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A GRADIENT METHOD FOR OPTIMIZING STOCHASTIC SYSTEMS

by

Robert John Fitzgerald

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Degree of Doctor of Philosophy

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ROBERT JOHN FITZGERALD

B.A.Sc., University of Toronto (1956)
S.M., Massachusetts Institute of Technology (1957)
Mech.E., Massachusetts Institute of Technology (1958)

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Signature of Author _____
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p. 12, par. 2, l. 4, change "eigher" to "either"

p. 60, 4th last line, change Ref. 11 to Ref. 23

p. 63, Eq. (3-96), change p to δp

p. 74, insert dt at end of Eq. (4-38)

p. 77, Sec. 4.4, last paragraph. First sentence is incorrect. More general transition matrices can be produced by pre- and post-multiplying the filter by gain matrices, and increasing its order. In the present case, with no constraints on A(t), the influence-function matrix M(t) may never vanish for finite A(t). With constraints, the optimality condition is the vanishing, not of $M_\phi(t)$, but of the gradient projection $M_\phi(t) - M_\psi(t)\, I_{\psi\psi}^{-1}\, \underline{\nu}_{\psi\phi}\, \underline{\nu}^T(t)$ in Eq. (2-47).

p. 111, after Eq. (A-13), change "to" to "of".

p. 117, first line after (B-21), change t_1- to $t-$.

p. 119, omit this page entirely. Incorrect conclusions are reached due to an error in Eq. (B-31), which actually reduces to the simple requirement $a \geq 0$.

p. 31, Sec. 2.6, 1st. par., last line, change "stricly" to "strictly"

p. 32, 2nd. par., 1st. line, after "types" insert Ref. no. 71.

p. 129, 3rd. last line, change "Theta" to "Sigma".

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ABSTRACT

A technique is presented for the optimization, by the method of steepest descent, of a stochastic system described by a set of nonlinear differential equations. It is assumed that the random disturbances are small enough that their effects can be adequately described by linearization of the system equations around some nominal solution.

The disturbances are assumed, in general, to be non-white, non-stationary random functions of the state variables and the control variables, as well as of the independent variable. In such cases reduction of the problem to the white-noise case, by the introduction of a "shaping filter", is impracticable. Methods of analysis based on perturbation of the appropriate variance equations are therefore not applicable, and a more general (and computationally more costly) approach becomes necessary.

The basic performance criterion is the minimization, in the mean-square sense, of the random perturbations in the final value of a prescribed scalar function of the state variables.

Some concepts from Hilbert-space theory are applied to a development of the gradient projection technique for the handling of constraints, and to a consideration of some abnormal situations which may arise in practice.

The technique is extended to several systems of specified configuration, and to a variety of performance criteria and constraints.

Numerical results are presented for several variations of a simple illustrative problem.

Thesis Supervisor: Wallace E. Vander Velde
Title: Associate Professor of
Aeronautics and Astronautics

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CONTENTS

ILLUSTRATIONS

SYMBOLS

a	Parameter in correlation functions, Eqs. (6-28, 6-39).
$A(t)$	Matrix of control functions. See Sec. 2.3 and Ch. 4.
b	Parameter in correlation functions, Eqs. (6-28, 6-39).
$\underline{b}$	Vector of influence coefficients, Eqs. (3-78, 3-80).
$\underline{b}_\phi$	Vector of influence coefficients, Eqs. (2-65, 3-46, 3-51, 4-17, 4-55, 4-62).
$\underline{b}_\omega$	Vector of influence coefficients, Eq. (3-50).
B_ψ	Matrix of influence coefficients, Eq. (2-66).
c	Parameter in correlation functions, Eqs. (6-28, 6-39).
c^2	Specified value for $\cos^2 \theta$, Eq. (2-79).
$\underline{c}_x$	Vector of influence coefficients, Sec. 3.5.
C	Covariance matrix of initial errors, Eq. (3-18.1).
C	Inequality constraint, Eq. (3-83).
C	The integral in Eq. (2-95).
d	Parameter in correlation function, Eq. (6-28).
d^2	Specified step-size, Eq. (2-3)
e	Error, Sec. 3.4.
$\underline{f}(\underline{x})$	Vector random field, Sec. 3.3.
$\underline{f}(\underline{x}, \underline{u}, \underline{n}, t)$	Right-hand side of system diff. eqn., Eq. (3-6).
F	Nonlinear element, Ch. 4.
$F(t, \tau)$	See Eq. (4-43).
$F_n(t)$	See p. 39.

$F_u(t)$	See p. 39.
$F_x(t)$	See p. 39.
$g(\underline{x}(T), T)$	See p. 39.
$g_e(x)$	See Eq. (6-25).
g_n	Nominal value of g, p. 62.
$g_y(x)$	See Eq. (6-26).
$g_\phi(x)$	See Eqs. (6-14, 6-19).
$g_\psi(x)$	See Eq. (6-23).
$\underline{g}$	Constant vector, Eq. (A-12).
$\underline{g}_\phi(t)$	Vector of influence functions, Eqs. (2-1, 3-42, 4-53, 4-60).
$\underline{g}_\omega(t)$	Vector of influence functions, Eq. (3-50).
$G(t)$	Measurement matrix, Ch. 4.
G_{xx}	See Eq. (3-33).
$G_\psi(t)$	Matrix of influence functions, Eqs. (2-2, 5-2).
h	Dimension of measurement noise vector, Sec. 4.3.
$\underline{h}(t)$	See Eq. (A-12).
$\underline{h}(\underline{x}(T), T)$	See pp. 59, 60.
$H(t)$	Hamiltonian, Eq. (3-13).
$H_{nu}(t)$	See Eq. (3-36).
$H_{nx}(t)$	See Eq. (3-37).
$H_{uu}(t)$	See Eq. (4-10).
$H_{xu}(t)$	See Eq. (3-30).
$H_{xx}(t)$	See Eq. (3-31).
I	Identity matrix.
I_i	Square matrix with one non-zero element, p. 16.
$I_{\psi\psi}$	See Eqs. (2-8, 2-31, 2-43, 2-62, 2-68).

k	Constant, Eq. (3-91).
k_i	Constant, Eq. (2-75).
$k_{\phi\phi}$	See Eqs. (2-24, 2-31.2, 2-45, 2-64, 2-70).
$\underline{\ell}(t)$	Vector of adjoint variables, p. 53.
L	Distance, p. 88.
$L(t)$	Adjoint matrix, Eqs. (A-20, A-23).
m	Dimension of control vector, p. 39.
$M(t)$	See Eqs. (4-14, 4-18, 4-41, 4-43.3).
$M(t, \tau)$	See Eq. (2-74).
$M_\phi(t)$	Matrix of influence functions, Eqs. (2-33, 4-13, 4-40, 4-54, 4-61).
$M_\psi(t)$	Matrix of influence functions, Eq. (2-34).
$M_{\psi i}(t)$	See Eq. (2-57).
n	Dimension of state vector, p. 39.
$\underline{n}(\underline{x}, \underline{u}, t)$	Noise p-vector, p. 39.
$\underline{n}_i$	Normal vector, p. 32; noise vector, Sec. 4.3.
$N_1(t, \tau)$	See Eq. (4-43.1).
$N_2(t)$	See Eq. (4-43.5).
$N_3(t)$	See Eq. (4-43.6).
p	Dimension of noise vector, p. 39.
p	A certain probability, Eq. (3-91).
p	Prediction time, Sec. 4.4.
$\underline{p}$	Vector of time-invariant parameters, Sec. 2.4.
$\underline{p}(t)$	Orthogonal projection vector, Eqs. (2-16, 2-23).
P	A certain probability, p. 101.
Pr	Probability.

q	Number of intermediate integrations, p. 84.
r	Dimension of constraint vector, p. 33.
r_i	A ratio, Eq. (2-78).
$\underline{r}$	Position vector, p. 32.
$\underline{r}(\tau, t)$	See Eq. (3-23).
$\underline{r}_I(t)$	See Eq. (3-24).
$\underline{r}'_I(t)$	See Eq. (3-67.3).
$\underline{r}_{Ii}(t)$	See Sec. 4.3.
$R(t, \tau)$	Correlation matrix, Eqs. (3-19, 3-19.1, 4-42).
$R(\underline{x}_1, \underline{x}_2)$	Correlation matrix, Eq. (3-1).
$R_i(t, \tau)$	See Sec. 4.3.
$R_{ii}(t, \tau)$	See Eq. 4-44.2.
$R_{nn}(t, \tau)$	Noise correlation matrix, Eq. (4-32).
$R_{ns}(t, \tau)$	Cross-correlation matrix, Eq. (4-33).
$R_{sn}(t, \tau)$	Cross-correlation matrix, Eq. (4-33).
$R_{ss}(t, \tau)$	Signal correlation matrix, Eq. (4-31).
$R_u(t)$	See Eq. (3-26).
$R_x(t)$	See Eqs. (3-25, 3-27).
s	dx/dy, App. C.
$\underline{s}(t)$	Signal vector, Sec. 4.4.
$S, S^{\perp}$	Subspaces, pp. 7, 8, 19.
$S(\omega, y)$	Power spectral density, Eq. (6-30).
t	Independent variable (usually time).
T	Final time.
$\underline{u}(t)$	Control m-vector, p. 39.
$U(t)$	Control matrix, Eq. A-6.

$\underline{v}(t)$	Vector of influence functions, Eqs. (2-34, 4-15).
$\underline{v}_{\psi\phi}$	See Eqs. (2-21, 2-31.1, 2-42, 2-63, 2-69).
$\underline{v}_i(t)$	See p. 71.
V	Velocity, Sec. 6.1.
$\underline{w}(t)$	See Eqs. (4-13, 4-16, 4-18.1).
$\underline{w}_1(t)$	See Eqs. (3-42.1, 3-42.3, 4-17.1, 4-56).
$\underline{w}_1'(t)$	See Eqs. (3-67.4, 3-67.5).
$\underline{w}_2(t)$	See Eqs. (3-42.2, 3-42.4, 4-57).
$\underline{w}_3(t)$	See Eq. (4-17.2).
$\underline{w}_4(t)$	See Eq. (4-17.3).
$\underline{w}_5(t)$	See Eq. (4-17.4).
W	Weighting matrix, Eq. (4-30).
W	Parameter in correlation functions, Eqs. (6-28, 6-39).
W(t)	Weighting matrix, Eq. (2-3).
$W(t, \tau)$	Controllability matrix, Eq. (2-96).
$W^{1/2}$	Square root of matrix W, p. 60.
$W_i(t)$	Weighting matrix, Eq. (2-35).
W_p	Weighting matrix, Eq. (2-67).
$\underline{x}(t)$	State-variable n-vector, p. 39.
$\underline{x}_n(t)$	Nominal state-variable vector, Sec. 3.11.
$\underline{x}_t$	Target state-vector, Sec. 3.11.
$\underline{x}_0$	Initial-condition state-vector, Eq. (3-6).
X(t)	Fundamental matrix, p. 42.
$X_n(t)$	Fundamental matrix, p. 78.

δ	Variation
θ	Angle, Sec. 2.5 and Ch. 6.
$\underline{\lambda}(t)$	Adjoint vector, p. 40 and App. A.
$\underline{\lambda}_n(t)$	Influence function vector, Eq. (3-17).
$\underline{\lambda}_{ni}(t)$	Influence function vector, Sec. 4.3.
$\underline{\lambda}_u(t)$	Influence function vector, Eq. (A-16).
$\Lambda_\psi(t)$	See Eq. (5-1).
μ	Lagrange multiplier, p. 15.
$\underline{\nu}$	Lagrange multiplier vector, p. 15.
σ	Variance, Eq. (3-95).
ϕ	Performance criterion.
ϕ_f	Final Value of ϕ, p. 49.
$\Phi(t, \tau)$	Transition matrix, p. 33 and Eq. (A-4).
$\Phi_a(t, \tau)$	Transition matrix of adjoint equation, Eq. (A-18).
$\underline{\psi}(\underline{x}(T), T)$	Constraint vector, p. 7.
Ψ_x	See Eq. (2-93).
$\omega(\underline{x}(T), T)$	Stopping criterion, Eq. (3-47).
ω_f	Final value of ω, p. 49.
Ω_{xx}	See Eq. (3-77).
$\underline{1}_i$	Unit vector, p. 14.

In general, matrices are represented by upper-case letters and vectors by underlined lower-case letters. The superscript T indicates the transpose of a matrix, and -T the transposed inverse. A tilde ($\tilde{}$) indicates the random perturbation of a quantity.

CHAPTER 1

INTRODUCTION

1.1 The Two-Point Boundary-Value Problem

One of the central problems in the field of system optimization is the solution of the so-called two-point boundary-value problem.

This question has attracted considerable attention in recent years in the study of rocket vehicles; it arises quite naturally, for example, if we attempt to minimize the rocket fuel required to perform a maneuver such as orbital insertion of a satellite.

Problems with markedly similar mathematical formulations may arise in entirely different fields. We may desire, for example, to choose the temperature profile in a flow-through chemical reactor in such a way as to maximize the concentration of a desirable constituent at the output. Numerous problems in business, economics and other fields may give rise to entirely equivalent formulations.

The time-honored Calculus of Variations approach to such problems[9, 11, 19, 36, 54] yields a set of differential and algebraic equations (the Euler-Lagrange equations) with boundary conditions specified at more than one value of the independent variable. Typically, the initial values of some of the state variables are given, and the trajectory must attain specified final values of some of the state variables and of the Lagrange multipliers with which the differential-equation constraints are adjoined to the performance criterion.

Until quite recently, solutions to significant problems of this type were largely non-existent. Only with the advent of high-speed computers did the requisite large-scale computations become practicable. The approach used, in most cases, is to guess values for the unknown initial conditions, solve the Euler equations, and compare the final values of all variables with the desired values.[11, 54] If the agreement is not satisfactory the computations are repeated with revised estimates of the initial values. Thus a series of trajectories is generated, each one being an optimal solution for a problem different from the one whose solution is desired. Hopefully, with judicious manipulation of the initial conditions, the sequence of trajectories will converge to one which solves the original problem.

The technique works well in many cases, but has often been found unmanageable due to extreme sensitivity of the trajectories to small changes in the initial values.[11, 49]

1.2 The Gradient Method

Efforts to overcome the inherent numerical difficulties of the "indirect" (calculus of variations) approach, led to the independent revival, by Kelley[48, 49, 52] and by Bryson and his colleagues,[14, 15, 16, 22, 28] of the gradient method, or "method of steepest descent", in function space.[47] This "direct" method has become widely used and has met with considerable success. Recent work has extended the theory to very general problem formulations[22] and, in particular, to procedures for the direct handling of "in-flight" inequality constraints.[22, 24, 25]

The technique depends basically on the determination of influence functions from which we can predict the effects, on the payoff function and constraint variables, of small perturbations in the control variables. These influence functions are determined by backward integration of the equations adjoint to the linearized perturbation equations of the system (see Appendix A and Ref. 15). The subsequent procedure, as described in Chapter 2, yields a sequence of non-optimal trajectories, each of which satisfies the boundary conditions and the original differential equations of the problem, but violates some of the Euler equations. The iterative technique ensures that each successive trajectory is closer to the optimum.

Dreyfus[28] refers to this procedure as "successive approximation to the solution", as opposed to the "successive approximation to the problem" of the indirect method.

Comparison of the two methods reveals that the adjoint equations of the direct method are identical with the Euler equations corresponding to the state variables. The Euler equations which have been ignored (the algebraic ones corresponding to the non-derivated,[19] or control, variables) are seen to be an expression of the requirement that the influence functions vanish (the "optimality conditions"), and will be satisfied only when a stationary solution is found.[11] Discarding the optimality conditions effectively

uncouples the adjoint equations from the differential equations of the system (the Euler equations corresponding to the Lagrange multipliers), with the result that the adjoint equations may be integrated backwards, after the system equations have been integrated forward with arbitrarily chosen control variables.

1.3 Second-Variation Methods

The method has been extended in several cases[12, 50, 53] by the application of second-variation techniques. By perturbing the optimality conditions along with the system equations and adjoint equations, it is possible to determine the changes necessary in the control program to re-optimize the trajectory after small perturbations in the initial conditions. The result is a "neighbouring-optimum" control scheme,[12, 31, 50] in which measured deviations of the state variables from their nominal values are continuously fed back to the control inputs through a set of pre-computed feedback gain coefficients ("lambda-matrices"[12]).

A similar derivation has been used[12] to provide a refinement to the indirect method of optimization. The effects of small perturbations in the unknown initial conditions, on the final values of the state variables and adjoint variables, is determined by the above approach. We can then generate a series of optimal trajectories whose terminal values converge, in small steps, to the desired values.

Kelley and his colleagues[53] have developed an optimization technique which improves the rate of convergence in the final stages of the optimization process. A second variation is performed around a non-optimal trajectory, in order to estimate the control-variable changes necessary to achieve the optimum in one iteration. Convergence is not at all certain when the trajectory differs greatly from the optimum, so the conventional techniques are used to limit step size in the early stages of the process.

1.4 Applications to Stochastic Systems

The method of approach to be used here is derived in part from some early work by Laning and Battin (Ref. 57, Sec. 6.6), who considered a linear system with an adjustable coefficient, subjected to a stochastic disturbance which was a non-white, non-stationary random function of time.

Subsequent work[21, 23, 26, 33, 41, 70] has been concentrated on systems in which the behaviour of the pertinent covariance matrices can be conveniently described by differential equations (variance equations). This implies that the random input is white noise, or is derivable from white noise by passage through a linear system. The statistical properties of the white noise can be time-varying and can depend on the state variables and control variables.

The approach used in these cases is to group the variance equations with the system differential equations and perturb the entire set, using the usual adjoint-equation technique.

In the problems to be considered here, however, the noise will be an arbitrary random function of several independent variables, in general non-white and non-stationary. In such problems the concepts of shaping filters and variance equations are impractical, and a more general approach is required.

1.5 Synopsis

The most general system to be considered here, is represented by a set of nonlinear, first-order differential equations. The system is subjected to disturbances which are non-white, non-stationary random functions of the independent variable, the state variables and the control variables. It is assumed that the disturbances are small, so that their effects may be determined on the basis of a linearization of the system equations around some nominal trajectory.

In the basic problem, we desire to minimize, in the mean-square sense, the random perturbations in a prescribed scalar function of the terminal state of the system. This is to be done by the manipulation of a set of control variables which determine the system trajectory. The trajectory, in turn, affects the random perturbations in two distinct ways: by determining the sensitivity of the system to the disturbances, and by determining the statistical properties of the disturbances themselves.

In Chapter 2 we present a derivation of some of the basic relations involved in the gradient method, from the point of view of Hilbert-space theory, on the assumption that the requisite influence functions are known. The same point of view is applied to a consideration of two distinct types of computational difficulties which may arise in practice.

In Chapter 3, after a discussion of the properties of the relevant types of stochastic processes, we derive the influence functions which apply to the basic problem. The succeeding sections extend these results to several types of performance criteria and constraints.

This is followed, in Chapter 4, by a similar derivation for several specific systems of fixed configuration.

Chapter 5 outlines the computational steps involved in the numerical determination of the influence functions.

In Chapter 6 the procedure is applied to several versions of a simple vehicle trajectory problem, and numerical results are presented.

Chapter 7 discusses the capabilities and limitations of the method, and suggests some further investigations which have yet to be carried out. This is followed, in the Appendices, by a brief discussion of some of the properties of linear systems, and of some interesting inequalities relating to matrices.

CHAPTER 2

THE GRADIENT METHOD IN HILBERT SPACE

2.1 Introduction

The basic procedures involved in the gradient method for optimization problems are: (1) the determination of influence functions which determine the effects of small control-variable changes on the cost function and on the constrained quantities; and (2) the application of these influence functions to determine the manner in which the control variables should be perturbed, in order to provide a rapid approach to the optimum without violating the constraints of the problem.

The central problem of this thesis is the first of these procedures, as applied to some general types of stochastic systems. In this chapter, however, we shall concern ourselves with the second phase of the technique, under the assumption that the influence functions have already been determined.

2.2 The Gradient Projection Technique in Function Space

The problem to be discussed here is the following: given the influence functions, or Green's functions, $\underline{g}_\phi(t)$ and $G_\psi(t)$ (a vector and a matrix respectively), such that

$$\delta\phi = \int_0^T \underline{g}_\phi^T(t)\, \delta\underline{u}(t)\, dt \tag{2-1}$$

and

$$\delta\underline{\psi} = \int_0^T G_\psi^T(t)\, \delta\underline{u}(t)\, dt \tag{2-2}$$

determine $\delta\underline{u}(t)$, $0 \leq t \leq T$, so as to maximize $\delta\phi$ with $\delta\underline{\psi}$ specified, with a prescribed value d^2 for the square of the norm of $\delta\underline{u}(t)$:

$$\left\|\delta\underline{u}\right\|^2 = \int_0^T \delta\underline{u}^T(t)\, W(t)\, \delta\underline{u}(t)\, dt = d^2 \tag{2-3}$$

In this formulation, $\phi(\underline{x}(T), T)$ is the payoff or cost function, a function of the final time T and the final state variables $\underline{x}(T)$. In problems of interest here, this corresponds to the Mayer formulation of the classical variational problem of Bolza.[9] Other types of performance criteria will be discussed in Chapter 5.

The vector function $\underline{\psi}(\underline{x}(T), T)$ is to be constrained to zero. In the usual steepest-descent procedure we set $\delta\underline{\psi} = -\underline{\psi}$ at each iteration, in order to correct for any wandering of $\underline{\psi}$ from zero due to numerical inaccuracies and the approximations involved in the linearization of the system equations.

The norm introduced in Eq. (2-3)serves as a measure of "step size", and the magnitude of d^2 is prescribed in such a way as to limit the step size to a region of reasonable linearity. The symmetric, positive definite weighting matrix W(t), if chosen properly for the problem in question, can be a valuable aid in improving convergence to the optimum.

The determination of $\delta\underline{u}(t)$ has usually been carried out by standard Lagrange-multiplier techniques[15]: d^2 and $\delta\underline{\psi}$ are adjoined to $\delta\phi$ by means of multipliers, the corresponding unconstrained maximization is carried out, and the multipliers are then determined by substitution in the constraint equations (2-2) and (2-3). This method is at best a mechanical one which obscures the physical significance of the multipliers. By the application of some of the basic concepts of the theory of Hilbert spaces, however, we can arrive at a derivation in which the significance of all quantities is obvious throughout.* Aside from its obvious pedagogical advantages, such a derivation can be helpful in avoiding basic errors in the application of the method.

Consider the control space to be divided into two orthogonal subspaces S and $S^{\perp}$. S is the subspace spanned by the columns of $W^{-1}(t)G_{\psi}(t)$, and $S^{\perp}$ (pronounced "S perp")[77, 69] is the orthogonal complement of S, i.e., the subspace of which every member is orthogonal to every member of S. By orthogo-

*After this development had been carried out, it was pointed out to the author that a somewhat similar approach had been outlined by R. Moroney in some internal memos at the Raytheon Company. Moroney's method, however, was based upon a procedure for orthonormalizing the columns of $W^{-1}(t)G_{\psi}(t)$. That this step is unnecessary will become apparent.

nality of two vector functions $\underline{v}(t)$ and $\underline{w}(t)$ we mean the vanishing of the inner product

$$(\underline{v}, \underline{w}) = \int_0^T \underline{v}^T(t) W(t) \underline{w}(t)\, dt \qquad (2\text{-}3.1)$$

For simplicity, we shall consider first the case $W(t) = I$.

Every vector function $\delta\underline{u}(t)$ can be expressed[77] as the sum of an element of S and an element of $S^{\perp}$:

$$\delta\underline{u}(t) = \underline{v}(t) + \underline{w}(t) \qquad (2\text{-}4)$$

where $\underline{v}(t) \in S$ and $\underline{w}(t) \in S$. Furthermore, the decomposition (2-4) is unique.[77] The characteristic property of $\underline{v}(t)$, as an element of S, is

$$\underline{v}(t) = G_{\psi}(t)\, \underline{c}, \qquad \underline{v}(t) \in S \qquad (2\text{-}5)$$

where $\underline{c}$ is a vector of constants. Thus $\underline{v}(t)$ is a linear combination of the columns of $G_{\psi}(t)$, the influence functions for the elements of $\underline{\psi}$. Since $\underline{w}(t)$ is a member of $S^{\perp}$, its characteristic property is that of orthogonality to every element of S (i. e., to every column of $G_{\psi}(t)$):

$$\int_0^T G_{\psi}^T(t)\, \underline{w}(t)\, dt = 0, \qquad \underline{w}(t) \in S^{\perp} \qquad (2\text{-}6)$$

Substituting (2-4) into (2-2) and using (2-6), we find

$$\delta\underline{\psi} = \int_0^T G_{\psi}^T(t)\, G_{\psi}(t)\, dt\, \underline{c} \qquad (2\text{-}7)$$

i. e., $\delta\underline{\psi}$ is not affected by $\underline{w}(t)$. Defining[15]

$$I_{\psi\psi} = \int_0^T G_{\psi}^T(t)\, G_{\psi}(t)\, dt \qquad (2\text{-}8)$$

we may determine the multipliers $\underline{c}$ as

$$\underline{c} = I_{\psi\psi}^{-1}\, \delta\underline{\psi} \tag{2-9}$$

(The possibility of a singular $I_{\psi\psi}$ will be discussed later.) Thus we have

$$\underline{v}(t) = G_{\psi}(t)\, I_{\psi\psi}^{-1}\, \delta\underline{\psi} \tag{2-10}$$

Because of the orthogonality of $\underline{v}(t)$ and $\underline{w}(t)$ (apparent from (2-5) and (2-6)), the Pythagorean relation holds:

$$\|\delta\underline{u}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2 \tag{2-11}$$

and hence the step-size constraint (2-3) becomes

$$\|\underline{v}\|^2 + \|\underline{w}\|^2 = d^2 \tag{2-12}$$

But $\underline{v}(t)$ has already been determined, so that

$$\|\underline{v}\|^2 = \int_0^T \delta\underline{\psi}^T\, I_{\psi\psi}^{-1}\, G_{\psi}^T(t)\, G_{\psi}(t)\, I_{\psi\psi}^{-1}\, \delta\underline{\psi}\, dt$$

$$= \delta\underline{\psi}^T\, I_{\psi\psi}^{-1}\, \delta\underline{\psi} \tag{2-13}$$

Since $\underline{v}(t)$ is not, in general, orthogonal to $\underline{g}_{\phi}(t)$, it will contribute to the payoff change $\delta\phi$. This contribution may or may not have the desired sign, but is unavoidable if the constraints are to be satisfied. It now remains to choose the control component $\underline{w}(t)$ in $S^{\perp}$, with norm specified by

$$\|\underline{w}\|^2 = d^2 - \delta\underline{\psi}^T\, I_{\psi\psi}^{-1}\, \delta\underline{\psi} \tag{2-14}$$

so as to maximize its contribution to $\delta\phi$, i.e., its inner product with $\underline{g}_{\phi}(t)$,

$$(\underline{g}_{\phi}, \underline{w}) = \int_0^T \underline{g}_{\phi}^T(t)\, \underline{w}(t)\, dt \tag{2-15}$$

We shall show that this objective is accomplished by making $\underline{w}(t)$ proportional to the orthogonal projection of $\underline{g}_{\phi}(t)$ on $S^{\perp}$. The orthogonal projection of a vector function $\underline{x}(t)$ on $S^{\perp}$ is defined[77] as the unique vector $\underline{p}(t)$ satisfying the relations

$$\left.\begin{aligned} \underline{x}(t) &= \underline{p}(t) + \underline{n}(t) \\ \underline{p}(t) &\in S^{\perp} \\ \underline{n}(t) &\in S \end{aligned}\right\} \quad (2\text{-}16)$$

We shall make use of the following lemma:

Lemma. The vector in $S^{\perp}$ with minimum norm, having unit inner product with $\underline{x}(t)$, is $\underline{r}(t) = \underline{p}(t)/(\underline{p}, \underline{x})$.

Proof. Consider any other vector $\underline{y}(t) \in S^{\perp}$ with $(\underline{y}, \underline{x}) = 1$. Then we may write

$$\underline{y}(t) = \underline{r}(t) + \underline{q}(t) \qquad \underline{q}(t) \in S^{\perp} \qquad (2\text{-}17)$$

$$(\underline{y}, \underline{x}) = 1 = \frac{(\underline{p}, \underline{x})}{(\underline{p}, \underline{x})} + (\underline{q}, \underline{x})$$

Hence $(\underline{q}, \underline{x}) = 0$, i.e., $\underline{q}(t)$ is orthogonal to $\underline{x}(t)$, $\underline{q}(t) \perp \underline{x}(t)$. But $(\underline{q}, \underline{x}) = (\underline{q}, \underline{p}) + (\underline{q}, \underline{n})$ and $(\underline{q}, \underline{n})$ vanishes because $\underline{q}(t) \subset S^{\perp}$ and $\underline{n}(t) \subset S$. Therefore it follows that $(\underline{q}, \underline{p}) = 0$, i.e., $\underline{q}(t) \perp \underline{r}(t)$, and from the Pythagorean relation

$$\|\underline{y}\|^2 = \|\underline{r}\|^2 + \|\underline{q}\|^2 \geq \|\underline{r}\|^2 \qquad (2\text{-}18)$$

(End of proof.) Our result follows as an obvious corollary.

From (2-16), the orthogonal projection $\underline{p}(t)$ of $\underline{g}_{\phi}(t)$ on $S^{\perp}$ satisfies

$$\underline{g}_{\phi}(t) = \underline{p}(t) + \underline{n}(t) \qquad (2\text{-}19)$$

$$\int_0^T G_{\psi}^T(t)\, \underline{p}(t)\, dt = 0 \qquad \underline{p}(t) \in S^{\perp} \qquad (2\text{-}20)$$

$$\underline{n}(t) = G_{\psi}(t)\, \underline{e} \qquad \underline{n}(t) \in S \qquad (2\text{-}21)$$

where $\underline{e}$ is a vector of constants. Combining these three relations we obtain

$$\int_0^T G_\psi^T(t)\, \underline{g}_\phi(t)\, dt = \int_0^T G_\psi^T(t)\, G_\psi(t)\, dt\, \underline{e} \tag{2-20}$$

Defining

$$\underline{v}_{\psi\phi} = \int_0^T G_\psi^T(t)\, \underline{g}_\phi(t)\, dt \tag{2-21}$$

and using (2-8), we obtain

$$\underline{e} = I_{\psi\psi}^{-1}\, \underline{v}_{\psi\phi} \tag{2-22}$$

so that finally

$$\underline{p}(t) = \underline{g}_\phi(t) - \underline{n}(t) = \underline{g}_\phi(t) - G_\psi(t)\, I_{\psi\psi}^{-1}\, \underline{v}_{\psi\phi} \tag{2-23}$$

Now $\underline{w}(t)$ must be made proportional to $\underline{p}(t)$, with a norm given by (2-14). Defining

$$k_{\phi\phi} = (\underline{g}_\phi, \underline{g}_\phi) \tag{2-24}$$

the squared norm of $\underline{p}(t)$ is

$$\|\underline{p}\|^2 = k_{\phi\phi} - \underline{v}_{\psi\phi}^T\, I_{\psi\psi}^{-1}\, \underline{v}_{\psi\phi} \tag{2-25}$$

so that $\underline{w}(t)$ should be chosen as

$$\underline{w}(t) = (\underline{g}_\phi(t) - G_\psi(t)\, I_{\psi\psi}^{-1}\, \underline{v}_{\psi\phi}) \sqrt{\frac{d^2 - \delta\underline{\psi}^T\, I_{\psi\psi}^{-1}\, \delta\underline{\psi}}{k_{\phi\phi} - \underline{v}_{\psi\phi}^T\, I_{\psi\psi}^{-1}\, \underline{v}_{\psi\phi}}} \tag{2-26}$$

For a minimization problem, in which a negative $\delta\phi$ is desired, $\underline{w}(t)$ is the negative of the expression in (2-26). We have now determined $\delta\underline{u}(t)$ completely:

$$\delta\underline{u}(t) = \pm\, (\underline{g}_\phi(t) - G_\psi(t)\, I_{\psi\psi}^{-1}\, \underline{v}_{\psi\phi}) \sqrt{\frac{d^2 - \delta\underline{\psi}^T I_{\psi\psi}^{-1}\, \delta\underline{\psi}}{k_{\phi\phi} - \underline{v}_{\psi\phi}^T\, I_{\psi\psi}^{-1}\, \underline{v}_{\psi\phi}}} + G_\psi(t) I_{\psi\psi}^{-1} \delta\underline{\psi} \tag{2-27}$$

By substitution in (2-1) we find

$$\delta\phi = \pm \sqrt{(d^2 - \delta\underline{\psi}^T I_{\psi\psi}^{-1} \delta\underline{\psi})(k_{\phi\phi} - \underline{v}_{\psi\phi}^T I_{\psi\psi}^{-1} \underline{v}_{\psi\phi})} + \underline{v}_{\psi\phi}^T I_{\psi\psi}^{-1} \delta\underline{\psi} \qquad (2\text{-}28)$$

In summary, we see that the second term in $\delta\underline{u}(t)$ is completely determined by the constraint corrections $\delta\underline{\psi}$. It adds an unavoidable contribution to $\delta\phi$, which may be beneficial or otherwise, and "uses up" a portion of the allotted step size d^2. The first term is then chosen so as to use the remaining portion of d^2 to best advantage while leaving the constraints unaffected.

