Robust Adaptive Flight Control Systems in the Presence of Time Delay

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Abstract

Adaptive control technology is a promising candidate to deliver high performance in aircraft systems in the presence of uncertainties. Currently, there is a lack of robustness guarantees against time delay with the difficulty arising from the fact that the underlying problem is nonlinear and time varying. Existing results for this problem have been quite limited, with most results either being local or at best, semi-global. In this thesis, robust adaptive control for a class of plants with global boundedness in the presence of time-delay is established. This class of plants pertains to linear systems whose states are accessible. The global boundedness is accomplished using a standard adaptive control law with a projection algorithm for a range of non-zero delays. The upper bound of such delays, i.e. the delay margin, is explicitly computed. The results of this thesis provide a highly desirable fundamental property of adaptive control, robustness to time-delays, a necessary step towards developing theoretically verifiable flight control systems.

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Chapter 1

Introduction

1.1 Motivation

1.1.1 Adaptive Controller for Safe Flight

One of the most important ingredients needed for achieving reliable flight is flight safety. Flight safety may be violated when an aircraft meets non-normal flight conditions such as failures, damages, or other upsets, and the flight controller on-board may not be adequate for stabilization and therefore fail to ensure safety. Therefore an advanced control method that can guarantee stability in the presence of these non-normal flight conditions is needed.

Adaptive control has been believed to be a strong candidate to achieve this goal with the potential to improve flight safety. In the context of the underlying dynamic model, most of the non-normal flight conditions can be directly mapped into parametric uncertainties, and adaptive control is the theoretical discipline which was developed with an aim to maintain stability against parametric uncertainties. Therefore, an adaptation-based reconfigurable flight controller is believed to maintain satisfactory performance when actuator failures, flight upsets, and other unforeseen changes in the system dynamics occur.

Adaptive control theory itself has been extensively studied over the past three decades, with its basic performance and robustness properties currently well understood [44, 34, 27, 60, 53, 3, 58]. With promising features, such as the stability against parametric uncertainties, adaptive control theory has been studied extensively in the context of adaptive flight control systems too after 90s and its potential has been verified both theoretically and numerically, [54, 18, 4, 19] for example. More recently, there has been significant interest [31, 35, 12, 11, 49] and success in applying adaptive methods to flight. In this thesis, we also demonstrate as well that introducing adaptation into the transport aircraft improves flight safety through high-fidelity simulation studies based on NASA Generic Transport Model (GTM). The improved safety against some failure scenarios will be discussed in Chapter 2, where adaptive controller achieves stable behavior in comparison to nominal controller in the presence of flight failures.

What remains to be shown is a rigorous demonstration of guarantees of robustness of these adaptive flight control systems in the presence of non-parametric perturbations such as unmodeled dynamics and more importantly, time-delays.

1.1.2 Time Delay

In a typical flight control problem, there almost always are perturbations which cannot be modeled as parametric uncertainties. Examples of such perturbations are unmodeled dynamics, time-delays, and nonlinearities. Reference [1] discusses challenges of adaptive control and shows that undesirable features such as instability and bursting can occur in their presence. Therefore what is important is to derive robustness of adaptive control systems. That is, a guarantee that even in the presence of these non-parametric uncertainties, the developed adaptive control systems remain to be stable needs to be established.

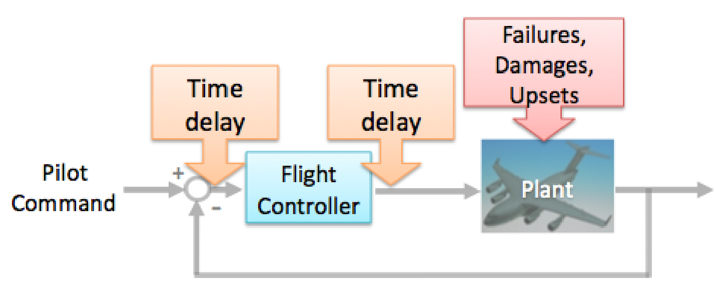


Figure 1-1: Time delays in flight control system

Time delay, a typical example of a non-parametric uncertainty, is critical when considering the robustness of flight control systems to be developed, since in flight systems there are usually present time delays (Figure 1-1) which are highly unknown due to signal processing, control computations, or telemetry. It is well known that control systems in general and adaptive control systems in particular can result in degraded performance or instability in the presence of time delays in closed-loop. We also demonstrate this undesirable property of adaptive systems against time delays in a flight example in Chapter 2, and rigorously analyze instability of standard Model Reference Adaptive Control (MRAC) in Chapter 3. Therefore it is imperative to develop a robust adaptive system which remains stable in the presence of time delays and to prove it rigorously.

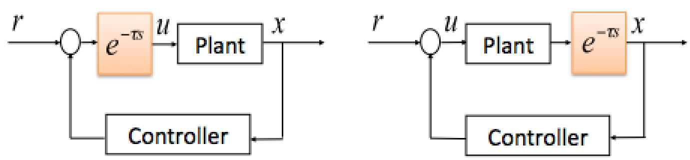


Figure 1-2: Time delay margins of control system

In order to state the problem concerning time delay formally, we begin with an introduction of a robustness metric: time delay margin. $\tau^\star$ is said to be time-delay margin of a closed-loop system if for all time-delays τ with $0 \leq \tau < \tau^\star$, the closed-loop system is guaranteed to be stable (See Figure 1-2 for an example). To-date, whether adaptive systems have a delay margin or not has not yet been answered.

Without a guaranteed delay margin, adaptive flight systems have not been allowed yet to be applied to commercial aviation, and noting the possible significant improvement in flight safety which can be introduced by adaptation, this can have a negative impact.

Our goal is to develop a robust adaptive control system in the presence of time delay, and provide analytically computable delay margins. In other words, the main goal is to solve a long standing problem in adaptive control for a class of plants with parametric uncertainties. The class corresponds to linear time-invariant plants with unknown parameters and subjected to certain unmodeled dynamics and time-delays, whose states are accessible for measurement. In this thesis we only consider the input delay, which corresponds to the left figure in Figure 1-2. Since the plant is linear, it can be noted that having a delay in the plant input or output has the same impact on stability and delay margins of the corresponding closed-loop systems should be identical.

1.2 Background and Previous Works

In this section we lay out the background and review the most relevant previous works. Time delay can be regarded as a special case of unmodeled dynamics, and therefore we start with reviewing the previous works on the robustness of adaptive systems against unmodeled dynamics.

1.2.1 Stability Margin for Unmodeled Dynamics

One of the first major milestones of robust adaptive control is robustness of uncertain linear plants to bounded disturbances in 1990 [42]. Several attempts have been made since then to extend the robustness properties of adaptive systems to the case when unmodeled dynamics are present. The most general result to date in this direction can be found in [44], [27] where semi-global stability is guaranteed for a certain class of unmodeled dynamics with a small parameter μ (see section 9.3 in [27], section 8.7 in [44]) and several papers published in the '90s (see [25] for example). The recently popular $\mathscr{L}_1$-adaptive controllers have also been shown to be only semi-global in the presence of unmodeled dynamics [8].

It should be noted that there have been also some results on global stability such as [41, 38], but are limited in their usefulness. In [41], the adaptive control problem for a continuous-time plant of arbitrary relative degree in the presence of bounded disturbances and a class of unmodeled dynamics is considered, and it is shown that global boundedness can be achieved by the usual gradient update law with parameter projection. However, the unmodeled dynamics analyzed in [41] and most of the other global results are again those which can be described with the small gain μ and it is not possible to apply the result to time delays straightforwardly. Furthermore, [41] is almost an existence type result and hence it is difficult to explicitly compute something that can be used in practice.

In [29] and [38], unmodeled dynamics described with a different form are studied so that the result can be applied to a Pade approximation of a delay. Using a Pade approximation, the problem with a delay is converted into one with a state-dependent disturbance which enables the results on more general unmodeled dynamics to be applied and demonstrate semi-global or global boundedness, respectively. However it turned out to be that such result on delays is too conservative to compute margins [29] or is applicable only to open-loop stable plants [38]. Therefore in spite of the rich results on unmodeled dynamics, a general, practically useful solution to our goal - delay margin, is unavailable to-date.

1.2.2 Global v.s. Local Results

In this subsection we mainly review the previous works on a delay margin of adaptive systems.

Due to the difficult nature of the problem, there have been only a handful of papers so far that have tackled this problem. Among them, the major works are [20, 45, 28, 29, 7, 9], and [17]. In [20] the authors studied MRAC system in the presence of time delay on the first-order plant. In [45], [28] and [29], the authors tried to provide computable delay margins for a standard MRAC possibly with σ-modification by taking approximations of a time delay. In [9], the authors proved stability of $\mathscr{L}_1$ adaptive controller in the presence of sufficiently small delays. In [17], newly proposed nonlinear robustness analysis tools are applied to adaptive flight control to analyze time delay margin based on local stability.

Even though these works provide a lot of insights into the adaptive systems which are subject to time delay, there is a critical limitation which is common throughout them. That is, all of the results are only either local or semi-global.

Since adaptive systems are nonlinear, the stability of states in the system depend on initial conditions. Given a finite time delay, the results so far guarantee stability or boundedness only for a finite set of initial conditions around the origin as in the left hand-side of Figure 1-3. This leads to the problem that we can compute a critical time delay only for a certain finite set of initial conditions, but may not be for any finite sets. Or, we may be only able to prove the existence of the critical time delay for any finite sets of initial conditions. Therefore those results do not provide a delay margin, which should exist for <u>any</u> initial conditions. In comparison to local results, global results do not have constraints on initial conditions and regardless of initial conditions the boundedness of trajectory is guaranteed as shown in Figure 1-3.

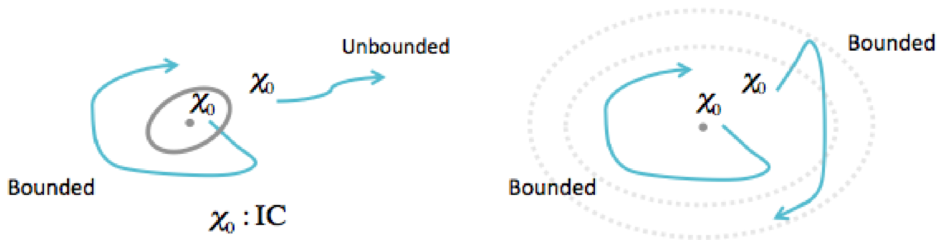


Figure 1-3: Local and global results

Noting the limitation in the previous works, our approach is the following;

- Develop a robust adaptive control system which ensures global boundedness in the presence of time delay.

Defining a delay margin $\tau^{\star}$ as in Definition 1, our goal is to compute a delay margin τ^{*} for adaptive control systems. This lays a foundation to guarantee the robustness of adaptive flight control systems (AFCS).

Definition 1. *Time delay margin $\tau^{\star} > 0$ is a positive number such that the adaptive control system exhibits global boundedness for all delay τ with $0 \leq \tau < \tau^{\star}$.*

We note that the previous works discussed above implicitly or explicitly define time delay margins through the existence of a stable region, or only for a finite set of initial conditions. On the other hand, the proposed research removes any ambiguities of the definition of delay margins or restrictions arising with initial condition dependent results.

1.2.3 Adaptive Controller for Time-Delay Systems

There are some notables works on adaptive control systems with a delay which obtain global results with respect to known / unknown time delays. While these insightful previous works developed useful and important techniques, these works may lack in implementability or their efficiency has not been verified.

We start with briefly reviewing previous results obtained for known time delays. There are many works which studied adaptive control systems for plants with known time delays, such as [59, 46, 10, 48, 62]. However they require knowing time delays in the system a priori, and therefore there is still a huge gap between our goal of developing a robust adaptive controller in the presence of a set of time delays which are usually not known.

There are also many other works which tackled unknown time delays, [22, 21, 64, 63, 33], for example. In [21], robust adaptive control is presented for a class of parametric-strict-feedback nonlinear systems where unknown time delays were compensated by using appropriate Lyapunov-Krasovskii functionals. It is proved that the proposed systematic backstepping design method is able to guarantee global uniform ultimate boundedness of all the signals in the closed-loop system. In [64], an adaptive controller is developed based on linear matrix inequality technique and it is shown that the controller can guarantee the state variables of the closed-loop system to converge, globally, uniformly and exponentially, to a ball in the state space with any pre-specified convergence rate. In [63], adaptive neural control is proposed for a class of uncertain multi-input multi-output nonlinear state time-varying delay systems in a triangular control structure with unknown nonlinear dead-zones and gain signs. The design is based on the principle of sliding mode control, the use of Nussbaum-type functions, and appropriate Lyapunov-Krasovskii functionals and proved to be semi-globally uniformly ultimately bounded. In [33], compensation of infinite-dimensional actuator and sensor dynamics were developed. A PDE backstepping approach was used to design delay adaptive systems for plants with infinite dimensional input dynamics.

Even though the above works successfully achieve global results with respect to unknown time delays and are insightful for developing robust adaptive systems with nonzero delay margins, their controller designs are often complicated or hard to implement. Also it should be noted that some of them are existence type of results with no explicit computation of delay margins. With these limitations of previous works, we hope to explore the robustness properties of simple adaptive laws. Such simple adaptive laws are also verified to be promising for flight control systems as shown in Chapter 2 as well as in literature.

1.3 Thesis Contributions

Currently the state of the art in robust adaptive control is that there is no rigorous demonstration of guarantee of robustness of adaptive flight control systems in the presence of time delays. The question frequently asked is; what is the time delay margin of the adaptive systems? The lack of a clear answer to this question is critical, and this is one of the major causes which prevent this promising technology from being certified in flight applications.

The main result that we establish in this thesis is that MRAC with projection algorithm *does* have a nonzero delay margin, which is analytically computable. This result theoretically verifies the adaptive control systems with time delay in general, and will enable us to get close to a certification for flight applications, which will significantly improve flight safety in commercial aviation.

Furthermore, analytic relations among control parameters, delay margins, and guaranteed bounds on states will allow us to study the dependencies among them and provide tuning capabilities of the control parameters so as to obtain better performance.

1.4 Thesis Layout

The remainder of this thesis is organized as follows. In Chapter 2, we discuss the actual development of adaptive control architecture for safe flight, in which a significant improvement from adaptation is observed as well as a robustness concern against time delays is confirmed. In Chapter 3, adaptive stabilizer systems with standard adaptive laws are studied to reveal their properties with respect to time delays (instability results). The chapter is concluded with proposing a simple solution to achieve global boundedness - modifying the standard adaptive law with a *projection algorithm*, as well as stating relevant definitions and lemmas of the algorithm. Chapter 4 then illustrates the analysis of the standard adaptive system with the projection algorithm, in the presence of unmodeled dynamics. The result is applied to Pade-approximated input delays. The result in Chapter 4, however, turns out to be overly conservative, and conclusions which can be drawn about time delays are quite restrictive. To overcome the limitation of the analysis method used in Chapter 4, Chapter

5 analyzes the identical adaptive system with a scalar plant, based on first-principles analysis and successfully provides a computable delay margin. Chapter 6 is the extension of Chapter 5 to higher-order plants and proposes a new adaptive law based on the projection algorithm and a nonsingular matrix transformation. Chapter 6 as well as Chapter 5 state the main results of the thesis. Finally, Chapter 7 presents concluding remarks and future works.

Chapter 2

Adaptive Control System for Safe Flight against Non-normal Conditions

In this chapter we focus on the development and implementation of adaptive control technology for safe flight. The developed control architecture consists of a nominal controller that provides satisfactory performance under nominal flying conditions; and a direct Model Reference Adaptive Controller (MRAC) that provides robustness to parametric uncertainty. The design, implementation and simulation studies with various uncertainties of both the nominal and augmented controllers are presented. The designing procedures which encompass both theoretical and practical considerations enable us to develop a controller suitable for flight.

The proposed adaptive control architecture is applied to the Generic Transport Model (GTM) developed by NASA Langley Research Center. Numerous simulation studies, which were conducted with various uncertainties and failures, indicate some advantages and drawbacks of adaptation. While a significant improvement in flight safety introduced by the suggested control design is observed with several failure and damage cases, an undesirable flight performance and robustness concerns also become apparent. The adverse conditions considered are grouped into four categories: aerodynamic uncertainties, structural damage, unknown time delays, and actuator failures. These failures include partial and total loss of control effectiveness, locked-in-place control surface deflections, and engine-out conditions.

Figure 2-1: NASA GTM test article and its concept of operations.

2.1 Generic Transport Model

The Generic Transport Model (GTM) is a model of a transport aircraft for which both a dynamically scaled flight-test article and a high-fidelity simulation are available. Figure 2.1 shows the flight test article and its concept of operations. References [30], [40], [16] provide details on the vehicle's configuration and characteristics, the concept of operations, and the flight experiments. The aircraft is piloted from a ground station via radio frequency links by using on-board cameras and synthetic vision technology.

The high-fidelity simulation uses non-linear aerodynamic models extracted from wind tunnel data and system identification for conditions that include high angles of attack and spins, and considers actuator dynamics with rate and range limits, engine dynamics, sensor dynamics along with analog-digital-analog latencies and quantization, sensor noise and biases, telemetry uplink and downlink time delays, turbulence, atmospheric conditions, etc. The open-loop system model has 278 state variables. As the actual vehicle itself, this model departs considerably from the Linear Time Invariant (LTI) system usually assumed for control design and therefore enables us to determine whether the improvements in stability, safety, and performance expected can be realized in practice.

