Geometric Langlands in Prime Characteristic

by

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Bachelor of Science, National Taiwan University (2004)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2012

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Abstract

Let C be a smooth projective curve over an algebraically closed field k of sufficiently large characteristic. Let G be a semisimple algebraic group over k and let $G^\vee$ be its Langlands dual group over k. Denote by Bun_G the moduli stack of G-bundles on C and $\mathrm{LocSys}_{G^\vee}$ the moduli stack of $G^\vee$-local systems on C. Let D_{Bun_G} be the sheaf of crystalline differential operator algebra on Bun_G. In this thesis I construct an equivalence between the derived category $D(\mathrm{QCoh}(\mathrm{LocSys}^0_{G^\vee}))$ of quasi-coherent sheaves on some open subset $\mathrm{LocSys}^0_{G^\vee} \subset \mathrm{LocSys}_{G^\vee}$ and derived category $D(D^0_{\mathrm{Bun}_G} -$ mod) of modules over some localization $D^0_{\mathrm{Bun}_G}$ of D_{Bun_G}. This generalizes the work of Bezrukavnikov-Braverman in the GL_n case.

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Acknowledgments

I would like to thank my advisor, Roman Bezrukavnikov, for his guidance, support and encouragement for the past five years. I have benefited a lot from his lectures, his papers and conversations with him. It is a great fortune for me to be his student.

I want to thank my friend and collaborator Xinwen Zhu for many helpful conversations and support throughout the years. Part of the thesis is based on a joint work with him. Our collaboration has been one of the most enjoyable experiences in my student life at MIT.

I want to thank my friend Zhiwei Yun. I have benefited a lot from several conversations with him. His mathematical insight and technical support play an important role in various stages of this thesis.

I want to thank Professor Dennis Gaitsgory for organizing the wonderful geometric representation theory seminar and sharing his point of view on many mathematical subjects.

I would like to express my gratitude to my thesis committee: Professor Pavel Etingof, Professor Zhiwei Yun and to many Professors at MIT for their help, namely, Professor David Vogan, Professor George Lusztig, Professor Haynes Miller.

I want to thank all my friends during the past five years for spending time together and bringing happiness to my life.

Financial support for the first four year was provided by MIT Department of Mathematics and for the last year by research assistantships of Professor Bezrukavnikov and Professor Lusztig. I want to use this opportunity to thank them.

I want to thank my parents for their continuous support, care and love. Finally, I want to thank my wife, Ruei-Lan, for sharing her life with me.

Contents

Chapter 1

Introduction

1.1 Geometric Langlands conjecture in prime characteristic

Let G be a reductive algebraic algebraic group over $\mathbb{C}$ and let $G^\vee$ be its Langlands dual group. Let C be a smooth projective curve over $\mathbb{C}$. Let $\mathrm{LocSys}_{G^\vee}$ be the stack of $G^\vee$-local systems on X and Bun_G be the stack of G-bundles on X. The geometric Langlands conjecture (GLC), as proposed by Beilinson and Drinfeld, is a conjectural equivalence between certain appropriate defined category of quasi-coherent sheaves on $\mathrm{LocSys}_{G^\vee}$ and certain appropriate defined category of D-module on Bun_G. A precise formulation of GLC (over $\mathbb{C}$) can be found in the recent work of Dennis Gaitsgory and Dima Arinkin [AG].

GLC has a quasi-classical limits which amounts to the duality of Hitchin fibrations. The classical duality is established "generically" in [DP] (see also §6 in this thesis.)

In this thesis, we consider the characteristic p version of geometric Langlands conjecture. Namely, we assume that the curve C is defined over an algebraically closed field k of sufficiently large characteristic and D-module are quasi-coherent sheaves with flat connections.

In [BB], Roman Bezrukavnikov and Alexander Braverman observed that, in this setting, the GLC can be thought as a twisted version of its classical limit. Since the classical duality is verified "generically", they proved a "generic" version of GLC in the case when $G = GL_n$.

In this thesis we generalized the construction in [BB] to any semisimple algebraic group G and proved a "generic" version of GLC for G.

The ideal of the proof/construction is based on the observation in [BB] in the case of GL_n: Azumaya property of sheaf of differential operator and its commutative group structure, torsor structure of moduli stack of rank n-local system and twisted Fourier-Mukai duality.

The main difficulty going from GL_n to a semisimple algebraic group G is that the construction of classical duality for G is more complicate and indirect than the case of GL_n. Therefore, new ingredients are needed in order to generalize the construction in [BB] to the case of semisimple algebraic groups.

The main ideal of the construction will be outlined in next section.

1.2 Summary of the construction

The main ideal of the construction of GLC is based on the the following twisted Fourier-Mukai duality:

Lemma 1.2.1 (Lemma 5.2.1). *Let S be a Noetherian scheme over k and $\pi : \mathscr{A} \to S$, and $\breve{\pi} : \breve{\mathscr{A}} \to S$ be two dual family of abelian variety (or Picard stack). Suppose we have a sheave of Azumaya algebra $\mathcal{A}$ over $\mathscr{A}$ with commutative group structure (see Section 5.1 for the definition of commutative group structure). Then there is a canonical $\breve{\mathscr{A}}$-torsor $\mathscr{T}_{\mathcal{A}}$ over S such that the derived category of quasi-coherent sheaves on $\mathscr{T}_D$ is equivalent to the derived category of $\mathcal{A}$-modules.*

We are going to apply above Lemma to the following setting:

1. The family of Picard stack will be $h' : \mathscr{P}' = T^*\operatorname{Bun}_G' \to B'$, $\breve{h}' : \breve{\mathscr{P}}' = T^*\operatorname{Bun}_{G^\vee}' \to B'$ the (Frobenius twist) Hitchin integrable systems for G and its Langlands dual group $G^\vee$. In fact, we have to restrict to an open subset B^0 of B, but we ignore this for the discussion here.

2. It is shown in [BB] that there is a sheaf of Azumaya algebra D_{Bun_G} on $\mathscr{P}'$ such that the category of D_{Bun_G}-module is the same as the category of D-module on Bun_G (It is the stacky analogy of the fact that the sheaf of differential operator $\mathcal{D}_X$ on a smooth algebraic variety X localizes to a sheaf of Azumaya algebra D_X on T^*X'.) We take $\mathcal{A}$ to be D_{Bun_G}.

Therefore the construction for GLC can be divided into three steps:

Step 1

Let $\mathrm{LocSys}_{G^\vee}$ be the stack of $G^\vee$-local system. Let $h_p : \mathrm{LocSys}_{G^\vee} \to B'$ be p-Hitchin map (see §7.1).

Theorem 1.2.2 (Theorem 7.2.1). *$\mathrm{LocSys}_{G^\vee}$ has a structure of $\breve{\mathscr{P}}'$-torsor.*

In the case of GL_n, this is proved by identifying LocSys_n, the stack of rank n bundle with connections, as splittings of certain Azumaya algebra on the Frobenius twist of the universal spectural curve $\widetilde{C}' \to B'$. Therefore it is a torsor over $\mathscr{P}ic(\widetilde{C}'/B')$ which is isomorphic to $\breve{\mathscr{P}}'$ in this case. For general semisimple algebraic group G, we give a complete different proof. We prove an "Abelianisation" Theorem for $\mathrm{LocSys}_{G^\vee}$, which roughly says the following: For any $(E, \nabla) \in \mathrm{LocSys}_{G^\vee}$ mapping to b' under the p-Hitchin map h_p, there is a canonical reduction of E to a $F^*J_{b'}$-torsor $E_{F^*J_{b'}}$ (equipped with a flat connection), where $J_{b'}$ is the group scheme of regular centralizer on C' corresponding to $b' \in B'$ and $F : C \to C'$ is the relative Frobenius map. Then the $\breve{\mathscr{P}}'$-torsor structure of $\mathrm{LocSys}_{G^\vee}$ follows from the fact that the fiber $\breve{\mathscr{P}}_{b'}$ of $\breve{\mathscr{P}}$ at b' is isomorphic to $\mathrm{Bun}_{J_{b'}}$ the stack of $J_{b'}$-torsors.

Step 2

Theorem 1.2.3 (Theorem 8.2.7). *The sheaf of Azumaya algebra D_{Bun_G} on $\mathscr{P}'$ has a commutative group structure.*

In the GL_n case, it is proved by showing certain identity on the canonical one form θ on $\mathscr{P}' = T^*\mathrm{Bun}_n$. The proof of this identity used the observation that the

Abel-Jacobi map for universal spectral curve $\widetilde{C} \to B$ is the cotangent morphism of the Hecke correspondence (see the proof of Theorem 4.12 in [BB]). We don't know how to generalize above observation to general G. Therefore, we used a different argument. We first consider the sheaf of Azumaya algebra $D_{\mathscr{P}'/B}$ on the relative cotangent $T^*(\mathscr{P}'/B)$ ($D_{\mathscr{P}'/B}$ corresponds to the sheaf of relative differential operators on $h : \mathscr{P} \to B$). It is shown in §5.5, $T^*(\mathscr{P}'/B)$ has a structure of Picard stack over $B \times B'$ (we denote the structure map by $\pi_{\mathscr{P}/B} : T^*(\mathscr{P}'/B) \to B \times B'$) and $D_{\mathscr{P}'/B}$ has a canonical commutative group structure. Now the key observation is that the Hitchin fibration $h' : \mathscr{P}' \to B'$ is the restriction of $\pi_{\mathscr{P}/B} : T^*(\mathscr{P}'/B) \to B \times B'$ to the "diagonal" embedding $\Delta : B' \to B \times B$. Moreover, we have an isomorphism of Azumaya algebra $D_{\mathrm{Bun}_G} \simeq D_{\mathscr{P}'/B}|_{\mathscr{P}'}$ (see Theorem 8.2.7 for the precise statement). This gives D_{Bun_G} a commutative group structure.

Step 3

Theorem 1.2.4 (Theorem 8.8.2). *The $\breve{\mathscr{P}}'$-torsor $\mathscr{T}_{D_{\mathrm{Bun}_G}}$ coming from Lemma 1.2.1 is isomorphic to* $\mathrm{LocSys}_{G^\vee}$.

In the GL_n case, this is done by the following construction: Points in $\mathscr{T}_D$ correspond to multiplicative splittings of the restriction of D to the corresponding Hitchin fibers $\mathscr{P}_{b'} \simeq \mathscr{P}ic(\widetilde{C}_{b'})$. The restriction of each splitting of $D|_{\mathscr{P}_{b'}}$ to the spectral curve $\widetilde{C}_{b'}$ via the Abel-Jacobi map defines a local system on C. This defines a map from $\mathscr{T}_D$ to $\mathrm{LocSys}_{G^\vee}$. This map is compatible with their torsor structure because the classical duality $\mathscr{P}ic(\widetilde{C}_{b'}) \simeq \mathscr{P}ic(\widetilde{C}_{b'})^\vee$ is exactly given by pull-back of multiplicative line bundle on $\mathscr{P}ic(\widetilde{C}_{b'})$ under the Abel-Jacobi map.

For general G, we use the same strategy, i.e., try to construct a map

$$\mathfrak{D} : \mathscr{T}_D \to \mathrm{LocSys}_{G^\vee}$$

which is compatible with their torsor structure. The construction for general G is more difficult than the case of GL_n and it is essentially because the construction of classical duality $\mathfrak{D}_0 : \mathscr{P}^\vee \simeq \breve{\mathscr{P}}$ is more complicate and indirect. Let me first give an outline of the construction of classical duality for details see §6. Let $\mathrm{Bun}_{T^\vee}^{\mathrm{W}}$ be the stack of strongly W-equivariant $T^\vee$-torsor on $\widetilde{C}$. Then using Abel-Jacobi map one can define a morphism

$$(\mathrm{AJ}^{\mathscr{P}})^\vee : \mathscr{P}^\vee \to \mathrm{Bun}_{T^\vee}^{\mathrm{W}}.$$

On the other hand, by using Galois description of $\check{\mathscr{P}}$ (see [DG] or §6.1), there is a natural morphism

$$\check{j} : \check{\mathscr{P}} \to \mathrm{Bun}_{T^\vee}^{\mathrm{W}}.$$

Moreover, the pair $(\check{j}, \check{\mathscr{P}})$ can be characterized as kernel of a morphism from $\mathrm{Bun}_{T^\vee}^{\mathrm{W}}$ to another Picard stack. Then using universal property of kernel, we show that there is a lifting of $(\mathrm{AJ}^{\mathscr{P}})^\vee$ to an isomorphism $\mathfrak{D}_0 : \mathscr{P}^\vee \simeq \check{\mathscr{P}}$, i.e. we have

$$\begin{array}{ccc} \mathscr{P}^\vee & \xrightarrow[\simeq]{\mathfrak{D}_0} & \check{\mathscr{P}}. \\ & {\scriptstyle (\mathrm{AJ}^{\mathscr{P}})^\vee} \searrow \quad \swarrow {\scriptstyle \check{j}} & \\ & \mathrm{Bun}_{T^\vee}^{\mathrm{W}} & \end{array}$$

The construction of $\mathfrak{D}$ can be thought as a "twist" version of above construction. Namely, in §8.6 we construct a morphism

$$\mathrm{Hk}_{\Delta}^{\mathscr{P}} : \mathscr{T}_D \to \mathrm{LocSys}_{T^\vee}^{\mathrm{W}}|_{\Delta}$$

where $\mathrm{LocSys}_{T^\vee}^{\mathrm{W}}|_{\Delta}$ is the $(\mathrm{Bun}_{T^\vee}^{\mathrm{W}})'$-torsor of W-equivariant $T^\vee$-local system on $\widetilde{C}$ with a fixed p-curvature. On the other hand, in §8.7 we construct a morphism

$$\mathrm{Ind}_{\Delta} : \mathrm{LocSys}_{G^\vee} \to \mathrm{LocSys}_{T^\vee}^{\mathrm{W}}|_{\Delta}.$$

Morphisms $\mathrm{Hk}_{\Delta}^{\mathscr{P}}$ and Ind_{Δ} can be thought as a twisted version of $(\mathrm{AJ}^{\mathscr{P}})^\vee$ and $\check{j}$ respectively.

Using a similar argument in the classical case, we showed that there is a lifting of

$\mathrm{Hk}^{\mathscr{P}}_{\Delta}$ to a morphism $\mathfrak{D} : \mathscr{T}_D \to \mathrm{LocSys}_{G^\vee}$, i.e., we have

$$\begin{array}{ccc} \mathscr{T}_D & \xrightarrow{\quad\mathfrak{D}\quad} & \mathrm{LocSys}_{G^\vee}\,. \\ & {\scriptstyle \mathrm{Hk}^{\mathscr{P}}_{\Delta}} \searrow \quad \swarrow {\scriptstyle \mathrm{Ind}_{\Delta}} & \\ & \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}|_{\Delta} & \end{array}$$

By the construction one can check that the morphism $\mathfrak{D} : \mathscr{T}_D \to \mathrm{LocSys}_{G^\vee}$ is compatible with their torsor structure vie the (Frobenius twist) classical duality $\mathfrak{D}_0 : \mathscr{P}^\vee \simeq \breve{\mathscr{P}}$. Therefore, $\mathfrak{D}$ is an isomorphism.

By applying Lemma 1.2.1 to our setting, we obtain the main Theorem in this thesis:

Theorem 1.2.5 (Imprecise version). *We have an equivalence of bounded derived category*

$$D(D_{\mathrm{Bun}_G} - \mathrm{mod}) \simeq D(\mathrm{QCoh}(\mathrm{LocSys}_{G^\vee})).$$

Remark. *As we mentioned before, we have to restrict everything to an open subset B^0 of B. For a precise version for each step and the the main Theorem see §6, §7 ,§8, and Theorem 8.0.2.*

1.3 Structure of the article

Let us now describe the contents of this paper in more detail. In §2 we introduce notations related to algebraic stack, algebraic groups. In §3 we collect some Theorems about Hitchin fibration that are used in this thesis. Main reference is [N1]. In §4 we study D-module on aglebraic stack in positive characteristic and Azumaya property of D-module of differential operators. In §5 we study GLC for abelian variety or more generally for Beilinson 1-motive. We proved Lemma 1.2.1 in this section and we applied it to the case of abelian variety. Notice that GLC for abelian variety is proved in [Lau] for arbitrary characteristic, and here we presented another proof in the case of positive characteristic. In §6 we study the classical duality for Hitchin moduli stack. In §7 we proved Theorem 1.2.2. In section §8 we proved Theorem 1.2.3

and Theorem 1.2.4. In §9 we study opers $\mathrm{Op}_{G^\vee}$ is positive characteristic and the compatibility of the construction in this thesis and Beilinson-Drinfeld construction of GLC for opers.

This thesis has two Appendixes. In §A we develop some basic theories about $\mathcal{G}$-local system for non-constant group scheme $\mathcal{G}$. This is a non-standard notion but it appears naturally in the argument. In §B we collect some basic fact about Beilinson's 1-motive and Duality on Beilinson's 1-motive.

Chapter 2

Notations

2.1 Notations related to algebraic stack

Let k be an algebraically closed field. Let S be an affine Noetherian scheme over k. In this note, an algebraic stack $\mathscr{X}$ over S is a stack such that the diagonal morphism

$$\Delta_S : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$$

is representable and quasi-compact and there exists a smooth presentation, i.e., a smooth, surjective morphism $X \to \mathscr{X}$ from a scheme X.

An algebraic stack $\mathscr{X}$ is called smooth if for every S-scheme U maps smoothly to $\mathscr{X}$, the structure morphism $U \to S$ is smooth.

For any algebraic stack $\mathscr{X}$, we denote by $\mathscr{X}_{sm}$ the smooth topology on $\mathscr{X}$, i.e., it is the site for which the underling category has objects consisting of S-scheme U together with a smooth morphism $U \to \mathscr{X}$ and for which morphisms are smooth 2-morphisms and for which covering maps are smooth surjective maps of schemes. Similarly, we denote by $\mathscr{X}_{et}$, the etale topology on $\mathscr{X}$.

Assuming that $\mathscr{X}/S$ is smooth and proper. Let $\mathscr{Y} \to \mathscr{X}$ quasi-projective morphism of algebraic stack. We denote by $\mathrm{Sect}_S(\mathscr{X}, \mathscr{Y})$ to be the algebraic stack of "sections" of $\mathscr{Y}$ over $\mathscr{X}$, i.e., for any $u : U \to S$ we have

$$\mathrm{Sect}_S(\mathscr{X}, \mathscr{Y})(U) = \mathrm{Hom}_{\mathscr{X}}(\mathscr{X}_U, \mathscr{Y}).$$

Let $\mathscr{X}$ be a smooth algebraic stack over S. We define the relative tangent stack $T(\mathscr{X}/S)$ to be the following stack: for any $\mathrm{Spec}(R) \to S$ we have

$$T(\mathscr{X}/S)(R) := \mathscr{X}(R[\epsilon]/\epsilon^2).$$

This stack is algebraic and the natural inclusion $R \to R[\epsilon]/\epsilon^2$ induces a morphism

$$\tau_{\mathscr{X}} : T(\mathscr{X}/S) \to \mathscr{X}.$$

One can show that $T(\mathscr{X}/S)$ is a relative Picard stack over $\mathscr{X}$, therefore, we can associate to it a complex in $D^{[-1,0]}(\mathscr{X}, \mathbb{Z})$ called the relative tangent complex:

$$T^{\bullet}_{\mathscr{X}/S} = \{T_{\mathscr{X}/S} \to T_{\mathscr{X}}\}.$$

The relative cotangent stack is then defined as

$$T^*(\mathscr{X}/S) := \mathrm{Spec}_{\mathscr{X}}(\mathrm{Sym}_{\mathcal{O}_{\mathscr{X}}} H^0(T^{\bullet}_{\mathscr{X}/S})).$$

Let $f : \mathscr{X} \to \mathscr{Y}$ be a (representable)morphism between two algebraic stacks over S. We have following cotangent morphism

$$\begin{array}{ccc} T^*(\mathscr{Y}/S) \times_{\mathscr{Y}} \mathscr{X} & \xrightarrow{f_d} & T^*(\mathscr{X}/S) . \\ \downarrow{\scriptstyle f_p} & & \\ T^*(\mathscr{Y}/S) & & \end{array}$$

2.2 Notations related Frobenius morphism

Let k be an algebraically closed field of char p. Let S be a Noetherian scheme over k and $\pi : \mathscr{X} \to S$ be an algebraic stack over S. Let $Fr_S : S \to S$ be the absolute

Frobenius map of S. We have the following commutative diagram

$$\begin{array}{ccccc} \mathscr{X} & \xrightarrow{F_{\mathscr{X}/S}} & \mathscr{X}' & \longrightarrow & \mathscr{X} \\ & \searrow & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ & & S & \xrightarrow{F_{r_S}} & S \end{array}$$

where the square is Cartesian. We called $F_{\mathscr{X}/S} : \mathscr{X} \to \mathscr{X}'$ the relative Frobenius morphism.

2.3 Notation related to reductive groups

Let k be an algebraically closed field of positive characteristic p. We assume that p is sufficiently large. Let C be an smooth projective curve over k. Let G be a reductive algebraic group over k. We denote by $\mathfrak{g}$ the Lie algebra of G. We fixed a Cartan and Borel subgroups $T \subset B \subset G$. We denote by $\mathfrak{t}$ and $\mathfrak{b}$ their Lie algebras. We denote by W the Weyl groups of G. We denote by ${}^L G$ its Langlands dual group over k. We denote by $\mathbb{X}^\bullet$ and $\mathbb{X}_\bullet$ the character and co-character group of T.

Let E be a G-torsor on C, we denote by $\mathrm{Ad}(E) = E \times^G G$ for the adjoint torsor and $\mathrm{ad}(E) = E \times^G \mathfrak{g}$ for the adjoint bundle. For any scheme S over k, we write $C_S = C \times S$.

Chapter 3

The Hitchin fibration

3.1 Hitchin map

Let $k[\mathfrak{g}]$ and $k[\mathfrak{t}]$ be the algebra of polynomial function on $\mathfrak{g}$ and $\mathfrak{t}$. We have an isomorphism $k[\mathfrak{g}]^G \simeq k[\mathfrak{t}]^W$. Let $\mathfrak{c} = \mathrm{Spec}(k[\mathfrak{t}]^W)$. $\mathfrak{c}$ carries a canonical $\mathbb{G}_m$ action that comes form the homotheties on $\mathfrak{g}$.

Let D be an invertible sheave on C and we denote by $\mathcal{L}_D$ the corresponding line bundle and ρ_D the corresponding $\mathbb{G}_m$-torsor. We denote by $\mathfrak{g}_D = \mathfrak{g} \times^{\mathbb{G}_m} \rho_D$ and $\mathfrak{c}_\omega = \mathfrak{c} \times^{\mathbb{G}_m} \rho_D$ the $\mathbb{G}_m$-twist of $\mathfrak{g}$ and $\mathfrak{c}$ with respect to the natural $\mathbb{G}_m$ action.

Let Higgs_D be the stack that associates to each k-scheme S the groupoid $\mathrm{Hitch}_D(S)$ consisting of (E, ϕ), where E is an G-torsor over $X \times S$ and ϕ is an element in $\Gamma(C \times S, ad(E) \otimes_C D)$. Let B_D be the scheme of section of $\mathfrak{c}_D$ over C. Namely, for each k-scheme S, $B_D(S)$ is the set of section

$$b : C \times S \to \mathfrak{c}_D$$

The natural G-invariant projection $\chi : \mathfrak{g} \to \mathfrak{c}$ induces a map

$$[\chi_D] : [\mathfrak{g}_D/G] \to \mathfrak{c}_D$$

We can think of $(E, \phi) \in \mathrm{Higgs}(S)$ as a section $h_{E,\phi} : C \times S \to [\mathfrak{g}_D/G]$ and the map

$[\chi_D]$ will give us map

$$h_D : \mathrm{Higgs}_D \to B_D$$

The natural map $\chi : \mathfrak{g} \to \mathfrak{c}$ admits a section due to Kostant $k : \mathfrak{c} \to \mathfrak{g}^{reg}$ and it will induce a section of the Hitchin fibration (need a choice of the square roots of D, say more about it)

$$\epsilon_D : B_D \to \mathrm{Higgs}_D .$$

Let Bun_G be the moduli stack of G-bundle on C. Let ω_C be the canonical line bundle. There is a canonical isomorphism between $\mathrm{Higgs}_{\omega_X}$ and $T^* \mathrm{Bun}_G$. In this case the fibration $h_{\omega_C} : T^* \mathrm{Bun}_G \to B_{\omega_C}$ constructed above is people usually called Hitchin Fibration. We denote by $B = B_\omega$, $\mathrm{Higgs} = \mathrm{Higgs}_\omega$, $h = h_\omega : \mathrm{Higgs} \to B$ and $\epsilon = \epsilon_\omega : B \to \mathrm{Higgs}$.

3.2 Centralizers

Consider the group scheme I over $\mathfrak{g}$ consisting of pair

$$I = \{(x, g) \in \mathfrak{g} \times G | ad(g)(x) = x\}.$$

We define $J = k^* I$. It is a smooth commutative group scheme over $\mathfrak{c}$. The following Proposition is proved in [N1]

Proposition 3.2.1. *We have an canonical isomorphism of group scheme* $\chi^* J|_{\mathfrak{g}^{reg}} \simeq I|_{\mathfrak{g}^{reg}}$*, moreover, this isomorphism extends to a morphism of group scheme* $\chi^* J \to I$.