From (2-25) we see that the denominator under the radical sign in (2-27) is always positive. The numerator will be negative if d^2 has been chosen too small to allow complete correction of the constraint violations. In such a case, we must eigher increase d^2 to a value at least as large as $\delta\underline{\psi}^T I_{\psi\psi}^{-1} \delta\underline{\psi}$, or else be content with only a partial correction of the constraints, achieved by reducing the magnitude of the specified $\delta\underline{\psi}$ until the radical becomes non-negative.

The vector function $\underline{p}(t)$ is known as the "gradient projection". The optimum has been reached when the constraints $\underline{\psi} = 0$ are satisfied and the gradient projection has vanished. The vanishing of the gradient projection means that $\underline{g}_\phi(t)$ has become an element of the subspace S (a linear combination of the columns of $G_\psi(t)$) and no longer contains a component orthogonal to all of the constraint influence functions. Hence no improvement in ϕ is possible without disturbing the constrained quantities $\underline{\psi}$.

The condition that the gradient projection vanish, along with the defining relations for the influence functions (the adjoint equations of Ref. 15 and Appendix A), can be shown to be identical with the Euler-Lagrange equations for the problem of Bolza, as derived by the classical techniques of the calculus of variations;[9, 11] the final values of the adjoint variables can be derived from the transversality conditions. The complete set of conditions is contained in the so-called multiplier rule.[9]

The analogy in a three-parameter problem is obvious: we wish to find the coordinates x, y, z of the point on the constraint surface $\psi(x, y, z) = 0$ for which $\phi(x, y, z)$ is maximum. The first step in improving an initial guess is to alter the coordinates in the direction of the gradient of ψ(the normal to the constraint surface) until the surface $\psi = 0$ is reached. We then

proceed tangent to the surface (normal to grad ψ) in a direction which gives the most rapid increase in ϕ. This direction is that of the orthogonal projection of grad ϕ on the constraining surface. The optimum has been reached when grad ϕ and grad ψ are parallel at a point on $\psi = 0$, for then ϕ cannot be varied without altering ψ.

For the case in which W(t) is not chosen to be the identity matrix, the analysis is altered only slightly. Using the general definition (2-3.1) for the inner product, and adopting the corresponding definitions of orthogonality and norm, the relations (2-1) and (2-2) may be expressed as

$$\delta\phi = (W^{-1}\underline{g}_{\phi},\ \delta\underline{u}) \tag{2-29}$$

and

$$\delta\underline{\psi} = (W^{-1}G_{\psi},\ \delta\underline{u}) \tag{2-30}$$

where the notation of (2-30) is used to represent a vector whose elements are the inner products of $\delta\underline{u}(t)$ with the columns of $W^{-1}(t)\,G_{\psi}(t)$. Redefining

$$I_{\psi\psi} = \int_0^T G_{\psi}^T(t)\,W^{-1}(t)\,G_{\psi}(t)\,dt \tag{2-31}$$

$$\underline{v}_{\psi\phi} = (W^{-1}G_{\psi},\ W^{-1}\underline{g}_{\phi}) = \int_0^T G_{\psi}^T W^{-1}\underline{g}_{\phi}\,dt \tag{2-31.1}$$

and

$$k_{\phi\phi} = (W^{-1}\underline{g}_{\phi},\ W^{-1}\underline{g}_{\phi}) = \int_0^T \underline{g}_{\phi}^T W^{-1}\underline{g}_{\phi}\,dt \tag{2-31.2}$$

the equations (2-27) and (2-28) remain valid if we premultiply the right hand side of Eq. (2-27) by $W^{-1}(t)$:

$$\delta\underline{u}(t) = \pm\, W^{-1}(t)\left[\underline{g}_{\phi}(t) - G_{\psi}(t)\,I_{\psi\psi}^{-1}\underline{v}_{\psi\phi}\right]\sqrt{\frac{d^2 - \delta\underline{\psi}^T I_{\psi\psi}^{-1}\,\delta\underline{\psi}}{k_{\phi\phi} - \underline{v}_{\psi\phi}^T I_{\psi\psi}^{-1}\,\underline{v}_{\psi\phi}}}$$

$$+\ W^{-1}(t)G_{\psi}(t)I_{\psi\psi}^{-1}\,\delta\underline{\psi} \tag{2-32}$$

2.3 The Case of a Matrix of Control Functions

We shall see that, in certain problems involving optimization by means of a matrix of control functions, the problem formulation of Sec. 2.3 is changed to the following:

Given the influence function matrices $M_\phi(t)$ and $M_\psi(t)$, and the vector $\underline{v}(t)$, such that

$$\delta\phi = \mathrm{tr} \int_0^T M_\phi^T(t)\, \delta A(t)\, dt \tag{2-33}$$

and

$$\delta\underline{\psi} = \int_0^T M_\psi^T(t)\, \delta A(t)\, \underline{v}(t)\, dt \tag{2-34}$$

find the control matrix variation, $\delta A(t)$, which maximizes $\delta\phi$ with $\delta\underline{\psi}$ specified, for a given value of a specified norm of $\delta A(t)$.

There are numerous forms which may be assumed for the norm of $\delta A(t)$, and the choice depends on the particular problem at hand. If we desire to employ the most general quadratic form in the elements of $\delta A(t)$, we require, if $A(t)$ has dimensions $n \times m$, $nm(nm+1)/2$ coefficients. The notation is greatly simplified if we assume a norm which weights only the squares of the elements. In this case, we require only nm coefficients, and the norm can be expressed by

$$\|\delta A\|^2 = \sum_{i=1}^{m} \int_0^T \underline{1}_i^T\, \delta A^T(t)\, W_i(t)\, \delta A(t)\, \underline{1}_i\, dt \tag{2-35}$$

where the matrices $W_i(t)$ are $n \times n$ diagonal matrices of positive elements and $\underline{1}_i$ is a vector of appropriate dimension (in this case m) with an i-th element of one and zeroes elsewhere. This expression weights the term δa_{ij}^2 by the coefficient $(W_j)_{ii}$. In the simpler case in which $W_i(t) = W(t)$ for all i, the expression may be written

$$\mathrm{tr} \int_0^T \delta A^T(t)\, W(t)\, \delta A(t)\, dt$$

It is possible, of course, by expanding the above expressions and redefining terms, to cast this problem in the form of that of Sec. 2.2, in which $\delta A(t)$ has been transformed into a vector by placing its columns end to end. It is felt that the present approach is both more straightforward and notationally simpler, and does not obscure the matrix character of the control function.

For purposes of demonstration, we shall derive the results by the usual Lagrange-multiplier technique, then indicate the steps involved in the derivation from Hilbert-space concepts. By means of the multipliers $\underline{\nu}$ and μ, we adjoin to $\delta\phi$ the constrained quantities of (2-34) and (2-35), and maximize the quantity

$$\begin{aligned} q &= \delta\phi - \underline{\nu}^T \delta\underline{\psi} - \mu \|\delta A\|^2 \\ &= \mathrm{tr} \int_0^T M_\phi^T \delta A \, dt - \underline{\nu}^T \int_0^T M_\psi^T \delta A \, \underline{v} \, dt \\ &\quad - \mu \sum_{i=1}^{n} \int_0^T \underline{1}_i^T \delta A^T W_i \, \delta A \underline{1}_i \, dt \end{aligned} \tag{2-36}$$

Noting that

$$\mathrm{tr}\, M = \sum_i \underline{1}_i^T M \underline{1}_i \tag{2-36.1}$$

and

$$\sum_i \underline{1}_i \, \underline{1}_i^T = I \tag{2-36.2}$$

we may write

$$\delta q = \int_0^T \sum_{i=1}^{m} \left[\underline{1}_i^T M_\phi^T - \underline{\nu}^T M_\psi^T \underline{1}_i^T \underline{v} - 2\mu \underline{1}_i^T \delta A^T W_i \right] \delta^2 A \underline{1}_i \, dt \tag{2-37}$$

which vanishes for arbitrary $\delta^2 A$ if

$$2\mu \underline{1}_i^T \delta A^T = \left[\underline{1}_i^T M_\phi^T - \underline{\nu}^T M_\psi^T \underline{1}_i^T \underline{v}\right] W_i^{-1} \tag{2-38}$$

Premultiplying by $\underline{1}_i$ and summing, and noting that the scalar $\underline{1}_i^T \underline{v}$ may be changed in position, we find, after transposing,

$$\delta A = \frac{1}{2\mu} \sum_{i=1}^{m} W_i^{-1} \left[M_\phi - M_\psi \underline{\nu}\, \underline{v}^T\right] I_i \tag{2-39}$$

where I_i is a square matrix with an ii-th element of one and zeroes elsewhere. If the matrices W_i are identical, this reduces to

$$\delta A = \frac{1}{2\mu} W^{-1} \left[M_\phi - M_\psi \underline{\nu}\, \underline{v}^T\right] \tag{2-40}$$

The multipliers must now be evaluated by substituting (2-39) into the boundary-value constraint (2-34) and the step-size constraint

$$\|\delta A\|^2 = d^2 \tag{2-41}$$

Substituting in (2-34) and defining

$$\underline{v}_{\psi\phi} = \int_0^T M_\psi^T \sum_{i=1}^{m} W_i^{-1} M_\phi I_i \underline{v}\, dt \tag{2-42}$$

and

$$I_{\psi\psi} = \int_0^T M_\psi^T \sum_{i=1}^{m} W_i^{-1} M_\psi (\underline{v}^T I_i \underline{v})\, dt \tag{2-43}$$

we find

$$\delta\underline{\psi} = \frac{1}{2\mu} \left[\underline{v}_{\psi\phi} - I_{\psi\psi} \underline{\nu}\right]$$

or

$$\underline{\nu} = I_{\psi\psi}^{-1} \left[-2\mu\, \delta\underline{\psi} + \underline{v}_{\psi\phi}\right] \tag{2-44}$$

Substitution of (2-40) and (2-44) into (2-41) yields a quadratic equation in μ in which the linear term vanishes. Defining

$$k_{\phi\phi} = \int_0^T \sum_{i=1}^{m} \underline{1}_i^T M_\phi^T W_i^{-1} M_\phi \underline{1}_i \, dt \tag{2-45}$$

we obtain, after considerable simplification,

$$4\mu^2 \left[d^2 - \delta\underline{\psi}^T I_{\psi\psi}^{-1} \delta\underline{\psi} \right] = k_{\phi\phi} - \underline{v}_{\psi\phi}^T I_{\psi\psi}^{-1} \underline{v}_{\psi\phi} \tag{2-46}$$

Finally we may write

$$\delta A(t) = \pm \sum_{i=1}^{m} W_i^{-1} \left[M_\phi(t) - M_\psi(t) I_{\psi\psi}^{-1} \underline{v}_{\psi\phi} \underline{v}^T(t) \right] I_i \sqrt{\frac{d^2 - \delta\underline{\psi}^T I_{\psi\psi}^{-1} \delta\underline{\psi}}{k_{\phi\phi} - \underline{v}_{\psi\phi}^T I_{\psi\psi}^{-1} \underline{v}_{\psi\phi}}}$$

$$+ \sum_{i=1}^{m} W_i^{-1} M_\psi(t) I_{\psi\psi}^{-1} \delta\underline{\psi} \underline{v}^T(t) I_i \tag{2-47}$$

Once again the sign of the first term is chosen according to whether the problem is one of maximization or minimization. The improvement in the payoff function is again given by (2-28).

The above procedure, in contrast with the Hilbert-space approach, guarantees only the stationarity of $\delta\phi$ with respect to δA. It is possible, however, to show that the δA of (2-47) does indeed give the global maximum of $\delta\phi$ (for the linearized problem) subject to the given constraints. To prove this, let $\delta A^*(t)$ denote the optimal $\delta A(t)$ as given by (2-39) or (2-47), and consider a variation $\delta^2 A(t)$ in $\delta A(t)$ (a second variation in $A(t)$). It follows from (2-33), (2-34), and (2-35) that with

$$\delta A(t) = \delta A^*(t) + \delta^2 A(t) \tag{2-48}$$

we will obtain

$$\delta^2 \phi = \mathrm{tr} \int_0^T M_\phi^T(t) \, \delta^2 A(t) \, dt \tag{2-49}$$

$$\delta^2 \underline{\psi} = \int_0^T M_\psi^T(t)\, \delta^2 A(t)\, \underline{v}(t)\, dt \tag{2-50}$$

and

$$\delta \| \delta A \|^2 = \| (\delta A^* + \delta^2 A) \|^2 - \| \delta A^* \|^2$$

$$= \sum_{i=1}^{m} \int_0^T \underline{l}_i^T\, \delta^2 A^T W_i\, \delta^2 A \underline{l}_i dt$$

$$+ 2 \sum_{i=1}^{m} \int_0^T \underline{l}_i^T\, \delta A^{*T} W_i\, \delta^2 A \underline{l}_i dt \tag{2-51}$$

The first term in (2-51) is the squared norm of $\delta^2 A(t)$, and hence must be positive. It follows, since $\delta \| \delta A \|^2$ must remain constant, that the second term is negative for any non-zero $\delta^2 A(t)$. Substituting from (2-39),

$$0 \geq \frac{1}{\mu} \sum_{i=1}^{m} \int_0^T \underline{l}_i^T \left[M_\phi^T - \underline{v}\, \underline{\nu}^T M_\psi^T \right] \delta^2 A \underline{l}_i dt$$

$$= \frac{1}{\mu} \left[\mathrm{tr} \int_0^T M_\phi^T\, \delta^2 A\, dt - \underline{\nu}^T \int_0^T M_\psi^T\, \delta^2 A\, \underline{v}\, dt \right]$$

$$= \frac{1}{\mu} \left[\delta^2 \phi - \underline{\nu}^T \delta^2 \underline{\psi} \right] \tag{2-52}$$

Now $\delta^2 \underline{\psi}$ must vanish if the constraints are to be maintained, so we conclude that

$$\frac{\delta^2 \phi}{\mu} \leq 0 \tag{2-53}$$

But the sign of μ is the desired sign of the payoff variation $\delta\phi$, so that $\delta^2 \phi$ must have the opposite sign, and the proof is complete.

The results of this section may also be derived by the methods of Sec. 2.2. In this case the elements of the control space are matrix functions. For simplicity, we shall consider only the case $W_i = I$. The inner product of matrices of similar dimensions is defined as

$$(A,\ B) = \mathrm{tr} \int_0^T A^T B\, dt \tag{2-54}$$

We now have

$$\delta\phi = (M_\phi,\ \delta A) \tag{2-55}$$

and

$$\delta\psi_i = (M_{\psi i},\ \delta A) \tag{2-56}$$

where

$$M_{\psi i} = M_\psi \underline{1}_i \underline{v}^T \tag{2-57}$$

The elements of the subspace S are linear combinations of the $M_{\psi i}$, i.e.,

$$V(t) = M_\psi(t)\, \underline{c}\ \underline{v}^T(t) \qquad\qquad V(t) \in S \tag{2-58}$$

where $\underline{c}$ is a vector of constants, and the subspace $S^\perp$ is defined by

$$(W,\ M_\psi \underline{c}\ \underline{v}^T) = 0 \text{ for all } \underline{c} \qquad\qquad W(t) \in S^\perp \tag{2-59}$$

which may be written

$$\int_0^T M_\psi^T W \underline{v}\, dt = 0 \qquad\qquad W(t) \in S^\perp \tag{2-60}$$

The development proceeds as before, leading to the final result

$$\delta A(t) = \pm \left[M_\phi(t) - M_\psi(t)\, I_{\psi\psi}^{-1} \underline{v}_{\psi\phi}\, \underline{v}^T(t) \right] \sqrt{\frac{d^2 - \delta\underline{\psi}^T I_{\psi\psi}^{-1} \delta\underline{\psi}}{k_{\phi\phi} - \underline{v}_{\psi\phi}^T I_{\psi\psi}^{-1} \underline{v}_{\psi\phi}}}$$

$$+ M_\psi(t)\, I_{\psi\psi}^{-1}\, \delta\underline{\psi}\, \underline{v}^T(t) \tag{2-61}$$

where

$$I_{\psi\psi} = \int_0^T M_\psi^T M_\psi \underline{v}^T \underline{v}\, dt \tag{2-62}$$

$$\underline{v}_{\psi\phi} = \int_0^T M_\psi^T M_\phi \underline{v}\, dt \tag{2-63}$$

and

$$k_{\phi\phi} = \left\| M_\phi \right\|^2 \tag{2-64}$$

Once again the payoff improvement is given by (2-28).

2.4 Adjustable Time-Invariant Parameters

Denham[22] has considered the case where a set of time-invariant parameters $\underline{p}$ is to be chosen along with a vector of control functions $\underline{u}(t)$, $0 \leq t \leq T$. $\underline{p}$ may include not only parameters of the system, but also such quantities as initial time and initial values of the state variables, if the starting conditions for the problem are not completely specified. In this section we shall state, without proof, the corresponding results in the case where a matrix of control functions A(t) is also to be adjusted.

Such a system cannot, in general, be optimized by a cyclic adjustment of the three sources of control. It is easy to imagine a system, for example, in which no adjustment of the initial conditions can be made without violating the constraints at the terminal time. It is possible, however, that such an adjustment, together with a change in the control functions, might result in an improvement in performance with no net change in the constrained quantities. It follows, therefore, that simultaneous adjustment of all the control quantities is necessary to ensure that the optimum will be reached.

The problem to be considered here is the following: given the appropriate influence functions such that

$$\delta\phi = \int_0^T \underline{g}_\phi^T(t)\, \delta\underline{u}(t)\, dt + \mathrm{tr} \int_0^T M_\phi^T(t)\, \delta A(t)\, dt + \underline{b}_\phi^T\, \delta\underline{p} \tag{2-65}$$

and

$$\delta\underline{\psi} = \int_0^T G_\psi^T(t)\,\delta\underline{u}(t)\,dt + \int_0^T M_\psi^T(t)\,\delta A(t)\,\underline{v}(t)\,dt + B_\psi^T\,\delta\underline{p} \tag{2-66}$$

determine $\delta\underline{u}(t)$ and $\delta A(t)$, $0 \le t \le T$, and $\delta\underline{p}$, so as to maximize $\delta\phi$ with $\delta\underline{\psi}$ specified, with a prescribed value d^2 for

$$\left\| (\delta\underline{u},\ \delta A,\ \delta\underline{p}) \right\|^2 = \int_0^T \delta\underline{u}^T(t)\,W(t)\,\delta\underline{u}(t)\,dt$$

$$+ \sum_{i=1}^{m} \int_0^T \underline{l}_i^T\,\delta A^T(t)\,W_i(t)\,\delta A(t)\,\underline{l}_i\,dt + \delta\underline{p}^T W_p\,\delta\underline{p} \tag{2-67}$$

where W_p is a constant positive-definite matrix.

Undoubtedly in many cases the vector $\delta\underline{u}(t)$ in (2-65) could be absorbed into the matrix $\delta A(t)$. In some problems, however (such as that of Sec. 4.5), the formulation used here is a more natural one. Furthermore, the norm defined in (2-67) provides the possibility of arbitrarily weighting the cross products of the elements of $\delta\underline{u}(t)$.

We shall redefine

$$I_{\psi\psi} = \int_0^T G_\psi^T W^{-1} G_\psi\,dt + \int_0^T M_\psi^T \sum_{i=1}^{m} W_i^{-1} M_\psi\,(\underline{v}^T I_i\,\underline{v})\,dt$$

$$+ B_\psi^T W_p^{-1} B_\psi \tag{2-68}$$

$$\underline{v}_{\psi\phi} = \int_0^T G_\psi^T W^{-1} \underline{g}_\phi\,dt + \int_0^T M_\psi^T \sum_{i=1}^{m} W_i^{-1} M_\phi\,I_i\,\underline{v}\,dt$$

$$+ B_\psi^T W_p^{-1} \underline{b}_\phi \tag{2-69}$$

$$k_{\phi\phi} = \int_0^T \underline{g}_\phi^T W^{-1} \underline{g}_\phi\,dt + \int_0^T \sum_{i=1}^{m} \underline{l}_i^T M_\phi^T W_i^{-1} M_\phi \underline{l}_i\,dt + \underline{b}_\phi^T W_p^{-1} \underline{b}_\phi \tag{2-70}$$

With these definitions, $\delta\underline{u}(t)$ and $\delta A(t)$ are determined by (2-32) and (2-47), and

$$\delta\underline{p} = \pm\, W_p^{-1}\left[\underline{b}_\phi - B_\psi I_{\psi\psi}^{-1}\underline{v}_{\psi\phi}\right]\sqrt{\frac{d^2 - \delta\underline{\psi}^T I_{\psi\psi}^{-1}\,\delta\underline{\psi}}{k_{\phi\phi} - \underline{v}_{\psi\phi}^T I_{\psi\psi}^{-1}\underline{v}_{\psi\phi}}} + W_p^{-1} B_\psi I_{\psi\psi}^{-1}\,\delta\underline{\psi} \qquad (2\text{-}71)$$

Equation (2-28) remains a valid expression for $\delta\phi$.

2.5 A Remedy for the "Ravine" Problem

A frequently-occurring difficulty in steepest-descent computations is the so-called "ravine" problem. The name arises by analogy with the two-parameter optimization problem. If the contour $\phi(x, y)$ exhibits a gradually-sloping, steep-sided "ravine", successive iterations may result in a jumping from side to side of the ravine with very slow payoff improvement. The result can be a considerable waste of computational effort.

Several methods have been devised for dealing with this difficulty, with varying degrees of success. Choice of the proper coordinate system in which to describe the problem can have the effect of distorting the "ravine" into a circular "bowl" in which convergence is more rapid. In most problems this transformation varies from one iteration to the next. A. A. Goldstein, using the analogy with the three-parameter case, refers to the "sausage" problem, and suggests a procedure by which a hyperellipsoid is fitted to the local payoff contour. A transformation of coordinates is then performed in such a way as to make the ellipsoid spherical. This method was applied in a two-parameter problem by Fadden and Gilbert,[32] but the extension to function-space is not immediately apparent.

Similar results can often be achieved in function-space problems by judicious choice of the weighting functions used in the definition of the inner product (2-3.1), but at the present time this procedure remains an art largely dependent on the experience and intuition of the programmer.

Kulakowski and Stancil[56] compare successive gradients in an attempt to estimate the appropriate second derivatives. The hoped-for result is a saving in computation time for gradient determination, which

in turn would allow the ravine problem to be solved by simply making smaller changes at each iteration. The method was improved upon by Beskind[6] and appears to give reasonably good results.

Other methods for improving convergence in finite-parameter problems have been proposed by Powell[66, 61] and by Fletcher and Powell.[34] The latter method seems to give particularly impressive results,[34, 62] and is based upon a transformation of the gradient vector which rotates it in the direction of the curvature of the path of steepest descent.

The approach to be outlined here is simple in principle and easily applied to function-space problems.

It is apparent in two- or three - parameter problems that the "zig-zagging" down a ravine will result from successive gradient vectors almost doubling back upon each other. If we determine the angle between successive gradient vectors at each step, this parameter can serve as an indicator of approaching difficulty, since an angle near 180° will result when a ravine is crossed. When this situation is detected, there are various procedures which may be followed. By estimating the rates of change of the gradient we may attempt to take a large step in the direction of the predicted optimum. In this case the nonlinearities of the problem may cause considerable difficulties. A more conservative approach, and a perfectly acceptable one, is to "backtrack" a portion of the last step to a point near the "bottom" of the ravine, where the gradient direction is more favorable. One method of accomplishing this is to search the line joining the two points in the parameter space, for the point of minimum cost. This is equivalent computationally to several iterations, and furthermore does not generally result in finding the best point from which to proceed, as will be seen presently. A simpler and more effective method, based upon a truncated Taylor series in Hilbert space, will be used here.

In two- or three - parameter problems the cosine of the angle between two gradients is their inner product normalized by dividing by the product of their norms. The analogous quantity in Hilbert-space problems is still referred to as the cosine of the "angle" between two elements of the space.

The gradient of interest is the gradient projection $\underline{p}(t)$ of (2-23), or, in the case of an arbitrary weighting matrix,

$$\underline{p}(t) = W^{-1}(t)\left[\underline{g}_{\phi}(t) - G_{\psi}(t)\, I_{\psi\psi}^{-1}\, \underline{v}_{\psi\phi}\right] \tag{2-72}$$

In other words, we are confining ourselves to the control subspace imposed by the constraints $\underline{\psi} = 0$.

For the more general problem discussed in Sec. 2.4, the development is entirely analogous.

Let two successive gradient projections be denoted $\underline{p}_i(t)$ and $\underline{p}_{i+1}(t)$. We define

$$\cos\,\theta_{i,\,i+1} = \frac{(\underline{p}_i,\ \underline{p}_{i+1})}{\|\underline{p}_i\|\ \ \|\underline{p}_{i+1}\|} \tag{2-73}$$

By the Schwarz inequality, this quantity never has magnitude greater than one, hence the "cosine" designation is reasonable. We will approximate the behaviour of $\underline{p}(t)$ in a small neighbourhood in the control space by

$$\underline{p}(t) = \underline{p}_i(t) + \int_0^T M(t,\ \tau)\,\delta\underline{u}(\tau)\,d\tau \tag{2-74}$$

and neglect higher order terms. The matrix $M(t,\ \tau)$ may be computed from second-variation considerations, assuming the necessary second derivatives exist, but the computation is rather involved and is, in fact, unnecessary.

Assuming that the "constraint-correcting" term $\underline{v}(t)$ in (2-4) is small compared to the "optimizing" term $\underline{w}(t)$, $\delta\underline{u}(t)$ will be proportional to $\underline{p}_i(t)$, and hence we have

$$\underline{p}_{i+1}(t) = \underline{p}_i(t) + k_i \int_0^T M(t,\ \tau)\ \underline{p}_i(\tau)\,d\tau \tag{2-75}$$

where k_i will be a known constant. If the succeeding step is to "backtrack" part way, we will have

$$\underline{p}_{i+2}(t) = \underline{p}_{i+1}(t) + k_{i+1} \int_0^T M(t,\ \tau)\,\underline{p}_i(\tau)\,d\tau \tag{2-76}$$

and the quantity to be determined is k_{i+1}. Defining

$$a_{ij} = (\underline{p}_i, \underline{p}_j) \tag{2-77}$$

we may attempt to choose

$$r_i = \frac{k_{i+1}}{k_i} \tag{2-78}$$

in such a way as to achieve a specified value c for $\cos \theta_{i, i+2}$. The requirement is then

$$\cos^2 \theta_{i, i+2} = \frac{a^2_{i, i+2}}{a_{ii}\, a_{i+2, i+2}} = c^2 \tag{2-79}$$

Taking the scalar products of (2-75) and (2-76) with $\underline{p}_i(t)$, and combining the results, we may eliminate $M(t, \tau)$ and obtain

$$r_i = \frac{a_{i, i+2} - a_{i, i+1}}{a_{i, i+1} - a_{ii}} \tag{2-80}$$

Noting that, from (2-75) and (2-76),

$$\underline{p}_{i+2} = - r_i \underline{p}_i + (1 + r_i) \underline{p}_{i+1} \tag{2-81}$$

and that therefore

$$a_{i+2, i+2} = r_i^2 a_{ii} + (1 + r_i)^2 a_{i+1, i+1} - 2r_i(1 + r_i) a_{i, i+1} \tag{2-82}$$

and

$$a_{i, i+2} = - r_i a_{ii} + (1 + r_i) a_{i, i+1} \tag{2-83}$$

we may now combine equations (2-79) to (2-83) to determine a quadratic equation for r_i:

$$r_i^2\left[(a_{ii} - a_{i,\,i+1})^2 - c^2 a_{ii}(a_{ii} + a_{i+1,\,i+1} - 2a_{i,\,i+1})\right]$$

$$+ 2r_i\left[a_{i,\,i+1}(a_{i,\,i+1} - a_{ii}) - c^2 a_{ii}(a_{i+1,\,i+1} - a_{i,\,i+1})\right]$$

$$+ a_{i,\,i+1}^2 - c^2 a_{ii} a_{i+1,\,i+1} = 0 \qquad (2\text{-}84)$$

In meaningful cases, both solutions will be between -1 and zero, and the most negative of the two should be chosen if $\cos \theta_{i,\,i+2}$ is to be made positive. If we require orthogonality of $\underline{p}_i$ and $\underline{p}_{i+2}$ (i.e., $c = 0$), (2-84) simplifies to

$$r_i = \frac{-1}{1 - \dfrac{a_{ii}}{a_{i,\,i+1}}} \qquad (2\text{-}85)$$

In a similar manner we may determine the value of r_i required for a specified value of $\cos \theta_{i+1,\,i+2}$. If we require orthogonality of $\underline{p}_{i+2}$ and $\underline{p}_{i+1}$ we obtain

$$r_i = \frac{-1}{1 - \dfrac{a_{i,\,i+1}}{a_{i+1,\,i+1}}} \qquad (2\text{-}86)$$

A particularly simple result, in comparison with (2-84), is obtained if we specify that $\underline{p}_{i+2}$ make equal "angles" with $\underline{p}_i$ and $\underline{p}_{i+1}$, i.e.,

$$\cos \theta_{i,\,i+2} = \cos \theta_{i+1,\,i+2} \qquad (2\text{-}87)$$

In this case we obtain*

*Or alternatively, letting $m = -r$, $(1 - m)/m = \sqrt{a_{ii}/a_{i+1,\,i+1}}$. In the case of time-invariant three-dimensional vectors, this reduces to the familiar geometric theorem that the bisector of one angle of a triangle divides the opposite side in the ratio of the lengths of the other two sides.

$$r_i = \frac{-1}{1 + \sqrt{\frac{a_{ii}}{a_{i+1,\,i+1}}}} \qquad (2\text{-}88)$$

A point worthy of note is that (2-88) depends only on the relative magnitudes of $\underline{p}_i$ and $\underline{p}_{i+1}$, and not on their inner product. The inner product is used only to determine when such a correction should be made.

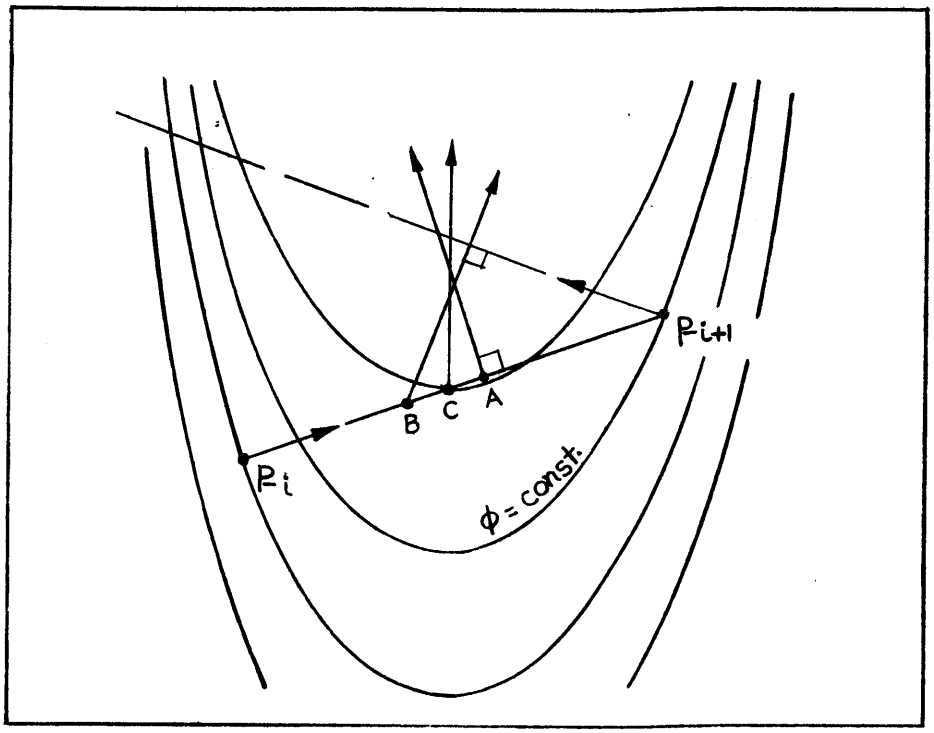

Figure 2.1 The Ravine Problem in the Two-Parameter Case.

