

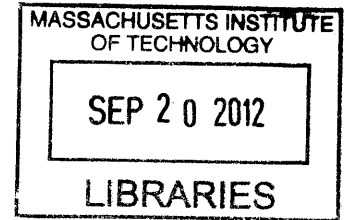
Optimal Intertemporal Decisions in Imperfect Capital Markets

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By

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To my parents, Esperanza and Rogelio for what they gave, my wife, Maria Esther for what we have shared and Rodrigo, my son, for what will be.

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Optimal Intertemporal Decisions in Imperfect Capital Markets

by

Ruben Ojeda Cardenas

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on February 26, 2012 in Partial Fulfillment of the
Requirements for the Degree of Doctor in Philosophy in Economics

Abstract:

Optimal intertemporal investment solution paths are derived for both, firms operating in perfect financial markets and those facing credit constraints, due to imperfect capital markets. However, as in these markets, saving and investment decision may not be separable, we obtain the optimal dynamic path of these decisions for agents that own capital but do not have any access to credit and extend the analysis when these agents have some access to credit but yet face credit constraints from financial intermediaries.

We next consider agents without physical capital and who derived their income from wages and/or financial assets. We study their optimal intertemporal decisions, among, consumption, financial assets and durable goods, first under perfect capital markets and then when credit constraints are present.

The above results may be useful for both, the comparative dynamic analysis of agents with different type of endowments and immersed in imperfect financial markets and also to derive, from micro foundations, the aggregate demand and supply functions of intertemporal macro models for developing economies. The distinction between different types of agents according to their endowment may also help to assess the wealth and income distribution implications of economic policy.

Thesis Supervisor: Robert M. Solow
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Introduction.

A common critique of Neoclassical Economic Theory is that this theory assumes perfectly competitive markets. In particular, it is argued that capital markets imperfections prevailing in developing countries may render this theory irrelevant for said economies. In addition, the assumption of equally endowed agents precludes the analysis of any income and wealth distribution effects of macroeconomic policy.

Using neoclassical growth and investment theories as a point of reference this thesis studies the optimal intertemporal decisions of agents in imperfect capital markets.

The deterministic neoclassical theory of investment by the firm is extended, first in Chapter I to obtain investment solution paths for convex but not necessarily quadratic investment expenditure functions. In Chapter II we study how the previously derived results should be modified when firms' access to credit is constrained. However in imperfect capital markets, saving and investment decision may not be separable, thus in Chapter III we derive optimal dynamic strategies for agents that own capital but do not have access to credit. Building on this background, in Chapter IV we analyze how these strategies should be modified when these agents have access to credit, but, due to imperfections in the capital markets, face quantity constraints from financial intermediaries.

If credit is rationed in proportion to the amount of physical capital that an agent has, a further implication of imperfect capital markets is that we may need to distinguish between agents that own capital and those who do not. Chapter V studies the optimal intertemporal decisions of agents without physical capital and who derived their income from wages and operate in perfect capital markets. This serves as a background for the analysis in Chapter VI of the same type of agents but immersed in imperfect capital markets.

In these last two chapters we also introduce interest earning financial assets, money, as well as durable goods as an alternative to the consumption of non-durables. Although not undertaken in this work, the analysis in Chapter IV can be readily extended to include these three assets by following the same techniques presented in Chapter V and VI.

The methodology developed in this work may be useful in analyzing the comparative dynamics of different types of individual agents in imperfect financial markets. In addition, the introduction of interest earning financial assets, money and durable goods enables the techniques developed here to provide the micro foundations for the aggregate demand and supply functions of intertemporal macro models aimed at analyzing diverse fiscal and monetary policy issues in economies with imperfect financial markets. Lastly, the distinction between agents that have physical capital and those that do not and who derive their income mostly from wages and/or financial assets may also allow us to use this analysis to address some

basic wealth and income distribution implications of macroeconomic policy in emerging economies where these issues are consider of strategic relevance.

I.- Neoclassical Intertemporal Production and Investment with Constant Returns to Scale.

I.-A. Introduction.

If firms could rent in any desired volume all their input services for an arbitrary short period with the corresponding transaction costs being either non-existent or in strict proportion to the volume of services rented, then, production decisions among different points in time would be independent of each other and consequently no need of a strategy through time for the hiring of input services would arise.

When in a closer approximation to reality we regard firms as owning stocks of production inputs which cannot be instantaneously adjusted to any level and with non-proportional costs of adjustment, production decisions are no longer time inter-independent. We will consider investment to be the process by which adjustments to the capital stock are made, encompassing both, the purchase and the installation of capital goods. If installation costs increase with the rate of investment then investment should be distributed through time in an optimal manner. The necessity of formulating an optimal investment plan may be furthered if this process is irreversible.

Before stating formally the decision problem of the neoclassical firm I shall make the following assumptions.

Output possibilities are a function of capital (K) assigned to the production process and labor (L) employed and are described for any time (t) by:

$$F(K(t), L(t)), F > 0, K(t) > 0, L(t) > 0$$

Thus, F () is assumed time invariant with,

$$F_k > 0 \quad F_l > 0$$

and

$$F_{kk} < 0; \quad F_{ll} < 0; \quad F_{kk}F_{ll} = (F_{kl})^2$$

To ease the analysis lets assure the existence of an interior solution by assuming Inada conditions,

$$\lim_{l \rightarrow 0} \left(\frac{\partial F}{\partial L} \right) = +\infty \quad \lim_{l \rightarrow \infty} \left(\frac{\partial F}{\partial L} \right) = 0$$

and

$$\lim_{k \rightarrow 0} \left(\frac{\partial F}{\partial k} \right) = +\infty \quad \lim_{k \rightarrow \infty} \left(\frac{\partial F}{\partial k} \right) = 0$$

Letting q be the price of new capital goods I, the investment process involves the disbursement of funds for the purchase of equipment by an amount qI plus an installation cost described by the function $G(I)$, thus total investment outlay H is:

$$H(q, I) = qI + G(I)$$

The function $G(I)$ will be assumed convex i.e., installation cost increase at an increasing rate. Since qI is a linear function, H exhibits similar properties as $G(I)$ see Figure I-1 thus $H' > 0$ and $H'' > 0$

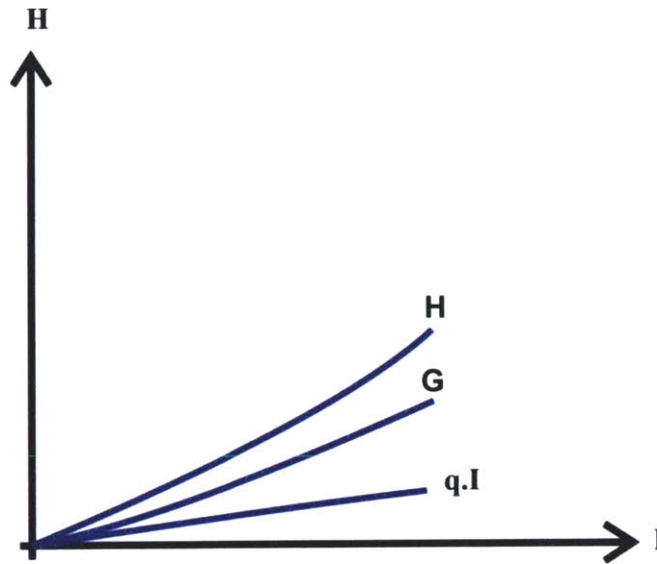


Figure I-1

The convexity of $H(I)$ provides an incentive to the firm to distribute its investment over time.

The production function $F(I)$ and the installation function $G(I)$ and $H(I)$ are assumed to be additively separable, thus $F(I)$ and $H(I)$ are likewise.

The firm lacks the ability to influence the price of the goods it produces p , nor can it influence the wage rate w or q . Thus $p(t)$, $w(t)$ and $q(t)$ are exogenous and will be assumed constant through time. The opportunity cost of funds to the shareholders r is assumed likewise. Furthermore, the labor, equipment and goods markets are of such nature that the firm can contract all the labor it wants at the given w , sell at the given p , all the output it cares to produce and purchase an unlimited amount of capital goods at its constant price q . Installed capital depreciates at a constant rate.

Given the recursive nature of the optimization problem, the optimal production and investment plans should maximize the present value of the firm over the planning horizon subject to the capital accumulated in the past, net of depreciation. Thus the objective will be to maximize for the current and subsequent periods - taken as a whole - the discounted cash-flow net of investment disbursements. Aware of the inadequacy of an arbitrarily limited time horizon and of the sensitivity of the results of the optimization problem to arbitrarily specified terminal conditions, the optimization horizon of the firm will be thought of as indefinitely long and formally modeled as infinite.

I.-B. The Formal Problem.

Formally, the neoclassical producer problem discussed in the preceding section may be stated as,

$$\text{Max. } \int_0^{\infty} \{pF(L, K) - wL - H(I)\} \exp -rt \, dt$$

Subject to, the transition law

$$\dot{K} = I - \delta K$$

The non-negative constraint on the state variable,

$$K \geq 0$$

The non-negative constraint on labor,

$$L \geq 0$$

The initial condition

$$K(0) = K_0$$

I.-C. Bibliographical Note.

The preceding problem with some variations has been address in the neoclassical theory of investment where the role of convex cost of adjusting the capital stock has been study by Eisner and Strotz (1963), Treadway (1971), Lucas (1967), Gould (1968) and Jorgenson (1973).

With a variety of heuristic arguments, calculus of variations or optimal control techniques, exact solutions paths have been obtained for the case of a quadratic investment cost function. Abel (1978) and Sargent (1979) exemplify such solutions.

The narrow purpose of this chapter is to generalize the solution to the above stated problem for a convex but not necessarily quadratic investment cost function. The solution obtained will be a useful point of reference for the work undertaken in the following chapters where a similar investment cost function will be used.

I.-D. Necessary and Sufficient Conditions.

Since it will turn out that the non-negative constraints on K and L do not become binding along optimal trajectories, we shall ignore them and write the current value Hamiltonian as

$$\mathcal{H} = pF(K, L) - wL H(I) + \lambda(I - \delta K)$$

From where we obtain the following set of necessary conditions:

$$\dot{\lambda} + pF_K(K, L) = (\delta + r)\lambda$$

$$H'(I) = \lambda$$

$$pF_L = w$$

$$\dot{K} = I - \delta K$$

As argued by Arrow and Kurz (1971, p. 49) if we complement the above set of necessary conditions with the following transversality conditions,

$$\lim_{t \rightarrow +\infty} \exp -rt\lambda \geq 0; \quad \lim_{t \rightarrow +\infty} \exp -rt\lambda K = 0$$

The resulting set of equations is both sufficient and necessary for optimality.

I.-E. Economic Interpretation of the Necessary Conditions.

The equation

$$\dot{\lambda} + pF_K(K, L) = (\delta + r)\lambda$$

states that the realized return of capital (left-hand side) which is the sum of the capital appreciation $\dot{\lambda}$ and the value of the marginal product of capital is equal to the required return (right-hand side).

The equation

$$H'(I) = \lambda$$

states that the shadow price of capital equals the marginal cost of installed capital.

The equation

$$F_L = w/p$$

implies that labor is utilized up to the point where its marginal product equals the exogenously fixed real wage.

I.-F. Solution Paths for the Constant Returns to Scale Case

Consider the two non-differential equations in the above set of necessary conditions, since $F_L > 0$ and $H' > 0$, we may solve for the control variables $I(t)$ and $L(t)$ to get

$$(1) \quad I(t) = H'^{-1}(\lambda)$$

$$(2) \quad L(t) = F_L^{-1}(w/p, K)$$

If we now substitute these expressions for I and L in the differential equations that conform the set of necessary conditions, we may re-express such set with the following system of two differential equations in λ and K :

$$(3.a) \quad \dot{\lambda} = (\delta + r)\lambda - pF_K [F_L^{-1}(w/p) ; K]$$

$$(3.b) \quad \dot{K} = H'^{-1}(\lambda) - \delta K$$

With constant returns to scale we may rewrite equation (2) as

$$(4) \quad L = F_L^{-1}(w/p) K$$

Since the derivative of a linear homogeneous function is homogeneous of degree zero and therefore, may be written in a multiplicative separable form.

Again, since $F(\cdot)$ is homogeneous of degree one, F_K is homogeneous of degree zero, thus for any constant b

$$F_k = F_k(bL, bK)$$

now let $b = 1/K$ then,

$$F_k = F_k(L/K, K/K)$$

In this last equation we now substitute the value of L given by equation (4) and since w/p is an exogenous constant we have,

$$F_k = F_k[F_l^{-1}(w/p); 1]$$

$$\text{Or, } F_k \equiv f$$

where f is a constant. If we now substitute this result in equation system (3) we see that the first equation is independent of the second and may be rewritten as

$$\dot{\lambda} - (\delta + r)\lambda = -pf$$

this has as general solution

$$\lambda(t) = \frac{pf}{(\delta + r)} + A \exp(\delta + r)t$$

where A is a constant. However for λ to meet the transversality conditions, A must be zero, hence

$$\lambda^*(t) = \frac{pf}{\delta + r}$$

Thus the shadow price of installed capital is constant. Now substituting the solution for $\lambda(t)$ obtained in the above paragraph, we have the solution for $I(t)$,

$$I^*(t) \equiv H'^{-1} \left(\frac{pf}{\delta + r} \right)$$

Again note I^* is constant.

Substitute this solution for $I(t)$ in the second equation of system (3) and the capital accumulation differential equation becomes

$$\dot{K} = I^* - \delta K$$

which now may be solved to obtain the solution path for $K(t)$

$$K^*(t) = \frac{I^*}{\delta} + \left\{ K(0) - \frac{I^*}{\delta} \right\} \exp(-\delta)t$$

From the solution path for $K(t)$, we notice that as time elapses the asymptotic value of $K(t)$ tends to I^*/δ i.e.,

$$K^\infty \equiv \lim_{t \rightarrow +\infty} K^*(t) = \frac{I^*}{\delta}$$

By transposing we may also express the constant rate of investment in terms of the steady state capital stock:

$$I^* = \delta K^\infty$$

Substituting this value of I^* in the third before last equation we may express the rate of capital accumulation as a function of the steady state capital stock,

$$\dot{K} = \delta \{K^\infty - K(t)\}$$

From the necessary conditions we solved for the control $L(t)$ obtaining equation (2) later shown susceptible to be rewritten as

$$L = F_l^{-1}(w/p) K(t)$$

Substituting in our solution path for $K(t)$ we have the employment solution path

$$L^*(t) = F_l^{-1}\left(\frac{w}{p}\right) \left\{ \frac{I^*}{\delta} + \left\{ K(0) - \frac{I^*}{\delta} \right\} \exp(-\delta)t \right\}$$

Hence,

$$L^\infty \equiv \lim_{t \rightarrow +\infty} L^*(t) = F_l^{-1}\left(\frac{w}{p}\right) \frac{I^*}{\delta}$$

Below we sketch the solution path $I(t)$, $K(t)$, $L(t)$ and $\lambda(t)$.

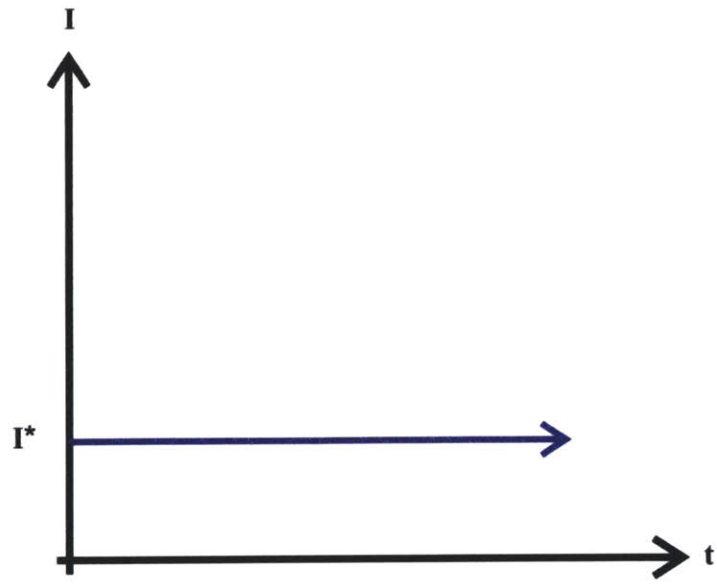


Figure I-2

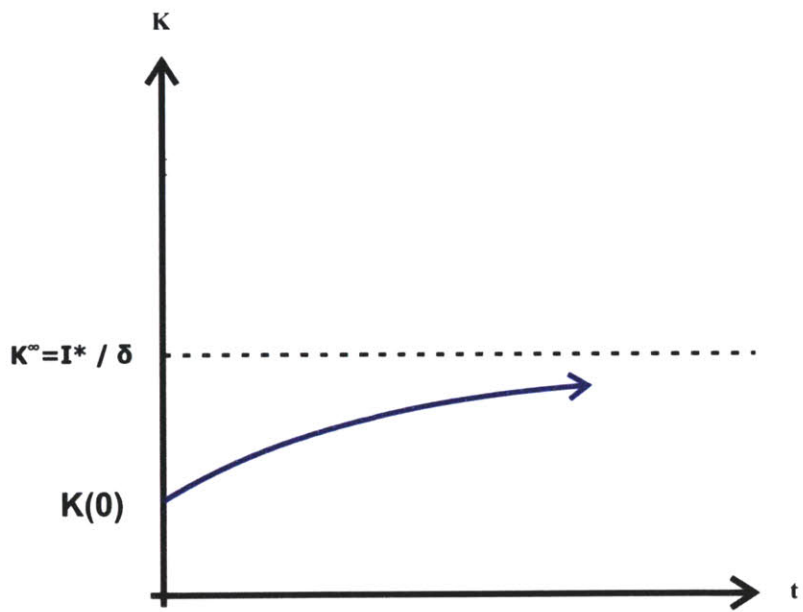


Figure I-3

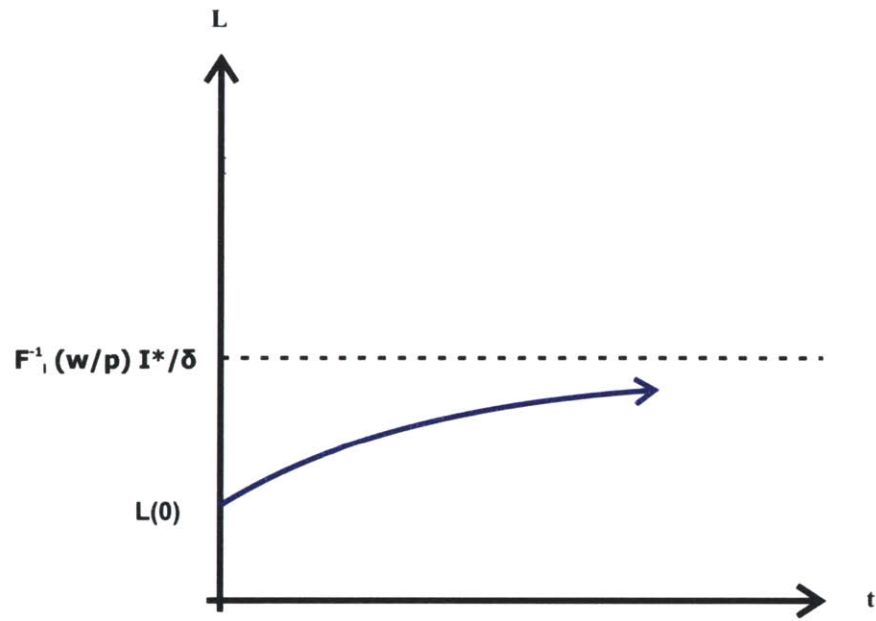


Figure I-4

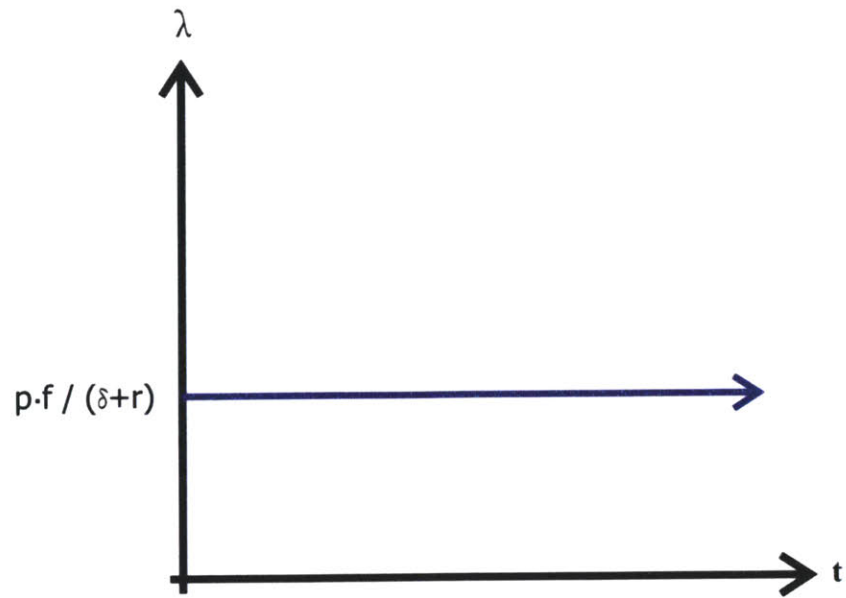


Figure I-5

I.-G. Phase Diagram for Constant Returns to Scale.

Since the system of equations that constitute the necessary conditions is time invariant and we have only two state variables, $K(t)$ and $\lambda(t)$, we can draw a phase diagram in this space.

From the necessary conditions of the optimization problem we obtained

$$\dot{\lambda} = (\delta + r)\lambda - pF_K$$

later rewritten as

$$\dot{\lambda} - (\delta + r)\lambda = -pf$$

Thus the locus for $\dot{\lambda} = 0$ is,

$$\lambda = pf/(\delta + r)$$

Since the right-hand side is a constant, $\frac{d\lambda}{dK} = 0$ and the $\dot{\lambda} = 0$ locus has zero slope in the K, λ space.