2.2 Control Architecture

The augmented control architecture consists of a nominal controller that provides satisfactory performance under nominal flying conditions; and a direct MRAC that provides robustness to parametric uncertainty. The nominal controller consists of a single-point longitudinal multivariable controller having the elevator and the throttle inputs to both engines as control inputs; and a single-point lateral/directional multivariable controller having the ailerons and rudders as control inputs. A fixed control allocation of this controller's outputs precludes using the engines for attitude control. On the other hand, the direct model-reference adaptive controller manipulates the control surfaces and throttle inputs independently; therefore, it is solely responsible for generating thrust differentials. In this section we discuss the design of the developed control architecture in detail.

The system dynamics can be represented as

$$\dot{X} = F(X, \Lambda U), \tag{2.1}$$

where F is a nonlinear function of the state vector X, the control input U, and $\Lambda > 0$ is the control effectiveness matrix. For control design purposes, this nonlinear plant is linearized about a trim point (X_0, U_0) satisfying $F(X_0, U_0) = 0$. Deviations from the trim values X_0 and U_0 will be written as lower-case letters hereafter, e.g., $X = X_0 + x_p$ and $U = U_0 + u$. Linearization of (2.1) about the trim point leads to the system

$$\dot{x_p} = A_p x_p + B_p u + h(x_p, u), \tag{2.2}$$

where

$$A_p = \left.\frac{\partial F}{\partial X}\right|_{X_0, U_0}, \quad B_p = \left.\frac{\partial F}{\partial U}\right|_{X_0, U_0}, \tag{2.3}$$

and $h(x_p, u)$ contains higher-order terms. In a sufficiently small neighborhood of the trim point the effect of the higher-order terms is negligible. The Linear Time Invariant (LTI) representation of the plant results from dropping the higher-order terms from Equation

(2.2). This LTI system can be written as

$$\dot{x}_p = A_p(\hat{p})x_p + B_p\Lambda(\hat{p})(R_s(u)+d) + B_2\hat{r}, \tag{2.4}$$

where A_p and Λ are unknown matrices that depend on the *uncertain parameter* $\hat{p}$, $d(t)$ is an exogenous disturbance, $\hat{r}(t)$ is the reference command generated by the pilot, and $R_s(u)$ is a saturation function that enforces range saturation limits. The vector $\hat{p}$, which parametrizes the adverse flying conditions (i.e., aerodynamic uncertainties, damage, unknown time delays and actuator failures), takes on the value $\bar{p}$ when the aircraft flies under nominal operating conditions.

The state x_p is given by

$$x_p = [\alpha\ \ \beta\ \ V\ \ p\ \ q\ \ r\ \ x\ \ y\ \ z\ \ \psi\ \ \theta\ \ \phi]^T \tag{2.5}$$

which are angle of attack, sideslip angle, true aerodynamic speed, roll rate, pitch rate, yaw rate, longitude, latitude, altitude, and the three Euler angles [57]. The control input u is

$$u = [\delta_e\ \ \delta_a\ \ \delta_r\ \ \delta_{thL}\ \ \delta_{thR}]^T \tag{2.6}$$

which are the elevators deflection, the ailerons deflection, the rudders deflection, the throttle input to the left engine and the throttle input to the right engine, respectively. The reference command $\hat{r}$ consist of angle of attack-, sideslip-, aerodynamic speed- and roll rate-commands

$$r = [\alpha_{\text{cmd}}\ \ \beta_{\text{cmd}}\ \ V_{\text{cmd}}\ \ p_{\text{cmd}}]^T. \tag{2.7}$$

These four commands are generated by the pilot to attain the desired flight maneuver. Both the nominal and adaptive controllers are based on a single trim point design.

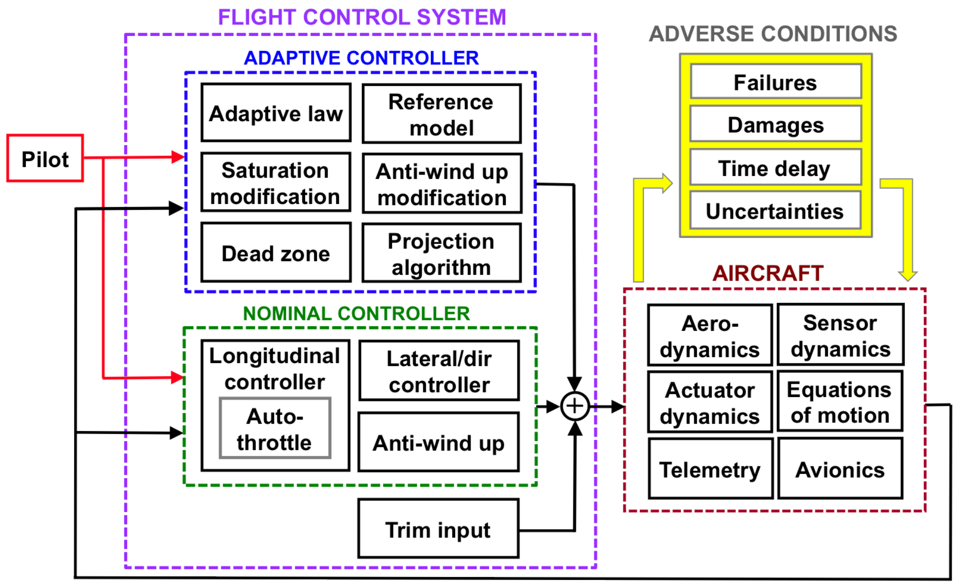


Figure 2-2: Control architecture.

2.2.1 Augmented Controller

Figure 2-2 shows the components of the augmented control architecture. The total control input is

$$u = u_{\text{nom}} + u_{\text{ada}}, \tag{2.8}$$

where u_{nom} is the output of the nominal controller and u_{ada} is the output of the adaptive controller. Any nominal controller, regardless of its structure and design methodology, can be augmented in the same fashion. Details of the structure of both controllers are presented below.

2.2.2 Nominal Controller

The nominal controller consists of independent controllers for the longitudinal and the lateral / directional dynamics. Both controllers assume a multivariate Linear-Quadratic-Regulator structure with Proportional and Integral (LQR-PI) terms having integral error states for each of the components of the reference command $\hat{r}$. Furthermore, strategies for preventing integration wind-up caused by input saturation are applied. A fixed control allocation matrix that correlates inputs of the same class is used to determine the ten main plant inputs: 4 elevators, 2 ailerons, 2 rudders and 2 throttles. As a result, out of these 10 inputs only 4 are independent.

Longitudinal Controller

The plant in the longitudinal axis takes the form

$$\dot{x}_{\text{lon}} = A_{\text{lon}} x_{\text{lon}} + B_{\text{lon}} u_{\text{lon}}, \tag{2.9}$$

where $A_{\text{lon}} \in \mathbb{R}^{3\times3}$ is the system matrix, $B_{\text{lon}} \in \mathbb{R}^{3\times2}$ is the input matrix, $x_{\text{lon}} = [\alpha\ q\ V]^T$ is the state and $u_{\text{lon}} = [\delta_e\ \delta_{th}]^T$ is the input. To enable command tracking for angle of attack and airspeed, the integral error states

$$e_\alpha = \int (\alpha - \alpha_{\text{cmd}}) dt \tag{2.10}$$

$$e_V = \int (V - V_{\text{cmd}}) dt \tag{2.11}$$

are added. This leads to the augmented plant

$$\begin{bmatrix} \dot{x}_{\text{lon}} \\ \dot{e}_\alpha \\ \dot{e}_V \end{bmatrix} = \begin{bmatrix} A_{\text{lon}} & 0 \\ H_1 & 0 \end{bmatrix} \begin{bmatrix} x_{\text{lon}} \\ e_\alpha \\ e_V \end{bmatrix} + \begin{bmatrix} B_{\text{lon}} \\ 0 \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_{th} \end{bmatrix} + \begin{bmatrix} 0 \\ -I \end{bmatrix} \begin{bmatrix} \alpha_{\text{cmd}} \\ V_{\text{cmd}} \end{bmatrix}, \tag{2.12}$$

where $H_1 = \left[[1,0]^T\ [0,0]^T\ [0,1]^T\right]$. A constant gain LQR-PI controller that minimizes (ignoring last term in 2.12)

$$J = \int_0^\infty \left(\left[x_{\text{lon}}^T\ e_\alpha\ e_V\right] Q_{\text{lon}} \left[x_{\text{lon}}^T\ e_\alpha\ e_V\right]^T + u_{\text{lon}}^T R_{\text{lon}} u_{\text{lon}} \right) dt, \tag{2.13}$$

where $Q_{\text{lon}} = Q_{\text{lon}}^T \geq 0$, $R_{\text{lon}} = R_{\text{lon}}^T > 0$ are weighting matrices, is designed. This leads to

$$\begin{bmatrix} \delta_e \\ \delta_{th} \end{bmatrix} = \left[K_{\text{lon}}\ K_{e\alpha}\ K_{eV} \right] \begin{bmatrix} x_{\text{lon}} \\ e_\alpha \\ e_V \end{bmatrix}. \tag{2.14}$$

This controller must attain ample stability margins so the inclusion of the low-pass- and anti-aliasing-filters from sensors and the delay caused by telemetry do not compromise stability. In particular, we use 6dB of gain margin and 60 deg of phase margin.

The plant's input is given by

$$(R_s(u))_i = \begin{cases} u_i & \text{if } u_{i,\min} < u_i < u_{i,\max}, \\ u_{i,\max} & \text{if } \ u_i \geq u_{i,\max}, \\ u_{i,\min} & \text{otherwise} \end{cases} \tag{2.15}$$

where u is the controller's output, μ_i denotes the i the component of vector μ, and $u_{i,\max}$ and $u_{i,\min}$ are the saturation limits of each actuator. The control deficiency caused by this saturation function is given by

$$u_\Delta = R_s(u) - u. \tag{2.16}$$

In the following we discuss a *resetting-based anti-windup* modification technique in detail. The aim of anti-windup compensation is to modify the dynamics of a control loop during control saturation so that an improved transient behavior is attained after desaturation. This practice mitigates the chance of having limit cycle oscillations and successive saturation. The anti-windup technique used prevents the occurrence of excessively large controller outputs by imposing virtual saturation limits and resetting to the integral error state used for feedback. Let $\langle e, \delta \rangle$ denote a strongly coupled pair of an integral error state e and a control input δ, e.g., e_α and δ_e. The anti-windup scheme proposed is governed by the saturation function R_e defined as follows:

$$R_e(e, \delta) = \begin{cases} e & \text{if } R_2 \leq e \leq R_1, \\ R_1 & \text{if } R_1 \leq e, \\ R_2 & \text{if } e \leq R_2 \end{cases} \tag{2.17}$$

where the limits R_1 and R_2 are time-varying functions, assuming the smallest value of e

for which the plant input is equal to any of its saturation values δ_{min} or δ_{max}. We note the similarities between (2.15) and (2.17). The integral error state is reset to the virtual saturation limit R_1 or R_2 when $\dot{e}(t) = 0$ and either $\delta < \delta_{min}$ or $\delta > \delta_{max}$. Magnitude saturation limits affect the plant inputs and the anti-wind up logic via Equations (2.15) and (2.17). Magnitude and rate saturation limits for all actuators are present in the nonlinear simulation distributed by Langley, therefore their effects are accounted for in the simulation studies with some failure scenarios (Section 2.3). Analogous to Equation (2.16), the error deficiency caused by the anti-windup logic is

$$e_\Delta = R_e(e, \delta) - e. \tag{2.18}$$

The saturated value of the integral error state $R_e(e, \delta)$, not the integral error state itself e, will be used for feedback. Additional details of this technique are available in [37, 39].

In the longitudinal controller case, we apply this strategy to the $\langle e_\alpha, \delta_e \rangle$ pair. The effectiveness of the anti-windup scheme is a function of how well the LTI model predicts saturation and desaturation. Nonlinearities such as control surface dead-band and hysteresis play a minor role in those predictions. In the case of $\langle e_V, \delta_{th} \rangle$, the highly nonlinear engine dynamics, where the thrust is a nonlinear function of the engine's RPM's, make the anti-windup scheme ineffective. The determination of whether such a scheme is effective or not is based on comparing the LTI predictions with those of the coupled, fully nonlinear GTM model.

The substitutions of u_{lon} with $R_s(u_{\text{lon}})$; and of e with $R_e(e, \delta)$ for $e = e_\alpha$, $\delta = \delta_e$ into Equation (2.12) lead to

$$\begin{bmatrix} \dot{x}_{\text{lon}} \\ \dot{e}_\alpha \\ \dot{e}_V \end{bmatrix} = \begin{bmatrix} A_{\text{lon}} + B_{\text{lon}} K_{\text{lon}} & B_{\text{lon}} K_e \\ H_1 & 0 \end{bmatrix} \begin{bmatrix} x_{\text{lon}} \\ e_\alpha \\ e_V \end{bmatrix} + \begin{bmatrix} B_{\text{lon}} \\ 0 \end{bmatrix} u_{\text{lon},\Delta} + \begin{bmatrix} B_{\text{lon}} \\ 0 \end{bmatrix} K_e \begin{bmatrix} e_{\alpha,\Delta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -I \end{bmatrix} \begin{bmatrix} \alpha_{\text{cmd}} \\ V_{\text{cmd}} \end{bmatrix}. \tag{2.19}$$

This linear time varying system prescribes the closed-loop longitudinal dynamics with anti-windup. The boundedness of the resulting system can be established for all initial conditions inside a bounded set [37]. This bounded set extends to the entire state-space when the open-loop plant is stable and there are no unmodeled dynamics.

Lateral/Directional Controller

An LTI model of the corresponding plant is

$$\dot{x}_{\text{lat}} = A_{\text{lat}} x_{\text{lat}} + B_{\text{lat}} u_{\text{lat}}, \tag{2.20}$$

where $A_{\text{lat}} \in \mathbb{R}^{3\times 3}$ is the system matrix, $B_{\text{lat}} \in \mathbb{R}^{3\times 2}$ is the input matrix, $x_{\text{lat}} = [\beta\ p\ r]^T$ is the state, and $u_{\text{lat}} = [\delta_a\ \delta_r]^T$ is the input. To enable satisfactory command following, integral error states for sideslip and roll rate, given by

$$e_\beta = \int (\beta - \beta_{\text{cmd}}) dt \tag{2.21}$$

$$e_p = \int (p - p_{\text{cmd}}) dt \tag{2.22}$$

are added. The integral error in sideslip was chosen over that of the yaw rate to facilitate the generation of commands for coordinated turns with non-zero bank angles and cross-wind landing. The augmented plant is given by

$$\begin{bmatrix} \dot{x}_{\text{lat}} \\ \dot{e}_\beta \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} A_{\text{lat}} & 0 \\ H_2 & 0 \end{bmatrix} \begin{bmatrix} x_{\text{lat}} \\ e_\beta \\ e_p \end{bmatrix} + \begin{bmatrix} B_{\text{lat}} \\ 0 \end{bmatrix} u_{\text{lat}} + \begin{bmatrix} 0 \\ -I \end{bmatrix} \begin{bmatrix} \beta_{\text{cmd}} \\ p_{\text{cmd}} \end{bmatrix}, \tag{2.23}$$

where $H_2 = \left[[1,0]^T\ [0,1]^T\ [0,0]^T\right]$. A LQR-PI control structure for the lateral controller is adopted. This leads to

$$\begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} = \left[K_{\text{lat}}\ K_{e_\beta}\ K_{e_p}\right] \begin{bmatrix} x_{\text{lat}} \\ e_\beta \\ e_p \end{bmatrix}, \tag{2.24}$$

As before, ample stability margins (e.g., 6 dB and 60 deg) should be attained to accommodate for the filters and time delays. The anti-windup technique presented earlier is applied to the $\langle e_\beta, \delta_r \rangle$ and $\langle e_p, \delta_a \rangle$ pairs. The anti-wind up scheme pairs an integral error state with a control input. In its present form, this scheme requires pairing only one integral error state to a single control input. The pairs chosen exhibit the strongest dependence between the control input and the dynamics of the integral error state, i.e., $\langle e_p, \delta_a \rangle$ is more important than $\langle e_p, \delta_r \rangle$.

Control Allocation

Equations (2.14) and (2.24) along with the three realizations of the anti-windup technique mentioned above, prescribe the pre-allocated input $u_n = [\delta_e \ \delta_a \ \delta_r \ \delta_{th}]^T$, where

$$u_n = K_n \left[x_{\text{lon}}^T \ e_\alpha \ e_V \ x_{\text{lat}}^T \ e_\beta \ e_p\right]^T \tag{2.25}$$

and $K_n \in \mathbb{R}^{4\times 10}$ is the feedback gain. This input along with a control allocation scheme fully determines the 10 control inputs of the aircraft. This relationship can be written as

$$u_{\text{nom}} = G_{\text{nom}} u_n, \tag{2.26}$$

where $G_{\text{nom}} \in \mathbb{R}^{10\times 4}$ is the control allocation matrix. The allocation of u_n enforced by G_{nom} makes the deflection of the four elevators equal, the thrust of both engines equal, the deflection of both rudders equal, and the deflection of both ailerons equal in magnitude having opposite directions.