The superiority of (2-88) over (2-85) and (2-86) is illustrated, in the two-parameter case, in Figure 2.1. The point A is the point which would be found by searching along ABC for the point of minimum cost (as indicated by the contour lines of constant cost). It is also the point attained approximately by the use of Eq. (2-85). . Equation (2-86) would yield point B, but point C, obtained as a result of the relation (2-88), lies somewhere between the two, and closer to the plane of symmetry of the

ravine. The cost at C is actually greater than that at A, but succeeding iterations can be expected to give better results because the sides of the ravine will not be climbed as quickly.

It should be noted that the procedure does not require the computation of the second variations involved in $M(t, \tau)$. The increase in computation is mainly in the determination of $a_{i, i+1}$, the inner product of two gradients. It is also not necessary to perform a search and make several payoff computations. This is a considerable advantage in the problems to be considered here, in which the computation of the payoff function is almost as costly as the determination of a new gradient.

Another advantage of the approach is the fact that the parameter $\cos \theta$ provides an excellent indicator of the linearity of the problem. If $\cos \theta \approx 1$, we can usually conclude that larger perturbations could be made on each iteration, thus saving computation time. If $\cos \theta$ approaches zero, it is probably wise to use somewhat smaller step sizes.

2.5.1 Numerical Results

A rather severe ravine problem was encountered in one of the problems described in Chapter 6, resulting in every second iteration being almost identical, although a gradual decrease in the cost function was still being obtained. The method described here was applied, and led to a considerable improvement in the behavior of the iteration procedure.

At every step the parameter $\cos \theta_{i, i+1}$ was determined, and a correction was applied when the parameter was less than -0.1. The value of r_i applied was the arithmetic mean of the quantities in (2-85) and (2-86). This is virtually identical with (2-88) when $\| \underline{p}_i \| \approx \| \underline{p}_{i+1} \|$, or when $\cos \theta_{i, i+1}$ is less than about -0.5.

Figure 2.2 illustrates the results achieved. The trajectory to be optimized joined the origin in the xy plane to the point x = 1000, y = 0. The values of y at x = 100 and x = 500 were chosen as parameters indicative of the general form of the trajectory. The first few cycles in the figure are the result of iterations with constant step size, showing the oscillatory nature of the trajectory variations. After each correction the oscillation disappears, but a divergence is again noted after a few cycles, at which time the correction procedure is applied once more.

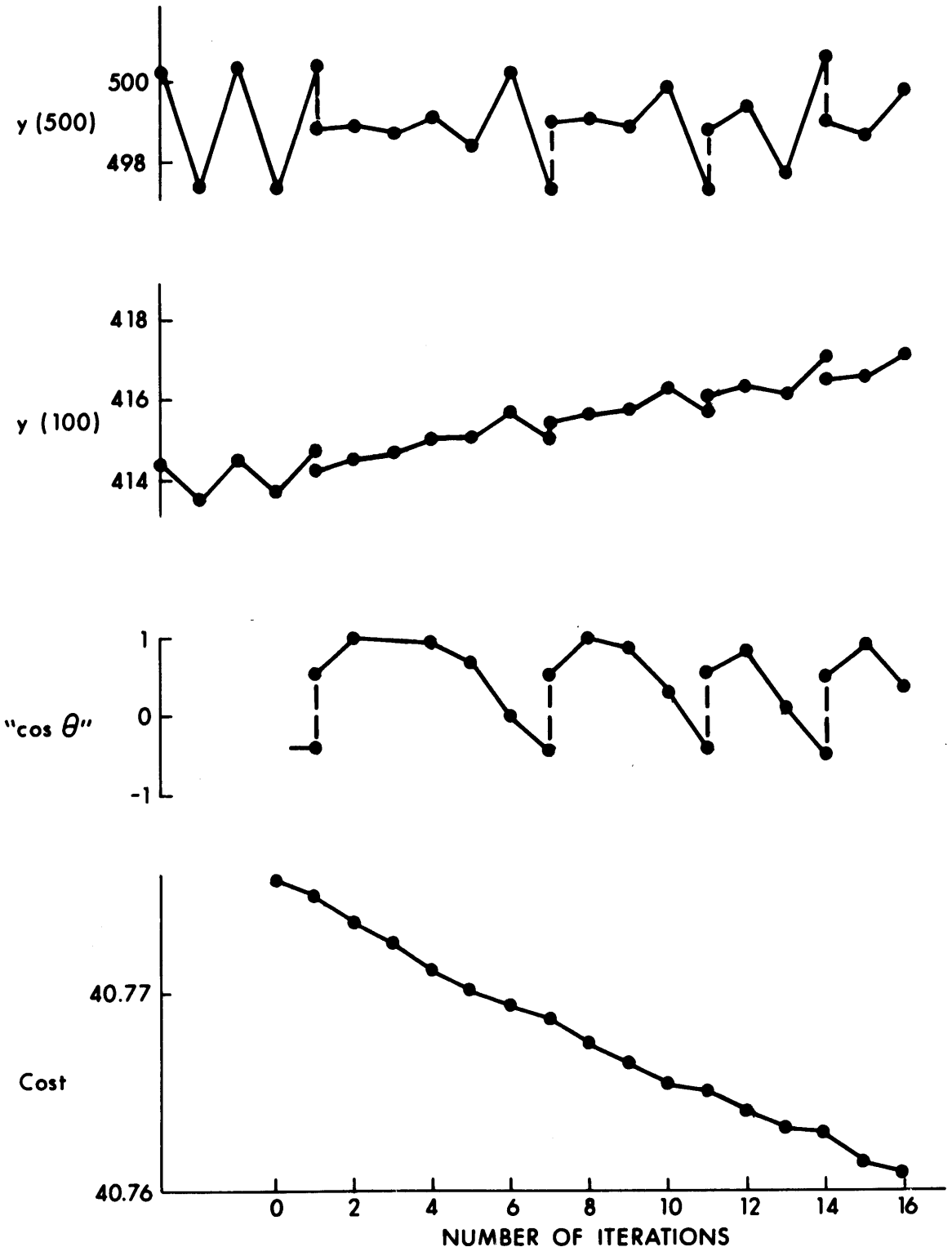

Figure 2.2 Correction Procedure for the Ravine Problem.

The divergence could be made less severe by correcting at a higher value of cos θ, but the more frequent corrections would be costly in terms of computation.

During the computation, each period of divergence was seen to be accompanied by a progressive increasing of the magnitude of the gradient projection, as indicated by the parameter $\|\underline{p}_i\|$. At each correction, however, this magnitude decreased considerably. In spite of this fact, and the fact that the step size was kept constant except during corrections, the cost function can be seen to decrease more rapidly immediately after each correction. This is exactly the type of behavior which would be predicted by analogy with the two-parameter case.

The values of cos θ serve to indicate whether the corrections have been applied in a desirable fashion. If so, we would expect the system to begin to operate in a more linear fashion, as we start along the "bottom" of the ravine. This would be indicated by values of cos θ near +1 following the first iteration after a correction. The figure shows that this is indeed the case.

Since in this problem "$\underline{p}(t)$" is not a vector but a single function p(x), it is of interest to compare successive gradient projections. Figure 2.3 shows two successive functions just before a correction, and the one immediately following, which is approximately the average of the other two. It can be concluded that the rapid variations near each end of the x

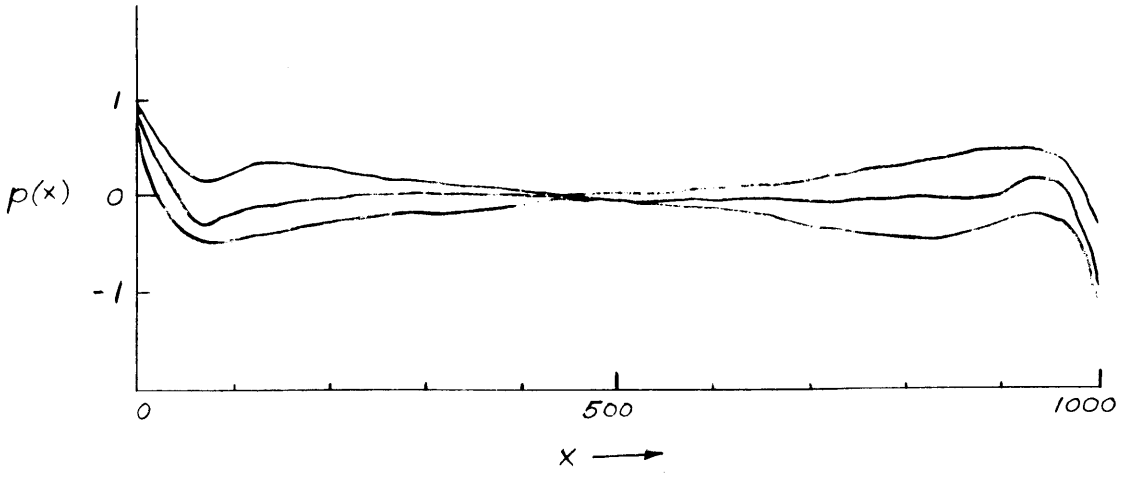

Figure 2.3 Effect of the Correction Procedure on the Gradient Projection.

scale, which are present in all three functions, represent, in some sense, the component of the gradient projection "along" the ravine. These are superimposed on an almost-linear function whose slope changes sign when the ravine is crossed, and which may be considered as representative of the gradient component "across" the ravine.

2.6 Local Uncontrollability of the Constraints

We shall now consider the possibility of singularity of the matrix $I_{\psi\psi}$ (variously defined in (2-8), (2-31), (2-43), (2-62) and (2-68)), and show that this occurrence may be interpreted as a loss of controllability of the constraint perturbations $\delta\underline{\psi}$. For simplicity we shall employ the definition (2-8). The extension to the case of (2-31) is trivial, and the other cases have strictly analogous interpretations.

The determinant of $I_{\psi\psi}$ is the Gramian of the vector functions which make up the columns of $G_{\psi}(t)$ (that is, the influence functions for the elements of $\underline{\psi}$). Hence singularity of $I_{\psi\psi}$ indicates linear dependence of those influence functions. That is, there exists a constant vector $\underline{k}$ such that

$$G_{\psi}(t)\,\underline{k} = \underline{0} \tag{2-89}$$

It follows from (2-2) that

$$\underline{k}^T \delta\underline{\psi} = \int_0^T \underline{k}^T G_{\psi}^T(t)\,\delta\underline{u}(t)\,dt = 0 \tag{2-90}$$

that is, there exists a linear combination of the elements of $\delta\underline{\psi}$ which must vanish regardless of the form of $\delta\underline{u}(t)$, and we are not free to specity $\delta\underline{\psi}$ arbitrarily. We no longer have the ability to "control" all components of $\delta\underline{\psi}$ simultaneously.

Considerable insight into this problem can be gained from a consideration of the three-parameter case. If two constraints are present, we are forced to consider only points along the intersection of the two constraint surfaces $\psi_1 = 0$ and $\psi_2 = 0$. But if these surfaces become mutually tangent at a point, their normal vectors will be proportional at that point, and if a small adjustment is to be made in this neighborhood, the perturbations in ψ_1 and ψ_2 must be in a prescribed ratio. If the two constraints are consistent,

we may regain the constraint surfaces by temporarily ignoring one constraint and adjusting the other. There then remains the problem of deciding in what direction to proceed, since, in the immediate vicinity of the point of tangency, any direction in the tangent plane is acceptable. There are only certain directions, however, in which we can proceed while remaining on both surfaces.

Such a point of tangency may be any one of three types,[71] of which the most critical is the so-called "hyperbolic" point exemplified by the intersection of a hyperbolic paraboloid with one of its tangent planes. The intersection consists of two lines which cross at the point of tangency. The two lines constitute the feasible directions from the critical point, and are the "asymptotic" lines, characterized by

$$d\underline{r} \cdot d\underline{n}_1 = d\underline{r} \cdot d\underline{n}_2 \, \mathrm{sgn}(\underline{n}_1 \cdot \underline{n}_2) \tag{2-91}$$

where $\underline{r}$ is the position vector in the parameter space and $\underline{n}_i$ is the unit vector in the direction of the gradient of ψ_i. The asymptotic lines are the directions in which the normal curvatures of the surfaces are equal, so that the surfaces coincide to second order in $d\underline{r}$. In problems of higher dimensions (when the excess of parameters over constraints is more than one), there are an infinite number of such feasible directions. A search procedure is then probably advisable, since, as can be seen in the three-parameter case, any feasible direction may lead to a false (local) optimum. In any case, one feasible direction will always be the direction along which the critical point was approached. A second reasonable choice would be to neglect one constraint for the next iteration, and hope that later iterations would converge back to one of the other asymptotic lines.

Returning to the controllability concept, we may write (see Chapter 3 and Ref. 15):

$$\delta\underline{\psi} = \Psi_x \delta\underline{x}(T) = \Psi_x \int_0^T \Phi(T, t) F_u(t) \delta\underline{u}(t) \, dt \tag{2-92}$$

where

$$\Psi_x = \left[\frac{\partial \psi_i}{\partial x_j} \right] \tag{2-93}$$

(considering, for simplicity, that $\underline{\psi}$ is independent of T), and $\Phi(t, \tau)$ is the transition matrix of the linearized system equations. It follows by comparison with (2-2) that

$$G_{\psi}(t) = F_u^T(t)\Phi^T(T, t)\Psi_x^T \tag{2-94}$$

and that, from (2-8),

$$I_{\psi\psi} = \Psi_x \int_0^T \Phi(T, t)\, F_u(t) F_u^T(t)\, \Phi^T(T, t)\, dt \Psi_x^T$$

$$= \Psi_x \Phi(T, 0) W(0, T) \Phi^T(T, 0) \Psi_x^T \tag{2-95}$$

where

$$W(0, T) = \int_0^T \Phi(0, t)\, F_u(t) F_u^T(t) \Phi^T(0, t)\, dt \tag{2-96}$$

The definition (2-96) is in accordance with that of Kalman.[44] In the terminology of Kelley,[49] the integral in (2-95) is defined as the matrix C, so that

$$I_{\psi\psi} = \Psi_x C \Psi_x^T \tag{2-97}$$

If $\underline{x}$ is an n-vector and $\underline{\psi}$ an r-vector ($r \leq n$), then Φ, W, and C are $n \times n$ matrices, Ψ_x is $r \times n$, and $I_{\psi\psi}$ is $r \times r$. In order to be nonsingular, therefore, $I_{\psi\psi}$ must have rank r. From Sylvester's Law of Nullity,[60] this requires that the factors of $I_{\psi\psi}$ each have rank $\geq r$. One necessary condition is therefore*

$$R(\Psi_x) = r \tag{2-98}$$

i.e., the constrained costates must be linearly independent. If $\underline{\psi}$ is a linear function of $\underline{x}(T)$, then Ψ_x will be constant throughout the computations, and (2-98) will always be satisfied if the constraints specified are not redundant. In other cases, however, (2-98) may be violated at some steps of the iteration process (i.e., at some points in the policy space). In such a case we must remove one of the (locally) redundant constraints for the next iteration.

A condition equivalent to (2-98)[60] is the one specified by Kelley,[49] namely that the $r \times r$ matrix $B = \Psi_x \Psi_x^T$ be nonsingular.

A second necessary condition is that

* where R(A) indicates the rank of A.

$$R(C) \geq r \tag{2-99}$$

If this condition holds, there exists an $r \times n$ matrix Ψ_x, for which $I_{\psi\psi}$ is nonsingular.* The condition (2-99) can therefore be interpreted as an indication that there exist at least r independently controllable costates. The condition (2-99) will certainly be satisfied if W(0, T) is nonsingular, in which case C is of rank n. This is Kalman's criterion[44] for "complete controllability", and indicates that every component of $\delta\underline{x}(T)$ is independently controllable. (Kalman specifies positive definiteness of W(0, T), which is equivalent to nonsingularity since W(0, T) is always at least positive semidefinite.) Equation (2-99) may be considered a condition for "partial" controllability.

Conditions (2-98) and (2-99) are not, however, sufficient for nonsingularity of $I_{\psi\psi}$. Since they were erroneously proffered as sufficient conditions by Kelley,[49] perhaps a simple counter-example would not be amiss:

Consider the system

$$\left.\begin{aligned} \delta\dot{x}_1 &= \delta u \\ \delta\dot{x}_2 &= 0 \end{aligned}\right\} \; 0 \leq t \leq T \tag{2-100}$$

with a single constraint $\psi = x_2(T) - \text{const.} = 0$. Obviously x_2 is not controllable, but nevertheless (2-98) and (2-99) are satisfied. In this problem $r = 1$; $\Psi_x = [0 \quad 1]$ and has rank r, so that (2-98) is satisfied; $\Phi(t, \tau) = I$ and $F_u^T(t) = [1 \quad 0]$, so that

$$C = \int_0^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 0] \, dt = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \tag{2-101}$$

Thus C has rank r. Nevertheless

---

* C is symmetric and has at least r non-zero eigenvalues. Hence we may let the rows of Ψ_x be a set of orthonormal eigenvectors of C corresponding to r of those eigenvalues. Then $I_{\psi\psi}$ is the diagonal matrix whose diagonal elements are those eigenvalues.

$$I_{\psi\psi} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \qquad (2\text{-}102)$$

which is, of course, a singular 1×1 matrix.

The insufficiency of (2-98) and (2-99) may be interpreted as follows: even if there are $c(\geq r)$ controllable costates, the r costates which constitute the rows of Ψ_x will not all be controllable unless they are equivalent to r of the controllable costates, i. e., unless there exists an $r \times c$ matrix M, of rank r, such that

$$\Psi_x = MS_c \qquad (2\text{-}103)$$

where S_c is the $c \times n$ matrix whose rows are the c controllable costates.

Kelley[49] points out that singularity of the matrix C will result from the presence, among the differential equations of the problem, of a holonomic condition, i. e., a differential equation derivable by differentiation of a functional relationship of the form $f(\underline{x}(t), t) = 0$. In general this should not result in singularity of $I_{\psi\psi}$ if the problem is well formulated.

The singularity of $I_{\psi\psi}$ is equivalent to the condition for abnormality in the problem of Bolza, as given by Bliss (Ref. 9, Theorem 77. 4), namely, that for variations which satisfy the differential equations of the problem, the maximum attainable number of independent constraint-vector perturbations $\delta\underline{\psi}$ be less than r, the dimension of $\underline{\psi}$. The defect of $I_{\psi\psi}$ is the order of abnormality.

CHAPTER 3

THE GRADIENT METHOD APPLIED TO STOCHASTIC SYSTEMS

3.1 Introduction

In the application of the gradient method to deterministic problems, a perturbation technique is employed to compute the influence functions which determine how control-variable perturbations will affect the system performance. In stochastic problems, however, these influence functions determine the effect on the system of the random disturbances to which it is subjected. They are the "noise sensitivities" of the system. The central problem of the stochastic gradient procedure, then, is the determination of the manner in which these sensitivities are affected by control perturbations. This involves a second perturbation of the relevant equations, and is therefore essentially a second-variation technique.

3.2 The Random Disturbances

The random processes to be considered here are of a very general nature. They are, in general, non-white and non-stationary. They may be multidimensional random functions[20, 37, 58, 67, 76] of several variables (vector random fields[7, 37, 67, 74, 76]), with the components cross-correlated in any fashion. In particular, we shall consider that the disturbances are functions, not only of the independent variable of the problem (usually time), but also of the state variables and the control variables.

Examples of such disturbances are not difficult to imagine. In a space vehicle re-entry problem, for example, the disturbance may be the variations in atmospheric density from some assumed model, a function of time, altitude, latitude and longitude. The buffeting encountered by a launch vehicle depends on air density and on vehicle velocity and attitude. An airplane may be subjected to disturbances due to air turbulence in regions of unsettled weather,[67] or due to the undulations of the surface of the earth if a terrain-following guidance system is in control of the vehicle.[68]

Disturbances which depend on the control variables are not rare. The errors in the application of corrective thrust to a space vehicle, for example, can be expected to depend upon the magnitude of thrust demanded.

In some cases it may be desirable to treat a problem in terms of an independent variable other than the time (for example, to avoid the complications induced by a variable end-point). In such a case the time becomes a state variable, and a disturbance in the form of a random time-function is now a function of that state variable.

The gradient method has already been applied to systems excited by white noise.[21, 23, 26, 33, 41, 70] It has generally been the practice to assume that the non-white noise case represents a simple extension, since the noise can be generated as the output of a linear filter excited by white noise.[10, 43] In the case of random fields, however, this approach is impracticable, and the problem is considerably more complicated.

3.3 Vector Random Fields

A random function $f(\underline{x})$, of several independent variables x_i, is known as a random field.[67, 76] A multidimensional process of this type, $\underline{f}(\underline{x})$ is a multidimensional, or vector, random field. We shall assume that the process has zero mean (i.e., every component of the random vector has zero mean), and shall concern ourselves mainly with second-order, or correlation, properties. We define the correlation matrix

$$R(\underline{x}_1, \underline{x}_2) = \overline{\underline{f}(\underline{x}_1)\underline{f}^T(\underline{x}_2)} \tag{3-1}$$

where the bar indicates the ensemble average. It is apparent that

$$R(\underline{x}_2, \underline{x}_1) = R^T(\underline{x}_1, \underline{x}_2) \tag{3-2}$$

A random vector $\underline{f}(x)$, which depends only on a scalar x, is termed stationary (in the wide sense) if its components are stationary and stationarily correlated in pairs, so that $R(x_1, x_2) = R(x_1 - x_2)$.

For a random field there are two concepts corresponding to that of stationarity;[76] thus $\underline{f}(\underline{x})$ is homogeneous if $R(\underline{x}_1, \underline{x}_2)$ depends only on the vector $\underline{r} = \underline{x}_1 - \underline{x}_2$; it is also isotropic if $R(\underline{r})$ depends only on the magnitude $r = |\underline{r}| = |\underline{x}_1 - \underline{x}_2|$. Isotropic random fields are of importance in the theory of turbulence.[3, 72]

If $R(\underline{x}_1, \underline{x}_2)$ is a diagonal matrix the random process $\underline{f}(\underline{x})$ is termed incoherent; otherwise it is coherent.[37] If $R(\underline{x}_1, \underline{x}_2)$ in non-zero only when $\underline{x}_1 = \underline{x}_2$ the process is "white". The term "rain on the roof" is a descriptive phrase sometimes applied to scalar random fields having this latter property.

The so-called covariance criterion, in the form appropriate to vector random fields, states that $R(\underline{x}_1, \underline{x}_2)$ is a correlation matrix if and only if it satisfies (3-2) and, for any vector-valued function $\underline{h}(\underline{x})$,

$$\int_A \int_A \underline{h}^T(\underline{x}_1)\, R(\underline{x}_1, \underline{x}_2)\, \underline{h}(\underline{x}_2)\, dV_1\, dV_2 \geq 0 \tag{3-3}$$

(where dV_i is the element of volume in the space of $\underline{x}_i$), for any region A of integration lying wholly within the region of variation of the vector argument $\underline{x}$ of the process.[59, 67]

One consequence of (3-3) is that, for all i and j,

$$r_{ij}^2(\underline{x}_1, \underline{x}_2) \leq r_{ii}(\underline{x}_1, \underline{x}_1)\, r_{jj}(\underline{x}_2, \underline{x}_2) \tag{3-4}$$

which is a well-known property of the cross-correlation function of two random processes.

Correlation matrices involving derivatives of $\underline{f}(\underline{x})$, with respect to the components of $\underline{x}$, may be derived from the relation:

$$\overline{\frac{\partial^p \underline{f}}{\partial x_i^p}(\underline{x}_1)\, \frac{\partial^q \underline{f}^T}{\partial x_j^q}(\underline{x}_2)} = \frac{\partial^{p+q} R(\underline{x}_1, \underline{x}_2)}{\partial x_{1,i}^p\, \partial x_{2,j}^q} \tag{3-5}$$

Some further properties of the correlation matrix, which may be regarded as generalizations of the Schwarz inequality, are derived in Appendix B.

3.4 The Basic Problem

Consider a system whose behaviour is described by the set of n nonlinear time-varying differential equations

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, \underline{n}, t) \qquad \underline{x}(0) = \underline{x}_0 \tag{3-6}$$

where $\underline{x}(t)$ is an n-vector of state variables; $\underline{u}(t)$ is an m-vector of control variables; $\underline{n}(\underline{x}, \underline{u}, t)$ is a p-vector of stochastic disturbances (noises) of the type discussed in Sec. 2.2; and t is the independent variable, which may or may not be the time. The dot indicates differentiation with respect to t.

It is assumed that the noises are small enough so that their effects may be adequately described by a linearized model of the system, and that they have zero mean. The latter assumption introduces no loss of generality, since any bias in the disturbances may be absorbed in the deterministic portion of the system description.

For any given $\underline{x}_0$, $\underline{u}(t)$, and $\underline{n}(\underline{x}, \underline{u}, t)$, the resulting succession of states $\underline{x}(t)$, through which the system passes, shall be considered as defining a trajectory in the state space.

Let $g(\underline{x}(T), T)$ be a prescribed function of the final state variables $\underline{x}(T)$ and the final time T. We shall define the error associated with the trajectory as

$$e(T) = \tilde{g}(\underline{x}(T), T) \tag{3-7}$$

where the tilde indicates the perturbation caused by the noise $\underline{n}(t)$. For the present we shall assume that the final time T is fixed.

The cost function which we are interested in minimizing is the mean-square value of e:

$$\phi = \overline{e^2} \tag{3-8}$$

We must consider two different perturbation equations corresponding to (3-6): one for the trajectory adjustment introduced by a variation of the control variables $\underline{u}(t)$, and one for the small perturbations around the nominal due to the random disturbances $\underline{n}(t)$:

$$\delta\dot{\underline{x}}(t) = F_x(t)\, \delta\underline{x}(t) + F_u(t)\, \delta\underline{u}(t) \tag{3-9}$$

$$\dot{\tilde{\underline{x}}}(t) = F_x(t)\, \tilde{\underline{x}}(t) + F_n(t)\, \underline{n}(t) \tag{3-10}$$

where $F_x(t)$, $F_u(t)$ and $F_n(t)$ are the Jacobian matrices of $\underline{f}(\underline{x}, \underline{u}, \underline{n}, t)$ with respect to $\underline{x}$, $\underline{u}$, and $\underline{n}$ respectively, and have dimensions $n \times n$, $n \times m$, and

$n \times p$ respectively. The ij-th component of $F_x(t)$ is

$$\left[F_x(t)\right]_{ij} = \frac{\partial f_i(\underline{x}, \underline{u}, \underline{n}, t)}{\partial x_j(t)} \tag{3-11}$$

with similar definitions for F_u and F_n. All coefficients are evaluated along some assumed nominal trajectory.

The adjoint equation associated with (3-9) and (3-10) (see Appendix A) is

$$\dot{\underline{\lambda}}(t) = - F_x^T(t)\, \underline{\lambda}(t) \tag{3-12}$$

In the Hamiltonian formulation, with

$$H = \underline{\lambda}^T \underline{f} \tag{3-13}$$

equations (3-6) and (3-12) become

$$\dot{\underline{x}} = \left[\frac{\partial H}{\partial \underline{\lambda}}\right]^T \tag{3-14}$$

$$\dot{\underline{\lambda}} = - \left[\frac{\partial H}{\partial \underline{x}}\right]^T \tag{3-14.1}$$

(Note that the derivative of a scalar with respect to a vector is defined as a row vector.) If we prescribe, as initial conditions for $\underline{\lambda}(t)$,

$$\underline{\lambda}(T) = \left[\frac{\partial e(T)}{\partial \tilde{x}(T)}\right]^T = \left[\frac{\partial g(T)}{\partial \underline{x}(T)}\right]^T \tag{3-15}$$

we obtain the useful result

$$e(T) = \underline{\lambda}^T(0)\tilde{\underline{x}}_0 + \int_0^T \underline{\lambda}^T(t)\, F_n(t)\, \underline{n}(t)\, dt = \underline{\lambda}^T(0)\, \tilde{\underline{x}}_0 + \int_0^T \underline{\lambda}_n^T(t)\, \underline{n}(t)\, dt \tag{3-16}$$

where $\underline{n}(t)$ represents $\underline{n}(\underline{x}, \underline{u}, t)$ along the nominal path, and the influence function $\underline{\lambda}_n(t)$ is defined as

$$\underline{\lambda}_n(t) = F_n^T(t)\,\underline{\lambda}(t) = \left[\frac{\partial H}{\partial \underline{n}}\right]^T \tag{3-17}$$

From (3-16), the mean-square error is

$$\overline{e^2} = \underline{\lambda}^T(0) C\,\underline{\lambda}(0) + \int_0^T \underline{\lambda}_n^T(t) \int_0^T R(t, \tau)\,\underline{\lambda}_n(\tau)\,d\tau\,dt \tag{3-18}$$

where C is the $n \times n$ covariance matrix of initial-condition ("injection") errors,

$$C = \overline{\underset{\sim}{\underline{x}}_0 \underset{\sim}{\underline{x}}_0^T} \tag{3-18.1}$$

and is symmetric, and $R(t, \tau)$ is the $p \times p$ correlation matrix of the random disturbances,

$$R(t, \tau) = \overline{\underline{n}(t)\,\underline{n}^T(\tau)} \tag{3-19}$$

or, more correctly,

$$R(\underline{x}(t), \underline{u}(t), t, \underline{x}(\tau), \underline{u}(\tau), \tau) = \overline{\underline{n}(\underline{x}(t), \underline{u}(t), t)\underline{n}^T(\underline{x}(\tau), \underline{u}(\tau), \tau)} \tag{3-19.1}$$

If we introduce a trajectory perturbation by means of a control variation $\delta\underline{u}(t)$, we obtain, from (3-16),

$$\delta e(T) = \delta\underline{\lambda}^T(0)\,\underset{\sim}{\underline{x}}_0 + \int_0^T \delta\underline{\lambda}_n^T\,\underline{n}\,dt + \int_0^T \underline{\lambda}_n^T\,\delta\underline{n}\,dt \tag{3-20}$$

The last term arises because, for every member of the ensemble of possible random disturbances, the value of $\underline{n}(t)$ varies as the trajectory is perturbed, due to the dependence of $\underline{n}(t)$ on $\underline{x}(t)$ and $\underline{u}(t)$. This variation is expressed by

$$\delta\underline{n}(t) = \left[\frac{\partial \underline{n}(t)}{\partial \underline{x}(t)}\right] \delta\underline{x}(t) + \left[\frac{\partial \underline{n}(t)}{\partial \underline{u}(t)}\right] \delta\underline{u}(t) \tag{3-21}$$

Noting that $\delta\overline{e^2} = 2\,\overline{e\delta e}$, we may combine (3-16) and (3-20), making use of (3-5), to obtain

$$\frac{\delta\overline{e^2}}{2} = \underline{\lambda}^T(0)C\,\delta\underline{\lambda}(0) + \int_0^T \underline{r}_I^T(t)\,\delta\underline{\lambda}_n(t)\,dt$$

$$+ \int_0^T \underline{\lambda}_n^T(t)R_x(t)\,\delta\underline{x}(t)\,dt + \int_0^T \underline{\lambda}_n^T(t)R_u(t)\,\delta\underline{u}(t)\,dt \qquad (3\text{-}22)$$

where we have introduced the following definitions:

$$\underline{r}(\tau, t) = R(t, \tau)\,\underline{\lambda}_n(\tau) \qquad (p \times 1) \qquad (3\text{-}23)$$

$$\underline{r}_I(t) = \int_0^T \underline{r}(\tau, t)\,d\tau \qquad (p \times 1) \qquad (3\text{-}24)$$

$$R_x(t) = \int_0^T \left[\frac{\partial \underline{r}(\tau, t)}{\partial \underline{x}(t)}\right] d\tau \qquad (p \times n) \qquad (3\text{-}25)$$

$$R_u(t) = \int_0^T \left[\frac{\partial \underline{r}(\tau, t)}{\partial \underline{u}(t)}\right] d\tau \qquad (p \times m) \qquad (3\text{-}26)$$

Thus, for example, the ij-th component of $R_x(t)$ is

$$\left[R_x(t)\right]_{ij} = \int_0^T \sum_{k=1}^{p} \lambda_{n_k}(\tau)\,\frac{\partial R_{ik}(t, \tau)}{\partial x_j(t)}\,d\tau \qquad (3\text{-}27)$$

In order to cast equation (3-22) into a form suitable for the application of the gradient method, it is necessary to express $\delta\overline{e^2}$ in terms of $\delta\underline{u}(t)$ only. This is done by replacing $\delta\underline{\lambda}(0)$, $\delta\underline{\lambda}_n(t)$ and $\delta\underline{x}(t)$ by their general solutions in terms of $\delta\underline{u}(t)$.