When we substituted the expression for $I(t)$ in terms of λ -equation (1)- in the transition law we obtained,

$$\dot{K} = H'^{-1}(\lambda) - \delta K$$

From where we may obtain the equation of the $\dot{K}=0$ locus.

$$K = 1/\delta H'^{-1}(\lambda)$$

or

$$\lambda = H'(\delta K)$$

Since we assumed $H'' > 0$, this locus has a positive slope and is strictly convex for H functions of higher than second degree - when H is quadratic then the loci $K=0$ is a positive sloped straight line as Figure 1 in Abel (op.cit., p. 111).

If we further assume $\lambda > H'(0)$, we would assure that $K^* > 0$ and we can draw the following phase diagram where the arrowed line shows the unique optimal Pontryagin path.

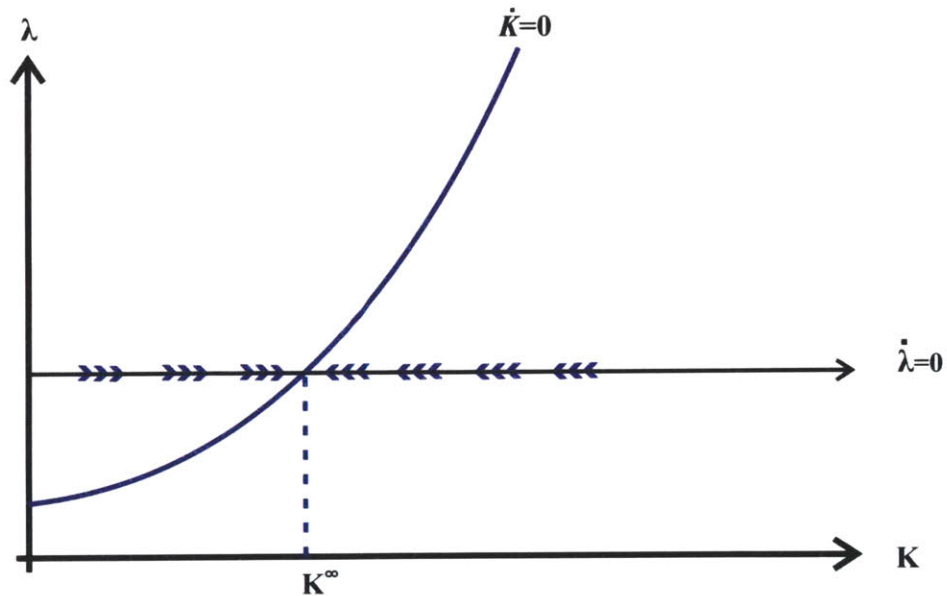


Figure I-6

I.-H. Summarizing Proposition.

We summarize the results obtained so far in the following proposition.

Proposition: A firm with constant returns to scale aiming to maximize the present value of its discounted cash-flow over an indefinitely long horizon, facing constant r , p , q , w , and with convex costs of adjusting its stock of capital will:

- a) Invest at a constant rate given by

$$I^*(t) \equiv H'^{-1} \left(\frac{pf}{\delta + r} \right)$$

b) Hire at each point in time labor up to the level where the value of the marginal product as determined by the current stock of capital equals the exogenous real wage.

c) Have K and L evolving according to the equations:

$$K^*(t) = \frac{I^*}{\delta} + \left\{ K(0) - \frac{I^*}{\delta} \right\} \exp(-\delta)t$$

$$L^*(t) = F_l^{-1} \left(\frac{w}{p} \right) K^*(t)$$

d) Approach asymptotically in time the following levels of capital stock and employment

$$K^\infty = \frac{1}{\delta} H'^{-1} \left(\frac{pf}{\delta + r} \right)$$

$$L^\infty = F_l^{-1} \left(\frac{w}{p} \right) \frac{I^*}{\delta}$$

I.-I. Concluding Remarks.

This chapter has served, both, as a review of the existent literature and as an introduction to the forthcoming analysis. Besides

presenting the results of previous work on the topic we have also shown how the assumption of a linear homogeneous production function can be exploited to reduce the set of necessary conditions to two differential equations, (3.a) and (3.b), which may be solved independently of each other to obtain exact solution paths for the more general case of convex but not necessarily quadratic investment cost functions. This eases the forthcoming work because unlike Abel (op.cit.) we did not need to take linear approximations to the steady state or unlike Sargent (op.cit.) we need not restrict the analysis to quadratic investment cost functions.

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II.- Intertemporal Production and Investment Decisions in Imperfect Capital Markets.

II.-A. Introduction.

The producer problem as formalized in Chapter I does not consider any constraint on the amount of funds available to the firm for investment purpose, hence, it tacitly assumes perfect financial markets in the sense that the firm can borrow as much as it considers optimal.

To assess the implications of capital market imperfections Appelbaum and Harris (1978) studied the effect on the time path of the capital stock when firms cannot float new debt and current earnings must be either distributed to shareholders or spent on current acquisition of capital goods but cannot be retained for future acquisition of capital goods. Schworm (1980) extends this analysis for the cases when the firm can retain earnings and when the firm can, both, retain earnings and borrow at a rate that depends on the amount of outstanding debt and which is paid on all outstanding debt. However no justification is given as to why capital market imperfections should constrain the firm in such way.

In order to have a theoretical base for the specification of the restrictions that capital market imperfections will pose on the economic agents it seems to me that first we must identify such imperfections and analyze their implications on the behavior of the existent

financial institutions. Only then will we be able to determine the terms and restrictions on the availability of credit to the individual agents.

Kurz (1976) studies in the context of a general equilibrium model the behavior of financial intermediaries when due to the incompleteness of insurance markets, individual agents cannot insure themselves against the possibility of bankruptcy and shows that under these circumstances "some quantity constraints will appear in the remaining functioning markets and these constraints will be determined endogenously like prices and provide essential allocative signals. ... the non existence of bankruptcy insurance induces financial intermediaries to undertake some insurance function and the difference between the borrowing and lending rates reflect the insurance premium". (op.cit. p. 45). However, in Kurz's analysis, individual borrowing agents do not possess physical capital, rather they have an endowment which may be either consumed or invested. It seems to me that if Kurz's analysis is extended to consider firms endowed with physical capital which cannot be sold without the consent of the financial institution with which the firm is indebted, the quantity credit constraint to the firm will be proportional to the capital stock it owns.

To extend Kurz's analysis in the above described fashion would take us too far afield, instead, lead by existent evidence we will proceed to consider firms that cannot insure themselves against the possibility of bankruptcy and that may finance their capital accumulation with either net revenue or credit, with the latter constrained not to exceed a certain proportion "a" of the value of the firm's current capital stock. We then have that the

maximization of the firms discounted cash-flow is subject to a liquidity constraint,

$$pF(K, L) - wL - H(I) + D - iD \geq 0$$

and a credit constraint,

$$aK - D \geq 0$$

We shall continue to hold the assumptions stated in the introduction to Chapter I and study now the effects of the above financial constraints on the firm with constant returns to scale.

II.-B. The Formal Problem.

The firm's problem now is:

$$\text{Max } V(0) = \int_0^{\infty} [pF(\cdot) - wL - H(I)] \exp(-r)t dt$$

Subject to:

The transition law:

$$\dot{K} = I(t) - \delta K(t)$$

The liquidity constraint:

$$pF(K, L) - wL - H(I) + \dot{D} - iD \geq 0$$

The credit rationing constraint:

$$aK(t) - D(t) \geq 0$$

The non-negative constraints on:

Instruments:

$$I(t) \geq 0 ; \quad L(t) \geq 0$$

State variable:

$$K(t) \geq 0$$

The initial conditions:

$$K(0) = K_0; \quad D(0) = D_0$$

We are interested in studying the effects on the optimal investment, employment, capital stock and output paths of the firm when either one or both of the liquidity and the credit constraints may be binding.

An initial conjecture for such circumstances is that none of the constraints

$$I(t) \geq 0 ; \quad L(t) \geq 0 ; \quad K(t) \geq 0$$

are effective along the optimal trajectories of the preceding problem and so will be assumed in order not to clutter the exposition. If for the cases below

considered, the solution we obtain violates anyone of the constraints we have just assumed ineffective, our analysis would have to be modified, but if all constraints are fulfilled then no further qualifications will be necessary.

II.-C. The Necessary Conditions.

II.-C.1 The current value Hamiltonian for boundary segments.

Our treatment of boundary segments follows Hadley and Kemp (1971). Along boundary segments we have

$$(i) \quad aK(t) - D(t) = 0$$

To deal with this constraint -unlike what will be done for the liquidity constraint- we cannot form a Hamiltonian by attaching the constraint to the objective function *a-la* Lagrange, because there would be no direct way in which the control variables could be affected to insure the constraint's fulfillment. However given constraint (i) it must be true that,

$$(ii) \quad a\dot{K} - \dot{D} = 0$$

while the constraint (i) is effective. Hence we can form the following Hamiltonian,

$$H = pF(\cdot) - wL - H(I) + \dot{D} - iD + \lambda(I - \delta K) - \eta[pF(K,L) - wL - H(I) + \dot{D} - iD] - v(a\dot{K} - \dot{D})$$

Since,

$$(iii) \quad \dot{K} = I(t) - \delta K(t)$$

the channel through which control variables influence may assure the satisfaction of the original constraint (i) is made explicit by substituting equation (iii) in (ii) obtaining,

$$a(I - \delta K) - \dot{D} = 0$$

And consequently the previous Hamiltonian may be rewritten as:

$$H = pF(\cdot) - wL - H(I) + \dot{D} - iD + \lambda (I - \delta K) - \eta [pF(K, L) - wL - H(I) + \dot{D} - iD] - v [(a(I - \delta K) - \dot{D})]$$

II.-C.2 The necessary conditions along boundary segments.

From the last Hamiltonian we may obtain the following set of necessary conditions:

$$-\dot{\lambda} + r\lambda = pF_k - \lambda\delta - \eta pF_k + va\delta$$

$$-\dot{v} + rv = \eta i$$

$$\dot{K} = I(t) - \delta K(t)$$

$$-\eta + v = 0$$

$$pF_l - w - \eta pF_l + \eta w = 0$$

$$-H'(I) + \lambda + \eta H'(I) - va = 0$$

$$\eta \{ pF(\cdot) - wL - H(I) + \dot{D} - iD \} = 0$$

$$pF(K, L) - wL - H(I) + \dot{D} - iD \geq 0$$

$$\eta \geq 0$$

II.-C.3 The current value Hamiltonian for interior segments.

For interior segments along which $aK - D > 0$, the constraint $aK - D \geq 0$ is being met as a strict inequality hence we could ignore it in forming the Hamiltonian corresponding to the periods when the credit constraint is not binding, from where we could derive a set of necessary conditions valid for interior segments. Thus the set of necessary conditions would vary according to the specific Hamiltonian from which they are derived.

II.-C.4 The general set of necessary conditions.

A general set of necessary conditions that would apply irrespective of whether the credit constraint is binding may be specified by attaching a Kuhn-Tucker like device to the set of necessary conditions derived for boundary segments that would reduce such set to the one corresponding to interior segments. Let such device be the following system of relations

$$v(aK - D) = 0$$

$$aK - D \geq 0$$

$$v \geq 0$$

then, when $aK - D \geq 0$, v is zero, the necessary conditions derived for boundary segments reduce to those valid along interior segments.

Thus rearranging the previous set of necessary conditions and attaching the above Kuhn-Tucker device we have the following set of general necessary conditions:

$$(1) \quad -\dot{\lambda} + r\lambda = pF_k - \lambda\delta - \eta pF_k + va\delta$$

$$(2) \quad -\dot{v} + rv = \eta i$$

$$(3) \quad \dot{K} = I(t) - \delta K(t)$$

$$(4) \quad -\eta + v = 0$$

$$(5) \quad L = F_l^{-1}\left(\frac{w}{p}\right)$$

$$(6) \quad I = H'^{-1}[(\lambda - va)/(1 - \eta)]$$

$$(7) \quad \eta\{pF(\cdot) - wL - H(I) + \dot{D} - iD\} = 0$$

$$(8) \quad pF(K, L) - wL - H(I) + \dot{D} - iD \geq 0$$

$$(9) \quad \eta \geq 0$$

$$(10) \quad v(aK - D) = 0$$

$$(11) \quad aK - D \geq 0$$

$$(12) \quad v \geq 0$$

Transversality conditions:

$$(13) \quad \lim_{t \rightarrow \infty} \exp(-r)t \lambda \geq 0; \quad \lim_{t \rightarrow \infty} (-r)t \lambda(t)K(t) = 0$$

II.-D. Optimal Growth Strategies.

We now discuss the way in which the set of general necessary conditions and the transversality conditions jointly characterize the solution.

II.D.1 The Case $i = r$

Assume initially we have

$$pF[K(0), L(0)] - wL(0) + aK(0) \leq H(I_i^*) = H(I_r^*)$$

where I_i^* and I_r^* denote, respectively the optimal investment level if either, the firm could borrow unrestrictedly at a rate (i) or if it had a revenue sufficiently large to finance, entirely from revenue, the optimal investment level that corresponds to a discount rate (r) and $H(I_{r=i}^*)$ is the associated level of expenditures. Thus, initially the un-constrained case solution is not feasible since it violates either the liquidity or the credit constraint or both.

Now consider the implications of equations (7) and (10), in each of these equations one of the two factors of the left-hand side must be zero. Suppose,

$$pF(K, L) - wL - H(I) + \dot{D} - iD = 0$$

and

$$aK - D = 0$$

then by transposing in the next to the last equation, the solution would have,

$$H(I) = pF() - wL + \dot{D} - iD$$

with

$$D = aK$$

and hence

$$\dot{D} = a\dot{K}$$

that is

$$H(I^*) = pF() - wL + a\dot{K} - iaK$$

and,

$$I^* = H'^{-1} [pF() - wL + a\dot{K} - iaK]$$

With the firm devoting at every time all of its revenue and credit resources to investment, the capital stock will become ever larger approaching ∞ as $t \rightarrow \infty$ thus violating the transversality conditions.

On the other hand if in equations (7) and (10), η and ν are zero then our solution becomes that of the unconstrained case, namely $H(I_{r=i}^*)$, which satisfies the transversality conditions but would violate either equation (8) or equation (11) or both.

The optimal solution will then be to initially set

$$H(I) = pF() - wL + a\dot{K} - iaK$$

and follow this policy until

$$pF() - wL + a\dot{K} - iaK$$

Reaches $H(I_{r=i}^*)$ then at that point in time neither the liquidity nor the credit constraint are effective and the solution is that of the unconstrained case i.e. $H(I_{r=i}^*)$ and,

$$I^* = H'^{-1} \left[\frac{(p F_k)}{\delta + i} \right] = H'^{-1} \left[\frac{(p F_k)}{\delta + r} \right]$$

Figure II-1 illustrates the solution when $(i) = (r)$ and

$$pF[K(0), L(0)] - wL(0) + aK(0) < H(I_i^*)$$

The firm initially devotes all of its revenue and credit to investment. As the optimal stock grows the firm will have, both, ever larger credit and revenue. Eventually, say beyond t_1 , the sum of revenue and available credit will be larger than $H(I^*_{r=i})$, although beyond t_1 , an investment rate higher than I^*_r is feasible it is not optimal. The firm will at t_1 stop increasing the level of investment setting it at the constant rate $I^*_{r=i}$.

From t_1 onwards, the firm has some lee-way in choosing the mixture of credit and revenue with which to finance its investment, with $(i) = (r)$ the firm would be indifferent among feasible mixtures. If we assume for definiteness that the firm prefers to finance investment entirely from its revenue and use credit only as needed, then, after t_1 the firm will be borrowing at a decreasing rate, finally at say, t_2 - see Figure II-1 - borrowing stops since it is possible to finance investment entirely from current revenue. Thus at t_2 the accumulated debt of the firm is the area in Figure II-1, from t_0 to t_2 between investment expenditures (continuous curve) and revenue (dotted curve) plus the initial stock of debt.

If, for say credit rating considerations, the firm starts paying back its debt as soon as its revenue exceeds its investment expenditures and continuously devoting all revenue in excess of investment expenditure to debt repayment, then until the area from t_2 , to say, t_3 between the revenue curve and the investment expenditures line equals accumulated debt repayment, inclusive of interest charges, will the firm start distributing dividends.

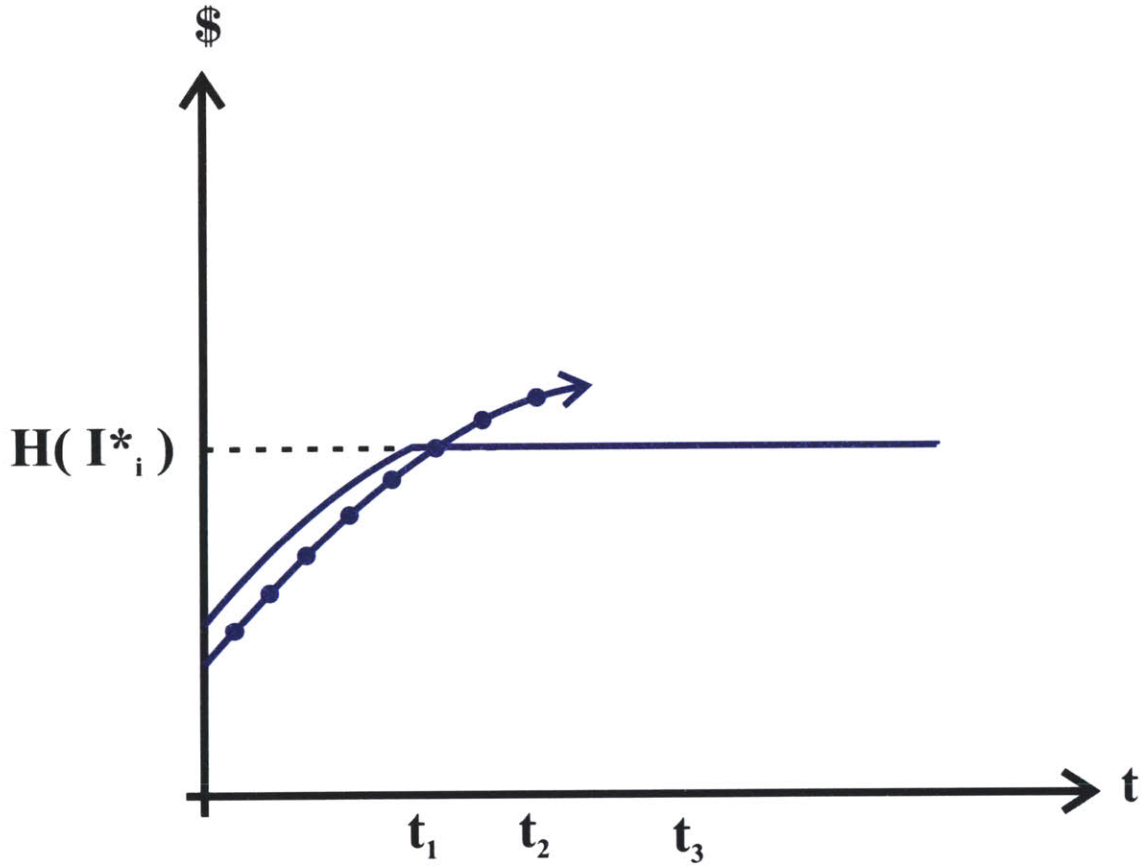


Figure II-1

To study the effects of the optimal investment policy above described on the evolution of the capital stock, we may portray such policy in the K, I , plane.

The $\dot{K} = 0$ locus in the K, I , space may be derived from the transition equation by setting $\dot{K} = 0$ to obtain,

$$I = \delta K$$

We draw this locus in Figure II-2. Any point above this locus would have $I > \delta K$ which implies $\dot{K} > 0$ since investment would be larger than what is necessary to make-up for depreciation. Similarly any point below the $\dot{K} = 0$ locus implies $\dot{K} < 0$. We denote this trend for K by placing arrowheads in the corresponding regions of the K, I , plane in Figure II-2.

The investment level which the firm would like to undertake if it could ignore the financial constraint is the solution of the unconstrained firm we obtained in Chapter I,

$$I^* = H'^{-1} \left[\frac{(p F_k)}{\delta + i} \right]$$

We mark this investment level with a broken line in Figure II-2

If at low initial K levels the firm does not have enough revenue to finance the optimal unconstrained investment level, the optimal policy of the firm is first, to borrow as much as possible and devote all of its revenue and credit to investment. Thus, with

$$D = aK$$

and

$$\dot{D} = a\dot{K}$$

The level of investment expenditures is

$$H(I) = pF(I) - wL + a\dot{K} - iaK$$

With constant returns to scale, the first two terms on the right hand side of this last equation equal $pF_k K$ and by substituting the transition law in the third term we have,

$$H(I) = pF_k K - a(I - \delta K) - iaK$$

Defining

$$E(I) \equiv H(I) - aI$$

We may write the one before last equation as

$$E(I) = [pF_k - a(\delta + i)] K$$

Since $H(I)$ is a convex and $H(I)$ is linear, $E(I)$ will also be convex, strictly monotonic and hence its inverse exist. Thus we may solve for I in the last equation to obtain,

$$I = E^{-1}\{ [pF_k - a(\delta + i)] K \}$$

Because $E(I)$ is convex, this last equation is a concave function as portrayed in Figure II-2.

Once the capital stock has grown to a level, say K_1 in Figure II-2, at which current revenue and credit suffice to finance the optimal unconstrained level of investment, the firm will use any further increase in revenue to distribute dividends and/or retire outstanding debt and

will continue with such level of investment indefinitely, hence the stock of capital will grow asymptotically in time towards K^∞ in Figure II-2.