2.2.3 Adaptive Controller

The second component of the architecture is an adaptive controller. The adaptive controller generates independent signals for the three main control surfaces as well as for each throttle input. This enables using the engines for attitude control. Because of the placement of the engines and the orientation of the thrust vector relative to the CG, changes in thrust create a pitching moment disturbance that must be canceled by the elevators. Auto-throttle designs that only depend on the aircraft velocity rely on the pilot's ability to generate a suitable set of pitch commands to attain the desired cancellation. The controller proposed pursues this cancellation automatically, thereby considerably reducing the pilot's workload. An immediate consequence of integrating the engines into the flight control system is the enlargement of the failure set where the vehicle remains controllable (e.g., the generation of thrust differentials to overcome a locked-in-place rudder).

We note that the LTI plants used for designing the nominal controller are good approximations of the aircraft dynamics as long as the longitudinal and lateral/directional dynamics are weakly coupled. However, for high angles of attack as well as for many adverse flying conditions this coupling is strong, e.g., when both left elevators are locked-in-place any deflection of the right elevators will excite the lateral/directional dynamics. In this case, the adaptive component of the controller, which is based on a coupled model, will be active.

Reference Model

The reference model is a component of the adaptive controller responsible for setting the desired closed-loop dynamics. These target dynamics are the same for both nominal and off-nominal flying conditions (e.g., those when physical failures and/or damage have occurred) regardless of the amount of control authority available. The reference model assumed herein is the linear closed-loop system corresponding to the nominal controller without anti-wind up modifications under nominal flying conditions. This leads to:

$$\dot{x}_m = \underbrace{\left(\begin{bmatrix} A_p(\bar{p}) & 0 \\ H & 0 \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} G_{\text{nom}} K_n \right)}_{A_m} x_m + B_m \hat{r} \tag{2.27}$$

where $A_m \in \mathbb{R}^{10\times 10}$, $B_m \in \mathbb{R}^{10\times 4}$,

$$x_m = [\alpha \ \beta \ V \ p \ q \ r \ e_\alpha \ e_V \ e_\beta \ e_p]^T$$

and

$$\hat{r} = [\alpha_{\text{cmd}} \ V_{\text{cmd}} \ \beta_{\text{cmd}} \ p_{\text{cmd}}]^T.$$

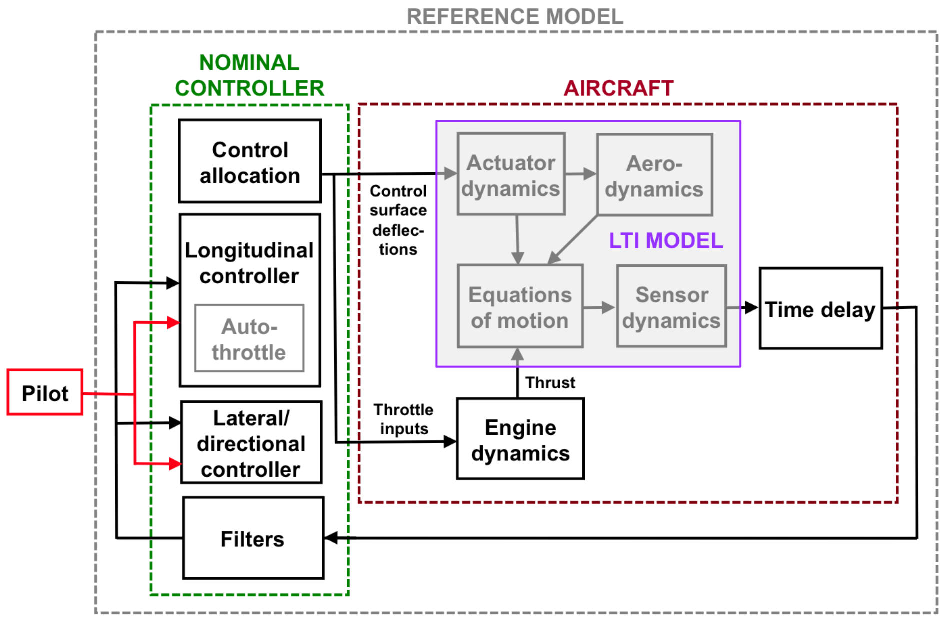


Figure 2-3: Implemented reference model.

This model will be used to design the adaptive controller, but not for calculating x_m during implementation. In the following we discuss our reference model implementation in detail.

The dynamics of the linear reference model in Equation (6.5) may differ considerably from those of the actual aircraft. The unmodeled linear dynamics and nonlinearities responsible for this will trigger undesired adaptation. Since the primary objective of adaptive control is to compensate for parametric uncertainties and not for nonlinear dynamics [1], this situation may seriously compromise the aircraft's stability and performance. In this section we examine alternatives for expanding the flight envelope where the reference model describes accurately the closed-loop dynamics corresponding to the nominal controller. A natural choice for the plant model in the reference model design is a full nonlinear model. Even though this will directly account for the main nonlinearities, the computational requirements associated with it may be exceedingly high. This complexity results from having to perform a high fidelity simulation in real time as well as from having to verify and validate software and hardware. The search for an accurate yet simple reference model led

[1] In general, the primary objective of adaptive control is to compensate for parametric uncertainties and unmodeled dynamics locally. The latter objective can be attained by identifying the coefficients of a radial basis function expansion of such dynamics. However, the effect of nonlinearities can not be perfectly compensated for globally. The controller proposed does not compensate for such nonlinearities.

us to the system in Figure 2-3. The main features of this system are as follows: (i) the underlying structure of the plant is LTI, (ii) there is an engine model to accurately describe the nonlinear dependency of the thrust on the engine's RPMs, (iii) there is uplink time delay between the controller and the plant capturing the effects of telemetry and signal processing, (iv) there is a down link time delay due to sensor dynamics, (v) there is a bank of low-pass filters for mitigating sensor noise, and (vi) there are anti-aliasing filters and command rate limiters as in the GTM. The states of the reference model implemented include those in the reference model used for design, x_m, the altitude, the three Euler angles, all the delayed states and those of the engine dynamics. We however note that only those in x_m affect the adaptive controller. The sum of the time delay in items (iii) and (iv) constitute a known time delay. In particular, they account for a 9ms downlink delay in all the states used for feedback and a 12ms uplink delay in the application of the controller's output to the plant. Note that the implementation of this reference model is a significant departure of the LTI framework supporting the theory; i.e., signal boundedness and asymptotic tracking cannot be guaranteed theoretically.

Adaptive Law

In this section we present an adaptive law that accounts for control saturation and integration anti-wind up. This anti-wind up scheme is independent of the anti-wind up scheme applied to the nominal controller.

The plant to be controlled assumes the LTI representation

$$\dot{x} = \begin{bmatrix} A_p(\hat{p}) & 0 \\ H & 0 \end{bmatrix} x + B_1 \Lambda(\hat{p}) \left(R_s(u) + d\right) + B_2 \hat{r} \tag{2.28}$$

where $A_p \in \mathbb{R}^{6\times 6}$, $B_1 \in \mathbb{R}^{10\times 5}$, $\Lambda = \text{diag}\{\lambda\} \in \mathbb{R}^{5\times 5}$ and $B_2 \in \mathbb{R}^{10\times 4}$. The states, inputs, and commands in (2.28) are

$$
\begin{aligned}
x &= \left[x_{\text{lon}}^T \; x_{\text{lat}}^T \; e_\alpha \; e_V \; e_\beta \; e_p\right]^T \\
u &= [\delta_e \; \delta_a \; \delta_r \; \delta_{th_L} \; \delta_{th_R}]^T \\
\hat{r} &= [\alpha_{\text{cmd}} \; V_{\text{cmd}} \; \beta_{\text{cmd}} \; p_{\text{cmd}}]^T
\end{aligned}
\tag{2.29}
$$

while $d \in \mathbb{R}^{5\times 1}$ is a vector of input disturbances.

The pre-allocated adaptive input is given by

$$
u_a = [\theta_x \; \theta_d] \begin{bmatrix} \hat{x} \\ 1 \end{bmatrix} = \theta^T \omega, \tag{2.30}
$$

where $\theta_x \in \mathbb{R}^{5\times 10}$ and $\theta_d \in \mathbb{R}^{5\times 1}$ are adaptive parameters, and

$$
\hat{x} = \left[x_{\text{lon}}^T \; x_{\text{lat}}^T \; f(e_\alpha) \; e_V \; e_\beta \; e_p\right]^T \tag{2.31}
$$

is the state being fed back. The function f, which is part of the adaptive anti-wind up logic, is defined as $f(e_\alpha) = R_e(e_\alpha, \delta_e)$. Adaptive laws without the anti-windup modification make f equal to its argument so $\hat{x} = x$. The adaptive input is

$$
u_{\text{ada}} = G_{\text{ada}} u_a, \tag{2.32}
$$

where $G_{\text{ada}} \in \mathbb{R}^{10\times 5}$ is a control allocation matrix. The allocation of u_a by G_{ada} makes the deflection of the four elevators equal, the deflection of both rudders equal, and the deflection of both ailerons equal in magnitude with opposite directions.

The adaptive laws are chosen as

$$
\dot{\theta} = \text{Proj}\left(\theta, -\Gamma_1 \omega e_u^T P B_1 \text{sign}(\Lambda) \Gamma_2\right) \tag{2.33}
$$

$$
\dot{\hat{\lambda}} = \Gamma_\lambda \text{diag}(\kappa) B_1^T P e_u \tag{2.34}
$$

$$
\dot{e}_\Delta = A_m e_\Delta + B_1 \text{diag}\left(\hat{\lambda}\right) \kappa \tag{2.35}
$$

$$
\kappa = u_\Delta + (K_{e_\alpha}^T + \theta_{e_\alpha}^T) e_{\alpha,\Delta} \tag{2.36}
$$

where Proj$(\cdot)$ is the projection operator [36], [51] which will be defined later in Chapter 3 in (3.21), $e_u = e - e_\Delta$, $P = P^T > 0$ satisfies $A_m^T P + PA_m = -Q$ for a fixed $Q = Q^T > 0$, $e = \hat{x} - x_m$, and u_Δ is the input deficiency given in Equation (2.16). While e_Δ is the error caused by the saturation of the control inputs and of the integral error state e_α, e_u can be considered as the error caused by parametric uncertainties. The variables $Q > 0$, $\Gamma_1 \in \mathbb{R}^{11\times 11} > 0$, $\Gamma_2 \in \mathbb{R}^{5\times 5} > 0$, $\theta_{\max} \in \mathbb{R}^{11\times 11} > 0$ and $\Gamma_\lambda \in \mathbb{R}^{5\times 5} > 0$ are design parameters.

The *anti-windup modification* to the adaptive law is enforced using the variable κ. κ depends on the column vectors of K_n and θ corresponding to e_α and $f(e_\alpha)$ respectively. In contrast to the anti-wind up modification of Section 2.2.2, this anti-windup modification not only modifies the integral error state used for feedback (i.e., ω) but also changes the controller gain (i.e., θ). The anti-windup modification for the $\langle e_\alpha, \delta_e \rangle$ pair is based on monitoring the total elevator input and modifying the integral error state of the adaptive controller when saturation occurs. The strong coupling between β and p, and the nonlinear engine dynamics made the anti-wind up modification for the $\langle e_\beta, \delta_a \rangle$, $\langle e_p, \delta_r \rangle$ and $\langle e_V, \delta_{th} \rangle$ pairs ineffective. This is the reason κ in Equation (2.36) only takes e_α into account[2].

The adaptive law in Equations (2.33-2.36) makes the plant's state track the state of the reference model, accommodates for control saturation, and mitigates the effects of integral windup in e_α. The Lyapunov stability analysis in reference [37] demonstrates that for a bounded set of commands, θ, x and e are semi-globally bounded. This result holds under the assumption that time-delays and unmodeled dynamics are not present and that both the plant and the reference model are LTI.

In the LTI framework supporting the theory, asymptotic tracking and stability are guaranteed for any adaptation rates satisfying $\Gamma_1 > 0$, $\Gamma_2 > 0$ and $\Gamma_\lambda > 0$. While excessively small adaptation rates nullify the advantages of adaptation by practically turning the adaptive controller off, excessively large ones, along with noise, saturation, time delay and/or unmodeled dynamics, induce high frequency oscillations that may not only degrade the system performance but can also lead to instability. The challenge from the control designer perspective is to balance these two attributes. A dead zone, where Γ_1 and Γ_λ are

[2]The modifications for the $\langle e_V, \delta_{th} \rangle$, $\langle e_\beta, \delta_r \rangle$ and $\langle e_p, \delta_a \rangle$ pairs, which are solely based on the developments of Section 2.2.2, are based on monitoring the control inputs generated by the nominal controller and modifying the integral error states of such a controller when saturation/desaturation occur.

made equal to zero (i.e., adaptation is switched off) depending on the state of the aircraft x_p, the pilot's command $\hat{r}$, and the tracking error due to uncertainties e_u, can be used to counteract some of the anomalies caused by unmodeled dynamics. In particular, we imposed dead zones when the deviation in V and bank angle from the trim state were large. In those regions, the significant discrepancies between the dynamics of the reference model and of the plant trigger unintended adaptation. The domain where the adaptive rates are non-zero and $\theta(t)$ is away from the projection boundaries in Equation (2.33) defines the range of adaptation.

2.3 Simulation Studies

In this section we showcase some of the advantages and disadvantages of adaptation using a set of batch simulations for various realizations of the uncertainty/failure.

2.3.1 Off-line Simulation

These are simulations where the reference commands are set a priori and for which the aircraft performs the desired maneuver under nominal flying conditions. For this we use the high-fidelity model described in Section 2.1, a set of representative flying maneuvers, a flight-validated nominal controller [14], and adaptive controllers with the structure as above but having various adaptation rates. These rates were prescribed according to the observed aircraft performance for a representative set of flying maneuvers and uncertainties among an extensive set of candidate designs. This is the most conventional tuning practice.

Figure 2-4 shows the closed-loop response of the nominal and augmented controllers to a set of command doublets when the effectiveness of the elevators is reduced to 50% and a severe degradation in pitch stiffness $C_{m\alpha}$ and roll damping C_{lp} occur. This case corresponds to 100% uncertainty in the nominal value of the aero-coefficients (open-loop marginal stability). Note that the nominal controller is unable to stabilize the pitch dynamics. The augmented controller on the other hand, not only stabilizes these dynamics but also exhibits a much better roll rate tracking. This is a situation where adaptation yields a significant improvement in performance. Against other types of parametric uncertainties

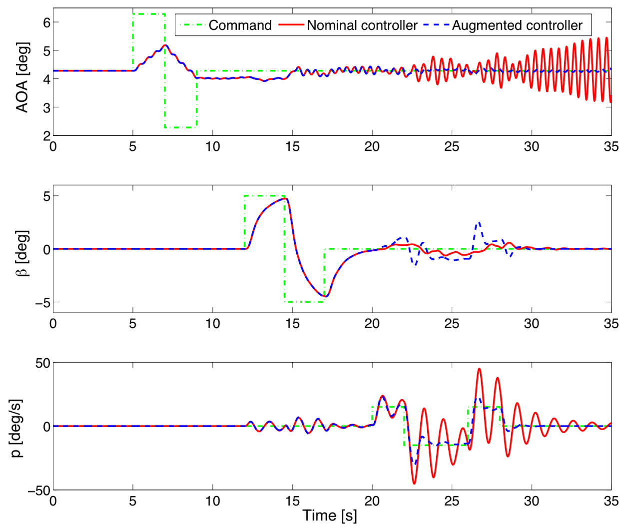


Figure 2-4: Closed-loop responses corresponding to the nominal and augmented controllers, where the effectiveness of the elevators is reduced to 50% and a severe degradation in pitch stiffness $C_{m\alpha}$ and roll damping C_{lp} occur.

and/or failures, non-trivial improvements in flight safety due to adaptation were demonstrated [15].

However, the adaptive component of this controller exhibits undesirable behavior for another type of uncertainties - unknown time delays. Figure 2-5 shows the closed-loop response for the same controllers when there is an uplink time delay of 60 ms. While the nominal controller achieves command tracking with minimal residual oscillations, the augmented controller yields a severely degraded response. The response to larger time delays, where the nominal system response is stable but the augmented one is not, demonstrates that adaptation itself can compromise safety.

While the first case demonstrates the promising ability of adaptation to achieve safe flight against some realistic failure scenarios, the second case raises a robustness concern against unknown time delays and unmodeled dynamics.

