat{\pi} \binom{p_H}{p_L} \Delta - \pi \binom{p_H}{p_L} \Delta) \\ &= \delta P(\underbrace{0 \cdots 0}_T 1 | \underbrace{0 \cdots 0}_T, \pi^1 M^T) (\hat{\pi} - \pi) \binom{p_H}{p_L} \Delta. \end{aligned}$$

From $p_L > 0$, we know that $\hat{\pi}_1 - \pi_1 < 0$, and the change in $V(\underbrace{0 \cdots 0}_T, \pi^1 M^T)$ is strictly negative. Since $h_t = 1$, the IC constraint doesn't bind at h^t , and the principal can make the adjustment for Δ sufficiently small. Therefore, the principal can lower the rent to the agent by raising $w(h^t)$ and lowering $w(h^t 1)$; if it is optimal to provide a

history-independent contract, p_L has to be zero.

Similarly, we can consider raising $w(1\underbrace{0\cdots 0}_T)$ and lowering $w(1\underbrace{0\cdots 0}_T 1)$. This will lower $V(1\underbrace{0\cdots 0}_{T-1}, \pi^2 M^{T-1})$, where π^2 is the agent's prior after the good outcome in period 1, and the principal can lower the payment $w(11)$.

When the agent updates priors π and $\hat{\pi}$ such that $\pi_1 > \hat{\pi}_1$, after the bad outcome, the priors become

$$\pi'' = \left(\frac{\pi_1(1-p_H)}{\pi(1-p_L)}, \frac{\pi_2(1-p_L)}{\pi(1-p_H)} \right)$$

and

$$\hat{\pi}'' = \left(\frac{\hat{\pi}_1(1-p_H)}{\hat{\pi}(1-p_L)}, \frac{\hat{\pi}_2(1-p_L)}{\hat{\pi}(1-p_H)} \right).$$

$\pi_1'' \geq \hat{\pi}_1''$, and the equality holds if and only if $p_H = 1$.

Let π and $\hat{\pi}$ be the priors of the agent in the period following $h^t = 1\underbrace{0\cdots 0}_T$ when he has deviated for T periods from period 2 and on the equilibrium path, respectively. If the principal lowers $w(1\underbrace{0\cdots 0}_T 1)$ by Δ and raises $w(1\underbrace{0\cdots 0}_T)$ by $\delta\hat{\pi}\binom{p_H}{p_L}\Delta$, the agent's continuation payoff on the equilibrium path doesn't change. On the other hand, the deviation payoff, $V(1\underbrace{0\cdots 0}_{T-1}, \pi^2 M^{T-1})$, changes by

$$\begin{aligned} & \delta P(1\underbrace{0\cdots 0}_T | 1\underbrace{0\cdots 0}_{T-1}, \pi^2 M^{T-1}) (\hat{\pi} \binom{p_H}{p_L} \Delta - \pi \binom{p_H}{p_L} \Delta) \\ &= \delta P(1\underbrace{0\cdots 0}_T | 1\underbrace{0\cdots 0}_{T-1}, \pi^2 M^{T-1}) (\hat{\pi} - \pi) \binom{p_H}{p_L} \Delta. \end{aligned}$$

Unless $p_H = 1$, $\hat{\pi}_1 - \pi_1 < 0$, and the principal can make the adjustment since the IC constraint doesn't bind. Therefore, if the optimal contract is history-independent, p_H must equal one.

For an optimal contract to be history-independent, it is necessary that $p_H = 1, p_L = 0$. \square

Lemma A.1. *Suppose $p_H = 1, p_L = 0$. The IC constraints for the one-shot deviations are sufficient conditions for all IC constraints.*

Proof of Lemma A.1. Randomizing the continuation contracts is the same as taking the linear combination of the IC constraints, and it is sufficient to prove the lemma for pure strategies. Consider the agent who deviated in every period he is induced to work from period $t + 1$ to period $t + T$. Denote this history by h^{t+T} . Without loss of generality, we only need to consider the case the principal doesn't take his outside option in period $t + T + 1$ after the given history. Suppose the principal takes his outside option for $k \geq 0$ times after history $h^{t+T}0$. The IC constraint for the one-shot deviation after h^{t+T} is given by

$$-c + \pi \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \geq w(h^{t+T}0) + \delta^{k+1}(-c + \pi M^{k+1} \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix}),$$

where π is the principal's prior after h^{t+T} when he believes that the agent worked in every period, and V_1 is the sum of the present compensation and the continuation value after $h^{t+T}1$. $V_2, V'_1,$ and V'_2 are defined analogously for $h^{t+T}0, h^{t+T}0 \underbrace{-1 \dots -1}_k 1$ and $h^{t+T}0 \underbrace{-1 \dots -1}_k 0$. Remember -1 refers to a period in which the principal takes his outside option.

By subtracting $V_2 = w(h^{t+T}0) + \delta^{k+1}(-c + M_2 M^k \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix})$ from both sides, we know that the IC constraint is equivalent to

$$-c + \pi_1(V_1 - V_2) \geq \delta^{k+1}(\pi M - M_2)M^k \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix},$$

which is again equivalent to

$$V_1 - V_2 \geq \frac{c}{\pi_1} + (\delta \det M)^{k+1}(V'_1 - V'_2).$$

When the agent has deviated from period $t + 1$ to $t + T$, his prior at the beginning of period $t + T + 1$ is given by $\pi^{t+1}M^T$. From the positive persistence, the agent assigns a strictly higher probability on the good state than the principal does, and we have

$$\pi^{t+1}M^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} > \pi_1.$$

After having deviated from period $t+1$ to $t+T$, the agent's IC constraint for working in period $t+T+1$ is given by

$$\begin{aligned} -c + \pi^{t+1} M^T \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} &\geq w(h^{t+T} \mathbf{0}) + \delta^{k+1} (-c + \pi^{t+1} M^{T+k+1} \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix}) \\ \Leftrightarrow V_1 - V_2 &\geq \frac{c}{\pi^{t+1} M^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}} + (\delta \det M)^{k+1} (V'_1 - V'_2). \end{aligned}$$

From

$$\pi^{t+1} M^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} > \pi_1,$$

the agent prefers to work in period $t+T+1$ even if he has deviated from period $t+1$ to $t+T$, as long as the IC constraint for the one-shot deviation after h^{t+T} is satisfied. Therefore, after each history h^t , it is sufficient to consider the IC constraint for the one-shot deviation. \square

Proof of Proposition 1.9. Suppose the principal wants the agent to work in every period. Let V_1^t, V_2^t be the sum of the present compensation and the continuation value after history $\underbrace{0 \cdots 0}_{t-1} 1$ and $\underbrace{0 \cdots 0}_t$. In period 1, the IC constraint for the one-shot deviation is given by

$$\begin{aligned} -c + \pi_1^1 V_1^1 + \pi_2^1 V_2^1 &\geq w(0) + \delta (-c + \pi^1 M \begin{pmatrix} V_1^2 \\ V_2^2 \end{pmatrix}) \\ \Leftrightarrow V_1^1 - V_2^1 &\geq \frac{c}{\pi_1^1} + \delta \det M (V_1^2 - V_2^2), \end{aligned}$$

and for $t \geq 2$, the IC constraint for the one-shot deviation is given by

$$\begin{aligned} -c + M_{21} V_1^t + M_{22} V_2^t &\geq w(\underbrace{0 \cdots 0}_t) + \delta (-c + M_2 M \begin{pmatrix} V_1^{t+1} \\ V_2^{t+1} \end{pmatrix}) \\ \Leftrightarrow V_1^t - V_2^t &\geq \frac{c}{M_{21}} + \delta \det M (V_1^{t+1} - V_2^{t+1}). \end{aligned}$$

We know from Proposition 2.2 that the principal can offer

$$w(1) = \frac{c}{\pi_1^1}, w(0) = 0,$$

$$w(h^t 1) = \frac{c}{M_{21}}, w(h^t 0) = 0, \forall h^t, t \geq 1.$$

to have the agent work in every period. Under the optimal contract, the IC constraints are strictly binding after $\underbrace{0 \cdots 0}_t, \forall t \geq 0$.