Let X(t) be a fundamental matrix for the homogeneous equation corresponding to (3-9) and (3-10).[77] Then the trajectory perturbation $\delta\underline{x}(t)$ can be expressed as

$$\delta\underline{x}(t) = X(t)X^{-1}(0)\,\delta\underline{x}_0 + X(t)\int_0^t X^{-1}(\tau)F_u(\tau)\delta\underline{u}(\tau)\,d\tau \qquad (3\text{-}28)$$

Similarly, $\delta\underline{\lambda}(t)$ is the solution of the perturbed adjoint equations

$$\delta\dot{\underline{\lambda}}(t) = - F_x^T(t)\delta\underline{\lambda}(t) - H_{xu}(t)\,\delta\underline{u}(t) - H_{xx}(t)\,\delta\underline{x}(t) \tag{3-29}$$

where

$$H_{xu} = \left[\frac{\partial^2 H}{\partial x_i \partial u_j}\right] \qquad (n \times m) \tag{3-30}$$

and

$$H_{xx} = \left[\frac{\partial^2 H}{\partial x_i \partial x_j}\right] \qquad (n \times n) \tag{3-31}$$

is the Hessian matrix of H with respect to $\underline{x}$. The initial conditions for (3-29) are

$$\delta\underline{\lambda}(T) = G_{xx}\delta\underline{x}(T) \tag{3-32}$$

where

$$G_{xx} = \left[\frac{\partial^2 g(T)}{\partial x_i(T)\partial x_j(T)}\right] \tag{3-33}$$

Now the fundamental matrix for (3-29) is the transposed inverse of that of (3-9).* Hence $\delta\underline{\lambda}(t)$ can be expressed as

$$\delta\underline{\lambda}(t) = X^{-T}(t)X^T(T)G_{xx}\delta\underline{x}(T) - X^{-T}(t)\int_T^t X^T(\tau)\left[H_{xu}(\tau)\,\delta\underline{u}(\tau) + H_{xx}(\tau)\,\delta\underline{x}(\tau)\right]d\tau \tag{3-34}$$

The vector $\delta\underline{\lambda}_n(t)$, from (3-17), is

$$\delta\underline{\lambda}_n(t) = H_{nu}(t)\,\delta\underline{u}(t) + H_{nx}(t)\,\delta\underline{x}(t) + F_n^T(t)\,\delta\underline{\lambda}(t) \tag{3-35}$$

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* See Ref. 77 and Appendix A.

where

$$H_{nu} = \left[\frac{\partial^2 H}{\partial n_i \partial u_j}\right] \qquad (p \times m) \tag{3-36}$$

and

$$H_{nx} = \left[\frac{\partial^2 H}{\partial n_i \partial x_j}\right] \qquad (p \times n) \tag{3-37}$$

We can now write (3-22) in the desired form. Assuming $\delta \underline{x}_0 = 0$, we have

$$\begin{aligned}
\frac{\overline{\delta e^2}}{2} = {} & \underline{\lambda}^T(0)CX^{-T}(0)X^T(T)G_{xx}X(T)\int_0^T X^{-1}(t)F_u(t)\,\delta\underline{u}(t)\,dt \\
& + \underline{\lambda}^T(0)CX^{-T}(0)\int_0^T X^T(t)H_{xu}(t)\,\delta\underline{u}(t)\,dt \\
& + \underline{\lambda}^T(0)CX^{-T}(0)\int_0^T X^T(t)H_{xx}(t)X(t)\int_0^t X^{-1}(\tau)F_u(\tau)\,\delta\underline{u}(\tau)d\tau\,dt \\
& + \int_0^T \underline{r}_I^T(t)H_{nu}(t)\,\delta\underline{u}(t)\,dt \\
& + \int_0^T \underline{r}_I^T(t)H_{nx}(t)X(t)\int_0^t X^{-1}(\tau)F_u(\tau)\,\delta\underline{u}(\tau)\,d\tau\,dt \\
& + \int_0^T \underline{r}_I^T(t)F_n^T(t)X^{-T}(t)\,dt\,X^T(T)G_{xx}X(T)\int_0^T X^{-1}(t)F_u(t)\,\delta\underline{u}(t)\,dt \\
& + \int_0^T \underline{r}_I^T(t)F_n^T(t)X^{-T}(t)\int_t^T X^T(\tau)H_{xu}(\tau)\,\delta\underline{u}(\tau)\,d\tau\,dt \\
& + \int_0^T \underline{r}_I^T(t)F_n^T(t)X^{-T}(t)\int_t^T X^T(s)H_{xx}(s)X(s)\int_0^s X^{-1}(\tau)F_u(\tau)\delta\underline{u}(\tau)d\tau\,ds\,dt \\
& + \int_0^T \underline{\lambda}_n^T(t)R_x(t)X(t)\int_0^t X^{-1}(\tau)F_u(\tau)\delta\underline{u}(\tau)\,d\tau\,dt + \int_0^T \underline{\lambda}_n^T(t)R_u(t)\delta\underline{u}(t)\,dt
\end{aligned} \tag{3-38}$$

By appropriate changes of order of integration (and corresponding changes of the limits), each term of (3-38) can be made to involve only a single integration of $\delta\underline{u}(t)$. For the double integrals we use the transformation

$$\int_0^T \int_0^t \underline{f}^T(t,\tau)\delta\underline{u}(\tau)d\tau dt = \int_0^T \int_\tau^T \underline{f}^T(t,\tau)dt\,\delta\underline{u}(\tau)\,d\tau \tag{3-39}$$

The term involving a triple integral is transformed thrice in succession, as follows:*

$$\begin{aligned}\int_0^T \int_t^T \int_0^s \underline{f}^T(t,s,\tau)\delta\underline{u}(\tau)d\tau\,ds\,dt &= \int_0^T \int_0^s \int_0^s \underline{f}^T(t,s,\tau)\delta\underline{u}(\tau)\,d\tau\,dt\,ds \\ &= \int_0^T \int_0^s \int_0^s \underline{f}^T(t,s,\tau)\,dt\,\delta\underline{u}(\tau)d\tau\,ds \\ &= \int_0^T \int_\tau^T \int_0^s \underline{f}^T(t,s,\tau)dt\,ds\,\delta\underline{u}(\tau)\,d\tau \end{aligned} \tag{3-40}$$

In this manner equation (3-38) may be reduced to the form

$$\delta\overline{e^2} = \int_0^T \underline{g}_\phi^T(t)\,\delta\underline{u}(t)\,dt \tag{3-41}$$

where

$$\begin{aligned}\frac{1}{2}\underline{g}_\phi(t) = F_u^T(t)X^{-T}(t)\Bigg[\int_t^T X^T(\tau)\Big[H_{nx}^T(\tau)\underline{r}_I(\tau) + R_x^T(\tau)F_n^T(\tau)\,\underline{\lambda}(\tau)\Big]d\tau \\ + X^T(T)G_{xx}X(T)\Big[X^{-1}(0)C\underline{\lambda}(0) + \int_0^T X^{-1}(\tau)F_n(\tau)\underline{r}_I(\tau)\,d\tau\Big] \\ + \int_t^T X^T(\tau)H_{xx}(\tau)X(\tau)\Big[X^{-1}(0)C\underline{\lambda}(0) + \int_0^\tau X^{-1}(s)F_n(s)\underline{r}_I(s)\,ds\Big]\,d\tau\Bigg]\end{aligned}$$

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* The inner differential corresponds to the inner integral sign, etc.

$$+ H_{xu}^T(t)X(t)\left[X^{-1}(0)C\underline{\lambda}(0) + \int_0^t X^{-1}(\tau)F_n(\tau)\,\underline{r}_I(\tau)\,d\tau\right]$$

$$+ H_{nu}^T(t)\,\underline{r}_I(t) + R_u^T(t)\,F_n^T(t)\,\underline{\lambda}(t) \qquad (3\text{-}42)$$

Equation (3-41) is now in a form amenable to the usual steepest-descent procedure, as outlined in Chapter 2.

Careful examination of equation (3-42) reveals the presence of expressions similar to that of equation (A-3) of Appendix A, indicating that some of these quantities may be determined economically as the solutions of differential equations. Let

$$\underline{w}_1(t) = X(t)X^{-1}(0)C\underline{\lambda}(0) + X(t)\int_0^t X^{-1}(\tau)F_n(\tau)\,\underline{r}_I(\tau)\,d\tau \qquad (3\text{-}42.1)$$

and

$$\underline{w}_2(t) = X^{-T}(t)X^T(T)G_{xx}\,\underline{w}_1(T) + X^{-T}(t)\int_t^T X^T(\tau)\left[H_{nx}^T(\tau)\,\underline{r}_I(\tau)\right.$$

$$\left. + R_x^T(\tau)F_n^T(\tau)\underline{\lambda}(\tau) + H_{xx}(\tau)\,\underline{w}_1(\tau)\right]d\tau \qquad (3\text{-}42.2)$$

In other words, $\underline{w}_1(t)$ and $\underline{w}_2(t)$ are solutions of the differential equations

$$\dot{\underline{w}}_1(t) = F_x(t)\,\underline{w}_1(t) + F_n(t)\,\underline{r}_I(t) \qquad (3\text{-}42.3)$$

and

$$\dot{\underline{w}}_2(t) = -F_x^T(t)\,\underline{w}_2(t) - \left[H_{nx}^T(t)\,\underline{r}_I(t)\right.$$

$$\left. + R_x^T(t)F_n^T(t)\,\underline{\lambda}(t) + H_{xx}(t)\,\underline{w}_1(t)\right] \qquad (3\text{-}42.4)$$

with initial conditions

$$\underline{w}_1(0) = C\underline{\lambda}(0) \qquad (3\text{-}42.5)$$

and

$$\underline{w}_2(T) = G_{xx}\,\underline{w}_1(T) \qquad (3\text{-}42.6)$$

With these substitutions equation (3-42) becomes simply

$$\frac{1}{2}\, g_{\phi}(t) = H_{xu}^{T}(t)\, \underline{w}_1(t) + F_u^{T}(t)\, \underline{w}_2(t) + H_{nu}^{T}(t)\, \underline{r}_I(t) + R_u^{T}(t)\, \underline{\lambda}_n(t) \qquad (3\text{-}42.7)$$

It should be noted that, if $g(\underline{x}(T), T)$ is a linear function of $\underline{x}(T)$, and the system itself is linear so that (3-6) may be written

$$\dot{\underline{x}}(t) = F_x(t)\underline{x}(t) + F_u(t)\, \underline{u}(t) + F_n(t)\, \underline{n}(t) \qquad (3\text{-}42.8)$$

then the influence function vanishes (no improvement can be made) unless the noise is a function of the state and/or control variables.

3.5 Adjustable Initial Conditions

If some of the initial state variables $\underline{x}_0$ can be varied so as to improve the performance criterion, it is necessary to determine the corresponding influence coefficients. If we consider initial-condition adjustments separately from the control-function perturbations, equation (3-28) becomes

$$\delta\underline{x}(t) = X(t)X^{-1}(0)\, \delta\underline{x}_0 \qquad (3\text{-}43)$$

If the initial-condition random errors $\tilde{\underline{x}}_0$ depend statistically on the initial conditions $\underline{x}_0$, then the equation (3-20) for δe will contain the additional term $\underline{\lambda}^{T}(0)\delta\tilde{\underline{x}}_0$, where

$$\delta\tilde{\underline{x}}_0 = \left[\frac{\partial\tilde{\underline{x}}_0}{\partial\underline{x}_0}\right] \delta\underline{x}_0 \qquad (3\text{-}44)$$

Hence the equation (3-22) for $\delta\overline{e^2}/2$ contains the additional term

$$\frac{1}{2}\, \underline{c}_x^{T}\, \delta\underline{x}_0 \qquad (3\text{-}44.1)$$

where $\underline{c}_x$ is the n-vector whose i-th element is

$$\underline{\lambda}^T(0)\left[\frac{\partial C}{\partial x_{0_i}}\right]\underline{\lambda}(0)$$

This last expression could be obtained directly by differentiation of the first term in (3-18). With this addition to (3-20), and using (3-43) for $\delta\underline{x}(t)$, we obtain, instead of (3-41),

$$\delta\overline{e^2} = \underline{b}_\phi^T \delta\underline{x}_0 \tag{3-45}$$

where $\underline{b}_\phi = 2\underline{w}_2(0) + \underline{c}_x$

$$= 2X^{-T}(0)X^T(T)G_{xx}X(T)\left[X^{-1}(0)C\underline{\lambda}(0) + \int_0^T X^{-1}(t)F_n(t)\,\underline{r}_I(t)\,dt\right]$$

$$+ 2X^{-T}(0)\int_0^T X^T(t)H_{xx}(t)X(t)\,dt\,X^{-1}(0)C\underline{\lambda}(0)$$

$$+ 2X^{-T}(0)\int_0^T X^T(t)\left[H_{nx}^T(t)\,\underline{r}_I(t) + R_x^T(t)\,\underline{\lambda}_n(t)\right]dt$$

$$+ 2X^{-T}(0)\int_0^T X^T(\tau)H_{xx}(\tau)X(\tau)\int_0^\tau X^{-1}(t)F_n(t)\,\underline{r}_I(t)\,dt\,d\tau + \underline{c}_x \tag{3-46}$$

The problem can now be stated in the form considered in Sec. 2.4, when both control-variable and initial-condition adjustments are possible, by simply adding the right hand sides of equations (3-41) and (3-45).

3.6 Variable Terminal Time: Case I

If the terminal time T of the problem is not specified, the termination of the trajectory is signalled by a "stopping condition" of the form[15]

$$\omega(\underline{x}(T), T) = 0 \tag{3-47}$$

In stochastic problems, since there are random inputs disturbing the nominal trajectory, we may recognize two distinct cases:

(I) The time T, determined from the nominal computations, is used as the stopping time of the trajectory, in which case the parameter $\omega(T)$ will not, in general, vanish, but will assume some random value $\tilde{\omega}(T)$, with zero mean.

(II) The real (noisy) trajectory is terminated when $\omega(\underline{x}(t), t)$ actually vanishes, in which case there is a random perturbation $\tilde{T}$ in the terminal time.

Case I is the simpler of the two, and will be considered first. First we generate a new set of adjoint variables $\underline{\ell}(t)$ satisfying (3-12):

$$\dot{\underline{\ell}}(t) = - F_x^T(t)\, \underline{\ell}(t) \tag{3-48}$$

with the initial conditions

$$\underline{\ell}(T) = \left[\frac{\partial \omega(T)}{\partial \underline{x}(T)} \right]^T \tag{3-49}$$

so that

$$\delta\omega(T) = \underline{\ell}^T(0)\, \delta\underline{x}_0 + \int_0^T \underline{\ell}^T(t) F_u(t)\, \delta\underline{u}(t)\, dt$$

$$= \underline{b}_\omega^T\, \delta\underline{x}_0 + \int_0^T \underline{g}_\omega^T(t)\, \delta\underline{u}(t)\, dt \tag{3-50}$$

where $\underline{b}_\omega = \underline{\ell}(0)$ and $\underline{g}_\omega(t) = F_u^T(t)\, \underline{\ell}(t)$. The corresponding result for ϕ, from (3-41) and (3-45), is

$$\delta\phi(T) = \underline{b}_\phi^T\, \delta\underline{x}_0 + \int_0^T \underline{g}_\phi^T(t)\, \delta\underline{u}(t)\, dt \tag{3-51}$$

When the variation δT in the final time is introduced, the total changes in the final values ϕ_f and ω_f of ϕ and ω are

$$\delta\phi_f = \delta\phi(T) + \dot{\phi}(T)\, \delta T \tag{3-52}$$

$$\delta\omega_f = \delta\omega(T) + \dot{\omega}(T)\,\delta T \tag{3-53}$$

where

$$\dot{\omega}(T) = \frac{\partial \omega(T)}{\partial T} + \left[\frac{\partial \omega(T)}{\partial \underline{x}(T)}\right] \underline{f}(T) \tag{3-54}$$

and similarly, $\dot{\phi}(T)$ is the total time derivative of the cost function, evaluated at t = T. It is determined as follows. From (3-7) we can write

$$e(T) = \tilde{g}(T) = \left[\frac{\partial g(T)}{\partial \underline{x}(T)}\right] \tilde{\underline{x}}(T) \tag{3-55}$$

from which

$$\dot{e}(T) = \left[\frac{\partial^2 g(T)}{\partial T \partial \underline{x}(T)}\right] \tilde{\underline{x}}(T) + \underline{f}^T(T)\, G_{xx}\, \tilde{\underline{x}}(T) + \left[\frac{\partial g(T)}{\partial \underline{x}(T)}\right] \dot{\tilde{\underline{x}}}(T) \tag{3-56}$$

where G_{xx} is defined in (3-33) and

$$\dot{\tilde{\underline{x}}}(t) = F_x(t)\, \tilde{\underline{x}}(t) + F_n(t)\, \underline{n}(t) \tag{3-57}$$

so that

$$\tilde{\underline{x}}(T) = X(T)X^{-1}(0)\, \tilde{\underline{x}}_0 + X(T) \int_0^T X^{-1}(t) F_n(t)\, \underline{n}(t)\, dt \tag{3-58}$$

Using (3-55) to (3-58) we obtain

$$\dot{\phi}(T) = \dot{\overline{e^2}}(T) = 2\,\overline{e\dot{e}}$$

$$= 2\left\{\left[\frac{\partial^2 g}{\partial T \partial \underline{x}}\right] + \underline{f}^T G_{xx} + \left[\frac{\partial g}{\partial \underline{x}}\right] F_x\right\} \overline{\tilde{\underline{x}}\, \tilde{\underline{x}}^T} \left[\frac{\partial g}{\partial \underline{x}}\right]^T$$

$$+ 2\left[\frac{\partial g}{\partial \underline{x}}\right] F_n\, \overline{\underline{n}\, \tilde{\underline{x}}^T} \left[\frac{\partial g}{\partial \underline{x}}\right]^T \tag{3-59}$$

where all quantities are evaluated at t = T and, assuming $\underline{n}(t)$ and $\underline{\tilde{x}}_0$ uncorrelated,

$$\overline{\underline{n}\,\underline{\tilde{x}}^T} = \int_0^T R(T,t)\,F_n^T(t)\,X^{-T}(t)\,dt\,X^T(T) \tag{3-60}$$

and

$$\overline{\underline{\tilde{x}}\,\underline{\tilde{x}}^T} = X(T)X^{-1}(0)\,CX^{-T}(0)X^T(T) + X(T)\int_0^T\int_0^T X^{-1}(t)F_n(t)R(t,\tau)F_n^T(\tau)X^{-T}(\tau)\,dt\,d\tau\,X^T(T) \tag{3-61}$$

Since $\underline{\lambda}_n(t)$ may be expressed as

$$\underline{\lambda}_n(t) = F_n^T(t)\,\underline{\lambda}(t) = F_n^T(t)\,X^{-T}(t)\,X^T(T)\left[\frac{\partial g(T)}{\partial \underline{x}(T)}\right]^T \tag{3-62}$$

we may, using (3-24), substitute the following expressions in (3-59):

$$\overline{\underline{\tilde{x}}\,\underline{\tilde{x}}^T}\left[\frac{\partial g}{\partial \underline{x}}\right]^T = \underline{w}_1(T) \tag{3-63}$$

$$\overline{\underline{n}\,\underline{\tilde{x}}^T}\left[\frac{\partial g}{\partial \underline{x}}\right]^T = \underline{r}_I(T) \tag{3-64}$$

to obtain

$$\dot{\phi}(T) = \dot{\overline{e^2}}(T) = 2\left[\frac{\partial g}{\partial \underline{x}}\right]F_n(T)\,\underline{r}_I(T) + 2\left[\frac{\partial^2 g}{\partial T\,\partial \underline{x}} + \underline{f}^T G_{xx} + \frac{\partial g}{\partial \underline{x}}F_x\right]\underline{w}_1(T) \tag{3-65}$$

Having evaluated $\dot{\omega}(T)$ and $\dot{\phi}(T)$ according to (3-54) and (3-65), if we require $\delta\omega_f = 0$ in (3-53) we obtain

$$\delta T = \frac{-1}{\dot{\omega}(T)}\,\delta\omega(T) = \frac{-1}{\dot{\omega}(T)}\left[\underline{b}_\omega^T\,\delta\underline{x}_0 + \int_0^T \underline{g}_\omega^T(t)\,\delta\underline{u}(t)\,dt\right] \qquad (3\text{-}65.1)$$

which may then be substituted in (3-52) to give

$$\delta\phi_f = \delta\phi(T) + \dot{\phi}(T)\,\delta T$$

$$= \left[\underline{b}_\phi^T - \frac{\dot{\phi}(T)}{\dot{\omega}(T)}\,\underline{b}_\omega^T\right]\delta\underline{x}_0 + \int_0^T \left[\underline{g}_\phi^T(t) - \frac{\dot{\phi}(T)}{\dot{\omega}(T)}\,\underline{g}_\omega^T(t)\right]\delta\underline{u}(t)\,dt \qquad (3\text{-}65.2)$$

which is of the same form as the general relation (3-51) for the fixed-terminal-time case, and may be treated by the same techniques.

Although the form of (3-65.2) is suggestive of the technique employed by Bryson and Denham[15] to handle variable terminal time in deterministic problems, it should be noted that in this case it is not possible to determine the bracketed quantities in one integration, by redefining the terminal values of the adjoint variables. The function $\underline{g}_\omega(t)$ is determined in a direct manner from the adjoint-equation solution $\underline{\ell}(t)$, as in (3-50), whereas $\underline{g}_\phi(t)$ is a result of the second-variation procedure of Sec. 3.4.

Alternatively, we may use (3-52) and (3-53) directly, evaluating $\delta\omega(T)$ and $\delta\phi(T)$ from (3-50) and (3-51), and $\dot{\omega}(T)$ and $\dot{\phi}(T)$ from (3-54) and (3-65). The problem is now in the form considered in Sec. 2.4, if we consider ω_f as simply one component of the constraint vector $\underline{\psi}(\underline{x}(T), T)$, and the final time T as one component (along with $\underline{x}_0$) of the adjustable constant-parameter vector $\underline{p}$ of that section. Then $\dot{\phi}(T)$ becomes one component of the vector $\underline{b}_\phi$ of (2-65), and $\dot{\omega}(T)$ becomes one element of the matrix B_ψ of (2-66). The methods of Chapter 2 can now be applied directly. This approach has the questionable advantage of a direct determination of the required end-point variation δT (which may be useful in simplifying computations) at the expense of introducing an additional control parameter.

In either case, since $\tilde{\omega}(T)$ is given by

$$\tilde{\omega}(T) = \left[\frac{\partial\omega(T)}{\partial\underline{x}(T)}\right] \tilde{\underline{x}}(T) \tag{3-66}$$

we can determine its mean-square value $\overline{\tilde{\omega}^2(T)}$ from

$$\overline{\tilde{\omega}^2(T)} = \left[\frac{\partial\omega(T)}{\partial\underline{x}(T)}\right] \overline{\tilde{\underline{x}}\,\tilde{\underline{x}}^T} \left[\frac{\partial\omega(T)}{\partial\underline{x}(T)}\right]^T \tag{3-67}$$

where $\overline{\tilde{\underline{x}}\,\tilde{\underline{x}}^T}$ is defined in (3-61).

Evaluation of equation (3-67) requires some additional computation. Since the vector $\underline{\ell}(t)$ of equation (3-48) may be expressed as

$$\underline{\ell}(t) = X^{-T}(t)X^T(T)\left[\frac{\partial\omega(T)}{\partial\underline{x}(T)}\right]^T \tag{3-67.1}$$

we may combine this expression with (3-61) to obtain

$$\overline{\tilde{\underline{x}}\,\tilde{\underline{x}}^T}\left[\frac{\partial\omega}{\partial\underline{x}}\right]^T = X(T)X^{-1}(0)CX^{-T}(0)X^T(T)\left[\frac{\partial\omega}{\partial\underline{x}}\right]^T + X(T)\int_0^T\int_0^T X^{-1}(t)F_n(t)R(t,\tau)F_n^T(\tau)\underline{\ell}(\tau)\,dt\,d\tau \tag{3-67.2}$$

Defining, by analogy with equations (3-23) and (3-24), the vector

$$\underline{r}_I'(t) = \int_0^T R(t,\tau)F_n^T(\tau)\underline{\ell}(\tau)\,d\tau \tag{3-67.3}$$

and substituting this expression into (3-67.2), we see that it is possible to write

$$\overline{\tilde{\underline{x}}\,\tilde{\underline{x}}^T}\left[\frac{\partial\omega}{\partial\underline{x}}\right]^T = \underline{w}_1'(T)$$

where

$$\underline{w}_1'(t) = X(t)X^{-1}(0)C\underline{\ell}(0) + X(t)\int_0^t X^{-1}(\tau)F_n(\tau)\underline{r}_I'(\tau)\,d\tau \tag{3-67.4}$$

is the solution of the differential equation

$$\dot{\underline{w}}_1'(t) = F_x(t)\,\underline{w}_1'(t) + F_n(t)\,\underline{r}_I'(t) \tag{3-67.5}$$

with the initial condition

$$\underline{w}_1'(0) = C\underline{\ell}(0) \tag{3-67.6}$$

It should be noted that the quantity $\overline{\tilde{\omega}^2}$ of (3-67) may itself be made the object of the minimization procedure, by simply defining $\omega(\underline{x}(T), T)$ and $g(\underline{x}(T), T)$ to be the same function. Then the trajectory is terminated at the time T when this function would vanish in the absence of noise, and the optimum trajectory is shaped so as to minimize the mean-square of the value which the function assumes at T in the presence of noise.

In the linear case of equation (3-42.8), with $g(\underline{x}(T), T)$ linear in $\underline{x}(T)$, the situation is now different from that of Sections 3.4 and 3.5. Even if the noise depends on t only, there still remains the possibility of improvement. This is as would be expected, since in most problems of this type the mean-square error varies with time.

3.7 Variable Terminal Time: Case II

In the second case of interest, the real trajectory is terminated in such a way as to force $\tilde{\omega}(\underline{x}_f, t_f)$ to zero. Thus, since $\tilde{T}$ may be non-zero, the final value of $\tilde{\omega}$ is

$$\tilde{\omega}_f = 0 = \left[\frac{\partial \omega}{\partial \underline{x}}\right] \tilde{\underline{x}}(T) + \dot{\omega}\tilde{T} \tag{3-68}$$

where $\dot{\omega}$ is defined in (3-54) as

$$\dot{\omega} = \left[\frac{\partial \omega}{\partial \underline{x}}\right] \underline{f}(T) + \frac{\partial \omega}{\partial T} \tag{3-69}$$

Hence

$$\tilde{T} = \frac{-1}{\dot{\omega}} \left[\frac{\partial \omega}{\partial \underline{x}}\right] \tilde{\underline{x}}(T) \tag{3-70}$$

and the mean-square value of $\tilde{T}$ may be obtained from

$$\overline{\tilde{T}^2} = \frac{1}{\dot{\omega}^2} \left[\frac{\partial \omega}{\partial \underline{x}}\right] \overline{\tilde{\underline{x}}\,\tilde{\underline{x}}^T} \left[\frac{\partial \omega}{\partial \underline{x}}\right]^T \tag{3-71}$$

with $\overline{\tilde{\underline{x}}\,\tilde{\underline{x}}^T}$ $\left[\partial\omega/\partial\underline{x}\right]^T$ determined as in Sec. 3.6.

The final error may be expressed as

$$e_f = \left[\frac{\partial g}{\partial \underline{x}}\right] \tilde{\underline{x}}(T) + \dot{g}\tilde{T} \tag{3-72}$$

where

$$\dot{g} = \left[\frac{\partial g}{\partial \underline{x}}\right] \underline{f}(T) + \frac{\partial g}{\partial T} \tag{3-73}$$

Combining (3-70) with (3-72), we obtain

$$e_f(T) = \left[\frac{\partial g}{\partial \underline{x}} - \frac{\dot{g}}{\dot{\omega}} \frac{\partial \omega}{\partial \underline{x}}\right] \tilde{\underline{x}}(T) \tag{3-74}$$

The error being a linear function of $\tilde{\underline{x}}(T)$ alone, its mean-square value is amenable to the minimization technique of Sections 3.4 and 3.5. The expression (3-18) for the mean-square error remains valid if we impose new terminal conditions on the adjoint solutions, namely

$$\underline{\lambda}(T) = \left[\frac{\partial g}{\partial \underline{x}} - \frac{\dot{g}}{\dot{\omega}} \frac{\partial \omega}{\partial \underline{x}}\right]^T \tag{3-75}$$

This is similar in form to the technique used by Bryson and Denham[15] for deterministic problems with variable terminal time. There are important

differences, however. Here we are altering $\underline{\lambda}(T)$ to take care of random variations $\tilde{T}$ in T rather than intentional adjustments δT. $\underline{\lambda}(t)$ is a measure of the sensitivity of the system to noise, rather than to control-variable adjustments.

The revised terminal conditions (3-75) for the adjoint equations necessitate a revision of the terminal conditions (3-32) for the perturbed adjoint equations. In particular, since $\dot{g}$ and $\dot{\omega}$ in (3-75) are explicit functions, through $\underline{f}(T)$, of the final control variables $\underline{u}(T)$, a formal variation of $\underline{\lambda}(T)$ will result in an expression involving $\delta\underline{u}(T)$:

$$\begin{aligned}\delta\underline{\lambda}(T) = {} & G_{xx}\delta\underline{x}(T) - \frac{\dot{g}}{\dot{\omega}}\,\Omega_{xx}\delta\underline{x}(T) \\ & - \frac{1}{\dot{\omega}}\left[\frac{\partial\omega}{\partial\underline{x}}\right]^T\left[\underline{f}^T G_{xx} + \frac{\partial g}{\partial\underline{x}}F_x + \frac{\partial^2 g}{\partial T\partial\underline{x}}\right]\delta\underline{x}(T) \\ & + \frac{\dot{g}}{\dot{\omega}^2}\left[\frac{\partial\omega}{\partial\underline{x}}\right]^T\left[\underline{f}^T\Omega_{xx} + \frac{\partial\omega}{\partial\underline{x}}F_x + \frac{\partial^2\omega}{\partial T\partial\underline{x}}\right]\delta\underline{x}(T) \\ & - \frac{1}{\dot{\omega}}\left[\frac{\partial\omega}{\partial\underline{x}}\right]^T\left[\frac{\partial g}{\partial\underline{x}} - \frac{\dot{g}}{\dot{\omega}}\frac{\partial\omega}{\partial\underline{x}}\right]F_u\,\delta\underline{u}(T)\end{aligned} \tag{3-76}$$

where

$$\Omega_{xx} = \left[\frac{\partial^2\omega(T)}{\partial x_i(T)\partial x_j(T)}\right] \tag{3-77}$$

and all quantities are evaluated on the nominal trajectory at time T.

Equation (3-76), instead of (3-32), must now be substituted into (3-34) and the succeeding equations. The final result, in the case in which the initial conditions are also adjustable, is

$$\delta\overline{e^2} = \underline{b}_\phi^T\,\delta\underline{x}_0 + \int_0^T \underline{g}_\phi^T(t)\,\delta\underline{u}(t)dt + \underline{b}^T\,\delta\underline{u}(T) \tag{3-78}$$

where $\underline{b}_\phi$ and $\underline{g}_\phi(t)$ are given by (3-46) and (3-42), except that for G_{xx} we substitute

$$\left[G_{xx} - \frac{\dot{g}}{\dot{\omega}}\Omega_{xx}\right] - \frac{1}{\dot{\omega}}\left[\frac{\partial\omega}{\partial\underline{x}}\right]^T\left\{\underline{f}^T\left[G_{xx} - \frac{\dot{g}}{\dot{\omega}}\Omega_{xx}\right]\right.$$

$$\left. + \left[\frac{\partial g}{\partial\underline{x}} - \frac{\dot{g}}{\dot{\omega}}\frac{\partial\omega}{\partial\underline{x}}\right]F_x + \left[\frac{\partial^2 g}{\partial T\partial\underline{x}} - \frac{\dot{g}}{\dot{\omega}}\frac{\partial^2\omega}{\partial T\partial\underline{x}}\right]\right\} \tag{3-79}$$

and $\underline{b}$ is given by

$$\underline{b}^T = \frac{-1}{\dot{\omega}}\left[\underline{\lambda}^T(0)CX^{-T}(0) + \int_0^T \underline{r}_I^T(t)F_n^T(t)X^{-T}(t)\,dt\right]X^T(T)\left[\frac{\partial\omega}{\partial\underline{x}}\right]^T\underline{\lambda}^T(T)\,F_u(T)$$

$$= -(1/\dot{\omega})\,\underline{w}_1^T(T)\left[\partial\omega/\partial\underline{x}\right]^T\underline{\lambda}^T(T)F_u(T) \tag{3-80}$$

The successive-improvement procedure may now be carried out in a formal manner, just as outlined in Chapter 2, provided that $\underline{u}(T)$ can be considered as a set of adjustable parameters entirely separate from the remainder of the control-variable history $\underline{u}(t)$, $0 \leq t < T$. This assumption, in turn, is justified only in the idealized case in which the perturbations $\tilde{T}$ are vanishingly small, since the linearizations used in analyzing the effects of $\tilde{T}$ are based on the assumption that $\underline{u}(t)$ makes no abrupt changes in some small region around the nominal final time.