This steady state level of K is the same as the one of the financially unconstrained firm, the only difference being that during the period when the financial constraint is effective the firm has a lower investment level and hence accumulates capital slower than the unconstrained firm.

Since if the firm begins with a $K > K^\infty$ the financial constraint would not be effective the optimal investment level would also be

$$I^* = H'^{-1} \left[\frac{(p F_k)}{r + \delta} \right]$$

and the capital stock would diminish asymptotically towards K^∞ . The optimal investment policy is denoted with the arrowed trajectory in Figure II-2.

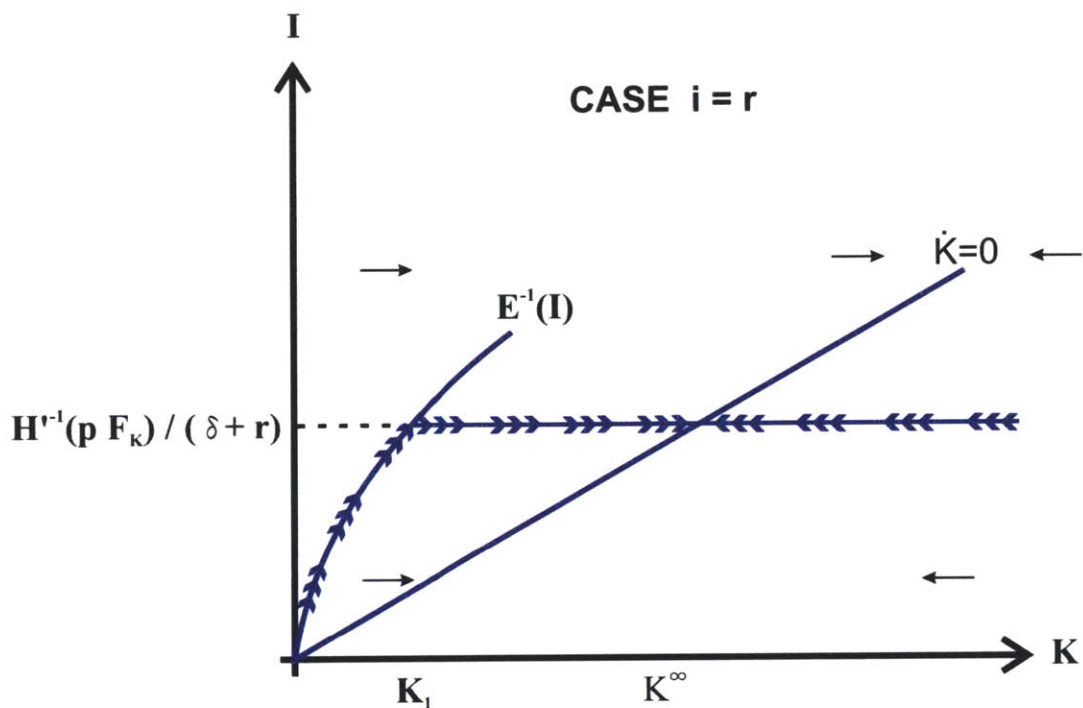


Figure II-2

II.-D.2 The Case $i > r$

If excluding the possibility of lending, the opportunity cost of funds of the shareholders is r such that $i > r$, the optimal investment policy is to first invest all the revenue and credit resources until at K_1 , in Figure III-3, when enough capital has been accumulated so that revenue and credit are sufficient to finance,

$$I_i^* = H'^{-1} \left[\frac{(p F_K)}{\delta + i} \right]$$

which is the optimal investment when the opportunity cost of funds is i . Since at investment levels higher than I_i^* the return on marginal investment expenditures would be lower than i , the firm would not be willing to finance such levels of investment. Rather, beyond K_1 the firm would use any revenue in excess of the expenditures necessary to finance I_i^* to first retire outstanding debt until say, at K_2 , all debt has been repaid. If as drawn in Figure II-3, $K_2 < K^\infty$ the firm would want to lend out any revenue in excess of $H(I_i^*)$ if it can do so at a rate i . The arrowed trajectory in Figure III-3 depicts the optimal investment policy when the firm can lend out funds and obtain an effective rate of return equal to i .

If for some reason the firm was not allowed to lend or due to risk and/or administrative cost the actual rate of return on lent out funds is lower than r , then, once all debt had been repaid the firm would rely only on the increases in revenue brought about by the growth of the capital stock to self-finance an increasing investment level until the latter is such that,

$$I = H'^{-1} \left[\frac{(p F_k)}{r + \delta} \right]$$

and the steady state capital stock would be higher than the K^∞ depicted in Figure III-3.

Similarly if the effective rate of return the firm could obtain on lent out funds is i' such that $i > i' > r$, then at K_2 , once the firm had repaid all its debt, it would use any revenue increase to self finance an increasing investment level until

$$I = H'^{-1} \left[\frac{(p F_k)}{i' + \delta} \right]$$

Once achieved this level the firm would lend out any excess of revenue over the expenditures necessary to finance this investment level.

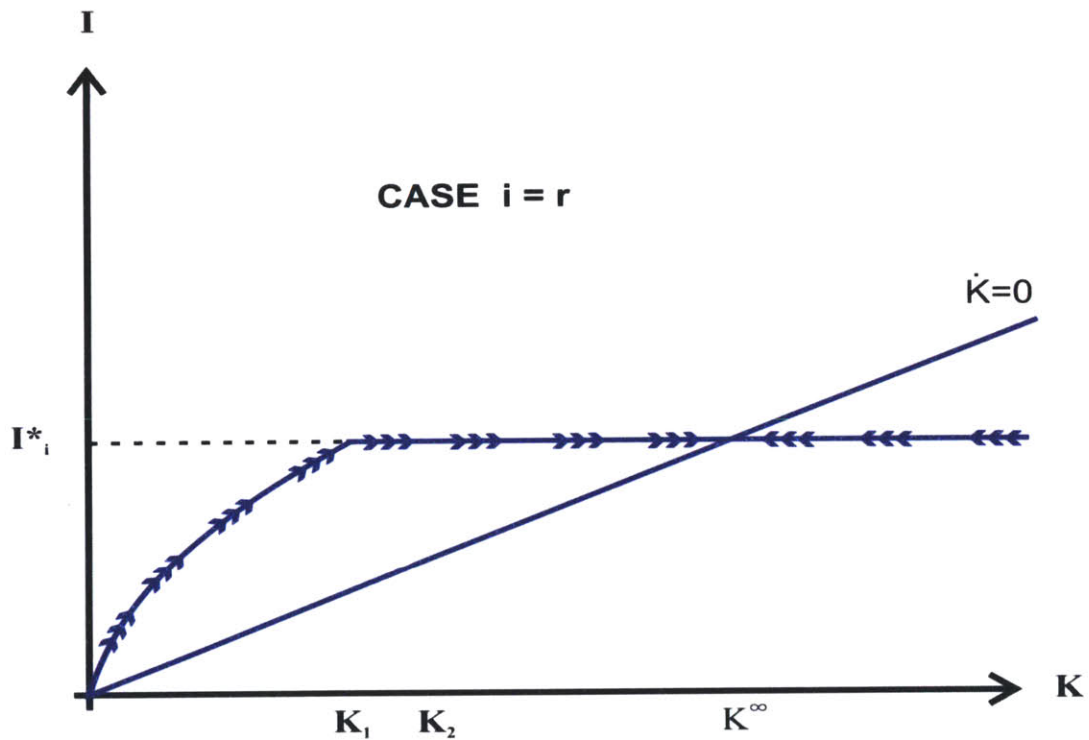


Figure II-3

II.-D.3 The Case $i < r$

When the interest rate on outstanding debt is less than the opportunity cost of funds to the shareholders, firms starting with a capital stock such that revenue and credit do not suffice to finance

$$I_r^* = H'^{-1} \left[\frac{(p F_k)}{r + \delta} \right]$$

would initially devote all of their revenue and credit resources to invest until the capital stock has grown to such level, say, K_1 , in Figure II-4, that current revenue and credit resources are sufficient to finance the optimal level I_r^* . Hereafter any increase in revenue would be used to distribute dividends to shareholders and in contrast with the case when $i > r$ the firm will not retire outstanding debt, rather it would increase indebtedness as the growth of the capital stock increases its credit ceiling. Thus after K_1 has been reached the amount of current net revenue devoted to investment expenditures diminishes as the new borrowings made possible by the growth of the capital stock are used to displace revenue as a source of finance. Again the arrowed trajectory in Figure II-4 depicts the optimal investment policy.

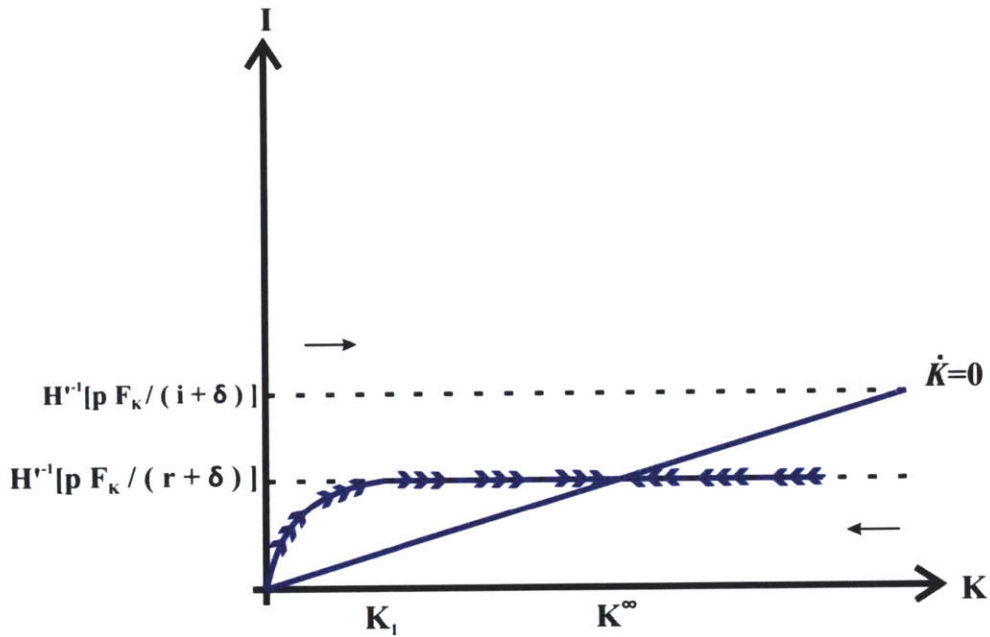


Figure II-4

Since with $i < r$ the firm always borrows as much as possible, the steady state level of indebtedness will be aK^{∞} . As in Chapter I, K^{∞} may be specified in terms of the parameters of the problem by substituting the optimal investment level

$$I_r^* = H'^{-1} \left[\frac{(p F_k)}{r + \delta} \right]$$

in the transition law, solving the resulting differential equation and finding the limit of this solution as t goes to infinity. Thus,

$$\lim_{t \rightarrow \infty} D = aK^{\infty} = \frac{a}{\delta} \left[\frac{(p F_k)}{r + \delta} \right]$$

II.-E. Concluding Remarks.

The discussion of the last two cases, that in which $i > r$ and that in which $i < r$, have suggested how in the presence of financial constraints, the firm's optimal strategy may be composed of several trajectories which in turn are optimal only for a limited range of values of the state variable. Furthermore, for the case $i = r$ we took the space to show that the path that connects such trajectories over the ranges for which they are optimal is in fact a global optimal policy, i.e. a policy that is optimal as a whole. For later use in the coming analysis we will state this result in the following paragraph.

Proposition:

If due to the presence of constraints one trajectory is optimal for a limited range of the state variable, then the path obtained by joining said trajectories over the range for which each individual trajectory is optimal yields a path that is optimal as a whole.

We showed this proposition to be true in the discussion of the case $i = r$. A general proof readily follows from Bellman's (1957) principle of optimality.

In this chapter we have analyzed the effects of capital market imperfections on the growth strategies of the firm for three purposes. First, to introduce our way of formalizing and dealing with financial constraints. Secondly, to study how the investment behavior of the

neoclassical firm as traditionally studied in the existent literature would be modified in the presence of financial constraints. Thirdly, although in the next chapter we question whether in the presence of financial constraints the firm is the appropriate economic agent for the study of investment behavior, the work presented in this chapter is meant to be a useful preliminary exercise that eases, both, the exposition of the techniques and the undertaking of the forthcoming work.

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III.- Intertemporal Production, Investment and Consumption by the Self Financed Agent.

III.-A. Introduction.

The problem studied in Chapter II is consistent with a scenario in which somehow the consumption and investment decisions of the shareholders are mutually independent. For instance if the firm is owned by a number of stockholders with diverse utility functions and/or wealth endowments and as a result, stockholders only agree to instruct management to maximize profits over the infinite horizon, or equivalently, to maximize the present value of the cash-flow.

Instead, let us now assume that the firm is owned by one or a few stockholders with similar utility functions and endowments. Under these circumstances, saving and investment decisions of the owner(s) are not mutually independent. Rather, the firm will be administered in such a way as to maximize the consumption utility of its shareholders. Again, we shall focus on the case of imperfect capital markets and assume that both the output and labor markets clear at the current price and wage respectively.

First we shall study the case in which credit is completely unavailable. Subsequently we will analyze how the results are modified when firms can borrow up to a constant proportion of their current capital stock.

III.-B. The Formal Problem.

Share-holders of the firm will,

$$Max = \int_0^{\infty} U(C) \exp(-r) t dt$$

Subject to:

The transition law:

$$\dot{K} = I(t) - \delta K(t)$$

The flow constraint on consumption:

$$C = pF() - wL - H(I)$$

The non-negative constraints on:

the instruments:

$$I(t) \geq 0 ; \quad L(t) \geq 0 ; \quad C(t) \geq 0$$

the state variable:

$$K(t) \geq 0$$

The initial conditions:

$$K(0) = K_0$$

III.-C. The Necessary and Sufficient Conditions.

If we assume Inada conditions for the utility function then it is natural to suppose that along the optimal path consumption will never be zero. Then, if output not invested is not storable, we would always want both, labor and capital to be positive. Although to keep a positive capital stock, investment need not be positive at all times, the presence of convex costs of adjusting the capital stock would strongly suggest, at least for the case in which the initial capital stock is below its steady state level, that along optimal paths, Investment is always positive. In any case, to ease the exposition we will assume that neither of the constraints

$$I(t) > 0; \quad L(t) > 0; \quad C(t) > 0; \quad K(t) > 0;$$

are binding along optimal trajectories. Should our optimal solution violate any of these constraints, we would have to modify our analysis accordingly.

The current-value Hamiltonian function for this problem can then be written as:

$$\mathcal{H} = U(C) + \lambda (I - \delta K) - \gamma [pF(K, L) - wL - H(I) - C]$$

From where we may derive the following set of necessary conditions:

$$-\dot{\lambda} + r\lambda = \lambda\delta - \gamma pF_k$$

$$\dot{K} = I(t) - \delta K(t)$$

$$C = pF(\cdot) - wL - H(I)$$

$$\lambda - \gamma H'(I) = 0$$

$$\gamma[pF_l - w] = 0$$

$$U'(C) - \gamma = 0$$

Replacing γ with the value given by the last equation and since under Inada conditions marginal utility is always positive, with rearrangements, we may rewrite the set of necessary conditions as:

$$(1 - a) \quad -\dot{\lambda} + (r + \delta)\lambda = U'(C)pF_k$$

$$(1 - b) \quad \dot{K} = I(t) - \delta K(t)$$

$$(1 - c) \quad C = pF(\cdot) - wL - H(I)$$

$$(1 - d) \quad \lambda = U'(C)H'(I)$$

$$(1 - e) \quad F_l = w/p$$

As in Chapter I, if we complement the above set of necessary conditions with, the following transversality conditions,

$$\lim_{t \rightarrow \infty} \exp(-r)t \lambda \geq 0 ; \quad \lim_{t \rightarrow \infty} (-r)t \lambda(t)K(t) = 0$$

The resulting set of equations is both, sufficient and necessary for optimality.

III.-D. Economic Interpretations of the Necessary Conditions.

Equation (1-a) may be rewritten as

$$U'(C)pF_k + \dot{\lambda} = (r + \delta) \lambda$$

and interpreted to state that the realized return to capital (left - hand side) which is the sum of the marginal revenue product of capital in utility units and the capital appreciation ($\dot{\lambda}$), is equal to the required return (right-hand side).

Equations (1-b) and (1-c) reproduce the transition law and the flow constraint on consumption respectively and do not require further comment.

Equation (1-d) states that the shadow price of capital equals the marginal cost of installed capital evaluated in utility units.

Finally equation (1-d) states that, again, labor is hired up to the point where its marginal product equals the real wage.

Now, since at $\dot{\lambda} = 0$

$$d^2\lambda/dK^2 = 1/(r + \delta) [U'(C)p F_{kkk} + U''(C)p F_k p F_{kk} + U'''(C)2pF_k p F_{kk} + U''''(C)pF_k(pF_k)^2]$$

We have that if, both, F_{kkk} and U''' were positive, the $\dot{\lambda} = 0$ locus would be a negatively sloped convex curve in the K, λ space, but actually since there is no reason why these third derivatives should be positive, the $\dot{\lambda} = 0$ locus while exhibiting a negative slope has an undetermined shape in regards to convexity. However, because of the also unrestricted possible shapes of the $\dot{K} = 0$ locus, as discussed below, it will be necessary to assume that the $\dot{\lambda} = 0$ locus is sufficiently convex in the neighborhood of the steady state in order to insure the uniqueness of the steady state equilibrium. Since we assumed Inada conditions for, both, the production and the utility functions, from equation (2) follows that the $\dot{\lambda} = 0$ locus is asymptotic to both the K and the λ axis.

We draw typical $\dot{\lambda} = 0$ loci in Figures III-1-a and III-1-b with shapes as characterized in the above discussion.

Next we obtain the $\dot{K} = 0$ locus. From equation (1-b) we have that $\dot{K} = 0$ when $I(t) = \delta K(t)$. Substituting this value of I in equation (1-d) we have

$$\lambda = U'(\cdot)H'(\delta K)$$

where the right-hand side is the marginal cost -in utility units- of installing an amount of capital sufficient to sustain a constant capital stock.

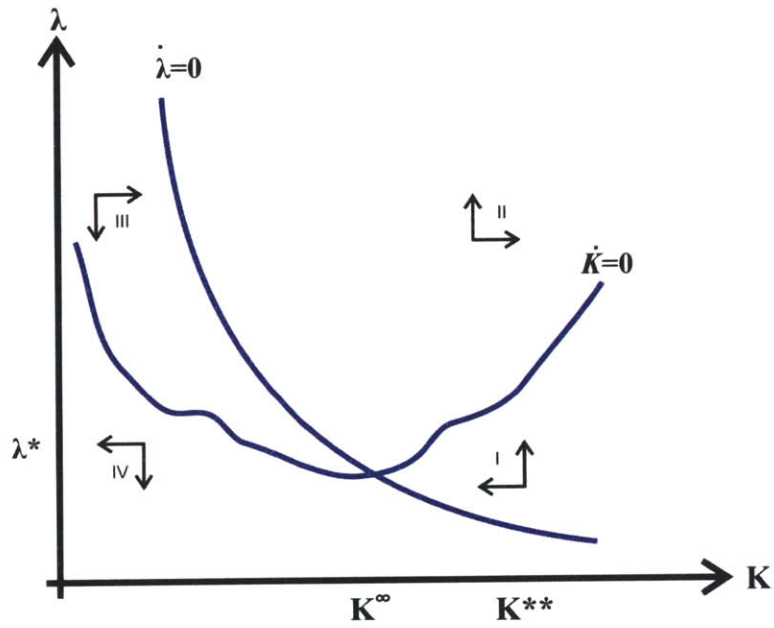


Figure III-1-a

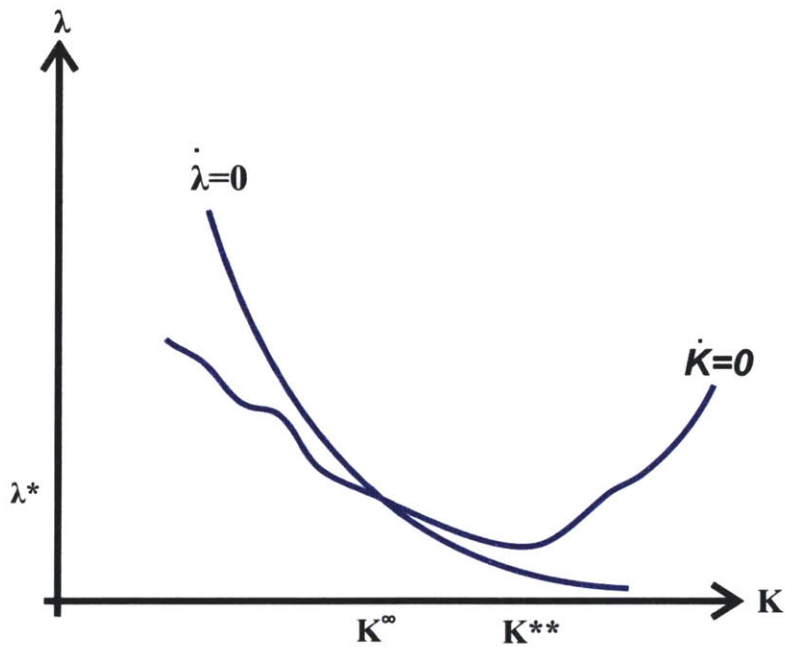


Figure III-1-b

To study the shape of the $\dot{K} = 0$ locus in the K, λ space we obtain from the last equation

$$(3) \quad \frac{d\lambda}{dK} = \frac{U' [] H'' \delta + U'' \frac{dC}{dK}}{dK} H'(\delta K) \quad \text{at} \quad \dot{K} = 0$$

in the right-hand-side, the algebraic signs of all but the second factor in the second term are known for all K , hence in order to determine the sign of $d\lambda / dK$ at $\dot{K} = 0$ we will examine said factor. To this end, consider that since along the $\dot{K} = 0$ locus we have $I = \delta K$ we may substitute this value of I in equation (1-c) to obtain

$$C = pF(K, L) - wL - H(\delta K)$$

or, assuming constant returns to scale,

$$C = pF_k K - H(\delta K)$$

As illustrated in Figures III-2 and III-3 the difference of the quasi-convex and the convex functions on the right-hand-side of this last equation results in a concave function so that initially C increases along with increases in K until the golden rule level of capital stock, denoted K^{**} , is reached. Any further increases in K beyond this level would cause a decrease in C .