I will now show that by taking his outside option after $h^{t_0} = \underbrace{0 \cdots 0}_{t_0}$, the principal can lower the payments $w(\underbrace{0 \cdots 0}_t 1)$ for $0 \leq t < t_0$, and therefore, the rent to the agent is reduced. If the reduction in rent is greater than the loss in outcome by taking the outside option, the principal will prefer to take his outside option after $h^{t_0} = \underbrace{0 \cdots 0}_{t_0}$.

From Lemma A.1, it is sufficient to consider the IC constraints for the one-shot deviations. Suppose the principal takes his outside option once after $h^{t_0} = \underbrace{0 \cdots 0}_{t_0}$. After taking the outside option, the principal continues to pay $w(h^t 1) = \frac{c}{M_{21}}, w(h^t 0) = 0$ for all histories $h^t = h^{t_0} \sqcup h^k, \forall k \geq 1, h^k$. Consider the IC constraint after $h^{t_0-1} = \underbrace{0 \cdots 0}_{t_0-1}$. Since the continuation games after the good outcomes are identical and the continuation games after the bad outcomes are identical, we have

$$V_1^{t_0} - V_2^{t_0} \geq \frac{c}{M_{21}} + (\delta \det M)^2 (\hat{V}_1^{t_0+2} - \hat{V}_2^{t_0+2}),$$

where $\hat{V}_1^{t_0+2}$ is the sum of the present compensation and the continuation value after $h^{t_0} - 11$. $\hat{V}_2^{t_0+2}$ is defined for $h^{t_0} - 10$. We also have

$$\hat{V}_1^{t_0+2} - \hat{V}_2^{t_0+2} = \tilde{V}_1^{t_0+1} - \tilde{V}_2^{t_0+1} = \frac{c}{M_{21}} \frac{1}{1 - \delta \det M},$$

where $\tilde{V}_1^{t_0+1}$ is the sum of the present compensation and the continuation value after the good outcome, from the contract in Proposition 2.2. $\tilde{V}_2^{t_0+1}$ is defined for the bad outcome.

Note that the IC constraint after h^{t_0-1} in the contract from Proposition 2.2 is

$$V_1^{t_0} - V_2^{t_0} \geq \frac{c}{M_{21}} + \delta \det M (\tilde{V}_1^{t_0+1} - \tilde{V}_2^{t_0+1}).$$

Therefore, by taking his outside option after $h^{t_0} = \underbrace{0 \cdots 0}_{t_0}$, the IC constraint after $h^{t_0-1} = \underbrace{0 \cdots 0}_{t_0-1}$ is relaxed by

$$\delta \det M (1 - \delta \det M) \frac{c}{M_{21}} \frac{1}{1 - \delta \det M} = \delta \det M \frac{c}{M_{21}}.$$

The principal can lower the payment $w(h^{t_0-1}1)$ by $\delta \det M \frac{c}{M_{21}}$. By an inductive argument, we have that $V_1^k - V_2^k$ for $1 \leq k \leq t_0$ can be reduced by

$$(\delta \det M)^{t_0-k+1} \frac{c}{M_{21}}.$$

V_1^k and V_2^k for $1 \leq k \leq t_0$ are each reduced by

$$\delta^{t_0-k+1} \det M \left(\frac{1 - (\det M)^{t_0-k}}{1 - \det M} + \frac{1}{M_{21}} (\det M)^{t_0-k} \right) c$$

and

$$\delta^{t_0-k+1} \det M \frac{1 - (\det M)^{t_0-k}}{1 - \det M} c.$$

Since

$$\delta^{t_0-k+1} \det M \left(\frac{1 - (\det M)^{t_0-k}}{1 - \det M} + \frac{1}{M_{21}} (\det M)^{t_0-k} \right) c$$

increases with k and, for $k \leq t_0$, is bounded from above by

$$\begin{aligned} & \delta^{t_0-k+1} \det M \left(\frac{1 - (\det M)^{t_0-k}}{1 - \det M} + \frac{1}{M_{21}} (\det M)^{t_0-k} \right) c \\ & \leq \delta \det M \frac{c}{M_{21}} \\ & < \frac{c}{M_{21}}, \end{aligned}$$

the principal can lower $w(\underbrace{0 \cdots 0}_k 1)$ for $1 \leq k \leq t_0$ by

$$\delta^{t_0-k+1} \det M \left(\frac{1 - (\det M)^{t_0-k}}{1 - \det M} + \frac{1}{M_{21}} (\det M)^{t_0-k} \right) c.$$

From period 1, the rent to the agent is reduced by

$$\begin{aligned} & \pi_1^1 \Delta V_1^1 + \pi_2^1 \Delta V_2^1 \\ &= \pi_1^1 \delta^{t_0} \det M \left(\frac{1 - (\det M)^{t_0-1}}{1 - \det M} + \frac{1}{M_{21}} (\det M)^{t_0-1} \right) c \\ & \quad + \pi_2^1 \delta^{t_0} \det M \frac{1 - (\det M)^{t_0-1}}{1 - \det M} c \\ &= \delta^{t_0} \det M \left(\frac{1 - (\det M)^{t_0-1}}{1 - \det M} + \frac{\pi_1^1}{M_{21}} (\det M)^{t_0-1} \right) c \\ &\geq \delta^{t_0} \det M \frac{1 - (\det M)^{t_0}}{1 - \det M} c. \end{aligned}$$

On the other hand, the loss in outcome from taking the outside option is

$$\delta^{t_0} \pi_2^1 (M_{22})^{t_0-1} (-c + M_{21} - \underline{u}).$$

Both the loss in outcome and the reduction in rent are discounted by δ^{t_0} . Apart from the discounting, the loss in outcome converges to zero as t_0 goes to infinity, while the reduction in the rent is bounded away from zero. Therefore, there exists t_0 such that

$$\frac{1 - (\det M)^{t_0}}{1 - \det M} \det M c > \pi_2^1 (M_{22})^{t_0-1} (-c + M_{21} - \underline{u}),$$

and for any discount factor $\delta > 0$, the principal strictly prefers to take his outside option after $h^{t_0} = \underbrace{0 \cdots 0}_{t_0}$ than to have the agent work in every period. \square

Proof of Proposition 1.11. Suppose the principal makes a positive payment for history h^t with $h_t = 0$. Let k be the maximum $k < t$ such that $h_k = 1$ in the history h^t . The principal can frontload the payment so that $\hat{w}(h^{k-1}1) = w(h^{k-1}1) + \delta^{t-k} M_{12} M_{22}^{t-k-1} w(h^t)$ and $\hat{w}(h^t) = 0$. If $k = 0$, lower the payment for h^t to $\hat{w}(h^t) = 0$.