2.3.2 Tuning of Control Parameters

It should be noted that the above cases also highlight the importance of prescribing adaptive rates that effectively compensate for uncertainties and failures without magnifying the

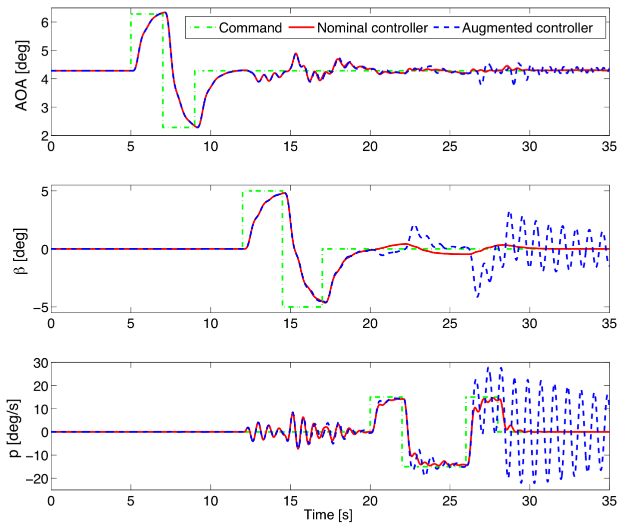


Figure 2-5: Closed-loop responses corresponding to the nominal and augmented controllers, with an uplink time delay of 60 ms.

adverse effects caused by unmodeled dynamics and time delays. Because these simulations (as all simulations) only give a local notion of the system's robustness, the framework proposed in [13] is used in [15] to evaluate robustness from a global perspective and tune the control parameters effectively. This framework enables sizing the set of deviations from nominal operating conditions for which the closed-loop requirements are met. The analysis is performed in a setting where most of the assumptions and simplifications supporting the control design procedure (e.g., decoupled longitudinal and lateral/directional dynamics, LTI plants, existence of matching conditions) do not hold. The specific adverse conditions considered are grouped into four categories: aerodynamic uncertainties (i.e., deviations in pitch stiffness, roll and yaw damping from nominal values), aspects of structural damage (e.g., situations where the Center of Gravity (CG) moves from its nominal location), unknown time delays, and actuator failures (e.g., situations where symmetric and asymmetric failures in control surfaces and engines occur). These failures include partial and total loss of control effectiveness, locked-in-place control surface deflections, and engine-out conditions. The requirements considered are fast pilot command tracking, bounded structural loading, bounded flight envelope (i.e., region in the state space where the aircraft dynamics are properly modeled and flying is safe); and satisfactory handling/ride qualities. We note that the controller's ability to satisfy these requirements depends on the aircraft's transient response, whose representation is mathematically intractable due to nonlinearities. Further these requirements define conflicting objectives. The application of this framework to a MRAC designed for the GTM illustrates some advantages and liabilities of this control architecture as well as the risks of over-tuning the controller's parameters based on point simulations.

In [15], a computational approach that integrates a design-optimization technique into this robustness analysis framework is used to search for the controller's parameters that yield optimal characteristics. We note that the adaptive controller's parameters, which have a significant influence on the system's response, are commonly set using trial and error procedures, or ad-hoc [18]. These procedures may not converge to a controller with the desired robustness characteristics. Furthermore, the determination of whether the convergence took place or not is based on computationally intensive Monte Carlo analyses. These analyses provide no guidance on how to tune the controller's parameters to achieve the desired objectives. The presence of conflicting design objectives (e.g., achieving a fast transient response and a sufficiently large time delay margin in the presence of uncertainty) further obscures the notion of causality required to deploy such methods effectively. The control tuning practice proposed in [13] compensates for these deficiencies by searching for controllers with improved characteristics in a systematic and automated fashion.

The framework supporting the analysis and the control tuning method based on it are out of the scope of this thesis. The readers refer to [15] for the details how we set up the framework for the proposed flight control architecture and tune the control parameters, as well as the analysis results of the proposed controller.

2.3.3 Real Time Simulation

With the optimized control parameters determined as in Section 2.3.2, extensive piloted simulations were performed as well for several sets of flying conditions. In those conditions, not only the trim point at which the controller is engaged (e.g., wings-level flight at various airspeeds), but also the desired maneuver was varied. These maneuvers included

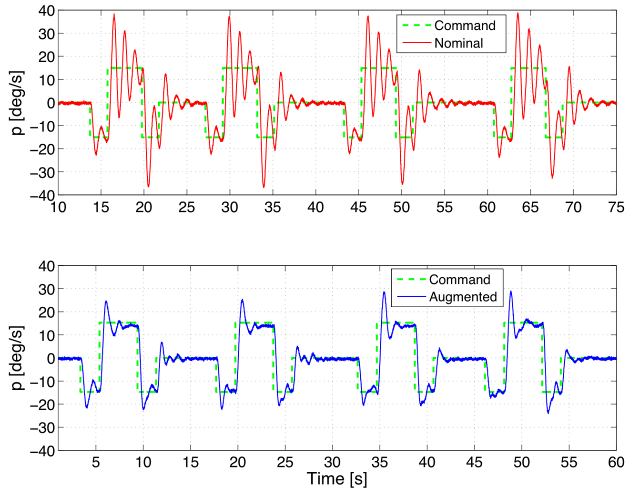


Figure 2-6: Closed-loop responses corresponding to a train of roll rate commands

coordinated turns, angle-of-attack captures, crab configurations and off-set landings. In the real time simulation, and in contrast to the simulations above, the pilot commands $\hat{r}$ are generated in real time according to the desired trajectory and the aircraft's response. Two FAA licensed commercial, multi-engine, and instrument-rated pilots performed the piloted simulations. They have served as research pilots on several NASA remotely piloted vehicle research programs [16]. Furthermore, there is a more accurate aerodynamic model, a surface dead-band in all control surfaces, sensor noise and moderate turbulence.

Figure 2-6 shows the closed-loop responses corresponding to the nominal controller and the augmented controller when δC_{lp} and $\delta C_{m\alpha}$ make the open-loop unstable. The improvement in the transient roll rate response attained by adaptation is apparent.

2.4 Summary

In this chapter, we designed an adaptive control architecture for safe flight, particularly for the GTM developed by NASA Langley Research Center. The high-fidelity simulation model of the GTM aircraft enables us to determine whether the improvements in stability, safety, and performance expected can be realized in practice. Extensive simulation studies using the model, which were conducted with various uncertainties and failures, indicate some advantages and drawbacks of adaptation. While a significant improvement in flight safety introduced by the proposed control design is observed in several realistic failure and damage scenarios, an undesirable flight performance and robustness concerns also became apparent. Particularly, the robustness concern against unknown time delays is crucial. It should be noted that it is one of the critical components usually tested in issuing a flight certification up to how much delay the flight control system remains stable. For the rest of the thesis, we will focus on developing robust adaptive control systems with respect to time delays as well as achieving analytically computable delay margins.

Chapter 3

Standard MRACs in the Presence of Time Delay

The study of adaptive control has enjoyed its theoretical development and maturity for over 30 years. It seemed that the recent technologies finally had catch up to make this theoretically mature discipline to be introduced into the real world. However, the recent trials and works in the fields, as well as the actual adaptive flight control architecture developed in the previous chapter, have revealed or confirmed several undesirable aspects of this technology. Some of those aspects have been already recognized as a concern in the community [52, 50, 26] and studied as a context of robust adaptive control (see [27] for example). Examples among them are bounded disturbances or a class of unmodeled dynamics, which have been solved by suggesting new theories, modifications or strategies. There are still some crucial issues remaining to be solved [1] before we bring adaptive controllers into the real world - which include time delays.

Time delay is the property of a physical system by which the response to an applied signal or delayed in its effect. Whenever material, information or energy is physically transmitted from one place to another, there always exists a delay. As a result, a large class of physical systems are modeled well by describing them with delays. The presence of large delays makes system analysis and control design much complex. A general concept of a feedback control system is to react immediately to errors such that the errors are reduced or eliminated in time. However, for a system with time delays, only after the inherent delays do the errors start to have its influence over the whole system. Therefore, the situation is changed significantly once large delays come into the system. In the worst case, delays are too large such that the system does not hold a desirable property such as stability, which is ensured by one without delays. Therefore, it is important to recognize the existence of delays in a system, and properly understand how they change the property of the whole system. In the end, we have to design a controller such that the delays do not trigger the system to suffer from poor transient or even to go unstable.

For some systems, it is rather easy to evaluate or estimate the time delays, such as telecommunication delays. If we know the exact or have a good estimate of a time delay, we can design a controller by applying the techniques suggested recently which guarantee stability with its presence. Examples of those adaptive controllers which achieve this goal are [59, 46, 10, 48, 62].

For some systems, however, it is difficult to determine how much time delays are present in the system. Even the systems which are constructed such that stability is guaranteed for a certain time delay which is known, they may become unstable if the delay changes - i.e. an unknown time delay exists. Especially for adaptive systems, in simulations, industrial applications and experiments, it has been observed that a large unknown time delay messes up adaptation and undesirably keeps giving energy to the system, as a result of which the system goes instability. In these cases, the question is what is the time delay margin, i.e. up to how much time delay deviation the system is guaranteed to be globally stable. In the following of the thesis we analyze and establish time delay margins for adaptive systems. In the analyses, any approximation like Pade approximation which are used in some of the previous works is not used, except for Chapter 4.

To begin with, in this chapter we first show that standard adaptive systems (standard MRAC, possibly with σ-modification) do not have a time delay margin. We also show that a delayed adaptive stabilizer modified only with a projection algorithm guarantees non-zero time delay margin.

The main claims to be appeared in this chapter are the followings.

Theorem 1. *A standard MRAC does not have a time delay margin, which means that for any positive time delay $\tau > 0$, global boundedness does not hold.*

Theorem 2. *A standard MRAC with σ-modification does not have a time delay margin.*

Theorem 3. *An adaptive stabilizer system with a projection algorithm does have a non-zero time delay margin, which is analytically computable.*

The theorems are stated precisely and discussed in details in the following sections.

3.1 Problem Statement

It has been long believed that adaptive controllers tend to result in undesirable behaviors in the presence of time delays. Furthermore, some people in the community have associated the reason to that since the one of the main features of adaptation is auto-tuning of feedback gains in the *time* domain. However, it has never explicitly analyzed what are the crucial factors of adaptive systems with respect to delays. In this section, we try to identify one of the dominant factors of instability induced by time delays, in order to get an insight and understand the issue of adaptive systems with delays.

As one of the most basic adaptive controllers known, a model reference adaptive controller (MRAC), possibly with some modifications, is studied in this section.

3.1.1 Problem Formulation

A first order plant with a scalar input and a parameter uncertainty is given by

$$\dot{x_p}(t) = a_p x_p(t) + b_p u(t) \tag{3.1}$$

where a_p is an unknown parameter. For the sake of simplicity we assume that $b_p = 1$ without loss of generality. Also, we assume that $a_p > 0$, which means the open-loop is unstable so definitely it needs a non-trivial control input to be stabilized. An adaptive controller is chosen as

$$u(t) = \theta(t)x_p(t) + r(t). \tag{3.2}$$

where $\theta(t)$ is time varying and $r(t)$ is a reference input.

The input $u(t)$ is additionally subject to the unknown time delay which is denoted τ.

$$\begin{aligned} \dot{x_p}(t) &= a_p x_p(t) + u(t-\tau) \\ &= a_p x_p(t) + u(t) + \underbrace{\big(u(t-\tau) - u(t)\big)}_{\eta(t)} \end{aligned} \tag{3.3}$$

Therefore the system subject to the input time delay is interpreted as a disturbed system with an unmodeled dynamics $\eta(t)$.

A reference model is chosen as

$$\dot{x}_m(t) = a_m x_m(t) + r(t) \tag{3.4}$$

where $a_m < 0$. A matching condition defines $\theta^\star$ as

$$a_p + \theta^\star = a_m. \tag{3.5}$$

Errors in the state and the gain are given as

$$\tilde{\theta}(t) = \theta(t) - \theta^\star(t), \quad e(t) = x_p(t) - x_m(t). \tag{3.6}$$

The closed-loop is then given by

$$\dot{x}_p(t) = a_m x_p(t) + \tilde{\theta}(t) x_p(t) + r(t) + \eta(t) \tag{3.7}$$

which gives the error equation

$$\dot{e}(t) = a_m e(t) + \tilde{\theta}(t) x_p(t) + \eta(t). \tag{3.8}$$

An adaptive law can be chosen as

$$\dot{\theta}(t) = -\gamma x_p(t) e(t) \tag{3.9}$$

based on which several modifications exist, such as

$$\dot{\theta}(t) = -\gamma x_p(t)e(t) - \sigma\theta(t) \quad \sigma\text{-modification} \tag{3.10}$$

$$\dot{\theta}(t) = \text{Proj}(\theta(t), -\gamma x_p(t)e(t)) \quad \text{projection algorithm} \tag{3.11}$$

where the projection operator is later defined in (3.21). With the adaptive law given by (3.9), a time derivative of the following Lyapunov function

$$V = \frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{3.12}$$

is obtained as

$$\dot{V} = a_m e^2 + e\eta. \tag{3.13}$$

When there is no time delay($\tau = 0$), (3.13) becomes $\dot{V} = a_m e^2$, and since $a_m < 0$, we can show $\dot{V} < 0$ and therefore the adaptive system is globally stable with the single equilibrium point at the origin. Therefore, the adaptive controller given by (3.2)(3.4)(3.5)(3.6), and (3.9) truly achieves the goal of stabilizing an uncertain unstable plant (3.1).

3.1.2 Robustness of Standard MRACs against Time Delay

The story is different when there is a time delay $\tau > 0$ in the system. The following figures describe how the system responses change when there is a delay in input.

Figure 3-1 shows two trajectories in $(e, \tilde{\theta})$ space with and without a delay. It can be seen that although starting the same initial condition the trajectories behave differently, and with a delay the system results in an unbounded solution.

Figure 3-2 shows a time response of signal $e(t) \times \eta(t)$. It can be seen that $\eta(t)$, which is a disturbance term due to the delay, belongs to nearly the opposite sign from that of $e(t)$. In (3.13), it can be implied that if the term $e(t)\eta(t)$ takes positive values and dominates, it leads the system to instability.

In the following we consider a stabilizer case, i.e. $x_m(t) = 0 \ \forall t$ and therefore

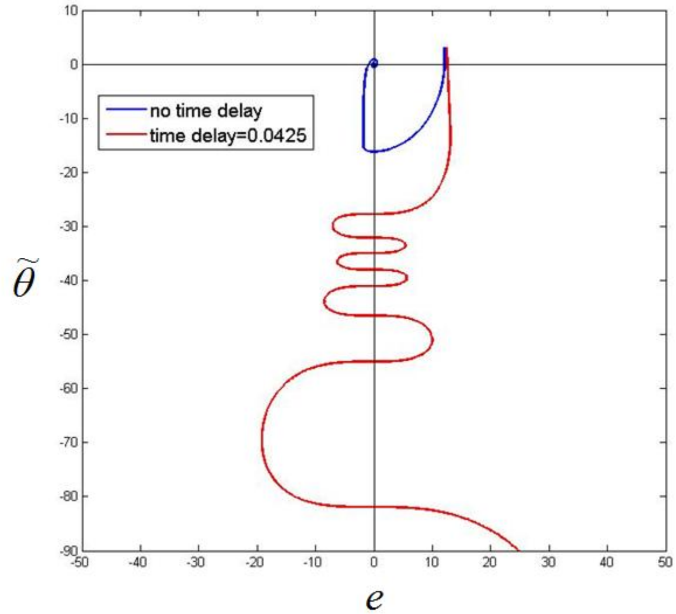


Figure 3-1: Trajectories with / without a delay, given the same initial condition

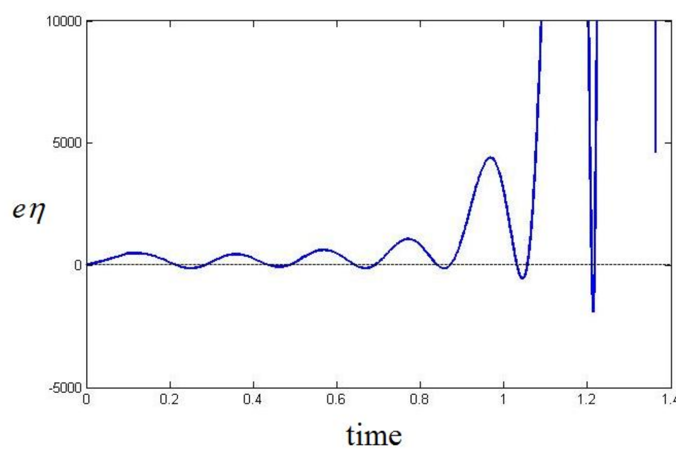


Figure 3-2: Time response of $e(t) \times \eta(t)$

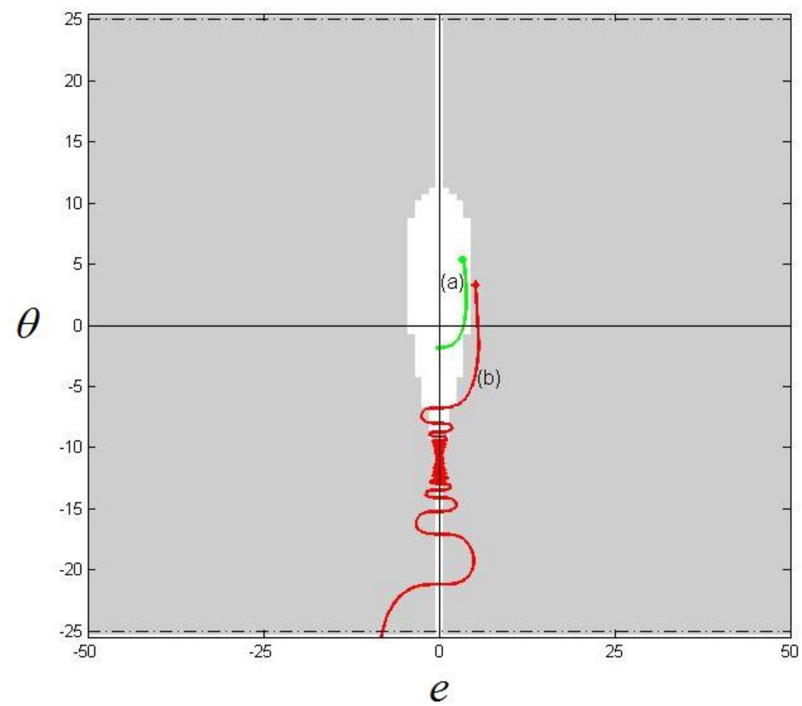
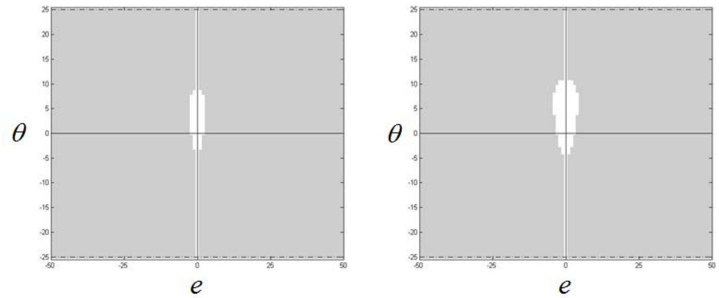
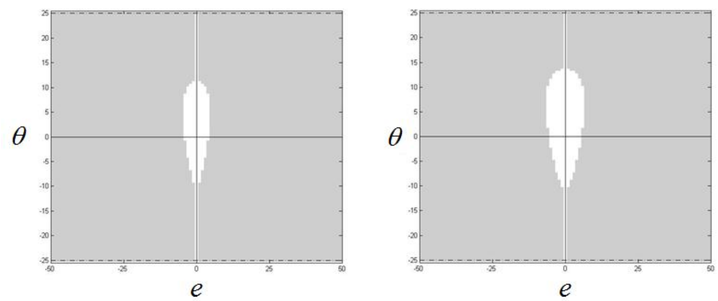