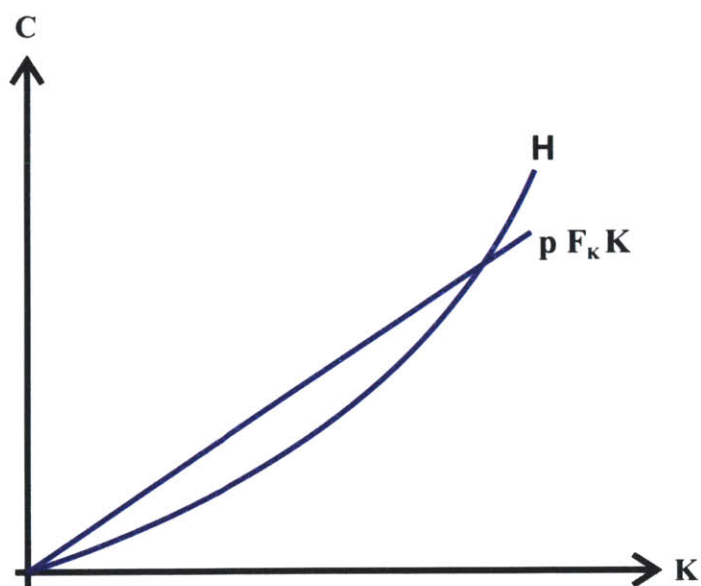


Figure III-2

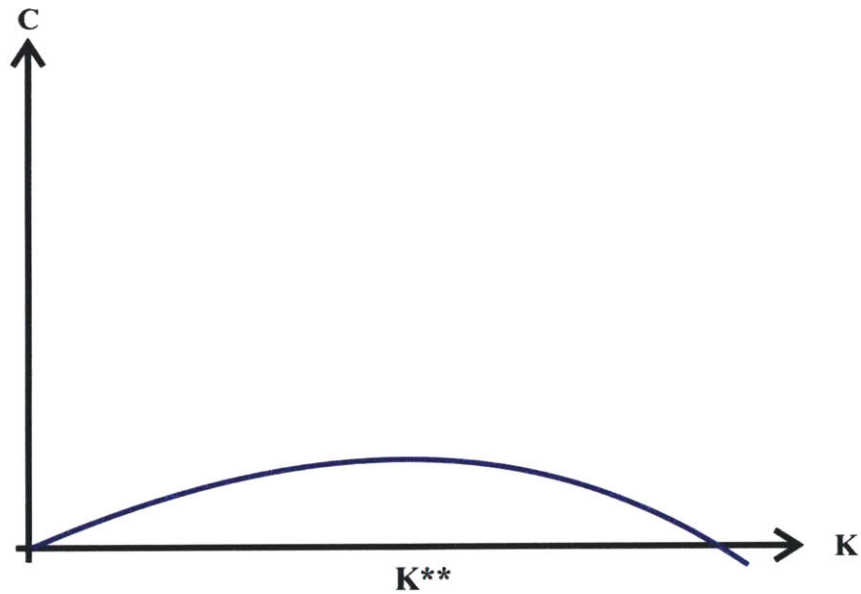


Figure III-3

Secondly, from the last equation we have that, at

$$\dot{K} = 0$$

$$(4) \quad \frac{dC}{dK} = pF_k - H'(\delta K)\delta$$

and, furthermore,

$$\frac{d^2C}{dK^2} = -H''(\delta K)\delta^2 \quad \text{at} \quad \dot{K} = 0$$

We read a negative algebraic sign on the right-hand-side and use this information to draw in Figure III-4 the shape of the one before last equation.

Since we have that at $\dot{K} = 0$

$$\frac{dC}{dK} = \begin{cases} > 0 & K < K^{**} \\ < 0 & K > K^{**} \end{cases}$$

from equation (3) we may now read that, at $\dot{K} = 0$

$$\frac{d\lambda}{dK} = \begin{cases} ? & K < K^{**} \\ > 0 & K \geq K^{**} \end{cases}$$

So that although the slope and shape of the $\dot{K} = 0$ locus is undetermined for $K < K^{**}$ we do know it has a positive slope for $K > K^{**}$. We draw typical $\dot{K} = 0$ loci in Figures III-1-a and III-1-b.

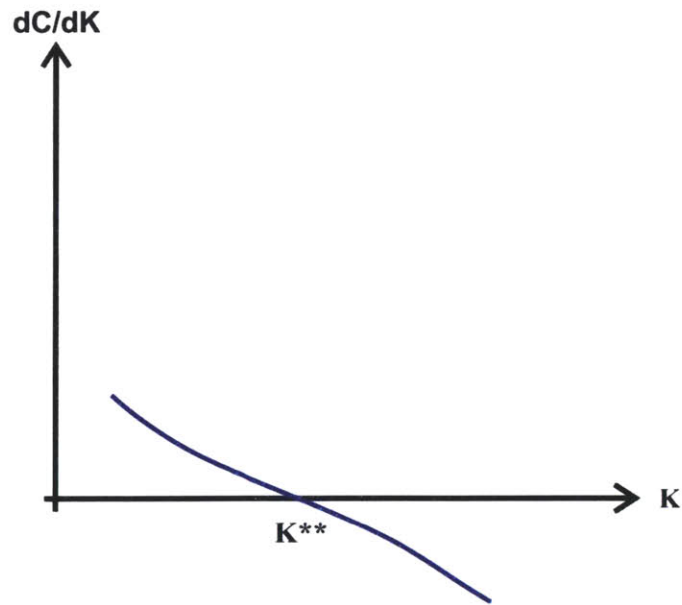


Figure III-4

The next two questions we investigate are the existence and uniqueness of a steady state equilibrium and the location of a “golden rule point” relative to the steady state.

For this purpose, consider the set of necessary conditions, if we substitute the values of λ given in equation (1-d) in equation (1-a) and set $\dot{\lambda} = 0$ in the latter equation we derive the following alternative equation for the $\dot{\lambda} = 0$ locus,

$$(r + \delta)H'(I) = pF_k$$

At the steady state we would have $I = \delta K^\infty$ substituting this value of I , the last equation becomes

$$(r + \delta)H'(\delta K^\infty) = pF_k$$

Solving for K^∞ from this last equation we obtain the unique K at which $\dot{\lambda}$ and \dot{K} are both equal to zero. Thus,

$$K^\infty = \frac{1}{\delta} H'^{-1}[pF_k/(r + \delta)]$$

Also observe that the one before last equation implies

$$pF_k - H'(\delta K)\delta > 0 \quad \text{at } K^\infty$$

or, in light of equation (4)

$$\frac{dC}{dK} > 0 \quad \text{at } K^\infty \quad \text{for } \dot{K} = 0$$

Using this result in equation (3) it follows that the sign of the slope of the $\dot{K} = 0$ locus at K^∞ is undetermined. However from the last inequality together with the information we have depicted in Figures III-2, III-3 and III-4 we have that since $\frac{dC}{dK}$ for $\dot{K} = 0$ can be positive only for $K < K^{**}$ it must be the case that, $K^\infty < K^{**}$

Thus it is clear, both that a "golden rule" point K^{**}, λ^{**} , can not exist for $K < K^\infty$ and that there must be one and only one for a $K > K^\infty$.

If the existence of a “golden rule” point had turned out to be possible to the left of K^∞ , then we would have had a way to distinguish between the case in which $K^{**} < K^\infty$ and the case $K^\infty < K^{**}$.

For the former case an optimal path would not exist and we would have had to restrict the analysis to the case $K^\infty < K^{**}$. As we have shown $K^\infty < K^{**}$ and since we will not assume in the forthcoming proposition neither that the minimum value of λ along the $\dot{K} = 0$ locus is either to the left or to the right of K^∞ , nor that the slope of said locus at K^∞ is either positive or non positive, there will be no need to distinguish between cases III.-1-a and III-1-b.

A brief digression is now in order. Earlier we mentioned that the problem under study may alternatively be interpreted as a neoclassical optimal growth model. Let us now view it in this fashion and note that it only differs from the latter type of models exemplified in the existent literature by, say Arrow and Kurz (1970; p. 64-81) in that it incorporates a convex cost of adjusting the capital stock. We then have that although the structure of the phase diagram in the model above presented is very similar to the one of the cited neoclassical optimal growth model it does differ in the important fact that the capital stock level corresponding to a golden rule point is always larger than the level of the steady state which implies that there will always exist an optimal path because the possibility that $K^{**} < K^\infty$, which is present in optimal growth models that do not include convex cost of adjusting K has been shown to disappear in the present formulation. See Arrow and

Kurz (op. cit; p. 67-70) for a description of the problem that arises when $K^{**} < K^{\infty}$.

III.-F. Typology of Paths in the K, λ Space.

To analyze the dynamics in the phase diagram, let's first solve for I from equation (1-d) to obtain

$$I = H'^{-1} \left(\frac{\lambda}{U'} \right)$$

and substitute this expression for I in equation (1-b) where upon the transition law may be rewritten as

$$\dot{K} = H'^{-1} \left(\frac{\lambda}{U'} \right) - \delta K(t)$$

Now, consider the phase diagram in Figure III-1 (a or b). Since $[H'^{-1}]' > 0$ the last equation implies that for a given K any point above the $\dot{K} = 0$ locus would have a larger λ and hence $\dot{K} > 0$. Similarly, $\dot{K} < 0$ for any points below such locus.

For any point to the right of the $\dot{\lambda} = 0$ locus there is a point on the $\dot{\lambda} = 0$ locus with the same $\bar{\lambda}$ and a smaller K for which $\dot{\lambda} = 0$. Since from equation (1-a) we have that

$$\frac{d\lambda}{dK} = -[U'(C)p F_{kk} + U''(C)(pF_k)^2] \text{ at } \lambda = \bar{\lambda}$$

where we read a positive sign, then, any point to the right of the $\dot{\lambda} = 0$ locus implies $\dot{\lambda} > 0$. By a similar argument any point to the left of the locus $\dot{\lambda} = 0$ implies $\dot{\lambda} < 0$.

The $\dot{\lambda} = 0$ and the $\dot{K} = 0$ loci divide the phase plane in four quadrants (“Q.”) which have been numbered in Figure IV-1 counter-clockwise. In each quadrant the direction of motion is the resultant of the small arrows given by the arguments in the last two paragraphs.

From the phase diagram it is intuitively clear that λ^∞, K^∞ ; the steady state, is a “saddle-point” which can only be approached by the paths beginning either on Q.III or Q.I. But furthermore, for a given initial K not any path beginning at an arbitrary $\lambda(0)$ in Q. III or Q.I will approach λ^∞, K^∞ . Hence we state the following:

PROPOSITION III.-1:

For any given initial $K(0)$ there exists a determined $\lambda(0)$ such that the solution of the system:

$$(1 - a) - \dot{\lambda} + (r + \delta) \lambda = U'(C)pF_k$$

$$(1 - b) \quad \dot{K} = I(t) - \delta K(t)$$

$$(1 - c) \quad C = pF(K) - wL - H(I)$$

$$(1 - d) \quad \lambda = U'(C)H'(I)$$

$$(1 - e) \quad F_L = w/p$$

with those initial values converges to the long run equilibrium values λ^∞ , K^∞ . If any other initial value $\lambda(0)$ were chosen, the resulting solution would either diverge to $+\infty$ in both variables and clearly be non-optimal or it would become non-feasible with the capital stock becoming zero in finite time.

III.-G. The Phase Diagram in the λ, I Space.

From the transition law -equation (1-b) - the $\dot{K} = 0$ locus in K, I is the positively sloped line given by

$$(8) \quad I = \delta K$$

In order to derive an equation for the $\dot{\lambda} = 0$ locus in K, I we first take the time derivative of both sides of equation (1—d) to obtain,

$$\dot{\lambda} = U'' p F_K \dot{K} H' + U'' (p F_L - w) \dot{L} H' + U'' (-H') \dot{I} H' + U' H'' \dot{I}$$

Secondly, substituting in the left-hand side of the last equation the value for $\dot{\lambda}$ given by equation (1-a) we have,

$$(r + \delta)\lambda - U'(C)pF_k = U''pF_k\dot{K}H' + U''(pF_l - w)\dot{L}H' + U''(-H')\dot{I}H' + U'H'\dot{I}$$

Thirdly, substituting equations (1-d) and (1-e) we have after collecting terms, transposing and multiplying by (-1),

$$(9) \quad [-U''H'^2 + U'H'']\dot{I} = U'[(r + \delta)H' - pF_k] - U''pF_kH'\dot{K}$$

Finally, setting $\dot{I} = 0$ we obtain the following equation for the $\dot{I} = 0$ locus in the K, I space.

$$(10) \quad U'[(r + \delta)H' - pF_k] - U''pF_kH'(I - \delta K) = 0$$

To study the position of this locus in the K, I space we could differentiate totally with respect to the latter set of variables. As this produces only ambiguous results, we examine each term in the last equation in order to determine an area within the K, I space where the $\dot{I} = 0$ locus may be located.

Since U' is always positive, the first left-hand side term in the last equation will equal zero whenever,

$$(11) \quad (r + \delta)H' = pF_k$$

By totally differentiating this last equation with respect to both K and I we have

$$\frac{dI}{dK} = p F_{kk} / [(r + \delta) H''] \quad \text{at} \quad (r + \delta) H' = p F_k$$

Assuming constant returns to scale F_{kk} equals zero hence we have that, in the K, I space, the locus along which (11) holds is a horizontal line as drawn in Figure III-5. Any point above $(r + \delta) H' = p F_k$ locus implies

$$(r + \delta) H' - p F_k > 0$$

and similarly any point below said locus implies

$$(r + \delta) H' - p F_k < 0$$

We have placed in Figure III-5 algebraic signs to denote the regions where each of the last two inequalities holds.

Consider now the second left hand side term in (10). Since $U'' < 0$ and assuming $H' > 0$ for all $I \geq 0$ it follows that

$$U'' p F_k H' (I - \delta K) \geq 0 \quad \text{if} \quad \dot{K} = I(t) - \delta K(t) \geq 0$$

and

$$U'' p F_k H' (I - \delta K) \leq 0 \quad \text{if} \quad \dot{K} = I(t) - \delta K(t) \leq 0$$

In Figure III-6 we have placed algebraic signs to denote the regions where these last inequalities hold.

Finally, using the loci $(r + \delta) H' - pF_k = 0$ and $\dot{K} = 0$ we may divide the K, I space in four quadrants and number them counterclockwise with roman numerals in Figure III-7. With the information depicted in Figure III-5 and III-6 we read from equation (10) that the $\dot{I} = 0$ locus may lay only in quadrants I and III. We draw in Figure III-7 a typical $\dot{I} = 0$ locus that lies in said quadrants.

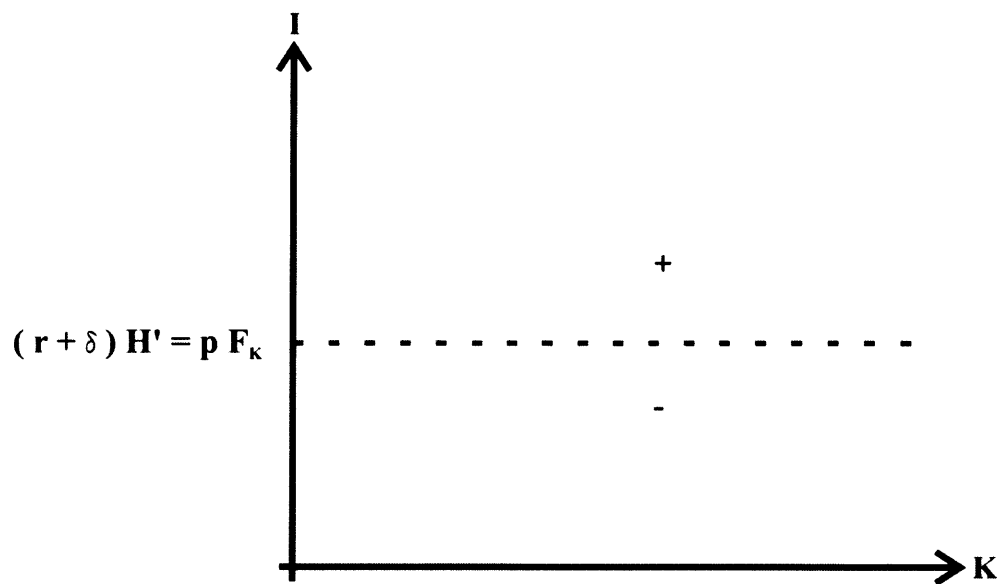


Figure III-5

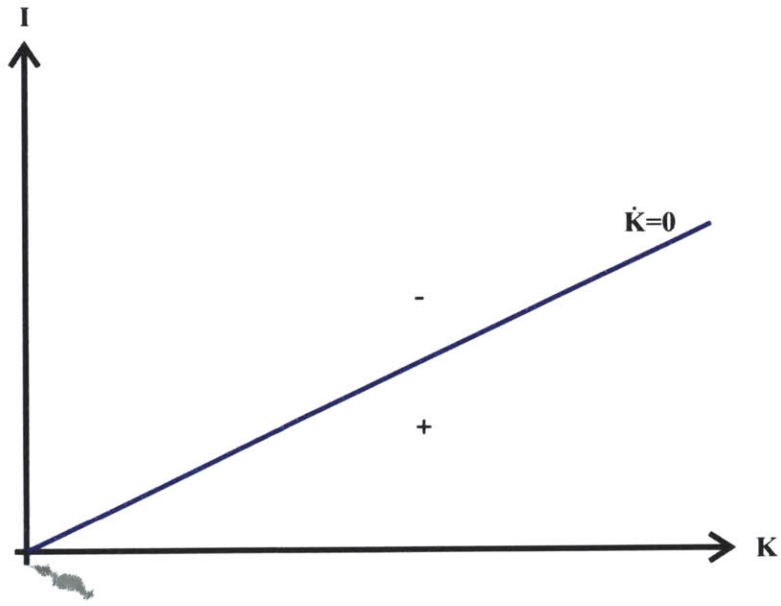


Figure III-6

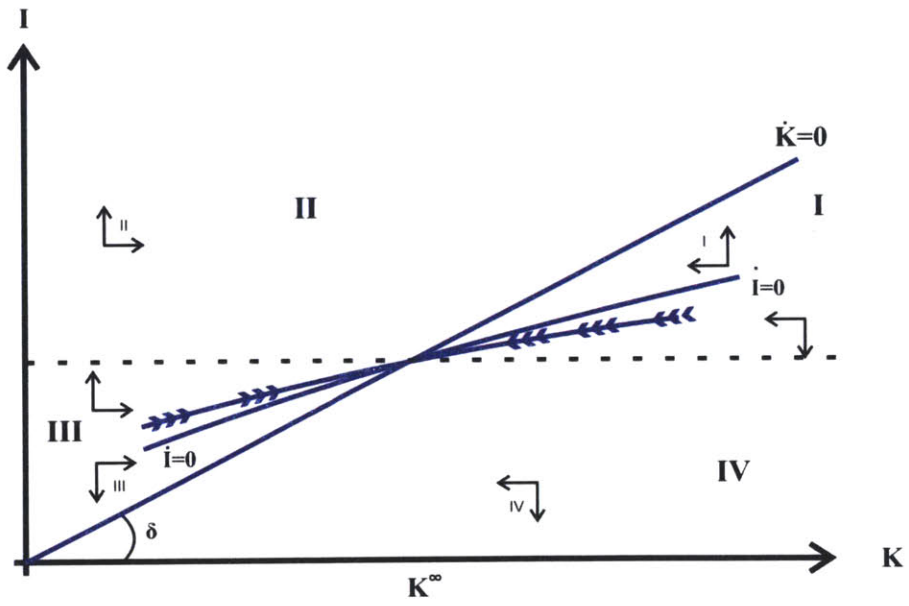


Figure III-7

III.-H. Typology of Paths in the K, I Space.

Consider the $\dot{K} = 0$ locus given by $I = \delta K$. For any point in the K, I space, above such locus we would have $I > \delta K$. Since for said points I is greater than necessary to have $\dot{K} = 0$, it follows that $\dot{K} > 0$ at any point above the $\dot{K} = 0$ locus. Similarly $\dot{K} < 0$ for any point below the $\dot{K} = 0$ locus. To depict this we place arrow heads in the different quadrants in Figure III-7.

The previous section argues that $\dot{I} = 0$ was possible only in Q.I and Q.III, the algebraic sign of \dot{I} in Q.II and IV where \dot{I} cannot be zero may be established by first noting that in the left hand side of equation (9) the bracketed factor is always positive hence the algebraic sign of \dot{I} will be determined by the right-hand side of said equation. Using the information depicted in Figures III -5 and III-6 and equation (9) we establish the sign of $\dot{I} = 0$ in quadrants II and IV and draw the corresponding arrowheads in Figure III-7.

A precise location for the $\dot{I} = 0$ locus can not be asserted in general, however any particular $\dot{I} = 0$ locus would itself divide each of quadrants I and III in two regions, one above the $\dot{I} = 0$ locus denoted Q.I-a and Q.III-a and one below said locus denoted Q.I-b and Q.III-b . From equation (9) and the information in Figures III-5 and III-6 we may determine that $\dot{I} > 0$ in Q.I-a and Q.III-a and that $\dot{I} < 0$ in Q.I.-b and Q.III.-b. We place arrow heads accordingly in Figure III-7 .