Since the composition of the continuation value after the good outcome doesn't matter for the agent's IC constraint, the IC constraints leading up to history h^k are not affected by the adjustment. On the other hand, the IC constraints after $h^k \sqcup h^l$ in the history h^t are relaxed under the new contract.

Under the new contract, the agent is induced to work after exactly the same set of histories as under the previous contract, and the agent's IC constraints are satisfied after every history after which the principal wants the agent to work. The rent to the agent is weakly lower under the new contract. Therefore, the principal can frontload the payment whenever he makes a positive payment for a bad outcome, and there is no loss of generality in assuming that the principal makes positive payments only for the good outcomes. \square

Proof of Proposition 1.12. The proof follows directly from Proposition 2.2, 1.10 and 1.11. \square

Proof of Proposition 1.14. The first step is to find the set of pairs of (V_1, V_2) with which the agent is induced to work in a given period. Define S to be the largest self-generating set of the form

$$S = \text{conv}(\{(\pi, V_1, V_2) | \exists T \geq 0, (\pi', V_1', V_2') \in S \text{ such that}$$

$$(i) \pi' = M_2 M^T,$$

$$(ii) V_2 = \delta^{T+1}(-c + \pi' \begin{pmatrix} V_1' \\ V_2' \end{pmatrix}),$$

$$(iii) V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1}(\det M)^{T+1}(V_1' - V_2')\}).$$

Since mixing the continuation contracts is the same as taking the linear combination of the IC constraints, we can focus on the pure strategies and take the convex hull. From Lemma A.1, it is sufficient to consider the one-shot deviations. Using Proposition 1.11, I'll consider contracts under which the principal makes positive payments only for the good outcomes.

Suppose the principal wants the agent to work after history h^t and he takes his outside option for $T \geq 0$ periods after history h^t . Let V_1 be the sum of the present

compensation and the continuation value after $h^t 1$. Define V_2, V_1' and V_2' similarly for $h^t 0, h^t 0 \underbrace{-1 \cdots -1}_T 1$ and $h^t 0 \underbrace{-1 \cdots -1}_T 0$. Let π be the prior on the state after history h^t . The IC constraint after history h^t is

$$-c + \pi \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \geq \delta^{T+1} (-c + \pi M^{T+1} \begin{pmatrix} V_1' \\ V_2' \end{pmatrix}).$$

Subtracting

$$V_2 = \delta^{T+1} (-c + \pi' \begin{pmatrix} V_1' \\ V_2' \end{pmatrix})$$

from both sides, we get

$$V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1} (\det M)^{T+1} (V_1' - V_2').$$

Conversely, if there exists $T \geq 0, V_1', V_2'$ such that

$$\begin{aligned} V_2 &= \delta^{T+1} (-c + \pi' \begin{pmatrix} V_1' \\ V_2' \end{pmatrix}), \\ V_1 - V_2 &\geq \frac{c}{\pi_1} + \delta^{T+1} (\det M)^{T+1} (V_1' - V_2') \end{aligned}$$

hold, and the agent is induced to work given prior $\pi' = M_2 M^T$ and V_1', V_2' , then given prior π and V_1, V_2 , the agent is induced to work.

Therefore, the set of feasible continuation values to induce work is given by the largest self-generating set

$$S = \text{conv}(\{(\pi, V_1, V_2) | \exists T \geq 0, (\pi', V_1', V_2') \in S \text{ such that}$$

$$(i) \pi' = M_2 M^T,$$

$$(ii) V_2 = \delta^{T+1} (-c + \pi' \begin{pmatrix} V_1' \\ V_2' \end{pmatrix}),$$

$$(iii) V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1} (\det M)^{T+1} (V_1' - V_2')\}).$$

The next step is to characterize the space of (R, L) for all incentive compatible contracts. Let X_π be the space of (R, L) for all incentive compatible contracts with

the initial prior π , where R is the rent to the agent and L is the loss in outcome under the contract. L is defined to be

$$L = (1 - \delta)(Y_{FB} - Y),$$

where Y_{FB} is the expected discounted sum of the outcome in the first best, and Y is the expected discounted sum of the outcome under the given contract. Given prior π , there exists a contract with the rent R and the loss L if the following is satisfied: the principal takes his outside option for $T \geq 0$ periods, and in the first period the agent is induced to work, the continuation contracts after the good outcome and the bad outcome have (R_1, L_1) and (R_2, L_2) , respectively. Specifically, X_π is given by

$X_\pi = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_{M_1}, (R_2, L_2) \in X_{M_2} \text{ such that}$

$$(i) \pi' = \pi M^T,$$

$$(ii) R = \delta^T(-c + \delta \pi' \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}),$$

$$(iii) L = (1 - \delta) \sum_{k=1}^T \delta^{k-1} (-c + \pi M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \underline{u}) + \delta^{T+1} \pi' \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

$$(iv) (\pi', \delta R_1, \delta R_2) \in S\}),$$

where Condition (ii) and (iv) use the fact that there is no loss of generality in delaying the payments.

I'll show that X_{M_1} and X_{M_2} can be found as limits of two sequences of sets. Once we find X_{M_1} and X_{M_2} , X_π is generated from X_{M_1} and X_{M_2} . Consider the sequences of sets, $\{X_1^n\}$ and $\{X_2^n\}$:

$$X_1^0 = \{(R, 0) | R \geq R_1^* \equiv \frac{\delta \det M c}{1 - \delta \det M} (\delta + (1 - \delta) \frac{M_{11}}{M_{21}})\},$$

$$X_2^0 = \{(R, 0) | R \geq R_2^* \equiv \frac{\delta \det M c}{1 - \delta \det M}\},$$

where R_1^* and R_2^* are the rents to the agent under the cost-minimizing contracts for

initial priors M_1 and M_2 , and

$X_1^{n+1} = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_1^n, (R_2, L_2) \in X_2^n \text{ such that}$

$$(i) \pi' = M_1 M^T,$$

$$(ii) R = \delta^T (-c + \delta \pi' \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}),$$

$$(iii) L = (1 - \delta) \sum_{k=1}^T \delta^{k-1} (-c + M_1 M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \underline{u}) + \delta^{T+1} \pi' \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

$$(iv) (\pi', \delta R_1, \delta R_2) \in S\},$$

$X_2^{n+1} = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_1^n, (R_2, L_2) \in X_2^n \text{ such that}$

$$(i) \pi' = M_2 M^T,$$

$$(ii) R = \delta^T (-c + \delta \pi' \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}),$$

$$(iii) L = (1 - \delta) \sum_{k=1}^T \delta^{k-1} (-c + M_2 M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \underline{u}) + \delta^{T+1} \pi' \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

$$(iv) (\pi', \delta R_1, \delta R_2) \in S\}.$$

Define

$$X_1^\infty = \lim_{n \rightarrow \infty} X_1^n,$$

$$X_2^\infty = \lim_{n \rightarrow \infty} X_2^n.$$

X_1^∞ and X_2^∞ are the sets of (R, L) we are looking for. Before proving $X_{M_1} = X_1^\infty$ and $X_{M_2} = X_2^\infty$, I'll first show that X_1^∞ and X_2^∞ are well-defined.