There is a $G \times \mathbb{G}_m$ action on I given by $(h, t)(x, g) = (ad(tx), hgh^{-1})$. Using the $\mathbb{G}_m$ action we can twist everything by a $\mathbb{G}_m$-torsor ρ_D and we set $J_D = J \times^{\mathbb{G}_m} \rho_D$, $I_D = I \times^{\mathbb{G}_m} \rho_D$. I_D is equivariant under the G-action, hence it descends to a group scheme $[I_D]$ over $[\mathfrak{g}_D/G]$. As in *loc. cit.*, they proved the following proposition

Proposition 3.2.2. *For* $(E, \phi) \in \mathrm{Higgs}_D(S)$. *Let* $h_{E,\phi} : C \times S \to [\mathfrak{g}_D/G]$ *be the morphism associated to it. We have* $h^*_{E,\phi}[I_D] = \mathrm{Aut}(E, \phi)$ *and there is a morphism* *of group scheme* $[\chi_D]^*J_D \to [I_D]$ *over* $[\mathfrak{g}_D/G]$ *such that it is an isomorphism over* $[\mathfrak{g}_D^{reg}/G]$.

For simplicity, we use $J = J_\omega$, $I = [I_\omega]$ and $\chi = [\chi_\omega]$.

3.3 Symmetries of Hitchin fibration

Let $b : S \to B$ be S-point of B. This is the same as giving a map $b : C \times S \to \mathfrak{c}$. Pulling back J using b we get a smooth groups scheme $J_b = b^*J$ over $C \times S$. Let Bun_{J_b} be the Picard category of J_b-torsors over $X \times S$. The assignment $b \to \mathrm{Bun}_{J_b}$ defines a Picard stack over B and we denote it by Bun_J. Let $b \in B(S)$. The map $\chi^*J \to I$ will induce a map

$$J_b \to h^*_{E,\phi}[I] = \mathrm{Aut}(E, \phi)$$

Using the above map, we can twist any $(E, \phi) \in \mathrm{Higgs}_b$ by a J_b-torsors. This defines an action of Bun_J on Higgs over B.

We define B^0 to be the open sub-scheme of B such that for $b \in B^0(k)$, the image of $b : C \to \mathfrak{c}$ in $\mathfrak{c}$ intersects the discriminant divisor transversally. The following Proposition is proved in [N1, Proposition 4.3]

Proposition 3.3.1. *Let* $\mathscr{M} = \mathrm{Higgs} \times_B B^0$ *and* $\mathscr{P} := \mathrm{Bun}_J^0 = \mathrm{Bun}_J \times_B B^0$. *Then* $\mathscr{M}$ *is a* $\mathscr{P}$*-torsor. Using Kostant section we can trivialize this* $\mathscr{P}$*-torsor, i.e., we have an isomorphism*

$$k : \mathscr{M} \simeq \mathscr{P}.$$

Moreover, for any $b \in B^0(k)$, *the isomorphism*

$$k_b : \mathscr{M}_b \simeq \mathscr{P}_b$$

is given by sending (E, ϕ) *to* $Isom((E_b, \phi_b), (E, \phi))$.

We are going to use the following Proposition many times in this paper:

Proposition 3.3.2.

1. *For any $b \in B(k)$, we have a canonical isomorphism*

$$\mathrm{Lie}(J_b) \simeq \mathfrak{c}_\omega^* \otimes \omega.$$

In particular, we have

$$T_e^* \mathscr{M}_b \simeq H^0((\mathscr{M}_e)_b, \omega_{(\mathscr{M}_e)_b}) \simeq H^1(C, \mathrm{Lie}\, J_b)^* \simeq H^0(C, \mathfrak{c}_\omega) = B.$$

where $\mathscr{M}_e$ is the connected component of $\mathscr{M}$ and $(\mathscr{M}_e)_b$ the fiber of

$$h|_{\mathscr{M}_e} : \mathscr{M}_e \to B$$

over b.

2. *We have*

$$T^*(\mathscr{M}_e/B^0) \times_{h, \mathscr{M}_e, \epsilon} B^0 \simeq B^0 \times B.$$

Proof. Part (1) is proved in [N1, Proposition 4.13.2]. For part (2), it is enough to show that

$$\epsilon^*(\omega_{\mathscr{M}_e/B}) \simeq \mathcal{O}_B \otimes_k B$$

where $\mathcal{O}_B \otimes_k B$ is the trivial bundle on B with fiber B (as a vector space). Since the Hitchin base B is isomorphic to $\mathbb{A}_k^n$ for some n, the locally free sheaf $\epsilon^*(\omega_{\mathscr{M}_e/B})$ is acturally free. Therefore, it is enough to show that the fiber of $\epsilon^*(\omega_{\mathscr{M}_e/B})$ (at any point of $B(k)$) is isomorphic to B. But it follows from part (1). This finished the proof. □

Chapter 4

$\mathcal{D}$-module on stacks and Azumaya property

In this section we review some basic facts about $\mathcal{D}$-modules on algebraic stack and Azumaya property of sheaf of differential operators. Standard reference for those materials are [BD] and [BB].

4.1 Category of $\mathcal{D}_{\mathscr{X}/S}$-modules

Let S be an affine Noetherian scheme over an algebraically closed field k of char $p > 0$. Let $\mathscr{X}$ be a smooth algebraic stack over S. A $\mathcal{D}_{\mathscr{X}/S}$-module M on $\mathscr{X}$ is an assignment for each $U \to \mathscr{X}$ in $\mathscr{X}_{sm}$, a $\mathcal{D}_{U/S}$-module M_U and for each morphism $f : V \to U$ in $\mathscr{X}_{sm}$ an isomorphim $\phi_f : f^*M_U \simeq M_V$ which satisfies cocycle conditions. We denote the category of $\mathcal{D}_{\mathscr{X}/S}$-modules by $\mathcal{D}_{\mathscr{X}/S} - \text{mod}$.

The forgetful functor $\mathcal{D}_{\mathscr{X}/S} - \text{mod} \to \text{QCoh}(\mathscr{X})$ has a left adjoint. We denote by $\mathcal{D}^{\sharp}_{\mathscr{X}/S}$ the image of $\mathcal{O}_{\mathscr{X}}$ under this left adjoint functor. Here is another description of $\mathcal{D}^{\sharp}_{\mathscr{X}/S}$. For any $U \to \mathscr{X}$ in $\mathscr{X}_{sm}$, we have

$$(\mathcal{D}^{\sharp}_{\mathscr{X}/S})_U = \mathcal{D}_{U/S}/I_U$$

where $I_U = \mathcal{D}_{U/S}T_{U/\mathscr{X}}$.

For any $U \in \mathscr{X}_{sm}$, we define

$$(\mathcal{D}^{\flat}_{\mathscr{X}/S})_U := \mathcal{E}nd_{\mathcal{D}_{U/S}}(\mathcal{D}^{\sharp}_{\mathscr{X}/S})_U.$$

Since $\mathcal{O}_{\mathscr{X}'}$ is in the center of $\mathcal{D}_{\mathscr{X}/S}$, we can regard $\mathcal{D}^{\flat}_{\mathscr{X}/S}$ as an $\mathcal{O}_{\mathscr{X}'}$-module. Moreover, for any $f : U \to V \in \mathscr{X}_{sm}$, we have

$$(\mathcal{D}^{\flat}_{\mathscr{X}/S})_U \simeq f'^*(\mathcal{D}^{\flat}_{\mathscr{X}/S})_V.$$

Therefore, $\mathcal{D}^{\flat}_{\mathscr{X}/S}$ is a quasi-coherent sheaf on $\mathscr{X}'$.

Let $\pi' : \mathscr{X}' \to S$ be the base change of $\mathscr{X} \to S$ using $Fr_S : S \to S$. Let $T^*(\mathscr{X}'/S)$ be the relative cotangent of $\pi' : \mathscr{X}' \to S$. We are going to construct a $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$ such that the category of $\mathcal{D}_{\mathscr{X}/S}$-modules is equivalent to the category of $\mathcal{G}_{\mathscr{X}/S}$-twisted quasi-coherent sheaves (see B.4 for the definition of twisted sheaves.)

In order to define a $\mathbb{G}_m$-gerbe on $T^*(\mathscr{X}/S)$ it is enough to supply a $\mathbb{G}_m$-gerbe $(\mathcal{G}_{\mathscr{X}/S})_U$ on $T^*(\mathscr{X}/S) \times_{\mathscr{X}'} U'$ for every $U \to \mathscr{X}$ in $\mathscr{X}_{sm}$ and compatible isomorphisms for any $\beta : U \to V$ in $\mathscr{X}_{sm}$. But for any $f : U \to \mathscr{X}$ in $\mathscr{X}_{sm}$ we have

$$(f'_U)_d : T^*(\mathscr{X}/S) \times_{\mathscr{X}'} U' \to T^*(U'/S).$$

We have a $\mathbb{G}_m$-gerbe $\mathcal{G}_{U/S}$ on $T^*(U'/S)$ corresponding to the sheaf of relative differential operators $\mathcal{D}_{U/S}$. We define a $\mathbb{G}_m$-gerbe $(\mathcal{G}_{\mathscr{X}/S})_U$ on $T^*(\mathscr{X}/S) \times_{\mathscr{X}'} U'$ to be the pull back of $\mathcal{G}_{U/S}$ by $(f'_U)_d$. One can check those gerbes $(\mathcal{G}_{\mathscr{X}/S})_U$ are compatible with pull back, therefore, they define a $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathscr{X}/S}$ on $\mathscr{X}$. The following Proposition is proved in [Trav]:

Proposition 4.1.1. *There is an equivalence of category*

$$\mathcal{D}_{\mathscr{X}/S} - \text{mod} \simeq \text{QCoh}(\mathcal{G}_{\mathscr{X}/S})_1.$$

Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism between two smooth algebraic stack. We have the following Lemma:

Lemma 4.1.2. *There is an equivalence of* $\mathbb{G}_m$*-gerbe on* $T^*(\mathscr{Y}'/S)\times_{\mathscr{X}'}\mathscr{X}'$

$$(f_p')^*\mathcal{G}_{\mathscr{Y}/S}\simeq(f_d')^*\mathcal{G}_{\mathscr{X}/S}.$$

4.2 Azumaya property of differential operators

Let us begin with a review of the basic theory of Azumaya algebras and the category of twisted sheaves. Let S be an Noetherian scheme. Let $\mathscr{X}$ be an algebraic stack over S. Recall that an Azumaya algebra $\mathcal{A}$ over $\mathscr{X}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-algebras, which is locally in smooth topology isomorphic to $\mathcal{E}nd(\mathcal{V})$ for some vector bundle $\mathcal{V}$ on $\mathscr{X}$. Such an isomorphism between $\mathcal{A}$ and the matrix algebra is called a splitting of $\mathcal{A}$. Given a sheaf of Azumaya alglebra $\mathcal{A}$ on $\mathscr{X}$, one can associate to it the $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathcal{A}}$ of splittings over $\mathscr{X}$, i.e., for any $U\to S$ we have

$$\mathcal{G}_{\mathcal{A}}(U)=\{(x,\mathcal{V},i)|x\in\mathscr{X}(U),i:\mathcal{E}nd(\mathcal{V})\simeq x^*(\mathcal{A})\}.$$

Recall that there is a bijection between isomorphism classes of $\mathbb{G}_m$-gerbes on $\mathscr{X}_{sm}$ and $H^2(\mathscr{X}_{sm},\mathbb{G}_m)$. Therefore, for any Azumaya algebra $\mathcal{A}$ on $\mathscr{X}$, by above construction, we can associate to it a cohomology class in $H^2(\mathscr{X}_{sm},\mathbb{G}_m)$. We denote this class by $c(\mathcal{A})$. The cohomology class $c(\mathcal{A})$ in $H^2(\mathscr{X}_{sm},\mathbb{G}_m)$ can be also described as following. For any Azumaya algebra $\mathcal{A}$ of rank n^2 on $\mathscr{X}$ we consider the following sheaf of sets

$$\mathcal{P}_{\mathcal{A}}:=Isom(\mathcal{A},M_{n\times n}(\mathcal{O}_{\mathscr{X}}))$$

on $\mathscr{X}_{sm}$: For any $U\to\mathscr{X}$ in $\mathscr{X}_{sm}$ we associate to it the set of isomorphisms $\mathcal{A}_U\simeq M_{n\times n}(\mathcal{O}_U)$. $\mathcal{P}_{\mathcal{A}}$ is a PGL_n-torsor on $\mathscr{X}$, therefore, it corresponds to a class in $H^1(\mathscr{X}_{sm},PGL_n)$. Using the following short exact sequence

$$0\to\mathbb{G}_m\to GL_n\to PGL_n\to 0$$

we get a boundary map

$$H^1(\mathscr{X}_{sm}, PGL_n) \to H^2(\mathscr{X}_{sm}, \mathbb{G}_m).$$

This sends the class of $\mathcal{P}_{\mathcal{A}}$ to an element in $H^2(\mathscr{X}_{sm}, \mathbb{G}_m)$ and one can show that this element is equal to $c(\mathcal{A})$.

We will use the following Proposition in the sequel

Proposition 4.2.1. *Let $\mathcal{A}$ be a sheaf of Azumaya algebra on $\mathscr{X}$. We have the following equivalence of categories*

$$\mathrm{Qcoh}(\mathcal{G}_{\mathcal{A}})_1 \simeq \mathcal{A} - \mathrm{mod}(\mathrm{Qcoh}(\mathscr{X})).$$

Recall that a smooth algebraic stack $\mathscr{X}$ over S of relative dimension d is called relative good iff it satisfied the following equivalent properties:

1. $\dim(T^*(\mathscr{X}/S)) = 2d$.

2. $\mathrm{codim}\{x \in \mathscr{X} \,|\, \dim Aut(x) = n\} \geqslant n$ for all $n > 0$.

3. For any $U \to \mathscr{X}$ in $\mathscr{X}_{sm}$, the complex

$$\mathrm{Sym}(T_{U/\mathscr{X}} \to T_{U/S})$$

has cohomology concentrated in degree 0 and

$$H^0(\mathrm{Sym}(T_{U/\mathscr{X}} \to T_{U/S})) \simeq \mathrm{Sym}(T_{U/S})/T_{U/\mathscr{X}}\,\mathrm{Sym}(T_{U/S}).$$

The following Proposition is proved in [BB] (see also [Trav])

Proposition 4.2.2. *Let $\mathscr{X}$ be a relative good stack. Let $\pi_{\mathscr{X}} : T^*(\mathscr{X}/S) \to \mathscr{X}$ be the natural projection and $\pi_{\mathscr{X}'}$ be its Frobenius twist. Let $T^*(\mathscr{X}'/S)^0$ be the maximal smooth open substack of $T^*(\mathscr{X}'/S)$. Then*

1. *There is a natural coherent sheaf of algebra $D_{\mathscr{X}/S}$ on $T^*\mathscr{X}'$ such that*

$$(\pi_{\mathscr{X}'})_*(D_{\mathscr{X}/S}) \simeq \mathcal{D}^{\flat}_{\mathscr{X}/S}.$$

2. *Let $D^0_{\mathscr{X}/S}$ be the restriction of $D_{\mathscr{X}/S}$ to $T^*(\mathscr{X}'/S)^0$. $D^0_{\mathscr{X}/S}$ is an Azumaya algebra on $T^*(\mathscr{X}'/S)^0$ of rank $p^{2\dim(\mathscr{X}/S)}$.*

3. *The gerbe $\mathcal{G}_{\mathscr{X}/S}|_{T^*(\mathscr{X}'/S)^0}$ is isomorphic to $\mathcal{G}_{D^0_{\mathscr{X}/S}}$, the gerbe of splittings of $D^0_{\mathscr{X}/S}$.*

Remark. *By applying the result in [Toen], one can show that for any smooth algebraic stack $\mathscr{X}$ there is a derived Azumaya algebra $D^{dr}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$ such that the $\mathbb{G}_m$-gerbe corresponding to it is isomorphic to $\mathcal{G}_{\mathscr{X}/S}$. Moreover, when $\mathscr{X}$ is "good", the restriction of $D^{dr}_{\mathscr{X}/S}$ to smooth open substack $T^*(\mathscr{X}'/S)^0$ is isomorphic to $D^0_{\mathscr{X}/S}$.*

4.3 Cohomology class of $D^0_{\mathscr{X}/S}$

In this section we review the results of [BB] and [OV] about the description of cohomology class

$$c_{et}(D_{\mathscr{X}/S}) \in H^2(T^*(\mathscr{X}'/S)^0_{et}, \mathbb{G}_m)$$

corresponding to the Azumaya algebra $D^0_{\mathscr{X}/S}$, or equivalently, the description of the $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathscr{X}/S}$ of splittings of $D^0_{\mathscr{X}/S}$.

Recall that for any smooth Deligne-Mumford stack $\mathscr{X}/S$ there is an exact sequence of sheaves on $\mathscr{X}_{et}$:

$$(a)\ 0 \longrightarrow (\mathbb{G}_m)_{\mathscr{X}'} \xrightarrow{F^*_{\mathscr{X}/S}} (F_{\mathscr{X}/S})_*(\mathbb{G}_m)_{\mathscr{X}} \xrightarrow{dlog} (F_{\mathscr{X}/S})_*\omega_{\mathscr{X}/S,cl} \xrightarrow{\pi_{\mathscr{X}/S}-C_{\mathscr{X}/S}} \omega_{\mathscr{X}'/S} \longrightarrow 0$$

where

$$C_{\mathscr{X}/S} : (F_{\mathscr{X}/S})_*\omega_{\mathscr{X}/S,cl} \to \omega_{\mathscr{X}'/S}$$

is the Cartier morphism and $\pi^*_{\mathscr{X}/S}$ is the morphism induced by the adjunction map

$$\omega_{\mathscr{X}/S} \to (\pi_{\mathscr{X}/S})_* \pi^*_{\mathscr{X}/S} \omega_{\mathscr{X}/S} \simeq (\pi_{\mathscr{X}/S})_* \omega_{\mathscr{X}'/S}.$$

The above exact sequence induces a morphism

$$\tau : H^0(\mathscr{X}', \omega_{\mathscr{X}'/S}) \to H^2(\mathscr{X}'_{et}, \mathbb{G}_m).$$

Proposition 4.3.1.

1. *Let $\beta' \in \Gamma(\mathscr{X}', \omega_{\mathscr{X}'/S})$ be a one form. For each etale morphism $u' : U' \to \mathscr{X}'$, let $U \to \mathscr{X}$ and let $\mathscr{P}ic^{\natural}_{\beta'}(U')$ be the groupoid of invertible sheaf $\mathcal{L}$ on U with integrable connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes \omega_{U/S}$ whose p-curvature is equal to $(u')^*\beta' \in H^0(U', \omega_{U'/S})$. Then, $\mathscr{P}ic^{\natural}_{\beta'}$ is a $\mathbb{G}_m$-gerbe over $\mathscr{X}_{et}$ and its class in $H^2(\mathscr{X}'_{et}, \mathbb{G}_m)$ is equal to $\tau(\beta')$.*

2. *Assuming that $\mathscr{X}/S$ is "good". Let $\theta' \in H^0(T^*(\mathscr{X}'/S), \omega_{T^*(\mathscr{X}'/S)})$ be the canonical one form on $T^*(\mathscr{X}'/S)$. Let $T^*(\mathscr{X}'/S)^0$ be the largest smooth Deligne-Mumford substack of $T^*(\mathscr{X}'/S)$ and let θ'^0 be the restriction of θ' to $T^*(\mathscr{X}'/S)^0$. Then the class of the splitting gerbe of the Azumaya algebra $D^0_{\mathscr{X}/S}$ on $T^*(\mathscr{X}/S)^0$ is equal to $\tau(\theta'^0)$.*

There is a geometric description of the $\mathbb{G}_m$-gerbe $\mathscr{P}ic^{\natural}_{\beta'}$ when $\mathscr{X}/S$ is a smooth Deligne-Mumford stack. The one form $\beta' \in H^0(\mathscr{X}', \omega_{\mathscr{X}'/S})$ defines a section

$$\beta' : \mathscr{X}' \to T^*(\mathscr{X}'/S).$$

We let $D_{\mathscr{X}/S,\beta'} := (\beta')^* D_{\mathscr{X}/S}$ be the pull-back of Azumaya algebra $D_{\mathscr{X}/S}$ under β'. We have the following proposition ([BB, Proposition 3.3]).

Proposition 4.3.2. *The gerbe of splittings of $D_{\mathscr{X}/S,\beta'}$ on $\mathscr{X}'_{et}$ is canonical equivalent to $\mathscr{P}ic^{\natural}_{\beta'}$.*

Remark. *The reason to restrict ourself to Deligne-Mumford stack is that we do not know (at this moment) whether a similar exact sequence (a) exists in smooth topology* *for a smooth algebraic stack $\mathscr{X}/S$. However, in this note we are mainly working on the smooth part $T^*(\mathscr{X}'/S)^0$ which is Deligne-Mumford, therefore, above discussion can be applied to our case.*

Chapter 5

Abelian duality

5.1 Azumaya algebra with group sturcture

In this section we review the definition of group structure and commutative group structure on Azumaya algebra. For more details see [OV].

Definition 5.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two Azumaya algebra over an algebraic stack $\mathscr{X}/S$.

1. A 1-morphism $M : \mathcal{A} \to \mathcal{B}$ is a splitting of the Azumaya algebra $\mathcal{A}^{op} \otimes_k \mathcal{B}$.

2. If M and N are two 1-morphism from $\mathcal{A}$ to $\mathcal{B}$, a 2-morphism between M and N is an isomorphism of $\mathcal{A}^{op} \otimes \mathcal{B}$-module $\gamma : M \simeq N$

Let $\mathscr{G}$ be a group stack over S. Let $\mathcal{A}$ be an Azumaya algebra on $\mathscr{G}$ and $\mathcal{G}_{\mathscr{G}}$ be the $\mathbb{G}_m$-gerbe of splitting . Let $m : \mathscr{G} \times_S \mathscr{G} \to \mathscr{G}$, $i : S \to \mathscr{G}$ be respectively the multiplication morphism and the unit morphism.

Definition 5.1.2. A group structure on $\mathcal{A}$ is the following structure:

1. A 1-morphism M between $\mathcal{A} \boxtimes \mathcal{A}$ and $m^*\mathcal{A}$ over $\mathscr{G} \times_S \mathscr{G}$;

2. A 2-morphism between the resulting two 1-morphisms between $\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}$ and $m^*\mathcal{A}$ over $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$, which satisfies the cocycle condition.

Similarly, we also have the notion of group structure on the gerbe $\mathcal{G}_{\mathcal{A}}$ (or more generally any $\mathbb{G}_m$-gerbe over $\mathscr{G}$). The fact that the group structure on $\mathcal{A}$ induces a group structure on $\mathcal{G}_{\mathcal{A}}$ is clear. All details can be found in [OV].

Lemma 5.1.3. *A group structure on $\mathcal{A}$ will make $\mathcal{G}_{\mathcal{A}}$ into a group over S. It is an extension of groups stacks:*

$$0 \to B\mathbb{G}_m \to \mathcal{G}_{\mathcal{A}} \to \mathscr{G} \to 0.$$

In the sequel we are going to need the following lemma [BB, Lemma 3.16].

Lemma 5.1.4. *Let $\mathscr{G}$ be a group stack over S, and let β' be a relative one-form on $\mathscr{G}'$. Assume that*

$$m^*\beta' = p_1^*\beta' + p_2^*\beta'$$

Then the algebra $D_{\mathscr{G}/S,\beta'}$ has a natural group structure.

Now, we further assume that $\mathscr{G}$ is a Beilinson's 1-motive (in particular a Picard stack). Let $\sigma : \mathscr{G} \times_S \mathscr{G} \to \mathscr{G} \times_S \mathscr{G}$ be the map $\sigma(x, y) = (y.x)$. We have the notion of commutative group structure on $\mathcal{A}$:

Definition 5.1.5. Let $\mathcal{A}$ be an Azumaya algebra on $\mathscr{G}$ with a group structure M. Then $\sigma^*(M)$ is another 1-morphism between $m^*\mathcal{A}$ and $\mathcal{A} \boxtimes \mathcal{A}$. A commutative group structure on $\mathcal{A}$ is a 2-morphism $i : \sigma^*M \simeq M$ such that $i^2 = id$.

We have the following Lemma:

Lemma 5.1.6. *Let $\mathscr{G}$ be a Beilinson's 1-motive, and let β' be a relative one-form on $\mathscr{G}'$. Assume that*

$$m^*\beta' = p_1^*\beta' + p_2^*\beta'$$

Then the algebra $D_{\mathscr{G}/S,\beta'}$ has a natural commutative group structure.

5.2 Formulation of abelian duality

In this section, we assume that S is a noetherian affine scheme. Let $\mathscr{B}$ be a Beilinson's 1-motive over S.

Let $\mathcal{A}$ be a sheaf of Azumaya algebra on $\mathscr{B}$ with commutative group structure and splits etale locally over S.