If we wish to take advantage of the independent optimizing capabilities of $\underline{u}(T)$ in a real problem, some restrictions must be placed on the behaviour of $\underline{u}(t)$ if the linear analysis is to remain valid. In other words, $\underline{u}(t)$ must be constrained to remain approximately constant in some small interval around $t = T$, the size of the interval being decided on the basis of an estimate of the amplitudes of the perturbation $\tilde{T}$ which can reasonably be expected.

It should be noted, if independent adjustment of $\underline{u}(T)$ is allowed, that in (3-69) and (3-73), and in the succeeding equations, $\dot{g}$ and $\dot{\omega}$ are to be evaluated using the $\underline{f}(T)$ determined from the isolated value of $\underline{u}(T)$.

3.8 Integral Constraints and Performance Criteria

In general, any scalar functional of the state-variable and control-variable histories (including ensemble properties in stochastic problems) is eligible as an equality or inequality constraint. If the functional possesses an extremum under the conditions imposed on the problem, it is also eligible as a performance criterion.

It is often desirable to constrain or extremize a functional of the form

$$\phi = g(\underline{x}(T), T) + \int_0^T f(\underline{x}(t), \underline{u}(t), t)\, dt \tag{3-81}$$

Appendix A indicates how the adjoint equations may be modified, by the addition of a forcing term, to handle criteria of this type when f does not depend on $\underline{u}(t)$. The dependence on $\underline{u}(t)$ (in total-power or total-fuel constraints, for example) can be included simply by adding to the expression for $\delta\phi$ a term

$$\int_0^T \left[\frac{\partial f}{\partial \underline{u}}\right] \delta\underline{u}\, dt \tag{3-82}$$

The nonhomogeneous adjoint equation (A-17) is identical with the Euler-Lagrange equations corresponding to the state variables, as they appear in the multiplier rule for the general problem of Bolza.[9, 11]

An alternative to modification of the adjoint equations is to eliminate the integral (i. e., put the problem in the Mayer form) by introducing an additional state variable whose derivative is $f(\underline{x}(t), \underline{u}(t), t)$. This procedure enlarges the dimension of the system, with attendant increases in the required computational effort.

In stochastic problems a possible criterion would be the mean-square value of the random perturbations of the quantity in equation (3-81). If the integral is handled by modification of the adjoint equations, there will be corresponding modifications of the perturbed adjoint equations and of the formulas for the influence functions. We encounter difficulties, however, if we attempt to include, in the criterion, the integral of the mean-square value of a function (as opposed to the mean-square value of an integral). This was attempted by the author for the linear filter problem of Sec. 4. 4. The occurrence of a series of successive integrations of the type encountered in steps (3) and (4) of Sec. 5. 2, seems to indicate that large amounts of computing time would be required in order to attain reasonable precision.

Finally, we note that inequality constraints on "terminal" quantities of the type (3-81) present no problem. We first predict the $\delta\underline{\psi}$ which will result from an unconstrained perturbation. If this will result in violation of the constraint, the constraint is made one of equality and the control perturbation recomputed for the next iteration. If it will not, the constraint is ignored for the next iteration.

3.9 In-Flight Inequality Constraints

A constraint of the type

$$C(\underline{x}(t), \underline{u}(t), t) \leq 0 \tag{3-83}$$

which is to be satisfied at all values of the independent variable t, has been termed an "in-flight" constraint.

Several approaches have been devised for handling such a constraint.[22, 25] Most of these fall into the category of "penalty function" methods, in which we define an auxiliary state variable whose final value provides a measure of the total constraint violation, thereby transforming the constraint into one of the "terminal" variety.[22] It is also possible to adjoin this terminal quantity to the performance criterion, rather than constraining it directly, an approach which allows only approximate satisfaction of the constraints.[6]

Denham[22, 24, 25] has devised a "direct" method, which involves an alteration of the differential equations in regions where the constrained quantity is at its boundary (i.e., where C = 0). Although conceptually not as simple as the penalty function methods, the technique appears to have several advantages. It does not require the introduction of an extra state variable; it allows direct determination of the control variables in regions where the constraint is active; and it exhibits a close correspondence to the necessary conditions obtained by extension of the classical variational theory to problems of this kind.[11, 17]

The principles of the direct method could undoubtedly be applied, with some modification, to stochastic problems of the type considered here.

3.10 General Second-Order Criteria

In Sec. 3.4 we considered the minimization of the mean-square value of a scalar function $\tilde{g}(\underline{x}(T), T)$. It may be desirable to consider the second-order statistical properties of more general functions. For example, we may define

$$\phi = \overline{\tilde{\underline{h}}^T W \tilde{\underline{h}}} = \mathrm{tr}\left[W \overline{\tilde{\underline{h}}\, \tilde{\underline{h}}^T}\right] \tag{3-84}$$

where $\underline{h}(\underline{x}(T), T)$ is a vector function of the final state variables and final time, and W is a symmetric, positive-definite matrix; ϕ is the most general linear function of the covariance matrix of the random perturbations of $\underline{h}$ (which may, of course, be a nonlinear function of $\underline{x}(T)$ and T).

If we let $W^{1/2}$ represent the symmetric, positive-definite square root matrix of W, and define

$$\underline{g}(\underline{x}(T), T) = W^{1/2}\,\underline{h}(\underline{x}(T), T) \tag{3-85}$$

the cost function becomes

$$\phi = \overline{\tilde{\underline{g}}^T \tilde{\underline{g}}} = \phi_1 + \phi_2 + \ldots \tag{3-86}$$

where

$$\phi_i = \overline{\tilde{g}_i^2} \tag{3-87}$$

so that we now have a sum of cost functions of the type previously considered. Since the influence function of the sum is the sum of the influence functions, this presents no difficulty other than a moderate increase in the amount of computation required.

An example of such a cost function is the mean-square length of the final perturbation vector (a purely random "miss distance"), in which case

$$\phi = \overline{\tilde{\underline{x}}^T(T)\,\tilde{\underline{x}}(T)} = \overline{\left|\tilde{\underline{x}}(T)\right|^2} \tag{3-88}$$

3.11 Partly-Deterministic Criteria

It is apparent that no difficulty is encountered if criteria of the kind already discussed are combined with deterministic properties of the nominal trajectory.[11] For example, a miss distance may include a random and a deterministic component, if the nominal trajectory does not result in zero miss. Letting $\underline{x}_n$ represent the nominal state vector and $\underline{x}_t$ the final target vector, we may wish to minimize the quantity

$$\phi = \overline{(\underline{x}_n(T) + \tilde{\underline{x}}(T) - \underline{x}_t)^T W(\underline{x}_n(T) + \tilde{\underline{x}}(T) - \underline{x}_t)}$$

$$= (\underline{x}_n(T) - \underline{x}_t)^T W(\underline{x}_n(T) - \underline{x}_t) + \overline{\tilde{\underline{x}}^T(T) W \tilde{\underline{x}}(T)} \qquad (3\text{-}89)$$

The perturbation in the first term is just

$$2(\underline{x}_n(T) - \underline{x}_t)^T W \delta \underline{x}_n(T)$$

for which the influence function is easily found. This is then added to the influence function for the second term, determined as in Sec. 3.10. This is in contrast with the usual approach of constraining the nominal in such a way that $\underline{x}_n(T) = \underline{x}_t$, then minimizing

$$\overline{\tilde{\underline{x}}^T W \tilde{\underline{x}}}$$

subject to this constraint. The present approach may allow large improvements if the disturbances depend on the controls and on the state variables, since satisfaction of the constraint $\underline{x}_n(T) = \underline{x}_t$ may require that the trajectory pass through regions of very great disturbances.

A simple example is the problem of aiming a gun so as to destroy a target which is hidden from the gun position by an intervening woods. A direct hit may require that the trajectory pass through the woods, where there is a high probability that the projectile will be deflected by the trees. It may be preferable to deliberately miss the target and allow the shell to explode somewhat above it. If this allows the trajectory to pass entirely above the trees, the aimed-for point may be reached with high precision, resulting in a net decrease in the mean-square miss distance.

3.12 Distribution-Dependent Criteria

Some other useful types of criteria may be handled if we have some additional information about the amplitude distributions of the perturbations. This usually implies the assumption of Gaussian disturbances, since the linearized system preserves the normal distribution in the output. If this assumption is valid, or approximately so, the mean and variance are sufficient to completely describe the distribution. Then, for example, minimization of the mean of the absolute value of the error is equivalent to

minimization of the mean-square, since for a normally distributed variable e,

$$\overline{|e|} = \sqrt{\frac{2}{\pi}\,\overline{e^2}} \tag{3-90}$$

A very useful criterion is the probability of a quantity exceeding a specified value (or, alternatively, the value which will be exceeded with a specified probability). If the nominal trajectory of a rocket is chosen so as to minimize the fuel expenditure, it is still necessary to apply a "factor of safety" in the fuel supply, based on an estimate of the perturbations, to ensure the desired probability of having sufficient fuel. It is possible, however, that a slightly "worse" nominal path would lessen the disturbances enough to cause a net decrease in the amount of fuel needed. A much more reasonable approach, therefore, would be to choose the nominal path so as to minimize the amount of fuel needed to ensure a specified probability of accomplishing the mission objectives.

Let p represent the probability that a function $g(\underline{x}(T), T)$ will exceed a value k, i.e.,

$$p = \Pr(g > k) \tag{3-91}$$

and let g_n represent the nominal value of g:

$$g = g_n + \tilde{g} \tag{3-92}$$

giving

$$p = \Pr(\tilde{g} > k - g_n) \tag{3-93}$$

Then, assuming Gaussian disturbances, $\tilde{g}$ will be normally distributed and[18]

$$p = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(k-g_n)/\sigma} e^{-x^2/2} dx \tag{3-94}$$

where

$$\sigma^2 = \overline{\tilde{g}^2} \tag{3-95}$$

The quantity σ^2 can be controlled by methods already described. If we are interested in extremizing p with k specified, we write

$$p = \frac{-1}{\sqrt{2\pi}} \exp\left[-\frac{(k - g_n)^2}{2\sigma^2}\right] \delta\left[\frac{k - g_n}{\sigma}\right] \tag{3-96}$$

Varying σ and g_n with k fixed, and noting that

$$\delta\sigma = \frac{1}{2\sigma}\, \delta(\sigma^2) \tag{3-97}$$

we obtain

$$\delta p = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(k - g_n)^2}{2\sigma^2}\right]\left[\delta g_n + \frac{k - g_n}{2\sigma^2}\, \delta(\sigma^2)\right] \tag{3-98}$$

The corresponding influence function is easily found, as in Sec. 3.11.

On the other hand, if we wish to fix p and extremize k by adjusting g_n and σ^2, the requirement, from (3-96), becomes

$$\delta\left[\frac{k - g_n}{\sigma}\right] = 0 = \frac{\delta k}{\sigma} - \frac{\delta g_n}{\sigma} - \frac{(k - g_n)}{\sigma^2}\, \delta\sigma \tag{3-99}$$

so that

$$\delta k = \delta g_n + \frac{k - g_n}{2\sigma^2}\, \delta(\sigma^2) \tag{3-100}$$

Either of the above points of view may be used in the case where the probability enters as a constraint rather than as a payoff function (i.e., p and k both prescribed).

CHAPTER 4

SOME SPECIAL SYSTEM CONFIGURATIONS

4.1 Introduction

In this chapter we shall derive the influence functions associated with certain specified system configurations. In some cases the results may appear more complex than the corresponding results for the general problem. Nevertheless, despite the proliferation of terms in the applicable equations, a computational saving is invariably realized when the problem formulation is such as to take into consideration the restrictions which have been placed on the system configuration.

For the case in which the noise is a random function of the time alone, certain related results have been obtained by Denham[21, 23] and, in the linear case, by Johansen.[41]

4.2 Linear Feedback around a Nonlinear System

The system to be considered is shown in Figure 4.1. The nonlinear element F behaves according to the differential equation

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}, \underline{u}, \underline{n}, t) \qquad \underline{x}(0) = \underline{x}_0 \tag{4-1}$$

as in the system of Chapter 3. In this case, however, the m-vector $\underline{u}(t)$ is generated by passing the state n-vector $\underline{x}(t)$ through the $h \times n$ "measurement matrix" $G(t)$ ($h \leq n$) and the $m \times h$ feedback-gain matrix $A(t)$. For the present we shall consider the case in which no measurement noise is present, so that the only random disturbances are represented by the noise p-vector $\underline{n}(\underline{x}, \underline{u}, t)$.

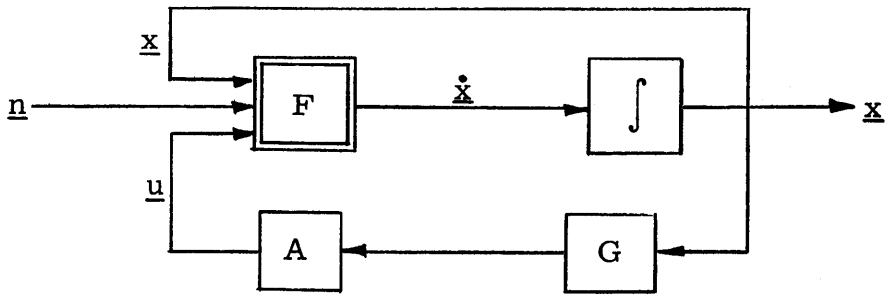

Figure 4.1 Linear Feedback around a Nonlinear System.

Assuming that the elements of A(t) can be adjusted at will, we wish to find the values of those elements which, for a given initial condition $\underline{x}_0$, will minimize the performance criterion of Section 3. 4, namely

$$\phi = \overline{e^2} = \overline{\tilde{g}^2}(\underline{x}(T), T) \tag{4-2}$$

Since we have restricted ourselves to inputs $\underline{u}(t)$ of the form

$$\underline{u}(t) = A(t)G(t)\,\underline{x}(t) \tag{4-3}$$

we find

$$\delta\underline{u}(t) = \delta A(t)\,G(t)\,\underline{x}(t) + A(t)\,G(t)\,\delta\underline{x}(t) \tag{4-4}$$

so that the perturbation equations (3-9) and (3-10) become

$$\delta\dot{\underline{x}}(t) = \left[F_x(t) + F_u(t)\,A(t)\,G(t)\right]\delta\underline{x}(t) + F_u(t)\,\delta A(t)\,G(t)\,\underline{x}(t) \tag{4-5}$$

and

$$\dot{\tilde{\underline{x}}}(t) = \left[F_x(t) + F_u(t)A(t)\,G(t)\right]\tilde{\underline{x}}(t) + F_n(t)\,\underline{n}(t) \tag{4-6}$$

The fundamental solutions X(t) and the adjoint variables $\underline{\lambda}(t)$ must now be determined from the coefficient matrix of (4-5) rather than from $F_x(t)$ alone. Equation (3-18) for the mean-square error remains unchanged, but we now have, instead of (3-28),

$$\delta\underline{x}(t) = X(t)\,X^{-1}(0)\,\delta\underline{x}_0 + X(t)\int_0^t X^{-1}(\tau)F_u(\tau)\,\delta A(\tau)G(\tau)\,\underline{x}(\tau)\,d\tau \tag{4-7}$$

Since the adjoint equation is now

$$\dot{\underline{\lambda}}(t) = -\left[F_x + F_uAG\right]^T\underline{\lambda}(t) \tag{4-8}$$

we replace (3-29) by

$$\delta\dot{\underline{\lambda}}(t) = -\left[F_x + F_u AG\right]^T \delta\underline{\lambda}(t) - H_{xu}\delta\underline{u}(t) - H_{xx}\delta\underline{x}(t)$$

$$-\left[F_u \delta AG\right]^T \underline{\lambda}(t) - G^T A^T H_{uu}\delta\underline{u}(t) - G^T A^T H_{xu}^T \delta\underline{x}(t) \qquad (4\text{-}9)$$

where

$$H_{uu}(t) = \left[\frac{\partial^2 H}{\partial u_i \partial u_j}\right] \qquad (m \times m) \qquad (4\text{-}10)$$

so that $\delta\underline{\lambda}(t)$ becomes

$$\delta\underline{\lambda}(t) = X^{-T}(t)X^T(T)G_{xx}\delta\underline{x}(T)$$

$$- X^{-T}(t)\int_T^t X^T(\tau)\Big\{\left[H_{xu}(\tau) + G^T(\tau)A^T(\tau)H_{uu}(\tau)\right]\delta\underline{u}(\tau)$$

$$+\left[H_{xx}(\tau) + G^T(\tau)A^T(\tau)H_{xu}^T(\tau)\right]\delta\underline{x}(\tau) + G^T(\tau)\,\delta A^T(\tau)\,F_u^T(\tau)\underline{\lambda}(\tau)\Big\}\,d\tau \qquad (4\text{-}11)$$

The vector $\delta\underline{\lambda}_n(t)$ is once more given by (3-35). With these alterations to the relationships of Sec. 3.4, and following procedures analogous to those outlined there, we may finally, after considerable simplification, derive the relationship for the payoff change in the form of equation (2-65), namely

$$\delta\phi = \mathrm{tr}\int_0^T M_\phi^T(t)\,\delta A(t)\,dt + \underline{b}_\phi^T\,\delta\underline{x}_0 \qquad (4\text{-}12)$$

The matrix $M_\phi^T(t)$ is given by

$$M_\phi^T(t) = M^T(t) + \underline{v}(t)\,\underline{w}^T(t) \qquad (4\text{-}13)$$

where

$$M^T(t) = 2G(t)X(t)\left[X^{-1}(0)C\underline{\lambda}(0) + \int_0^t X^{-1}(\tau)F_n(\tau)\underline{r}_I(\tau)\,d\tau\right]\underline{\lambda}^T(t)\,F_u(t) \qquad (4\text{-}14)$$

$$\underline{v}(t) = 2G(t)\,\underline{x}(t) \tag{4-15}$$

and

$$\begin{aligned}
\underline{w}^T(t) = {} & \underline{\lambda}^T(0)CX^{-T}(0)\Big\{X^T(T)G_{xx}X(T)X^{-1}(t)F_u(t) \\
& + X^T(t)\left[H_{xu} + G^TA^TH_{uu}(t)\right]\Big\} \\
& + \underline{\lambda}^T(0)CX^{-T}(0)\int_t^T X^T(\tau)\Big\{\left[H_{xu} + G^TA^TH_{uu}\right]AG + H_{xx} \\
& + G^TA^TH_{xu}^T\Big\}X(\tau)\,d\tau\,X^{-1}(t)\,F_u(t) \\
& + \int_t^T \underline{r}_I^T\left[H_{nx} + H_{nu}AG\right]X(\tau)\,d\tau\,X^{-1}(t)\,F_u(t) \\
& + \int_0^T \underline{r}_I^T(\tau)F_n^T(\tau)X^{-T}(\tau)\,d\tau\,X^T(T)G_{xx}X(T)X^{-1}(t)\,F_u(t) \\
& + \int_0^t \underline{r}_I^T\,F_n^TX^{-T}(\tau)\,d\tau\,X^T(t)\left[H_{xu}(t) + G^TA^TH_{uu}(t)\right] \\
& + \int_t^T\left[\int_0^\tau \underline{r}_I^TF_n^TX^{-T}(s)\,ds\right]X^T(\tau)\Big\{\left[H_{xu} + G^TA^TH_{uu}\right]AG(\tau) \\
& + H_{xx}(\tau) + G^TA^TH_{xu}^T(\tau)\Big\}X(\tau)\,d\tau\,X^{-1}(t)\,F_u(t) \\
& + \int_t^T \underline{\lambda}_n^T(\tau)\left[R_x + R_uAG\right]X(\tau)\,d\tau\,X^{-1}(t)\,F_u(t) \\
& + \underline{r}_I^T(t)H_{nu}(t) + \underline{\lambda}_n^T(t)R_u(t)
\end{aligned} \tag{4-16}$$

The vector of influence coefficients for the initial-condition adjustments is given by

$$\frac{1}{2}\underline{b}_{\phi}^{T} = \underline{\lambda}^{T}(0)CX^{-T}(0)X^{T}(T)G_{xx}X(T)X^{-1}(0)$$

$$+ \underline{\lambda}^{T}(0)CX^{-T}(0)\int_{0}^{T} X^{T}(t)\Big\{\Big[H_{xu} + G^{T}A^{T}H_{uu}\Big]AG$$

$$+ H_{xx} + G^{T}A^{T}H_{xu}^{T}\Big\}X(t)\,dt\,X^{-1}(0)$$

$$+ \int_{0}^{T} \underline{r}_{I}^{T}F_{n}^{T}X^{-T}(t)\,dt\,X^{T}(T)G_{xx}X(T)X^{-1}(0)$$

$$+ \int_{0}^{T}\Big\{\underline{r}_{I}^{T}\Big[H_{nx} + H_{nu}AG\Big] + \underline{\lambda}_{n}^{T}R_{x}\Big\}X(t)\,dt\,X^{-1}(0)$$

$$+ \int_{0}^{T}\Bigg[\int_{0}^{\tau} \underline{r}_{I}^{T}(t)\,F_{n}^{T}(t)X^{-T}(t)\,dt\Bigg]X^{T}(\tau)\Big\{\Big[H_{xu} + G^{T}A^{T}H_{uu}\Big]AG(\tau)$$

$$+ H_{xx}(\tau) + G^{T}A^{T}H_{xu}^{T}(\tau)\Big\}X(\tau)\,d\tau X^{-1}(0) \qquad (4\text{-}17)$$

Once again we may simplify the expressions for the influence functions by recognizing, as in Chapter 3, that certain terms represent solutions of certain differential equations. We redefine the vector $\underline{w}_1(t)$ by

$$\dot{\underline{w}}_1(t) = \Big[F_x(t) + F_u(t)A(t)G(t)\Big]\underline{w}_1(t) + F_n(t)\,\underline{r}_I(t) \qquad (4\text{-}17.1)$$

with the initial condition $\underline{w}_1(0) = C\underline{\lambda}(0)$ as before. We also define three vector solutions of the adjoint differential equation, as follows:

$$\dot{\underline{w}}_3 = -\Big[F_x + F_uAG\Big]^{T}\underline{w}_3 - \Big[(H_{nx} + H_{nu}AG)^{T}\underline{r}_I$$

$$+ (R_x + R_uAG)^{T}\underline{\lambda}_n + (H_{xx} + G^{T}A^{T}H_{xu}^{T} + H_{xu}AG + G^{T}A^{T}H_{uu}AG)\,\underline{w}_1\Big]$$

(4-17.2)

$$\dot{\underline{w}}_4 = -\left[F_x + F_u AG\right]^T \underline{w}_4 - \left[(H_{nx} + H_{nu}AG)^T \underline{r}_I + R_x^T \underline{\lambda}_n + H_{xx}\underline{w}_1\right] \quad (4\text{-}17.3)$$

$$\dot{\underline{w}}_5 = -\left[F_x + F_u AG\right]^T \underline{w}_5 - (G^T A^T H_{xu}^T + H_{xu}AG + G^T A^T H_{uu}AG)\underline{w}_1 \quad (4\text{-}17.4)$$

with the boundary conditions

$$\underline{w}_3(T) = \underline{w}_4(T) = G_{xx}\underline{w}_1(T) \quad (4\text{-}17.5)$$

$$\underline{w}_5(T) = 0 \quad (4\text{-}17.6)$$

With these substitutions it may be shown that the equations for the influence functions become:

$$M^T(t) = 2G(t)\underline{w}_1(t)\underline{\lambda}^T(t)F_u(t) \quad (4\text{-}18)$$

$$\underline{w}(t) = \left[H_{xu}^T + H_{uu}AG\right]\underline{w}_1 + F_u^T \underline{w}_3 + H_{nu}^T \underline{r}_I + R_u^T \underline{\lambda}_n \quad (4\text{-}18.1)$$

and

$$\underline{b}_\phi = 2\underline{w}_4(0) + 2\underline{w}_5(0) \quad (4\text{-}18.2)$$

4.3 Linear Feedback with Measurement Noise

We shall now generalize the results of the preceding section to the case where the measurements include additive noise, as shown in Figure 4.2.

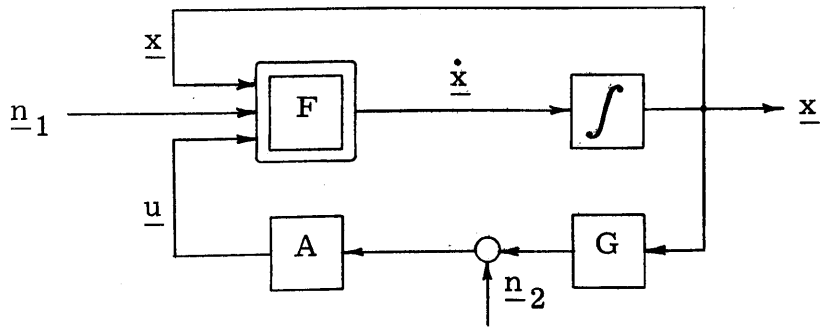

Figure 4.2 Linear Feedback with Measurement Noise

We shall consider the measurement noise to constitute an h-vector $\underline{n}_2(\underline{x}, t)$, dependent on the state variables $\underline{x}$ as well as the time t, and rename the original noise p-vector $\underline{n}_1(\underline{x}, \underline{u}, t)$. All quantities derived from the properties of $\underline{n}_1$ shall be given the subscript 1(e.g., $\underline{\lambda}_{n1}$, R_1, $\underline{r}_{I1}$), and the corresponding properties of $\underline{n}_2$ shall have a subscript 2. We assume no correlation between $\underline{n}_1$ and $\underline{n}_2$.

Equation (4-6) is now replaced by

$$\dot{\tilde{\underline{x}}}(t) = \left[F_x + F_u AG\right] \tilde{\underline{x}}(t) + F_n \underline{n}_1(t) + F_u A \underline{n}_2(t) \qquad (4\text{-}18)$$

and we now have two vector functions corresponding to $\underline{\lambda}_n(t)$:

$$\underline{\lambda}_{n1}(t) = F_n^T(t)\, \underline{\lambda}(t) \qquad (4\text{-}19)$$

and

$$\underline{\lambda}_{n2}(t) = A^T(t)\, F_u^T(t)\, \underline{\lambda}(t) \qquad (4\text{-}20)$$

so that $\delta\underline{\lambda}_{n1}(t)$ is given by (3-35) and

$$\delta\underline{\lambda}_{n2}(t) = \delta A^T(t) F_u^T(t)\, \underline{\lambda}(t) + A^T(t) H_{uu}(t)\, \delta\underline{u}(t)$$
$$+ A^T(t) H_{xu}^T(t)\, \delta\underline{x}(t) + A^T(t) F_u^T(t) \delta\underline{\lambda}(t) \qquad (4\text{-}21)$$

The error is given by

$$e(T) = \underline{\lambda}^T(0)\, \tilde{\underline{x}}_0 + \int_0^T \underline{\lambda}_{n1}^T \underline{n}_1\, dt + \int_0^T \underline{\lambda}_{n2}^T \underline{n}_2\, dt \qquad (4\text{-}22)$$

and its perturbation by

$$\delta e(T) = \delta\underline{\lambda}^T(0) \tilde{\underline{x}}_0 + \underline{\lambda}^T(0)\, \delta\tilde{\underline{x}}_0 + \int_0^T \delta\underline{\lambda}_{n1}^T \underline{n}_1\, dt$$
$$+ \int_0^T \underline{\lambda}_{n1}^T\, \delta\underline{n}_1\, dt + \int_0^T \delta\underline{\lambda}_{n2}^T \underline{n}_2 dt + \int_0^T \underline{\lambda}_{n2}^T\, \delta\underline{n}_2\, dt \qquad (4\text{-}23)$$

The expression for $\delta \overline{e^2}$ is found to be the sum of (3-44.1) and (3-22), with appropriate subscripts for the noise $\underline{n}_1$, plus the corresponding terms for $\underline{n}_2$:

$$\frac{\delta \overline{e^2}}{2} = \frac{1}{2}\,\underline{c}_x^T\,\delta\underline{x}_0 + \underline{\lambda}^T(0)C\,\delta\underline{\lambda}(0)$$

$$+ \int_0^T \underline{r}_{I1}^T\,\delta\underline{\lambda}_{n1}\,dt + \int_0^T \underline{\lambda}_{n1}^T R_{1x}\,\delta\underline{x}\,dt$$

$$+ \int_0^T \underline{\lambda}_{n1}^T R_{1u}\,\delta\underline{u}\,dt + \int_0^T \underline{r}_{I2}^T\,\delta\underline{\lambda}_{n2}\,dt$$

$$+ \int_0^T \underline{\lambda}_{n2}^T R_{2x}\,\delta\underline{x}\,dt \qquad (4\text{-}24)$$

The final results can be derived from those of Sec. 4.2 by two types of alterations:

First, to the expression (4-18) for $M^T(t)$ we add the term

$$2\underline{r}_{I2}(t)\,\underline{\lambda}^T(t)F_u(t) \qquad (4\text{-}25)$$

which is present because of the first term in (4-21).

Second, in the equations (4-17.1) to (4-18.1) we make the substitutions indicated in the following table:

Replace	by
$\underline{r}_I^T H_{nu}$	$\underline{v}_1^T = \underline{r}_{I1}^T H_{nu} + \underline{r}_{I2}^T A^T H_{uu}$
$\underline{r}_I^T H_{nx}$	$\underline{v}_2^T = \underline{r}_{I1}^T H_{nx} + \underline{r}_{I2}^T A^T H_{xu}^T$
$\underline{r}_I^T F_n^T$	$\underline{v}_3^T = \underline{r}_{I1}^T F_n^T + \underline{r}_{I2}^T A^T F_u^T$
$\underline{\lambda}_n^T R_x$	$\underline{v}_4^T = \underline{\lambda}_{n1}^T R_{1x} + \underline{\lambda}_{n2}^T R_{2x}$

4.3.1 A Special Case: Linear Systems

It should be noted that in the case in which the element F is completely linear, so that (4-1) becomes

$$\dot{\underline{x}}(t) = F_x(t)\underline{x}(t)+F_u(t)\,\underline{u}(t) + F_n(t)\,\underline{n}(t) \tag{4-26}$$

all of the H variables vanish in (4-16) and (4-17), so that most of the terms disappear. It is reasonable to expect that some of these terms will vanish in many nonlinear problems as well.

Furthermore, if the function $g(\underline{x}(T), T)$ (whose mean-square random perturbation is to be minimized) is linear in $\underline{x}$, then $G_{xx} = 0$ as well; if the noise is a function of t only, we then have

$$M_\phi(t) = M(t) \tag{4-27}$$

and

$$\underline{b}_\phi = 0 \tag{4-28}$$

with corresponding results for the system with measurement noise.

4.3.2 A Special Case: Time-Invariant Gains

If the matrix of feedback gains A(t) is constrained to be time-invariant, equation (4-12) may be replaced by

$$\delta\phi = \mathrm{tr} \int_0^T M_\phi^T(t)\,dt\,\delta A + \underline{b}_\phi^T\,\delta\underline{x}_0 \tag{4-29}$$

The techniques of Chapter 2 are again applicable with little modification.