For all $n \geq 1$, we have

$$L \leq (1 - \delta) \sum_{k=1}^{\infty} \delta^{k-1} (-c + M_1 M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \underline{u}) \equiv L_1^*, \forall (R, L) \in X_1^n,$$

$$L \leq (1 - \delta) \sum_{k=1}^{\infty} \delta^{k-1} (-c + M_2 M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \underline{u}) \equiv L_2^*, \forall (R, L) \in X_2^n.$$

Each X_i^n is of the form

$$X_i^n = \{(R, L) | R \geq f_i^n(L), 0 \leq L \leq L_i^*\}$$

for some function

$$f_i^n : [0, L_i^*] \rightarrow [0, R_i^*].$$

For each i and n , $f_i^n(\cdot)$ is strictly decreasing in L , and from $X_i^n \subset X_i^{n+1}, \forall n \geq 0, i = 1, 2$, we know that

$$f_i^{n+1}(L) \leq f_i^n(L), \forall 0 \leq L \leq L_i^*, n \geq 1, i = 1, 2.$$

Together with $R \geq 0$, the monotone convergence theorem gives that the limits

$$\bar{X}_i^\infty \equiv \{(R, L) | R = \lim_{n \rightarrow \infty} f_i^n(L), 0 \leq L \leq L_i^*\}, i = 1, 2,$$

are well-defined and downward-sloping. Therefore,

$$X_i^\infty = \{(R, L) | R \geq \lim_{n \rightarrow \infty} f_i^n(L), 0 \leq L \leq L_i^*\}, i = 1, 2$$

are well-defined.

That any $(R, L) \in X_1^\infty$ and X_2^∞ are feasible given priors M_1 and M_2 can be shown as the following. Let Y_i be the set generated by X_1^∞ and X_2^∞ for the initial prior M_i . Given $\epsilon > 0$ and $(R, L) \in Y_i$, there exists n such that $(R, L + \epsilon) \in X_i^n$. Therefore, Y_i lies in the limit of X_i^n , and we have

$$Y_i \subset X_i^\infty, i = 1, 2.$$

On the other hand, we know that each Y_i is closed, and

$$Y_i \supset X_i^\infty \setminus \bar{X}_i^\infty.$$

Together, $Y_i = X_i^\infty$ for $i = 1, 2$ and X_1^∞ and X_2^∞ are jointly self-generating.

Conversely, if (R, L) is feasible given M_1 or M_2 , we can show that it's in X_1^∞ or X_2^∞ , respectively. Note that if the principal takes his outside option for T blocks under the given contract, $(R, L) \in X_i^T \subset X_i^\infty$. For contracts under which the principal takes his outside option for an infinite number of times, we can construct a truncated contract as the following. Given T , for each history h^T , pay the sum of the present compensation and the continuation value after history h^T . From period $T + 1$ and on, take the outside option forever. This replacement contract weakly relaxes the IC constraints of the agent, and it provides exactly the same amount of rent to the agent. The loss in outcome under the contract differs from the original contract by at most $\delta^T L_i^*$, depending on whether $h_T = 1$ or 0 . For given $\epsilon > 0$, the principal can choose T sufficiently large so that the replacement contract lies within ϵ from the original contract in the space of (R, L) . This implies that any feasible (R, L) lies in the limits X_1^∞ and X_2^∞ .

Together, we get $X_{M_1} = X_1^\infty$ and $X_{M_2} = X_2^\infty$.

Once we have $X_{M_1} = X_1^\infty$ and $X_{M_2} = X_2^\infty$, X_π for any prior π can be constructed from X_{M_1} and X_{M_2} . The second best contract given the prior π is the contract that minimizes $R+L$ in X_π , and it can be constructed as the following. Given $(R, L) \in X_\pi$, there exist $T \geq 0$, $(R_1, L_1) \in X_{M_1}$, $(R_2, L_2) \in X_{M_2}$ supporting (R, L) . The principal takes the outside option for T periods, and after the first period the agent is induced to work, the continuation contract is determined by (R_1, L_1) and (R_2, L_2) . The contract continues in a probationary period if the outcome is good and $0 < R_1 < R_1^*$ or if the outcome is bad and $0 < R_2 < R_2^*$. The continuation contract is (R_1, L_1) if the outcome is good, and it's (R_2, L_2) if the outcome is bad. If the outcome is bad and $R_2 = 0$, the contract terminates, and the principal takes his outside option forever. If the outcome is good and $R_1 \geq R_1^*$, the agent is tenured, and $\delta(R_1 - R_1^*)$ is provided as the initial payment. From the following period, the contract continues with $(R_1^*, 0)$, and the payments are given by the contract in Proposition 1.12. If the outcome is

bad but $R_2 = R_2^*$ given h^t0 , again, the agent is tenured, and the principal provides

$$w(h^t0 \sqcup h^k1) = \frac{c}{M_{21}}, w(h^t0 \sqcup h^k0) = 0, \forall h^k, k \geq 0,$$

from the following period. □

Proof of Proposition 2.7. I'll show the proposition by constructing a review contract that allows the principal to approximate his first best payoff. Consider the following review contract: the contract specifies a review block of T periods, a quota, Q , and a lump sum transfer, X . A quota is on the number of successful outcomes from the block. If the agent meets the quota, the principal pays the agent the discounted sum of the outcome subtracted by the lump sum transfer at the end of the review block, and the contract continues. If the agent fails to meet the quota, the principal pays the agent the discounted sum of the outcome, and the contract terminates.

First, consider the principal's payoff. Let s be the agent's strategy in the equilibrium and $p(s)$ be the minimum probability he meets the quota. In general, the probability the agent meets the quota with a strategy depends on the prior on the state at the beginning of the review block, but we can take the minimum of the probabilities over the priors. Then the principal's average per period payoff is at least

$$\begin{aligned} & (1 - \delta)\delta^{T-1}p(s)X(1 + \delta^T p(s) + \delta^{2T} p(s)^2 + \dots) \\ &= \frac{\delta^{T-1}p(s)(1 - \delta)X}{1 - \delta^T p(s)} \\ &= \frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^T p(s)} \frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta}. \end{aligned}$$

Since the states exhibit positive persistence, the expected discounted sum of the outcome in the first best is the maximum when the pair starts with $\pi^1 = (1, 0)$. Let \bar{y} be the average expected discounted sum of the outcome over the infinite horizon in the first best when $\pi^1 = (1, 0)$. When the following two inequalities hold,

$$\frac{\delta^T(1-\delta^T)p(s)}{1-\delta^T p(s)} \geq 1 - \frac{\epsilon}{2},$$

$$\frac{1-\delta}{1-\delta^T} \frac{X}{\delta} \geq (1 - \frac{\epsilon}{2})\bar{y} - c,$$

the principal's payoff is at least $(1-\epsilon)\bar{y} - c \geq \bar{y} - c - \epsilon$. Note that the first inequality implies that both δ^T and $p(s)$ are greater than $1 - \epsilon/2$.

The second step is to verify the agent's incentives that the agent will pass the quota with $p(s)$ close to one. Let $V(\pi)$ be the agent's continuation value when the review block starts with the prior π . Since the agent can always choose to work in every period, letting s be the strategy of working in every period, we have

$$V(\pi) \geq (1-\delta)(Y(\pi) - \frac{1-\delta^T}{1-\delta}c) + \delta^T p(s)(\mathbb{E}[V(\hat{\pi})] - \frac{1-\delta}{\delta}X),$$

where $Y(\pi)$ is the expected discounted sum of the outcome from working in every period from a block with the initial prior π and $\hat{\pi}$ is the prior in the beginning of the next block.