Lemma 5.2.1. *There is a $\mathscr{B}^\vee$-torsor $\mathscr{T}_\mathcal{A}$ over S called multiplicative-splittings of $\mathcal{A}$ such that the bounded derived category of $\mathcal{A}$-module is equivalent to the bounded derived category of quasi-coherent sheaves on $\mathscr{T}_\mathcal{A}$, i.e. we have*

$$D(\mathcal{A}-\text{mod}) \simeq D(\text{QCoh}(\mathscr{T}_\mathcal{A})).$$

Proof. Let me first give the construction of $\mathscr{T}_\mathcal{A}$. Let $\mathcal{G}_\mathcal{A}$ be $\mathbb{G}_m$-gerbe of splitting of $\mathcal{A}$. Since $\mathcal{A}$ is group-like and splits etale locally over S, we have a short exact sequence of Picard stack

$$0 \to \mathbb{G}_m \to \mathcal{G}_\mathcal{A} \to \mathscr{B} \to 0$$

Applying duality of Picard stack (see B.2) on a above sequence we get

$$0 \to \mathscr{B}^\vee \to \mathcal{G}_\mathcal{A}^\vee \to \mathbb{Z} \to 0$$

We define $\mathscr{T}_\mathcal{A}$ to be the $\mathscr{B}^\vee$-torsor associates to above extension. By Proposition 4.2.1, we have

$$D(\mathcal{A}-\text{mod}) \simeq D(\text{QCoh}(\mathcal{G}_\mathcal{A}))_1.$$

On the other hand, by duality for torsors (see B.4.1), we have

$$D(\text{QCoh}(\mathcal{G}_\mathcal{A}))_1 \simeq D(\text{QCoh}(\mathscr{T}_\mathcal{A})).$$

Therefore, we have

$$D(\mathcal{A}-\text{mod}) \simeq D(\text{QCoh}(\mathscr{T}_\mathcal{A})).$$

□

Let A be an abelian scheme over S. We let $A^\natural$ be the universal extension of $A^\vee$ by vector groups, which is the scheme classifying $(\mathcal{L}, \nabla)$, where $\mathcal{L}$ is a multiplicative line bundle on A, together with a flat connection relative to S. We have a natural map $A^\natural \to A^\vee$ which sends $(\mathcal{L}, \nabla)$ to $\mathcal{L}$. Since we assume S is affine, above map is surjective (see [MM] Prop 3.2.3 (a) at section 2.6). Thus we have the following short exact sequece

$$1 \to H^0(A, \omega_{A/S}) \to A^\natural \to A^\vee \to 1.$$

The p-curvature construction gives rise to a morphism

$$\psi : A^\natural \to H^0(A', \omega_{A'/S'}).$$

More precisely, for every $T \to A^\natural$ given by $(\mathcal{L}, \nabla)$, the p-curvature of $\mathcal{L}$ is $\psi_p(\nabla) \in H^0(A', \omega_{A'/S'})$.

Lemma 5.2.2. *The map ψ is proper and surjective.*

Proof. We first show that this map is proper. Let $R \subset K$ be a valuation ring. By valuative criterion of properness, we need to show that for any diagram

$$\begin{array}{ccccc} \operatorname{Spec} K & \xrightarrow{f} & A^\natural & \longrightarrow & A^\vee \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & H^0(A', \omega_{A'/S'}) & \longrightarrow & S' \end{array}$$

There is a unique map $s : \operatorname{Spec} R \to A^\natural$ making above diagram commutes. As $A^\vee$ is proper over S', there is an unique map $s^\vee : \operatorname{Spec} R \to A^\vee$ making the diagram commutes. Since $A^\natural(R) \to A^\vee(R) \to H^1(R, H^0(A, \omega_{A/S})) = 0$ is surjective, choose any lifting of $s^\vee$, we denote it by $s^\natural$. There is a unique point in $g \in (A^\vee)'(K)$ that transform $s^\natural|_{\operatorname{Spec} K}$ to f. Extends g to $g_R \in (A^\vee)'(R)$, then $s := g_R \cdot s^\vee \in A^\natural(R)$ will be the desired map.

The surjectivity of this map follows from the fact that the image of ψ is closed and has the same dimension as $H^0(A', \omega_{A'/S'})$. □

We give another proof of surjectivity of ψ in the case $S = \operatorname{Spec} k$. Recall that for any smooth variety M over k, we have an exact sequence of etale sheaves on M:

$$0 \longrightarrow \mathcal{O}_M^* \xrightarrow{Fr} \mathcal{O}_M^* \xrightarrow{dlog} \omega_{M,cl} \xrightarrow{C-1} \omega_M \longrightarrow 0$$

where C is the Cartier operator closed one forms. Moreover, giving a closed one form $\omega \in H^0(M, \omega_{M,cl})$, the p-curvature of $(\mathcal{O}_M, d+\omega)$ is given by $(C-1)(\omega) \in H^0(M, \omega_M) \simeq H^0(M', \omega_{M'})$. (The last isomorphism is induced by the natural morphism $M' \to M$.) Therefore, to show that ψ is surjective, it is enough to show that the map

$$C-1 : H^0(M, \omega_{M,cl}) \to H^0(M, \omega_M)$$

is surjective when $M = A$ is an Abelian variety. But if $M = A$ is an abelian variety, we have $H^0(A, \omega_{A,cl}) = H^0(A, \omega_A)$, therefore the map:

$$C-1 : H^0(A, \omega_A) \to H^0(A, \omega_A)$$

becomes an endomorphism (additive but not k-linear) of finite dimensional k-vector space. Since the Cartier morphism C is a p-linear map, the following Lemma implies $C-1$ is surjective and this implies ψ is surjective.

Lemma 5.2.3. *[Milne, Lemma 4.13] Let V be a finite dimensional vector space over an algebraically closed field k of char $p > 0$. Let $\phi : V \to V$ be a p-linear additive map, i.e., $\phi(av) = a^p v$ for any $a \in k$ and $v \in V$. Then $\phi - 1 : V \to V$ is surjective.*

5.3 Abelian duality over $\operatorname{Spec} k$

Let us assume $S = \operatorname{Spec} k$, for some algebraically closed field k of positive characteristic. Set $B' = H^0(A', \omega_{A'})$. By Cartier descent we have a short exact sequence

$$0 \to (A^\vee)' \to A^\natural \to B' \to 0$$

Let T^*A' be the cotangent bundle of A'. Since A is a group T^*A' is trivial and it is isomorphic to $A' \times B'$. We have the trivial fibration $T^*A' = A' \times B' \to B'$ giving by projection on the second factor and a natural isomorphism $(T^*A')^\vee \simeq (A^\vee)' \times B'$. The above short exact sequence shows that $A^\natural$ is a $(T^*A')^\vee$-torsor over B'. Let D_A be the sheaf of Azumaya algebra on T^*A' coming form the sheaf of differential operator on A.

Theorem 5.3.1.

1. *The sheaf of Azumaya algebra D_A has a natural group structure and splits etale locally over B'.*

2. *Let $\mathscr{T}_{D_A}$ be the $(T^*A')^\vee$-torsor of multiplicative-splittings of D_A (see Lemma 5.2.1). We have an isomorphism of $(T^*A')^\vee$-torsors*

$$\mathscr{T}_{D_A} \simeq A^\natural$$

Proof. Proof of the first statement. Let us first prove that the Azumaya algebra $\mathcal{D}_A$ has a natural group structure. By Lemma 5.1.6, it is enough to show that

$$m^*\theta = \theta + \theta$$

where $m : T^*A \times_B T^*A \to T^*A$ and θ is the canonical one form of T^*A.

Let me first recall the definition of θ. For any $(a, b) \in T^*A = A \times B$ and $(u, w) \in T_{a,b}(T^*A) = T_aA \times T_bB \simeq (B)^* \times (B)^*$, we have $\theta|_{(a,b)}(u, w) =< b, u >$ here $<,>$ is the natural paring between B and $(B)^*$. Let $A \times A \times B \simeq T^*A \times_B T^*A$ be the natural isomorphism. Let $(v_1, v_2, v_3) \in T_{a_1,a_2,b}(A \times A \times B)$. Then we have $m^*\theta|_{a_1,a_2,b}(v_1, v_2, v_3) = \theta|_{a_1+a_2,b}(v_1 + v_2, v_3) =< b, v_1 + v_2 >=< b, v_1 > + < b, v_2 >= \theta|_{a_1,b}(v_1, v_3) + \theta|_{a_2,b}(v_2, v_3)$. This implies $m^*\theta = \theta + \theta$.

Above description of θ also shows that the restriction of θ to fibers of $p_1 : T^*A \to B$ is constant and is given by $\theta|_{p_1^{-1}(b)} = b \in H^0(A, \omega_A)$.

To show that $\mathcal{D}_A$ splits etale locally on B', recall from Lemma 5.2.2, we have a surjective map $\psi : A^\natural \to B'$. On the other hand, by Lemma 4.3.2, any object in

$\psi^{-1}(b')$ defines a splitting of $\mathcal{D}_A|_{p_1^{-1}(b)}$ (here we used the fact the restriction of θ to $p_1^{-1}(b')$ is $b' \in H^0(A', \omega_{A'})$). Since ψ is smooth (see Theorem 4.14 in [OV]), it admits section etale locally and it implies $\mathcal{D}_A$ splits etale locally on B'.

Proof of (2). Let me first give a description of $\mathscr{T}_{\mathcal{D}_A}$. Let $\mathcal{G}_{\mathcal{D}_A}$ be the gerbe of splittings of $\mathcal{D}_A$. By definition the dual of $\mathcal{G}_{\mathcal{D}_A}$ is

$$\mathcal{G}_{\mathcal{D}_A}^{\vee} = \mathrm{Hom}_{\mathrm{gp}}(\mathcal{G}_{\mathcal{D}_A}, B\mathbb{G}_m)$$

An element $f \in \mathcal{G}_{\mathcal{D}_A}^{\vee}$ belongs to $\mathscr{T}_{\mathcal{D}_A} := (\mathcal{G}_{\mathcal{D}_A}^{\vee})_1$ if and only if the composition

$$B\mathbb{G}_m \xrightarrow{i} \mathcal{G}_{\mathcal{D}_A} \xrightarrow{f} B\mathbb{G}_m$$

is equal to identity.

Any $f \in \mathscr{T}_{\mathcal{D}_A}$ gives a splitting of the exact sequence

$$0 \longrightarrow B\mathbb{G}_m \underset{f}{\overset{i}{\rightleftarrows}} \mathcal{G}_{\mathcal{D}_A} \longrightarrow T^*A' \longrightarrow 0$$

Hence, for any $b' \in B'(k)$, an object in the fiber of $\mathscr{T}_{\mathcal{D}_A} \to B'$ over b' defines a splitting of $\mathcal{D}_A|_{A' \times b'}$ which is compatible with the group structure on $A' \times b' \simeq A'$. By Proposition 4.3.2, such a splitting corresponds to a character line bundle $\mathcal{L}$ on A whose p-curvature is equal to $\theta|_{A' \times b'} = b' \in B' = H^0(A', \omega_{A'})$. Above discussion can be done for any S' point of B' and this defines a morphism from $\mathscr{T}_{\mathcal{D}_A}$ to $A^{\natural}$ over B' and it is easy to see that it is an equivalence. □

Corollary 5.3.2. *There is an equivalence of categories*

$$D(\mathcal{D}_A - \mathrm{mod}) \simeq D(\mathrm{QCoh}(A^{\natural}))$$

5.4 Abelian duality over general base S

Let $\pi : A \to S$ be an Abelian scheme over S. Let

$$B'_S := \mathrm{Sect}_S(A', T^*(A'/S))$$

be the S-scheme of sections of $T^*(A'/S)$ over A'. We have a natural isomorphism

$$T^*(A'/S) \simeq A' \times_S B'_S.$$

We denote by $\pi_S : T^*(A'/S) \to B'_S$ the natural projection. Under the morphism π_S, $T^*(A'/S)$ becomes an Abelian scheme over B'_S and we denote by m_S the multiplication map:

$$m_S : T^*(A'/S) \times_{B'_S} T^*(A'/S) \to T^*(A'/S).$$

Let $\pi_S^\vee : A^\natural \to B'_S$ be the universal extension of A be vector group.

Let $D_{A/S}$ be the Azumaya algebra on $T^*(A'/S)$. The following Proposition is the relative version of Abelian duality, whose proof is essentially the same as the case of Abelian variety.

Proposition 5.4.1.

1. *The sheave of Azumaya algebra $D_{A/S}$ has a natural group structure and splits etale locally over B'_S.*

2. *Let $\mathscr{T}_{D_{A/S}}$ be the $T^*(A'/S)^\vee$-torsor corresponding to $D_{A/S}$. Then we have a natural isomorphism*

$$\mathscr{T}_{D_{A/S}} \simeq A^\natural.$$

Corollary 5.4.2. *There is an equivalence of category*

$$D(\mathcal{D}_{A/S} - \mathrm{mod}) \simeq D(\mathrm{QCoh}(A^\natural)).$$

Remark. *Let $(\mathcal{L}_{univ}, \nabla)$ be the Poincare line bundle on $A \times_S A^\natural$, equipped with an* *integrable connection along A, i.e.,*

$$\nabla : \mathcal{L} \to \mathcal{L} \otimes \omega_{A \times_S A^\natural / A^\natural}.$$

Then the Fourier-Mukai type functor

$$\Phi : D(\mathcal{D}_{A/S} - \mathrm{mod}) \to D(\mathrm{QCoh}(A^\natural)), \quad Rp_+(q^*\mathcal{E} \otimes \mathcal{L}_{univ}),$$

is an equivalence. One can show that this equivalence is isomorphic to the equivalence above.

5.5 Abelian duality for Beilinson 1-motive

Above results have an analogy for good Beilinson 1-motive. Let $\mathscr{A}$ be a good Beilinson 1-motive over S, i.e. the BG part of $\mathscr{A}$ is finite group of multiplicative type. In this case, $\mathscr{A}$ is a good algebraic stack, therefore, we have a sheaf of Azumaya algebra $D_{\mathscr{A}/S}$ on $T^*(\mathscr{A}'/S)^0$. We want to show that this Azumaya algebra has a commutative group structure.

The only difference between good Beilinson 1-motive and abelian scheme is that $\mathscr{A}$ might be disconnected. But the description of canonical one form θ in Theorem 5.3.1 works in the disconnected case. Therefore, the canonical one form $\theta_{\mathscr{A}/S}$ is still multiplicative with respect to the map

$$m : T^*(\mathscr{A}/S)^0 \times_{B'} T^*(\mathscr{A}/S)^0 \to T^*(\mathscr{A}/S)^0$$

where $B' = H^0((\mathscr{A}')_e, \omega_{\mathscr{A}'/S})$ and $(\mathscr{A}')_e$ is the neutral component of $\mathscr{A}'$. Rest of the arguments in Theorem 5.3.1 work in the same way, therefore we get

Proposition 5.5.1. *Let $\mathscr{A}$ be a good Beilinson 1-motive and let $\mathscr{A}^\natural$ be the universal extension of $\mathscr{A}$ by vector group.*

1. *The sheaf of Azumaya algebra $D_{\mathscr{A}/S}$ has a natural commutative group structure. Equivalently, the splitting gerbe $\mathcal{G}_{D_{\mathscr{A}/S}}$ on $T^*(\mathscr{A}'/S)^0$ is a Beilinson 1-motive.*

2. *The* $(T^*(\mathscr{A}'/S)^0)^\vee$*-torsor* $\mathscr{T}_{D_{\mathscr{A}/S}}$ *is isomorphism to* $\mathscr{A}^\natural$.

3. *We have an equivalence of categories*

$$D(\mathrm{QCoh}(\mathscr{A}^\natural)) \simeq D(\mathcal{D}_{\mathscr{A}/S} - \mathrm{mod})$$

For a general Beilinson 1-motive $\mathscr{A}$, since it is not "good" in our terminology, we can not apply above Proposition. However, we expect that in this case, there should be DG-version of above result which is related to the derived Azumaya algebra on $T^*(\mathscr{A}'/S)$ corresponds to the $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathscr{A}/S}$.

On the other hand, in some nice cases, we can do a little modification and still get an equivalence. To elaborate it more, let us consider the case of Picard stack $\mathscr{A} = \mathscr{P}ic(C)$. It is a $\mathbb{G}_m$-gerbe over the Picard scheme $\underline{\mathscr{P}ic}(C)$. Let $\pi : \mathscr{P}ic(C) \to \underline{\mathscr{P}ic}(C)$ be the natural projection. It follows from above that the cotangent stack $T^*\mathscr{P}ic(C)$ is a $\mathbb{G}_m$-gerbe over $T^*\underline{\mathscr{P}ic}(C)$. Let $D_{\underline{\mathscr{P}ic}(C)}$ be the Azumaya algebra on $T^*\underline{\mathscr{P}ic}(C)'$ and we denote by $D_{\mathscr{P}ic(C)}$ the pull back of $D_{\underline{\mathscr{P}ic}(C)}$ to $T^*\mathscr{P}ic(C)'$.

Now, instead of considering the category of $\mathcal{D}$-modules on $\mathscr{P}ic(C)$ we consider the category of modules over the sheaf of Azumaya algebra $D_{\mathscr{P}ic(C)}$. One can check that the multiplicative splitings torsor $\mathscr{T}_{D_{\mathscr{P}ic(C)}}$ for $D_{\mathscr{P}ic(C)}$ is isomorphic to $\mathscr{P}ic(C)^\natural$, therefore, by applying abelian duality, we have the following equivalence

Lemma 5.5.2. *There is an equivalence*

$$D(D_{\mathscr{P}ic(C)} - \mathrm{mod}) \simeq D(\mathrm{QCoh}(\mathscr{A}^\natural)).$$

We have Abel-Jacobi map $AJ : C \to \mathscr{P}ic(C)$. It is well known that the pull back of AJ defines an auto-equivalence of $\mathscr{P}ic(C)$

$$\mathscr{P}ic(C) \simeq \mathscr{P}ic(C)^\vee.$$

The pull back of AJ defines a morphism form $\mathscr{P}ic(C)^\natural$ to $\mathrm{LocSys}_{\mathbb{G}_m}(C)$, the stack of $\mathbb{G}_m$-local systems. We have the following

Lemma 5.5.3. *The pull back of AJ defines an isomorphism*

$$\mathscr{P}ic(C)^{\natural} \simeq \mathrm{LocSys}_{\mathbb{G}_m}(C).$$

The following Corollary is immediately, which is a special case of [BB]:

Corollary 5.5.4 (Geometric Langlands for $\mathbb{G}_m$). *We have an equivalence of derived categories*

$$D(D_{\mathscr{P}ic(C)} - \mathrm{mod}) \simeq D(\mathrm{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}(C)))$$

Chapter 6

Classical duality

In this section, we show that the $\check{\mathscr{P}} \simeq \mathscr{P}^\vee$ as Picard stacks over B. In fact, it is enough to prove the statement over B^0, and therefore, we write B^0 for B for simplicity, and so on and so forth.

6.1 Galois description of $\mathscr{P}$

We first introduce several auxiliary Picard stacks.

Let $\widetilde{C} \to B$ be he universal cameral curve. There is a natural action of W on $\widetilde{C}$. For a T-torsor E_T on $\widetilde{C}$, and an element $w \in \mathrm{W}$, there are two ways to produce a new T-torsor. Namely, the first is via the pullback $w^*E_T = \widetilde{C} \times_{w,\widetilde{C}} E_T$, and the second is via the induction $E_T \times^{T,w} T$. We denote

$$w(E_T) = ((w^{-1})^*E_T) \times^{T,w} T.$$

Let us decribe $w(E_T)$ more explicitly in the case $G = \mathrm{SL}_2$. Let s be the unique nontrivial element in the Weyl group, acting on the spectral curve $s : \widetilde{C}_b \to \widetilde{C}_b$. If we identify $T = \mathbb{G}_m$-torsors with invertible sheaves $\mathcal{L}$, then

$$s(\mathcal{L}) = s^*\mathcal{L}^{-1}.$$

Clearly, the assignment $E_T \mapsto w(E_T)$ defines an action of W on $\mathrm{Bun}_T(\widetilde{C}/B)$, i.e. for every $w, w' \in \mathrm{W}$, there is a canonical isomorphism $w(w'(E_T)) \simeq (ww')(E_T)$ satisfying the usual cocycle conditions.

Let $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$ be the Picard stack of strongly W-equivariant T-torsors on $\widetilde{C}/B$. By definition, for a B-scheme S, $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)(S)$ is the groupoid of $(E_T, \gamma_w, w \in W)$, where E_T is a T-torsor on $\widetilde{C}_S$, and $\gamma_w : w(E_T) \simeq E_T$ is an isomorphism, satisfying the natural compatibility conditions. Another way to formulate these compatibility conditions is provided in [DG]. Namely, for a T-torsor E_T, let $\mathrm{Aut}_{\mathrm{W}}(E_T)$ be the group consists of (w, γ_w), where $w \in \mathrm{W}$ and $\gamma_w : w(E_T) \simeq E_T$ is an isomorphism. Then there is a natural projection $\mathrm{Aut}_{\mathrm{W}}(E_T) \to \mathrm{W}$. Then an object of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)(S)$ is a pair (E_T, γ), where $\gamma : \mathrm{W} \to \mathrm{Aut}_{\mathrm{W}}(E_T)$ is a splitting of the projection.

Let $\mathscr{P}^1$ be the Picard stack over B classifying J^1-torsors on $C \times B$. First, we claim that there is a canonical morphism

$$j^1 : \mathscr{P}^1 \to \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B). \tag{6.1}$$

To construct j^1, recall that $J^1 = (\pi_*(T \times \widetilde{C}))^W$ and therefore, for any J^1-torsor E_{J^1} on $C \times S$ (where $b : S \to B$ is a test scheme), one can form a T-torsor on $\widetilde{C}_S$ by $E_T := \pi^* E_{J^1} \times^{\pi^* J^1} T$. Clearly, E_T carries on a strongly W-equivariant structure γ, and $j^1(E_{J^1}) = (E_T, \gamma)$ defines the morphism j^1.

The morphism j^1, in general, is not an isomorphism. Let us describe the image. Let $\alpha \in \Phi$ be a root and let $i_\alpha : \widetilde{C}_\alpha \to \widetilde{C}$ be the inclusion of the fixed point subscheme of the reflection s_α. Let $T_\alpha = T/(s_\alpha - 1)$ be the torus of coinvariants of the reflection s_α. Then $s_\alpha(E_T)|_{\widetilde{C}_\alpha} \times^T T_\alpha$ is canonically isomorphic to $E_T|_{\widetilde{C}_\alpha} \times^T T_\alpha$ and therefore $\gamma_{s_\alpha}|_{\widetilde{C}_\alpha}$ induces an automorphism of the T_α-torsor $E_T \times^T T_\alpha$. In other words, there is a natural map

$$r = \prod_{\alpha\in\Phi} r_\alpha : \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B) \to (\prod_{\alpha\in\Phi} T_\alpha(\widetilde{C}_\alpha))^W.$$

It is easy to see that rj_1 is trivial, and one can show that

Lemma 6.1.1. $\mathscr{P}^1 \simeq \ker r$. *In other words,* $\mathscr{P}^1(S)$ *consists of those strongly* W-

equivariant T-torsors (E_T, γ) such that the induced automorphism of $E_T \times^T T_\alpha|_{\widetilde{C}_\alpha}$ is trivial for every $\alpha \in \Phi$.