4.4 A Linear Filter and Predictor

In the system of Figure 4.3, a random signal and a random noise, correlated with each other, and with known statistical properties, are summed and passed through a linear system. The feedback matrix A(t) is to be chosen in the interval $0 \leq t \leq T$ so as to minimize

$$\phi = \overline{\left[\underline{x}(T) - \underline{s}(T + p)\right]^T W \left[\underline{x}(T) - \underline{s}(T + p)\right]} \tag{4-30}$$

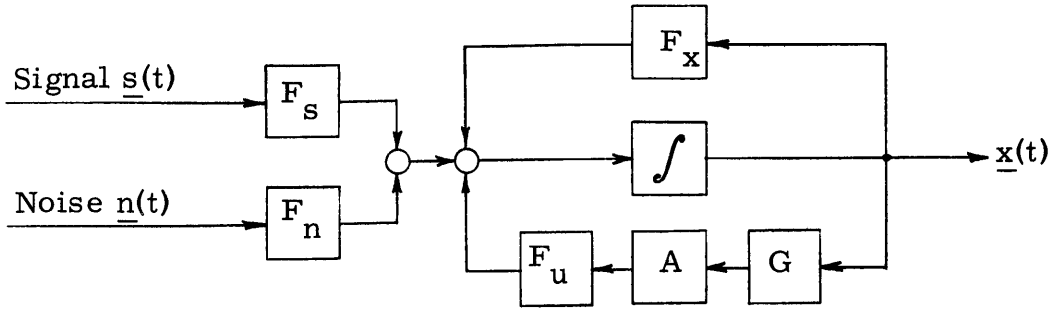

Figure 4.3 A Linear Filter and Predictor

where W is an arbitrary positive-definite weighting matrix, and p is the desired prediction time.*

We shall assume that the initial condition $\underline{x}(0)$ is the result of the infinite past history of $\underline{s}(t)$ and $\underline{n}(t)$, with A(t) known for $t < 0$ (e.g., $A(t) = 0$); $\underline{s}(t)$ and $\underline{n}(t)$ are random functions of t only, and the following statistical properties are known:

$$R_{ss}(t, \tau) = \overline{\underline{s}(t)\,\underline{s}^T(\tau)} \tag{4-31}$$

$$R_{nn}(t, \tau) = \overline{\underline{n}(t)\,\underline{n}^T(\tau)} \tag{4-32}$$

$$R_{ns}^T(\tau, t) = R_{sn}(t, \tau) = \overline{\underline{s}(t)\,\underline{n}^T(\tau)} \tag{4-33}$$

From (4-30) we may see that

$$\delta\phi = 2\overline{\left[\underline{x}(T) - \underline{s}(T + p)\right]^T W\, \delta\underline{x}(T)} \tag{4-34}$$

where, with the given assumptions about $\underline{x}(0)$, we may write

* The criterion requires that the matrix F_s be square, so that $\underline{x}(t)$ and $\underline{s}(t)$ have the same dimension.

$$\underline{x}(t) = X(t) \int_{-\infty}^{t} X^{-1}(\tau)\left[F_s(\tau)\,\underline{s}(\tau) + F_n(\tau)\,\underline{n}(\tau)\right] d\tau \tag{4-35}$$

in which X(t) is a fundamental solution of the system differential equation

$$\dot{\underline{x}}(t) = \left[F_x(t) + F_u(t)A(t)G(t)\right]\underline{x}(t) + F_s(t)\,\underline{s}(t) + F_n(t)\,\underline{n}(t) \tag{4-36}$$

From the perturbation equation

$$\delta\dot{\underline{x}}(t) = \left[F_x(t) + F_u(t)A(t)G(t)\right]\delta\underline{x}(t) + F_u(t)\,\delta A(t)G(t)\underline{x}(t) \tag{4-37}$$

with $\delta\underline{x}(0) = 0$, we obtain

$$\begin{aligned}\delta\underline{x}(T) &= X(T) \int_0^T X^{-1}(t)\,F_u(t)\delta A(t)G(t)\,\underline{x}(t)\,dt \\ &= X(T)\int_0^T X^{-1}(t)\,F_u(t)\delta A(t)G(t)X(t)\int_{-\infty}^{t} X^{-1}(\tau)\left[F_s(\tau)\underline{s}(\tau)\right. \\ &\quad \left. + F_n(\tau)\,\underline{n}(\tau)\right] d\tau\end{aligned} \tag{4-38}$$

Substitution of these relations into (4-34) yields

$$\delta\phi = \mathrm{tr} \int_0^T M_\phi^T(t)\,\delta A(t)\,dt \tag{4-39}$$

where

$$M_\phi^T(t) = 2G(t)M(t)WX(T)X^{-1}(t)F_u(t) \tag{4-40}$$

and

$$\begin{aligned}M(t) &= X(t)\int_{-\infty}^{t} X^{-1}(\tau)F(\tau)\int_{-\infty}^{T} R(\tau, v)F^T(v)X^{-T}(v)X^T(T)\,dv\,d\tau \\ &\quad -X(t)\int_{-\infty}^{t} X^{-1}(\tau)\left[F_s(\tau)R_{ss}(\tau, T+p) + F_n(\tau)R_{ns}(\tau, T+p)\right] d\tau\end{aligned} \tag{4-41}$$

where we have employed the definitions

$$R = \begin{bmatrix} R_{ss} & R_{sn} \\ R_{ns} & R_{nn} \end{bmatrix} \tag{4-42}$$

and

$$F = \begin{bmatrix} F_s & F_n \end{bmatrix} \tag{4-43}$$

Equation (4-41) can be expressed in terms of differential-equation solutions. If we let

$$N_1(T, \tau) = X(T) \int_{-\infty}^{T} X^{-1}(v)F(v)R(v, \tau)\, dv \tag{4-43.1}$$

it becomes apparent that $N_1(T, \tau)$ is the value at t = T of the solution $N_1(t, \tau)$ of the matrix differential equation

$$\frac{d}{dt}N_1(t, \tau) = \left[F_x(t) + F_u(t)A(t)G(t)\right]N_1(t, \tau) + F(t)R(t, \tau) \tag{4-43.2}$$

with the "initial" condition $N_1(-\infty, \tau) = 0$.

If we assume stability of the system for negative time, it is only necessary to impose the initial condition at some large negative value of t, rather than at $t = -\infty$. It is evident that, in general, it will be necessary to evaluate $N_1(T, \tau)$ by repeated integration of (4-43.2) for several selected values of τ.

With this definition of $N_1(T, \tau)$, equation (4-41) becomes

$$M(t) = X(t) \int_{-\infty}^{t} X^{-1}(\tau) \left[F(\tau)N_1^T(T, \tau) - F_s(\tau)R_{ss}(\tau, T+p) - F_n(\tau)R_{ns}(\tau, T+p)\right] d\tau \tag{4-43.3}$$

so we see that M(t) is the solution of the matrix differential equation

$$\dot{M}(t) = \left[F_x + F_u AG\right]M(t) + \left[F(t)N_1^T(T, t) - F_s(t)R_{ss}(t, T+p) - F_n(t)R_{ns}(t, T+p)\right] \tag{4-43.4}$$

with the "initial" condition $M(-\infty) = 0$, where the previous remarks concerning the initial conditions apply here as well.

Finally, we note that the quantity, $N_2(t) = X^{-T}(t)X^T(T)W$, whose transpose appears in (4-40), is the solution of the matrix adjoint differential

equation

$$\dot{N}_2(t) = -\left[F_x(t) + F_u(t)A(t)G(t)\right]^T N_2(t) \qquad (4\text{-}43.5)$$

with the boundary condition $N_2(T) = W$.

An expression for the performance criterion ϕ may be derived by substituting (4-35) into (4-30). If we then define $N_3(t)$ to be the solution of

$$\dot{N}_3(t) = \left[F_x + F_u AG\right]N_3(t) + \left[F_n(t)R_{ns}(t, T+p) + F_s(t)R_{ss}(t, T+p)\right] \qquad (4\text{-}43.6)$$

with the "initial" condition $N_3(-\infty) = 0$, it may be shown that ϕ is given by

$$\phi = \text{tr } W\left[R_{ss}(T+p, T+p) - N_3^T(T) + M(T)\right] \qquad (4\text{-}43.7)$$

For the case in which $\underline{x}$ is set to zero at $t = 0$ (i.e., the quiescent system is "switched in" to the signal and noise sources at $t = 0$), all lower limits of integration in (4-35), (4-38) and (4-41) are zero. In this case the initial conditions for (4-43.2), (4-43.4) and (4-43.6) are imposed at $t = 0$.

If no additional constraints are placed on the elements of A(t), the condition for stationarity of ϕ with respect to A(t) is the vanishing of $M_\phi(t)$ in the interval $0 \leq t \leq T$. It is evident from (4-40), then, that a sufficient condition for stationarity is the vanishing of M(t), which is assured if

$$F_s(t)R_{ss}(t, T+p) + F_n(t)R_{ns}(t, T+p) = F(t)\int_{-\infty}^{T} R(t, \tau)F^T(\tau)\Phi^T(T, \tau)\,d\tau \qquad (4\text{-}44)$$

If the signal and noise have equal dimensions and $F_s = F_n = I$, this becomes

$$R_{ss}(t, T+p) + R_{ns}(t, T+p) = \int_{-\infty}^{T} R_{ii}(t, \tau)\Phi^T(T, \tau)\,d\tau \qquad (4\text{-}44.1)$$

where we have defined the correlation matrix of the total input

$$R_{ii} = R_{ss} + R_{sn} + R_{ns} + R_{nn} \qquad (4\text{-}44.2)$$

Equation (4-44.1) is the non-stationary multi-dimensional version of the Wiener filter-predictor[57, 63] for infinite sampling time.* For the case $\underline{x}(0) = 0$ (finite sampling time), the lower integration limit is zero.

Depending on the form of G(t) and $F_u(t)$, the adjustment of A(t) may be inadequate to provide the transition matrix required by (4-44) or (4-44.1). In such a case $M_\phi(t)$ must vanish for some non-zero M(t) if there is to be a stationary solution. Since the Wiener filter is known to be the optimum linear filter, this constrained optimum must result in a larger mean-square error.

If G(t) and $F_u(t)$ are non-singular square matrices, manipulation of A(t) is capable of providing any desired system transition matrix $\Phi(T, t)$. In this case it is evident from (4-40) that the vanishing of M(t) is a necessary condition for optimality. This result holds regardless of the form of W, as long as W is positive definite.† If W is only positive semidefinite (i.e., if only some components of the signal are to be predicted), it is the matrix product M(t)W which must vanish.

4.5 Nominal-Plus-Feedback Optimization

We shall now consider a system which affords the possibility of two distinct types of optimizing adjustments.

In the system of Figure 4.4, the nominal control input $\underline{u}(t)$ and the initial conditions $\underline{x}_0$ determine how the system would behave in the absence of random disturbances. In addition, noisy measurements of the actual state variables are compared with the corresponding values predicted for the noise-free trajectory. The differences are fed back through a feedback gain matrix A(t) as a perturbation $\underline{\tilde{u}}$ to the control input $\underline{u}(t)$. The quantities capable of adjustment are $\underline{u}(t)$ and A(t) ($0 \leq t \leq T$), and the initial conditions $\underline{x}_0$. Once again we shall seek to minimize the performance criterion of equation (4-2):

* See J. S. Bendat, Principles and Applications of Random Noise Theory, Wiley, New York, 1958, p. 150.

† See J. E. Potter and R. G. Stern, "Statistical Filtering of Space Navigation Measurements," AIAA preprint 63-333, presented at AIAA Guidance and Control Conference, M.I.T., Cambridge, Mass., Aug. 12-14, 1963.

$$\phi = \overline{e^2} = \overline{\tilde{g}^2}(\underline{x}(T), T) \tag{4-45}$$

with the assumption that $\underline{n}_1$ depends on $\underline{x}$, $\underline{u}$, and t, and $\underline{n}_2$ only on $\underline{x}$ and t.

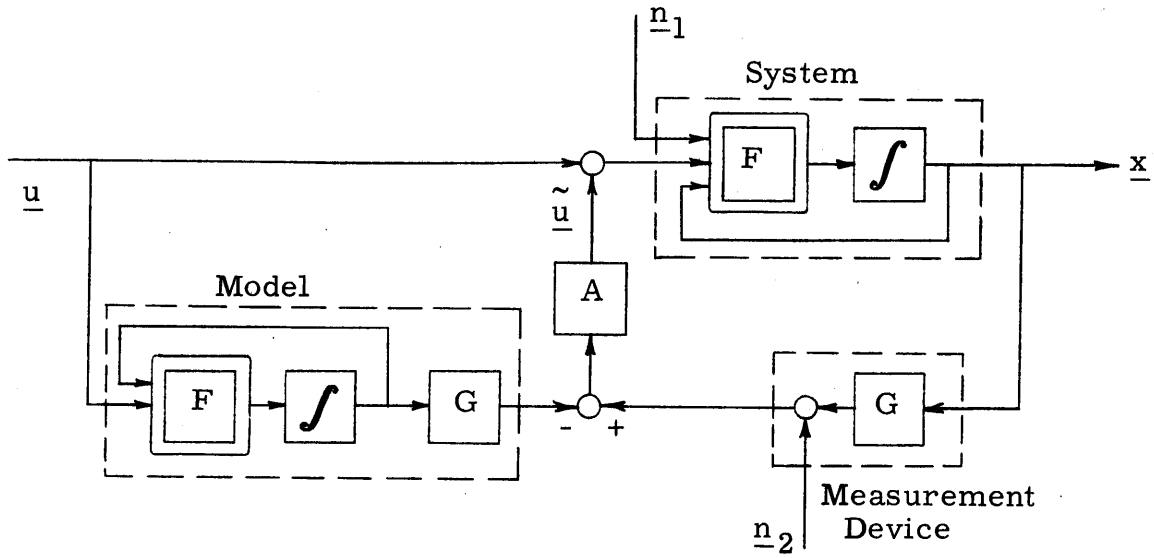

Figure 4.4 System with Nominal-Plus-Feedback Control

The perturbation equation (4-18) still describes the effects of small random disturbances:

$$\dot{\tilde{\underline{x}}}(t) = \left[F_x + F_u A G\right] \tilde{\underline{x}}(t) + F_n \underline{n}_1(t) + F_u A \underline{n}_2(t) \tag{4-46}$$

but perturbations in the nominal trajectory are now described by

$$\delta\dot{\underline{x}}(t) = F_x(t)\, \delta\underline{x}(t) + F_u(t)\, \delta\underline{u}(t) \tag{4-47}$$

It is now necessary to introduce two distinct fundamental solutions for these differential equations. That which corresponds to equation (4-46) will be given the symbol X(t) as before, and the fundamental solution of (4-47) will be designated $X_n(t)$. Just as in Sec. 4.3, we have

$$e(T) = \underline{\lambda}^T(0)\,\underline{\tilde{x}}_0 + \int_0^T \underline{\lambda}_{n1}^T\,\underline{n}_1\,dt + \int_0^T \underline{\lambda}_{n2}^T\,\underline{n}_2\,dt \tag{4-48}$$

where $\underline{\lambda}_{n1}(t)$ and $\underline{\lambda}_{n2}(t)$ are defined in (4-19) and (4-20). $\underline{\lambda}(t)$ is again the solution of (4-8) with the initial conditions (3-15). $\delta e(T)$ is given by (4-23), with

$$\delta\underline{\lambda}_{n1}(t) = H_{nu}(t)\,\delta\underline{u}(t) + H_{nx}(t)\,\delta\underline{x}(t) + F_n^T(t)\,\delta\underline{\lambda}(t) \tag{4-49}$$

$$\begin{aligned}\delta\underline{\lambda}_{n2}(t) = {} & \delta A^T(t)F_u^T(t)\,\underline{\lambda}(t) + A^T(t)H_{uu}(t)\,\delta\underline{u}(t) \\ & + A^T(t)H_{xu}^T(t)\,\delta\underline{x}(t) + A^T(t)F_u^T(t)\,\delta\underline{\lambda}(t)\end{aligned} \tag{4-50}$$

From (4-47), $\delta\underline{x}(t)$ can be written as

$$\delta\underline{x}(t) = X_n(t)X_n^{-1}(0)\,\delta\underline{x}_0 + X_n(t)\int_0^t X_n^{-1}(\tau)F_u(\tau)\,\delta\underline{u}(\tau)\,d\tau \tag{4-51}$$

and $\delta\underline{\lambda}(t)$ is again the solution (4-11) of equation (4-9), in which $\delta\underline{x}(t)$ is now given by (4-51) instead of (4-7).

The perturbation $\delta\overline{e^2}$ in the mean-square error is found by combining (4-48) with (4-23), giving the result (4-24). With the substitutions indicated by (4-11) and (4-49) to (4-51), followed by the appropriate changes of order of integration, we may derive the final result in the standard form of equation (2-65):

$$\begin{aligned}\delta\phi = \delta\,\overline{e^2} = {} & \int_0^T \underline{g}_\phi^T(t)\,\delta\underline{u}(t)\,dt \\ & + \mathrm{tr}\int_0^T M_\phi^T(t)\,\delta A(t)\,dt + \underline{b}_\phi^T\,\delta\underline{x}_0\end{aligned} \tag{4-52}$$

For brevity, we shall employ the definitions introduced at the end of Sec. 4.3. With these abbreviations we can derive the following expressions for the influence functions:

$$\frac{1}{2} g_{\phi}^{T}(t) = \underline{\lambda}_{nl}^{T}(t)R_{lu}(t) + \underline{v}_{1}^{T}(t)$$

$$+ \underline{\lambda}^{T}(0)CX^{-T}(0)X^{T}(T)G_{xx}X_{n}(T)X_{n}^{-1}(t)F_{u}(t)$$

$$+ \underline{\lambda}^{T}(0)CX^{-T}(0)X^{T}(t)\left[H_{xu}(t) + G^{T}(t)A^{T}(t)H_{uu}(t)\right]$$

$$+ \underline{\lambda}^{T}(0)CX^{-T}(0)\int_{t}^{T} X^{T}(\tau)\left[H_{xx}(\tau)\right.$$

$$\left. + G^{T}(\tau)A^{T}(\tau)H_{xu}^{T}(\tau)\right]X_{n}(\tau)\,d\tau\,X_{n}^{-1}(t)F_{u}(t)$$

$$+ \int_{t}^{T}\left[\underline{v}_{2}(\tau) + \underline{v}_{4}(\tau)\right]^{T} X_{n}(\tau)\,d\tau\,X_{n}^{-1}(t)\,F_{u}(t)$$

$$+ \int_{0}^{T} \underline{v}_{3}^{T}(\tau)X^{-T}(\tau)\,d\tau\,X^{T}(T)G_{xx}X_{n}(T)X_{n}^{-1}(t)\,F_{u}(t)$$

$$+ \int_{0}^{t} \underline{v}_{3}^{T}(\tau)X^{-T}(\tau)\,d\tau\,X^{T}(t)\left[H_{xu}(t) + G^{T}(t)A^{T}(t)H_{uu}(t)\right]$$

$$+ \int_{t}^{T}\left[\int_{0}^{\tau} \underline{v}_{3}^{T}(s)X^{-T}(s)\,ds\right]X^{T}(\tau)\left[H_{xx}(\tau)\right.$$

$$\left. + G^{T}(\tau)A^{T}(\tau)H_{xu}^{T}(\tau)\right]X_{n}(\tau)\,d\tau\,X_{n}^{-1}(t)F_{u}(t) \qquad (4\text{-}53)$$

$$M_{\phi}^{T}(t) = 2\underline{r}_{I2}(t)\,\underline{\lambda}^{T}(t)F_{u}(t)$$

$$+ 2G(t)X(t)X^{-1}(0)C\underline{\lambda}(0)\,\underline{\lambda}^{T}(t)\,F_{u}(t)$$

$$+ 2G(t)\,X(t)\int_{0}^{t} X^{-1}(\tau)\,\underline{v}_{3}(\tau)\,d\tau\,\underline{\lambda}^{T}(t)\,F_{u}(t) \qquad (4\text{-}54)$$

$$\frac{1}{2}\underline{b}_{\phi}^{T} = \frac{1}{2}\underline{c}_{x}^{T} + \underline{\lambda}^{T}(0)CX^{-T}(0)X^{T}(T)G_{xx}X_n(T)X_n^{-1}(0)$$

$$+ \underline{\lambda}^{T}(0)CX^{-T}(0)\int_0^T X^{T}(t)\left[H_{xx}(t) + G^{T}(t)A^{T}(t)H_{xu}^{T}(t)\right]X_n(t)\,dt\,X_n^{-1}(0)$$

$$+ \int_0^T \left[\underline{v}_2(t) + \underline{v}_4(t)\right]^{T}X_n(t)\,dt\,X_n^{-1}(0)$$

$$+ \int_0^T \underline{v}_3^{T}(t)X^{-T}(t)\,dt\,X^{T}(T)G_{xx}X_n(T)X_n^{-1}(0)$$

$$+ \int_0^T \underline{v}_3^{T}(t)X^{-T}(t)\int_t^T X^{T}(\tau)\left[H_{xx}(\tau)\right.$$

$$\left. + G^{T}(\tau)A^{T}(\tau)H_{xu}^{T}(\tau)\right]X_n(\tau)\,d\tau\,dt\,X_n^{-1}(0) \qquad (4\text{-}55)$$

As in the cases considered previously, the computation of the influence functions is facilitated by the introduction of the solutions of some particular vector differential equations. We redefine $\underline{w}_1(t)$ and $\underline{w}_2(t)$ by

$$\dot{\underline{w}}_1(t) = \left[F_x(t) + F_u(t)A(t)G(t)\right]\underline{w}_1(t) + \underline{v}_3(t) \qquad (4\text{-}56)$$

and

$$\dot{\underline{w}}_2(t) = -F_x^{T}(t)\underline{w}_2(t) - \left[\underline{v}_2(t) + \underline{v}_4(t) + (H_{xx} + H_{xu}AG)\underline{w}_1(t)\right] \qquad (4\text{-}57)$$

with

$$\underline{w}_1(0) = C\underline{\lambda}(0) \qquad (4\text{-}58)$$

and

$$\underline{w}_2(T) = G_{xx}\underline{w}_1(T) \qquad (4\text{-}59)$$

With these definitions, the influence functions may be written:

$$\frac{1}{2}\underline{g}_{\phi}(t) = \underline{v}_1(t) + R_{1u}^{T}(t)\underline{\lambda}_{nl}(t) + F_u^{T}(t)\underline{w}_2(t) + \left[H_{xu}^{T} + H_{uu}AG\right]\underline{w}_1(t) \qquad (4\text{-}60)$$

$$M_{\phi}(t) = 2F_u^T(t)\underline{\lambda}(t)\left[G(t)\,\underline{w}_1(t) + \underline{r}_{I2}(t)\right]^T \tag{4-61}$$

$$\underline{b}_{\phi} = \underline{c}_x + 2\,\underline{w}_2(0) \tag{4-62}$$

CHAPTER 5

SOME COMPUTATIONAL CONSIDERATIONS

5.1 Introduction

In previous chapters we have developed formulas for the influence functions corresponding to a certain type of performance criterion, and have specialized these formulas for some specific system configurations. We have also considered the alterations necessary for the handling of various other types of performance criteria and constraints. We shall now consider the computational steps required for the determination of the influence functions and the subsequent application of the steepest-descent procedure.

5.2 The Computational Procedure

The "basic problem" of Sec. 3.4, in which the initial conditions $\underline{x}_0$ are fixed, will serve to illustrate the computations involved. The computation is devoted principally to the determination of the influence function $\underline{g}_\phi(t)$ of equation (3-42.7).The matrix $G_\psi(t)$ of constraint influence functions (equation (2-2)) is determined in a simple manner from a set of r n-vector solutions of the adjoint equations, as in Appendix A and Ref. 15.

The additional steps required for the steepest-descent procedure are essentially the same as those described by other authors.[15, 22]

The nature and extent of the computations may be expected to vary from one problem to another, depending on the degree of linearity of the system, the possibility of analytic solution of the differential equations, the properties of the random disturbances, etc. The basic steps, however, will be substantially the same.

Assuming given initial conditions $\underline{x}_0$, and a "first guess" control program $\underline{u}(t)$, we proceed as follows:

(1) Integrate the equations of motion in the forward direction, and store the resulting state-variable time-histories (an n-dimensional vector function).

(2) Integrate the adjoint equation (3-12) backward from the boundary condition (3-15), with the coefficients evaluated along the nominal trajectory stored in (1). Store the n-vector solution $\underline{\lambda}(t)$.

An analogous adjoint vector is required for each element of the constraint vector $\underline{\psi}$. Hence at this point we also perform a backward integration of the n × r matrix adjoint equation

$$\dot{\Lambda}_{\psi}(t) = -F_x^T(t)\Lambda_{\psi}(t) \tag{5-1}$$

from the boundary condition $\Lambda_{\psi}(T) = \Psi_x^T$ (see equation (2-93)).

Each column of the m × r matrix $G_{\psi}(t)$ is the m-vector influence function for one of the r components of $\underline{\psi}$. It is computed by post-multiplying the matrix $F_u^T(t)$ by the appropriate column of $\Lambda_{\psi}(t)$, as indicated by equation (A-16). Hence as we perform the integration of (5-1) we compute and store the m × r matrix

$$G_{\psi}(t) = F_u^T(t)\Lambda_{\psi}(t) \tag{5-2}$$

(3) Compute the p-vector $\underline{r}_I(t)$, using (3-24), and store it. Since this involves the integration of the vector $\underline{r}(\tau, t)$ with respect to τ, in most cases it will be necessary to perform the integration several (say q) times, for several selected values of t. However, the number of values to be stored will not be large. The number, q, of t-values required will depend upon the smoothness of $\underline{r}_I(t)$, and can only be determined, in general, after examination of some of the computed results.

The integration is between fixed limits, hence reasonable accuracy may be attained by the use of a simple integration rule, such as Simpson's.

(4) Compute the p × n matrix $R_x(t)$ and the p × m matrix $R_u(t)$ from (3-25) and (3-26), in a manner similar to that used for $\underline{r}_I(t)$, and store them. These tables may be interpolated as needed for later steps in the computation. It is reasonable to expect that in many cases some of the terms involved here will vanish, so that we seldom have to integrate the full p(n+m) equations.

(5) Integrating equation (3-42. 3) in the forward direction from the initial condition (3-42. 5), compute and store the n-vector function $\underline{w}_1(t)$. Since this requires interpolation of the values of $\underline{r}_I(t)$, it is probably advisable to use a fairly coarse integration step, followed by interpolation of $\underline{w}_1(t)$.

(6) Integrating equation (3-42. 4) in the backward direction from the boundary condition (3-42. 6), compute the n-vector function $\underline{w}_2(t)$. These values need not be stored. Instead we simultaneously perform the additional computations prescribed by equation (3-42. 7) and store the resulting m-vector function $\underline{g}_\phi(t)$.

(7) By means of a simple integration rule, evaluate the scalar $k_{\phi\phi}$, the r-vector $\underline{v}_{\psi\phi}$, and the $r \times r$ matrix $I_{\psi\psi}$. For the basic problem being considered here, these are given by equation (2-31), (2-31. 1) and (2-31. 2). For the most general case see equations (2-68), (2-69) and (2-70).

(8) Select a value for the step-size parameter d^2. A reasonable procedure is to use a constant step-size until no further improvement is being obtained, then reduce the step-size if a closer approach to the optimum is desired. The original value is based on an estimate of the amount of improvement which can reasonably be expected.

If $d^2 - \underline{\psi}^T I_{\psi\psi}^{-1} \underline{\psi}$ is negative, we must increase d^2 or relax the requirement that $\delta\underline{\psi} = -\underline{\psi}$. This is especially likely on the first few iterations, if the "first guess" trajectory exhibits large constraint violations.

(9) Determine the control-variable perturbations $\delta\underline{u}(t)$ from equation (2-32).

In problems with variable terminal time, it may be advisable to compute the predicted change δT from (3-65. 1). If δT appears large enough to seriously violate the linearity assumptions, $\delta\underline{u}(t)$ may be scaled down accordingly. Alternatively, we may treat T as an adjustable parameter, as discussed in Sec. 3. 6, and control the magnitude of its change by adjusting the weighting coefficient with which δT enters the definition (2-67) of the control-perturbation norm.

(10) Add the perturbation $\delta\underline{u}(t)$ to the nominal control program $\underline{u}(t)$ to determine a new set of controls, and return to (1).

The procedure is repeated until a trajectory is found for which $\underline{\psi} = 0$ and the squared norm of the gradient projection, $k_{\phi\phi} - \underline{v}_{\psi\phi}^{T} I_{\psi\psi}^{-1} \underline{v}_{\psi\phi}$, is sufficiently small.

An examination of the computational steps reveals that the method may be expected to require a considerable amount of computer time and storage capacity, probably involving tape operations in most cases. The most time-consuming operations are the repeated integrations of steps (3) and (4). These require the integration (perhaps by a simplified process such as Simpson's rule) of at most $p(m + n + 1)$ functions for each of the q chosen values of the second independent variable.

It should be noted that in white-noise problems (such as that of Sec. 6.2), the integrations of steps (3) and (4) are trivial, and require no computer time at all. If the noise is a function of t only, it is sometimes possible for the integrations to be carried out analytically, without resorting to the repetitive integration described in steps (3) and (4).

In the simple problem of Sec. 6.3, $n = 2$ and $m = p = r = 1$. Step (2) is eliminated completely, $R_u(t) = 0$ and $R_x(t)$ has only one non-zero component. The parameter q was effectively set equal to 26 (including some extra values inserted in order to examine more closely the behaviour of the functions). The 2q integrations of steps (3) and (4) were performed by a Runge-Kutta procedure with the interval divided into 100 increments. The computations required approximately 80 seconds per iteration on the Minneapolis-Honeywell H1800, a machine roughly comparable in speed to the IBM 7090.

5.3 Weighting Functions and Coordinate Transformations

The weighting functions $W(t)$, $W_i(t)$ and W_p of equation (2-67), although they do not affect the trajectory to which the method converges, can have a powerful effect on the rate of convergence to that trajectory. Methods for determining the weighting functions are, at present, so rudimentary that the entire steepest-descent procedure remains somewhat of an art. Denham[22] states that, in general, the weighting function should be made small, to encourage reasonable control perturbations, in regions

where the influence functions become extremely small. More general rules would undoubtedly have to be based on second-variation considerations.

The introduction of a weighting function is equivalent to a linear transformation of coordinates in the linearized system. Other types of transformations may also be expected to have profound effects on the rate of convergence. Some of these approaches have already been mentioned in Sec. 2.5. Another possible transformation is a change of independent variable, such as was performed in the problem of Chapter 6 in order to avoid the complications of a variable terminal time.

A transformation of the control variable may be introduced in order to prevent violation of an inequality constraint. For example, if the constraint is $a \geq u(t) \geq b$, we may introduce the transformation

$$u(t) = \frac{a+b}{2} + \frac{a-b}{2} \sin \theta(t) \tag{5-3}$$

and employ $\theta(t)$ as the control variable. In the problem of Chapter 6, the use of $u(x) = \tan \theta(x)$ as the control variable implies the constraint $90^{\circ} \geq \theta \geq -90^{\circ}$. This in turn makes y a single-valued function of x, and ensures that x may be used as the independent variable. The constraint causes no loss of generality, since we are reasonably sure that it will be obeyed by the true optimum trajectory. Another transformation whose effect is comparable to that of (5-3) is

$$u(t) = b + (a - b)\, e^{-\theta^2(t)} \tag{5-4}$$

which again maps the infinite line $\infty \geq \theta(t) \geq -\infty$ into the finite line $a \geq u(t) \geq b$.

CHAPTER 6

NUMERICAL EXAMPLES

6.1 Introduction

In this chapter we shall demonstrate the application of the techniques of the previous chapters to several related versions of a particularly simple and instructive problem:

A vehicle is required to travel from point A to point B (Figure 6.1), a distance L = 1000 miles, at a constant velocity V of 200 miles per hour. It is constrained to remain in the xy plane, the axes being chosen so that the coordinates of A are (0, 0) and those of B are x = L, y = 0. During the journey the vehicle is subjected to a random disturbance n(x, y, t) which is, in general, a function of x, y, and the time t, and whose correlation function $r(x_1, y_1, t_1, x_2, y_2, t_2)$ is assumed known.

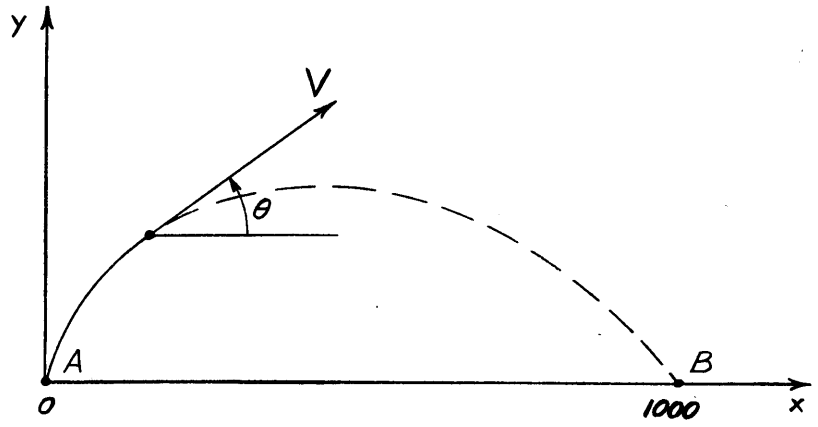

Figure 6.1 Vehicle in a Disturbing Medium.