With this information it is intuitively clear from the phase diagram that only paths beginning in quadrant I-b and quadrant III-a could possibly converge to, K^∞, I^∞ . We draw these paths with arrowed curves in Figure III-7. Note the saddle point nature of K^∞, I^∞ .

We could now assert a proposition similar to Proposition III-1 with λ replaced by I .

Although all paths of the system of equation (1-a through e) satisfy the necessary conditions, only the two paths that converge to K^∞, I^∞ would satisfy the transversality conditions and thus qualify as optimal paths.

Hence we may state the following proposition:

Proposition III-2

If $r > 0$ there exists an optimal policy characterized by a particular solution of the system:

$$[-U''H'^2 + U'H''] \dot{I} = U' [(r + \delta) H' - pF_k] - U''pF_k H' \dot{K}$$

$$\dot{K} = I(t) - \delta K(t)$$

For any given $K(0)$ there is only one $I(0)$ for which a solution to the above system with such initial values will converge to a limit and has a positive capital stock everywhere. This solution defines the optimal path for $K(t)$ and $C(t)$.

The limit of $I(t)$ along the optimal Pontryagin path is,

$$I^* = H'^{-1}[pF_K/(r + \delta)]$$

Since in the steady state,

$$I^* = \delta K^\infty$$

the limit for the capital stock is

$$K^\infty = \left(\frac{1}{\delta}\right)H'^{-1}[pF_K/(r + \delta)]$$

Using the above limits for K and I , from the consumption flow constraint we obtain the consumption limit

$$C^\infty = pF(K^\infty, L) - wL - H(I^*)$$

Since constant returns to scale have been assumed, also

$$C^\infty = pF_k K^\infty - H(I^*)$$

III.-I. The Neoclassical Phase Diagram in the K, I Space.

We now flash-back to Chapter I and construct a phase diagram for the neoclassical investment analysis so that we may more easily compare the neoclassical investment solution path with the self financed house-hold solution as analyzed earlier in this chapter.

Consider the set of necessary conditions in section I-D for convenience reproduced below

$$\dot{\lambda} = (\delta + r)\lambda - pF_K(K, L)$$

$$(12) \quad H'(I) = \lambda$$

$$pF_L = w$$

$$\dot{K} = I - \delta K$$

In the above differential equation for λ , using equation (12) we first substitute out λ . Then, by taking the time derivative of both side of equation (12) we obtain

$$\dot{\lambda} = H''(I) \dot{I}$$

and now use this to substitute out $\dot{\lambda}$ in said differential equation for λ to obtain

$$H''(I) \dot{I} = (\delta + r)H'(I) - pF_K$$

From where we may obtain the following equation of the $\dot{I} = 0$ locus in the K, I space,

$$H'(I) = \frac{pF_K}{(\delta + r)}$$

or,

$$(13) \quad I = H'^{-1} \left[\frac{pF_K}{(\delta + r)} \right]$$

We draw this $\dot{I} = 0$ locus with a broken line in the K, I space in Figure III-8 where we have also drawn the $\dot{K} = 0$ locus, the equation of which, is again, obtained from the transition law.

Since equation (13) is also the investment solution path for the constant returns to scale case we obtained in section I-E in Chapter I, we may denote the neoclassical solution path with arrow heads along the $\dot{I} = 0$ locus in Figure III-8

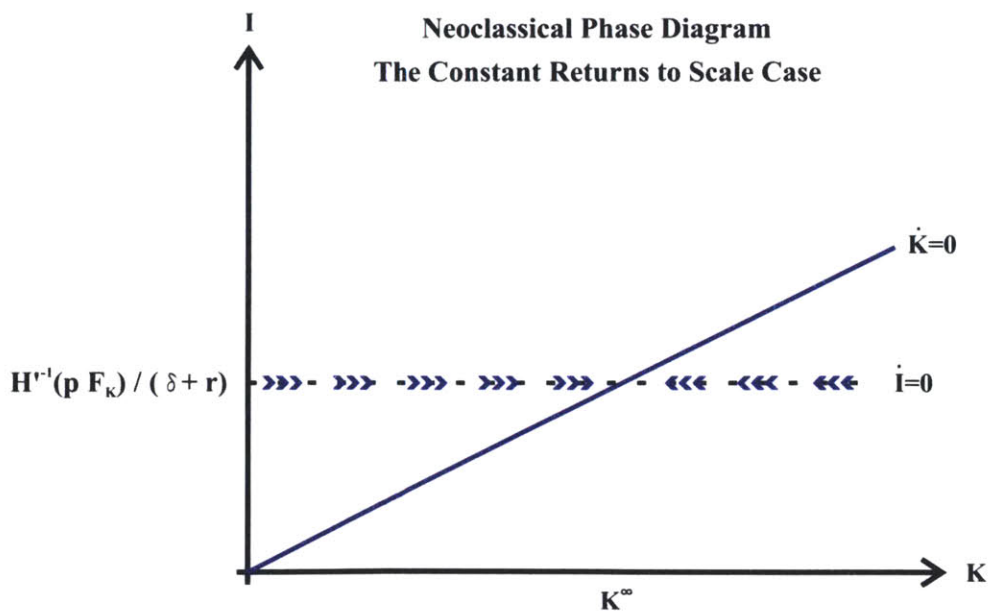


Figure III-8

III.-J. Comparison between the Neoclassical Firm and the Self-Financed Capitalist.

The utility discount rate of the house-hold is the time preference rate. While the cash-flow discount rate of the neoclassical firm in Chapter I is the opportunity cost of funds of the share-holders. However, if the firm is immersed in a “well-behaved” neoclassical economy where the rates on financial instruments and other investment opportunities have been equated with the rate of time preference, then, both discount rates are the same. Thus we may proceed to compare our previous results.

In Figure III-9 we omit the $\dot{I} = 0$ in order not to clutter the phase diagram and we draw in the K, I space the solution paths, when returns to scale are constant, obtained in the preceding sections. The broken arrowed lines are the solution paths for the neoclassical firm while the continuous arrowed curves are the self-financed house-hold solution paths.

Let us first compare paths beginning with $K(0) < K^\infty$. Since the house-hold lacks any financial instruments, both, C and I are financed entirely from current revenue which in turn is proportional to K , hence the lower the $K(0)$, the lower the feasible $C(0)$ levels, and thus the higher the opportunity cost of investment funds to the house-hold. This results in a lower $I(0)$, the lower the initial K . As the house-hold accumulates capital, revenue grows and, both, higher I and higher C become feasible, the latter in turn decrease the house-hold opportunity cost of investment funds, allowing the house-hold to reach levels of investment closer to those of the

neoclassical firm, which, unlike the house-hold, faces a constant opportunity cost of funds (r). However the convexity of H makes higher I increasingly more expensive, furthermore, as K grows, more I is needed just to make-up for depreciation. These two effects combine to dampen capital accumulation so that the house-hold approaches I^* and K^∞ asymptotically in time.

Both, the firm and the house-hold approach asymptotically the same level of capital stock, K^∞ . However since the firm is always investing more than the house-hold, the firm approaches K^∞ faster.

For paths beginning with $K(0) > K^\infty$ we have that the larger the initial K is, the larger will the house-hold's revenue be and so will the C it can afford and hence the lower the opportunity cost of investment funds to the house-hold.

Thus the excess of the house-hold level of I over that of the neoclassical firm with constant opportunity cost of funds, will be larger the larger $K(0)$ is.

Since paths to the right of K^∞ have $I < \delta K$, K is diminishing and so in revenue. Thus diminishing the level of affordable C which by increasing the opportunity cost of investment funds reduce I , bringing it closer to the level of the neoclassical firm.

As shown in Figure III-9 for any initial capital stock larger than K^∞ , both, the house-hold and the neoclassical firm approach the same steady state capital stock. Because the level of I of the firm is always

below that of the house-hold the firm approaches such K^∞ faster than the house-hold.

If for any $K(0) > K^\infty$ the house-hold were to set its investment level at the level of the neoclassical firm, the lack of any financial instrument to allow savings would force a temporary consumption glut caused by the depletion of its “excess capital stock”, $K(0) - K^\infty$, over a time horizon shorter then optimal. Precisely, to avoid this, the house-hold chooses an $I(t)$ path above that of the firm.

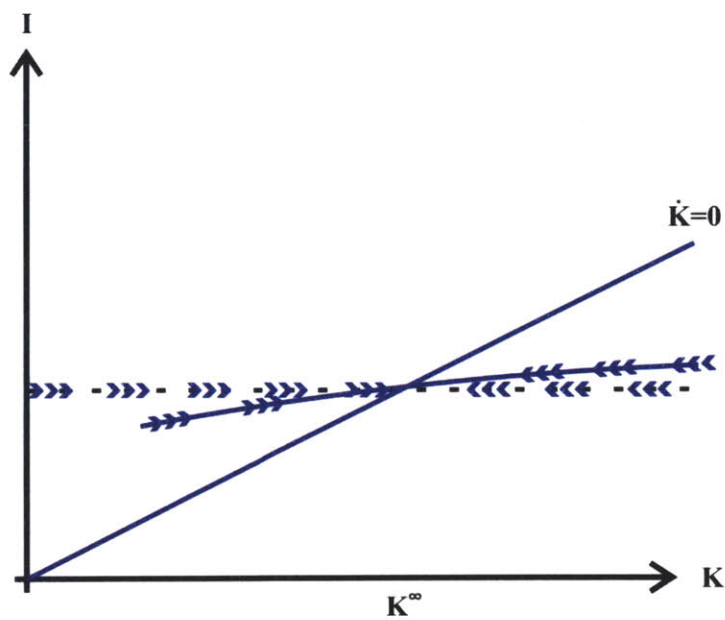


Figure III-9

III.-K. Treasury Bills and the Self-Financed Capitalist.

Now suppose that the government issues a treasury bill with a rate of return (s). How would this modify the behavior of the self-financed house-hold?

First assume $s = r$

For paths with $K(0) < K^\infty$, the house-hold solution path for $I(t)$, above obtained, never reached the level of I of the neoclassical firm with constant returns to scale and constant opportunity cost of funds (r). Since the house-hold approached such I level asymptotically from below its rate of return on physical capital is always larger than (r).

Thus for $K(0) < K^\infty$ and $s = r$, the treasury bill is never a superior investment alternative and the investment solution path will remain the same than prior to the introduction of the treasury bill since the house-hold needs to borrow not to lend. For $K(0) > K^\infty$ the previously obtained investment solution path for the self financed house-hold had a higher investment level than those profitable for the neo-classical firm with $s = r$. Thus the house-hold marginal return on investment was always less than s .

The introduction of a treasury bill with a rates $s = r$ by providing a saving instrument with a rate of return higher than along the house-hold previous solution path, allows the house-hold to separate consumption and investment decisions.

Hence upon the introduction of a treasury bill with $s = r$, the house-hold will reduce investment on physical capital to the level optimal for the neoclassical firm with $s = r$, denote such investment level I_s^* , there it will obtain a return equal to that of the treasury bill.

To determine the optimal consumption path, the house-hold will solve a separate problem where it will take into account the stream of revenue generated by pursuing a constant level of investment in physical capital as above described.

The flow demand for treasury bills will equal the excess of current earnings over the sum of current consumption and current investment expenditures in physical capital, $H(I_s^*)$.

In Figure III-10 we draw the house-hold solution path for investment when there exists a treasury bill with $s = r$.

The results for the case $s > r$ and $s < r$ readily follows from the above analysis.

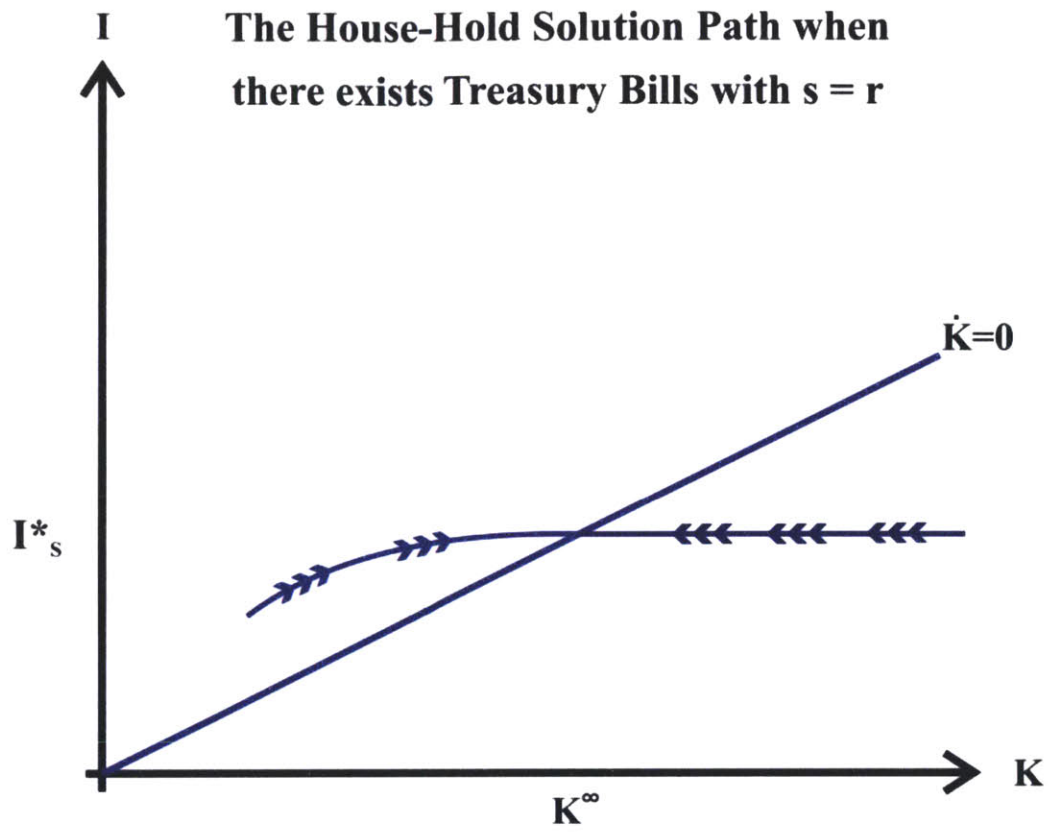


Figure III-10

IV. - Intertemporal Production, Investment and Consumption by the Capitalist in Imperfect Capital Markets.

IV.-A Introduction.

Now allowing the existence of financial intermediaries we extend the analysis of the preceding chapter and derive the output, investment and consumption paths when a house-hold may borrow for investment purposes, up to a constant proportion, denoted “a” of the value of its current capital stock.

Letting “D” denote outstanding debt the credit constraint is

$$D(t) \leq a q K(t)$$

where $q \equiv$ *price of equipment* .

Taking account of borrowings and interest payments, the flow constraint on consumption now is

$$C = pF() - wL - H(I) + \dot{D} - i D$$

and hence, formally the problem of the house-hold may be stated in the following way.

IV.-B. The Formal Problem.

$$Max = \int_0^{\infty} U(C) \exp(-r) t dt$$

Subject to:

The transition law:

$$\dot{K} = I(t) - \delta K(t)$$

The flow constraint on consumption:

$$C = pF(\cdot) - wL - H(I) + \dot{D} - iD$$

The credit constraint:

$$D(t) \leq a q K(t)$$

The non-negative constraints on the controls:

$$I(t) \geq 0; \quad L(t) \geq 0; \quad C(t) \geq 0$$

The non-negative constraints on the state variables:

$$K(t) \geq 0; \quad D \geq 0$$

The initial conditions:

$$K(0) = K_0 \quad ; \quad D(0) = D_0$$

IV.-C. The Necessary and Sufficient Conditions.

In the above house-hold problem both sources of finance, revenue and credit, are an increasing function of the capital stock level, hence it is natural to conjecture that if the credit constraint is ever binding along the optimal path, it will be at “low” capital stock levels since it is for these levels that credit is more attractive because the opportunity cost of investment funds in utility terms is “high”. Also, at “low” capital stock levels, as illustrated in the preceding Chapter, the optimal investment level is “low” and this in turn implies that only “moderate” expenditures are needed to achieve marginal additions to installed capacity.

We will first solve the above problem assuming that the house-hold must always borrow as much as possible so that the credit constraint is met with equality at all times and then determine the period when it would be optimal to do so. Only for this period will the solution derived in this manner be optimal.

Paths for periods when it is no optimal to borrow as much as possible will then have to be studied.

Thus, when the credit constraint is binding we have

$$D(t) = a q K(t)$$

and consequently, also

$$\dot{D}(t) = a q \dot{K}(t)$$

Substituting these last two equations in the consumption flow equation we have

$$C = pF(K, L) - wL - H(I) + a q \dot{K}(t) - i a q K(t)$$

As in Chapter III we assume that all of the non-negative constraints on, both, the controls and the state variables are never binding along the optimal trajectories and since the credit constraint is already embodied in the consumption flow constraint, the current value Hamiltonian for this problem is:

$$H = U(C) + \lambda(I - \delta K) + \gamma[pF(K, L) - wL - H(I) + aq(I - \delta K) - i a q K - C]$$

where we have used the transition law to substitute out \dot{K} in the consumption flow constraint stated in the one before last equation.

From the preceding Hamiltonian we may obtain the following set of necessary conditions:

$$-\dot{\lambda} + r\lambda = -\lambda\delta + \gamma p F_k - \gamma a q \delta - \gamma i a q$$

$$\dot{K} = I(t) - \delta K(t)$$

$$C = pF(\cdot) - wL - H(I) + aq(I - \delta K) - iaqK$$

$$0 = \lambda - \gamma H'(I) + \gamma aq$$

$$0 = \gamma[pF_l - w]$$

$$0 = U'(C) - \gamma$$

Replacing γ with the value given in the last equation, the preceding set of necessary conditions may be rearranged as,

$$(1 - a) \quad -\dot{\lambda} + (r + \delta)\lambda = U'(C)[pF_k - aq(\delta + i)]$$

$$(1 - b) \quad \dot{K} = I(t) - \delta K(t)$$

$$(1 - c) \quad C = pF(\cdot) - wL - H(I) + aq(I - \delta K) - iaqK$$

$$(1 - d) \quad \lambda = U'(C)[H'(I) - aq]$$

$$(1 - e) \quad F_l = w/p$$

Again, complementing the above set of necessary conditions with the following transversality conditions

$$\lim_{t \rightarrow \infty} \exp(-rt)\lambda \geq 0; \quad \lim_{t \rightarrow \infty} (-r)t\lambda(t)K(t) = 0$$

we now have a set of equations that is both, necessary and sufficient for optimality.

IV.-D. The Phase Diagram in the K, I Space.

An advantage of constructing a phase diagram in the K, λ space is that since the direction of movement of both K and λ may be unambiguously determined for all points in the K, λ space, the saddle point nature of the critical points K^∞, λ^∞ is more clearly seen.

However, from the set of necessary conditions in section IV.-C it is clear that such diagram would be very similar to that in Figure III -1, thus, we sidestep the analysis of paths in such diagram taking for granted the saddle point nature of the steady state K^∞, λ^∞ and continue the analysis in the K, I space where other aspects are more easily perceived.

Since the algebraic developments in this section parallel those of section III.-G, the exposition will be less detailed.

From equation (1-b) we obtain the following

$\dot{K} = 0$ locus

$$(2) \quad I(t) = \delta K$$

which we draw in Figure IV-3.

To obtain the $\dot{I} = 0$ locus first we take the time derivative of both sides in equation (1-d), replace $\dot{\lambda}$ in equation (1-a), where we also use equation (1-d) to replace λ and thus obtain,

$$(3) \quad [-U(H' - aq)^2 + U'H] \dot{I} = U' [(r + \delta)(H' - aq) - (pF_k - aq\delta - aqi)] - U''[pF_k - aq(\delta + i)](H' - aq)\dot{K}$$

Setting $\dot{I} = 0$ in the last equation we obtain the $\dot{I} = 0$ locus in the K, I space,

$$(4) \quad U'[(r + \delta)(H' - aq) - (pF_k - aq(\delta + i))] - U''[pF_k - aq(\delta + i)](H' - aq)\dot{K} = 0$$

As in section III.-G, to determine the areas of the K, I space where the $\dot{I} = 0$, is situated we examine each of both terms in the left hand side of the last equation. By determining the areas of the K, I space in which each term is positive, zero or negative we may determine the sub regions of the K, I space where equation (4) may possibly hold. The $\dot{I} = 0$ locus would extend along such sub-regions of the K, I space.

Since U' is always positive the first left-hand side term in equation (4) will equal zero whenever,

$$[(r + \delta)(H' - aq)] = pF_k - aq(\delta + i)$$

We solve out for I in the last equation by first transposing,

$$(r + \delta)H'(I) = pF_k + aq(r - i)$$

Then, dividing by $(r + \delta)$ and inverting H'

$$(5) \quad I = H'^{-1} [pF_k + aq(r - i)] / (r + \delta)$$

Since with constant returns $pF_{kk} = 0$, equation (5) will be a horizontal line in the K, I space which we represent by the dotted line in Figure IV-1. The first left-hand side term in equation (4) will be positive for all points above such line and negative for all points below.