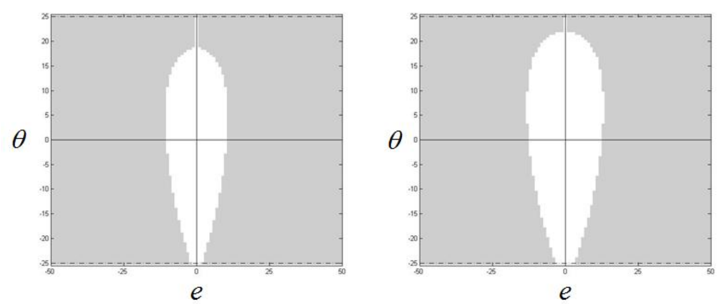
Figure 3-3: Trajectories with different initial conditions; implying local stability, and stable(*S*):white / unbounded(*F*):gray domains with non-zero delay



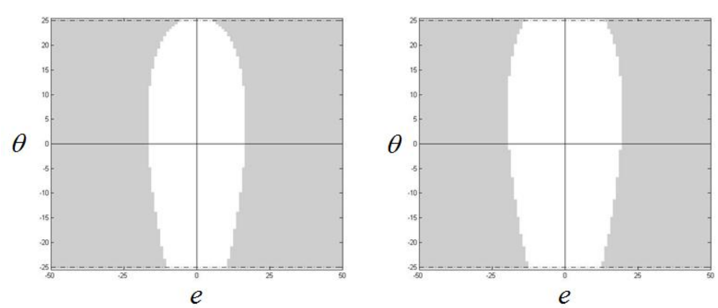
(a) $\tau = 0.135$



(b) $\tau = 0.090$



(c) $\tau = 0.045$



(d) $\tau = 0.030$

Figure 3-4: Stable(S):white / unbounded(F):gray domains with delays. Left - standard MRAC, Right - with σ-modification.

$x_p(t) = e(t)\ \forall t$. Although the analyses conducted in this chapter are only on stabilizer systems and therefore restricted, it will be shown that we can identify one of the crucial sources of instability of adaptive systems with the simplest adaptive laws (3.9) and (3.10).

We now widen our scope to the whole state-space to capture the relation between initial conditions and stability of adaptive systems with a delay. Here we introduce the definition of domains as follows;

$$F = \Big\{ (x_0, \theta_0) \in \mathbb{R}^2 \Big| \text{if } x_p(t) = \chi_x(t), \theta(t) = \chi_\theta \forall t \in [t_0 - \tau,\ t_0], \\ \text{then } \forall r \in \mathbb{R}\ \ \exists t \in (t_0,\ \infty) \text{ s.t. } |x_p(t)| + |\theta(t)| > r \Big\}$$

where $\chi_x(t)$ and $\chi_\theta(t)$ specify the initial conditions

$$\left\{ \begin{array}{c} \chi_x(t) = 0\ \forall t \in [t_0 - \tau,\ t_0), \quad \chi_x(t_0) = x_0 \\ \chi_x(t) = \theta_0\ \forall t \in [t_0 - \tau,\ t_0] \end{array} \right\}.$$

In other words, a failure domain F is a set of initial conditions with which if the system starts at $t = t_0$, then either x_p or θ, or both become unbounded. We further define the compliment of F as S, a safe domain.

Given a time delay $\tau = 0.090$, Figure 3-3 shows two trajectories starting at different initial conditions. Since trajectory (a) in Figure 3-3 converges to a point and achieves $e(t) \to 0$ as $t \to \infty$,[1] it is believed that this initial condition belongs to the set S. It is also seen that most likely trajectory (b) belongs to the set F. According to these observations, schematic shape of two domains are added over Figure 3-3 as white (S) and gray (F), respectively. [2]

Figure 3-3 gives us an important insight. Compared to adaptive systems without a time delay for which *global stability* is guaranteed, the system with a delay is only *locally stable*.

Figure 3-4 shows some rough numerical simulation results with different size of time delays. The plots are obtained from numerous point simulations, by determining whether

[1] Trajectory (a) however does not converge to the origin. This is because that in the adaptive stabilizer case, $\dot{\theta}$ never takes positive values as later noted in (3.18).

[2] We note that actually it can be never concluded only from the simulation studies that a certain trajectory is unstable or not, since it may still finally cease its growth and be bounded. In the next section, we prove the instability rigorously.

the trajectory results in stable or unbounded solutions, with the point in state space set to be an initial condition. The size and the shape of the domain S (or F) changes depending on the system parameters such as the speed of adaptation γ, size of the time delay τ, and plant parameter a_p. With a smaller delay τ, the system allows larger initial conditions which ensure boundedness. However, it is seen from the figures that the system always only results in local stability with delays of these four different values. It can be actually observed from the similar simulation studies that for any non-zero delays, we can always find the failure domain which results in unbounded solutions.

According to our definition, this corresponds to that the system does not have a non-zero delay margin, as stated in Theorem 1. Actually the same theorem also holds for MRAC with σ-modification, which is one of the simplest modification in robust adaptive control and known to be globally stable in the presence of bounded disturbances. This is stated as Theorem 2.

In the following section we rigorously prove these Theorems.

3.2 Instability of Standard MRAC in the Presence of Time Delay

In this section, we prove Theorem 1, the first of the main claims in this chapter. We start with the following theorem for a delayed LTI system.

Theorem 4. *(Instability of Delayed LTI System)*
For any non-zero time delay $\tau > 0$, there exists $k^\star$ such that for all $k > k^\star$ a system

$$\dot{x}_p(t) = a_p x_p(t) - k x_p(t-\tau), \quad a_p > 0 \tag{3.14}$$

is unstable.

Proof. Including a term of disturbance,

$$\dot{x}_p(t) = a_p x_p(t) - k x_p(t-\tau) + d(t),$$

where we can assume boundedness as $|d(t)| \leq d_{\max}$. Taking Laplace transform gives us

$$X(s) = \frac{D(s)}{s - a_p + k\exp(-\tau s)}$$

where $X(s), D(s)$ are the Laplace transform of $x_p(t), d(t)$, respectively. Figure 3-5 shows the trajectory of s which satisfies $s - a_p + k\exp(-\tau s) = 0$ as k changes from 0 to ∞. The specific values we take to obtain Figure 3-5 is $a_p = .1$ and $\tau = 1$. It corresponds to a root locus plot of the corresponding open-loop system with a fixed time delay. Note that there are infinite number of poles for the delayed system. Other examples of root locus diagrams for time-delay systems may be found in [47].

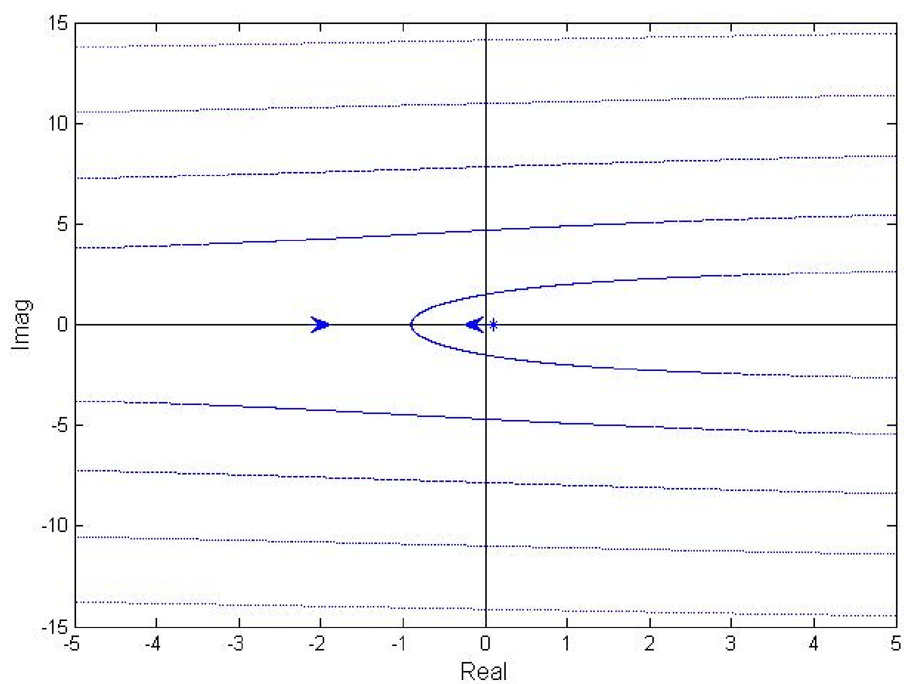


Figure 3-5: Root locus with a delay

If we take $k^\star$ which satisfies

$$\tan(\omega^\star \tau) = \frac{\omega^\star}{a_p}$$
$$k^\star = \sqrt{a_p^2 + \omega^{\star 2}},$$

then with $k > k^\star$, $s - a_p + k\exp(-\tau s) = 0$ has at least two solutions which lie strictly in the right half plane. Therefore, we can see that the system is unstable, proving the theorem. □

$k^\star$ is the function of τ and it can be further seen that given a constant feedback gain, the system becomes unstable beyond a certain value of the time delay. We now extend Theorem 4 to a broader class of systems, where the feedback gain is now time varying.

Theorem 5. *(Instability of Delayed LTV System)*
For any non-zero time delay $\tau > 0$, there exists $k^\star(\tau)$ such that for all $k(t)$ which is an element of the class of systems $K; [t_0 - \tau \ \ \infty) \to \mathbb{R}^1$ and satisfies $\lim_{t\to\infty} k(t) \to k_l$ where $k_l > k^\star(\tau)$, a system

$$\dot{x}_p(t) = a_p x_p(t) - k(t-\tau) x_p(t-\tau) \tag{3.15}$$

is unstable.

Proof. We first note that since $k(t)$ is a converging analytic function, the real parts of all the characteristic roots λ_i $i = 1, 2, ...$ of $k(t)$ satisfies $Re(\lambda_i) \leq 0$. Taking Laplace transform of (3.15), including a term of disturbance, gives

$$sX(s) = a_p X(s) - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(p)X(s-p)\exp(-\tau s)dp + D(s), \tag{3.16}$$

where the integration is done along the vertical line $\Re(p) = c$ that lies entirely within the region of convergence of $K(s)$, which is the Laplace transform of $k(t)$. (3.16) can be rewritten as

$$X(s)(s - a_p + k_l\exp(-\tau s)) = \sum_{i,\lambda_i \neq O} k_i X(s-\lambda_i)\exp(-\tau s) + D(s). \tag{3.17}$$

where k_is are the constants each of which corresponds to each pole λ_i. O denotes the origin.

Since $k_l > k^\star$, $s - a_p + k_l \exp(-\tau s) = 0$ has at least two solutions which lie strictly in the right half plane. Noting that λ_i in the summand have strictly negative real part, it can be proven from (3.17) that $X(s)/D(s)$ has unstable poles. □

Finally, we provide the proof for Theorem 1. We start with reformulating the theorem statement.

Theorem 1'. *For any non-zero time delay $\tau > 0$, there exists a reference command $r(t)$ $t \in [t_0 \ \ \infty)$ and an initial condition $x_p(t_0) = x_0 \in \mathbb{R}^1$ such that an adaptive system given by (3.3), the control law (3.2), and the adaptive law (3.9) with (3.4) is unstable.*

Proof. The statement can be proven with the simple reference command $r(t) = 0 \quad \forall t \geq t_0$ (stabilizer), which then lets us to assume $x_m(t) = 0 \quad \forall t \geq t_0$ given $x_m(t_0) = 0$. We note that with setting $x_m = 0$, (3.9) becomes

$$\dot{\theta} = -\gamma x_p^2. \tag{3.18}$$

The theorem is proven by contradiction. We define the state of the system as $z = \begin{bmatrix} x_p & \theta \end{bmatrix}^T$. Suppose given τ, the system has a bounded solution $z(t)$ $\forall t \in [t_0 \ \ \infty)$ for all initial conditions $\forall z_0 \in \mathbb{R}^2$ where $z(t_0) = z_0 = \begin{bmatrix} x_0 & \theta_0 \end{bmatrix}^T$. The assumption of boundedness of $\theta(t)$ immediately leads to $\exists M > 0$ such that $\theta(t) \geq -M \quad \forall t \geq t_0$. Since $\theta(t)$ is a monotonically decreasing function from (3.18), we can conclude that there exists $k_l \in (-\infty \ \ M]$ which satisfies $\theta(t) \to -k_l \quad$ as $t \to \infty$. The following lemma is then useful.

Lemma 1. *For any $M' > 0$, there exists $z_0 = \begin{bmatrix} x_0 & \theta_0 \end{bmatrix}^T$ which leads to $z(t_0 + \tau) = \begin{bmatrix} x_\tau & \theta_\tau \end{bmatrix}^T$ where $x_\tau \neq 0$ and $\theta_\tau < -M'$.*

Proof. Consider the case when the system stays at equilibrium $\forall t \in [t_0 - \tau \ \ t_0)$, i.e., $x_p(t) = 0$ $\forall t \in [t_0 - \tau \ \ t_0)$. Then let the sudden state change happens at $t = t_0$, that is $x_p(t_0^+) = x_0$. We note that $x_p(t_0^-) = 0$. This state change can be due to a gust, or some disturbances, which happen in the physical world. There is no discontinuity in θ at $t = t_0$. Due to the time delay, the control signal stays at zero $u(t-\tau)=0\ \forall t \in [t_0\ t_0+\tau)$. The state response is then given as

$$x_p(t) = \exp(a_p t) x_0 \quad \forall t \in [t_0\ t_0+\tau),$$

the magnitude of the system state is lower bounded as $|x_p(t)| \geq |x_0|\ \forall t \in [t_0\ t_0+\tau)$. By substituting into (3.18) and taking integral, $\theta(t_0+\tau)$ is then upper bounded by $\theta(t_0+\tau)-\theta(t_0) \leq -\gamma|x_0|^2\tau$. If we chose the initial condition which satisfies $|x_0| > \sqrt{\frac{M'+\theta_0}{\gamma\tau}}$, then $\theta_\tau < -M'$. □

Since $\theta(t)$ is a monotonically decreasing function, if we take sufficiently large $M' = k^\star$, $k_l > k^\star$. From Theorem 5, it can be then proven that the state $x_p(t)$ is unstable.

This contradicts the assumption that the system has bounded solution for all $x_p(t_0) = x_0$, completing the proof. □

3.3 Applications

In this section, we show several applications of the proof for Theorem 1 which is discussed in the previous section.

3.3.1 Sigma Modification

In the previous section, we prove instability due to a time delay for the simplest adaptive law given by (3.9). It is shown that given a finite time delay, once the gain $\theta(t)$ reaches the critical value, it leads the plant state $x_p(t)$ to be unstable. Several modifications have been proposed as robust adaptive laws to prevent $\theta(t)$ from blowing up, the most famous one of which would be σ-modification.

In this section we show that the same discussion in Section 3.2 is also applied to the adaptive law with σ-modification given by (3.10), and that the system has no time delay margin either.