The error associated with a given trajectory is defined as the time integral of n(x, y, t) over the trajectory, i.e.,

$$e = \int_0^T n(x(t),\ y(t),\ t)\, dt \tag{6-1}$$

A path is to be chosen from A to B in such a way as to minimize the mean square value $\overline{e^2}$ of the error e, where the expectation is taken over the ensemble of possible noise functions n(x, y, t).

The problem formulation is capable of many interpretations. If the vehicle is an airplane flying in a horizontal plane, the noise may represent some measure of air turbulence and the error e a measure of passenger discomfort. Or the noise may represent variations in cloud cover and the error may be considered as indicative of the probability of visual detection. We shall assume that the noise represents a small additive velocity in the y direction due to winds, so that the error to be minimized is the lateral displacement of the vehicle, in miles, when the 1000-mile distance has been covered in the x direction. The pilot is to control the vehicle by means of the compass heading θ. We shall consider the control variable, for the present, to be $\theta(t)$.

The differential equations of the system are

$$\dot{x} = V \cos \theta \tag{6-2}$$

$$\dot{y} = V \sin \theta + n \tag{6-3}$$

Because the final value of x is fixed, we may avoid the complications induced by a variable terminal time, by using x as the independent variable, since we are reasonably certain that x will behave in a monotonic fashion. The equations then become

$$\frac{dt}{dx} = \frac{1}{V \cos \theta} \tag{6-4}$$

$$\frac{dy}{dx} = \tan \theta + \frac{n}{V \cos \theta} \tag{6-5}$$

with initial conditions t(0) = y(0) = 0, and the final condition y(L) = 0.

The equations (3-9) for perturbation of the nominal become

$$\frac{d}{dx}(\delta t) = \frac{\sin\theta}{V\cos^2\theta}\,\delta\theta(x) \tag{6-6}$$

$$\frac{d}{dx}(\delta y) = \frac{1}{\cos^2\theta}\,\delta\theta(x) \tag{6-7}$$

and the equations (3-10) for random disturbances are

$$\frac{d\tilde{t}}{dx} = 0 \tag{6-8}$$

$$\frac{d\tilde{y}}{dx} = \frac{1}{V\cos\theta}\,n(x) \tag{6-9}$$

It is seen that, due to the simple form of the original differential equations, all transition matrices are the identity matrix and the adjoint solutions are constant:

$$\underline{\lambda}(x) = \underline{\lambda}(L) = \begin{bmatrix} \partial e/\partial t_f \\ \partial e/\partial y_f \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6-10}$$

Furthermore,

$$H = \underline{\lambda}^T \underline{f} = \tan\theta + \frac{n}{V\cos\theta} \tag{6-11}$$

hence $H_{xu} = H_{xx} = H_{nx} = 0$ and the perturbed adjoint equation (3-29) becomes

$$\frac{d}{dx}(\delta\underline{\lambda}) = 0 \tag{6-12}$$

and since $G_{xx} = 0$, we have

$$\delta\underline{\lambda}(x) = 0 \tag{6-13}$$

Finally, since the noise does not depend on $\theta(x)$, it follows that $R_u = 0$ and equation (3-42) reduces to

$$\underline{g}_\phi(x) = 2F_u^T(x) \int_x^L R_x^T(z)\, \underline{\lambda}_n(z)\, dz + 2H_{nu}^T(x)\, \underline{r}_I(x) \tag{6-14}$$

Now we have, from (3-17) and (6-11),

$$\underline{\lambda}_n(x) = \frac{1}{V \cos \theta(x)} \tag{6-15}$$

so that, from (3-23) and (3-24)

$$\underline{r}_I(x) = \int_0^L \frac{r(z, x)}{V \cos \theta(z)}\, dz \tag{6-16}$$

From (3-36) and (6-11)

$$H_{nu}(x) = \frac{\sin \theta(x)}{V \cos^2 \theta(x)} \tag{6-17}$$

and from the perturbation equations (6-6) and (6-7):

$$F_u^T(x) = \left[\frac{\sin \theta(x)}{V \cos^2 \theta(x)} \quad \frac{1}{\cos^2 \theta(x)} \right] \tag{6-18}$$

Using the symbol $r(x_1, x_2)$ to represent $r(x_1, y(x_1), t(x_1), x_2, y(x_2), t(x_2))$, equation (6-14) becomes finally

$$\begin{aligned} g_\phi(x) = {} & \frac{2 \sin \theta(x)}{V \cos^2 \theta(x)} \int_x^L \int_0^L \frac{\partial r(x_1, x_2)/\partial t(x_1)}{V \cos \theta(x_2)}\, dx_2\, \frac{dx_1}{V \cos \theta(x_1)} \\ & + \frac{2}{\cos^2 \theta(x)} \int_x^L \int_0^L \frac{\partial r(x_1, x_2)/\partial y(x_1)}{V \cos \theta(x_2)}\, dx_2\, \frac{dx_1}{V \cos \theta(x_1)} \\ & + \frac{2 \sin \theta(x)}{V \cos^2 \theta(x)} \left[\int_0^L \frac{r(x_1, x_2)}{V \cos \theta(x_2)}\, dx_2 \right]_{x_1 = x} \end{aligned} \tag{6-19}$$

Furthermore, from (3-18) we can see that

$$\overline{e^2} = \int_0^L \int_0^L \frac{r(x_1, x_2)}{V \cos \theta(x_2)} dx_2 \frac{dx_1}{V \cos \theta(x_1)} \tag{6-20}$$

The problem has a single end-point constraint, $y(L) = 0$. Since

$$y(L) = \int_0^L \frac{dy}{dx} dx = \int_0^L \tan \theta(x) dx \tag{6-21}$$

it follows that

$$\delta y(L) = \int_0^L \frac{\delta\theta(x)}{\cos^2 \theta(x)} dx \tag{6-22}$$

so that the influence function for $\psi(L) = y(L)$ is

$$g_\psi(x) = \frac{1}{\cos^2 \theta(x)} \tag{6-23}$$

In the actual formulation of the problem for the computations to be described here, it was found convenient to consider $\tan \theta(x)$ as the control variable, rather than $\theta(x)$, i. e.,

$$u(x) = \tan \theta(x) = \frac{dy}{dx}(x) \tag{6-24}$$

Now since $\delta\theta(x) = \cos^2 \theta(x)\, \delta u(x)$, the influence functions of (6-19) and (6-23) must be multiplied by $\cos^2 \theta(x)$, giving the new influence functions

$$g_e(x) = \cos^2 \theta(x)\, g_\phi(x) \tag{6-25}$$

and

$$g_y(x) = 1 \tag{6-26}$$

Because of the constancy of $g_y(x)$, the procedure of Sec. 2. 2, for orthogonalizing $g_e(x)$ with respect to $g_y(x)$, is very simple. The gradient projection

p(x) is formed from $g_e(x)$ by simply subtracting its average value:

$$p(x) = g_e(x) - g_{e(av)} \tag{6-27}$$

6.2 A White-Noise Problem

We consider first the problem for which the noise n(x, y, t) has a correlation function of the form

$$r = a\delta(t_1 - t_2)\left[b + (1 - b)\left(\frac{y_1 + y_2}{2W} - 1\right)^2\right] \exp\left[-c\,|x_1 - x_2| - d\,|y_1 - y_2|\right] \tag{6-28}$$

Such a noise is white in time and exponentially correlated in x and y. But since along our trajectory $x_1 = x_2$ and $y_1 = y_2$ whenever $t_1 = t_2$, the values of c and d have no effect and we may assume c = d = 0. We shall also assume $0 \leq b \leq 1$.

The noise is stationary in x and t, but non-stationary in y. At a fixed point $y_1 = y_2 = y$, $x_1 = x_2$, the random time function has the correlation function

$$r' = a\delta(t_1 - t_2)\left[b + (1 - b)\left(\frac{y}{W} - 1\right)^2\right] \tag{6-29}$$

which represents a white noise with power spectral density

$$S(\omega, y) = \frac{a}{\pi}\left[b + (1 - b)\left(\frac{y}{W} - 1\right)^2\right] \tag{6-30}$$

which, considered as a function of x and y, forms a parabolic "valley" parallel to the x-axis, with minimum along the line y = W.

Because of the delta function in (6-28), the double integrals of (6-19) and (6-20) become, in effect, single integrals, since the first integration is trivial. This is a considerable advantage in the machine computation of solutions. With c = d = 0 in (6-28) we find

$$\int_0^L \frac{r(x_1, x_2)}{V \cos \theta(x_2)}\, dx_2 = \int_0^{t(L)} r(t_1, t_2)\, dt_2 = a\left[b + (1 - b)\left(\frac{y_1}{W} - 1\right)^2\right] \tag{6-31}$$

and

$$\int_0^L \frac{\partial r(x_1, x_2)/\partial y(x_1)}{V \cos \theta(x_2)}\, dx_2 = \frac{a(1 - b)}{W}\left(\frac{y_1}{W} - 1\right) \tag{6-32}$$

Using integration by parts we find

$$\int_0^L \frac{\partial r(x_1, x_2)/\partial t(x_1)}{V \cos \theta(x_2)}\, dx_2 = a\int_0^{t(L)} \frac{\partial\left[\delta(t_1 - t_2)\right]}{\partial t_1}\left[b + (1 - b)\left(\frac{y_1 + y_2}{2W} - 1\right)^2\right] dt_2$$

$$= -a\int_0^{t(L)} \frac{\partial\left[\delta(t_1 - t_2)\right]}{\partial t_2}\left[b + (1 - b)\left(\frac{y_1 + y_2}{2W} - 1\right)^2\right] dt_2$$

$$= -a\delta(t_1 - t_2)\left[b + (1 - b)\left(\frac{y_1 + y_2}{2W} - 1\right)^2\right]\Bigg|_{t_2=0}^{t_2=t(L)}$$

$$+ a\int_0^{t(L)} \delta(t_1 - t_2)\frac{\partial}{\partial t_2}\left[b + (1 - b)\left(\frac{y_1 + y_2}{2W} - 1\right)^2\right] dt_2$$

$$= -a\delta(t_1 - t_2)\left[b + (1 - b)\left(\frac{y_1 + y_2}{2W} - 1\right)^2\right]\Bigg|_{t_2=0}^{t_2=t(L)}$$

$$+ a\int_0^{t(L)} \delta(t_1 - t_2)\left[\frac{2(1 - b)}{2W}\left(\frac{y_1 + y_2}{2W} - 1\right)\frac{dy_2}{dt_2}\right] dt_2$$

$$= -a\delta(t_1 - t(L))\left[b + (1 - b)\left(\frac{y_1 + y(L)}{2W} - 1\right)^2\right]$$

$$+ a\delta(t_1)\left[b + (1 - b)\left(\frac{y_1 + y(0)}{2W} - 1\right)^2\right] + \frac{a(1 - b)}{W}\left(\frac{y_1}{W} - 1\right)\frac{dy_1}{dt_1} \tag{6-33}$$

where it may be seen that the last term is just

$$\frac{a(1 - b)}{2}\ \frac{d}{dt_1}\left[\left(\frac{y_1}{W} - 1\right)^2\right]$$

The first double integral of (6-19) is the integral of (6-33) with respect to t_1, between the limits t(x) and t(L), i. e.,

$$-\frac{a}{2}\left[b + (1-b)\left(\frac{y(L)}{W} - 1\right)^2\right] + 0$$

$$+\frac{a(1-b)}{2}\left[\left(\frac{y(L)}{W} - 1\right)^2 - \left(\frac{y(x)}{W} - 1\right)^2\right] = -\frac{a}{2}\left[b + (1-b)\left(\frac{y(x)}{W} - 1\right)^2\right] \quad (6\text{-}34)$$

Thus we find

$$g_e(x) = \frac{a}{V}\sin\theta(x)\left[b + (1-b)\left(\frac{y(x)}{W} - 1\right)^2\right]$$

$$+ 2\frac{a}{W}(1-b)\int_x^L \left(\frac{y(x_1)}{W} - 1\right)\frac{dx_1}{V\cos\theta(x_1)} \quad (6\text{-}35)$$

and the mean-square error, from (6-20) and (6-31), is

$$\overline{e^2} = \int_0^L a\left[b + (1-b)\left(\frac{y(x)}{W} - 1\right)^2\right]\frac{dx}{V\cos\theta(x)}$$

$$= \int_0^L a\left[b + (1-b)\left(\frac{y(x)}{W} - 1\right)^2\right]\sqrt{1 + y'^2(x)}\,dx \quad (6\text{-}36)$$

6.2.1 The Euler Equation

The simple form (6-36) for the cost function in the white-noise case, facilitates application of the standard variational techniques to derive the corresponding Euler equation. With x as the independent variable we obtain the single Euler equation:

$$\frac{1}{\sqrt{1 + y'^2}}\left\{\frac{2(1-b)}{W}\left(\frac{y}{W} - 1\right) - \frac{y''}{(1 + y'^2)}\left[b + (1-b)\left(\frac{y}{W} - 1\right)^2\right]\right\} = 0 \quad (6\text{-}37)$$

Because of the simple relation (6-27) for the gradient projection, it can be seen that the condition for a stationary solution is that the influence function $g_e(x)$ be constant. By differentiation of (6-35) we can see that this is indeed true when (6-37) is satisfied.

The Weierstrass-Erdmann corner condition for (6-36) dictates that the quantity

$$F_{y'} = \left[b + (1 - b) \left(\frac{y}{W} - 1 \right)^2 \right] \frac{y'}{\sqrt{1 + y'^2}} \qquad (6\text{-}38)$$

be continuous at any discontinuity in y'. This is impossible unless the bracketed quantity vanishes, which it never does unless $b = 0$ and $y = W$. We conclude that corners are impossible for $b \neq 0$. Furthermore, it is obvious that the optimum trajectory will not exceed $y = W$. But if $y = W$ is the maximum y, there will be a point $y = W$, $y' = 0$. If $b \neq 0$, (6-37) then becomes simply $y'' = 0$, so that y is constant and the boundary conditions cannot be met. Hence for $b \neq 0$ we must always have $y < W$.

For the case $b = 0$ (so that the noise actually vanishes along $y = W$), these conclusions do not hold. In fact, it may be shown that in this case the rectangular trajectory bounded by the lines $x = 0$, $y = W$, $x = L$, always represents at least a local minimum for $\overline{e^2}$ (see Appendix C). It is apparent (especially for $W >> L$) that this solution will not always represent the global minimum, so that we can expect multiple extrema in some cases.

This has been verified by computation for the case $b = 0$, $W = 1500$ miles, $L = 1000$ miles, $a = 20$ mph^2. Besides the rectangular trajectory with cost 100.00 mi.^{2}, two other Euler-equation solutions were found which satisfied the boundary conditions. One solution, with a maximum excursion of 980 miles in the y direction, corresponded to a cost of 103.26 mi.^{2}, and represented a local maximum. The true minimum cost of 91.04 mi.^{2} was attained by a solution with $y_{max} = 210$ miles.

6.2.2 Numerical Results

Using the influence function as given by (6-35), optimum trajectories were sought, by the method of steepest descent, for two values of the parameter b. W was fixed at 500 miles and a at 20 mph.^{2} The results are shown in Figure 6.2.

For the case $b = 0.5$, the approximate optimum trajectory reaches a maximum y of 248.76 miles and attains a mean-square error of 85.4084 mi.^{2}

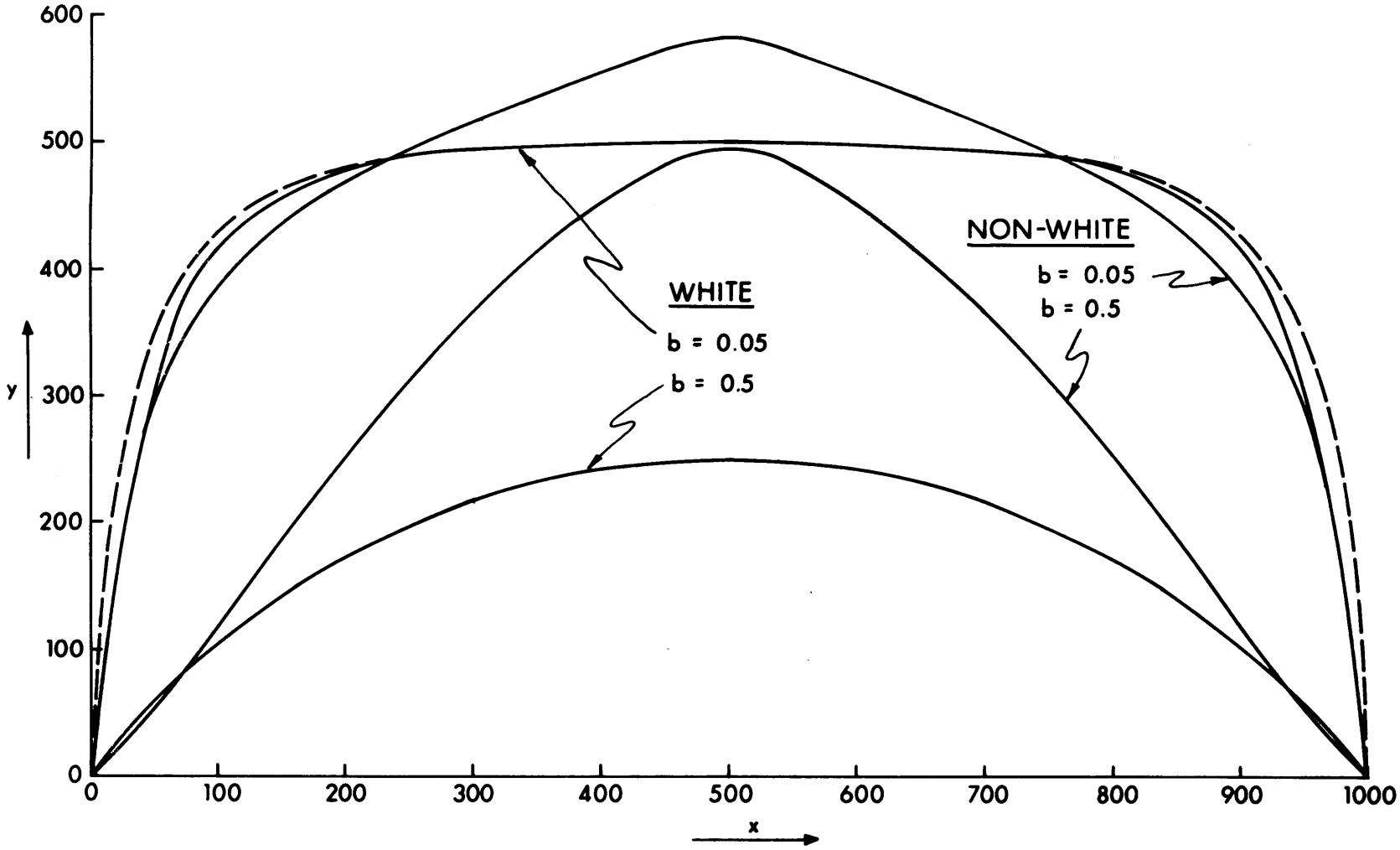

Figure 6.2 Extremal Trajectories for White and Non-White Noise Problems

For purposes of comparison, the optimum was also found from the Euler equation (6-37), by varying the initial slope $y'(0)$ until the boundary condition $y(L) = 0$ was satisfied. The result was indistinguishable from the steepest-descent solution. The y values agreed at all points to within one part in 10^5, and the mean-square error was 85.4075 $mi.^2$

It is interesting to examine the behaviour of the influence function $g_e(x)$ as the optimization proceeds. For the straight-line trajectory $y = 0$ (which was the initial guess), $g_e(x)$ is a linear function of x, varying from -0.2 at $x = 0$ to zero at $x = 1000$. For the approximate optimum it has become very nearly constant, as predicted, with maximum and minimum values of -0.0779622 and -0.0779638.

The second case considered was that of $b = 0.05$. It was expected, and verified by the computations, that the deeper "valley" would cause the trajectory to approach more nearly the rectangular trajectory which provides a minimum for $b = 0$.

A certain amount of computational difficulty was encountered in this case due to the "ravine" problem discussed in Sec. 2.5. This was probably due in large measure to the rapid variations of y with x, near the ends of the x scale (a relatively coarse integration step of 5 miles was used in the computations). The computations were halted while improvement was still being obtained (see Figure 2.2). Nevertheless, although the trajectory agrees only approximately with the Euler-equation solution (broken line in Figure 6.2), the mean-square error of 40.761 is within 0.07% of the true minimum value of 40.733 $mi.^2$

The Euler-equation solution yields a value of $y = 499.02$ miles at $x = 500$. Figure 2.2, which shows the results of the ravine correction procedure for this problem, indicates that the steepest-descent procedure will converge to a value very close to this.

For $b = 0.05$, the influence function for the trajectory $y = 0$ varies linearly from $g_e(0) = -0.38$ to $g_e(1000) = 0$. Once again it becomes almost constant as the optimum is approached. The influence function associated with the corrected iteration 14 of Figure 2.2, varies from -0.1009 to -0.0994, in a fashion similar to that of the median curve of Figure 2.3.

For purposes of comparison we note that in both cases the path $y=0$ has a cost of 100 $mi.^2$, while the rectangular trajectory yields 116.67 $mi.^2$ for $b = 0.5$, and 41.67 $mi.^2$ for $b = 0.05$.

6.3 A Non-White Noise Problem

As an example of a random disturbance which does not exhibit the "whiteness" of the noise of Sec. 6.2, we shall consider a noise function n(y) whose correlation function has the form

$$r(y_1, y_2) = a\left[b + (1 - b)\left(\frac{y_1 + y_2}{2W} - 1\right)^2\right]\exp\left[-c\,|y_1 - y_2|\right] \qquad (6\text{-}39)$$

The noise is thus a non-white, non-stationary random function of y, and is constant with respect to t and x. If plotted as a surface above the xy plane, the function would be represented by a series of ridges parallel to the x-axis, and invariant with time. This could be visualized, for example, as an idealized representation of the surface contours of the earth, in a region which exhibits very high correlation in the x direction.

The invariance of n(y) with respect to x may represent a rather unrealistic situation, but serves to accentuate the effect of the y-dependence on the numerical results, by separating it from other effects. The inclusion of x- and t-dependence of the noise would also complicate the computations somewhat, though not to a great extent.

The nonstationary mean-square value of n(y) is of the same form as the parabolic "valley" represented by equation (6-30), which serves as a basis for comparison of the results:

$$\overline{n^2}(y) = a\left[b + (1 - b)\left(\frac{y}{W} - 1\right)^2\right] \qquad (6\text{-}40)$$

In the evaluation of $g_e(x)$ the first term of equation (6-19) now vanishes, but the second term, since $r(y_1, y_2)$ contains no delta function, must now be evaluated by actually integrating twice numerically. The function

$$f_1(x_1) = \int_0^L \frac{\partial r(y(x_1), y(x_2))/\partial y(x_1)}{V \cos \theta(x_2)}\, dx_2 \qquad (6\text{-}41)$$

was evaluated for specified values of x_1, spaced at 50-mile intervals from zero to 1000 miles, followed by an interpolation procedure for the integration with respect to x_1. The derivative to be integrated is

$$\frac{\partial r}{\partial y_1} = \exp\left[-c\,|y_1 - y_2|\right]\left\{\frac{a(1-b)}{W}\left(\frac{y_1 + y_2}{2W} - 1\right)\right.$$

$$\left. - ac\left[b + (1-b)\left(\frac{y_1 + y_2}{2W} - 1\right)^2\right]\operatorname{sgn}(y_1 - y_2)\right\} \qquad (6\text{-}42)$$

evaluated at $y_1 = y(x_1)$ and $y_2 = y(x_2)$. The function

$$f_2(x_1) = \int_0^L \frac{r(y(x_1),\, y(x_2))}{V \cos\theta(x_2)}\, dx_2 \qquad (6\text{-}43)$$

was evaluated in a similar manner.

6.3.1 A Simple Analog

The behaviour of the optimum trajectory, when the system is subjected to the noise n(y) just described, is not immediately obvious. In particular, when b = 1 (so that the "valley" is flat), will the optimum trajectory be the minimum-time path y = 0, or will the vehicle seek a path along which y varies, in order that successive values of n(y) may partially cancel each other? A simple analogous system will demonstrate that there is, indeed, a tendency to avoid regions of small slope $y'(x)$, in which y remains nearly constant over some finite time interval.

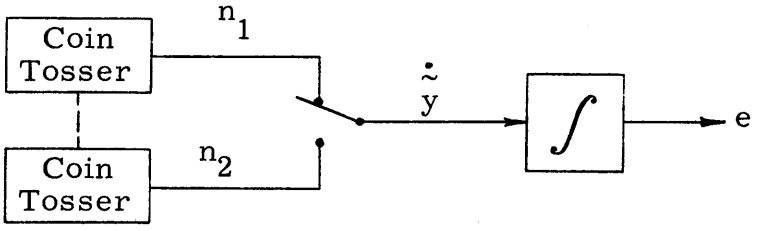

Figure 6.3 A Two-State System

The system illustrated in Figure 6.3 has two states, represented by the switch positions 1 and 2. The random inputs n_1 and n_2 have values ± 1 with equal probability, determined by the outputs of two coin-tossers. Each input remains constant during the experiment, but varies in a random

fashion from one experiment to another. The error e is the integral of the switch output $\dot{\tilde{y}}$ over a one-second interval. We wish to minimize the mean-square value $\overline{e^2}$ by choice of the proper switching strategy.

For generality, we shall consider the coin-tossers to be correlated. The correlation may be expressed by the probability P of the event $n_1 = n_2$. This is equivalent to a correlation coefficient $\rho = \overline{n_1 n_2}$ = 2P-1, as will be seen presently.

We shall consider two switching strategies: (A), the switch remains on position 1; and (B), the switch spends 1/2 sec. on each of positions 1 and 2.

The results are shown in Table 6.1:

Input State Number	Inputs			Probability	Strategy A		Strategy B	
	n_1	n_2	$n_1 n_2$		e	e^2	e	e^2
1	1	1	1	P/2	1	1	1	1
2	1	-1	-1	(1-P)/2	1	1	0	0
3	-1	1	-1	(1-P)/2	-1	1	0	0
4	-1	-1	1	P/2	-1	1	-1	1
Mean	0	0	2P-1		0	1	0	P

Table 6.1 Results of the Two-State Experiment

Since $P \leq 1$, it is seen that strategy B results in a smaller mean-square error. It can, in fact, be shown that this is the best of all strategies, for any strategy which spends s and 1-s seconds on positions 1 and 2 respectively yields

$$\overline{e^2} = P + (1 - P)(2s - 1)^2 \tag{6-44}$$

which exhibits a minimum at $s = 1/2$ for all $P < 1$.

For our trajectory problem, we may conclude by analogy that the vehicle will tend to avoid trajectories on which considerable time is spent at almost-equal values of y (i.e., trajectories with extended regions of small slope).

6.3.2 Numerical Results

With a = 20, c = 0.01, and W = 500, approximate optimum trajectories were found for three different values of b.

The results for b = 0.5 and b = 0.05 are plotted in Figure 6.2. It is seen that the effect predicted in Sec. 6.3.1 is indeed present. In comparison with the corresponding paths for the white-noise case, the trajectories show a definite tendency to maintain large slopes as much as possible, reversing slope rapidly so that regions of small slope are passed through quickly. In the case b = 0.05 this tendency is strong enough to cause the trajectory to exceed the value y = 500 miles and deliberately enter regions of greater noise. The associated cost values are 202.23 when b = 0.5 and 62.22 when b = 0.05.

Some interesting results were obtained for the case b = 1 (stationary n(y)). It can be seen that, if the path deviates from the line y = 0, a deviation in the direction of negative y is as good as one in the positive direction, so that there is some question as to the form of the influence function for this path. It is found that $g_e(x)$ vanishes (if we "round off" the "peak" of the exponential term in (6-39)), and it becomes evident that this path represents a maximum.

It is found that the method converges, in this case, to different paths, depending on the path chosen as the initial guess. Figure 6.4 shows two such paths, resulting from initial guesses having the general shapes, respectively, of a half-cycle and a full cycle of a sine wave. Although no effort was made to refine the optimization to any great extent, the associated cost values, 296.74 and 296.66 respectively, are seen to be essentially equal.

This result can be explained as follows. Because the noise depends on y alone, and in a stationary manner, the total cost of any path is unchanged if any part of the curve is translated in the x direction, or if the entire curve is translated in the y direction. It follows that, given any

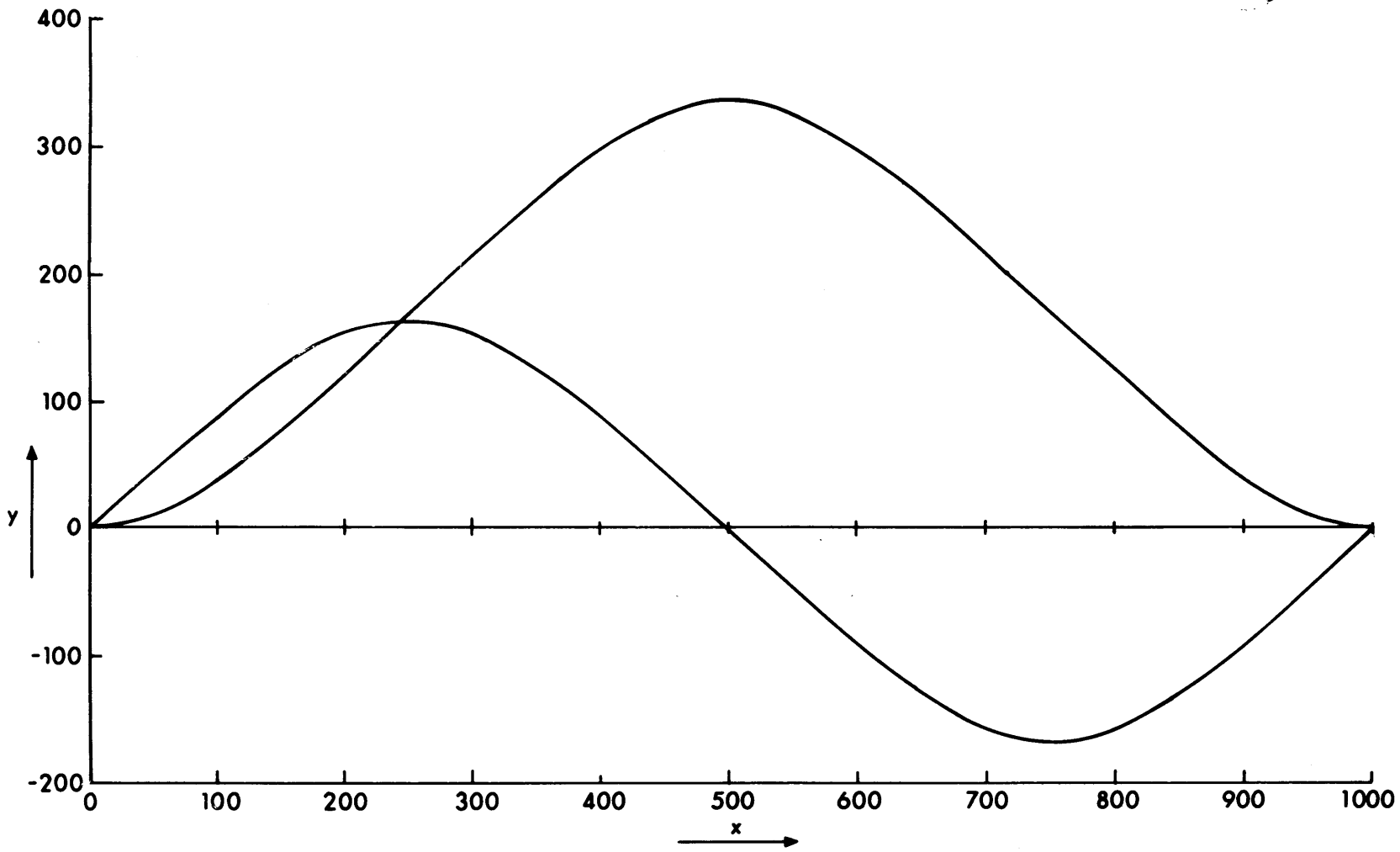

Figure 6-4. Extremal Trajectories for the Case b = 1 (Stationary Noise)

such trajectory with initial and final points at the same value of y, we can construct an infinite number of admissible trajectories with identical cost. There exists a periodic function y(x), any full cycle of which, regardless

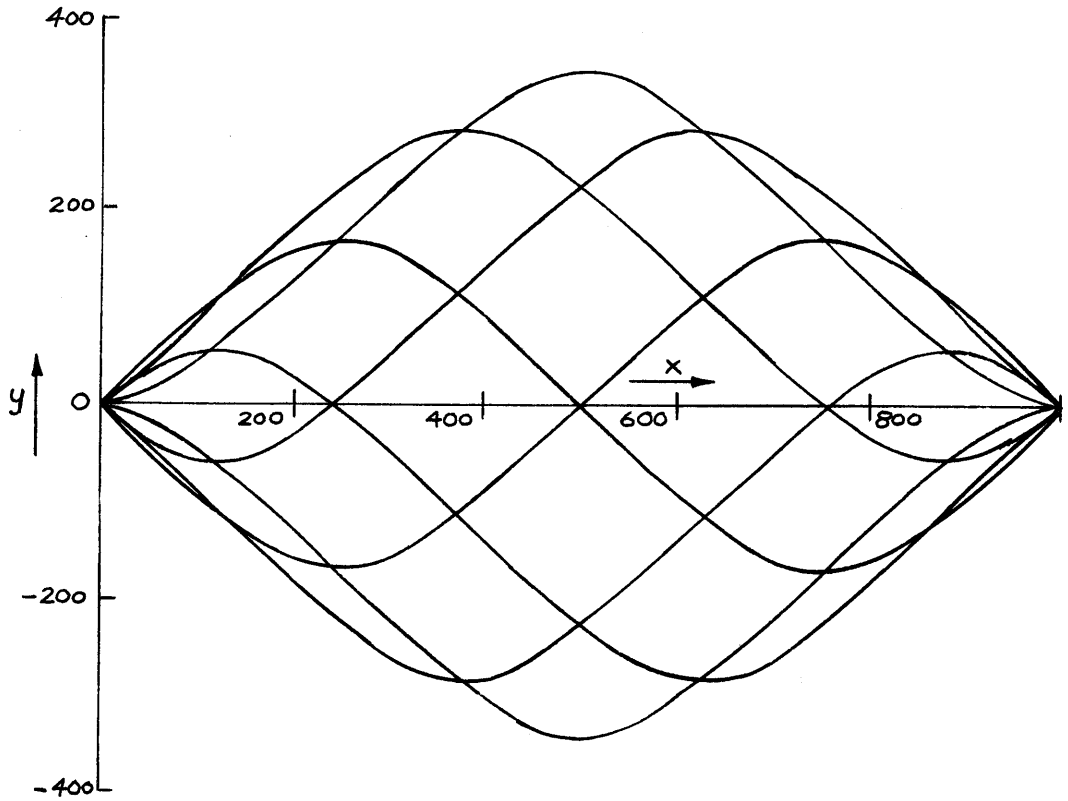

Figure 6.5 Family of Extremals for the Stationary-Noise Problem

of phase, provides an optimum path for our problem. In the present problem that function is seen to be very close to a sine wave, with period 1000 miles. Figure 6.5 shows several members of the family of extremal trajectories. In the terminology of the calculus of variations, the end-points of the trajectory are conjugate to each other. This situation is a consequence of the particular boundary values, y(0) = y(1000) = 0, of the problem.