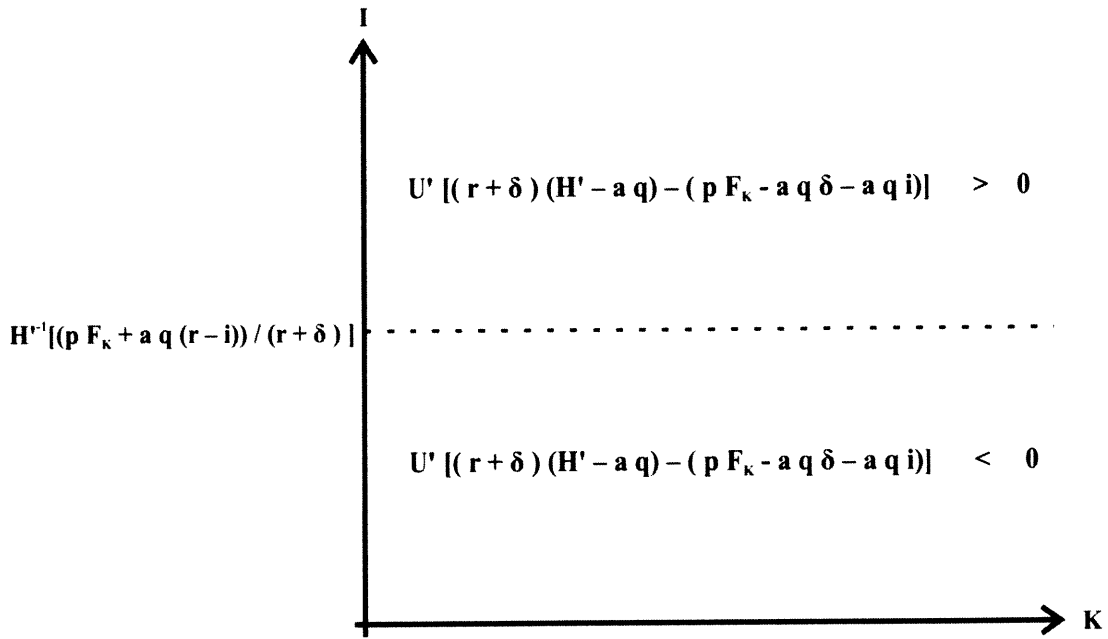


Figure IV-1

Consider now the second left-hand side term in equation (4). We now direct our attention to the algebraic sign of each factor in said term.

By assumption,

$$U'' \leq 0$$

We will assume the value of the parameters are such that,

$$pF_k - aq(\delta + i) > 0$$

for otherwise the problem under consideration would not be very interesting.

Considering the third factor in the second left-hand side term of equation (4), we note that by definition H is the cost of purchase and installation of equipment i.e.,

$$H(I) = G(I) + qI$$

where $G(I)$ is the equipment installation cost function.

From the last equation we obtain,

$$H'(I) = G'(I) + q$$

With $H'(I) > 0$ for all $I \geq 0$ we have

$$H'(I) > q$$

and since by assumption $a < 1$, we must have,

$$H'(I) > aq, \text{ for all } I$$

From the last few paragraphs it follows that,

$$U''[pF_k - aq(\delta + i)][H'(I) - aq]\dot{K} \geq 0 \text{ if } \dot{K} \leq 0$$

and

$$U''[pF_k - aq(\delta + i)][H'(I) - aq]\dot{K} \leq 0 \text{ if } \dot{K} \geq 0$$

In Figure IV-2 we draw the locus $\dot{K} = 0$, given by equation (2) and denote the region of the K, I space in which the second left-hand side term in equation (4) is either positive or negative.

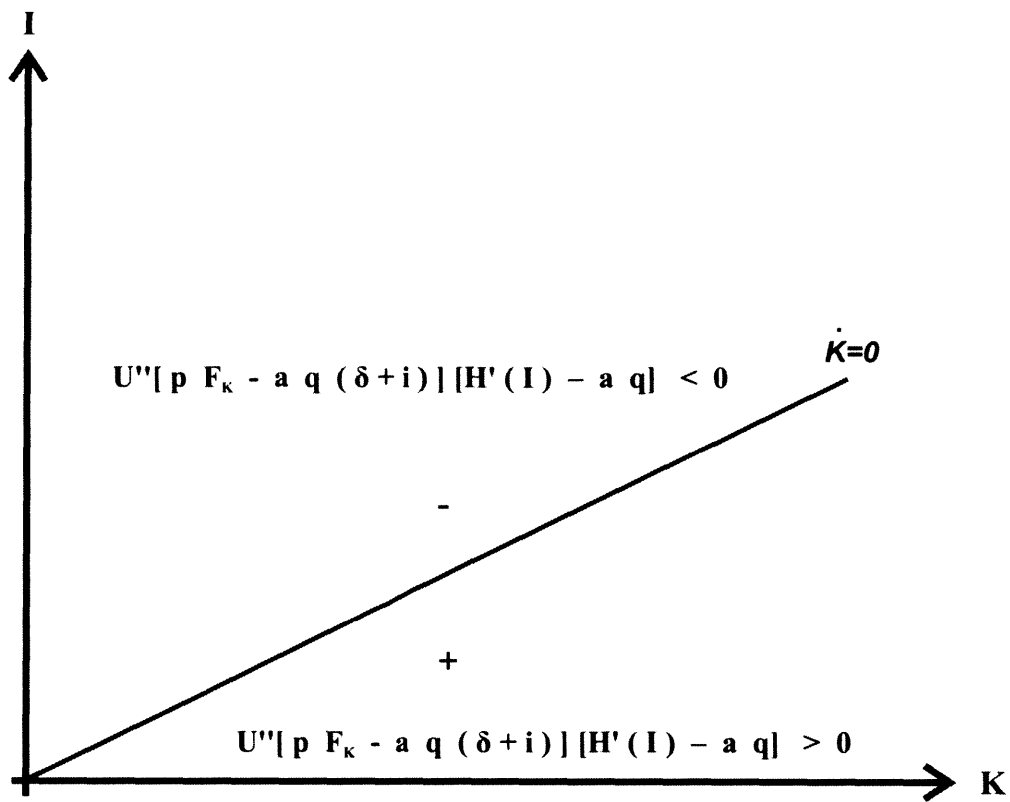


Figure IV-2

Now by using the information depicted in Figures IV-1 and IV-2 together with equation (4) we draw a typical $\dot{I} = 0$ locus in Figure IV-3

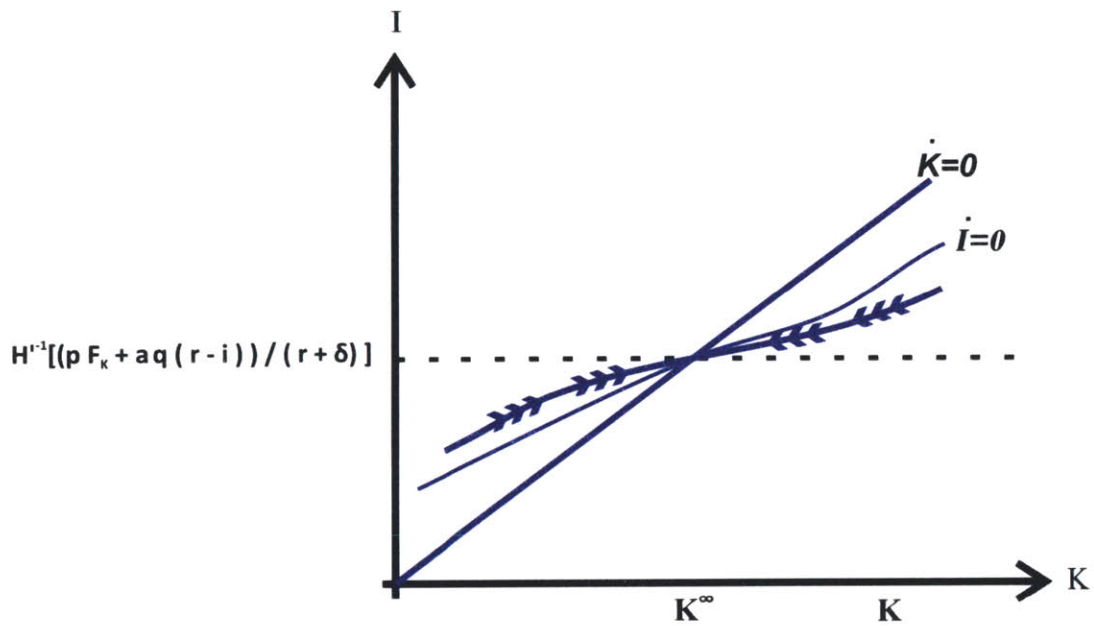


Figure IV - 3

An analysis similar to that of section III.- H would yield paths, in the various quadrants of the K, I space similar to those in Figure III-7, hence we will also have similar optimal solution paths that approach a steady state. We draw these optimal paths with arrowhead curves in Figure IV-3.

However, such solution paths were derived assuming that the house-hold would always borrow as much as possible i.e. that

$$D = aqK$$

was compulsory at all times.

Next we will examine the circumstances in which it is optimal for the house-hold to set $D = aqK$. Only for such circumstances would the solution paths sketched in Figure IV-3 be optimal for a house-hold that could have $D \leq aqK$.

IV.-E. The Phase Diagram in the K, I Space for the Capitalist in Imperfect Markets.

To derive an equation for an iso-rate-of-return curve in the K, I space above which the rate of return on marginal investment expenditures is lower than i , consider a neoclassical firm with an opportunity cost of funds i , according to the analysis in Chapter I, such firm would have,

$$I^* = H'^{-1} [pF_k / (i + \delta)]$$

For any K the last equation gives that investment level at which the firm obtains a rate of return on its marginal investment expenditures just equal to i . This may be seen explicitly by solving for i in the last equation, to obtain,

$$i = [pF_k / H'(I^*)] - \delta$$

For any $I > I^*$ the rate of return would be lower than i . Ceteris paribus, for either the house-hold or the firm, higher investment levels imply higher marginal investment expenditures, H' , and hence, lower rates of return.

The optimal investment path derived in the preceding section for the house-hold that compulsory sets $D = aqK$, and which we sketched in Figure IV-3 would be optimal for a house-hold that could have $D \leq aqK$, only for as long as

$$I < H'^{-1} [pF_k / (i + \delta)]$$

Investment level higher than the right-hand side of the last inequality along the path sketched in Figure IV-3 would not be optimal for the latter house-hold since they imply a rate of return on marginal investment expenditures lower than i .

To elaborate this argument more thoroughly first consider the case with $i < r$. Then provided that, as previously assumed,

$$pF_k > aq(\delta + i)$$

we would have,

$$\frac{pF_k}{r + \delta} < \frac{[pF_k + aq(r - i)]}{r + \delta} < pF_k / (i + \delta)$$

and consequently

$$(6) \quad H^{-1} \left[\frac{pF_k}{r+\delta} \right] < H^{-1} \left[\frac{[pF_k + aq(r-i)]}{r+\delta} \right] < H^{-1} [pF_k/(i+\delta)]$$

Thus when $i < r$ the level of investment below which the house-hold would be willing to borrow is higher than the steady state investment level of the self financed house-hold but lower than the steady state investment level of the house-hold that must set $D = aqK$.

We mark each of the investment level in inequality (6) on the vertical axis in Figure IV-4 where we also draw the $\dot{K} = 0$ locus and for visual ease omit the $\dot{I} = 0$ loci.

Consider an initial capital stock level, higher then say, K_1 in Figure IV-4, such that the self-financed house-hold optimal path would have,

$$I > H^{-1} \left[\frac{pF_k}{i+\delta} \right]$$

If at such $K(0) > K_1$ we now allow the house-hold to borrow at an interest rate i , the house-hold would not want to borrow because the return on marginal investment expenditures would be less than i , hence the house-hold would continue with the path optimal for the self-financed house-hold derived in Chapter III and which we draw in Figure IV-4 for $K > K_1$.

However, since along such path investment is declining asymptotically towards,

$$I = H'^{-1} \left[\frac{pF_k}{r + \delta} \right]$$

while the capital stock is diminishing, eventually - and before reaching a steady state - say, at K_1 , the self-financed optimal path must have,

$$I = H'^{-1} \left[\frac{pF_k}{i + \delta} \right]$$

at this point borrowings will be considered, and the optimal self-financed path that calls for a decreasing I and C with the simultaneous decrease in the capital stock and revenue will be abandoned, instead the house-hold will avoid such decrease in C and I by borrowing. Thus once K_1 is reached the house-hold will be able to continue with I and C set constant at the level they were at K_1 by making-up the revenue decreases, due to decreasing K , with decreasing borrowings. This process will continue until the house-hold reaches its credit ceiling, say at K_2 , where we have,

$$I = H'^{-1} \left[\frac{pF_k}{i + \delta} \right]$$

and where,

$$D = aqK$$

The house-hold is now borrowing as much as possible and thus the investment path from K_2 onwards is the one derived for a house-hold for whom the last equation holds at all times and which we previously sketched in Figure IV-3

For $K < K_2$ we draw again such paths in Figure IV-4

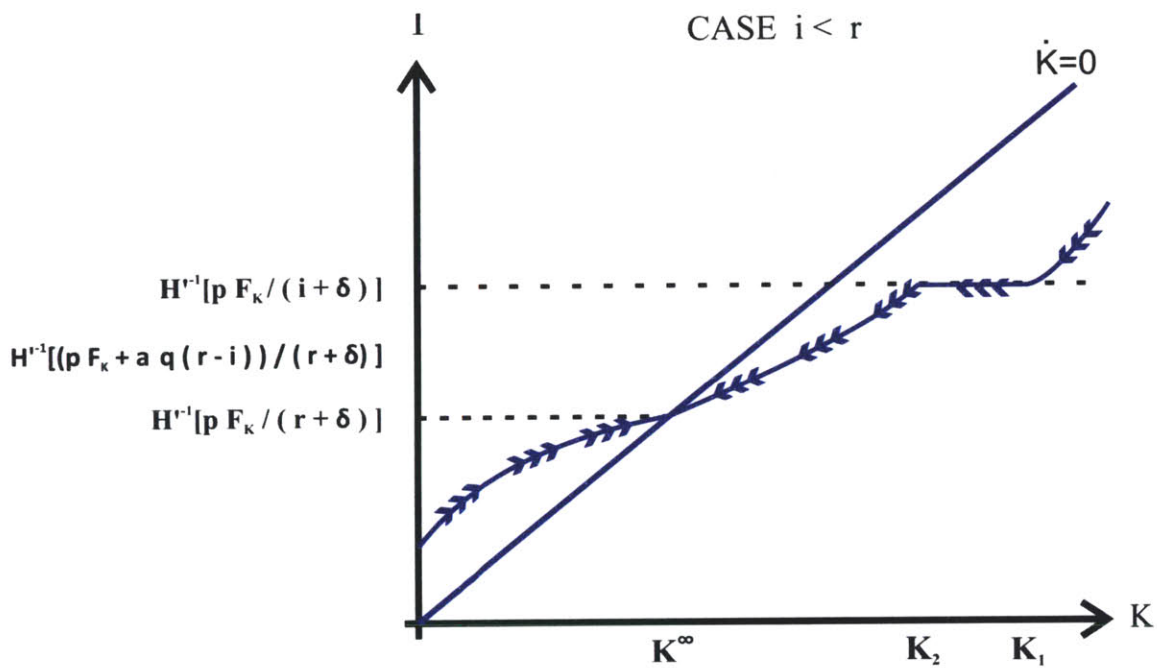


Figure IV-4

For the case with $i > r$ and assuming $pF_k > aq(i + \delta)$ we would have,

$$H'^{-1} \left[\frac{pF_k}{i + \delta} \right] < H'^{-1} \left[\frac{[pF_k + aq(r - i)]}{r + \delta} \right] < H'^{-1} [pF_k / (r + \delta)]$$

Thus, when $i > r$ the investment level below which the house-hold would be willing to borrow is lower than the steady state investment level of, both, the

self-financed house-hold and the house-hold that must set $D = aqK$ at all times.

In Figure V-5 we mark each of the investment levels in the last set of inequalities, in the vertical axis, as before we also draw the $\dot{K} = 0$ locus but omit the $\dot{I} = 0$ loci.

Consider initial capital stock levels lower than, say, K_1 in Figure IV-5, at which the optimal path for the house-hold that must always borrow as much as possible, sketched in Figure V-3 , has

$$I < H'^{-1} \left[\frac{pF_k}{i + \delta} \right]$$

For as long as the above inequality holds, such path will be optimal for a house-hold that could but did not have to borrow as much as possible i.e, a house-hold whose credit constraint is

$$D \leq aqK$$

This is so because as long as the one before last inequality holds the house-hold with a credit constraint, as stated in the last inequality, would borrow as much as possible because its rate of return on marginal investment expenditures is higher than i .

Along the optimal path for the house-hold that always must set $D = aqK$ investment is increasing asymptotically towards

$$I = H'^{-1} \left[\frac{[pF_k + aq(r - i)]}{r + \delta} \right]$$

and hence it must eventually, say at K_1 , have

$$I = H'^{-1} \left[\frac{pF_k}{i + \delta} \right]$$

At this point, the path sketched in Figure V-3 will be abandoned because such a path call for increasing investment expenditures on which the marginal return is lower than i .

But because,

$$H'^{-1} \left[\frac{pF_k}{i + \delta} \right] < H'^{-1} [pF_k / (r + \delta)]$$

the house-hold will want to pursue higher investment levels by financing them entirely from revenue. However, since on such investment expenditures the rate of return is lower than i , the house-hold would find it more advantageous to first retire outstanding debt. Thus, at K_1 the house-hold would set both investment and consumption fixed at their current level and as the capital stock increases it will use the increasing revenue to retire outstanding debt. Only once all debt has been repaid, say at K_2 , will the house-hold pursue higher investment levels by financing them entirely from revenue.

Hence, from the time the house-hold reaches K_2 onwards the optimal path will be identical to that of a self-financed house-

hold, studied in Chapter III, that begins with a capital stock equal to K_2 . We draw such a path for $K^\infty > K > K_2$ in Figure IV-5.

The optimal path for a house-hold beginning with $K > K^\infty$ would also be identical to that of the self-financed house-hold with such initial capital stock levels. See Figure IV-5.

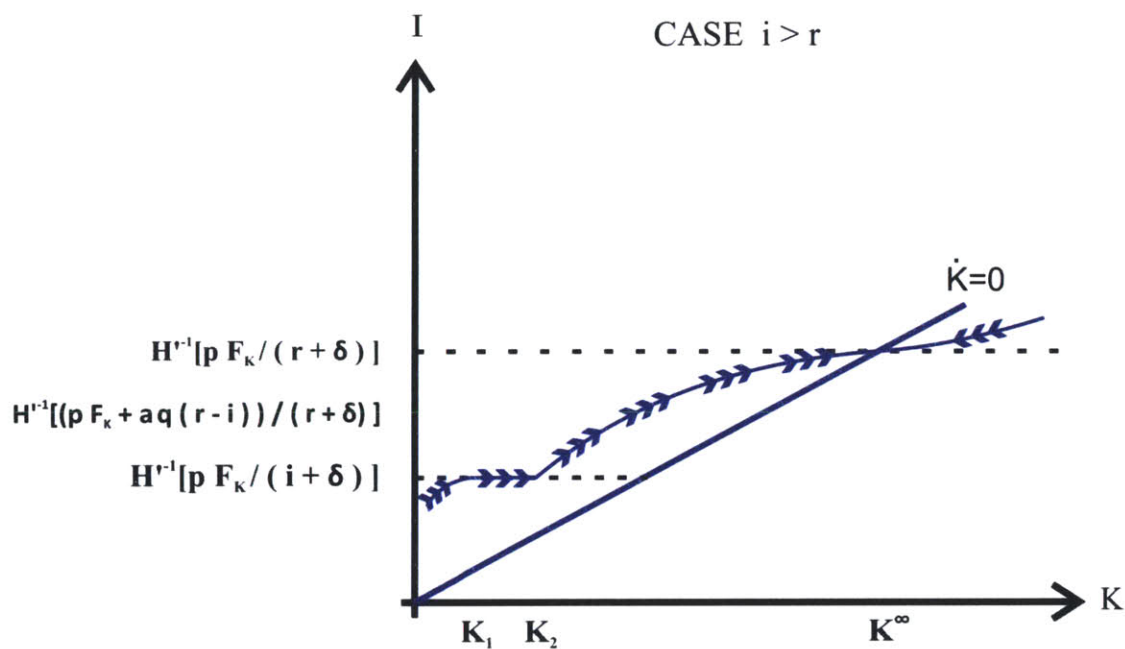


Figure IV-5

V.- Neoclassical Intertemporal Consumption and Saving Behavior.

V.-A. Introduction.

In perfect capital markets consumption and physical capital investment decisions are separable, hence there is no need to distinguish between the economic behavior of those households that own claims to physical capital and those households who possess no physical capital derive their income from wages and/or financial assets, hereafter referred to as wage earners.

However, when the absence of a market for shares makes firms investment plans dependent on the credit they may obtain from financial intermediaries and if the latter set limits to the amount of credit they are willing to supply to firms, we have that consumption and investment decisions of shareholders are in general no longer independent of each other, thus the need to distinguish between households with physical capital and wage earners. In Chapter IV we analyzed the decisions of the former households, in this chapter we first study the behavior of wage earners operating in perfect capital markets in order to have a point of reference when in the next chapter we consider the effect of imperfect capital markets on the decisions of the wage earners.

At any given time wage earners may own a certain amount of financial assets and cash balances. The financial assets could be treasury bills of short maturity valued at par, for simplicity we shall consider them interest bearing time deposits, denoted T , which yield a real interest rate, “ s ”. The yield on real cash balances, M/P , is equal to the negative of the rate of inflation, $-\dot{P}/P$. In addition, the household is assumed to supply labor inelastically for which it earn a wage, W .

If C denotes current consumption, the wage earners transition law will be

$$\frac{\dot{T}}{P} = s \frac{T}{P} + \frac{W}{P} - \frac{\dot{P}}{P} \frac{M}{P} - C - \frac{\dot{M}}{P}$$

It seems natural to regard as finite the time horizon over which the wage earners optimize their economic behavior and as it may be anticipated, the bequest motive plays a crucial role in determining appropriate terminal conditions. As argued by Barro (1974) when the current generation regards the welfare of future generations equal to its own, it is equivalent to assume infinitely lived households. Thus we shall consider the wage earners optimization horizon as of infinite length.