Let \underline{V} be the minimum of $V(\pi)$ over all priors π . Together with the fact that $Y(\pi)$ increases with π_1 , we get the following inequality:

$$\underline{V} \geq \frac{(1-\delta)(Y((0,1)) - \frac{1-\delta^T}{1-\delta}c) - \delta^T p(s) \frac{1-\delta}{\delta}X}{1-\delta^T p(s)}.$$

If $\underline{V} \geq \frac{1-\delta}{\delta}X$, the agent always prefers to increase the probability of meeting the quota. Since the principal pays the agent the discounted sum of the outcome, subtracted by X on meeting the quota, the agent works in every period on the equilibrium path.

When the lump sum transfer is specified to

$$X \leq \delta(Y((0,1)) - \frac{1-\delta^T}{1-\delta}c),$$

the inequality $\underline{V} \geq \frac{1-\delta}{\delta}X$ is always satisfied. The last condition is to ensure that the discounted sum of the outcome on meeting the quota is weakly greater than X so

that the principal can actually take away the lump sum transfer. A slightly stronger condition is

$$Q \geq X.$$

Therefore, when a review contract satisfies

$$\frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^T p(s)} \geq 1 - \frac{\epsilon}{2}, \quad (\text{A.1})$$

$$\frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta} \geq (1 - \frac{\epsilon}{2})\bar{y} - c, \quad (\text{A.2})$$

$$X \leq \delta(Y((0, 1)) - \frac{1 - \delta^T}{1 - \delta}c),$$

$$Q \geq X,$$

the agent chooses to work in every period, and the principal's payoff is within ϵ of his first best payoff.

Since the Markov chain is irreducible and there are two states, there exist δ_0 and T_0 such that for all $\delta \geq \delta_0, T \geq T_0$ and the initial prior π , we have

$$|\frac{1 - \delta}{1 - \delta^T}Y(\pi) - \bar{p}| < \frac{\epsilon}{4}\bar{p},$$

where

$$\bar{p} = \left(\frac{M_{21}}{M_{12} + M_{21}}, \frac{M_{12}}{M_{12} + M_{21}} \right) \begin{pmatrix} p_H \\ p_L \end{pmatrix}$$

is the probability of the good outcome from the ergodic distribution of the Markov chain.

Let

$$X = \frac{\delta(1 - \delta^T)}{1 - \delta} \left((1 - \frac{\epsilon}{4})\bar{p} - c \right).$$

For $\delta \geq \delta_0, T \geq T_0$, Inequality (A.2) is satisfied as

$$\begin{aligned}
\frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta} &= \left(1 - \frac{\epsilon}{4}\right) \bar{p} - c \\
&> \left(1 - \frac{\epsilon}{2}\right) \left(1 + \frac{\epsilon}{4}\right) \bar{p} - c \\
&\geq \left(1 - \frac{\epsilon}{2}\right) \bar{y} - c.
\end{aligned}$$

Lastly, Inequality (A.1) can be rearranged as a quadratic equation of δ^T . We get

$$1 - \frac{\epsilon}{4} - \sqrt{\left(1 - \frac{\epsilon}{4}\right)^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \leq \delta^T \leq 1 - \frac{\epsilon}{4} + \sqrt{\left(1 - \frac{\epsilon}{4}\right)^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}}.$$

There exists $p < 1$ such that for $p(s) \geq p$, we have

$$\sqrt{\left(1 - \frac{\epsilon}{4}\right)^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \geq \frac{\epsilon}{8}.$$

Let

$$Q = \lceil X \rceil,$$

where $\lceil X \rceil$ is the smallest integer greater than or equal to X . I'll now show that we can find T_1 such that for $T \geq T_1$, the agent meets the quota with a probability higher than p by working in every period.

From $Q = \lceil X \rceil$, the quota is satisfied whenever

$$\frac{\hat{Q}}{T} \geq \frac{X + 1}{T},$$

where \hat{Q} is the number of good outcomes from the block. By the strong law of large numbers, \hat{Q}/T converges to \bar{p} for all initial priors π . Since the right hand side of the inequality is bounded from above by

$$\left(\left(1 - \frac{\epsilon}{4}\right) \bar{p} - c\right) + \frac{1}{T},$$

we can pick $T_1 \geq \frac{1}{c}$, and the right hand side is strictly bounded away from \bar{p} for all

$\delta, T \geq T_1$. Therefore, there exists T_1 such that for $T \geq T_1$, we have

$$\Pr\left(\frac{\hat{Q}}{T} \geq \frac{X+1}{T}\right) \geq p.$$

We can find T_1 that holds uniformly for all initial priors π , since \hat{Q}/T for the given prior π can be written as

$$\frac{\hat{Q}}{T} = 1_{\{Z=1\}}X_1 + 1_{\{Z=2\}}X_2.$$

X_1 is \hat{Q}/T for the prior $(1, 0)$, and X_2 is \hat{Q}/T for the prior $(0, 1)$. Z is a random variable with $\Pr(Z = 1) = \pi_1$ and $\Pr(Z = 2) = \pi_2$.

Let $\bar{T} = \max\{T_0, T_1\}$ and define $\bar{\delta}$ to be

$$\bar{\delta} = \max\left\{\delta_0, \frac{1 - \frac{3}{8}\epsilon}{1 - \frac{\epsilon}{8}}, \sqrt[3]{1 - \frac{3}{8}\epsilon}\right\}.$$

Then for any $\delta \geq \bar{\delta}$, there exist T, Q , and X for the review contract that allows the principal to approximate his first best payoff. \square

Proof of Proposition 1.16. Consider the following review contract. Each review block lasts T periods, and there exist a quota, Q , and a lump sum transfer, X . The quota is on the discounted sum of the outcome, and if the agent meets the quota, the principal pays him the discounted sum of the outcome from the review block, subtracted by the lump sum transfer, and the contract continues. If the agent fails to meet the quota, the principal pays the discounted sum of the outcome from the review block to the agent, and the contract terminates.

Given $\epsilon > 0$, the expected discounted sum of the outcome converges to μ , and there exists δ_0, T_0 such that for any $k, (x_0, x_1, \dots, x_k), \delta \geq \delta_0, T \geq T_0$, we have

$$\left| \frac{1 - \delta}{1 - \delta^T} \left(\sum_{t=1}^T \delta^{t-1} \mathbb{E}[X_{k+t}] \right) - \mu \right| < \frac{\epsilon}{4} \mu.$$

Let

$$Q \equiv X \equiv \frac{\delta(1 - \delta^T)}{1 - \delta} \left(\left(1 - \frac{\epsilon}{4}\right) \mu - c \right).$$

Denote by $V((x_0, x_1, \dots, x_k))$ the agent's continuation value given history (x_0, x_1, \dots, x_k) .

We have the following expression for $\underline{V} = \min_{\mathcal{H}} V((x_0, x_1, \dots, x_k))$, where \mathcal{H} is the set of all histories:

$$\underline{V} \geq \frac{(1 - \delta) (\min_{\mathcal{H}} (\sum_{t=1}^T \delta^{t-1} \mathbb{E}[X_{k+t}]) - \frac{1 - \delta^T}{1 - \delta} c) - \delta^T p(s) \frac{1 - \delta}{\delta} X}{1 - \delta^T p(s)}.$$

For $\delta \geq \delta_0, T \geq T_0$, we have

$$\underline{V} \geq \frac{1 - \delta}{\delta} X,$$

and the agent always prefers to increase the probability of meeting the quota. Since the agent is paid the discounted sum of the outcome, the agent is induced to work in every period under the contract.