Theorem 2'. *For any non-zero time delay $\tau > 0$, there exists a reference command $r(t)$*

$t \in [t_0\ \ \infty)$ and an initial condition $x_p(t_0) = x_0 \in \mathbb{R}^1$ such that an adaptive system given by (3.3), the control law (3.2), and the adaptive law (3.10) with (3.4) is unstable.

Proof. As in Theorem 1, a stabilizer case with the reference command $r(t) = 0\ \ \forall t \geq t_0$ and $x_m(t) = 0\ \ \forall t \geq t_0$ is considered. An adaptive law in (3.18) is replaced by

$$\dot{\theta} = -\gamma x_p^2 - \sigma\theta.$$

Defining $\Theta(t)$ as $\theta(t) = \exp(-\sigma t)\Theta(t)$, we obtain

$$\dot{\Theta} = -\exp(\sigma t)\gamma x_p^2$$

which again ensures $\Theta(t)$ be a monotonically decreasing function. It is also straightforward to see that the closed loop system is given by

$$\dot{x}_p(t) = a_m x_p(t) + \exp(-\sigma(t-\tau))\Theta(t-\tau)x_p(t-\tau). \tag{3.19}$$

By taking Laplace transform of (3.19), it can be seen that a new pole $s = -\sigma$ is introduced into the system. The same discussion then holds as in the proof of Theorem 5. Furthermore, we can find critical initial conditions by proceeding in a similar way as in Lemma 1. Therefore we can prove zero delay margin for the system with σ-modification. □

3.3.2 Upper-bound of Local Stability

In the previous section, zero time delay margins of the standard adaptive systems are proven. In other words, it is shown that for any nonzero time delay τ, these systems always only guarantee *local* stability. Then the next question is, given a value of the delay τ, what is the region where the systems are guaranteed to be stable?

It is challenging to exactly solve the boundary, however by using the part of the discussion in the previous section, an upper bound of the boundary can be obtained in a certain region. Specifically, $|x_0| > \sqrt{\frac{M'+\theta_0}{\gamma\tau}}$ can be regarded as a bound of the unstable domain with the given time delay. Figure 3-6 shows one such example. Here we consider the adaptive system which is defined in (3.1)-(3.9), where $a_p = 5$, $a_m = -1$ and the control parameter is chosen as $\gamma = 2$. A time delay considered is $\tau = .090$.

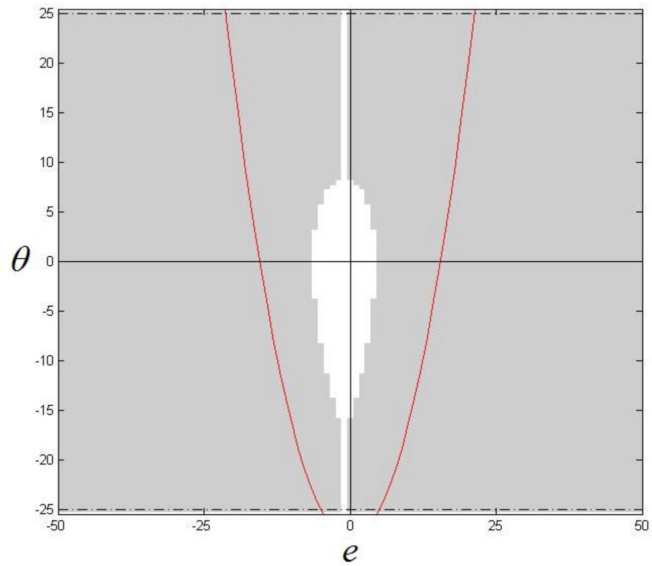


Figure 3-6: Numerically obtained stable(S):white / unbounded(F):gray domains and analytic bound of unstable domain

In this figure numerically computed domains and the analytic boundary (red line), are shown. We note that the analytic result only gives us a super set of S, outside which (below the red line) it is guaranteed that the system becomes *unstable* (domain F) with the given time delay. Note however that it is seen that the actual and estimated boundaries are only differed by the factor of two, or at least of the same order. As a conclusion, this method is considered to be a good estimate of the boundary of stable/unstable domains.

3.4 Projection Algorithm as a Tool to Achieve Global Boundedness

In the previous sections, it was seen that once the parameter being adapted $\theta(t)$ reaches the crucial boundary $k^{\star}(\tau)$, it leads to unstable solutions. Also it is actually straightforward to see that $k^{\star}$ is a monotonically decreasing function of τ. A question which naturally arises is that, given an upper bound of the possible delay in the system $\tau_{\max}$, what if we design the adaptive law such that the parameter being adapted never exceeds the critical boundary $k^\star(\tau_{\max})$?

A projection algorithm is the one which can be used to attain such properties, by enforcing the feedback gain $\theta(t)$ to stay in a certain region. The projection algorithm we consider is formulated below, with general nth order vector θ.

We first state a few definitions and two lemmas. Let sets Ω_0 and Ω_1 be defined as

$$\begin{aligned} \Omega_0 &= \left\{\theta \in \mathbb{R}^n | f(\theta) \leq 0\right\} \\ \Omega_1 &= \left\{\theta \in \mathbb{R}^n | f(\theta) \leq 1\right\}, \end{aligned} \tag{3.20}$$

where $f(\cdot)$ is a convex function. A Projection function, denoted as Proj, is defined as follows:

$$\text{Proj}(\theta, y) = \begin{cases} y - \frac{\nabla f(\theta)(\nabla f(\theta))^T}{\|\nabla f(\theta)\|^2} y f(\theta) \\ \qquad\qquad \text{if} \quad [f(\theta) > 0 \wedge y^T \nabla f(\theta) > 0] \\ y \qquad\qquad \text{otherwise.} \end{cases} \tag{3.21}$$

This is the definition of the projection operator which will be used throughout the thesis.

Lemma 2. *Let $\theta \in \Omega_1$ and $\theta^\star \in \Omega_0$. Then for any vector y, the following inequality holds:*

$$(\theta - \theta^\star)^T (Proj(\theta, y) - y) \leq 0. \tag{3.22}$$

The reader is referred, for the proof of Lemma 2 to [51] and to [36].

Lemma 3. *For any time varying piecewise continuous vector y, if $\theta(t_0) \in \Omega_1$ and*

$$\dot{\theta} = Proj(\theta, y) \tag{3.23}$$

where $Proj(\theta, y)$ is given by (3.21), then $\theta(t) \in \Omega_1$, for all $t \geq t_0$.

Proof. It is sufficient to show that $[f(\theta(t_0)) \leq 1] \Rightarrow [f(\theta(t)) \leq 1]$ for all $t \geq t_0$. Towards this end, compute time derivative of $f(\theta)$ along the trajectories of (3.23):

$$\dot{f}(\theta) = (\nabla f(\theta))^T \dot{\theta} = (\nabla f(\theta))^T \text{Proj}(\theta, y). \tag{3.24}$$

Substituting (3.21) into (3.24) results in

$$\dot{f}(\theta) = \begin{cases} (\nabla f(\theta))^T y(1-f(\theta)) & \text{if } [f(\theta) > 0 \bigwedge y^T \nabla f(\theta) > 0] \\ (\nabla f(\theta))^T y & \text{otherwise.} \end{cases}$$

Consequently,

$$\begin{cases} \dot{f}(\theta) > 0 & \text{if } [0 < f(\theta) < 1 \bigwedge y^T \nabla f(\theta) > 0] \\ \dot{f}(\theta) = 0 & \text{if } [f(\theta) = 1 \bigwedge y^T \nabla f(\theta) > 0] \\ \dot{f}(\theta) < 0 & \text{if } [f(\theta) \le 1 \bigvee y^T \nabla f(\theta) \le 0]. \end{cases} \tag{3.25}$$

The first and the second relations in (3.25) imply that $f(\theta)$ monotonically increases but never exceeds 1, while the third condition states that the function is monotonically decreasing. In other words, if $f(\theta(t_0)) \le 1$ then $f(\theta(t)) \le 1$, for all $t \ge t_0$. This completes the proof of the Lemma. □

Remark 1. *The Projection operator in* (3.23) *modifies the velocity vector y only in the annulus region* $\Omega_1 \backslash \Omega_0$*, such that* $\theta(t)$ *will never leave* Ω_1*, for all future times. This is the main benefit of the Projection operator.*

Now we consider a specific convex function $f(\theta)$ and establish the following lemma.

Lemma 4. *Consider the the dynamics with Projection algorithm in (3.23) with*

$$f(\theta) = \frac{\parallel \theta \parallel^2 - \theta_{max}'^2}{\varepsilon^2 + 2\varepsilon\theta_{max}'} \tag{3.26}$$

where θ_{max}' *and* ε *are arbitrary positive constants. Then,*

$$\parallel \theta(t_0) \parallel \le \theta_{\max} \implies \parallel \theta(t) \parallel \le \theta_{\max} \forall t \ge t_0 \tag{3.27}$$

where $\theta_{\max} = \theta_{\max}' + \varepsilon$.

The proof of Lemma 4 follows immediately from Lemma 3.

If the adaptive law is chosen as in Lemma 3 (3.23) with

$$y = -\gamma x_p e, \tag{3.28}$$

then irrespective of the boundedness of $x_p(t)$ or $e(t)$, it follows that $\theta(t)$ is bounded and lies in Ω_1 if $\theta(t_0)$ and θ^* belong to Ω_1, Ω_0 respectively. With the special choice of a convex function (3.26), this corresponds to that (3.27) is satisfied.

The next theorem states that an adaptive control system with the projection algorithm, where projection bounds are chosen appropriately, has a nonzero delay margin. In this chapter we only discuss about the adaptive stabilizer of a scalar plant. More general controller cases, with either first-order or higher-order plants, will be discussed in the rest of the thesis.

Theorem 3'. *Consider the closed-loop adaptive system given by (3.3), the control law (3.2), the adaptive law (3.23) with y and $f(\theta)$ chosen as in (3.28) and (3.26). If $x_m(t) = 0$ $\forall t \geq t_0$ (stabilizer) and θ'_{max}, ε are such that θ^* in* (3.5) *belongs to Ω_0, there exists $\tau^\star > 0$ such that for any initial conditions $x_p(t_0) \in \mathbb{R}^1$ and $\theta(t_0) \in \Omega_1$, the closed-loop adaptive system has bounded solutions for all $\tau \in [0\ \tau^\star)$.*

Actually, with $x_m(t) = 0$ $\forall t \geq t_0$, the origin is globally asymptotically stable. $\tau^\star$ can be considered as a time delay margin of the system, and obtained by analyzing the LTI system which corresponds to the closed-loop system with $\theta(t) = -\theta_{\max}$.

First, let us define the regions A and B in the system as in Figure 3-7.

Definition 2. *We define the region A and the boundary region B as follow:*

$$A = \left\{ z \in \mathbb{R}^2 | -\theta_{\max} < \theta \leq \theta_{\max} \right\}$$
$$B = \left\{ z \in \mathbb{R}^2 | \theta = -\theta_{\max} \right\}$$

where $z(t) = [x_p(t)\ \theta(t)]^T$. These regions are illustrated in Figure 3-7.

Proof. Without loss of generality, let the initial condition is $z_0 = \begin{bmatrix} x_0 & \theta_0 \end{bmatrix} \in A$.

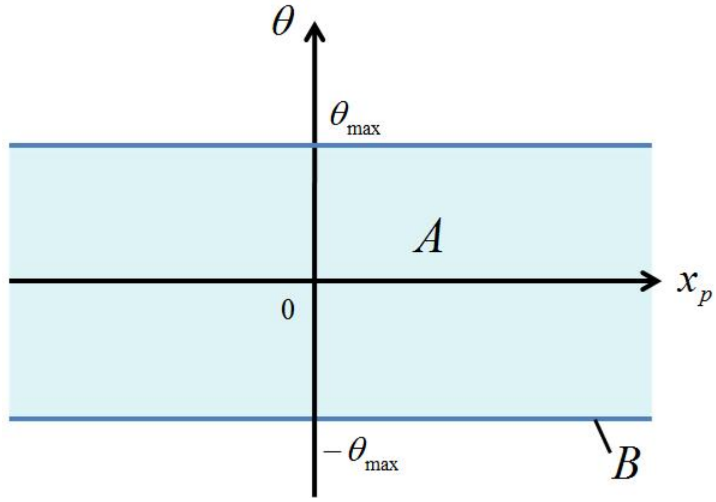


Figure 3-7: Definition of regions

The boundedness of $\theta(t)$ is straightforward from the projection algorithm and it is proven in Lemma 4 that $|\theta(t)| \leq \theta_{\max}$.

First consider the case when $z(t) \in A \; \forall t$. We prove the boundedness of $x_p(t)$ by contradiction. Suppose that $x_p(t)$ is unbounded. Define

$$\begin{aligned} \Psi_1 &= \{t \,|\, |x_p(t)| \leq \varsigma\} \\ \Psi_2 &= \{t \,|\, |x_p(t)| > \varsigma\} \end{aligned} \tag{3.29}$$

where $\varsigma = \sqrt{\frac{\theta_{\max}+|\theta_0|}{\gamma \delta_M}}$. Since $x_p(t)$ is unbounded and an analytic function, $M(\Psi_2)$ is finite where $M(\cdot)$ denotes the measure. Let $0 < \delta_M \leq M(\Psi_2)$. We note that $y = -\gamma x_p^2$ from (3.28), since $x_m(t) = 0 \; \forall t \geq t_0$ immediately implies that $x_p(t) = e(t)$. Then from (3.23),

$$\begin{aligned} \theta(t) &= -\gamma \int_{\Psi_1} x_p(s)^2 ds - \gamma \int_{\Psi_2} x_p(s)^2 ds + \theta_0 \\ &< -\gamma \varsigma^2 \delta_M + |\theta_0| \\ &= -\theta_{\max}, \end{aligned}$$

which contradicts the condition of the case. Therefore it is proven that if $z(t) \in A \; \forall t$, then there exists $\bar{x} \in \mathbb{R}^1$ s.t. $|x_p(t)| < \bar{x} \; \forall t$.

Next, we consider the case when there exists $t_b \in (0 \;\; \infty)$ s.t. $z(t_b) \in B$. From (3.20) and

(3.26), it is straightforward to see that $f(\theta(t_b)) = 1$. Together with the fact that $y \leq 0$ in the stabilizer case, it can be derived from the definition of the projection operator in (3.21) that $\dot{\theta}(t_b) = 0$. This leads to $z(t) \in B$ for all $t \geq t_b$. In other words, once the system reaches the projection boundary $\theta(t) = -\theta_{\max}$ for the first time at $t = t_b$, the system behaves as a linear time invariant system with the fixed feedback gain afterwards ($t \geq t_b$). Therefore the following theorem can be applied to study the stability of the system.

Theorem 6. *(Stability of Delayed LTI System) For any $a_p > 0$, there exists $\bar{\tau}^\star > 0$ such that for all time delays $\tau \in [0\ \bar{\tau}^\star)$, there exists $k_n(\tau)$, $k^\star(\tau)$ with which a system*

$$\dot{x}_p(t) = a_p x_p(t) - k x_p(t-\tau) \quad a_p > 0$$

is stable if $k \in (k_n\ k^\star)$.

Similar to Theorem 4, the proof for Theorem 6 can be obtained straightforwardly by studying the root locus of the delayed LTI system (Figure 3-5). Especially, it can be seen that such k_n, $k^\star$ are given by

$$\begin{aligned} k_n &= a_p, \\ k^\star &= \sqrt{a_p^2 + \omega^{\star 2}} \end{aligned}$$

where $\omega^\star$ is the minimum of all positive real numbers which satisfy

$$\tan(\omega^\star \tau) = \frac{\omega^\star}{a_p}.$$

It can be also proven that $k^\star(\tau)$ is a strictly monotonically decreasing function with respect to τ. Taking $\tau^\star$ which satisfies $k^\star(\tau^\star) = \theta_{\max}$, from Theorem 6 we can show that any trajectory of the system is guaranteed to be bounded with an input delay $\tau \in [0\ \tau^\star)$. Non-zero time delay margin is therefore proven and given as $\tau^\star$, proving Theorem 3. It can be also seen that $\tau^\star < \bar{\tau}^\star(a_p)$.

□

In this section, we introduced the projection algorithm into the adaptive law which immediately enforces θ to stay in a bounded region, and showed that it ensures non-zero time delay margin of the stabilizer system.

Then, what are the drawbacks of having the projection algorithm in the adaptive law? We note that the original purpose of introducing adaptation is to stabilize the closed-loop system in the presence of parametric uncertainty in a_p. Projection algorithm restricts the range of values which θ can take, i.e. $|\theta| \leq \theta_{\max}$. If $a_p \geq \theta_{\max} > 0$, then $a_p - \theta \geq 0$ $\forall |\theta| \leq \theta_{\max}$ and therefore the closed-loop system can not be stabilized even there is no input delay ($\tau = 0$). Actually, the existence of $\theta^\star \in \Omega_0$ ($|\theta^\star| \leq \theta'_{\max}$) which satisfies the matching condition (3.5) is necessary to guarantee the stability of a delay-free adaptive system with the projection.

Therefore in order to apply the projection algorithm, it is necessary to know the size of uncertainty (the bounds on the unknown parameter a_p) so that we can choose the projection parameters $\theta'_{\max}$, ε appropriately. This is in contrast to the standard adaptive law without projection, where any uncertain plant ($\forall a_p \in \mathbb{R}^1$) without a delay is guaranteed to be stabilized.