Proof. One needs to show that every strongly W-equivaraint T-torsor (E_T, γ) such that $r(E_T, \gamma) = 1$ is étale locally on $\widetilde{C}$ isomorphic to the trivial one, i.e., the trivial T-torsor together with the canonical W-equivariance structure. Following the argument as in [DG, Proposition 16.4], one reduces to prove the statement for a neighborhood around a point $x \in \cap_\alpha \widetilde{C}_\alpha$. By replacing $\widetilde{C}$ by the local ring around x, one can assume that E_T is trivial. Pick up a trivialization, then the W-equivariance structure on E_T amounts to a 1-cocycle $W \to T(\widetilde{C})$. By evaluating $T(\widetilde{C})$ at the unique closed point x, there is a short exact sequence $1 \to K \to T(\widetilde{C}) \to T(k) \to 1$. The condition $r(E_T, \gamma) = 1$ would mean that the cocycle takes value in K. As K is an $\mathbb{F}_p$-vector space and $p \nmid |\mathrm{W}|$, this cocycle is trivial. □

Recall that in [DG], an open embedding $J \to J^1$ is constructed. To describe the cokernel, we need some notations. Let $\check{\alpha} \in \check{\Phi}$ be a coroot. Let $\mu_{\check{\alpha}} : \ker(\check{\alpha} : \mathbb{G}_m \to T)$. This is either trivial, or μ_2, depending on where $\check{\alpha}$ is primitive or not. Let $\mu_{\check{\alpha}} \times \widetilde{C}_\alpha$ be the constant group scheme over $\widetilde{C}_\alpha$, regarded as a sheaf of groups over $\widetilde{C}_\alpha$, and let $(i_\alpha)_*(\mu_{\check{\alpha}} \times \widetilde{C}_\alpha)$ be its push forward to $\widetilde{C}$. Now, the result of [DG] can be reformulated as

We have

$$1 \to J \to J^1 \to \pi_*(\bigoplus_{\alpha \in \Phi} (i_\alpha)_*(\mu_{\check{\alpha}} \times \widetilde{C}_\alpha))^{\mathrm{W}} \to 1. \tag{6.2}$$

As a result, we obtain a short exact sequence

$$1 \to (\prod_{\alpha \in \Phi} \mathrm{Res}_{\widetilde{C}_\alpha/B}(\mu_{\check{\alpha}} \times \widetilde{C}_\alpha))^{\mathrm{W}} \to \mathscr{P} \to \mathscr{P}^1 \to 1. \tag{6.3}$$

Consider the composition $j : \mathscr{P} \to \mathscr{P}^1 \to \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$. Then Lemma 6.1.1 and (6.3) allows us to give a description of $\mathscr{P}$ in terms of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$. Namely, given a strongly W-equivariant T-torsor (E_T, γ), one obtains a canonical trivialization

$$(E_T|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \times^{\mathbb{G}_m, \check{\alpha}} T \simeq E_T^0|_{\widetilde{C}_\alpha}, \tag{6.4}$$

as $(E_T|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \times^{\mathbb{G}_m,\check{\alpha}} T \simeq E_T|_{\widetilde{C}_\alpha} \otimes s_\alpha(E_T^{-1})|_{\widetilde{C}_\alpha}$. Then it is easy to see that the condition $r_\alpha(E,\gamma) = 1$ is equivariant to say that (6.4) comes from a trivialization $c_\alpha : (E_T|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \simeq \mathbb{G}_m \times \widetilde{C}_\alpha$. Therefore, we can describe $\mathscr{P}(S)$ as the Picard groupoid of triples (E_T, γ, c_α), where (E_T, γ) is a strongly W-equivariant T-torsor on $\widetilde{C}$, and $c_\alpha : (E_T|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \simeq \mathbb{G}_m \times \widetilde{C}_\alpha$ is a trivialization, which induces (6.4), and is compatible with the W-equivariant structure.

Finally, let us denote $\mathscr{P}^0$ the Picard stack of J^0-torsors over B. Recall that there is the following exact sequence

$$1 \to J_b^0 \to J_b \to \pi_0(J_b) \to 1,$$

where $\pi_0(J_b)$ is the étale sheaf on C of the group of connected components of J_b.

6.2 The Abel-Jacobi map

Observe there is the norm map

$$\mathrm{Nm} : \mathrm{Bun}_T(\widetilde{C}/B) \to \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B), \quad E_T \mapsto (\bigotimes_{w \in W} w(E_T), \gamma_{\mathrm{can}}).$$

We claim that Nm admits a canonical lifting

$$\mathrm{Nm}^{\mathscr{P}} : \mathrm{Bun}_T(\widetilde{C}/B) \to \mathscr{P}$$

To show this, we need exhibit a canonical trivialization

$$c_\alpha : \bigotimes_{w \in W} w(E_T)|_{\widetilde{C}_\alpha} \times^{T,\alpha} \mathbb{G}_m \simeq \mathbb{G}_m \times \widetilde{C}_\alpha.$$

compatible with the strongly W-equivariant structure. However, for any T-torsor E_T, there is a canonical isomorphism $(E_T|_{\widetilde{C}_\alpha} \otimes s_\alpha(E_T)|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m \simeq \mathbb{G}_m \times \widetilde{C}_\alpha$, and therefore, we obtain c_α by write

$$\bigotimes_{w \in W} w(E_T)|_{\widetilde{C}_\alpha} \times^{T,\alpha} \mathbb{G}_m \simeq \bigotimes_{w \in s_\alpha \backslash W} (w(E_T)|_{\widetilde{C}_\alpha} \otimes s_\alpha w(E_T)|_{\widetilde{C}_\alpha}) \times^{T,\alpha} \mathbb{G}_m.$$

The compatibility of the collection $\{c_\alpha\}$ with the W-equivariant structure is clear.

Now, let

$$\mathrm{AJ} : \widetilde{C} \times \mathbb{X}_\bullet(T) \to \mathrm{Bun}_T(\widetilde{C}/B)$$

be the Abel-Jacobi map given by $(x, \check{\lambda}) \mapsto \mathcal{O}(\check{\lambda} x) := \mathcal{O}(x) \times^{\mathbb{G}_m, \check{\lambda}} T$.

Therefore we obtain

$$\mathrm{AJ}^{\mathscr{P}} : \widetilde{C} \times \mathbb{X}_\bullet(T) \to \mathscr{P}.$$

Observe that for any $x \in \widetilde{C}_\alpha$, $\mathrm{AJ}^{\mathscr{P}}(x, \check{\alpha})$ is the unit in $\mathscr{P}$. This follows from

$$\bigotimes_{w \in W} w\mathcal{O}(\check{\alpha} x) \simeq \bigotimes_{w \in W/s_\alpha} w\mathcal{O}(\check{\alpha} x + s_\alpha(\check{\alpha}) x)$$

is canonically trivialized, and the trivialization is compatible with the W-equivariant structure.

By pulling back the line bundles, we thus obtain

$$(\mathrm{AJ}^{\mathscr{P}})^\vee : \mathscr{P}^\vee \to \mathrm{Pic}(\widetilde{C} \times \mathbb{X}_\bullet(T))^{\mathrm{W}}.$$

Observe that there is the canonical morphism $\mathrm{Bun}_{\check{T}}^{\mathrm{W}}(\widetilde{C}/B) \to \mathrm{Pic}(\widetilde{C} \times \mathbb{X}_\bullet(T))^{\mathrm{W}}$ given by $(E_{\check{T}}, s) \mapsto \mathcal{L}$, where $\mathcal{L}|_{(x,\check{\lambda})} = E_{\check{T}}^{\lambda}|_x$. We claim that $(\mathrm{AJ}^{\mathscr{P}})^\vee$ canonically lifts to

$$\mathfrak{D}_0 : \mathscr{P}^\vee \to \check{\mathscr{P}}.$$

Let $\mathcal{L}$ be a multiplicative line bundle on $\mathscr{P}$. We thus need to show that

$$(\mathrm{AJ}^{\mathscr{P}})^* \mathcal{L}|_{(\widetilde{C}_\alpha, \check{\alpha})}$$

admits a canonical trivialization, which is compatible with the W-equivariance structure. However, this follows from $\mathrm{AJ}^{\mathscr{P}}((x,\breve{\alpha}))$ is the unit of $\mathscr{P}$ and a multiplicative line bundle on $\mathscr{P}$ is canonically trivialized over the unit.

Now, the classical duality theorem reads as

Theorem 6.2.1. *$\mathfrak{D}_0$ is an isomorphism and we have the following commutative diagram*

$$\begin{array}{ccc} \mathscr{P}^\vee & \xrightarrow{\quad\mathfrak{D}_0\quad} & \breve{\mathscr{P}} \\ & {\scriptstyle (\mathrm{AJ}^{\mathscr{P}})^\vee} \searrow \qquad \swarrow {\scriptstyle \breve{j}} & \\ & \mathrm{Bun}^{\mathrm{W}}_{T^\vee}(\widetilde{C}/B) & \end{array}$$

Chapter 7

The stack of $\mathrm{LocSys}_{G^\vee}$

7.1 The Hitchin map for $G^\vee$-local systems

Let $\mathrm{LocSys}_{G^\vee}$ be the stack of $G^\vee$-local system on C, i.e. for every scheme S over k, $\mathrm{LocSys}_{G^\vee}(S)$ is the groupoid of all $G^\vee$-torsors E on $C \times S$ together with a connection $\nabla : T_{C\times S/S} \to \widetilde{T}_E$. As X is a curve, the p-curvature operation will give us a morphism

$$f : \mathrm{LocSys}_{G^\vee} \to \mathcal{M}_{\omega^p}.$$

Combining this map with h_{ω^p}, we get a morphism from $\mathrm{LocSys}_{G^\vee}$ to B_{ω^p}

$$\tilde{h}_p : \mathrm{LocSys}_{G^\vee} \to B_{\omega^p}.$$

Let B' is the Forbenius twist of B.

Lemma 7.1.1. *We have* $B' = \Gamma(C', (\mathfrak{c}_\omega)')$

Proof. Let S be a k-scheme. An S-point of B' is a k-morphism $S \times C \to \mathfrak{c}_\omega$ over X. Here when we formulate the fiber product $S \times C$, we change the k-structure of S by $S \to \operatorname{Spec} k \xrightarrow{F} \operatorname{Spec} k$. Therefore, $S \times C \simeq S \times C'$, regarded as a k-scheme via $S \times C' \to \operatorname{Spec} k \xrightarrow{F} \operatorname{Spec} k$. The lemma follows from the following diagram where all squares are Cartesian.

$$
\begin{array}{ccccc}
 & & (\mathfrak{c} \times^{\mathbb{G}_m} D)' & \longrightarrow & \mathfrak{c} \times^{\mathbb{G}_m} D \\
 & & \downarrow & & \downarrow \\
S \times C' & \longrightarrow & C' & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k
\end{array}
$$

□

Lemma 7.1.2. *We have a natural map $f : B' \to B_{\omega^p}$.*

Proof. We have the following diagram which are all Cartesian squares.

$$
\begin{array}{ccccc}
\mathfrak{t}_{\omega^p} & \longrightarrow & (\mathfrak{t}_\omega)' & \longrightarrow & \mathfrak{t}_\omega \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{c}_{\omega^p} & \longrightarrow & (\mathfrak{c}_\omega)' & \longrightarrow & \mathfrak{c}_\omega \\
\downarrow{\scriptstyle \pi^p} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
C & \longrightarrow & C' & \longrightarrow & C \\
 & & \downarrow & & \downarrow \\
 & & k & \longrightarrow & k
\end{array}
$$

From above diagram we see that giving a section b' of π', it will induce a section b^p of π^p. Thus by previous lemma we obtain a map

$$f : B' \to B_{\omega^p}$$

□

Lemma 7.1.3. *Let $b' \in B'(S')$. We denote by b^p be the image of b' under the map $f : B' \to B_{\omega^p}$. We have $C_{b^p} = C \times_{C'} C_{b'}$ and $J_{b^p} = C \times_{C'} J_{b'}$.*

Proof. By definition we have $C_{b'} = C' \times_{(\mathfrak{c}_\omega)', b'} (\mathfrak{t}_\omega)'$. Thus, $C \times_{C'} C_{b'} = C_{b^p}$ follows from above diagram. The proof of $J_{b^p} = C \times_{C'} J_{b'}$ is exactly the same. □

Theorem 7.1.4 ([LP]). *The p-curvature morphism $\tilde{h}_p : \mathrm{LocSys}_{G^\vee} \to B_{\omega^p}$ factors through an unique morphism*

$$h_p : \mathrm{LocSys}_{G^\vee} \to B'.$$

We called this map the p-Hitchin map.

Unlike the usual Hitchin map, h_p does not admit a section.

7.2 $\mathrm{LocSys}_{G^\vee}$ is a $(J')^\vee$-gerbe

Set $J'^0 = J' \times_{B'} B'^0$, $\mathcal{P}'^0 = \mathcal{P}' \times_{B'} B'^0$ and $\mathrm{LocSys}^0_{G^\vee} = \mathrm{LocSys}_{G^\vee} \times_{B'} B'^0$. Our main Theorem in this section is the following

Theorem 7.2.1. $\mathrm{LocSys}^0_{G^\vee}$ *is a torsor over the Picard stack $\mathcal{P}'^0$.*

The rest of this section is devoted to prove Theorem 7.2.1. Recall that we have a map

$$f : \mathrm{LocSys}_{G^\vee} \to \mathcal{M}_{\omega^p}$$

given by $(E, \nabla) \to (E, \Psi(\nabla))$. Let $f^0 : \mathrm{LocSys}^0_{G^\vee} \to \mathcal{M}_{\omega^p}$ be the restriction of f to $\mathrm{LocSys}^0_{G^\vee}$. Let me recall the following Theorem in [OV]

Theorem 7.2.2 (Theorem 4.1 in [OV]). *Let $\mathcal{M}$ be a coherent sheaf with an F-Higgs field $\Psi : E \to E \otimes_X F^*\omega_X$. Then etale locally on C, the following are equivalent:*
1) There exists a connection on E whose p-curvature is Ψ.
2) There exists a coherent sheaf E' with a Higgs field (E', Ψ') on X' and an isomorphism

$$(E, \Psi) \simeq (F^*E', F^*\Psi').$$

Using above Theorem we deduce the following lemma:

Lemma 7.2.3. *The map f^0 factors through $\mathcal{M}^{reg}_{\omega^p}$, here $\mathcal{M}^{reg}_{\omega^p}$ is the open substack of regular F-Higgs bundle.*

Proof. Let $(E, \nabla) \in \mathrm{LocSys}^0_{G^\vee}$. Let $ad(E)$ be the adjoint bundle of E and

$$ad(\nabla) : ad(E) \to ad(E) \otimes_C \omega$$

be the induced connection. The p-curvature of $ad(\nabla)$ is given by

$$ad(\Psi(\nabla)) : ad(E) \to ad(E) \otimes_C F^*\omega_{C'}$$

which sends $v \to [v, \Psi(\nabla)]$. By above Theorem, etale locally, there exists a Higgis field $(M', \Psi' : M' \to M' \otimes_{C'} \omega_{C'})$ such that $(ad(E), ad(\Psi(\nabla))) = (F^*M', F^*\Psi')$. In particular, it implies etale locally the F-Higgis field $\Psi(\nabla)$ is a pull back of a Higgs field $\Psi(\nabla)'$. Since the Higgs field $\Psi(\nabla)'$ maps to $b' \in B'^0$, it implies $\Psi(\nabla)'$ is regular, hence $\Psi(\nabla)$ is regular. □

Corollary 7.2.4. *For any $b' \in B'^0(S)$ and $(E, \nabla) \in \mathrm{LocSys}^0_{G^\vee}(b')$, we have*

$$\mathrm{Aut}(E, \Psi(\nabla)) = J_{b^p}$$

where b^p is the image of b' under the map $f : B' \to B_{\omega^p}$.

7.3 Scheme of horizontal sections

Let X be any smooth connected scheme over an algebraic closed field of non-zero characteristic. A $\mathcal{D}_X$-scheme is an X-scheme equipped with a flat connection along X. A $\mathcal{D}_X$-algebra is a sheave of $\mathcal{O}_X$-algebra(quasi-coherent as $\mathcal{O}_X$-module) equipped with a flat connection which is compatible with the algebra structure. $\mathcal{D}_X$-schemes affine over X are spectra of commutative $\mathcal{D}_X$-algebras. Let $F : X \to X'$ be the relative Frobenius map.

Lemma 7.3.1 (Cartier descent).

1) For any quasi-coherent sheaf $\mathcal{L}$ on X', there is a canonical $\mathcal{D}_X$-module structure on $F^*(\mathcal{L})$ and the assignment $\mathcal{L} \to F^*(\mathcal{L})$ defines an equivalence between the category *of quasi-coherent sheaves on X' and the category of $\mathcal{D}_X$-modules on X with zero p-curvature. Moreover, the inverse functor is given by taking flat sections, which we denote it by ∇.*

2) Let $\mathcal{L}$ be a $\mathcal{O}_{X'}$-algebra on X'. Then there is a canonical $\mathcal{D}_X$-algebra structure on $F^*(\mathcal{L})$ and the assignment $\mathcal{L} \to F^*(\mathcal{L})$ defines an equivalence between the category of $\mathcal{O}_{X'}$-algebras on X' and the category of $\mathcal{D}_X$-algebra on X with zero p-curvature.

Proof. Part 1) is the standard Cartier descent. For part 2), it is enough to show that the functor F^* sends algebra objects to algebra objects. Let $(\mathcal{L}, m : \mathcal{L} \otimes_{X'} \mathcal{L} \to \mathcal{L})$ be an algebra object in Qcoh(X'). Then the back of m under F

$$F^*m : F^*\mathcal{L} \otimes_X F^*\mathcal{L} \to F^*\mathcal{L}$$

gives us an algebra structure on $F^*\mathcal{L}$. To show that F^*m is compatible with the $\mathcal{D}_X$-module structure, it is the same to show that for any $d \in \mathcal{D}_X$ we have

$$d(F^*m) = F^*m(d).$$

Let $a \otimes f_1$, $b \otimes f_2$ be elements in $F^*\mathcal{L} = \mathcal{L} \otimes_{X'} \mathcal{O}_X$ and $d \in \mathcal{D}_X$. We have

$$F^*m((a \otimes l_1) \otimes_X (b \otimes l_2)) = m(a \otimes b) \otimes l_1 l_2.$$

Therefore, $F^*m(d((a\otimes l_1)\otimes_X(b\otimes l_2))) = F^*m((a\otimes dl_1)\otimes_X(b\otimes l_2)+(a\otimes l_1)\otimes_X(b\otimes dl_2)) = m(a\otimes b)\otimes dl_1 l_2 + m(a\otimes b)\otimes l_2 dl_2 = m(a\otimes b)\otimes d(l_1 l_2) = d(F^*m((a\otimes l_1)\otimes_X(b\otimes l_2)))$. The proof is finished.

□

The following lemma is the characteristic p analogy of Proposition 2.6.2 in [BD] about scheme of horizontal sections of a $\mathcal{D}$-scheme.

Lemma 7.3.2.

1) The functor $\mathcal{M}' \to F^*\mathcal{M}'$ admits a left adjoint functor, i.e. for a $\mathcal{D}_X$-algebra $\mathcal{N}$

there is a $\mathcal{O}_{X'}$-algebra $H_\nabla(\mathcal{N})$ such that

$$\mathrm{Hom}_{\mathcal{D}_\mathrm{X}-\mathrm{alg}}(\mathcal{N}, \mathrm{F}^*\mathcal{M}') = \mathrm{Hom}_{\mathrm{X}'-\mathrm{alg}}(\mathrm{H}_\nabla(\mathcal{N}), \mathcal{M}')$$

for any $\mathcal{O}_{X'}$-algebra $\mathcal{M}'$.

2) The canonical map $\mathcal{N} \to F^*(H_\nabla(\mathcal{N}))$ is surjective.

Proof. Let ∇ be the connection of $\mathcal{N}$. We denote by Ψ the p-curvature of ∇. We can think of Ψ as a map $\Psi : F^*T_{X'} \otimes_X \mathcal{N} \to \mathcal{N}$. Let $\Psi(\mathcal{N})$ be the ideal of $\mathcal{N}$ generated by the image of Ψ and we defined $\mathcal{N}_\Psi = \mathcal{N}/\Psi(\mathcal{N})$. Since ∇ commutes with Ψ, the $\mathcal{O}_X$-algebra $\mathcal{N}_\Psi$ carries a connection and we define the following $\mathcal{O}_{X'}$-algebra

$$H_\nabla(\mathcal{N}) = (\mathcal{N}_\Psi)^\nabla$$

Let us show that $H_\nabla(\mathcal{N})$ satisfies our requirement. For any $\mathcal{O}_{X'}$-algebra $\mathcal{M}'$, the p-curvature of $F^*\mathcal{M}'$ is zero hence we have

$$\mathrm{Hom}_{\mathcal{D}_\mathrm{X}-\mathrm{alg}}(\mathcal{N}, \mathrm{F}^*\mathcal{M}') = \mathrm{Hom}_{\mathcal{D}_\mathrm{X}-\mathrm{alg}}(\mathcal{N}_\Psi, \mathrm{F}^*\mathcal{M}')$$

On the other hand, since the p-curvature of $\mathcal{N}_\Psi$ is zero, by Lemma 7.3.1, we have

$$\mathrm{Hom}_{\mathcal{D}_\mathrm{X}-\mathrm{alg}}(\mathcal{N}_\Psi, \mathrm{F}^*\mathcal{M}') = \mathrm{Hom}_{\mathrm{X}'-\mathrm{alg}}((\mathcal{N}_\Psi)^\nabla, \mathcal{M}') = \mathrm{Hom}_{\mathrm{X}'-\mathrm{alg}}(\mathrm{H}_\nabla(\mathcal{N}), \mathcal{M}').$$

Therefore

$$\mathrm{Hom}_{\mathcal{D}_\mathrm{X}-\mathrm{alg}}(\mathcal{N}, \mathrm{F}^*\mathcal{M}') = \mathrm{Hom}_{\mathrm{X}'-\mathrm{alg}}(\mathrm{H}_\nabla(\mathcal{N}), \mathcal{M}').$$

This proved part 1).

By Lemma 7.3.1 again, we have $\mathcal{N}_\Psi = F^*(H_\nabla(\mathcal{N}))$ and it implies the canonical map $\mathcal{N} \to \mathcal{N}_\psi = F^*(H_\nabla(\mathcal{N}))$ is surjective. This proved part 2) □

Remark. *Assume $\mathcal{N}$ is commutative. Then the $\mathcal{O}_{X'}$-algebra $H_\nabla(\mathcal{N})$ is commutative by 2). We called the X'-scheme*

$$N^\nabla := \mathrm{Spec}(\mathrm{H}_\nabla(\mathcal{N}))$$

the scheme of horizontal sections. Let N and N^{Ψ} be the $\mathcal{D}_X$-scheme associated to the commutative $\mathcal{D}_X$-algebra $\mathcal{N}$ and $\mathcal{N}_{\Psi}$. Above Lemma implies

$$N^{\Psi} = X \times_{X'} N^{\nabla}.$$

Recall that the affine smooth group scheme $J^{\vee}_{b^p}$ is isomorphic to $C \times_{C'} J^{\vee}_{b'}$, in particular, it is a $\mathcal{D}_C$-scheme. From above remark, the following corollary is immediate

Corollary 7.3.3. *The C'-scheme of horizontal sections of $J^{\vee}_{b^p}$ is isomorphic to $J^{\vee}_{b'}$, i.e. we have*

$$(J^{\vee}_{b^p})^{\nabla} = J^{\vee}_{b'}$$

For any $(b' : S' \to B'^0)$ and $(E, \nabla) \in \mathrm{LocSys}^0_{G^{\vee}}(b')$. Consider the sheaf of automorphism $\underline{\mathrm{Aut}}(E, \nabla)$ on C'_{et} associated to every $(f : U' \to C') \in C'_{et}$, the group $\underline{\mathrm{Aut}}(E, \nabla)(U') = \mathrm{Aut}(E_U, \nabla_U)$, where $(E_U, \nabla_U) = (f^*_U(E), f^*_U(\nabla))$ and $f_U : U_S \to C_S$.

On the other hand, let $J^{\vee}_{b'} \to S' \times C'$ be the commutative group scheme over $S' \times X'$. Let

$$\pi_{S' \times C'/C'} : S' \times C' \to C'$$

be the natural projection and we denote by

$$J^{\vee}_{b',et} := \mathrm{Res}_{S' \times C'/C'}(J^{\vee}_{b'})$$

to be the Weil restriction of $J^{\vee}_{b'}$ with respect to $\pi_{S' \times C'/C'}$. We regard it as a sheaf on C'_{et}.

The goal of this section is to show that $\underline{\mathrm{Aut}}(E, \nabla)$ is isomorphic to $J^{\vee}_{b',et}$ as sheaf on C'_{et}. In order to prove this, let me introduce another sheaf $\underline{\mathrm{Ad}}(E)^{\nabla}$ on C'_{et} associated to every $(f' : U' \to C') \in C'_{et}$ the set

$$\underline{\mathrm{Ad}}(E)^{\nabla}(U') = \underline{\mathrm{Hom}}_{U_S-alg}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})^{\nabla}(U_S) = \mathrm{Hom}_{\mathcal{D}_{U_S-alg}}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})$$

where ∇ is the induced connection on sheaf-hom $\underline{\mathrm{Hom}}_{U_S}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})$.

The following lemma is immediate

Lemma 7.3.4. *Let $b' \in B'^0(S')$. Let $(E, \nabla) \in \mathrm{LocSys}_{G^\vee}(b')$. Let $Ad(E)$ be the group scheme of automorphism of E over $S' \times C$. The connection ∇ on E will give $Ad(E)$ a $\mathcal{D}_{S' \times C/S'}$-scheme structure. Let $Ad(E)^\nabla$ be the $S' \times C'$-scheme of horizontal sections defined in Remark 7.3. Then the etale sheaf $\mathrm{Res}_{S' \times C'/C'}(Ad(E)^\nabla)$ on C' is isomorphic to $\underline{\mathrm{Ad}}(E)^\nabla$, here $\mathrm{Res}_{S' \times C'/C'}$ is the Weil restriction.*

Lemma 7.3.5. *$\underline{\mathrm{Ad}}(E)^\nabla$ is isomorphic to $\underline{\mathrm{Aut}}(E, \nabla)$ as sheaves on C'_{et}.*

Proof. Let $(f' : U' \to C') \in C'_{et}$. Since $\mathrm{Aut}(E) = \mathrm{Ad}(E)$, we can identify $\mathrm{Aut}(E)(U)$ with $\mathrm{Ad}(E)(U) = \mathrm{Hom}_{U_S-alg}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})$. So we have a natural inclusion

$$i : \mathrm{Ad}(E)^\nabla(U') \hookrightarrow \mathrm{Ad}(E)(U') = \mathrm{Aut}(E)(U')$$

Our goal is to show that i factors through $\underline{\mathrm{Aut}}(E, \nabla)(U')$ and induces an isomorphism with $\underline{\mathrm{Aut}}(E, \nabla)(U')$.