Assume, (a) that wage earners derive utility from consumption and from holding real cash balances which provide no pecuniary services, (b) that $U(C, M/P)$ has the following standard properties:

$$U_C > 0, \quad U_{M/P} > 0,$$

$$U_{CC} < 0, \quad U_{M^2/P^2} < 0$$

$$U_{CC} U_{\left(\frac{M}{P}\right)^2} > (U_{CM/P})^2$$

(c) that, to assure the existence of an interior solution,

$$\lim_{C \rightarrow 0} \frac{\partial U}{\partial C} = +\infty; \quad \lim_{C \rightarrow \infty} \frac{\partial U}{\partial C} = 0$$

$$\lim_{M/P \rightarrow 0} \frac{\partial U}{\partial M/P} = +\infty; \quad \lim_{M/P \rightarrow \infty} \frac{\partial U}{\partial M/P} = 0$$

In this chapter, wage earners are regarded to operate in perfect financial markets by which we mean: (a) that the individual may borrow and lend at the same rate, s ; (b) that their debt or credit ceiling is determined only by its net worth, which is equal to the present value of its future income stream discounted at a rate s ; (c) however, at any point in time, the short run rate of interest may be greater, equal or less than the time preference rate, r .

Hence the debt constraint in perfect credit markets, as characterized above is,

$$-\frac{T}{P} \leq \frac{1}{s} \frac{W}{P}$$

where $-\frac{T}{P}$ denotes real debt.

After the foregoing discussion we may now state the wage earner household problem formally.

V.-B. The Wage Earner Formal Problem.

$$\text{Max. } \int_0^{\infty} U(C, M/P) \exp. -r t dt$$

Subject to:

The flow constraint on consumption,

$$\frac{\dot{T}}{P} = s \frac{T}{P} + \frac{W}{P} - \frac{\dot{P}}{P} \frac{M}{P} - C - \frac{\dot{M}}{P}$$

The indebtedness constraint,

$$-\frac{T}{P} \leq \frac{1}{s} \frac{W}{P}$$

The non-negative constraint on the instruments,

$$C(t) \geq 0; \quad \frac{M(t)}{P} \geq 0$$

The initial condition,

$$\frac{T(0)}{P} = \frac{T}{P}o$$

V.-C. Bibliographical Note.

The study of the optimal economic behavior of the household when consumption is the only argument of the utility function emphasizing its intertemporal nature by considering wealth and not income as the main determinant of consumption has been previously undertaken by Cass and Yaari (1967) for the case of a finite horizon with a bequest motive and by Arrow and Kurz (1970) for an infinite horizon. When cash balance are introduced as an additional argument of the utility function the following complication arises, the flow constraint is not determined independently of the choice of money balances and thus we do not have a well defined problem. Mussa (1974) solves the above stated problem by reformulating it in such a way that the flow constraint depends only on variables beyond the household control time t . The advantages of providing in this chapter an alternative specification are that, (a) it will allow us to readily extend the solution obtained for the capital owner household problem of Chapter IV to the case when money and other financial assets are additional arguments of the utility

function and appear in the portfolio; (b) it will allow us to consider a claim made by Arrow and Kurz (op.cit.) relevant to the analysis of wage earners in imperfect capital markets; (c) by preserving the concavity of the resulting Hamiltonian function, we ease the proof of optimality of the solution since the necessary conditions will also be sufficient.

No new results will be obtained in this chapter. We only present a different approach and show that it yields the same results than those previously obtained by Mussa (op.cit.). The purpose is to propose and illustrate the approach which we shall use in the next chapter for the analysis of wage earners in imperfect capital market.

V.-D. The Alternative Approach.

We will break-up the wage earner household problem in two stages: one that deals with the intertemporal choice and one that deals with the atemporal aspect of the problem.

For these purposes we shall assume that the non-pecuniary services of cash-balances are valued by the household at their opportunity cost, s , so that we may define total expenditure, Z as

$$Z \equiv C + s \frac{M}{P}$$

First we will solve for the optimal intertemporal total expenditure path. Then, in a second stage, we solve a static optimization

problem at each point in time in order to determine the allocation of optimal total expenditure level, Z , between its two components, consumption and services of cash balances assuming a utility function with a constant elasticity of marginal utility of total expenditure, denoted σ , the set of preferences that satisfy this condition may then be characterized by,

$$U (C, M/P) = \frac{1}{1 - \sigma} \varphi (C, M/P)^{1-\sigma} \quad \text{for } \sigma \neq 1$$

Using our previous definition of total expenditure we may rewrite the last equation as,

$$U (C, M/P) = \varepsilon(r)^{-(1-\sigma)} \frac{1}{1 - \sigma} Z^{1-\sigma}$$

where ε is a function of s , and represents the cost associated with a unit of utility of φ . Since for any homothetic function φ , ε is not a function of $Z(t)$, the first stage of the wage earner household problem may now be formalized as in the following section.

V.-E. Alternative Specification.

Wage earners will,

$$\text{Max. } \int_0^{\infty} \frac{1}{1 - \sigma} Z^{1-\sigma} \exp. -r t dt$$

Subject to:

The flow constraint on consumption,

$$\frac{\dot{T}}{P} = s \frac{T}{P} + \frac{W}{P} - Z$$

The indebtedness constraint,

$$-\frac{T}{P} \leq \frac{1}{s} \frac{W}{P}$$

The non-negative constraint on the control,

$$Z \geq 0$$

The initial condition,

$$\frac{T(0)}{P} = \frac{T}{P}^0$$

V.-F. The Necessary Conditions.

The first stage of the household problem is now formally identical to the problem by Arrow and Kurz (op.cit.p.155). Its solution follows.

Since we assumed Inada conditions for the marginal utility function we would expect that along the optimal path, total expenditure will always be positive so that the non-negative constraint on Z will not be binding and hence we will ignore it. We shall also neglect the net indebtedness constraint as it will turn out that this constraint is always satisfied. Thus we may write the following Hamiltonian,

$$\mathcal{H} = \frac{1}{1-\sigma} Z^{1-\sigma} + \varphi \left(s \frac{T}{P} + \frac{W}{P} - Z \right)$$

From where we may obtain the following set of necessary conditions

$$(1 - a) \quad Z^{-\sigma} = \varphi$$

$$(1 - b) \quad \dot{\varphi} - r \varphi = -s \varphi$$

$$(1 - c) \quad \frac{\dot{T}}{P} = s \frac{T}{P} + \frac{W}{P} - z$$

Although the indebtedness constraint may never be actually binding, it always holds, hence we may complement the above set of necessary conditions with the following transversality conditions that holds in perfect financial markets as characterized above

$$(1 - d) \quad \lim_{t \rightarrow \infty} -\frac{T}{P} \leq \frac{1}{s} \frac{W}{P}$$

V.-G. Solution Paths.

Solving (1-b) we obtain,

$$\varphi(t) = \varphi(0) \exp. (r - s) t$$

Substituting this solution of $\varphi(t)$, in the right hand side of (1-a) and solving out for $Z(t)$ we obtain,

$$Z(t) = \varphi(0)^{-1/\sigma} \exp. - \frac{(r-s)}{\sigma} t$$

Defining,

$$Z(0) \equiv \varphi(0)^{-1/\sigma}$$

we have,

$$(2) \quad Z(t) = Z(0) \exp. - \frac{(r-s)}{\sigma} t$$

To determine the value of $Z(0)$ we first solve (1-c) with $Z(t)$ as given in the last equation to obtain,

$$(3) \quad \frac{T}{P} = \left[\frac{T(0)}{P} + \frac{1}{s} \frac{W}{P} - \frac{Z(0)}{(r-s)/\sigma + s} \right] \exp s t - \frac{1}{s} \frac{W}{P} + \frac{Z(0)}{(r-s)/\sigma + s} \exp. -(r-s)/\sigma t$$

Secondly, we use the transversality condition (1-d) to solve for $Z(0)$.

For this purpose, note that as long as

$$(4) \quad s > -\frac{(r-s)}{\sigma}$$

The first r.h.s. term in (3) will be dominant, so that if

$$Z(0) > \left[\frac{T(0)}{P} + \frac{1}{S} \frac{W}{P} \right] \cdot [(r-s)/\sigma + s]$$

The dominant term approaches $-\infty$ and the transversality condition will not be fulfilled. On the other hand, if,

$$Z(0) < \left[\frac{T(0)}{P} + \frac{1}{S} \frac{W}{P} \right] \cdot [(r-s)/\sigma + s]$$

The transversality condition is always satisfied but $T(t)$ would still be always non-negative if $Z(0)$ were increased by a sufficiently small amount and this would provide a higher consumption path at all points in time, Thus optimality requires

$$Z(0) = [(r-s)/\sigma + s] \left[\frac{T(0)}{P} + \frac{1}{S} \frac{W}{P} \right]$$

Substituting the value of $Z(0)$ in (2) we obtain the optimal expenditure path,

$$(5) \quad Z^*(0) = [(r-s)/\sigma + s] \left[\frac{T(0)}{P} + \frac{1}{S} \frac{W}{P} \right] \cdot \exp -\frac{r-s}{\sigma} t$$

When,

$$s < -(r - s)/\sigma$$

the preceding argument does not hold and we conclude from (3) that there is no optimal policy. However, this last inequality is not possible if $\sigma > 1$

We now proceed to the second stage of the household problem and solve a simple static optimization problem to obtain the optimal paths for consumption and money balances. If for simplicity we let $\varphi(C, M/P)$ be a Cobb-Douglas function, the problem is

$$\text{Max } \varphi = C^\alpha (M/P)^{1-\alpha} \quad 0 < \alpha < 1$$

Subject to,

$$Z^* = C + s(M/P)$$

and the solution is

$$C^*(t) = \alpha Z^*(t)$$

$$\frac{M^*}{P}(t) = (1 - \alpha) \frac{1}{s} Z^*(t)$$

Substituting the value of Z stated in (5) we may express the optimal path of C and M/P as functions of initial wealth,

$$(6) \quad C^*(t) = \alpha[(r-s)/\sigma + s] \left[\frac{T(0)}{P} + \frac{1}{s} \frac{W}{P} \right] \exp - (r-s)/\sigma t$$

$$(7) \quad \frac{M^*}{P}(t) = \frac{1-\alpha}{s} [(r-s)/\sigma + s] \left[\frac{T(0)}{P} + \frac{1}{s} \frac{W}{P} \right] \exp - (r-s)/\sigma t$$

These equations are the same solutions as those obtained by Mussa (p.91; op.cit.) and cross-checks our alternative approach.

V.-H. The Role of the Rate of Return on Financial Assets.

When $r = s$ we read from equation (5) the total expenditure equals current income, i.e.,

$$Z^*(t) = s \frac{T}{P} + \frac{W}{P}$$

So that the amount of financial assets remains constant and the credit constraint is never approached. From (6) and (7) it follows that expenditures on $C(t)$ and $M(t)/P$ will equal a constant proportion, α and $(1-\alpha)/s$, respectively, of current income.

If $s > r$, again from equation (5) we have that Z is less than current income. The difference allows for the accumulation of financial assets which in turn permits expenditures on consumption and

services of cash-balances to grow exponentially. Since financial assets are being accumulated the indebtedness constraint will not be approached.

Lastly, when $s < r$ again from equation (5) we have that total expenditure is larger than income and is declining exponentially while the indebtedness constraint is being approached asymptotically in time and from above so that it never becomes actually binding. This last claim can be verified by substituting the solution for $Z^*(0)$ in (3) to obtain,

$$\frac{T(t)}{P} = -\frac{1}{s} \frac{W}{P} + \left[\frac{T(0)}{P} + \frac{1}{s} \frac{W}{P} \right] \exp. -(r - s)t/\sigma$$

so that when $s < r$ we have,

$$\lim_{t \rightarrow \infty} T(t) = -\frac{1}{s} \frac{W}{P}$$

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VI. - Intertemporal Consumption and Savings of Wage Earners in Imperfect Capital Markets.

VI.-A. Introduction.

The preceding chapter proposed an approach in which the household economic behavior was analyzed in two steps, one that dealt with the intertemporal choice and another that dealt with the atemporal aspects. Using and extending this approach, this chapter studies the economic behavior of household without physical capital, who derived their income from wages and/or assets and are immersed in imperfect financial markets which are characterized by the following properties: (a) households may borrow only up to a fraction, denoted g , of the present value of their income stream; (b) the lending rate s -which is the return on financial assets- and the borrowing rate j are such that $j \geq s$; (c) s may be greater, equal or smaller than the pure time preference rate, r .

Thus under imperfect financial markets the household without physical capital faces the following pair of indebtedness constraints

$$(1) \quad T \geq 0$$

$$(2) \quad jD \leq g \frac{W}{P}$$

It is now useful to consider the case when g is zero so that D is also zero at all times. Then, the problem of the wage earner

household in imperfect capital markets may be formalized as in section V.-E but with the indebtedness constraints replaced by (1).

Thus if as in section V.-F we again ignore the indebtedness constraint in the formulation of the Hamiltonian we would, of course, obtain the same set of necessary conditions, which would now be complemented by the following transversality condition

$$\lim_{t \rightarrow \infty} T(t) \geq 0$$

We now use this transversality condition to determine $Z^*(0)$ from equation V - 3, again as long as $s > -(r - s)/\sigma$ the dominant term is the first r.h.s. term in V - 3 thus by an analogous argument as that given for the perfect capital markets case we must set

$$Z^*(0) = [(r - s)/\sigma + s] \left[\frac{T(0)}{P} + \frac{1}{S} \frac{W}{P} \right]$$

This causes the first r.h.s. term in V -3 to vanish, leaving the long run limit of $T(t)$ to be determined by the remaining terms. Now when V-4 holds and $s > r$ the dominant term in V-3 will be the third r.h.s. term so that $T(t)$ will be growing monotonically and consequently the indebtedness constrain will never be approached, hence it turns out innocuous to have ignored it in the formulation of the Hamiltonian. When $s < r$ the dominant term in V- 3 becomes the second r.h.s. term so that $T(t)$ converges asymptotically in time and from above to $\frac{1}{s} \frac{W}{P}$.

In the perfect capital markets case this presents no problem as this is precisely the credit constraint, but for the imperfect capital markets case this solution of $T(t)$ would eventually violate the credit constraint (1) and therefore is not feasible, let alone optimal.

Arrow and Kurz (p.155-156 Op. Cit.) claim that for the problem presented in section VI.-E if $s < r$ and if the transversality condition is

$$\lim_{t \rightarrow \infty} T(t) \geq 0$$

“there would be no optimal policy”.

Below we will first argue that an optimal policy does in fact exist for this case and characterize such policy. Then in light of this result we will extend the approach proposed in the preceding chapter in order to analyze the wage earners decisions in imperfect capital markets.

To show the existence of said solution we first consider an optimal consumption policy for a household who receives a constant salary but does not possess financial assets, operating in imperfect capital markets so that the interest rate on available financial assets s , the borrowing rate j and the pure time preference rate r are such that

$$s \leq j < r$$

Such household would certainly not want to devote any of its income to the acquisition of financial assets; rather it would prefer to borrow. But, if under imperfect capital markets the household is not permitted to borrow against its future salary stream the best it can do is to devote the entire salary to current consumption and thus its optimal total expenditure path would be,

$$Z^*(t) = \frac{W}{P}$$

We now ask how the consumption policy stated in the last equation should be modified when the household, still facing $s \leq j < r$ and unable to borrow against its future wage stream, is endowed with financial assets. For this purpose consider the class of utility functions used by Arrow and Kurz (op.cit.) in which the marginal utility of total expenditure has a constant elasticity, σ . Since U' must be proportional to $Z^{-\sigma}$ we have,

$$U' = Z^{-\sigma}$$

Integrating both sides we obtain

$$U(Z) = \frac{1}{1-\sigma} Z^{1-\sigma} + A$$

where A is a constant.

Since there is no reason why the possession of financial assets should by itself modify the desire of the household to spend all

its salary on current consumption, its expenditure level will equal its salary plus expenditures financed from its financial endowment, denoted \tilde{Z} .

We thus have,

$$(3) \quad Z(t) \equiv \tilde{Z}(t) + W/P$$

Using this expression of $Z(t)$ the transition law of section V.- E now becomes

$$\frac{\dot{T}}{P} = s \frac{T}{P} - \tilde{Z}$$

If we identify A with the constant amount of utility from spending the entire salary the utility level is

$$U(Z) = \frac{1}{1 - \sigma} \tilde{Z}^{1-\sigma} + A$$

We may now state the household problem in the following way

VI.-B. A formalization with a Non-negative Constraint on Assets.

The household will,

$$Max. \int_0^{\infty} \left\{ \frac{1}{1 - \sigma} \tilde{Z}^{1-\sigma} + A \right\} exp. -rt dt$$

Subject to:

$$\frac{\dot{T}}{P} = s \frac{T}{P} - \tilde{Z}$$

The indebtedness constraint,

$$\frac{T}{P} > 0$$

The non-negative constraint on the control

$$\tilde{Z} + A \geq 0$$

The initial condition

$$\frac{T(0)}{P} = \frac{T}{P}^0$$

VI.-C. The Necessary Conditions.

Since the Inada conditions imply that the constraint on the control will never be binding along an optimal path and since we will show that along an optimal path the indebtedness constraint is approached asymptotically in time and from above, we may ignore said constraints and write the following Hamiltonian,

$$\mathcal{H} = \frac{1}{1-\sigma} \tilde{Z}^{1-\sigma} + A + \varphi \left(s \frac{T}{P} - \tilde{Z} \right)$$

From where we obtain the following set of necessary conditions,

$$(3 - a) \quad \tilde{Z}^{-\sigma} = \varphi$$

$$(3 - b) \quad \dot{\varphi} - r \varphi = -s \varphi$$

$$(3 - c) \quad \frac{\dot{T}}{P} = s \frac{T}{P} - \tilde{Z}$$

VI.-D. Solution Paths.

Solving equation (3 – b) , substituting this result in equation (3-a) and solving out for $\tilde{Z}(t)$ we obtain,

$$\tilde{Z}(t) = \varphi^{-1/\sigma} (0) \exp. - \frac{(r-s)}{\sigma} t$$

Letting,

$$\tilde{Z}(0) = \varphi^{-1/\sigma} (0)$$

We have

$$(4) \quad \tilde{Z}^*(t) = \tilde{Z}(0) \exp. - \frac{(r-s)}{\sigma} t$$

Substituting this solution for \tilde{Z} in the transition law, (3-c), we obtain

$$\frac{\dot{T}}{P} = s \frac{T}{P} - \tilde{Z}(0) \exp. - \frac{(r-s)}{\sigma} t$$

Which solution is

$$(5) \quad \frac{T(t)}{P} = \left[\frac{T(0)}{P} - \frac{\tilde{Z}(0)}{(r-s)/\sigma + s} \right] \exp s t + \frac{\tilde{Z}(0)}{(r-s)/\sigma + s} \exp. \{-(r-s)/\sigma\}t$$

Again restricting the analysis to the case

$$s > - \frac{(r-s)}{\sigma}$$

by an argument similar to that given in section V.-G for the determination of

$\tilde{Z}^*(0)$, we have,

$$\tilde{Z}^*(0) = [(r-s)/\sigma + s] \left[\frac{T(0)}{P} \right]$$

Substituting the last equation in (4) using (2), we may now characterize the optimal expenditure strategy,

$$(6) \quad \tilde{Z}^*(t) = [(r - s)/\sigma + s] \left[\frac{T(0)}{P} \right] \exp.\{- (r - s)/\sigma\}t + \frac{W}{P}$$

This is the solution that Arrow and Kurz (p. 156; op. cit.) claim does not exist.

To obtain the solution path for $\frac{T}{P}(t)$ substitute the value of $\tilde{Z}^*(0)$, as given in

the one before last equation in (5) to obtain

$$\frac{T(t)}{P} = \left[\frac{T(0)}{P} \right] \exp.\{- (r - s)/\sigma\}t$$

When $s > r$, $\frac{T(t)}{P}$ is growing monotonically and the indebtedness constraint, $\frac{T(t)}{P} \geq 0$, is not being approached. When $s < r$ the indebtedness constraint is being approached asymptotically in time and from above, hence regardless of whether $s \geq r$ or $s \leq r$, the indebtedness constraint never becomes actually binding and it is innocuous to ignore it in the formulation of the Hamiltonian.

If we believe $r > 0$, how could we explain the empirical evidence of growing $T(t)$ with $s \leq 0$?

Among a host of possible explanation, money illusion, lagged adjustment, irrational behavior, etc., we want to study the holdings of T/P as mean for the household to accumulate purchasing power for later use in the acquisition of durable goods, denoted R. The household

prefers to accumulate purchasing power by holding T/P instead of M/P since $s > -\frac{\dot{P}}{P}$

The real yield on durables, denoted n , will be the sum of the value of the services rendered by the durable - as a percentage of its value- plus real net appreciation (i.e., the rate of real appreciation minus the rate of depreciation which we assumed to be a constant percentage of its real value). The value imputed to the services rendered by durables is the cost of hiring an equivalent service.

If $n > s$, the household would prefer to hold durables instead of financial assets and the household transition law becomes

$$\frac{\dot{R}}{P} = n \frac{R}{P} + \frac{W}{P} - z$$

However, this transition laws has a different interpretation than the transition equation given for the accumulation of financial assets in section V.-E. In general the markets for the services of durables will not be developed to an extent such that the household can rent-out a durable good R and obtain an income $n R$, rather direct use of the durable is needed to obtain a yield n . But as long as the services provided by the durables can be perfectly substituted for current expenditure on non-durables goods and services, the last equation may be used as the transition law equation and the problem of the household may be formalized as in Chapter V.

We shall first assume such perfect substitutability but at the end of this section we shall consider some alternatives.

For many wages earner household, durables may be priced above what they can save out of current income, also, for a large number of durables there does not exist a continuum of values over which a smooth accumulation could take place. For this and other reasons, in practice the accumulation of durables is bound to be discrete and not smooth. Even when such continuum is provided by a well developed market for used durables, transaction cost, from which we have abstracted, would still make for a discrete acquisition of durables.