3.5 Summary

In this chapter we demonstrated instability with the standard MRACs, which confirms the necessity of developing a robust adaptive system with respect to time delay. We also introduced the simple modification based on the projection algorithm, and showed that it leads to a nonzero delay margin of the adaptive stabilizer. Although the analyses conducted in this chapter are only a stabilizer case with a scalar plant and therefore very simple, they enlighten the potential of projection algorithm as a tool to develop a robust adaptive system with respect to time delay. In the following of the thesis, we study more general controller cases.

Chapter 4

Robust MRAC with Projection Algorithm for Global Results

In this thesis, we will use the projection algorithm to achieve a robust adaptive system which ensures global boundedness in the presence of delay, i.e. guarantees nonzero delay margin. In this chapter we conduct some initial analyses on MRAC with projection algorithm with respect to a class of unmodeled dynamics. Even though the result in this chapter is quite restrictive and applicable only to a small class of plants with input delay using Pade approximation, it sheds light on the potential of the projection algorithm as a tool to achieve desirable robustness properties of adaptive systems.

4.1 Projection Algorithm for Global Results

Theorems 1 and 2 demonstrate that neither the standard MRAC nor the MRAC with σ-modification can ensure a delay margin for adaptive systems. A different modification, based on projection algorithm, was also discussed for a stabilizer of a scalar plant and a desirable property stated as Theorem 3 was shown for this simple case. In this section, we focus more on the projection algorithm and discuss our recent work which demonstrates the potential of parameter projection in achieving global results.

Projection algorithms started being used in late 80s in continuous adaptive systems as in [56] so that it enforces parameters to stay in certain appropriate ranges. Later in 90s projection algorithms were studied extensively in the context of robust adaptive control. Among the works, notable results are Pomet *et al.*[51] and Naik *et al.*[41]. In [51] and [41], the authors proved the global boundedness of adaptive systems with projection algorithm against bounded disturbances and a class of unmodeled dynamics, which are however not equivalent to time delays. As discussed in Chapter 3, the idea of protecting parameters being adapted from blowing up due to disturbances or non-parametric uncertainties naturally arises, and its potential was demonstrated with the simple adaptive stabilizer.

Given the above, it is seen that the projection algorithm is a promising tool to achieve our goal of developing a robust adaptive system against time delay and obtaining delay margin. In the following we revisit our recent work [2, 38] which demonstrates the potential of the projection, where global results were obtained by modifying adaptive systems only with projection algorithm [51, 36] in the presence of a class of unmodeled dynamics (see Figure 4-1).

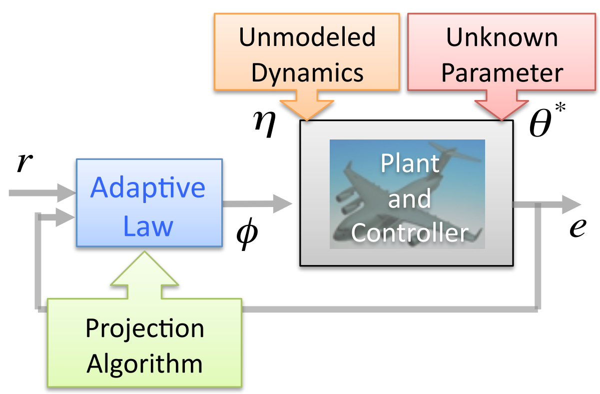


Figure 4-1: Robust adaptive control system with projection algorithm in the presence of unmodeled dynamics

We show that boundedness can be guaranteed for linear plants whose states are accessible for measurement, when subjected to parameter uncertainties and unmodeled dynamics, for arbitrary initial conditions of the plant states. It is assumed that the parameter uncertainties lie in a bounded hypercube, enabling the use of an adaptive law with the parameter projection formulated in Chapter 3 using which the robustness result is established.

4.2 Robust Adaptive Control Revisited

One of the very first problems where stable adaptive control was solved was for the case when states are accessible [43], with the plant given by[1]

$$\dot{x}_p = A_p x_p + b\lambda u \tag{4.1}$$

where $A_p \in \mathbb{R}^{n\times n}$ and the scalar λ are unknown parameters with b and the sign of λ known, and (A_p, b) controllable. It is well known that an adaptive controller of the form

$$u = \theta_x^T(t)x_p + \theta_r(t)r, \tag{4.2}$$

adaptive laws

$$\dot{\theta} = -\Gamma\omega b^T Pe, \tag{4.3}$$

where $\Gamma = \Gamma^T > 0$, $\omega = \begin{bmatrix} x_p & r \end{bmatrix}^T$, $\theta = \begin{bmatrix} \theta_x^T & \theta_r \end{bmatrix}^T$, $e = x_p - x_m$, and x_m is the state of a reference model

$$\dot{x}_m = A_m x_m + br \tag{4.4}$$

with A_m Hurwitz, and P is the solution of the Lyapunov equation $A_m^T P + PA_m = -Q$, $Q > 0$, guarantees stability when the matching conditions

$$A_p + b\lambda\theta_x^{\star T} = A_m, \qquad \lambda\theta_r^* = 1 \tag{4.5}$$

are satisfied for some $\theta^* = [\theta_x^{*T}, \theta_r^*]^T$. The controller in (4.2) and (4.3) also ensures that $x_p(t)$ tracks $x_m(t)$. The underlying Lyapunov function is quadratic in e and the parameter error $\tilde{\theta} = \theta - \theta^*$, with a negative semi-definite time-derivative $\dot{V}$ [44].

When a bounded disturbance d is present, with the plant dynamics changed as

$$\dot{x}_p = A_p x_p + b\lambda(u + d(t)) \tag{4.6}$$

[1] The argument t is suppressed for the sake of convenience, except for emphasis.

robust adaptive laws need to be designed that modify (4.3) as

$$\dot{\theta} = -\Gamma\omega b^T Pe - \sigma g(\theta, e) \tag{4.7}$$

where $g(\theta,e) = \theta$, $||e||\theta$, or of the form[2]

$$g(\theta,e) = \theta\left(1 - \frac{||\theta||}{\bar{\theta}_{\max}}\right)^2 \tag{4.8}$$

where $\bar{\theta}_{\max}$ is a known bound on the parameter θ. While the boundedness of the overall adaptive systems is well known and was established several decades ago, we briefly describe it below. Without loss of generality, we assume that $\lambda > 0$.

A quadratic positive definite function is chosen as

$$V = \frac{1}{2}e^T Pe + \frac{1}{2}\lambda\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{4.9}$$

which yields a time-derivative

$$\dot{V} \leq -\frac{1}{2}e^T Qe + k_1\|e\|\|d\| - \frac{1}{2}\lambda\sigma\|\tilde{\theta}\|g(\theta,e), \;\; k_1 > 0. \tag{4.10}$$

The property of $g(e,\theta)$, together with the fact that d is bounded, ensures that $\dot{V} < 0$ outside a compact set Ω in the $(e,\tilde{\theta})$ space. This ensures the global boundedness of both e and $\tilde{\theta}$. Boundedness of x_p follows.

In all of the above methods, the idea behind adding the term $g(e,\theta)$ is this: The parameter θ can drift away from the correct direction due to the term $k_1\|e\|\|d\|$, and the construction of $g(e,\theta)$ is such that it counteracts this drift and keeps the parameter in check, by adding a negative quadratic term in $\tilde{\theta}$. The boundedness of both e and θ are simultaneously assured in the above since V has a time-derivative $\dot{V}$ that is non-positive outside a compact set in the $(e,\tilde{\theta})$ space. It should be noted however that this was possible to a large extent because d was bounded and as a result, the sign-indefinite term remained linear in $\|e\|$.

An alternate procedure, originally proposed in [51] and revised and refined in [55]

[2]One can choose to set γ to zero if $\|\theta\| \leq \bar{\theta}_{\max}$, as is done in [27, 32] and many other references in the literature.

proceeds in a slightly different manner. Here, the boundedness of θ is first established, independent of the error equation. It should be noted that a similar approach is adopted in the context of output feedback in plants with higher relative degree by using normalization and an augmented error approach[44, 27]. In [55] and [38], no normalization is used but a projection algorithm. In the following section, we present the adaptive law based on the projection introduced in Section 3.4, which is of interest in this chapter, as well as the proof of boundedness for the sake of completeness.

4.2.1 Robust Adaptive Control in the Presence of a Projection Algorithm

The projection algorithm considered in this chapter is identical with the one defined in Section 3.4 and given by (3.20) and (3.21) together with (3.26).

The implications of Lemma 3 on robust adaptive control are obvious. If the adaptive law is chosen as in Lemma 3 with

$$y = -\omega b^T Pe, \tag{4.11}$$

then irrespective of the boundedness of e, it follows that $\theta(t)$ is bounded and lies in Ω_1 if $\theta(t_0)$ and θ^* belong to Ω_1, Ω_0 respectively. This is summarized in (3.27), with the special selection of the convex function $f(\theta)$ chosen as (3.26) in Lemma 4.

Remark 2. *We can also apply the projection algorithm to an adaptive law in a slightly different way. Instead of treating the vector θ as a whole, it is also possible to implement the projection algorithm by parts, for θ_x and θ_r independently. The design parameters in this case will be $\theta'_{x,\max}$, ε_x, and $\theta'_{r,\max}$, ε_r, respectively. The boundedness of the norm of θ_x by $\theta_{x,\max} = \theta'_{x,\max} + \varepsilon_x$ and that of θ_r by $\theta_{r,\max} = \theta'_{r,\max} + \varepsilon_r$ are guaranteed as in Lemma 4.*

With the boundedness of θ established using Lemma 4, boundedness of e follows by the application of the Gronwall-Bellman Lemma. This is summarized in the Theorem below.

Throughout the chapter, we use the following notations. Let

$$\underline{s}_A = \min_i \left|\Re\left(\lambda_i(A)\right)\right|$$

$$\bar{s}_A = \max_i \left|\Re\left(\lambda_i(A)\right)\right|$$

where λ_i is ith eigenvalue of a matrix A and $\Re(\lambda_i)$ is its real part.

Theorem 7. *Consider the closed-loop adaptive system given by (4.6), the control law (4.2), the adaptive law (3.23) with y and $f(\theta)$ chosen as in (4.11) and (3.26). If the reference model in* (4.4) *and θ_{max} are such that θ^* in* (4.5) *belongs to Ω_0, then for any initial conditions $x_p(t_0)$ and $x_m(t_0)$, and $\theta(t_0) \in \Omega_1$, the closed-loop adaptive system has bounded solutions, with $\theta(t)$ remaining in Ω_1 for all $t \geq t_0$.*

Proof. Lemma 4 implies that $\theta(t) \in \Omega_1$ with a bound as in (3.27). With a V as in (4.9), we obtain

$$\begin{aligned}\dot{V} = & -\frac{1}{2}e^T Qe + e^T Pb\lambda d \\ & + \left(e^T Pb\lambda\tilde{\theta}^T\omega + \lambda\tilde{\theta}^T \text{Proj}(\theta, -\omega b^T Pe)\right).\end{aligned} \tag{4.12}$$

Equation (3.22) in Lemma 2 together with (4.11) implies that the term within the parentheses in Eq. (4.12) is non-positive. This in turn implies that

$$\dot{V} \leq -\frac{1}{2}e^T Qe + k_1\|e\|\|d\|. \tag{4.13}$$

From (4.9) and (4.13) and the fact that $\theta(t)$ is bounded, it can be shown that

$$\dot{V} \leq -k_2\left(V - k_3\right) + k_4\sqrt{V} \tag{4.14}$$

where

$$k_1 = \|Pb\|\lambda, \quad k_2 = \frac{\underline{s}_Q}{\bar{s}_P}, \quad k_3 = \frac{\lambda\theta_{\max}^2}{2\underline{s}_\Gamma}, \quad k_4 = \frac{k_1 d_{\max}}{\sqrt{\underline{s}_P}}. \tag{4.15}$$

For positive constants Δ_1, Δ_2 such that $\Delta_1 < k_2$ and $4\Delta_1\Delta_2 \geq k_4^2$, it can be shown that for any V,

$$\Delta_1 V + \Delta_2 \geq k_4 \sqrt{V} \tag{4.16}$$

through a straight forward completion of squares. Inequalities (4.14) and (4.16) imply that

$$\dot{V} \leq -(\underbrace{k_2 - \Delta_1}_{K_0})V + \underbrace{k_2 k_3 + \Delta_2}_{K_1}. \tag{4.17}$$

From the application of the Gronwall-Bellman Lemma [24] [5] to (4.17), it follows that

$$V(t) \leq \left(V(t_0) - \frac{K_1}{K_0}\right) exp(-K_0 t) + \frac{K_1}{K_0} \tag{4.18}$$

and therefore $V(t)$ is bounded. This in turn implies the boundedness of $e(t)$ and therefore $x_p(t)$ for any initial conditions in $e(t_0)$. □

4.3 Robustness of Adaptive Systems to Unmodeled Dynamics

We now consider an LTI plant in the presence of a disturbance that may not be known to be bounded a-priori, such as a state-dependent disturbance η given by

$$\dot{\zeta} = A_\eta \zeta + b_\eta u, \;\; \eta = c_\eta^T \zeta \tag{4.19}$$

where A_η is a Hurwitz matrix. For ease of exposition, we assume that the plant has the form

$$\dot{x}_p = A_m x_p + b\lambda(u - \theta_x^{*T} x_p + \eta) \tag{4.20}$$

where λ and θ_x^* are unknown, and A_m and b are known. With the same reference model and definitions as in Section 4.2, we obtain the error dynamics

$$\dot{e} = A_m e + b\lambda(\tilde{\theta}^T \omega + \eta). \tag{4.21}$$

We now show that an identical result as in Theorem 7 can be derived in this case even though the disturbance η is not known to be bounded a-priori.

We introduce a few definitions. P and P_η are the solutions of the Lyapunov equations

$$\begin{aligned} A_m^T P + P A_m &= -q_1 I \\ A_\eta^T P_\eta + P_\eta A_\eta &= -q_2 I \end{aligned} \tag{4.22}$$

where q_1 and q_2 are positive scalars. Since A_m and A_η are Hurwitz, P and P_η exist and are positive definite and symmetric. Let

$$\begin{aligned} & x_{m_0} = \theta_{x,\max} \max_{t \geq t_0} \|x_m(t)\|, \\ & c_1 = x_{m_0},\ c_2 = \theta_{r,\max} \max_{t \geq t_0} |r(t)| \\ & p_b = \|Pb\|,\ p_\eta = ||P_\eta b_\eta|| \\ & c_3 = (\lambda p_b \|c_\eta\| + p_\eta \theta_{x,\max}),\ c_4 = 2 p_\eta (c_1 + c_2) \end{aligned} \tag{4.23}$$

$$F(e,\zeta) = q_1 \|e\|^2 + q_2 \|\zeta\|^2 - 2 c_3 \|e\| ||\zeta|| - c_4 \|\zeta\|. \tag{4.24}$$

Theorem 8. *Consider the closed-loop adaptive system given by (4.20), the unmodeled dynamics by (4.19), the control law (4.2), the adaptive law (3.23) with y and $f(\theta)$ chosen as in (4.11) and (3.26). If the reference model in* (4.4) *and θ_{max} are such that θ^* in* (4.5) *belongs to Ω_0, then for any initial conditions $x_p(t_0)$, $x_m(t_0)$, and $\theta(t_0) \in \Omega_1$, the closed-loop adaptive system has bounded solutions, with $\theta(t)$ remaining in Ω_1 for all $t \geq t_0$ if*

$$q_1 q_2 > c_3^2. \tag{4.25}$$

Proof. Let a Lyapunov function candidate be chosen as

$$V = e^T P e + \lambda \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \zeta^T P_\eta \zeta. \tag{4.26}$$

Taking the time derivative

$$\dot{V} \leq -q_1||e||^2 - q_2||\zeta||^2 + 2e^T Pb\lambda\eta + 2\zeta^T P_\eta b_\eta u$$

with some simplifications leads to

$$\begin{aligned}\dot{V} \leq -q_1||e||^2 - q_2||\zeta||^2 + 2||e||p_b\lambda||c_\eta||\,||\zeta|| \\ +2p_\eta||\zeta||(\theta_{\text{x,max}}||x|| + \theta_{\text{r,max}}|r|).\end{aligned} \tag{4.27}$$

Noting that $e = x_p - x_m$ and x_m is bounded, using the definitions in (4.23) and (4.24), (4.27) can be simplified as

$$\dot{V} \leq -F(e,\zeta).$$

It can be shown that $F(e,\zeta) = 0$ is an ellipse in the (e,ζ) space if (4.25) holds. Defining $z_\zeta = [e^T, \zeta^T]^T$, and

$$M = \begin{bmatrix} q_1 & -c_3 \\ -c_3 & q_2 \end{bmatrix},$$

(4.27) can be rewritten as

$$\dot{V} \leq -z_\zeta^T M z_\zeta + 2c_4||z_\zeta|| \tag{4.28}$$

where M is positive definite due to (4.25), and $\|\zeta\| \leq \|z_\zeta\|$. We note that the form of the inequality (4.28) is identical to that of (4.13), and that V is a function of z_ζ and θ with θ bounded. Therefore, identical arguments to that of Theorem 7 can be used to conclude the boundedness of z_ζ for any initial conditions $e(t_0)$ and $\zeta(t_0)$. Boundedness of $x_p(t)$ follows in a straight forward manner. □

Remark 3. *It should be noted that the global nature of the above result was possible primarily because boundedness of the parameter was established independent of the error dynamics. The former allowed the sign-definiteness terms to be bounded by a quadratic function, thereby leading to boundedness of all signals in the system with arbitrary initial conditions in the state. In other words, the parameter projection algorithm allowed the* *overall adaptive system, by virtue of Lemma 3, to be treated as a linear time-varying system, thereby leading to a global result. This could not have been accomplished by other adaptive laws with robustness–based modifications than the projection algorithm discussed above.*

We now show that a class of unmodeled dynamics (A_η, b_η, c_η) as in (4.19) exists for any A_m, b, λ, and θ^* in (4.20). The following lemma is useful in this regard.