Let $m : G \times G \to G$ be the multiplication map. G carries a natural left G-action and there is a G-action on $G \times G$ defined by $g(x, y) = (gxg^{-1}, gy)$. The multiplication map m is G-equivariant under those actions and twisting m by E_U we get the action map

$$a : \mathrm{Ad}(E_U) \times_{U_S} E_U \to E_U$$

Let $a^* : \mathcal{O}_{E_U} \to \mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{\mathrm{Ad}_{E_U}}$ be the corresponding coaction map.

Under the identification $\mathrm{Aut}(E)(U') \simeq \mathrm{Hom}_{U_S-alg}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})$, saying that an element $g \in \mathrm{Aut}(E)(U')$ belongs to $\mathrm{Aut}(E, \nabla)(U')$ is equivalant to the following condition: The map $a(g) := g \circ act^* : \mathcal{O}_{E_U} \to \mathcal{O}_{E_U}$ preserves connection, i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{E_U} & \xrightarrow{a(g)} & \mathcal{O}_{E_U} \\ \downarrow \nabla & & \downarrow \nabla \\ \mathcal{O}_{E_U} \otimes_{U_S} \omega_{U_S/S} & \xrightarrow{a(g)\otimes 1} & \mathcal{O}_{E_U} \otimes_{U_S} \omega_{U_S/S} \end{array} \tag{7.1}$$

So we have to show that an element $g \in \mathrm{Hom}_{U_S}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})$ lies in the image of i if and only if $a(g)$ preserves connection.

Step 1$(\Rightarrow)$

Since m is G-equivariant, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{E_U} & \xrightarrow{a^*} & \mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{\mathrm{Ad}_{E_U}} \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla\otimes 1 + 1\otimes\nabla} \\ \mathcal{O}_{E_U} \otimes_{U_S} \omega_{U_S/S} & \longrightarrow & (\mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{\mathrm{Ad}_{E_U}}) \otimes_{U_S} \omega_{U_S/S} \end{array} \tag{7.2}$$

Let $g \in \mathrm{Hom}_{\mathrm{D}_{U_S}}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})$. Using the fact that g is a D_{U_S}-morphisim, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{\mathrm{Ad}_{E_U}} & \xrightarrow{1\otimes g} & \mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{U_S} \\ \downarrow{\scriptstyle \nabla\otimes 1 + 1\otimes\nabla} & & \downarrow{\scriptstyle \nabla\otimes 1 + 1\otimes d} \\ (\mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{\mathrm{Ad}_{E_U}}) \otimes_{U_S} \omega_{U_S/S} & \xrightarrow{1\otimes g} & (\mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{U_S}) \otimes_{U_S} \omega_{U_S/S} \end{array} \tag{7.3}$$

If we identify $\mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{U_S}$ with $\mathcal{O}_{E_U}$, the map $\nabla \otimes 1 + 1 \otimes d$ will become ∇, i.e. we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{\mathcal{O}_{E_U}} & \longrightarrow & \mathcal{O}_{E_U} \\ \downarrow{\scriptstyle \nabla\otimes 1 + 1\otimes d} & & \downarrow{\scriptstyle \nabla} \\ (\mathcal{O}_{E_U} \otimes_{U_S} \mathcal{O}_{\mathcal{O}_{E_U}}) \otimes_{U_S} \omega_{U_S/S} & \longrightarrow & \mathcal{O}_{E_U} \otimes_{U_S} \omega_{U_S/S} \end{array} \tag{7.4}$$

Composing (1),(2),(3) together we get the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{E_U} & \xrightarrow{a(g)} & \mathcal{O}_{E_U} \\ \downarrow \nabla & & \downarrow \nabla \\ \mathcal{O}_{E_U} \otimes_{U_S} \omega_{U_S/S} & \xrightarrow{a(g)\otimes 1} & \mathcal{O}_{E_U} \otimes_{U_S} \omega_{U_S/S} \end{array} \tag{7.5}$$

which shows that $a(g) \in \mathrm{Aut}(E, \nabla)(U)$

Step 2$(\Leftarrow)$

Let $g \in \mathrm{Hom}_{U_S}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})$ and assume that $a(g)$ preserves connection. Since diagram (1) always commutes, thus diagram (5) commutes will imply diagram (3) commutes and this means exactly $g \in \mathrm{Hom}_{\mathrm{D}_{U_S}}(\mathcal{O}_{\mathrm{Ad}(E_U)}, \mathcal{O}_{U_S})$.

□

We have a similar story for the corresponding F-Higgs bundle (E, Ψ), i.e., let $\mathrm{Ad(E)}^{\Psi}$ be the $\mathcal{D}_{S\times C}$-scheme associated to the $\mathcal{D}_{S\times C}$-algebra $(\mathcal{O}_{Ad(E)})_{\Psi}$ (see Remark 7.3 for the definition). We have a F-Higgs bundle analogy for Lemma 7.3.5:

Lemma 7.3.6. *Let $b' \in B'^0(S')$. Let $(E, \nabla) \in \mathrm{LocSys}^0_{G^\vee}(b')$. Let $\mathrm{Aut}(E, \Psi)$ be the group scheme of automorphism of (E, Ψ) over $S' \times C$. Then $\mathrm{Ad}(E)^{\Psi}$ is isomorphic to $\mathrm{Aut}(E, \Psi) = J^\vee_{a^p}$ as $S' \times C$-scheme.*

Remark. *Unlike the case of (E, ∇), where it only makes sense to define $\underline{\mathrm{Aut}}(E, \nabla)$ as a sheaf on X'_{et}. For a F-Higgs field (E, Ψ), the sheaf of automorphism of (E, Ψ) is in fact represented by a group scheme $\mathrm{Aut}(E, \Psi)$.*

Proof. The proof of the isomorphism between $\mathrm{Ad}(E)^{\Psi}$ and $\mathrm{Aut}(E, \Psi)$ is basically the same as the case (E, ∇). The isomorphism $\mathrm{Aut}(E, \Psi) \simeq J^\vee_{b^p}$ follows from Corollary 7.2.4. □

Proposition 7.3.7. *Let $b' \in B'^0(S')$ and $(E, \nabla) \in \mathrm{LocSys}^0_{G^\vee}(b')$. There is an isomorphism between $\underline{\mathrm{Aut}}(E, \nabla)$ and $J^\vee_{b',et}$ as sheaves on C'_{et}.*

Proof. Let $Ad(E)$ be the $\mathcal{D}_{S'\times C/S'}$-scheme of automorphism of E and let $Ad(E)^\nabla$ be the $(S' \times C')$-scheme of horizontal sections. By Lemma 7.3.5, we have

$$\underline{\mathrm{Aut}}(E, \nabla) = \mathrm{Res}_{S'\times C'/C'} Ad(E)^\nabla$$

as sheaves on C'_{et}. Thus it is enough to show that $Ad(E)^\nabla = J^\vee_{b'}$ as $(S' \times C')$-scheme. Notice that we have $Ad(E)^\nabla = (Ad(E)^\Psi)^\nabla$ as $(S' \times C')$-scheme, where $Ad(E)^\Psi$ is the $\mathcal{D}_{S'\times C/S'}$-scheme corresponds to the $\mathcal{D}_{S'\times C/S'}$-algebra $(\mathcal{O}_{Ad(E)})_\Psi$. Therefore, Lemma 7.3.6 and Corollary 7.3.3 implies

$$Ad(E)^\nabla = (Ad(E)^\Psi)^\nabla = (J^\vee_{b^p})^\nabla = J^\vee_{b'}.$$

□

7.4 Prove of Theorem 7.2.1

We are going to deduce Theorem 7.2.1 form the following lemma:

Lemma 7.4.1. *[DG, Lemma 14.2] Let X be a smooth scheme over k. Let $\mathcal{Q}$ be a sheaf of categories on X_{Et} (the big etale site of X), and $\mathcal{F}$ be a sheaf of abelian groups on X_{Et}. Suppose that for every $U \to X \in X_{Et}$ and every $C \in \mathcal{Q}(U)$, we are given an isomorphism $\mathrm{Aut}_{\mathcal{Q}(C)}(C) \simeq \mathcal{F}(U)$ such that the following conditions hold:*
0) There exists a covering $U \to X$ such that $\mathcal{Q}(U)$ is not empty.
1) If $C_1 \to C_2$ is an isomorphism between two object in $\mathcal{Q}(U)$, then the indued isomorphism $\mathrm{Aut}_{\mathcal{Q}(U)}(C_1) \simeq \mathrm{Aut}_{\mathcal{Q}(U)}(C_2)$ is compatible with the identification of both sides with $\mathcal{F}(U)$.
2) If $f : U' \to U$ is a morphism in X_{Et} and $C \in \mathcal{Q}(U)$, then the map

$$f^* : \mathrm{Aut}_{\mathcal{Q}(U)}(C) \to \mathrm{Aut}_{\mathcal{Q}(U')}(f^*C)$$

is compatible with the restriction map $\mathcal{F}(U) \to \mathcal{F}(U')$.
3) For any $U \to X \in X_{Et}$ and two $C_1, C_2 \in \mathcal{Q}(U)$, there exist a covering $f : U' \to U$

such that the objects f^*C_2 and f^*C_1 are isomorphic.

Then $\mathcal{Q}$ has a canonical structure of a gerbe over $\mathcal{F}$-torsors.

Proof of Theorem 7.2.1

Let us apply above Lemma to the case $X = B^{'0}$, $\mathcal{Q} = \mathrm{LocSys}^0_{G^\vee} \to B^{'0}$ and $\mathcal{F} = (J^\vee)^{'0} \to B^{'0}$. We claim that $\mathrm{LocSys}^0_{G^\vee} \to B^{'0}$ satisfied the conditions of the Lemma. Since our sheaf of categories $\mathrm{LocSys}^0_{G^\vee}$ is an algebraic stack, in particular, it is non-empty. This check 0). Condition 1) and 2) follow from Proposition 7.3.7. Let us check condition 3). Let $b' : S' \to B^{'0} \in (B^{'0})_{Et}$ and $(E_1, \nabla_1), (E_2, \nabla_2) \in \mathrm{LocSys}^0_G(b')$. We have to show that they are locally isomorphic, i.e., there exists an etale cover $f : U' \to S'$ such that $f^*((E_1, \nabla_1))$ is isomorphic to $f^*((E_2, \nabla_2))$. This condition is the same as saying that $\mathrm{LocSys}^0_{G^\vee}(b')$ is a $\mathrm{Bun}_{J^\vee_{b'}}$-torsor, where $\mathrm{Bun}_{J^\vee_{b'}}$ is the Picard category of $J^\vee_{b'}$-torsors. To prove this, it is enough to show that etale locally on $S' \times X'$, (E_1, ∇_1) and (E_2, ∇_2) are isomorphic. We can assume E_1 and E_2 are isomorphic to each other and let

$$Isom(E_1, E_2)$$

be the scheme over $S' \times X$ defines as follows: for a scheme $Y \to S' \times X$, we have

$$Isom(E_1, E_2)(Y) = Isom_{S' \times Y}(E_1, E_2).$$

Let $\underline{Isom}((E_1, \nabla_1), (E_2, \nabla_2))$ be the sheaf on $(S' \times X')_{et}$ defines by

$$\underline{Isom}((E_1, \nabla_1), (E_2, \nabla_2))(Y') = Isom_Y((E_1, \nabla_1), (E_2, \nabla_2)).$$

for any $Y' \to S' \times X' \in (S' \times X')_{et}$. Similarly, let $Isom((E_1, \Psi_1), (E_2, \Psi_2))$ be the $S' \times X$-scheme such that for any $Y \to S' \times X$ we have

$$Isom((E_1, \Psi_1), (E_2, \Psi_2))(Y) = Isom_Y((E_1, \Psi_1), (E_2, \Psi_2)).$$

Connections on E_1 and E_2 will induce a $\mathcal{D}_{S' \times C/S'}$-scheme structure on $Isom(E_1, E_2)$.

Let $Isom(E_1, E_2)^{\nabla_{isom}}$ be the $S' \times C'$-scheme of horizontal sections. We have

$$Isom(E_1, E_2)^{\nabla_{isom}} \simeq \underline{Isom}((E_1, \nabla_1), (E_2, \nabla_2))$$

as sheaves on $(S' \times C')_{et}$ and

$$Isom(E_1, E_2)^{\Psi_{isom}} \simeq Isom((E_1, \Psi_1), (E_2, \Psi_2))$$

as $S' \times C$-scheme where Ψ_{isom} is the p-curvature of ∇_{isom}. Thus, it is enough to show that, locally on $(S' \times C')_{et}$, the scheme $Isom(E_1, E_2)^{\nabla_{isom}}$ admits sections. By Lemma 7.3.1, we have the following Cartesian diagram

$$\begin{array}{ccc} Isom(E_1, E_2)^{\Psi_{isom}} & \longrightarrow & Isom(E_1, E_2)^{\nabla_{isom}} \\ \downarrow & & \downarrow \\ S' \times C & \xrightarrow{id \times F} & S' \times C' \end{array}$$

By the result in [DG], the scheme $Isom(E_1, E_2)^{\Psi_{isom}} = Isom((E_1, \Psi_1), (E_2, \Psi_2))$ is a torsor over $J^{\vee}_{b^p}$. In particular, it is smooth over $S' \times C$. Since $S' \times C \to S' \times C'$ is faithfully flat, it implies $Isom(E_1, E_2)^{\nabla_{isom}}$ is smooth over $S' \times C'$, therefore it admits sections locally on $S' \times C'$. This checked 3).

Chapter 8

Main result

Let D_{Bun_G} be sheaf of algebra on Higgs$'$ constructed in §4. Let $\mathscr{M}' = \text{Higgs}' \times_{B'} B'^0$ and $\text{LocSys}^0_{G^\vee} = \text{LocSys}_{G^\vee} \times_{B'} B'^0$. Let $D^0_{\text{Bun}_G} := D_{\text{Bun}_G}|_{\mathscr{M}'}$ be the restriction of D_{Bun_G} to $\mathscr{M}'$. Our goal is to prove the following Theorem:

Theorem 8.0.2. *We have a canonical equivalence of derived categories*

$$D(D^0_{\text{Bun}_G} - \text{mod}) \simeq D(\text{QCoh}(\text{LocSys}^0_{G^\vee}))$$

8.1 Proof of Theorem 8.0.2

The main Theorem will follow from following two propositions.

Proposition 8.1.1. *Let $k : \mathscr{P}' \simeq \mathscr{M}'$ be the trivialization given by Kostant section. We denote by $D := k^* D^0_{\text{Bun}_G}$. The sheaf of Azumaya algebra D on $\mathscr{P}'$ has a natural commutative group structure with respect to the multiplication morphism*

$$m : \mathscr{P}' \times_{B'^0} \mathscr{P}' \to \mathscr{P}'.$$

Proposition 8.1.2. *The $\breve{\mathscr{P}}' \simeq (\mathscr{P}')^\vee$-torsor $\mathscr{T}_D$ of multiplicative splittings of D is isomorphic to* $\text{LocSys}^0_{G^\vee}$.

The main Theorem follows immediately from above two propositions. Indeed, by

Lemma 5.2.1 we have

$$D(D^0_{\mathrm{Bun}_G} - mod) \simeq D(\mathrm{QCoh}(\mathscr{T}_D)) \simeq D(\mathrm{QCoh}(\mathrm{LocSys}^0_{G^\vee})).$$

8.2 Proof of Proposition 8.1.1

In this section we show that the sheaf of Azumaya algebra D on $\mathscr{P}'$ has a natural commutative group structure.

Let us fixed $b' \in B'^0(k)$ and let $b \in B'^0(k)$ be its image in B. Considering the natural projection

$$v_b : \mathscr{P}_b \to \mathrm{Bun}_G .$$

We have the (Frobenius twist) cotangent morphism of v_b

$$\begin{array}{ccc} (v_b')_d : T^* \mathrm{Bun}_G' \times_{\mathrm{Bun}_G'} \mathscr{P}_{b'} & \longrightarrow & T^* \mathscr{P}_{b'} . \\ \big\downarrow (v_b')_p & & \\ T^* \mathrm{Bun}_G' & & \end{array}$$

Let $i_{b'} : \mathscr{P}_{b'} \to T^* \mathrm{Bun}_G' \times_{\mathrm{Bun}_G'} \mathscr{P}_{b'}$ be the natural embedding. Considering the restriction of above cotangent morphism to $\mathscr{P}_{b'}$:

$$\begin{array}{ccc} \mathscr{P}_{b'} & \xrightarrow{(v_b')_d \circ i_{b'}} & T^* \mathscr{P}_{b'} \\ \big\downarrow (v_b')_p \circ i_{b'} & & \\ T^* \mathrm{Bun}_G' & & \end{array}$$

It is clear that the composition $(v_b')_p \circ i_{b'}$ is the natural embedding.

Proposition 8.2.1. *Under the natural isomorphism $T^* \mathscr{P}_{b'} \simeq \mathscr{P}_{b'} \times B'$ the cotangent morphism $(v_b')_d \circ i_{b'}$ is the graph of a constant map*

$$\mathscr{P}_{b'} \longrightarrow k \xrightarrow{\tilde{b}'} B' .$$

Equivalently, we have the following Cartesian diagram .

$$\begin{array}{ccc} \mathscr{P}_{b'} & \xrightarrow{(v'_b)_d \circ i_{b'}} & \mathscr{P}_{b'} \times B' \, . \\ \big\downarrow & & \big\downarrow \\ k & \xrightarrow{\tilde{b}'} & B' \end{array}$$

Definition 8.2.2. We define $\Delta : B'^0 \to B^0 \times B'$ to be the "diagonal" morphism given by $\Delta(b') = (b, \tilde{b}')$.

Remark. *We believe that $\tilde{b}' = b'$, therefore Δ is really the diagonal morphism. But at this moment, we can not prove this.*

Corollary 8.2.3.

1. *There is an natural isomorphism of Azumaya algebra on $\mathscr{M}_{b'}$:*

$$D|_{\mathscr{P}_{b'}} \simeq D_{\mathscr{P}_b}|_{\mathscr{P}_{b'} \times b'}$$

2. *$D|_{\mathscr{P}_{b'}}$ has a natural commutative group structure.*

Proof. Part (1) follows from Lemma 4.1.2. Part (2) follows form above Proposition. Indeed, since the Azumaya algebra $D_{\mathscr{P}_b}$ on the total space of the abelian fibration $T^*\mathscr{P}_{b'} \to B'$ has a commutative group structure, in particular, its restriction to the fiber $\mathscr{P}_{b'} \times b'$ has a commutative group structure. $\square$

Proof of Proposition 8.2.1

For any $x \in \mathscr{P}_b$ we denote by P_x the corresponding J_b-torsor and (E, ϕ) the corresponding element in $\mathscr{M}_b$ under the isomorphism $\mathscr{P}_b \simeq \mathscr{M}_b$. Considering the differential of v_b at x:

$$(v_b)_{d,x} : T^*_E \operatorname{Bun}_G \to T^*_x \mathscr{P}_b = B.$$

We have to show that

$$(v_b)_d(i_b(x)) = (v_b)_{d,x}(\phi) \in B$$

is independent of $x \in \mathscr{P}_b$.

We first give a description of dual of $(v_b)_{d,x}$. Recalling that $\mathscr{P}_b = \mathrm{Bun}_{J_b}$ and we have a natural isomorphism between $T\mathscr{P}_b$ and $\mathscr{P}_b \times H^1(C, LieJ_b)$. We denote by $d\pi_x$ the dual of $(v_b)_{d,x}$:

$$d\pi_x : H^1(C, LieJ_b) \to H^1(C, ad(E)) \simeq T_E \,\mathrm{Bun}_G\,.$$

Let $(E_b, \phi_b) \in \mathscr{M}_b$ be the Higgs bundle given by Kostant section. By definition the corresponding J_b-torsor $x_b \in \mathscr{P}_b$ is the trivial J_b-torsor.

Then by the result in [N2], we have the following short exact sequence

$$0 \longrightarrow LieJ_b \xrightarrow{j_b} ad(E_b) \xrightarrow{ad(\phi_b)} ad(E_b) \otimes_C \omega_C.$$

Moreover, for any $x = (E, \phi) \in \mathscr{P}_b$ we have an isomorphism

$$E \simeq E_b \times^{J_b} P_x$$

and the Higgs field

$$\phi : C \to ad(E) \times_C^{\mathbb{G}_m} \omega_C$$

is obtained from

$$\phi_b : C \to ad(E_b) \times_C^{\mathbb{G}_m} \omega_C$$

by twisting with P_x (J_b acts trivially on C).

Twisting j_b by P_x, we get a morphism

$$LieJ_b \xrightarrow{j_x} ad(E)\,.$$

Lemma 8.2.4.

1. *The morphism $d\pi_x : H^1(C, LieJ_b) \to H^1(C, ad(E))$ is induced by the morphism $j_x : LieJ_b \to ad(E)$.*

2. *Let $(v_b)_{d,x} : T_E^* \mathrm{Bun}_G = H^0(C, ad(E) \otimes_C \omega_C) \to H^0(C, LieJ^* \otimes_C \omega_C) \simeq B$ be the dual of $d\pi_x$. Then the image of $\phi \in H^0(C, ad(E) \otimes_C \omega_C)$ in $H^0(C, (LieJ_b)^* \otimes_C$*

$\omega_C)$ *is independent of* $x = (E, \phi) \in \mathscr{P}_b$.

Proof. Part (1) follows from the definition of $d\pi_x$ and the isomorphism $T\mathscr{P}_b \simeq \mathscr{P}_b \times H^1(C, LieJ_b)$. Let us prove part (2). By definition, $(v_b)_{d,x}(\phi_b) \in B$ corresponds to the following map

$$C \xrightarrow{\phi_b} ad(E_b) \times^{\mathbb{G}_m} \omega_C \xrightarrow{j_b^* \otimes \omega_C} (LieJ_b)^* \times^{\mathbb{G}_m} \omega_C. \tag{8.1}$$

On the other hand, for any $x \in \mathscr{P}_b$, the morphism

$$C \xrightarrow{\phi} ad(E) \times^{\mathbb{G}_m} \omega_C \xrightarrow{j_x^* \otimes \omega_C} (LieJ_b)^* \times^{\mathbb{G}_m} \omega_C. \tag{8.2}$$

corresponding to $(v_b)_{d,x}(\phi) \in B$ is obtained by twisting (8.1) by the J_b-torsor P_x. Since the action of J_b on C and $(LieJ_b)^* \times^{\mathbb{G}_m} \omega_C$ is trivial it implies $(v_b)_{d,x_b}(\phi_b) = (v_b)_{d,x}(\phi)$. □

Remark. *If the group G is semisimple and simply connected, then above Corollary also follows from the fact that the Hitchin fiber $\mathscr{P}_b$ is connected and proper, therefore, any morphism from $\mathscr{P}_b$ to the affine scheme B is constant.*

Considering the Hitchin fibration

$$h : \mathscr{P}^0 \to B^0.$$

Let $S := \mathrm{Sect}_B(\mathscr{P}'_e, T^*(\mathscr{P}'_e/B^0))$ be the B^0-scheme of section of $T^*(\mathscr{P}'_e/B)$ over $\mathscr{P}'^0_e$. We have a natural isomorphism

$$T^*(\mathscr{P}'/B^0) \simeq \mathscr{P}' \times_{B'^0} S$$

and a sheaf of Azumaya algebra $D_{\mathscr{P}'/S}$ on the total space of the abelian fibration

$$T^*(\mathscr{P}'/B) \to S.$$

Lemma 8.2.5. *There is a natural isomorphism*

$$S \simeq B^0 \times B'.$$

Proof. Let us consider the following general setting. Let $\pi : Y \to A$ be an abelian scheme with section $\epsilon : A \to Y$. Let $T^*(Y/A) \to Y$ be the relative cotangent. By the result of [Lau, Lemma 1.1.3], we have

$$\mathrm{Res}_{Y/A}(T^*(Y/A)) \simeq \epsilon^*(T^*(Y/A)) = T^*(Y/A) \times_Y A$$

Therefore, for any $U \to A$, we have

$$\mathrm{Sect}_A(Y, T^*(Y/A))(U) = \mathrm{Hom}_Y(Y_U, T^*(Y/A)) = \mathrm{Hom}_A(U, \mathrm{Res}_{Y/A} T^*(Y/A)) = \mathrm{Hom}_A(U, T^*(Y/A) \times_Y A)$$

Applying above setting to our abelian fibration $h'|_{\mathscr{P}'_e} : \mathscr{P}'_e \to B$ and ϵ is the kostant section, we get

$$S(U) = \mathrm{Hom}_B(U, T^*(\mathscr{P}'_e/B) \times_{\mathscr{P}'} B).$$

By Proposition 3.3.2, we have

$$T^*(\mathscr{P}'_e/B) \times_{\mathscr{P}'} B^0 \simeq B^0 \times B'$$

Thus, $\mathrm{Hom}_B(U, T^*(\mathscr{P}'_e/B^0) \times_{\mathscr{P}'} B^0) = \mathrm{Hom}_B(U, B^0 \times B')$. This finished the proof. □

Considering the morphism over B^0

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{v} & \mathrm{Bun}_G \times B^0 \\ & \searrow^{h} & \downarrow \\ & & B^0 \end{array}$$

given by $v(E, \phi) = E \times h(\phi)$.