It remains to show that the principal's payoff under the contract is close to his first best payoff. Let $p(s)$ be the infimum of the probability of meeting the quota by working in every period, where the infimum is taken over the set of histories \mathcal{H} . The principal's payoff is at least

$$\begin{aligned} & (1 - \delta) \delta^{T-1} p(s) X (1 + \delta^T p(s) + \delta^{2T} p(s)^2 + \dots) \\ &= \frac{\delta^{T-1} p(s) (1 - \delta) X}{1 - \delta^T p(s)} \\ &= \frac{\delta^T (1 - \delta^T) p(s)}{1 - \delta^T p(s)} \frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta}. \end{aligned}$$

Similarly as in the proof of Proposition 2.7, the principal's payoff is within ϵ of his first best payoff if the following two inequalities hold:

$$\begin{aligned} \frac{\delta^T (1 - \delta^T) p(s)}{1 - \delta^T p(s)} &\geq 1 - \frac{\epsilon}{2}, \\ \frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta} &\geq \left(1 - \frac{\epsilon}{2}\right) \bar{y} - c, \end{aligned}$$

where \bar{y} is the supremum of the expected discounted sum of outcome over the infinite

horizon in the first best, and the supremum is taken over all initial conditions x_0 . The second inequality is satisfied for any $\delta \geq \bar{\delta}_0, T \geq T_0$, and we need to show that the first inequality also holds. By the uniform weak law of large numbers, there exist δ_1, T_1 such that for $\delta \geq \delta_1, T \geq T_1$, we have

$$\Pr\left(\left|\frac{1-\delta}{1-\delta^T}\left(\sum_{t=1}^T \delta^{t-1} X_{k+t}\right) - \mu\right| > \epsilon'\right) < \epsilon'.$$

Choose ϵ' such that

$$\epsilon' < \frac{\epsilon}{4}\mu,$$

and that for $p(s) \geq 1 - \epsilon'$,

$$\sqrt{\left(1 - \frac{\epsilon}{4}\right)^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \geq \frac{\epsilon}{8}.$$

Let $\bar{T} = \max\{T_0, T_1\}$ and define $\bar{\delta}$ to be

$$\bar{\delta} = \max\left\{\delta_0, \frac{1 - \frac{3}{8}\epsilon}{1 - \frac{1}{8}\epsilon}, \sqrt[{\bar{T}}]{1 - \frac{3}{8}\epsilon}\right\}.$$

Then for any $\delta \geq \bar{\delta}$, there exist T, Q , and X for the review contract that allows the principal to approximate his first best payoff. \square

Proof of Proposition 1.17. Suppose there are n states and M is the Markov transition matrix. When there are a finite number of states following an irreducible Markov chain, the prior on the state is a sufficient static for the distribution of future states, and it is sufficient to show the following: there exists $\mu > 0$ such that (i) for given $\epsilon > 0$, there exist δ_0, T_0 such that for any prior π , $\delta \geq \delta_0, T \geq T_0$,

$$\left|\frac{1-\delta}{1-\delta^T}\left(\sum_{t=1}^T \delta^{t-1} \mathbb{E}[X_t(\pi)]\right) - \mu\right| < \epsilon, \quad (\text{A.3})$$

and (ii) for given $\epsilon, \epsilon' > 0$, there exists T_1 such that for any prior $\pi, \delta, T \geq T_1$ with $\delta^T \geq 1 - \frac{\epsilon}{2}$,

$$\Pr\left(\frac{1-\delta}{1-\delta^T}\left(\sum_{t=1}^T \delta^{t-1} X_t(\pi)\right) < \left(1 - \frac{\epsilon}{4}\right)\mu - c\right) < \epsilon', \quad (\text{A.4})$$

where $X_t(\pi)$ is the stochastic process for the outcome of working in period t given the initial prior π . Without loss of generality, I assume $X_t(\pi)$ is non-negative for all t, π .

I'll first show why Inequalities (A.3) and (A.4) are sufficient conditions for the principal to be able to approximate his first best payoff. Consider the following review contract: each review block lasts T periods, and there exist a quota, Q , and a lump sum transfer, X . The quota is on the discounted sum of the outcome, and if the agent meets the quota, the principal pays him the discounted sum of the outcome from the review block, subtracted by the lump sum transfer, and the contract continues. If the agent fails to meet the quota, the principal pays the discounted sum of the outcome from the review block to the agent, and the contract terminates.

Suppose Inequalities (A.3) and (A.4) hold. Given $\epsilon > 0$, the expected discounted sum of the outcome converges to μ , and there exists δ_0, T_0 such that for any prior π , $\delta \geq \delta_0$, and $T \geq T_0$, we have

$$\left| \frac{1-\delta}{1-\delta^T} \left(\sum_{t=1}^T \delta^{t-1} \mathbb{E}[X_t(\pi)] \right) - \mu \right| < \frac{\epsilon}{4} \mu.$$

Let

$$Q \equiv X \equiv \frac{\delta(1-\delta^T)}{1-\delta} \left(\left(1 - \frac{\epsilon}{4}\right)\mu - c \right).$$

Denote by $V(\pi)$ the agent's continuation value given the initial prior π . We have the following expression for $\underline{V} = \min_{\pi} V(\pi)$:

$$\underline{V} \geq \frac{(1-\delta) \left(\min_{\pi} \left(\sum_{t=1}^T \delta^{t-1} \mathbb{E}[X_t(\pi)] \right) - \frac{1-\delta^T}{1-\delta} c \right) - \delta^T p(s) \frac{1-\delta}{\delta} X}{1 - \delta^T p(s)}.$$

For $\delta \geq \delta_0, T \geq T_0$, we have

$$\underline{V} \geq \frac{1-\delta}{\delta} X,$$

and the agent always prefers to increase the probability of meeting the quota. Since

the agent is paid the discounted sum of the outcome, the agent is induced to work in every period under the contract.

We can also show that the principal's payoff under the contract is close to his first best payoff. Let $p(s)$ be the minimum of the probability of meeting the quota by working in every period, where the minimum is taken over the initial priors. The principal's payoff is at least

$$\begin{aligned} & (1 - \delta)\delta^{T-1}p(s)X(1 + \delta^T p(s) + \delta^{2T} p(s)^2 + \dots) \\ &= \frac{\delta^{T-1}p(s)(1 - \delta)X}{1 - \delta^T p(s)} \\ &= \frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^T p(s)} \frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta}. \end{aligned}$$

Similarly as in the proof of Proposition 2.7, the principal's payoff is within ϵ of his first best payoff if the following two inequalities hold:

$$\begin{aligned} \frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^T p(s)} &\geq 1 - \frac{\epsilon}{2}, \\ \frac{1 - \delta}{1 - \delta^T} \frac{X}{\delta} &\geq (1 - \frac{\epsilon}{2})\bar{y} - c, \end{aligned}$$

where \bar{y} is the maximum expected discounted sum of outcome over the infinite horizon in the first best. The second inequality is satisfied for any $\delta \geq \delta_0, T \geq T_0$, and we need to show that the first inequality also holds.