To illustrate this point, assume that there is only one type of durable (e.g. automobiles) for which there exists a second-hand market which provides a continuum of values and for which $R(t)$ in Figure VI-1-a represents the optimal smooth accumulation path in the absence of transactions costs. Now, if step increases in $R(t)$ by a constant amount \bar{T}/P are required in order to make it worthwhile to the household to incur in the associated transaction cost, then the actual discreet accumulation path would be $R'(t)$, in Figure VI-1-a. Thus, at $t = 1$, the household whose associated optimal smooth accumulation path is $R(t)$ would begin accumulation financial assets until, $t = 2$, when it would sell the old durable and use this revenue together with accumulated financial assets to purchase a durable of value $R(2)/P$. At $t=3$, this process would repeat itself.

The time profile of the holdings of T/P associated with the above described discreet accumulation process of the durable is depicted in Figure VI-1-b.

So that once obtained the optimal path $R^*(t)$ and with a given \bar{T}/P we could obtain, both $R'^*(t)$ and an associated path of financial assets to be held for the acquisition of durables.

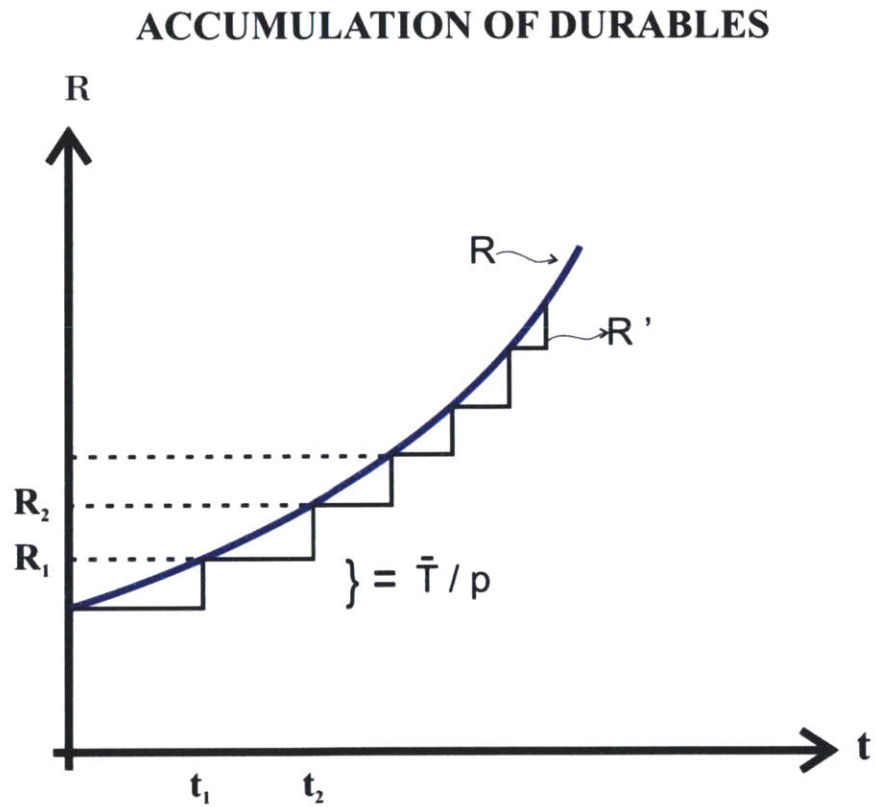


Figure VI - 1 - a

HOLDINGS OF FINANCIAL ASSETS

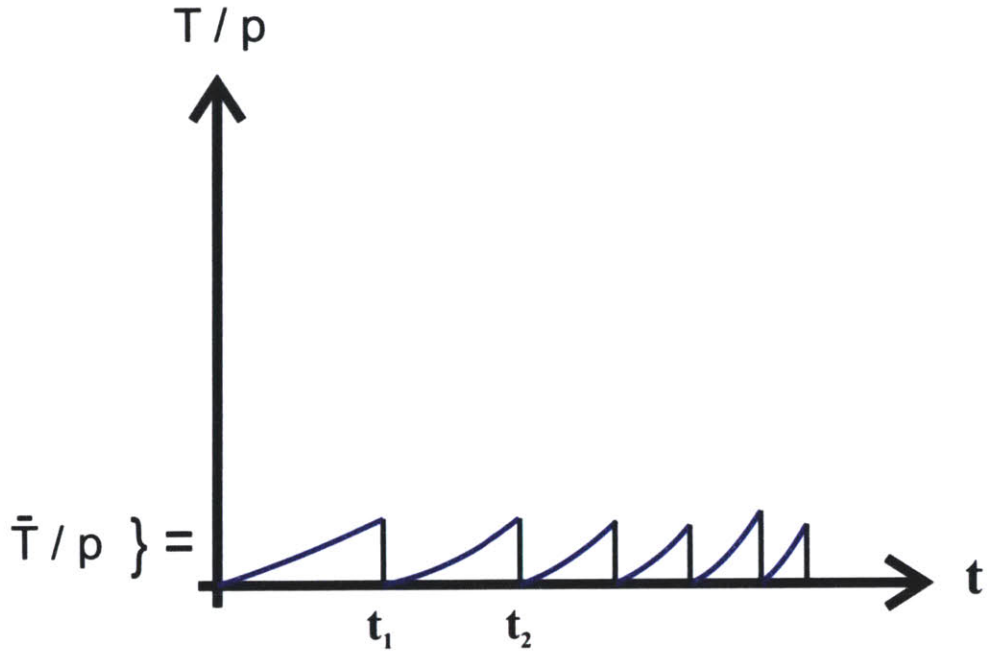


Figure VI - 1 - b

VI.-E. Demand for Quasi-liquidity and the Household Atemporal Problem.

Even when financial assets have a rate of return lower than that of durables, if the latter are not easily marketable, the household may still demand a certain amount of T/P for liquidity purposes. Then, if holdings of T/P provide less liquidity than money but more liquidity than durables we may regard holdings of T/P as providing quasi-liquidity.

To obtain the optimal allocation of a given current total expenditure, $Z(t)$, among nondurable goods and services C , cash balances services, M/P and quasi-liquidity services T/P , we need to specify the instantaneous utility function of the household and in this context the issue of substitutability among the several goods and services plays an important role. One possibility is to have an homogenous instantaneous utility function with an elasticity of substitution equal to one among the three different goods, the household atemporal problem could then be written as,

$$\text{Max } \varphi = C^{\alpha_1} M^{\alpha_2} T^{1-\alpha_1+\alpha_2}$$

where $0 < \alpha_1, \alpha_2, \{1 - (\alpha_1 + \alpha_2)\} < 1$

Subject to ,

$$Z(t) = C + n \frac{M}{P} + (n - s) \frac{T}{P}$$

However this particular φ function implies an equal elasticity of substitution among its three arguments which would not exist if, as it seems likely, the household regards T/P and M/P to be more similar between each other than with regard to C . To overcome this deficiency we want to specify a simple instantaneous utility function, yet one that captures this important aspect. For this purpose we group T/P and M/P in one category of goods, name it financial services denoted B.

Now we can form a Cobb-Douglas instantaneous utility function with the arguments being consumption goods and services, C , and the household instantaneous problem is,

$$\text{Max } C^\alpha B^{1-\alpha} \quad 0 \leq \alpha \leq 1$$

Subject to

$$Z^* = C + B$$

Which solution is,

$$C^*(t) = \alpha Z^*(t)$$

$$B^*(t) = (1 - \alpha)Z^*(t)$$

To decide the allocation between cash balances and time deposits of the share of total expenditure to be spent on financial services the household then solves the following subsidiary problem,

$$\text{Max } B = \left[\frac{\delta T^\rho}{P} + \frac{(1 - \delta)M^{-\rho}}{P} \right]^{-1/\rho}$$

where, $-1 < \rho < \infty$

$$0 < \delta < 1$$

Subject to, $(1-\alpha) Z^*(t) = n \frac{M}{P} + (n-s) \frac{T}{P}$

which solution is,

$$\frac{M}{P} = \frac{1}{n} \beta_{M/P} (1-\alpha) Z^*(t)$$

$$\frac{T}{P} = \frac{1}{n-s} \beta_{T/P} (1-\alpha) Z^*(t)$$

and where

$$\beta_{M/P} = (\delta^{1/(1+\rho)}) / [(1-\delta)^{1/(1+\delta)} (n)^{-\rho/(1+\rho)} + \delta^{1/(1+\rho)}]$$

$$\beta_{T/P} = (1-\delta)^{1/(1+\rho)} / [\delta^{1/(1+\rho)} (n)^{\rho/(1+\rho)} + (1-\delta)^{1/(1+\rho)}]$$

We may now assume that the elasticity of substitution between T/P and M/P is greater than unity so that $-1 < \rho < 0$

Thus $\beta_{T/P}$ will be an increasing function of s and since it can be shown that $\beta_{T/P} + \beta_{M/P} = 1$ we have that an increase in s would cause an increase in T/P and a decrease in M/P.

However, a change in s does not affect either of the shares of Z(t) devoted to financial services or current consumption. This is, of

course, due to our assumption of an elasticity of substitution between consumption and financial services equal to one.

Alternatively, we may consider the case in which an increase in s would not only cause an increase in T/P and a reduction in M/P but also a reduction in C . This would happen if, for example, the instantaneous utility function exhibits an elasticity of substitution greater than one among its three components so that φ may be written as,

$$\varphi = \left[\delta_1 \frac{T^{-\rho}}{P} + \delta_2 \frac{M^{-\rho}}{P} + (1 - \delta_1 - \delta_2) C^{-\rho} \right]^{-1/\rho}$$

where, $\rho < 1$

and, $0 < \delta_1, \delta_2, (1 - \delta_1 - \delta_2) < 1$

However, written in this way the above instantaneous utility function implies an equal elasticity of substitution among the three arguments which as mentioned before would not obtain if the household, as it seems likely, regards T/P and M/P to be closer substitutes of each other than of C .

Nevertheless, both properties: (a) an elasticity of substitution greater than one between consumption and financial services which would cause C to decrease if s increases and (b) an elasticity of substitution between T/P and M/P greater than that between B and C , may be

portrayed with a two-level C.E.S. function in which first, T/P and M/P are as before grouped together in an argument, denoted B, of a two argument C.E.S. function where C is the other argument and where the elasticity of substitution is a constant greater than one but less than that of a subsidiary C.E.S. function which serves to determine the allocation between M/P and T/P of the share of total expenditure to be spend on financial services. The household atemporal problem would then be,

$$\text{Max } \varphi = [\delta B^{-\rho_2} + (1 - \delta) C^{-\rho_2}]^{-1/\rho_2}$$

where

$$0 > \rho_2 > \rho_1 > -1$$

$$0 < \delta < 1$$

Subject to,

$$Z^*(t) = C + n \frac{M}{P} + (n - s) \frac{T}{P}$$

Setting $\rho_2 > \rho_1$ will make the elasticity of substitution between B and C less than that between T/P and M/P.

The budget shares that appear in the solution for C, and M/P and T/P in the above problem will be cumbersome expressions. But, since at this point we are interested only in noting some properties of the

resulting demand functions which are peculiar to this last formalization of the household atemporal problem, we express said solutions by noting only the parameters and variables that are arguments of the budget shares. Thus the solutions are,

$$C(t) = \bar{\alpha}_C (\partial, \rho_1, \rho_2, n, s) Z^*(t)$$

$$\frac{M}{P}(t) = \frac{1}{n} \bar{\alpha}_{M/P} (\partial, \rho_1, \rho_2, n, s) Z^*(t)$$

$$\frac{T}{P}(t) = \frac{1}{n-s} \bar{\alpha}_{T/P} (\partial, \rho_1, \rho_2, n, s) Z^*(t)$$

An increase in s would again cause a decrease in M/P and an increase in T/P but now C decreases. Thus, because of the relative values we assumed for the different elasticities involved, the effect of an increase in s would be such that,

$$\frac{dT/P}{ds} > 0 > \frac{dC}{ds} > \frac{dM/P}{ds} \text{ all derivatives evaluated at } Z = Z^*$$

VI.-F. Credit Availability to Wage Earner Household in Imperfect Financial Markets.

The last two preceding sections have addressed the role of durable goods as an alternative mean of holding wealth and have study the holding of financial assets for the purpose of, both, supporting the process of accumulation of durables and providing quasi-liquidity services in the context of the household atemporal problem.

Before studying the intertemporal problem of wage earner household who hold their wealth mainly in material assets such as durables and real estate, we need to establish their credit constraint. For this purpose we shall consider households that may borrow up to certain fraction of their wealth which at any given time is

$$\omega(t) = \frac{T}{P} + \frac{1}{j} \frac{W}{P} + \frac{R}{P}$$

It seems reasonable to assume that credit granted by financial intermediaries will not be a fraction of total wealth as a whole but influenced by its composition. If we denote with f , g , and h the fractions of, respectively, financial assets, discounted wage income and durable goods up to which the household may barrow, the credit constraint may then be written as,

$$\frac{D}{P} \leq f \frac{T}{P} + \frac{g}{j} \frac{W}{P} + h \frac{R}{P}$$

where $f \neq g \neq h$ if financial intermediaries perceive the various wealth components, $\frac{T}{P}$, $\frac{1}{j} \frac{W}{P}$ and $\frac{R}{P}$ as providing different degrees of guarantee against credit granted.

Because in general ownership of financial assets may be easy to change without necessarily incurring in large capital losses or transaction costs and without financial intermediaries being able to control such ownership, we will regard f as zero. Similarly if credit granting institutions regard R to be more secure collateral than expected wage income we would expect $g < h$, however in what follows we shall not assume that this last inequality holds.

Thus, the debt constraint of the wage earner household shall hereafter work with shall be,

$$\frac{D}{P} \leq \frac{gW}{jP} + h \frac{R}{P}$$

For future reference we note that since this inequality holds at all times, it will also hold in the limit, i.e.,

$$\lim_{t \rightarrow \infty} \frac{D(t)}{P} \leq \frac{gW}{jP} + h \frac{R}{P}$$

Taking account of current borrowings and of interest payments on outstanding debt, the transition law for the household that holds wealth mainly in the form of durable goods becomes,

$$\frac{\dot{R}}{P} = n \frac{R}{P} + \frac{W}{P} - Z + \frac{\dot{D}}{P} - \frac{jD}{P}$$

We may now set the wage earner household intertemporal problem in the following way

VI.-G. The Formal Intertemporal Problem with Material Assets.

$$\text{Max. } \int_0^{\infty} U(Z) \exp. -r t dt$$

Subject to,

The transition law,

$$\frac{\dot{R}}{P} = n \frac{R}{P} + \frac{W}{P} - Z + \frac{\dot{D}}{P} - \frac{jD}{P}$$

The credit constraint,

$$\frac{D}{P} \leq \frac{gW}{jP} + h \frac{R}{P}$$

The non-negative constraint on the state variable,

$$\frac{R}{P} \geq 0$$

The non-negative constraint on the control variable

$$Z \geq 0$$

The initial conditions

$$\frac{R(0)}{P} = \frac{R_0}{P} \quad ; \quad \frac{D(0)}{P} = \frac{D_0}{P}$$

VI.-I. The Necessary Conditions.

Before proceeding to solve the above stated problem we note some relative magnitude relations among the various rates, s , n , j , which will allow us to express the transition law in a more convenient form.

First note that the household problem as formalized above implicitly assumes $n > s$ since otherwise $\frac{T}{P}$ would be a dominant asset in the sense that it offers both, a higher return and more liquidity than material assets and would thus displace the latter as a means of holding wealth.

It seems also reasonable to assume $j \geq s$, since credit granting institutions are not likely to have rates such that by depositing borrowed funds in time deposits the households would be making a profit at the expense of the credit institutions.

However there seems to be no reason why we should assume either $j > s$ or $j < s$. If $j > s$ the households will never borrow since neither in material assets nor in time deposits (since $n > s$) would they obtain a return large enough to compensate for interest payments. Similarly if $n > j$ the household would borrow as much as possible. Lastly when $n = j$ the household would in principle be indifferent between borrowing as much as possible and nothing at all, for definiteness we assume that to establish and/or consolidate credit history the household borrows as much as possible. Hence,

$$D = \begin{cases} 0 & \text{for } j > n \\ \frac{gW}{jP} + h\frac{R}{P} & \text{for } j \leq n \end{cases}$$

and consequently,

$$\dot{D} = \begin{cases} 0 & \text{for } j > n \\ h\frac{\dot{R}}{P} & \text{for } j \leq n \end{cases}$$

Substituting the last two equations in the transition law of the preceding section we obtain,

$$\dot{R} = \begin{cases} nR + \frac{W}{P} - Z & \text{for } j > n \\ \frac{n-jh}{1-h}R + \frac{1-g}{1-h}\frac{W}{P} - \frac{Z}{1-h} & \text{for } j \leq n \end{cases}$$

For ease of notation define,

$$\begin{aligned}
m_1 &= \begin{cases} n & \text{for } j > n \\ \frac{n-jh}{1-h} & \text{for } j \leq n \end{cases} \\
m_2 &= \begin{cases} 1 & \text{for } j > n \\ \frac{1-g}{1-h} & \text{for } j \leq n \end{cases} \\
m_3 &= \begin{cases} 1 & \text{for } j > n \\ \frac{1}{1-h} & \text{for } j \leq n \end{cases}
\end{aligned}$$

So that the last transition law may be rewritten as

$$\frac{\dot{R}}{P} = m_1 \frac{R}{P} + m_2 \frac{W}{P} - m_3 Z$$

In the solutions of wage earner household problem we have previously study, when the return on assets was greater than the time preference rate, the credit constraint never became binding since the optimal trajectories we obtained were characterized by asset accumulation. However for the case in which $n < r$ the conjecture that our previous work suggests is that along the optimal path, the stock of material assets will be depleted and the concern about whether the credit constraint might become binding is justified. We shall focus on the case $n > r$ and disregard the credit constraint, although we will verify whether it ever becomes binding along the optimal trajectories we obtain when $n > r$.

As before, we also ignore the nonnegative constraint on Z because we have assumed Inada conditions for the marginal utility function.

Thus we may write the following Hamiltonian,

$$\mathcal{H} = U(Z) + \varphi \left(m_1 \frac{R}{P} + m_2 \frac{W}{P} - m_3 Z \right)$$

From where we obtain the following set of necessary conditions:

$$(9-a) \quad U'(Z) = \varphi$$

$$(9-b) \quad \dot{\varphi} - r\varphi = -m_1 \varphi$$

$$(9-c) \quad \frac{\dot{R}}{P} = m_1 \frac{R}{P} + m_2 \frac{W}{P} - m_3 Z$$

VI.-J. Solution Paths.

Solving equation (9-b), substituting this result in equation (9-a), assuming a constant elasticity of the marginal utility of total expenditure and solving out for $Z(t)$ we obtain

$$Z^*(t) = \varphi^{-1/\sigma}(0) \exp. - \frac{(r - m_1)}{\sigma} t$$

Defining,

$$Z^*(0) \equiv \varphi^{-1/\sigma}(0)$$

we have,

$$(10) \quad Z^*(t) = Z^*(0) \exp. - \frac{(r-m_1)}{\sigma} t$$

Substituting this solution for $Z^*(t)$ in (9-c) and solving the resulting differential equation we obtain,

(11)

$$\frac{R(t)}{P} = \left[\frac{R(0)}{P} + \frac{m_2 W}{m_1 P} - \frac{m_3 Z(0)}{(r-m_1)/\sigma + m_1} \right] \exp.m_1 t - \frac{m_2 W}{m_1 P} + \frac{m_3 Z(0)}{(r-m_1)/\sigma + m_1} \exp. - (r-m_1)/\sigma t$$

To solve for $Z(0)$, note that as long as $m_1 > -(r-m_1)/\sigma$ the first r.h.s. term in the last differential equation will be dominant so that if

$$Z(0) > \frac{1}{m_3} \left[\frac{R(0)}{P} + \frac{m_2 W}{m_1 P} \right] [(r-m_1)/\sigma + m_1]$$

the dominant term approaches $-\infty$ and the constraint on the state variable would be violated.

$$\text{If, } Z(0) < \frac{1}{m_3} \left[\frac{R(0)}{P} + \frac{m_2 W}{m_1 P} \right] [(r-m_1)/\sigma + m_1]$$

the non-negative constraint on the state variable, R , would not be violated but said constraint would still be always satisfied if $Z(0)$ were increased by a sufficiently small amount and this would provide a higher consumption at all points in time. Thus, for optimality we need,

$$Z^*(0) = \frac{1}{m_3} \left[\frac{R(0)}{P} + \frac{m_2 W}{m_1 P} \right] [(r - m_1)/\sigma + m_1]$$

Substituting this value of $Z(0)$ in (9) we obtain the optimal expenditure path,

$$(12) \quad Z^*(t) = \frac{1}{m_3} \left[\frac{R(0)}{P} + \frac{m_2 W}{m_1 P} \right] [(r - m_1)/\sigma + m_1] \exp - (r - m_1)/\sigma t$$

Substituting the value for $Z^*(0)$ given in the one before last equation in to (11) we have the solution path for durable goods,

$$\frac{R^*(t)}{P} = \frac{R(0)}{P} - \frac{m_2 W}{m_1 P} + \frac{m_2 W}{m_1 P} \exp. - (r - m_1)/\sigma t$$

To obtain the solution paths for $C(t)$, $\frac{M(t)}{P}$, and $\frac{T(t)}{P}$, we need only substitute the solution for total expenditure given by (12) in the atemporal solutions previously obtained.

Adopting, for instance the last formalization of the household atemporal problem presented in section VI-E we have,

$$C^*(t) = \bar{\alpha}_C Z^*(t)$$

$$\frac{M^*}{P}(t) = \frac{1}{n} \bar{\alpha}_{M/P} Z^*(t)$$