We have

$$T^*((\mathrm{Bun}_G \times B^0)'/B^0) \simeq \mathscr{P}' \times B^0$$

and the (relative) cotangent morphism of v:

$$\begin{array}{ccc} (\mathscr{P}' \times B^0) \times_{\mathrm{Bun}'_G \times B} \mathscr{P}' & \xrightarrow[v'_d]{} & T^*(\mathscr{P}'/B^0)\,. \\ \downarrow{\scriptstyle v'_p} & & \\ \mathscr{P}' \times B^0 & & \end{array}$$

Let $i : \mathscr{P}' \to (\mathscr{P}' \times B^0) \times_{\mathrm{Bun}'_G \times B} \mathscr{P}'$ be the embedding given by $i(E, \phi) = (E, \phi, h'(\phi), E, \phi)$. By restriction of the cotangent morphism to $\mathscr{P}'$, we obtained

$$v'_d \circ i : \mathscr{P}' \to T^*(\mathscr{P}'/B^0)$$

$$v'_p \circ i : \mathscr{P}' \to \mathscr{P}' \times B^0.$$

Proposition 8.2.6.

1. *The morphism $v'_p \circ i$ is the graph of the Hitchin morphism .*

2. *We have the following Cartesian diagram*

$$\begin{array}{ccc} \mathscr{P}' & \xrightarrow{v'_d \circ i} & T^*(\mathscr{P}'/B^0) \\ {\scriptstyle h'}\downarrow & & \downarrow \\ B'^0 & \xrightarrow{\Delta} & S = B^0 \times B' \end{array}$$

   i.e., the Hitchin morphism $h' : \mathscr{M}' \to B'$ is the restriction of the abelian fibration $T^*(\mathscr{P}'/B^0) \to B^0 \times B'$ to the "diagonal".

Proof. Part one follows from definition. The proof of part (2) is a relative version of the argument in Lemma 8.2.1.

□

Theorem 8.2.7. *We have a natural isomorphism of sheaves of Azumaya algebra on $\mathscr{P}'$*

$$D \simeq (v'_d \circ i)^*(D_{\mathscr{P}'/B^0})$$

and the commutative group structure on $D_{\mathscr{P}'/B^0}$ (constructed in Section 5.5) induces a commutative group structure on D.

Proof. Considering $D_{\mathrm{Bun}_G \times B/B} = D \boxtimes \mathcal{O}_B$ on $T^*((\mathrm{Bun}_G \times B)'/B) = T^* \mathrm{Bun}'_G \times B$. By Lemma 4.1.2, we have

$$D_{\mathrm{Bun}_G} \simeq (v'_p \circ i)^*(D \boxtimes \mathcal{O}_B) \simeq (v'_d \circ i)^*(D_{\mathscr{P}'/B^0}).$$

This prove the first claim. The commutative group structure on D follows from part (2) of previous Proposition. □

We denote by

$$\pi : T^*(\mathscr{P}'/B^0) \to B^0 \times B' \tag{8.3}$$

the natural projection and

$$\pi^\vee : (\mathscr{P}^0)^\natural \to B^0 \times B' \tag{8.4}$$

be the p-curvature morphism where $(\mathscr{P})^\natural$ be the universal extension of $\mathscr{P}$. We denote by $(\mathscr{P})^\sharp := (\mathscr{P})^\natural|_\Delta$ be its restriction to the "diagonal".

Corollary 8.2.8. *We have an isomorphism of $(\mathscr{P}')^\vee$-torsors*

$$(\mathscr{P})^\sharp \simeq \mathscr{T}_D$$

where $\mathscr{T}_D$ is the $(\mathscr{P}')^\vee$-torsor of multiplicative splittings of D. In particular, we have an equivalence

$$D(D_{\mathrm{Bun}_G} - \mathrm{mod}) \simeq D(\mathrm{QCoh}((\mathscr{P})^\sharp))$$

8.3 Proof of Proposition 8.1.2

8.4 The stack $\mathrm{LocSys}_{F^*J^\vee}$

Let $b' \in B'^0(U)$ and $b^p \in B_{\omega^p}(U)$ be its p-the power. We denote by $(E_{b'}, \phi_{b'}) \in \mathscr{M}_{b'}$ the element given by Kostant section.

By Lemma 7.1.3, we have $J^\vee_{b^p} = F^* J^\vee_{b'}$, therefore, the group scheme $J^\vee_{b^p}$ carries a connection. By Appendix A, we can define $\mathrm{LocSys}_{J^\vee_{b^p}}$ the stack of $J^\vee_{b^p}$-local systems.

Considering the p-curvature map

$$h_{J^\vee_{b^p}} : \mathrm{LocSys}_{J^\vee_{b^p}} \to B_{J^\vee_{b^p}}$$

where

$$B_{J^\vee_{b^p}} = \Gamma(C, \mathrm{Lie}(J^\vee_{b^p}) \otimes_C \omega_C^p).$$

Since we have

$$Lie(J^\vee_{b^p}) \otimes_C \omega_C^p \simeq F^*(Lie(J^\vee_{b'}) \otimes_C \omega_C)$$

and the image of f_{b^p} is flat for the induced connection on $F^*(Lie(J^\vee_{b'}) \otimes_X \omega_X)$, we see that the map $h_{J^\vee_{b^p}}$ factors through a map

$$h_{J^\vee_{b'}} : \mathrm{LocSys}_{J^\vee_{b^p}} \to B_{J^\vee_{b'}}$$

where $B_{J^\vee_{b'}} = \Gamma(C', Lie(J^\vee_{b'}) \otimes_{C'} \omega_{C'})$.

We have the following lemma:

Lemma 8.4.1. *1. $B_{J^\vee_{b'}}$ is one dimensional.*

2. Let $l : B_{J^\vee_{b'}} \to H^0(C', \mathrm{ad}(E_{b'}) \otimes_{C'} \omega_{C'})$ be the map induced by

$$Lie(J^\vee_{b'}) \otimes_{C'} \omega_{C'} \to \mathrm{ad}(E_{b'}) \otimes_{C'} \omega_{C'}.$$

Then the Higgs field $\phi_{b'}$ is in the image of l.

Proof. Let of first part (1). We have an isomorphism

$$Lie(J^\vee_{b'}) \otimes_{C'} \omega_{C'} \simeq \oplus \omega_{C'}^{-e_i^\vee+2}.$$

Since the first exponent $e_1^\vee = 2$ and $e_i^\vee > 2$ for $i > 1$, we have

$$B_{J_{b'}} \simeq H^0(C', \oplus \omega_{C'}^{-e_i^\vee+2}) \simeq H^0(C', \mathcal{O}_{C'}) = k.$$

For part (2), recall that we have an exact sequence

$$0 \longrightarrow Lie(J_{b'}^{\vee}) \longrightarrow \mathrm{ad}(E_{b'}) \xrightarrow{\mathrm{ad}(\phi_{b'})} \mathrm{ad}(E_{b'}) \otimes_{C'} \omega_{C'} .$$

Therefore, $Lie(J_{b'}^{\vee}) \otimes \omega_{C'}$ is the kernel of

$$\mathrm{ad}(\phi_{b'}) \otimes \omega_{C'} : \mathrm{ad}(E_{b'}) \otimes \omega_{C'} \to \mathrm{ad}(E_{b'}) \otimes_{C'} \omega_{C'}^2.$$

It implies the kernel of $\Gamma(\mathrm{ad}(\phi_{b'}) \otimes \omega_{C'})$ is equal to $B_{J_{b'}^{\vee}}$. However, it is clear that $\phi_{b'}$ is in the kernel $\Gamma(\mathrm{ad}(\phi_{b'}) \otimes \omega_{C'})$, hence it is in the image of l. □

For any $x \in B_{J_{b'}^{\vee}}$, we denote by

$$\mathrm{LocSys}_{J_{b^p}^{\vee}}(x) = (f_{b'})^{-1}(x)$$

the fiber of $f_{b'}$ over x.

Let $(E', \Psi') \in (\mathscr{M}_{G^{\vee}})_{b'}$ be the element given by Kostant section. Let $(E, \Psi) \in (\mathscr{M}_{G^{\vee}})_{b^p}$ be its Frobenius pullback. By the result in A.5, E has a structure of $(\mathrm{Aut}(E) \times G^{\vee})$-bitorsor and it carries a canonical connection ∇_E. Recall that we have a natural isomorphism $J_{b^p}^{\vee} \simeq \mathrm{Aut}(E, \phi)$. Thus we have a natural map of group scheme

$$k_{b^p} : J_{b^p}^{\vee} \to \mathrm{Aut}(E).$$

This map is horizontal because it is the Frobenius pullback of the natural map

$$k_{b'} : J_{b'}^{\vee} \to \mathrm{Aut}(E').$$

By the formalism in A.5, we can form the induction functor

$$(k_{b^p})_* : \mathrm{LocSys}_{J_{a^p}^{\vee}} \to \mathrm{LocSys}_{G^{\vee}}$$

from $J_{b^p}^{\vee}$-torsors to $G^{\vee}$-torsors using the $(\mathrm{Aut}(E) \times G^{\vee})$-bitorsor E. The following Lemma follows from the result in A.6

Lemma 8.4.2. *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{LocSys}_{J^\vee_{b^p}} & \xrightarrow{(k_{b^p})_*} & \mathrm{LocSys}_{G^\vee} \\ \downarrow{\scriptstyle h_{J^\vee_{b'}}} & & \downarrow{\scriptstyle h_p} \\ B_{J^\vee_{b'}} & \xrightarrow{\quad i \quad} & B' \end{array} .$$

where i is induced by the morphism

$$Lie(J^\vee_{b'}) \otimes \omega_{C'} \longrightarrow \mathrm{ad}(E_{b'}) \otimes \omega_{C'} \longrightarrow \mathfrak{c}_{\omega_{C'}} .$$

Proposition 8.4.3.

1. *The map $i : B_{J^\vee_{b'}} \to B'$ is injective and the image of $\phi_{b'}$ is equal to b'.*

2. $\mathrm{LocSys}_{J^\vee_{b^p}}(\phi_{b'})$ *is non-empty and the induction map $(k_{b^p})_*$ induced an equivalence of $(\mathscr{M}_{G^\vee})_{b'}$-torsors*

$$(k_{b^p})_* : \mathrm{LocSys}_{J^\vee_{b^p}}(\phi_{b'}) \to \mathrm{LocSys}_{G^\vee}(b')$$

Proof. Let us prove (1). Let me first show that i is injective. Notice that there are $\mathbb{G}_m$-action on $B_{J^\vee_{b'}}$ and B' and the map i is $\mathbb{G}_m$-equivariant. Since $B_{J^\vee_{b'}}$ is one dimensional by Lemma 8.4.1, there are exactly two $\mathbb{G}_m$-orbits on $B_{J^\vee_{b'}}$. Therefore, to show that i is injective it is enough to show that i is not a zero map. But since $\phi_{b'}$ is given by Kostant section, it is clear that the image of $\phi_{b'}$ in B' is b', hence i is not a zero map.

Proof of part (2). It is enough to show that $\mathrm{LocSys}_{J^\vee_{b^p}}(\phi_{b'})$ is non-empty. Indeed, since it implies $\mathrm{LocSys}_{J^\vee_{b^p}}(\phi_{b'})$ and $\mathrm{LocSys}^\vee_G(b')$ are torsors over $(\mathscr{M}_{G^\vee})_{b'}$ and the induction map $(k_{b^p})_*$ is compatible with the action of $(\mathscr{M}_{G^\vee})_{b'}$.

We will construct an object (P_1, ∇_1) in $\mathrm{LocSys}_{J^\vee_{b^p}}$ such that the $G^\vee$-local system $(k_{b^p})_*(P_1, \nabla_1)$ maps to b' under the map h_p. By part (1), it implies the image of (P_1, ∇_1) under $h_{J^\vee_{b'}}$ is $\phi_{b'}$.

Let (P, ∇_P) be any element in $\mathrm{LocSys}_{G^\vee, b'}$. We define P_1 to be the following scheme

$$P_1 := Isom((E, \Psi), (P, \Psi_P))$$

Since the pair (P, Ψ_P) is regular by Lemma 7.2.3, the scheme P_1 has a natural structure of $J^{\vee}_{b^p} = \mathrm{Aut}(E, \nabla)$-torsor and there is an canonical isomorphism of $G^{\vee}$-torsors

$$P_1 \times^{J^{\vee}_{b^p},k} E \simeq P$$

Claim:

The connection ∇ on P descends to a connection ∇_1 on $P_1 = Isom((E, \Psi), (P, \Psi_P))$.

Assuming this claim, we have constructed an element $(P_1, \nabla_1) \in \mathrm{LocSys}_{J^{\vee}_{b^p}}$ satisfying the the property we want. The proof is completed. □

Lemma 8.4.4 (Proof of above claim). *The connection ∇ on P descends to a connection ∇_1 on $P_1 = Isom((E, \Psi), (P, \Psi_P))$.*

Proof. It is clear that the flat connections ∇_E, ∇_P on E and P induce a flat connection ∇_1 on the scheme $Isom(E, P)$. Since ∇_E commutes with Ψ(because Ψ is a Frobenius pullback of some Higgs field Ψ') and ∇_P commutes with its p-curvature Ψ_P (standard property of p-curvature), we see that the flat connection ∇_1 induces a flat connection on $P_1 = Isom((E, \Psi), (P, \Psi_P)) \subset Isom(E, P)$, which we still denote it by ∇_1. Using Grothendieck perspective of flat connection (see A.3), it is clear that the induced connection on $P \simeq P_1 \times^{J^{\vee}_{b^p},k} E$ is the original flat connection ∇_P. This finished the proof. □

Let $J^{\vee} \to C \times B$ be the universal regular centralizer. Considering the following Cartesian diagram

$$\begin{array}{ccc}
F^*J^{\vee} & \longrightarrow & J \\
\downarrow & & \downarrow \\
C_{B'} = C \times B' & \longrightarrow & C_B = C \times B \\
\downarrow & & \downarrow \\
C & \xrightarrow{Fr} & C
\end{array} .$$

Let $\widetilde{\mathrm{LocSys}}_{F^*J^\vee}$ be the stack of $F^*J^\vee$-torsors on $C_{B'}$ with flat connection along C. We have a natural morphism

$$h_{J^\vee} : \widetilde{\mathrm{LocSys}}_{F^*J^\vee} \to B'.$$

Moreover, in the proof of the claim we have constructed an embedding from $\mathrm{LocSys}^0_{G^\vee}$ to $\widetilde{\mathrm{LocSys}}^0_{F^*J^\vee}$ over $B^{0'}$:

$$\begin{array}{ccc} \mathrm{LocSys}^0_{G^\vee} & \xrightarrow{\mathrm{Ind}_1} & \widetilde{\mathrm{LocSys}}^0_{F^*J^\vee} \\ & {\scriptstyle h_p} \searrow & \downarrow {\scriptstyle \tilde{h}_{J^\vee}} \\ & & B^{0'} \end{array}.$$

We denote by $\mathrm{LocSys}^0_{F^*J^\vee}$ the image of Ind_1 and let

$$h_{J^\vee} : \mathrm{LocSys}_{F^*J^\vee} \to {B'}^0$$

be the restriction of $\tilde{h}_{J^\vee}$ on $\mathrm{LocSys}_{F^*J^\vee}$. The fiber of $b' \in {B'}^0$ over $h_{J^\vee}$ is isomorphic to $\mathrm{LocSys}_{J^\vee_{b^p}}(\phi_{b'})$.

Theorem 8.4.5. *The functor* Ind_1 *induces an isomorphism of* $(\breve{\mathscr{P}})'^0$*-torsors*

$$\mathrm{LocSys}^0_{G^\vee} \simeq \mathrm{LocSys}^0_{F^*J^\vee}.$$

By abuse of notation we still denote the isomorphism by Ind_1.

8.5 The stack $\mathrm{LocSys}^{\mathrm{W}}_{T^\vee}$

Let $\mathrm{LocSys}^{\mathrm{W}}_{T^\vee} := \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}(\widetilde{C}^0/B^0)$ be the stack of relative W-equivariant $T^\vee$-local system, i.e., for any affine k-scheme U, the groupioid $\mathrm{LocSys}^{\mathrm{W}}_{T^\vee}(U)$ is consisting of triple (b, P, ∇_P), where $b : U \to B^0$ and (P, ∇_P) is a W-equivariant $T^\vee$ bundle on $\widetilde{C}_b$ with a W-equivariant flat connection ∇_P.

For any $(b, P, \nabla_P) \in \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}(U)$, the p-curvature of ∇_P, we denote it by $\Psi(\nabla_P)$, is an element in

$$H^0(\widetilde{C}'_U, \mathfrak{t}^\vee \otimes \omega_{\widetilde{C}'_U/U})^{\mathrm{W}}.$$

The assignment $(b, P, \nabla_P) \to \Psi(\nabla_P)$ defines a functor

$$\Psi : \mathrm{LocSys}^{\mathrm{W}}_{T^\vee} \to \mathrm{Sect}^{\mathrm{W}}_{B^0}(\widetilde{C}'^0, \mathrm{Tot}(\mathfrak{t}^\vee \otimes \omega_{\widetilde{C}'^0/B^0}))$$

where $\mathrm{Sect}^{\mathrm{W}}_{B^0}(\widetilde{C}'^0, \mathrm{Tot}(\mathfrak{t}^\vee \otimes \omega_{\widetilde{C}'^0/B^0}))$ is the B^0-scheme of W-invariant section of $\mathrm{Tot}(\mathfrak{t}^\vee \otimes \omega_{\widetilde{C}'^0/B^0})$ over $\widetilde{C}'^0$. We are going to show that $\mathrm{Sect}^{\mathrm{W}}_{B^0}(\widetilde{C}'^0, \mathrm{Tot}(\mathfrak{t}^\vee \otimes \omega_{\widetilde{C}'^0/B^0}))$ is isomorphic to $B^0 \times B'$.

Let $\pi_B : \widetilde{C} \to B$ be the universal cameral curve over B. and let $\pi'_B : \widetilde{C} \to B$ be its Frobenius twist. Let π_{B^0} and π'_{B^0} be their restriction to B^0 respectively.

Lemma 8.5.1. *We have a natural isomorphism*

$$(\pi'_{B^0})_*(\mathfrak{t}^\vee \otimes_k \omega_{\widetilde{C}'^0/B^0})^{\mathrm{W}} \simeq \mathcal{O}_{B^0} \otimes_k B'.$$

Proof. Let me write $I = (\pi'_{B^0})_*(\mathfrak{t}^\vee \otimes_k \omega_{\widetilde{C}'^0/B^0})^{\mathrm{W}}$. Let us first show that I is locally free on B^0. By cohomology of base chang, it is enough to show that the dimension of fibers of I is constant. For any $b \in B^0(k)$, using the isomorphism $\mathfrak{t}^\vee \simeq \mathfrak{t}^*$, we have

$$I_b = H^0(\widetilde{C}_{b'}, \mathfrak{t}^\vee \otimes \omega_{\widetilde{C}_{b'}})^{\mathrm{W}} \simeq (H^1(\widetilde{C}_{b'}, \mathfrak{t})^*)^{\mathrm{W}} \simeq H^1(C', Lie(J_{b'}))^* \simeq B'$$

which is constant. Therefore it is locally free on B^0.

Since $B \simeq \mathbb{A}^n_k$. To show that I is free over B^0, it is enough to show that there is a locally free extension $\widetilde{I}$ of I to B. We claim that

$$\widetilde{I} := ((R^1\pi'_B)_*(\mathfrak{t})^{\mathrm{W}})^\vee$$

will do the job. We first show that $\widetilde{I}$ is locally free. It is enough to show that $(R^1\pi'_B)_*(\mathfrak{t})^{\mathrm{W}}$ is locally free. But for any $b \in B(k)$, we have

$$\widetilde{I}_b = H^1(\widetilde{C}_b, \mathfrak{t})^{\mathrm{W}} \simeq H^1(C, (\pi_b)_*(\mathfrak{t})^{\mathrm{W}}) \simeq H^1(C, Lie(J_b))$$

which is independent of b by Proposition 3.3.2. Therefore, $\widetilde{I}$ is locally free.

Then we show that $\widetilde{I}|_{B^0} \simeq I$. But it follows from (relative) Grothedieck-Serre duality:

$$\widetilde{I}|_{B^0} \simeq ((R^1\pi'_B)_*(\mathfrak{t})^{\mathrm{W}})^\vee|_{B_0} \simeq (\pi'_{B^0})_*(\mathfrak{t}^* \otimes \omega_{\widetilde{C}^0/B^0})^{\mathrm{W}} \simeq I.$$

This finished the proof. □

Corollary 8.5.2. *We have a natural isomorphism*

$$\mathrm{Sect}^{\mathrm{W}}_{B^0}(\widetilde{C}'^0, \mathrm{Tot}(\mathfrak{t}^\vee \otimes \omega_{\widetilde{C}'^0/B^0})) \simeq B^0 \times B'.$$

Proof. For any $b : U \to B^0$ we denote by $\widetilde{C}'_U$ to be the fiber product

$$\begin{array}{ccc} \widetilde{C}'_U & \longrightarrow & \widetilde{C}'^0 \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi'_{B^0}} \\ U & \xrightarrow{b} & B^0 \end{array}.$$

Using above lemma we see that

$$\mathcal{O}_U \otimes_k B' \simeq b^*((\pi'_{B^0})_*(\mathfrak{t}^\vee \otimes_k \omega_{\widetilde{C}'^0/B^0})^{\mathrm{W}}) \simeq (\pi_U)_*(\mathfrak{t}^\vee \otimes \omega_{\widetilde{C}_U/U})^{\mathrm{W}}.$$

Therefore, we have

$$\mathrm{Sect}^{\mathrm{W}}_{B^0}(\widetilde{C}'^0, \mathrm{Tot}(\mathfrak{t}^\vee \otimes \omega_{\widetilde{C}'^0/B^0}))(U) = H^0(U, (\pi_U)_*(\mathfrak{t}^\vee \otimes \omega_{\widetilde{C}_U}/U)^{\mathrm{W}}) = H^0(U, \mathcal{O}_U) \times B' = \mathrm{Sect}_{B^0}(U, B^0 \times B').$$

□

Using above Corollary, we have the following map

$$\Psi : \mathrm{LocSys}^{\mathrm{W}}_{T^\vee} \to B^0 \times B'.$$

8.6 The functor $\mathrm{Hk}_{\Delta}^{\mathscr{P}}$

Recall that we have a morphism

$$\mathrm{AJ}^{\mathscr{P}} : \widetilde{C}^0 \times \mathbb{X}^{\bullet} \to \mathscr{P}.$$

For any $(\mathcal{L}, \nabla_{\mathcal{L}}) \in (\mathrm{Bun}_J)^{\natural}$ the pull back $(\mathrm{AJ}^{\mathscr{P}})^*(\mathcal{L})$ of $\mathcal{L}$ along $\mathrm{AJ}^{\mathscr{P}}$ defines an element in $\mathrm{LocSys}_{T^{\vee}}^{\mathrm{W}}$. This defines a functor

$$\mathrm{Hk}^{\mathscr{P}} : (\mathscr{P})^{\natural} \to \mathrm{LocSys}_{T^{\vee}}^{\mathrm{W}}.$$

We have the following Lemma

Lemma 8.6.1. *Let $\pi^{\vee} : (\mathscr{P})^{\natural} \to B^0 \times B'$ be the dual abelian fibration (see (8.4)). The morphism $\mathrm{Hk}^{\mathscr{P}}$ is a morphism over $B^0 \times B'$, i.e., the following diagram commutes*

$$\begin{array}{ccc} (\mathscr{P})^{\natural} & \xrightarrow{\mathrm{Hk}^{\mathscr{P}}} & \mathrm{LocSys}_{T^{\vee}}^{\mathrm{W}} \\ & \searrow^{\pi^{\vee}} & \downarrow \Psi \\ & & B^0 \times B'. \end{array}$$

Proof. Since the morphism $\mathrm{Hk}^{\mathscr{P}}$ is given by pulling back of line bundel under the Abel-Jacobi map $\mathrm{AJ}^{\mathscr{P}}$. The lemma follows from the functorial property of p-curvature morphism developed in A.7. □

Let $\Delta : B'^0 \to B^0 \times B'$ be the "diagonal" embedding. Let

$$\mathrm{Hk}_{\Delta}^{\mathscr{P}} : (\mathscr{P})^{\sharp} \to \mathrm{LocSys}_{T^{\vee}}^{\mathrm{W}} |_{\Delta}$$

be the restriction of $\mathrm{Hk}^{\mathscr{P}}$ to the "diagonal".