Let p be

$$p = \frac{1 - \frac{\epsilon}{2}}{1 - \frac{\epsilon}{2} + \frac{3}{64}\epsilon^2}.$$

For any $p(s) \geq p$, we have

$$\sqrt{(1 - \frac{\epsilon}{4})^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \geq \frac{\epsilon}{8}.$$

By Inequality (A.4), there exist T_1 such that for $T \geq T_1$ with $\delta^T \geq 1 - \frac{\epsilon}{2}$, we have

$$\Pr\left(\frac{1-\delta}{1-\delta^T}\left(\sum_{t=1}^T \delta^{t-1} X_t(\pi)\right) < \left(1-\frac{\epsilon}{4}\right)\mu - c < 1-p.\right)$$

Then, the agent meets the quota with $p(s) \geq p$ by working in every period.

We can rearrange

$$\frac{\delta^T(1-\delta^T)p(s)}{1-\delta^T p(s)} \geq 1 - \frac{\epsilon}{2}$$

as

$$1 - \frac{\epsilon}{4} - \sqrt{\left(1 - \frac{\epsilon}{4}\right)^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \leq \delta^T \leq 1 - \frac{\epsilon}{4} + \sqrt{\left(1 - \frac{\epsilon}{4}\right)^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}}.$$

Let $\bar{T} = \max\{T_0, T_1\}$ and define $\bar{\delta}$ to be

$$\bar{\delta} = \max\left\{\delta_0, \frac{1 - \frac{3}{8}\epsilon}{1 - \frac{1}{8}\epsilon}, \sqrt[3]{1 - \frac{3}{8}\epsilon}\right\}.$$

Then for any $\delta \geq \bar{\delta}$, there exist T, Q , and X for the review contract that allows the principal to approximate his first best payoff.

I'll next show that Inequalities (A.3) and (A.4) are satisfied. Define X^i to be the stochastic process for the outcome of working in the state i , and we have

$$\mathbb{E}[X_t(\pi)] = \pi M^{t-1} \cdot (\mathbb{E}[X^1], \dots, \mathbb{E}[X^n])$$

for all $t \geq 1$.

Let π_0 be the invariant distribution of the Markov chain, and define $\mu = \mathbb{E}[X_1(\pi_0)]$. Since the Markov chain is irreducible and aperiodic, π_0 and μ are well-defined. We also know that for any prior π , $\mathbb{E}[X_t(\pi)]$ converges to μ as t goes to infinity.

Given any prior π and T , we can rewrite $\sum_{t=1}^T \delta^{t-1} X_t(\pi)$ as

$$\sum_{t=1}^T \delta^{t-1} X_t(\pi) = \sum_{i=1}^n \mathbf{1}_{\{Z=i\}} \left(\sum_{t=1}^T \delta^{t-1} X_t(e_i) \right),$$

where e_i is the indicator vector for the i -th coordinate and Z is a random variable with $\Pr(Z = i) = \pi_i$. Since there are a finite number of states, it is sufficient to show that Inequality (A.3) is satisfied for each e_i , $1 \leq i \leq n$. By symmetry, we only need

to prove the statement for $\pi = e_1$.

For given $\epsilon > 0$, we have

$$\mathbb{E}[X_t(e_1)] \rightarrow \mu \text{ as } t \rightarrow \infty,$$

and there exists N such that

$$|\mathbb{E}[X_t(e_1)] - \mu| < \frac{\epsilon}{2}, \forall t \geq N.$$

From the fact that

$$\frac{1 - \delta}{1 - \delta^T}$$

is decreasing in both δ and T , there exist $\hat{\delta}_1, \hat{T}_1$ such that for $\delta \geq \hat{\delta}_1, T \geq \hat{T}_1$,

$$\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^N \delta^{t-1} |\mathbb{E}[X_t(e_1)] - \mu| < \frac{\epsilon}{2}.$$

Therefore, for given prior e_1 and $\epsilon > 0$, we can always find $\hat{\delta}_1, \hat{T}_1$ such that for $\delta \geq \hat{\delta}_1, T \geq \hat{T}_1$,

$$\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^T \delta^{t-1} |\mathbb{E}[X_t(e_1)] - \mu| < \epsilon.$$

Similarly, we can find $\hat{\delta}_i, \hat{T}_i$ for $2 \leq i \leq n$ such that for $\delta \geq \hat{\delta}_i, T \geq \hat{T}_i$, we have

$$\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^T \delta^{t-1} |\mathbb{E}[X_t(e_i)] - \mu| < \epsilon.$$

Let $\hat{\delta} = \max_i \hat{\delta}_i, \hat{T} = \max_i \hat{T}_i$, and we get

$$\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^T \delta^{t-1} |\mathbb{E}[X_t(\pi)] - \mu| < \epsilon. \tag{A.5}$$

for all $\pi, \delta \geq \hat{\delta}, T \geq \hat{T}$.

On the other hand, from

$$\sum_{t=1}^T \delta^{t-1} X_t(\pi) = \sum_{i=1}^n 1_{\{Z=i\}} \left(\sum_{t=1}^T \delta^{t-1} X_t(e_i) \right),$$

it is sufficient to show Inequality (A.4) for ϵ, ϵ' and each $e_i, 1 \leq i \leq n$. By symmetry, we can show the inequality for e_1 . Without loss of generality, assume $\epsilon < \frac{c}{2\mu}$. Since the Markov chain has an ergodic distribution and the outcome of working is bounded, the strong law of large numbers holds, and there exists \tilde{T}_1 such that for $T \geq \tilde{T}_1$, we have

$$\Pr\left(\frac{1}{T} \sum_{t=1}^T X_t(e_1) \geq \mu(1 - \epsilon)\right) \geq 1 - \epsilon'. \quad (\text{A.6})$$

For δ and T such that $\delta^T \geq 1 - \frac{\epsilon}{2}$, we have

$$\frac{(1 - \delta)\delta^{T-2}}{1 - \delta^T} \geq \left(1 - \frac{\epsilon}{2}\right) \frac{1}{T}.$$

Therefore, for $T \geq \tilde{T}_1$ and δ such that $\delta^T \geq 1 - \frac{\epsilon}{2}$, we know that

$$\begin{aligned} \Pr\left(\sum_{t=1}^T \delta^{t-1} X_t(e_1) \geq Q\right) &\geq \Pr\left(\frac{(1 - \delta)\delta^{T-2}}{1 - \delta^T} \sum_{t=1}^T X_t(e_1) \geq \left(1 - \frac{\epsilon}{4}\right)\mu - c\right) \\ &\geq \Pr\left(\left(1 - \frac{\epsilon}{2}\right) \frac{1}{T} \sum_{t=1}^T X_t(e_1) \geq \left(1 - \frac{\epsilon}{4}\right)\mu - c\right) \\ &\geq 1 - \epsilon', \end{aligned}$$

where the last inequality follows from Inequality (A.6). Rearranging the inequality, we get

$$\Pr\left(\frac{1 - \delta}{1 - \delta^T} \left(\sum_{t=1}^T \delta^{t-1} X_t(e_1)\right) < \left(1 - \frac{\epsilon}{4}\right)\mu - c\right) < \epsilon'.$$

Similarly, we can find \tilde{T}_i for $2 \leq i \leq n$ such that for $T \geq \tilde{T}_i, \delta^T \geq 1 - \frac{\epsilon}{2}$, we have

$$\Pr\left(\frac{1 - \delta}{1 - \delta^T} \left(\sum_{t=1}^T \delta^{t-1} X_t(e_i)\right) < \left(1 - \frac{\epsilon}{4}\right)\mu - c\right) < \epsilon'.$$

Take $\tilde{T} = \max_i \tilde{T}_i$, and we have Inequality (A.4).