It would be interesting, if sufficient computer time were available, to repeat the computations for the case b = 1, for several values of the parameter c. As c approaches zero the noise becomes invariant with y

(i. e., the disturbance is a constant for any trajectory, but varies randomly from one trajectory to another). In this case the optimum trajectory will be the minimum-time path $y = 0$.

As c becomes very large, the character of the disturbance approximates that of a stationary white-noise function of y. In this case, vanishingly small deviations of the trajectory from the straight-line path are sufficient to take full advantage of the whiteness of the noise. Hence we would expect the trajectory once more to approach the straight-line path $y = 0$ in the limit. It follows that there is an intermediate value of c which causes the maximum excursions in the y direction (i. e., which produces the largest amplitude for the periodic function from which the optimal paths are derived).

CHAPTER 7

CONCLUSIONS

7.1 Capabilities of the Method

A method has been presented for the iterative optimization of a nonlinear system operating in a stochastic environment. The random disturbances must be small enough to allow a linearized analysis, but are otherwise quite arbitrary. They may, in general, be non-white, non-stationary random functions of the state variables and the control variables, as well as of the independent variable of the problem. The method may thus be applied to problems in which analysis based on the associated variance equations is impracticable. The only statistical properties which must be known, for the quadratic criteria considered here, are the associated correlation functions.

The basic performance criterion is the minimization, in the mean-square sense, of the random perturbations in the final value of a prescribed scalar function of the state variables. This is extended to allow minimization of the mean-square length of a vector with random components, and to criteria of a partly-deterministic nature. With some additional information about the amplitude distributions of the perturbations, it is possible to minimize such quantities as the probability of a variable lying outside a specified range of values.

As examples of the variety of system configurations to which the method may be applied, we have considered the optimization of a set of linear feedback gains (with and without measurement noise), a linear filter-predictor problem, and a system in which we may adjust both the nominal trajectory and a set of feedback gains which control deviations from that trajectory.

7.2 Limitations of the Method

We have seen that the application of the method to a problem of reasonable generality would require considerable amounts of computing time and computer storage capacity. This is probably largely unavoidable, since the problem is an order of magnitude more difficult than the corresponding deterministic problems. The problem is, moreover,

certainly a significant one, so that its consideration is in keeping with the general tendency to consider ever more complex and computationally demanding problems, concurrently with the continuing expansion of the capabilities of modern computing machines.

Although we have demonstrated the application of this approach to the optimization of time-varying feedback gains, nevertheless the method is basically of a non-adaptive character, in that it leads to a set of control variables which are completely precomputed, and are not readjusted in accordance with the random disturbances actually encountered by the system. A suboptimal type of adaptive operation could be attained by repetitive re-optimization of the remaining portion of the trajectory (which would probably involve prohibitive amounts of computation), or by the application of "neighbouring-optimum" techniques[12] (which would require the introduction of third-variation considerations). Either of these approaches would fall into the class referred to by Dreyfus[29] as "open-loop-optimal feedback".

One limitation in the application of the method, as in most methods of this type, is the basic assumption that the appropriate correlation functions of the random processes are known. In reality, these correlation functions may be quite difficult to determine. This limitation becomes even more serious for the types of disturbances considered here. Nevertheless, it is reasonable to expect that in many cases intelligent guessing will provide a mathematical model which will result in a reasonable approximation to the optimum system. In any case, linearized analyses of this type represent only a first approximation to the noise response of the system, and considerably more statistical information is necessary if a more exact analysis is to be carried out.

7.3 Possibilities for Further Investigation

There are several areas in which extensions of this work, or investigations related to it, could provide useful results.

In the general area of second-variation methods for deterministic systems, there is a need for methods of improving convergence of the steepest-descent procedure by the proper choice of weighting functions. The second-variation approach could also provide ways of determining the "sharpness" of the optimum, which would be helpful in deciding how much effort should be expended in an iterative optimization scheme.

Both of the above concepts, when applied to the stochastic systems considered here, would lead naturally to third-variation considerations, as would the "neighbouring-optimum" approach mentioned above.

As mentioned in Sec. 3.9, it may be possible to apply, to the stochastic problems discussed here, the principles of Denham's "direct" method for handling inequality constraints.[22, 24, 25] This could conceivably result in greater economy of computation time.

The most time-consuming computations involved in the method are the repeated integrations of steps (3) and (4) of Sec. 5.2. The method could be greatly improved if these operations could be largely eliminated. One possibility is the representation of the relevant functions by series of terms separable in the two independent variables. To date efforts to achieve this have been unsuccessful.

Other computational economies can undoubtedly be realized in specific problems and for specific systems. Such savings would make more practical the consideration of the integrated mean-square type of criterion mentioned in Sec. 3.8. Alteration of the procedure to handle such criteria is a straightforward, but somewhat tedious, procedure.

An interesing area of investigation could be a consideration of to what extent, and for what types of systems and problems, analog-computer techniques could be employed. Analog methods for the types of computations involved here have been discussed at some length by Laning and Battin.[57] The principal difficulty would appear to be the requirements for storage of the computed functions. On the other hand, considerable increases in computing speed may be realized by the use of analog machines, assuming that their precision is adequate for the problem in question. The logical and decision-making requirements of the iterative procedure, together with the requirements for storage and integration speed, make hybrid analog-digital machines very attractive for applications of this kind.*

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* See Ref. 62. See also E. G. Gilbert, "The Application of Hybrid Computers to the Iterative Solution of Optimal Control Problems", paper presented at the Conference on Computing Methods in Optimization Problems, U. C. L. A., January 30-31, 1964.

APPENDIX A

TRANSITION MATRICES AND ADJOINT EQUATIONS

We shall present here, in a brief and incomplete form, some of the significant results concerning fundamental solutions and adjoint differential equations. Further results and proofs may be found in Ref. 77.

Consider the linear vector differential equation

$$\dot{\underline{x}}(t) = F_x(t)\,\underline{x}(t) + F_u(t)\,\underline{u}(t) \tag{A-1}$$

A non-singular square matrix solution X(t) of the corresponding homogeneous equation,

$$\dot{X}(t) = F_x(t)\,X(t) \tag{A-2}$$

is known as a fundamental matrix of the homogeneous system. If X(t) is nonsingular at any time t_0, it is nonsingular for all finite time.[5]

Any solution of (A-1) may be written

$$\underline{x}(t) = X(t)X^{-1}(t_0)\,\underline{x}(t_0) + X(t)\int_{t_0}^{t} X^{-1}(\tau)F_u(\tau)\,\underline{u}(\tau)\,d\tau \tag{A-3}$$

and the product

$$\Phi(t,\tau) = X(t)\,X^{-1}(\tau) \tag{A-4}$$

is known as the transition matrix of the system. It should be noted that the fundamental matrix, from which the transition matrix is derived, may be computed by solving (A-2) for increasing or decreasing t, and that the initial condition for (A-2) may be any nonsingular matrix prescribed at any value of t.

For a matrix differential equation

$$\dot{Z}(t) = F_z(t)\,Z(t) + F_u(t)\,U(t) \tag{A-5}$$

the general solution is similar in form to (A-3). For the more general case

$$\dot{Z}(t) = A(t)Z(t) + Z(t)B^{T}(t) + F_u(t)U(t) \qquad \text{(A-6)}$$

(where Z and $F_u U$ are $n \times m$, A is $n \times n$, and B is $m \times m$), we may define two fundamental matrices X and Y such that

$$\dot{X}(t) = A(t)X(t) \qquad (n \times n) \qquad \text{(A-7)}$$

and

$$\dot{Y}(t) = B(t)Y(t) \qquad (m \times m) \qquad \text{(A-8)}$$

Using the fact that the time derivative of the inverse of a matrix M is given by $-M^{-1}\dot{M}M^{-1}$, we may show that

$$\frac{d}{dt}\left[X^{-1}(t)Z(t)Y^{-T}(t)\right] = X^{-1}(t)F_u(t)U(t)Y^{-T}(t) \qquad \text{(A-9)}$$

so that

$$X^{-1}(t)Z(t)Y^{-T}(t) = X^{-1}(t_0)Z(t_0)Y^{-T}(t_0) + \int_{t_0}^{t} X^{-1}(\tau)F_u(\tau)U(\tau)Y^{-T}(\tau)\,d\tau \qquad \text{(A-10)}$$

or, in terms of the transition matrices Φ_A and Φ_B of (A-7) and (A-8),

$$Z(t) = \Phi_A(t, t_0)Z(t_0)\,\Phi_B^{T}(t, t_0) + \int_{t_0}^{t} \Phi_A(t, \tau)F_u(\tau)U(\tau)\,\Phi_B^{T}(t, \tau)\,d\tau \qquad \text{(A-11)}$$

For the homogeneous case (U = 0), this result was given by Bellman (ref. 5, p. 175, Ex. 2).

Consider now the quantity*

$$\phi = \underline{g}^T \underline{x}(T) + \int_0^T \underline{h}^T(t)\underline{x}(t)\,dt \tag{A-12}$$

and suppose we desire to express ϕ in terms of $\underline{x}(0)$ and $\underline{u}(t)$, $0 \leq t \leq T$. Using (A-3) we may write

$$\begin{aligned}
\phi &= \underline{g}^T X(T) X^{-1}(0)\underline{x}(0) + \underline{g}^T X(T) \int_0^T X^{-1}(t) F_u(t)\,\underline{u}(t)\,dt \\
&\quad + \int_0^T \underline{h}^T(t) X(t)\,dt\, X^{-1}(0)\,\underline{x}(0) \\
&\quad + \int_0^T \underline{h}^T(t) X(t) \int_0^t X^{-1}(\tau) F_u(\tau)\,\underline{u}(\tau)\,d\tau\,dt \\
&= \left[X^{-T}(0) X^T(T)\underline{g}\right]^T \underline{x}(0) + \int_0^T \left[X^{-T}(t) X^T(T)\underline{g}\right]^T F_u(t)\underline{u}(t)\,dt \\
&\quad - \left[X^{-T}(0) \int_T^0 X^T(t)\,\underline{h}(t)\,dt\right]^T \underline{x}(0) \\
&\quad - \int_0^T \left[X^{-T}(t) \int_T^t X^T(\tau)\,\underline{h}(\tau)\,d\tau\right]^T F_u(t)\,\underline{u}(t)\,dt
\end{aligned} \tag{A-13}$$

where we have changed the order to the double integration. If we write

$$\underline{\lambda}(t) = X^{-T}(t) X^T(T)\,\underline{g} - X^{-T}(t) \int_T^t X^T(\tau)\,\underline{h}(\tau)\,d\tau \tag{A-14}$$

then (A-13) becomes

$$\phi = \underline{\lambda}^T(0)\,\underline{x}(0) + \int_0^T \underline{\lambda}_u^T(t)\,\underline{u}(t)\,dt \tag{A-15}$$

where

$$\underline{\lambda}_u(t) = F_u^T(t)\,\underline{\lambda}(t) \tag{A-16}$$

---

* ϕ may represent a perturbation in the performance criterion or in one element of the constraint vector $\underline{\psi}$.

When (A-14) is compared with (A-3) it becomes apparent that $\underline{\lambda}(t)$ is the solution of a differential equation for which $X^{-T}(t)$ is a fundamental matrix, with initial condition $\underline{\lambda}(T) = \underline{g}$ and forcing term $-\underline{h}(t)$, provided that such an equation exists. Such an equation can indeed be found by differentiating $X^{-T}(t)$:

$$\frac{d}{dt}(X^{-T}) = -X^{-T}\dot{X}^{T}X^{-T} = -F^{T}X^{-T} \qquad \text{(A-16.1)}$$

The required differential equation for $\underline{\lambda}(t)$ is therefore

$$\dot{\underline{\lambda}}(t) = -F^{T}(t)\underline{\lambda}(t) - \underline{h}(t) \qquad \underline{\lambda}(T) = \underline{g} \qquad \text{(A-17)}$$

Equation (A-17) (or, more correctly, the corresponding homogeneous equation) is known as the adjoint equation for the system of equation (A-1).[8, 73, 77] If we represent its transition matrix by $\Phi_a(t, \tau)$, we have

$$\Phi_a(t, \tau) = \Phi^{-T}(t, \tau) = \Phi^{T}(\tau, t) \qquad \text{(A-18)}$$

The forcing term $-\underline{h}(t)$ in the adjoint equation, although not included in most developments, can be quite useful in optimization computations. It allows us to employ performance criteria involving integrals, as in equation (A-12) (see Sec. 5.4). This is the most general formulation of the problem of Bolza.[9]

We may derive an analogous adjoint equation for the matrix differential system (A-6).[21, 41] If we assume a general scalar performance criterion of the form

$$\phi' = \text{tr } G^{T}Z(T) + \text{tr} \int_0^T H^{T}(t)Z(t)\,dt \qquad \text{(A-19)}$$

and define the adjoint matrix L(t) by

$$\dot{L}(t) = -A^{T}(t)L(t) - L(t)B(t) - H(t) \qquad L(T) = G \qquad \text{(A-20)}$$

we can prove quite easily that

$$\phi' = \text{tr } L^{T}(0)Z(0) + \text{tr} \int_0^T L^{T}(t)F_u(t)\,U(t)\,dt \qquad \text{(A-21)}$$

For the slightly more general case

$$\dot{Z}(t) = A(t)Z(t) + Z(t)B^T(t) + C(t)Z(t)D^T(t) + F_u(t)U(t) \qquad \text{(A-22)}$$

the adjoint equation becomes

$$\dot{L}(t) = -A^T(t)L(t) - L(t)B(t) - C^T(t)L(t)D(t) - H(t) \qquad L(T) = G \qquad \text{(A-23)}$$

and the relation (A-21) remains valid.

In dealing with the matrix adjoint equations (A-20) and (A-23) we have made extensive use of the identity

$$\text{tr } AB = \text{tr } BA \qquad \text{(A-24)}$$

APPENDIX B

A GENERALIZATION OF THE SCHWARZ INEQUALITY

Let A, B, M, and N be matrices of dimensions $p \times m$, $q \times m$, $n \times p$ and $n \times q$ respectively, so that the quantity MA + NB is defined and represents an $n \times m$ matrix. Then

$$[MA + NB][MA + NB]^T \geq 0 \tag{B-1}$$

where we have used the inequality sign in the usual matrix sense (i.e., $A \geq B$ means A - B is positive semidefinite). Thus we have

$$MAA^TM^T + MAB^TN^T + NBA^TM^T + NBB^TN^T \geq 0 \tag{B-2}$$

Letting $M = -BA^T(AA^T)^{-1}$ and N = I we obtain*

$$BB^T \geq BA^T(AA^T)^{-1}AB^T \tag{B-3}$$

Alternatively, with $M = -(AA^T)^{-1/2}$ and $N = (BB^T)^{-1/2}$ (where the superscript $^{-1/2}$ indicates the inverse of the unique positive semidefinite symmetric square root matrix), we obtain

$$I \geq (AA^T)^{-1/2}(AB^T)(BB^T)^{-1/2} \tag{B-4}$$

Equation (B-4) may be considered a consequence of (B-3), for if we pre- and post-multiply (B-3) by $(BB^T)^{-1/2}$ (which leaves the inequality still valid) we obtain

$$I \geq C^TC \tag{B-5}$$

where

$$C = (AA^T)^{-1/2}(AB^T)(BB^T)^{-1/2} \tag{B-6}$$

* The result (B-3) was obtained by Grenander and Rosenblatt (Ref. 37, pp. 87-88) for the case p = q.

Equation (B-5) means that $|C\underline{x}| \leq |\underline{x}|$ for all $\underline{x}$. But by the Cauchy inequality[4] $\underline{x}^T C\underline{x} \leq |\underline{x}|\,|C\underline{x}| \leq |\underline{x}|^2$, hence $C \leq I$ and (B-4) is again demonstrated.

If A and B have random elements and we let N = I and

$$M = -\overline{BA^T}\,(\overline{AA^T})^{-1}$$

we obtain, instead of (B-3) and (B-4),

$$\overline{BB^T} \geq \overline{BA^T}\,(\overline{AA^T})^{-1}\,\overline{AB^T} \tag{B-7}$$

and

$$I \geq (\overline{AA^T})^{-1/2}(\overline{AB^T})(\overline{BB^T})^{-1/2} \tag{B-8}$$

The corresponding integral versions, when A and B are functions of a variable t, are

$$\int BB^T\,dt \geq \int BA^T\,dt \left[\int AA^T\,dt\right]^{-1} \int AB^T\,dt \tag{B-9}$$

and

$$I \geq \left[\int AA^T\,dt\right]^{-1/2} \left[\int AB^T\,dt\right]\left[\int BB^T\,dt\right]^{-1/2} \tag{B-10}$$

Equation (B-9), in the case q = 1, provides another proof of the positivity of the expression in equation (2-25). If p = q = m = 1 all quantities are scalars, and both (B-9) and (B-10) reduce to the usual integral form of the Schwarz inequality.

Furthermore, if we consider a multidimensional random process $\underline{n}(t)$ and let $B = \underline{n}(t_1)$ and $A = \underline{n}(t_2)$, and define the correlation matrix

$$R(t_1, t_2) = \overline{\underline{n}(t_1)\,\underline{n}^T(t_2)} \tag{B-11}$$

we obtain from (B-7) and (B-8)

$$R(t_2, t_2) \geq R(t_2, t_1)\,R^{-1}(t_1, t_1)\,R(t_1, t_2) \tag{B-12}$$

and

$$I \geq R^{-1/2}(t_1, t_1)\, R(t_1, t_2)\, R^{-1/2}(t_2, t_2) \tag{B-13}$$

or in the stationary case, with $t_1 - t_2 = \tau$ and $R(t_1, t_2) = R(\tau)$:

$$R(0) \geq R(-\tau) R^{-1}(0)\, R(\tau) = R^{T}(\tau)\, R^{-1}(0)\, R(\tau) \tag{B-14}$$

and

$$I \geq R^{-1/2}(0)\, R(\tau) R^{-1/2}(0) \tag{B-15}$$

or

$$R(0) \geq R(\tau) \tag{B-16}$$

In the one-dimensional case, this is a familiar result, as is the scalar version of (B-13) for the non-stationary case:[59]

$$\sigma(t_1)\, \sigma(t_2) \geq r(t_1, t_2) \tag{B-17}$$

In the case q = 1, as we have seen in Sec. 2.2, equation (B-9) can be interpreted as expressing the fact that the norm of the vector function $B^{T}(t)$ is greater than that of its perpendicular to the subspace S, for which the rows of A form a basis; or alternatively, that the norm of the orthogonal projection of $B^{T}(t)$ on S is positive. An analogous interpretation is valid for the cases q > 1, and for the equations (B-3) and (B-7). In the latter case, the concepts of orthogonality and projection are defined in the probabilistic sense.[59]

The inequality (B-3) becomes an equality when B = QA (where Q is an arbitrary rectangular matrix of suitable dimensions), and that of (B-4) when B = A. Analogous relations hold for the other inequalities.

In the scalar version of (B-16) it is obvious that

$$\left.\begin{aligned} &\left.\frac{dr(\tau)}{d\tau}\right|_{\tau=0+} \leq 0 \\ \text{and} \quad &\left.\frac{dr(\tau)}{d\tau}\right|_{\tau=0-} \geq 0 \end{aligned}\right\} \tag{B-18}$$

Similarly, from (B-12), for a one-dimensional non-stationary process, we may define

$$f(t_1, t_2) = r^2(t_1, t_2) - r(t_1, t_1)r(t_2, t_2) \leq 0 \tag{B-19}$$

Then since $f(t_1, t_1) = 0$,

$$\left.\frac{\partial f(t_1, t_2)}{\partial t_2}\right|_{t_2=t_1+} \leq 0 \tag{B-20}$$

and similarly for $t_2 = t_1-$. Thus we have

$$2r(t_1, t_1) \left.\frac{\partial r(t_1, t_2)}{\partial t_2}\right|_{t_2=t_1+} - r(t_1, t_1) \left.\frac{dr(t_2, t_2)}{dt_2}\right|_{t_2=t_1+} \leq 0$$

or equivalently

$$2 \left.\frac{\partial r(t, t_2)}{\partial t_2}\right|_{t_2=t+} - \frac{d}{dt}\left[\sigma^2(t)\right] \leq 0 \tag{B-21}$$

with the opposite inequality at $t_2 = t_1-$, and corresponding results for the t_1-derivatives. These results reduce to (B-18) in the stationary case. Applying the same procedure to (B-12) in the multidimensional case, we find the corresponding relation for the correlation matrix:

$$\left[\frac{\partial R(t_1, t_2)}{\partial t_2} + \frac{\partial R(t_2, t_1)}{\partial t_2} - \frac{dR(t_2, t_2)}{dt_2}\right]_{t_2=t_1+} \leq 0 \tag{B-22}$$

or equivalently

$$\left.\frac{\partial}{\partial t_2}\left[R(t, t_2) + R^T(t, t_2)\right]\right|_{t_2=t+} - \frac{d}{dt} R(t, t) \leq 0 \tag{B-23}$$

Similarly in (B-13), defining

$$F(t_1, t_2) = R^{-1/2}(t_1, t_1)R(t_1, t_2)R^{-1/2}(t_2, t_2) - I \leq 0 \tag{B-24}$$

we have

$$R^{-1/2}(t_1, t_1) \left.\frac{\partial R(t_1, t_2)}{\partial t_2}\right|_{t_2=t_1+} R^{-1/2}(t_1, t_1)$$

$$- R^{-1/2}(t_1, t_1)R(t_1, t_1)R^{-1/2}(t_1, t_1) \left.\frac{dR^{1/2}(t_2, t_2)}{dt_2}\right|_{t_2=t_1+} R^{-1/2}(t_1, t_1) \leq 0 \tag{B-25}$$

or equivalently

$$\left.\frac{\partial R(t, t_2)}{\partial t_2}\right|_{t_2=t+} - R^{1/2}(t, t)\frac{d}{dt}R^{1/2}(t, t) \leq 0 \tag{B-26}$$

It is possible to derive (B-23) from (B-26). For corresponding to (B-13) is the relation

$$I \geq R^{-1/2}(t_2, t_2)\,R(t_2, t_1)\,R^{-1/2}(t_1, t_1) \tag{B-27}$$

which, upon differentiation will respect to t_2 as before, yields

$$\left.\frac{\partial R^T(t, t_2)}{\partial t_2}\right|_{t_2=t+} - \left[\frac{d}{dt}R^{1/2}(t, t)\right] R^{1/2}(t, t) \leq 0 \tag{B-28}$$

Addition of (B-26) and (B-28) yields the result (B-23).

The inequalities derived by differentiation of the Schwarz inequality and some of its generalizations, may be seen to impose constraints on the rates of change of the elements of R(t, t) with respect to t, in terms of the rates of change of the correlation functions with respect to the difference between the arguments. This is in accord with intuition, for we would not expect a highly correlated process to have rapidly changing mean-square values.

To demonstrate the effect of these constraints, suppose the function

$$r(t_1, t_2) = \exp\left[-a|t_1 - t_2| - b|t_1 + t_2|\right] \tag{B-29}$$

is postulated as the correlation function of a non-stationary random process for $t_1, t_2 \geq 0$. Then (B-21) must be satisfied. In particular, for $t_2 > t_1 > 0$ we may write

$$r(t_1, t_2) = \exp\left[a(t_1 - t_2) - b(t_1 + t_2)\right] \tag{B-30}$$

from which, using (B-21), and noting that $\sigma^2(t) = e^{-2bt}$,

$$-2a \exp\left[-2bt\right] + 2b \exp\left[-2bt\right] \leq 0 \tag{B-31}$$

which requires simply $a \geq b$. In other words, the characteristic time of the exponential correlation must be shorter than twice the time constant of decay of the mean-square value.

APPENDIX C

THE MINIMIZING PROPERTY OF A PARTICULAR TRAJECTORY*

We wish to demonstrate that, for the problem of Sec. 6.2.1 with b = 0, the rectangular trajectory always provides a local minimum. From physical considerations it may be seen that the optimum trajectory will be symmetrical, with each half behaving monotonically. Hence we may consider the following equivalent system:

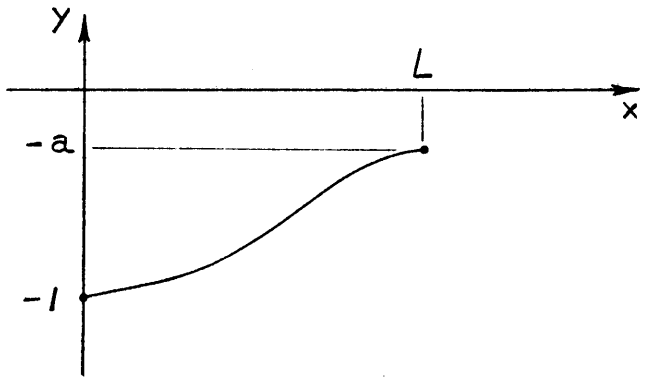

Figure C.1

The trajectory y(x) joins the points (0, -1) and (L, -a), and $y'(x)$ is always positive. The cost associated with the trajectory is

$$\phi = \int_0^L y^2 \sqrt{1 + y'^2}\, dx \qquad \text{(C-1)}$$

Now since

$$\int_0^L dx = L \qquad \text{(C-2)}$$

we may write

$$\phi = a^2 L + \int_0^L \left[y^2 \sqrt{1 + y'^2} - a^2 \right] dx \qquad \text{(C-3)}$$

---

* The proof given here is due in large part to Dr. James E. Potter of M. I. T.

Let $s = dx/dy$. Then

$$\phi = a^2 L + \int_0^L \left[y^2 \sqrt{s^2 + 1} - a^2 s \right] \frac{dx}{s} \tag{C-4}$$

By differentiation we can show that, for a given value of y, the bracketed quantity is minimum when

$$s^2 = \frac{1}{(\frac{y}{a})^4 - 1} \tag{C-5}$$

and has the minimum value $\sqrt{y^4 - a^4}$. Hence

$$\phi \geq a^2 L + \int_0^L \sqrt{y^4 - a^4}\, \frac{dx}{s}$$

$$= a^2 L + \int_{-1}^{-a} \sqrt{y^4 - a^4}\, dy$$

$$= a^2 L + \int_a^1 \sqrt{y^4 - a^4}\, dy \tag{C-6}$$

Defining this quantity as f(a) and letting $u = a/y$, we have

$$\frac{df}{da} = 2a \left[L - \int_a^1 \frac{a^2 dy}{\sqrt{y^4 - a^4}} \right]$$

$$= 2a \left[L - a \int_a^1 \frac{du}{\sqrt{1 - u^4}} \right]$$

$$\geq 2a \left[L - a \int_0^1 \frac{du}{\sqrt{1 - u^4}} \right]$$

$$= 2a \left[L - 1.311\, a \right] \tag{C-7}$$

where the integral is evaluated from a table of elliptic integrals* as $1.8541 \div \sqrt{2} = 1.311$. Finally, we may integrate (C-7), with $f(0) = 1/3$, to obtain

$$\phi(a) \geq f(a) = \frac{1}{3} + La^2 - 0.874\,a^3 \qquad \text{(C-8)}$$

Since the cost of the rectangular trajectory is 1/3, we may conclude that the rectangular trajectory is the best trajectory with $a = 0$; and since (C-8) indicates that $f'(a) = 0$ and $f''(a) > 0$ at $a = 0$, all neighbouring trajectories, for small positive values of a, must be more costly than the rectangular one. Hence our statement is proved.

* See Jahnke and Emde, "Tables of Functions", Dover, New York, 1945, pp. 59, 65.

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BIOGRAPHY

Robert John Fitzgerald was born in Hamilton, Ontario, Canada on February 9, 1934. He was raised in Peterborough, Ontario, and entered the University of Toronto in 1951. After one year in a liberal arts course, he registered in the Mechanical Engineering Department of the School of Practical Science, and in 1956 received the degree of Bachelor of Applied Science. He was also awarded the Gold Medal of the Association of Professional Engineers of the Province of Ontario.

In September 1956 he entered M. I. T. on a Whitney Fellowship, and one year later received the degree of Master of Science in the field of Mechanical Engineering. His thesis, written under the supervision of Professor Stephen H. Crandall, was in the field of vibrations and was entitled "Determination of Mode Shapes from Driving-Point Admittance Measurements".

He received the degree of Mechanical Engineer in June, 1958. From 1957 to 1960 he was a Teaching Assistant in the Mechanical Engineering Department, where his duties consisted mainly of lecturing in sophomore courses in applied mechanics.

In 1960-61, as the recipient of a Rotary Fellowship for International Understanding, he spent one year at the École Nationale Supérieure de l'Aéronautique in Paris, France, where he studied automatic control and its applications to aviation.

Since 1961 he has been a Research Assistant in the Space Guidance Analysis group of the Instrumentation Laboratory, under Dr. Richard H. Battin.

His industrial experience includes three summers with the Canadian General Electric Company, one summer with the Ramo Wooldridge Division of Thompson Ramo Wooldridge, and one summer with the Autonetics Division of North American Aviation.

He is a member of Zeta Psi, Theta Xi, Tau Beta Pi, the American Society of Mechanical Engineers, the Institute of Electrical and Electronics Engineers, and the American Institute of Aeronautics and Astronautics.