Corollary 8.6.2. *The morphism*

$$\mathrm{Hk}_{\Delta}^{\mathscr{P}} : (\mathscr{P})^{\sharp} \to \mathrm{LocSys}_{T^{\vee}}^{\mathrm{W}} |_{\Delta}$$

respects their torsor structure via

$$\mathrm{AJ}^{\mathscr{P}'} : (\mathscr{P}')^\vee \to \mathrm{Bun}^{\mathrm{W}}_{T^\vee}(\widetilde{C}'^0/B'^0).$$

8.7 The functor Ind_Δ

In this section we construct a morphism Ind_Δ:

$$\mathrm{Ind}_\Delta : \mathrm{LocSys}^0_{G^\vee} \to \mathrm{LocSys}^0_{T^\vee}|_\Delta.$$

Considering the following diagram

$$\begin{array}{ccccc}
\widetilde{C} & \xrightarrow{F_{\widetilde{C}/C}} & F^*\widetilde{C} & \xrightarrow{W_{\widetilde{C}/C}} & \widetilde{C} \\
 & \searrow^{\pi'} & \downarrow & & \downarrow \pi \\
 & & C_{B'} & \xrightarrow{F} & C_B \\
 & & \downarrow & & \downarrow \\
 & & C & \longrightarrow & C
\end{array}$$

Recall that we have a morphism of group scheme over $\widetilde{C}$:

$$n : \pi^* J^\vee \to T^\vee \times \widetilde{C}.$$

Pulling back via $Fr_{\widetilde{C}}$, it induces a morphism of group scheme over $\widetilde{C}$:

$$(Fr_{\widetilde{C}})^* n : (\pi')^* F^* J^\vee \simeq (Fr_{\widetilde{C}})^* \pi^* J^\vee \to T^\vee \times \widetilde{C}.$$

For any $P \in \mathrm{LocSys}^0_{F^*J^\vee}$, $(\pi')^* P$ is a $(\pi')^* F^* J^\vee \simeq (Fr_{\widetilde{C}})^* \pi^* J^\vee$ local system. Let

$$P_{T^\vee} := (\pi')^* P \times^{(\pi')^* F^* J^\vee, (Fr_{\widetilde{C}})^* n} T^\vee$$

be the induced W-equivariant $T^\vee$-torsor.

Since the map $(Fr_{\widetilde{C}})^* n$ is horizontal, the W-equivariant $T^\vee$-torsor carries a connection, therefore, the assignment $P \to P_{T^\vee}$ defines a functor

$$\mathrm{Ind}_2 : \mathrm{LocSys}^0_{F^*J^\vee} \to \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}.$$

Let Ind be the following morphism

$$\mathrm{Ind} :=: \mathrm{Ind}_2 \circ \mathrm{Ind}_1 : \mathrm{LocSys}^0_{G^\vee} \simeq \mathrm{LocSys}^0_{F^*J^\vee} \to \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}.$$

Proposition 8.7.1. *The following diagram commutes (not a Cartesian diagram in general)*

$$\begin{array}{ccc} \mathrm{LocSys}^0_{G^\vee} & \xrightarrow{\mathrm{Ind}} & \mathrm{LocSys}^{\mathrm{W}}_{T^\vee} \\ \downarrow{\scriptstyle h_p} & & \downarrow{\scriptstyle \Psi} \\ B'^0 & \xrightarrow{\Delta} & B^0 \times B' \end{array}.$$

Let $\mathrm{LocSys}^{\mathrm{W}}_{T^\vee}|_\Delta$ be the restriction of $\mathrm{LocSys}^{\mathrm{W}}_{T^\vee}$ to the "diagonal", i.e.,

$$\begin{array}{ccc} \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}|_\Delta & \longrightarrow & \mathrm{LocSys}^{\mathrm{W}}_{T^\vee} \\ \downarrow & & \downarrow \\ B'^0 & \longrightarrow & B^0 \times B' \end{array}.$$

Then above Lemma shows that the morphism Ind factors through a morphism

$$\mathrm{Ind}_\Delta : \mathrm{LocSys}^0_{G^\vee} \to \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}|_\Delta.$$

8.8 Final Construction

Let

$$For : \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}|_\Delta \to \mathrm{Bun}^{\mathrm{W}}_{T^\vee}$$

be the be the functor of forgetting connections. Let

$$r_\nabla := r \circ For : \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}|_\Delta \to \mathrm{Bun}^{\mathrm{W}}_{T^\vee} \to (\prod_{\alpha\in\Phi} T_\alpha(\widetilde{C}_\alpha))^W.$$

Lemma 8.8.1.

1. *We have the following Cartesian Diagram:*

$$\begin{array}{ccc} \mathrm{LocSys}^{0}_{G^\vee} & \longrightarrow & e \\ \Big\downarrow{\scriptstyle \mathrm{Ind}_\Delta} & & \Big\downarrow \\ \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}|_\Delta & \xrightarrow{r_\nabla} & (\prod_{\alpha\in\Phi} T_\alpha(\widetilde{C}_\alpha))^W \end{array}$$

2. *We have the following Cartesian Diagram:*

$$\begin{array}{ccc} (\mathscr{P})^\sharp & \longrightarrow & e \\ \Big\downarrow{\scriptstyle \mathrm{Hk}^{\mathscr{P}}_\Delta} & & \Big\downarrow \\ \mathrm{LocSys}^{\mathrm{W}}_{T^\vee}|_\Delta & \xrightarrow{r_\nabla} & (\prod_{\alpha\in\Phi} T_\alpha(\widetilde{C}_\alpha))^W \end{array}$$

Proof. We first prove part (1). We have the following short exact sequence

$$0 \longrightarrow \breve{\mathscr{P}}' \longrightarrow \mathrm{Bun}^{\mathrm{W}}_{T^\vee}(\widetilde{C}'^0/B'^0) \xrightarrow{r} (\textstyle\prod_{\alpha\in\Phi} T^\vee_\alpha(\widetilde{C}'_\alpha))^W .$$

Since morphisms Ind_Δ and r_∇ are compatible with their torsor structures via above sequence, result follows.

Proof of part (2) is similar to part (1). By classical duality, we have the following short exact sequence

$$0 \longrightarrow (\mathscr{P}')^\vee \xrightarrow{(\mathrm{AJ}^{\mathscr{P}'})^*} \mathrm{Bun}^{\mathrm{W}}_{T^\vee}(\widetilde{C}'^0/B'^0) \xrightarrow{r} (\textstyle\prod_{\alpha\in\Phi} T^\vee_\alpha(\widetilde{C}'_\alpha))^W .$$

Since morphism $\mathrm{Hk}^{\mathscr{P}}_\Delta$ and r_∇ are compatible with their torsor structures via above morphism, result follows. □

Proposition 8.1.2 follows from Lemma 8.8.1:

Theorem 8.8.2 (Proposition 8.1.2). *We have an isomorphism of $\breve{\mathscr{P}}'$-torsors*

$$\mathrm{LocSys}^{0}_{G^\vee} \simeq (\mathscr{P}^0)^\sharp \simeq \mathscr{T}_D.$$

Therefore, we have an equivalence of category

$$D(D^0_{\mathrm{Bun}_G} - \mathrm{mod}) \simeq D(\mathrm{QCoh}(\mathrm{LocSys}^0_{G^\vee})).$$

Chapter 9

Opers

In this section we will study Opers and compatibility of our result with Beilinson-Drinfeld's approach. In this section, we assume that G is adjoint.

9.1 Definition of Opers

Let me recall the construction and results about Opers following [BD].

Let G be connected adjoint group over k and let $\mathfrak{g}$ be its Lie algebra. There is a canonical decreasing Lie algebra filtration $\{\mathfrak{g}^k\}$ of $\mathfrak{g}$

$$\cdots \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots$$

such that $\mathfrak{g}^0 = \mathfrak{b}$, $\mathfrak{g}^1 = \mathfrak{n}$ and for any $k > 0$ (resp. < 0) weights of the action of $\mathfrak{h} = gr^0(\mathfrak{g})$ on $gr^k(\mathfrak{g})$ are sums of k simple positive (resp. negative) roots. In particular, we have $gr^{-1}(\mathfrak{g}) = \oplus \mathfrak{g}_\alpha$, where α is a simple negative root and $\mathfrak{g}_\alpha$ is the corresponding root space.

Let $\mathcal{F}$ be a B-torsor on C and $\mathcal{F}_G$ be the induced G-torsor on C. Let $\mathfrak{b}_\mathcal{F}$ and $\mathfrak{g}_{\mathcal{F}_G} \simeq \mathfrak{g}_\mathcal{F}$ be the associated bundles. Let $\widetilde{T}_\mathcal{F}$ and $\widetilde{T}_{\mathcal{F}_G}$ be the Lie algebroid of infinitesimal symmetries of and $\mathcal{F}$ and $\mathcal{F}_G$. There is a natural embedding $\widetilde{T}_\mathcal{F} \to \widetilde{T}_{\mathcal{F}_G}$ and we have a canonical isomorphism

$$\widetilde{T}_{\mathcal{F}_G}/\widetilde{T}_\mathcal{F} \simeq (\mathfrak{g}/\mathfrak{b})_\mathcal{F}$$

For any connection ∇ on $\mathcal{F}_G$, we denote by $\bar{\nabla}$ to be the composition

$$\bar{\nabla}: T_C \xrightarrow{\nabla} \widetilde{T}_{\mathcal{F}_G} \longrightarrow \widetilde{T}_{\mathcal{F}_G}/\widetilde{T}_{\mathcal{F}} \simeq (\mathfrak{g}/\mathfrak{b})_{\mathcal{F}}$$

Definition 9.1.1. An G-Oper on C is a B-torsor $\mathcal{F}$ together with a connection ∇ on $\mathcal{F}_G$ such that

1. The image of $\bar{\nabla}$ lands in $(\mathfrak{g}^{-1}/\mathfrak{b})_{\mathcal{F}} \subset (\mathfrak{g}/\mathfrak{b})_{\mathcal{F}}$.

2. The composition

$$T_C \xrightarrow{\bar{\nabla}} (\mathfrak{g}^{-1}/\mathfrak{b})_{\mathcal{F}} \xrightarrow{pr_\alpha} (\mathfrak{g}_\alpha)_{\mathcal{F}}$$

is an isomorphism for every simple negative root α. Here

$$pr_\alpha : (\mathfrak{g}^{-1}/\mathfrak{b})_{\mathcal{F}} = \oplus(\mathfrak{g}_\beta)_{\mathcal{F}} \to (\mathfrak{g}_\alpha)_{\mathcal{F}}$$

is the natural projection.

We denote by Op_G the stack of G-Opers on C.

9.2 h-connection and its p-curvature

For any $h \in k$, a h-connection on a G-torsor P is a $\mathcal{O}_C$-linear map

$$\nabla_h : T_C \to \widetilde{T}_P$$

such that the composition

$$\sigma \circ \nabla_h : T_C \to T_C$$

is equal to $h \cdot id_{T_C}$ (where $\sigma : \widetilde{T}_P \to T_C$). If $h = 1$ we recover the definition of connection. We denote by $\mathrm{LocSys}_{G,h}$ be the stack of G-bundles on C with h-connections.

One defines a (G, h)-Oper as above replacing connection ∇ by h-connection ∇_h. We denote by $\mathrm{Op}_{G,h}$ the stack of (G, h)-Opers . As shown in [BD, 3.1.14] We have $\mathrm{Op}_{G,1} \simeq \mathrm{Op}_G$ and $\mathrm{Op}_{G,0} \simeq B$.

Let $(P, \nabla_h) \in \mathrm{LocSys}_{G,h}$. The p-curvature of ∇_h is defined by

$$\Psi(\nabla_h) : T_C \to \mathfrak{g}_P, \quad v \to \nabla_h(v)^p - h^{p-1}\nabla_h(v^p).$$

Using p-curvature for h-connections one can construct p-Hitchin map for $\mathrm{LocSys}_{G,h}$:

$$h_{p,h} : \mathrm{LocSys}_{G,h} \to B'.$$

We denote by

$$h_{p,h}^{Op} : \mathrm{Op}_{G,h} \to B'$$

the restriction of $h_{p,h}$ to $\mathrm{Op}_{G,h}$.

There is a flat family

$$f : \widetilde{\mathrm{Op}_G} \to \mathbb{A}^1$$

such that for any $h \in \mathbb{A}^1(k)$ the fiber of $\widetilde{\mathrm{Op}_G}$ over h is $\mathrm{Op}_{G,h}$. Moreover, there a $\mathbb{G}_m$-action on $\widetilde{\mathrm{Op}_G}$ such that the morphism $f : \widetilde{\mathrm{Op}_G} \to \mathbb{A}^1$ is $\mathbb{G}_m$-equivariant. We defined the following morphism:

$$\widetilde{h}_p^{Op} : \widetilde{\mathrm{Op}_G} \to B' \times \mathbb{A}^1, \quad (P, \nabla_h) \to (h_{p,h}^{Op}(P, \nabla_h), h).$$

We have the following commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Op}_G} & \xrightarrow{\widetilde{h}_p^{Op}} & B' \times \mathbb{A}^1 \\ & \searrow & \downarrow \\ & & \mathbb{A}^1 \end{array}$$

Recall that there is an $\mathbb{G}_m$-action on B' and one can check that all morphisms appearing in above diagram are $\mathbb{G}_m$-equivariant (where $\mathbb{G}_m$ acts on $B' \times \mathbb{A}^1$ through the diagonal embedding $\mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$).

The base change of $\widetilde{h}_p^{Op}$ to $B' \times \{1\}$ is the usual p-Hitchin map $h_p^{Op} : \mathrm{Op}_G \to B'$

and the base change to $B' \times \{0\}$ is the relative Frobenius morphism

$$F_{B/k} : \mathrm{Op}_{G,0} \simeq B \to B'.$$

From the results in [BD, 3.1.14], above discussion is equivalent to the following Lemma:

Lemma 9.2.1. *Let R_{Op} and $R_{B'}$ be the ring of functions on Op_G and B' (there are both affine schemes). Let $\eta : R_{B'} \to R_{Op}$ be the homomorphism corresponding to $h_p^{Op} : \mathrm{Op}_G \to B'$. Then there are filtrations $\{R_{Op}^i\}$, $\{R_{B'}^i\}$ on R_{Op} and $R_{B'}$ such that*

1. *$gr(R_{Op}) \simeq R_B$ and $gr(R_{B'}) \simeq R_{B'}$.*
2. *η is compatible with those filtrations.*
3. *Then induced morphism*
$$gr(\eta) : R_{B'} \to R_B$$
is the relative Frobenius map, i.e., the corresponding map from B to B' is the relative Frobenius map.

Corollary 9.2.2. *The p-Hitchin map*

$$h_p^{Op} : \mathrm{Op}_G \to B'$$

is dominate.

Proof. We will show that the corresponding map η on ring of function is injective. But it follows from above Lemma that the associated grade of η is the Frobenius map which is injective (since R_B is a domain). □

Let $\mathrm{Op}_G^0 := \mathrm{Op}_G \cap \mathrm{LocSys}_G^0$. and let $(h_p^{Op})^0 : \mathrm{Op}_G^0 \to B'^0$ be the base change of p-Hitchin map to B'^0.

Corollary 9.2.3. *The morphism*

$$(h_p^{Op})^0 : \mathrm{Op}_G^0 \to B'^0$$

is finite and faithfully flat of degree $p^{dim(B)}$.

Proof. Since $(h_p^{Op})^0$ is dominate and both Op_G^0 and B'^0 are smooth of the same dimension. It is enough to show that $(h_p^{Op})^0$ is proper and quasi-finite. $(h_p^{Op})^0$ is proper because the p-Hitchin map $h_p : \mathrm{LocSys}_G^0 \to B'^0$ is proper and Op_G^0 is closed in LocSys_G^0. $(h_p^{Op})^0$ is quasi-finite because the fibers of $(h_p^{Op})^0$ are affine and complete, therefore consisting of finite many points.

□

Appendix A

The Stack of $\mathrm{LocSys}_{\mathcal{G}}$

In the section we review the notion of the G-local systems and their p-curvatures. We will fix a smooth morphism $X \to S$ of noetherian schemes. For any scheme $Y \to S$ smooth over S, we denote $T_{Y/S}$ (resp. $\Omega_{Y/S}$) its tangent (resp. cotangent) sheaf relative to S, or sometimes by T_Y (resp. Ω_Y) if no confusion will likely arise.

A.1 Connection on $\mathcal{G}$-torsor

Let $\mathcal{G}$ be a smooth affine group scheme over X. We do not assume that $\mathcal{G}$ is constant over X. The main examples in the paper are the universal centralizer group scheme J_{a^p}.

In order to talk about a connection on $\mathcal{G}$-torsor, we need to assume that $\mathcal{G}$ itself is a D_X-group scheme. Then it carries a flat connection which is compatible with the group structure, i.e. there is a connection

$$\nabla_{\mathcal{G}} : \mathcal{O}_{\mathcal{G}} \to \mathcal{O}_{\mathcal{G}} \otimes_{\mathcal{O}_X} \Omega_X$$

which is compatible with multiplication and co-multiplication on $\mathcal{O}_{\mathcal{G}}$. Given a $\mathcal{G}$-torsor E, we define a flat connection on E to be the following data

1) A connection $\nabla : \mathcal{O}_E \to \mathcal{O}_E \otimes_{\mathcal{O}_X} \Omega_X$ which is compatible with the multiplication of $\mathcal{O}_E$, i.e. ∇ makes $\mathcal{O}_E$ into a D_X-algebra.

2) We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_E & \xrightarrow{a} & \mathcal{O}_E \otimes_X \mathcal{O}_{\mathcal{G}} \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla\otimes 1 + 1\otimes\nabla_{\mathcal{G}}} \\ \mathcal{O}_E \otimes_X \Omega_X & \xrightarrow{a} & (\mathcal{O}_E \otimes_X \mathcal{O}_{\mathcal{G}}) \otimes_X \Omega_X \end{array}$$

where $a : \mathcal{O}_E \to \mathcal{O}_E \otimes_X \mathcal{O}_{\mathcal{G}}$ is the co-action map.

We denote by the stack of $\mathcal{G}$-torsors with flat connections by $\mathrm{LocSys}_{\mathcal{G}}$.

Example A.1.1. In the case of constant group scheme $\mathcal{G} = G \times X$, there is a canonical connection on $\mathcal{O}_{\mathcal{G}} = \mathcal{O}_G \otimes_k \mathcal{O}_X$ coming from $\mathcal{O}_X$. One can easily see that in this case the above definition of connection on a G-torsor is equivalent to the standard one.

Example A.1.2. Let $F : X \to X^{'}$ be the relative Frobenius map and $\mathcal{G}^{'}$ be a smooth affine group scheme over $X^{'}$. Then group scheme $F^*\mathcal{G}^{'}$ has a canonical connection, and for any $\mathcal{G}^{'}$-torsor $E^{'}$, there is a canonical connection on the $F^*\mathcal{G}^{'}$-torsor $F^*E^{'}$. Indeed, we have $\mathcal{O}_{F^*\mathcal{G}} = \mathcal{O}_X \otimes_{\mathcal{O}_{X^{'}}} \mathcal{O}_{\mathcal{G}^{'}}$, and therefore the connection on $\mathcal{O}_X$ will induce a connection on $\mathcal{O}_{F^*\mathcal{G}}$ which is compatible with group structure. For the same reason, $F^*E^{'}$ also carries a connection and one can check that this connection satisfied above requirements.

A.2 Lie algebroid definition

We give a Lie algebroid definition of connection. Let $\mathcal{G}$ be a affine group scheme with connection. Let E be a $\mathcal{G}$-torsor. Let us define $\widetilde{T}_E$ to be the Lie algebroid of infinitesimal symmetry of E: a section of $\widetilde{T}_E$ is a pair $(v, \tilde{v})$, where $v \in T_X$ and $\tilde{v} \in T_E$ is a vector field on E such that

1) The restriction on $\tilde{v}$ to $\mathcal{O}_X \subset \mathcal{O}_E$ is equal to v (i.e. $\tilde{v}$ is a lifting of v).

2) $\tilde{v}$ is $\mathcal{G}$-invariant, i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_E & \xrightarrow{a} & \mathcal{O}_E \otimes_X \mathcal{O}_{\mathcal{G}} \\ \downarrow{\scriptstyle \tilde{v}} & & \downarrow{\scriptstyle \tilde{v}\otimes 1 + 1\otimes\nabla_{\mathcal{G}}(v)} \\ \mathcal{O}_E & \xrightarrow{a} & \mathcal{O}_E \otimes_X \mathcal{O}_{\mathcal{G}} \end{array}$$

Let $\pi : \widetilde{T}_E \to T_X$ be the projection map $(v, \tilde{v}) \to v$. We have the following short exact sequence

$$0 \to \mathrm{ad}E \to \widetilde{T}_E \xrightarrow{\sigma} T_X \to 0. \tag{A.1}$$

A connection ∇ on E is splitting of above short exact sequence, i.e. ∇ is a map $\nabla : T_X \to \widetilde{T}_E$ such that $\pi \circ \nabla = \mathrm{id}$. If in addition that ∇ is a Lie algebroid homomorphism, we say that ∇ to be a flat connection.

A.3 The Grothendieck perspective

Let X be a smooth scheme over an algebraic closed field k. Let $\Delta : X \to X \times_k X$ be the diagonal embedding and let $X^{(1)}$ be the first infinitesimal neighborhood of Δ. We denote by $p_1, p_2 : X^{(1)} \to X$ to be the first and second projections. Let G be a smooth affine group scheme over X. A flat connection on $\mathcal{G}$ is an isomorphism of group schemes $\nabla_{\mathcal{G}} : p_1^*(\mathcal{G}) \simeq p_2^*(\mathcal{G})$ such that
1) The restriction of $\nabla_{\mathcal{G}}$ to Δ is the identity map.
2) Let $\Delta_3 : X \to X \times X \times X$ be the diagonal embedding. Let $X_3^{(1)}$ be the first infinitesimal neighborhood of $\Delta_3(X)$. We denote by π_1, π_2 and π_3 three projections from $X \times X \times X$ to X. The isomorphism $\nabla_{\mathcal{G}}$ induces isomorphisms $f_{ij} : \pi_i^*(\mathcal{G})|_{X_3^{(1)}} \simeq \pi_j^*(\mathcal{G})|_{X_3^{(1)}}$. We require $f_{13} = f_{23} \circ f_{12}$.

Let E be a $\mathcal{G}$-torsor over X. A flat connection on E is the following data:
1) An isomorphism $\nabla : p_1^*(E) \simeq p_2^*(E)$ of $\mathcal{G}$-torsors, i.e. the following diagram commutes

$$\begin{array}{ccc} p_1^*\mathcal{G} \times p_1^*E & \longrightarrow & p_1^*E \\ \downarrow{\scriptstyle \nabla_{\mathcal{G}} \times \nabla} & & \downarrow{\scriptstyle \nabla} \\ p_2^*\mathcal{G} \times p_2^*E & \longrightarrow & p_2^*E \end{array}$$

2) The restriction of ∇ to $\Delta(X)$ is the identity map.
3) Let $g_{ij} : \pi_i^*(E)|_{X_3^{(1)}} \simeq \pi_j^*(E)|_{X_3^{(1)}}$ be isomorphisms induced by ∇. We have $g_{13} = g_{23} \circ g_{12}$.

A.4 The p-curvature

Let us now assume that $p\mathcal{O}_S = 0$. Let $(E, \nabla) \in \mathrm{LocSys}_{\mathcal{G}}$. Let E be a G-local system. We have an operation of taking pth power on vector fields $v \to v^p$, and the p-curvature $\Psi(\nabla)$ of a flat connection ∇ is by definition an $\mathcal{O}_{X'}$-linear mapping

$$\Psi(\nabla) : T_X \to \widetilde{T}_E, \qquad v \to \nabla(v^p) - \nabla(v)^p$$

By (A.1), we see that the element $v \to \nabla(v^p) - \nabla(v)^p$ is in $\mathrm{ad}(E)$, hence we can view $\Psi(\nabla)$ as a $\mathcal{O}_{X'}$-linear mapping

$$\Psi(\nabla) : T_X \to \mathrm{ad}(E) \tag{A.2}$$

or equivalently, as an element in $\Gamma(X, \mathrm{ad}(E) \otimes_{\mathcal{O}_X} F^*\Omega_{X'})$.

Example A.4.1. Consider the case of Example A.1.2. The $F^*\mathcal{G}'$-torsor F^*E' carries a connection and one can easily check that the p-curvature of this connection is zero.

A.5 Bitorsors and connection on bitorsors

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two smooth affine group schemes with flat connections. We called a scheme E over X a $(\mathcal{G}_1 \times \mathcal{G}_2)$-bitorsor if is has an $(\mathcal{G}_1 \times \mathcal{G}_2)$-action and this action makes E into a left $\mathcal{G}_1$-torsor and a right $\mathcal{G}_2$-torsor. We can define a similar notion of a flat connection of a $(\mathcal{G}_1 \times \mathcal{G}_2)$-bitorsor E, i.e. a flat connection on E is an isomorphism $\nabla : p_1^*(E) \simeq p_2^*(E)$ such that

1) The following diagram commutes

$$\begin{array}{ccc} p_1^*\mathcal{G}_1 \times p_1^*\mathcal{G}_2 \times p_1^*E & \longrightarrow & p_1^*E \\ \downarrow{\scriptstyle \nabla_{\mathcal{G}_1 \times \mathcal{G}_2} \times \nabla} & & \downarrow{\scriptstyle \nabla} \\ p_2^*\mathcal{G}_1 \times p_2^*\mathcal{G}_2 \times p_2^*E & \longrightarrow & p_2^*E \end{array}$$

2) The restriction of ∇ to $\Delta(X)$ is the identity map.