Therefore, when there are a finite number of states following an irreducible, aperi-

odic first-order Markov chain, and the outcome of working is bounded, the principal can approximate his first best payoff. Given $\epsilon > 0$, there exists $\bar{\delta}$ such that for $\delta \geq \bar{\delta}$, the principal's average per period payoff in the second best is within ϵ of his first best payoff. \square

A.2 Proofs for Chapter 2

Proof of Proposition 2.5. Consider the relational contract that provides s . The principal offers in the initial period $w(\theta_0), b(\phi_0)$, and if the agent accepts, he exerts effort $e(\theta_0)$. The continuation payoffs under the contract are denoted by $u(\phi_0)$ and $\pi(\phi_0)$, and the expected payoffs from the contract are u_0 and π_0 . Without loss of generality, we can assume that off the equilibrium path, the parties revert to the static equilibrium of $(\bar{u}, \bar{\pi})$. The first period payment W is a function of ϕ_0 .

The contract is self-enforcing if and only if the following conditions hold:

$$\begin{aligned}
 (i) \quad & u_0 \geq \bar{u}, \pi_0 \geq \bar{\pi}, \\
 (ii) \quad & e(\theta_0) \in \arg \max_e \mathbb{E}_{\theta_1} [(1 - \delta)W(\phi_0) + \delta u(\phi_0) | e, \theta_0] - c(e, \theta_0), \\
 (iii) \quad & b(\phi_0) + \frac{\delta}{1 - \delta} u(\phi_0) \geq \frac{\delta}{1 - \delta} \bar{u}, \\
 & -b(\phi_0) + \frac{\delta}{1 - \delta} \pi(\phi_0) \geq \frac{\delta}{1 - \delta} \bar{\pi},
 \end{aligned}$$

and (iv) each continuation contract is self-enforcing.

Given any (u, π) such that $u \geq \bar{u}, \pi \geq \bar{\pi}, u + \pi = s$, the principal can offer the same $b(\phi_0)$ and continuation contracts and adjust $w(\theta_0)$ to

$$\hat{w}(\theta_0) \equiv w(\theta_0) + \frac{\pi - \pi_0}{1 - \delta}.$$

The conditions are satisfied with the new contract, and it provides (u, π) as the expected payoffs. \square

Proof of Proposition 2.6. Suppose a contract that maximizes the joint surplus provides w_t, b_t and the agent chooses e_t . The first step is to construct an alternative

contract \hat{w}_t, \hat{b}_t under which the agent chooses the same level of effort e_t and his expected payoff is constant in every state.

When the states are observable, the distribution of the states from period $t + 1$ only depends on θ_{t+1} , which is observed before the principal makes payments in period t . The principal can adjust the contingent payment b_t and keep the expected payoff in each state constant. Specifically, consider the following contract. Let $u_t(h^t, \phi_t)$ be the continuation value of the agent under the given contract, and define \hat{w}_t, \hat{b}_t as the following:

$$\begin{aligned}\hat{b}_t(h^t, \phi_t) &\equiv b_t(h^t, \phi_t) + \frac{\delta}{1-\delta}(u_t(h^t, \phi_t) - \bar{u}), \\ \hat{w}_t(h^t, \theta_t) &\equiv \bar{u} - \mathbb{E}_{y_t}[\hat{b}_t(h^t, \phi_t)|e_t(h^t, \theta_t)].\end{aligned}$$

From

$$\hat{b}_t(h^t, \phi_t) + \frac{\delta}{1-\delta}\bar{u} = b_t(h^t, \phi_t) + \frac{\delta}{1-\delta}u_t(h^t, \phi_t),$$

the agent chooses the same level of effort e_t under the new contract. The agent's expected payoff is \bar{u} for all t, h^t, θ_t .

The next step is to show that we can choose $\tilde{w} : \Theta \rightarrow \mathbb{R}, \tilde{b} : \Phi \rightarrow \mathbb{R}$ such that the principal offers \tilde{w}, \tilde{b} in every period. Consider \hat{w}_t and \hat{b}_t . The agent's expected payoff is constant over all t, h^t , and θ_t , which implies that the agent's IC constraint is determined by the within period compensation scheme. Specifically, the agent chooses e such that

$$e_t(h^t, \theta_t) \in \arg \max_e \mathbb{E}_{\theta_{t+1}}[\hat{b}_t(h^t, \phi_t)|e, \theta_t] - c(e, \theta_t).$$

When the agent's IC constraints are myopic, the principal can replace a compensation scheme for any given period with another compensation scheme without affecting the incentives. Under an optimal contract, $s_t(\theta_t)$ is constant for given state θ_t . If there's multiplicity of the compensation schemes, we can pick one without loss of generality.

Given $\tilde{b} : \Phi \rightarrow \mathbb{R}$, the agent chooses $e : \Theta \rightarrow \mathcal{E}$ such that

$$e(\theta_t) \in \arg \max_e \mathbb{E}_{\theta_{t+1}} [\tilde{b}(\phi)|e, \theta_t] - c(e, \theta_t).$$

Define \tilde{w} as

$$\tilde{w}(\theta_t) \equiv \bar{u} - \mathbb{E}_{\theta_{t+1}} [\tilde{b}(\phi)|e(\theta_t), \theta_t],$$

and we have a stationary contract that maximizes the expected joint surplus. By construction, it is self-enforcing, and it provides the same expected payoff to the agent in all t, h^t, θ_t . \square

Proof of Proposition 2.7. (\Rightarrow) Suppose $e(\theta)$ is implementable with a stationary contract that provides $\hat{u} \geq \bar{u}$ to the agent in every state. Let $\pi(\theta')$ be the continuation value for the principal when the realized productivity is θ' . The IC constraint has to be satisfied, and we know that

$$\frac{\delta}{1-\delta}(\pi(\theta') - \bar{\pi}) \geq b(\theta, y, \theta'), \forall \theta, \theta', \quad (\text{A.7})$$

$$\frac{\delta}{1-\delta}(\hat{u} - \bar{u}) \geq -\inf_{\theta'} b(\theta, y, \theta'), \forall \theta \quad (\text{A.8})$$

have to hold. Adding the two inequalities, we have the dynamic enforcement constraint.

(\Leftarrow) Suppose $W(\phi)$ and $e(\theta)$ satisfy the IC constraint and the dynamic enforcement constraint. Define

$$b(\phi) = W(\phi) - \inf_{\phi} W(\phi),$$

$$w(\theta) = \bar{u} - \mathbb{E}_{\theta'} [W(\phi)|e(\theta), \theta],$$

and consider the stationary contract with $w(\theta)$, $b(\phi)$ and $e(\theta)$. The parties revert to the static equilibrium if a deviation occurs. The agent receives \bar{u} as expected payoff in each state, and the principal receives $\pi(\theta) = s(\theta) - \bar{u}$ if the productivity is θ . By the dynamic enforcement constraint, $s(\theta) \geq \bar{s}$ and $\pi(\theta) \geq \bar{\pi}$ for all θ . From the IC constraint, the agent chooses $e(\theta)$ in each state θ , and it can be verified that Inequality (A.7) and (A.8) are satisfied. \square