3) Let $g_{ij} : \pi_i^*(E)|_{X_3^{(1)}} \simeq \pi_j^*(E)|_{X_3^{(1)}}$ be isomorphisms induced by ∇. We have $g_{13} = g_{23} \circ g_{12}$.

Definition A.5.1. We denote by $\mathrm{LocSys}_{\mathcal{G}_1,\mathcal{G}_2}$ the stack of $(\mathcal{G}_1 \times \mathcal{G}_2)$-bitorsors with flat connections.

Example A.5.2. Let E be $\mathcal{G}$-torsor. Let Aut(E) be the group scheme of automorphism of E (as $\mathcal{G}$-torsor). Then E has a natural structure of $(\mathrm{Aut(E)} \times \mathcal{G})$-bitorsor.

Example A.5.3. The constant group scheme $G_X = G \times X$ is an example of $(G_X \times G_X)$-bitorsor and there is a canonical connection on G_X induced from the standard connection on X.

Example A.5.4. Let $E^{'}$ be a $\mathcal{G}^{'}$-torsor over $X^{'}$. The $(\mathrm{Aut(E)} \times \mathcal{G})$-bitorsor $E = F^*E^{'}$ has a canonical connection.

A.6 Induction functor

Given two smooth affine group schemes $\mathcal{G}_1$ and $\mathcal{G}_2$ with connections and a groups scheme homomorphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ which is compatible with the connection. For any $(E, \nabla) \in \mathrm{LocSys}_{\mathcal{G}_1}$ we can form the fiber product $h_*E = E \times_X^{\mathcal{G}_1,h} \mathcal{G}_2$, which is a $\mathcal{G}_2$-torsor. Moreover, one can show that the connection on E will induce a connection $h_*\nabla$ on h_*E. Thus the assignment $(E, \nabla) \to (h_*E, h_*\nabla)$ defines a functor

$$h_* : \mathrm{LocSys}_{\mathcal{G}_1} \to \mathrm{LocSys}_{\mathcal{G}_2}$$

Considering the case $\mathcal{G}_1 = G_1 \times X$ and $\mathcal{G}_2 = G_2 \times X$ are constant group schemes. Let $(E, \nabla) \in \mathrm{LocSys}_{G_1}$. Then the p-curvature of the induced flat connection $h_*\nabla$ is given by $dh_E(\Psi(\nabla))$ where

$$dh_E : \Gamma(X^{'}, \mathrm{Lie(G_1)_E} \otimes \Omega^{'}_{\mathrm{X}}) \to \Gamma(\mathrm{X}^{'}, \mathrm{Lie(G_2)_E} \otimes \Omega^{'}_{\mathrm{X}})$$

is the map induced from the Lie algebra map $dh : \mathrm{Lie(G_1)} \to \mathrm{Lie(G_2)}$.

More generally, let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ be three smooth group schemes with connections and let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be a horizontal map of group scheme. Let $E \in \mathrm{LocSys}_{\mathcal{G}_1}$ and $F \in \mathrm{LocSys}_{\mathcal{G}_2,\mathcal{G}_3}$. We can form the fiber product $E \times_X^{\mathcal{G}_1,h} F$, which is an $\mathcal{G}_3$-torsor. Moreover, connections on E and F will induce a connection on $E \times_X^{\mathcal{G}_1,h} F$. Thus, giving an element $F \in \mathrm{LocSys}_{\mathcal{G}_2,\mathcal{G}_3}$, the assignment $E \to E \times_X^{\mathcal{G}_1,h} F$ defines functor

$$h_* : \mathrm{LocSys}_{\mathcal{G}_1} \to \mathrm{LocSys}_{\mathcal{G}_3}$$

Considering the special case when $\mathcal{G}_1$ is commutative and $\mathcal{G}_3 = X \times G$ is a constant group scheme for a reductive algebraic group G. Let $(E, \nabla) \in \mathrm{LocSys}_{\mathcal{G}_1}$. We claim that in this case there is a well defined map $j : \Gamma(X', \mathrm{Lie}(\mathcal{G}_1) \otimes \Omega'_{\mathrm{X}}) \to \Gamma(\mathrm{X}', \mathrm{ad}(\mathrm{h}_*\mathrm{E}) \otimes \Omega_{\mathrm{X}'})$ such that the p-curvature of $h_*\nabla$ is equal to $j(\Psi(\nabla))$. Let me give a construction of j. Let $s(x)$ be a local section of F. The section $s(x)$ induces a map $a : \mathcal{G}_2 \to \mathcal{G}_3$ characterized by $g_2 \cdot s(x) = s(x) \cdot a(g_2)^{-1}$ for all $g_2 \in \mathcal{G}_2$. We denote by $h_a = a \circ h : \mathcal{G}_1 \to \mathcal{G}_3$. If s_1 is another section, then there exists a section $g(x)$ of $\mathcal{G}_3$ such that $a(x) = a_1(x) \cdot g(x)$ and the induced map $h_{a_1} : \mathcal{G}_2 \to \mathcal{G}_3$ is equal to $g(h_a)g^{-1}$. Hence, the map $j_1 : \mathrm{Lie}(\mathcal{G}_1) \times_{\mathrm{X}} \mathrm{E} \to \mathrm{ad}(\mathrm{h}_*\mathrm{E}) := \mathfrak{g} \times^{\mathrm{G}} (\mathrm{E} \times_{\mathrm{X}}^{\mathcal{G}_1,\mathrm{h}} \mathrm{F})$ given by $j_1(v, e) = (dh_a(v), e, s(e))$ does not depend on the section s, moreover it factors through a unique map

$$j_2 : \mathrm{Lie}(\mathcal{G}_1) \to \mathrm{ad}(\mathrm{h}_*\mathrm{E})$$

and induces a map $j : \Gamma(X', \mathrm{Lie}(\mathcal{G}_1) \otimes \Omega'_{\mathrm{X}}) \to \Gamma(\mathrm{X}', \mathrm{ad}(\mathrm{h}_*\mathrm{E}) \otimes \Omega_{\mathrm{X}'})$ on sheaves of sections. This finished the construction of the map j and one can check that this map is the induced map on p-curvature.

A.7 Pullback functor

Let $f : Y \to X$ be morphism between smooth schemes and let $\mathcal{G}_X$ be a smooth group scheme on X with flat connection. Let $\mathcal{G}_Y := f^*\mathcal{G}_X$. Then it is easy to see that $\mathcal{G}_X$ also carries a flat connection and the pullback of f defines a functor

$f^* : \mathrm{LocSys}_{\mathcal{G}_X} \to \mathrm{LocSys}_{\mathcal{G}_Y}$. For any $(P, \nabla_P) \in \mathrm{LocSys}_{\mathcal{G}_X}$, let

$$\Psi_P \in \Gamma(X, ad(\mathrm{Lie}\,\mathcal{G}_X) \otimes F^*\omega_X)$$

be the p-curvature of ∇_P. We have a map sheaves on Y:

$$f^*(ad(\mathrm{Lie}\,\mathcal{G}_X) \otimes F^*\omega_X) \simeq ad(\mathrm{Lie}\,\mathcal{G}_Y) \otimes f^*F^*\omega_X \to ad(\mathrm{Lie}\,\mathcal{G}_Y) \otimes F^*\omega_Y.$$

This map induces a map

$$f_p^* : H^0(X, ad(\mathrm{Lie}\,\mathcal{G}_X) \otimes F^*\omega_X) \to H^0(Y, ad(\mathrm{Lie}\,\mathcal{G}_Y) \otimes F^*\omega_Y)$$

and the p-curvature of $f^*(P, \nabla_P)$ is equal to $f_p^*(\Psi_P)$.

A.8 Cartier descent

Let $\mathcal{G}'$ be a smooth affine group scheme over X' Set $\mathcal{G} := F^*\mathcal{G}'$. By example A.1.2, we have a functor

$$F : \mathrm{Bun}_{\mathcal{G}'} \to \mathrm{LocSys}_{\mathcal{G}}, E' \to F^*E'$$

Theorem 1. *The functor F induces an equivalence between the category of $\mathcal{G}'$-torsros and the category of $\mathcal{G}'$-local systems with zero p-curvature.*

Proof. Using the definition of flat connection in A.1, the Theorem follows from the original Cartier descent (here we used the fact that Cartier descent is a tensor equivalence).

□

Corollary A.8.1. *Considering the case when $\mathcal{G}'$ is commutative. Then the category of $\mathcal{G}$-local system with a fixed p-curvature is a torsor over the Picard stack* $\mathrm{Bun}_{\mathcal{G}'}$.

Appendix B

Beilinson's 1-motive

In this section, we review the duality theory of Beilinson's 1-motives.

B.1 Picard Stack

Let us first review the theory of Picard stacks. The standard reference is [Del, §1.4]. Let $\mathcal{T}$ be a given topos. Recall that a Picard Stack is a stack $\mathcal{P}$ in $\mathcal{T}$ together with a bifunctor

$$\otimes : \mathcal{P} \times \mathcal{P} \to \mathcal{P},$$

and the associativity and commutative constraints

$$a : \otimes \circ (\otimes \times 1) \simeq \otimes \circ (1 \times \otimes), \qquad c : \otimes \simeq \otimes \circ \mathrm{flip},$$

such that for every $U \in \mathcal{T}$, $\mathcal{P}(U)$ form a Picard groupoid (i.e. symmetrical monoidal groupoid such that every object has a monoidal inverse). The Picard stack is called strictly commutative if $c_{x,x} = \mathrm{id}_x$ for every $x \in \mathcal{P}$. In the paper, Picard stacks will always mean strictly commutative ones.

Let us denote $\mathcal{PS}/\mathcal{T}$ to be the 2-category of Picard stacks over $\mathcal{T}$. This means that if $\mathcal{P}_1, \mathcal{P}_2$ are two Picard stacks over $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{PS}/\mathcal{T}}(\mathcal{P}_1, \mathcal{P}_2)$ form a category. Indeed, $\mathcal{PS}/\mathcal{T}$ is canonically enriched over itself. For $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{PS}/\mathcal{T}$, we use $\underline{\mathrm{Hom}}(\mathcal{P}_1, \mathcal{P}_2)$ to denote the Picard stack of 1-homorphisms from $\mathcal{P}_1$ to $\mathcal{P}_2$ over $\mathcal{T}$ (cf. [Del] §1.4.7). On the other hand, let $C^{[-1,0]}$ be the 2-category of 2-term complexes of sheaves of abelian groups $d : \mathcal{K}^{-1} \to \mathcal{K}^0$ with $\mathcal{K}^{-1}$ injective and 1-morphisms are morphisms of chain complexes (and 2-morphisms are homotopy of chain complexes). Let $\mathcal{K} \in C^{[-1,0]}$. We associate to it a Picard prestack $\mathrm{pch}(\mathcal{K})$ whose U point is the following Picard category

1. Objects of $\mathrm{pch}(\mathcal{K})(\mathrm{U})$ are equal to $\mathcal{K}^0(U)$.

2. 2)If $x, y \in \mathcal{K}^0(U)$, a morphism from x to y is an element $f \in \mathcal{K}^{-1}(U)$ such that $df = y - x$.

Let $\mathrm{ch}(\mathcal{K})$ be the stackification of $\mathrm{pch}(\mathcal{K})$. Then a theorem of Deligne says that the functor

$$\mathrm{ch} : \mathrm{C}^{[-1,0]} \to \mathcal{PS}/\mathcal{T}$$

is an equivalence of 2-categories.

Let us fix an inverse functor $()^\flat$ of the above equivalence. So for $\mathcal{P}$ a Picard stack, we have a 2-term complex of sheaves of abelian groups $\mathcal{P}^\flat := \mathcal{K}^{-1} \to \mathcal{K}^0$. The following result of Deligne is suitable for computations.

$$(\underline{\mathrm{Hom}}(\mathcal{P}_1, \mathcal{P}_2))^\flat \cong \tau_{\leq 0} \,\mathrm{R}\, \underline{\mathrm{Hom}}(\mathcal{P}_1^\flat, \mathcal{P}_2^\flat). \tag{B.1}$$

B.2 The duality of Picard stacks

Let S be a noetherian scheme. We consider the category Sch/S of schemes over S. We will endow Sch/S with *fppf* topology in the following discussion. The precise choice of the topology will be immaterial for the following discussion.

Definition B.2.1. For a Picard stack $\mathcal{P}$, we define the dual Picard stack to be

$$\mathcal{P}^\vee := \underline{\mathrm{Hom}}(\mathcal{P}, B\mathbb{G}_m)$$

where $B\mathbb{G}_m$ is the classifying stack of $\mathbb{G}_m$.

Example B.2.2. Let $A \to S$ be an abelian scheme over S. Then by definition $A^\vee := \underline{\mathrm{Hom}}(A, B\mathbb{G}_m) = \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ classifies the multiplicative line bundles on A. If A is projective over S, then $A^\vee$ is represented by a projective abelian scheme over S, called the dual abelian scheme of A.

Example B.2.3. Let Γ be a finitely generated abelian group over S. By definition, this means locally on S, Γ is isomorphic to the constant sheaf M_S, where M is a finitely generate abelian group (in the usual sense). Recall that the Cartier dual of Γ, denoted by $\mathrm{D}(\Gamma)$ is the sheaf which assigns every scheme U over S the group $\mathrm{Hom}(\Gamma \times_S U, \mathbb{G}_m)$, which is represented by an affine group scheme over S. We claim that $\Gamma^\vee \simeq B\,\mathrm{D}(\Gamma)$. By (B.1), it is enough to show that $\mathrm{R}^i\,\underline{\mathrm{Hom}}(\Gamma, \mathbb{G}_m[1]) = 0$ if $i > 0$. By this is clearly since locally on S, Γ is represented by a 2-term complex $\mathbb{Z}_S^m \to \mathbb{Z}_S^n$.

Example B.2.4. Let G be a group of multiplicative type over S, i.e. $G = D(\Gamma)$ for some finitely generated abelian group Γ over S. Let $\mathcal{P} = BG$, the classifying stack of G. We have

$$\mathcal{P}^\vee \simeq \tau_{\leq 0}\,\mathrm{R}\,\underline{\mathrm{Hom}}(BG, B\mathbb{G}_m) \simeq \underline{\mathrm{Hom}}(G, \mathbb{G}_m) \simeq \Gamma.$$

Definition B.2.5. Let $\mathcal{P}$ be a Picard stack. We say that $\mathcal{P}$ is dualizable if the canonical 1-morphism $\mathcal{P} \to \mathcal{P}^{\vee\vee}$ is an isomorphism.

By the above examples, projective abelian schemes, finitely generated abelian groups, and the classify stacks of groups of multiplicative type are dualizable.

Let $\mathcal{P}$ be a dualizable Picard stack. There is the Poincare line bundle $\mathcal{L}_\mathcal{P}$ over $\mathcal{P} \times_S \mathcal{P}^\vee$. Let $\mathrm{Qcoh}(\mathcal{P})$ denote the derived category of quasicoherent sheaves on $\mathcal{P}$. We define the Fourier-Mukai funtor

$$\Phi_\mathcal{P} : \mathrm{Qcoh}(\mathcal{P}) \to \mathrm{Qcoh}(\mathcal{P}^\vee), \quad \Phi_\mathcal{P}(F) = (p_2)_*(p_1^* F \otimes \mathcal{L}_\mathcal{P}).$$

Here $p_1 : \mathcal{P} \times_S \mathcal{P}^\vee \to \mathcal{P}$ and $p_2 : \mathcal{P} \times_S \mathcal{P}^\vee \to \mathcal{P}^\vee$ denote the natural projections. It is easy to see in the case when $\mathcal{P}$ is of the form given in the above examples, $\Phi_\mathcal{P}$ is an equivalence of categories. Indeed, the case when $\mathcal{P} = A$ follows from the results of Mukai; the case when $\mathcal{P} = \Gamma$ or BG is clear.

On the other hand, $\Phi_{\mathcal{P}}$ is not an equivalence for general dualizable Picard stacks. Indeed, let $\mathcal{P} = \mathbb{G}_m$. Then $\mathcal{P}^\vee = B\mathbb{Z}$, as $\underline{\mathrm{Ext}}^1(\mathbb{G}_m, \mathbb{G}_m) = 0$, and therefore $\mathbb{G}_m$ is dualizable. But clearly, $\mathrm{Qcoh}(\mathbb{G}_m) \not\simeq \mathrm{Qcoh}(B\mathbb{Z})$.

In the following subsection, we select out a particular class of Picard stacks, called the Beilinson's 1-motive, for which the Fourier-Mukai transforms are equivalences.

B.3 Beilinson's 1-motives

Let $\mathcal{P}_1, \mathcal{P}_2$ be two Picard stacks. We say that $\mathcal{P}_1 \subset \mathcal{P}_2$ if there is a 1-morphism $\phi : \mathcal{P}_1 \to \mathcal{P}_2$, which is a full embedding.

Definition B.3.1. We called a Picard stack $\mathcal{P}$ a Beilinson's 1-motive if it admits a two step filtration $W_\bullet \mathcal{P}$:

$$W_{-1} = 0 \to W_0 \to W_1 \to W_2 = \mathcal{P}$$

such that (i) $\mathrm{Gr}_0^W \simeq BG$ is the classifying stack of a group G of multiplicative type; (ii) $\mathrm{Gr}_1^W \simeq A$ is a projective abelian scheme; and (iii) $\mathrm{Gr}_2^W \simeq \Gamma$ is a finitely generated abelian group.

Lemma B.3.2. *The dual of a Beilinson's 1-motive is a Beilinson's 1-motive and Beilinson's 1-motive are dualizable.*

Proof. This is proved via the induction on the length of the filtration. We use the following fact. Let

$$0 \to \mathcal{P}' \to \mathcal{P} \to \mathcal{P}'' \to 0$$

be a short exact sequence of Picard stacks. Then

$$0 \to (\mathcal{P}'')^\vee \to \mathcal{P}^\vee \to (\mathcal{P}')^\vee$$

with the right arrow surjective if $\mathrm{R}^2\,\underline{\mathrm{Hom}}((\mathcal{P}'')^\flat, \mathbb{G}_m) = 0$.

If $\mathcal{P} = W_0\mathcal{P}$, this is given by Example B.2.4. If $\mathcal{P} = W_1\mathcal{P}$, we have the following exact sequence

$$0 \to BG \to \mathcal{P} \to A \to 0.$$

Then by the fact that $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m) = 0$, we know that $\mathcal{P}$ is also a Beilinson's 1-motive. In general, we have

$$0 \to W_1\mathcal{P} \to \mathcal{P} \to \Gamma \to 0,$$

and the lemma follows from the fact $\underline{\mathrm{Ext}}^2(\Gamma, \mathbb{G}_m) = 0$. □

Remark. *In [Del], Deligne introduced another notion of 1-motives, where he required* Gr_0^W *to be a group of multiplicative type, and* Gr_2^W *to be the classifying stack of a finitely generated abelian category. But as pointed out in the previous subsection, this is not suitable for the duality.*

Lemma B.3.3. *Let $\mathcal{P}$ be a Beilinson's 1-motive. Then locally on S,*

$$\mathcal{P} \simeq A \times BG \times \Gamma.$$

Proof. It is enough to prove that $\underline{\mathrm{Ext}}^1(\Gamma, BG) = \underline{\mathrm{Ext}}^1(\Gamma, A) = \underline{\mathrm{Ext}}^1(A, BG) = 0$. Clearly, $\underline{\mathrm{Ext}}^1(\Gamma, BG) = \underline{\mathrm{Ext}}^2(\Gamma, G) = 0$. To see that $\underline{\mathrm{Ext}}^1(\Gamma, A) = 0$, we can assume that $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Then it follows that $A \xrightarrow{n} A$ is surjective in the flat topology that $\underline{\mathrm{Ext}}^1(\Gamma, A) = 0$.

To see that $\underline{\mathrm{Ext}}^1(A, BG) = 0$, let $\mathcal{P}$ to the Beilinson's 1-motive corresponding to a class in $\underline{\mathrm{Ext}}^1(A, BG)$. Taking the dual, we have $0 \to A^\vee \to \mathcal{P}^\vee \to \mathrm{D}(G) \to 0$. Therefore, locally on S, $\mathcal{P}^\vee \simeq A^\vee \times \mathrm{D}(G)$, and therefore locally on S, $\mathcal{P}^{\vee\vee} \simeq A \times BG$. □

Now the following result is clear.

Theorem B.3.4. *Let $\mathcal{P}$ be a Beilinson's 1-motive. Then the functor $\Phi_{\mathcal{P}}$ is an equivalence of categories.*

B.4 Duality for torsors

Let us return to the general set-up. Let $\mathcal{T}$ be a fixed topos and let $\mathcal{P}$ be a Picard stack over $\mathcal{T}$. A torsor of $\mathcal{P}$ is a stack $\mathcal{Q}$ over $\mathcal{T}$, together with a bifunctor

$$m : \mathcal{P} \times \mathcal{Q} \to \mathcal{Q},$$

$\mathcal{P}$ acts on $\mathcal{Q}$ as a tensor category and for any object $C \in \mathcal{Q}$ the functor given by

$$P \in \mathcal{P} \Rightarrow \text{Action}(P, C) \in \mathcal{Q}$$

is an equivalence.

If $\mathcal{P}$ is a Picard stack and $\mathcal{Q}$ is another stack, we say that $\mathcal{Q}$ is a torsor over the Picard category $\mathcal{P}$, if the following holds:
1) For every $(U \to X) \in X_{et}$, $\mathcal{Q}(U)$ is a gerbe over the Picard category $\mathcal{P}(U)$
2) There exists a covering $U \to X$, such that $\mathcal{Q}(U)$ is non-empty.

In the case when $\mathcal{P}$ is the Picard stack of A-torsors for some sheaf of abelian gorups A, people usually called a $\mathcal{P}$-torsor $\mathcal{Q}$ a A-gerbe.

Let $\mathcal{Y}$ be a Picard stack. Let $\widetilde{\mathcal{Y}}$ be a $\mathbb{G}_m$-gerbe over $\mathcal{Y}$, i.e. it is a torsor over the Picard stack $\text{Pic}(\mathcal{Y})$. $\widetilde{\mathcal{Y}}$ is called split if it is isomorphic to $\mathcal{Y} \times B\mathbb{G}_m$.

Let $D(\mathcal{O}_{\widetilde{\mathcal{Y}}} - \text{mod})$ be the bounded derived category of coherent sheaves on $\widetilde{\mathcal{Y}}$. If $\widetilde{\mathcal{Y}}$ is split, there is a decomposition

$$D(\mathcal{O}_{\widetilde{\mathcal{Y}}} - \text{mod}) = \oplus_{\text{n} \in \mathbb{Z}} \text{D}(\mathcal{O}_{\widetilde{\mathcal{Y}}} - \text{mod})_{\text{n}}$$

according to the character of $\mathbb{Z}$. If $\widetilde{\mathcal{Y}}$ is not split we still have a decomposition of $D(\mathcal{O}_{\widetilde{\mathcal{Y}}})$ into direct sum like above. This decomposition is described as following. Let $a : B\mathbb{G}_m \times \widetilde{\mathcal{Y}} \to \widetilde{\mathcal{Y}}$ be the action map. Then $\mathcal{M} \in D(\mathcal{O}_{\widetilde{\mathcal{Y}}} - \text{mod})_{\text{n}}$ if only if $a^*(\mathcal{M}) \in D(\mathcal{O}_{B\mathbb{G}_m \times \widetilde{\mathcal{Y}}} - \text{mod})_{\text{n}}$

Let $\mathcal{P}$ be a Beilinson's 1-motive. Let $\widetilde{\mathcal{P}}$ be an extension of Picard stack of $\mathbb{Z}_S$ by

$\mathcal{P}$

$$0 \longrightarrow \mathcal{P} \longrightarrow \widetilde{\mathcal{P}} \xrightarrow{f} \mathbb{Z}_S \longrightarrow 0$$

Let $\widetilde{\mathcal{P}}_1 = f^{-1}(1)$ be the $\mathcal{P}$-torsor associated to above extension. This construction yields an equivalence of the category of extensions of $\mathbb{Z}_S$ to $\mathcal{P}$ and the category of $\mathcal{P}$-torsors. The inverse functor is given as follows: for a $\mathcal{P}$-torsor $\mathcal{T}$, $\widetilde{\mathcal{P}} = \{(n,t) \mid n \in \mathbb{Z}, t \in \mathcal{T}^{\otimes n}\}$.

Now let $\widetilde{\mathcal{P}}^\vee$ be the dual Picard stack. It fits into the short exact sequence

$$0 \to B\mathbb{G}_m \to \widetilde{\mathcal{P}}^\vee \to \mathcal{P}^\vee \to 0$$

Clearly, both Picard stacks $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}^\vee$ are Beilinson's 1-motives and we have the following

Theorem B.4.1. *The Fourier-Mukai functor $\Phi_{\widetilde{\mathcal{F}}}$ restricts to an equivalence*

$$\Phi_{\widetilde{\mathcal{F}}} : \mathrm{Qcoh}(\widetilde{\mathcal{P}}_1) \simeq \mathrm{Qcoh}(\widetilde{\mathcal{P}}^\vee)_1.$$

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