Lemma 5. *Let P be the solution of the Lyapunov equation $A^T P + PA = -qI$ for a matrix A that is Hurwitz. Then*

$$\bar{s}_P = \frac{q}{2\underline{s}_A}. \tag{4.29}$$

Proof. Since A is Hurwitz,

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt. \tag{4.30}$$

If λ_i and v_i are ith eigenvalue and corresponding normalized eigenvector of A, respectively, it follows that

$$\begin{aligned} Pv_i &= \left(q \int_0^\infty e^{\lambda_i^* t} e^{\lambda_i t} dt \right) v_i \\ &= \frac{q}{2|\Re(\lambda_i)|} v_i \end{aligned} \tag{4.31}$$

since $Av_i = \lambda_i v_i$, $A^T v_i = \lambda_i^* v_i$, and $e^{At} v_i = e^{\lambda_i t} v_i$. Therefore we can derive (4.29). □

We note using Lemma 5 that we can express c_3 in (4.23) as

$$c_3 \leq q_1 \frac{\|b\| \|c_\eta\| \lambda_{\max}}{2\underline{s}_{A_m}} + q_2 \frac{\|b_\eta\| \theta_{x,\max}}{2\underline{s}_{A_\eta}}. \tag{4.32}$$

Defining

$$\alpha = \sqrt{\frac{q_1}{q_2}}, \qquad \beta_m = \frac{\|b\| \|c_\eta\|}{2\underline{s}_{A_m}}, \qquad \beta_\eta = \frac{\|b_\eta\|}{2\underline{s}_{A_\eta}} \tag{4.33}$$

and

$$g(\alpha, \beta_m, \beta_\eta) = \beta_m \lambda_{\max} \alpha + \frac{\beta_\eta \theta_{x,\max}}{\alpha}, \tag{4.34}$$

it follows that the sufficient condition (4.25) is satisfied if

$$g(\alpha, \beta_m, \beta_\eta) < 1 \tag{4.35}$$

or equivalently, since $\alpha > 0$, if

$$\beta_m \lambda_{\max} \alpha^2 - \alpha + \beta_\eta \theta_{x,\max} < 0. \tag{4.36}$$

For ease of exposition, we set $\|c_\eta\| = 1$. This implies that the known parameters A_m and b determine β_m and the parameters of the unmodeled dynamics determine β_η. The question that needs to be answered can be posed as follows: Given β_m and $\theta_{x,\max}$, does a β_η exist such that (4.36) is satisfied? The answer is affirmative, since it can be derived that there exists $\alpha > 0$ with which (4.36) is satisfied if

$$4\beta_m \beta_\eta \theta_{x,\max} \lambda_{\max} < 1, \tag{4.37}$$

and α, defined in (4.33), is a free parameter. The above discussions are summarized in the following proposition:

Proposition 1. *If β_η satisfies the inequality (4.37), then the sufficient condition (4.25) in Theorem 8 is satisfied.*

Proposition 1 implies that for any A_m, b, λ, and θ^*, a class of unmodeled dynamics always exists for which the sufficient condition (4.25) is satisfied. This conclusively demonstrates that the closed-loop adaptive system described in this section is robust with respect to a class of unmodeled dynamics that satisfies (4.37) with the relevant quantities defined in (4.33).

4.4 Robustness of Adaptive Systems to Time Delay Based on Pade Approximation

Suppose the input into the plant is delayed so that the plant equation is given by

$$\dot{x}_p = A_m x_p + b\lambda(u(t-\tau) - \theta_x^{*T} x_p). \tag{4.38}$$

Equation (4.38) can be rewritten as

$$\dot{x}_p = A_m x_p + b\lambda\big((u(t) + \eta(t)) - \theta_x^{*T} x_p\big) \tag{4.39}$$

where

$$\eta(t) = [G(s)]u(t), \tag{4.40}$$

and $G(s)$ is an operator defined by $G(s) = [e^{-\tau s} - 1]$, whose rational approximation of order $2N$ (where $N \in \mathbb{Z}_{>0}$) can be obtained by using the Pade approximation of $e^{-\tau s}$:

$$e^{-\tau s} \approx \frac{\sum_{k=0}^{2N} (-1)^k c_k \tau^k s^k}{\sum_{k=0}^{2N} c_k \tau^k s^k} \tag{4.41}$$

where the coefficients are

$$c_k = \frac{(4N-k)!(2N)!}{(4N)!k!(2N-k)!}, \quad k = 0, 1, \ldots, 2N. \tag{4.42}$$

It is easy to see that the rational approximation, $G_{Pade}(s)$, of $G(s)$ admits a state-space representation (4.19), with the parameters

$$A_\eta = \frac{1}{\tau}\underbrace{\begin{bmatrix} -w_1 & 1 & 0 & \cdots \\ -w_2 & 0 & 1 & \cdots \\ -w_3 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{A_N}, \quad b_\eta = \frac{1}{\tau}\underbrace{\begin{bmatrix} -v_1 \\ -v_2 \\ -v_3 \\ \vdots \end{bmatrix}}_{b_N}, \quad c_\eta^T = \begin{bmatrix} 1 & 0 & 0 & \cdots \end{bmatrix} \tag{4.43}$$

where $w_1, w_2, \cdots, w_{2N}$ and $v_1, v_2, \cdots, v_{2N}$ are positive constants which are obtained from analyzing (4.41) and (4.42). It is important to note that in (4.43), while the $2N \times 2N$ dimensional matrix A_η and the $2N \times 1$ dimensional matrix b_η depend on τ, the matrix A_N, b_N, and c_η are independent of τ, with A_N Hurwitz. This allows us to conclude from Theorem 8 that there exists a family of the adaptive controller given by (4.2) and (3.23) with y and $f(\theta)$ chosen as in (4.11) and (3.26) which guarantees boundedness for A_η, b_η, and c_η. This is summarized in Theorem 9, with the introduction of additional parameters $\beta_m^\tau, \beta_\eta^\tau$ as

$$\beta_m^\tau = \frac{\|b\|}{2\underline{s}_{A_m}}, \qquad \beta_\eta^\tau = \frac{\|b_N\|}{2\underline{s}_{A_N}}.$$

Theorem 9. *Consider the closed-loop adaptive system given by the plant (4.39), the disturbance η due to time delay which is represented by (4.19) with parameters (4.43), the control law (4.2), the adaptive law (3.23) with y and $f(\theta)$ chosen as in (4.11) and (3.26). If the reference model in* (4.4) *and θ_{max} are such that θ^* in* (4.5) *belongs to Ω_0, then for any initial conditions $x_p(t_0)$, $x_m(t_0)$, and $\theta(t_0) \in \Omega_1$, the closed-loop adaptive system has bounded solutions, with $\theta(t)$ remaining in Ω_1 for all $t \geq t_0$, if*

$$\beta_\eta^\tau < \frac{1}{4\beta_m^\tau \theta_{x,\max} \lambda_{\max}}. \tag{4.44}$$

Proof. From the definitions of β_m^τ, β_η^τ and since $A_\eta = (1/\tau)A_N$, $b_\eta = (1/\tau)b_N$, it follows that $\beta_m = \beta_m^\tau$ and $\beta_\eta = \beta_\eta^\tau$. Therefore condition (4.44) immediately implies that (4.37) holds. Theorem 8 and Proposition 1 imply that if (4.37) is satisfied, then boundedness of the overall adaptive system follows, which proves Theorem 9. □

Remark 4. *As in Section 4.3, whether it exists a class of Pade approximations for which β_η^τ satisfies (4.44) remains to be shown. Unlike (4.37), we note that β_η^τ depends on b_N and A_N both of which are independent of τ. In other words, β_η^τ is a fixed constant. Therefore the class of reference models and θ^* that satisfy the matching condition* (4.5) *are more limited in this case compared to those in Section 4.3, for a given Pade approximation $G_{Pade}(s)$. In fact, it is possible to show that the sufficient condition (4.44) essentially requires the uncertain open-loop plant to be stable. The main reason for this limitation is the nature of* *"unmodeled dynamics" of $G_{Pade}(s)$, where both the zeros and poles diverge as τ becomes smaller, which makes the condition (4.44) quite restrictive.*

Remark 5. *Another point to note is that the sufficient condition is independent of τ. That is, if for a given A_m and θ^*, condition (4.44) is satisfied, then it continues to hold for any τ. This counterintuitive result comes from the fact that the error of Pade approximation $(|\angle G_{Pade}(s) - \angle G(s)|)$ becomes larger as τ increases. This is another critical limitation of this analysis, with which one can not develop any methods to compute a delay margin of the adaptive system, despite Pade approximation is used.*

4.5 Summary

In this chapter, we show that the closed-loop adaptive system with MRAC and projection algorithm has globally bounded solutions for any initial conditions in the presence of a class of unmodeled dynamics. This class includes the Pade approximation of time delay. However the sufficient condition obtained as (4.44) is quite restrictive, and limits the class of plants for which robustness to time delay can be guaranteed. This conservative result is partially due to the fact that the proof of boundedness solely relies on just a single Lyapunov function. Moreover, the analysis uses Pade approximation, which obviously remains some concerns about how reliable the result is.

Therefore a different approach and tools are necessary to prove robustness of the adaptive control system with projection algorithm to time delay.

Chapter 5

Guaranteed Delay Margins for Adaptive Control of Scalar Plants

In Chapter 3, the Lipschitz continuous projection algorithm is formulated, and it was demonstrated that by introducing the algorithm a scalar adaptive stabilizer system obtains a non-zero delay margin, which was not accomplished by the standard adaptive laws without projection. In Chapter 4, it was shown that with a modified adaptive law based on projection, global results are obtained in the presence of unmodeled dynamics or a Pade-approximated time delay. However the result with respect to a time delay in Chapter 4 is quite restrictive, being only applicable to a small class of plants and not yet providing a computable delay margin. In contrast to the above results, in this chapter we show that global boundedness can be derived for a first-order plant with a guaranteed delay margin using an adaptive law which includes a modification based on projection.

The adaptive law used in this chapter is exactly the same as in Chapter 4, which was originally proposed in [56], rigorously analyzed in [41, 51], and revised and refined in [38, 55]. As in Chapter 4, this allows us to establish the boundedness of the parameters, independent of the plant state. Unlike the standard practice of Lyapunov function based arguments which suffice when states are measurable, which was also followed in Chapter 4, extensive first-principles based arguments are employed in order to prove the boundedness.

The problem is stated in Section 5.1. The main result is stated in Section 5.2.3 and proved in Section 5.3. A flight control example is used to illustrate the order of magnitude of the analytically computable delay margin. Extensions to a higher dimensional plant where states are accessible are discussed in Chapter 6.

5.1 Problem Statement

The problem is the adaptive control of a first-order plant

$$\dot{x_p}(t) = a_p x_p(t) + b_p u(t-\tau) \tag{5.1}$$

where a_p is an unknown parameter and $\tau \geq 0$ is an unknown time delay. For ease of exposition, we assume that $b_p = 1$. It is also assumed that

$$|a_p| \leq \bar{a}, \tag{5.2}$$

where $\bar{a}$ is a known positive constant.

The standard adaptive control solution is to choose a control input

$$u(t) = \theta(t)x_p(t) + r(t) \tag{5.3}$$

where

$$\dot{\theta}(t) = -\gamma x_p(t)e(t) \tag{5.4}$$

$$e(t) = x_p(t) - x_m(t). \tag{5.5}$$

and $x_m(t)$ is specified by a reference model

$$\dot{x}_m(t) = a_m x_m(t) + r(t) \qquad a_m < 0. \tag{5.6}$$

The problem is to ensure globally bounded solutions with the control input and adaptive law as in (5.3) and (5.4).

The fundamental difficulty in solving this problem stems from the fact that the ubiquitous Lyapunov function candidate

$$V = \frac{1}{2}e^2 + \frac{1}{2\gamma}\phi^2 \tag{5.7}$$

where $\gamma > 0$,

$$\phi(t) = \theta(t) - \theta^\star, \tag{5.8}$$

$$\theta^\star = a_m - a_p \tag{5.9}$$

yields a time-derivative

$$\dot{V} = a_m e^2 + e\eta \tag{5.10}$$

where

$$\eta(t) = u(t-\tau) - u(t). \tag{5.11}$$

While uniform asymptotic stability in the large of the errors e and ϕ to zero can be assured when $\eta(t) \equiv 0$, global boundedness in the presence of η has eluded researchers for the past several years. In Chapter 4, it was shown that modifying the adaptive law with the simple projection algorithm guarantees the global boundedness in the presence of a class of unmodeled dynamics which includes the Pade approximation of time delay. However in addition to the fact that the analysis in Chapter 4 relies on taking approximation, the result is restrictive and does not provide a computable delay margin.

In the following, we provide a complete solution to this problem for scalar plants of the form (5.1).

5.2 Boundedness in the Presence of Time Delay

In this chapter, the modification of the standard MRAC adaptive law based on projection algorithm is considered as robust adaptive control to establish boundedness. The adaptive law and the definition of the projection algorithm are identical with those in Chapter 4, however we again describe them here for the sake of completeness. The adaptive law is given by

$$\dot{\theta}(t) = \gamma \mathrm{Proj}(\theta(t), -x_p(t)e(t)) \tag{5.12}$$

where

$$\mathrm{Proj}(\theta, y) = \begin{cases} \frac{\theta_{\max}^2 - \theta^2}{\theta_{\max}^2 - \theta_{\max}'^2} y & \text{if } [\theta \in \Omega_1 \backslash \Omega_0 \wedge y\theta > 0] \\ y & \text{otherwise} \end{cases} \tag{5.13}$$

with

$$\begin{aligned} \Omega_0 &= \left\{\theta \in \mathbb{R}^1 | -\theta_{\max}' \leq \theta \leq \theta_{\max}'\right\} \\ \Omega_1 &= \left\{\theta \in \mathbb{R}^1 | -\theta_{\max} \leq \theta \leq \theta_{\max}\right\}. \end{aligned} \tag{5.14}$$

We note that (5.13) and (5.14) are a scalar version of (3.21) and (3.20), i.e. θ is a scalar, with the special choice of the convex function $f(\cdot)$ as in (3.26).

We will show that this projection algorithm leads to a nonzero delay margin in adaptive systems. The overall adaptive controller in this chapter is specified by (5.3), (5.12), and (5.13).

5.2.1 Properties of the Lipschitz Continuous Projection Algorithm

The projection algorithm guarantees the boundedness of the parameter estimate independent of the system dynamics. We note the important lemmas regarding the projection algorithm Lemma 2 - Lemma 4 appeared in Chapter 4. If the adaptive law is chosen as in Lemma 3 with $y = -\gamma x_p e$ and the convex function $f(\theta)$ given by (3.26) in Lemma 4, then irrespective of the boundedness of y and e, it follows that $\theta(t)$ is bounded and lies in Ω_1

$$|\theta(t_0)| \leq \theta_{\max} \implies |\theta(t)| \leq \theta_{\max} \forall t \geq t_0. \tag{5.15}$$

5.2.2 Choice of Projection Algorithm Parameters

The adaptive law based on projection algorithm (5.12) requires $\theta_{\max}$ and ε to be specified, whose selections are discussed below.

We note that the size of parametric uncertainty is assumed to be known as given by (5.2). The control parameters $\theta_{\max}$ and ε are then chosen such that

$$\theta_{\max} - \varepsilon \geq \bar{a} + |a_m|, \tag{5.16}$$

$$0 < \varepsilon < |a_m|. \tag{5.17}$$

The condition (5.16) is necessary in order to guarantee $\theta^\star$ which satisfies the matching condition (5.9) to lie in Ω_0, i.e. $\theta^\star \in \Omega_0$.

5.2.3 Main Result

Theorem 10. *There exists a* $\tau^\star$ *such that the closed-loop adaptive system with the plant in* (5.1)*, control input in* (5.3)*, reference model in* (5.6)*, and adaptive law in* (5.12)*,* (5.13) *and* (5.14)*, together with the projection parameters as in* (5.16) *and* (5.17)*, has globally bounded solutions for any initial conditions*

$$x_p(t) = \chi(t), \quad \theta(t) = \chi_\theta(t) \qquad t \in [t_0 - \tau, t_0] \tag{5.18}$$

where $\chi(t) : \Re \to \Re$, $\chi_\theta(t) : \Re \to \Omega_1$, *and* $\forall \tau \in [0, \tau^\star)$.

A few definitions are stated before proceeding to the proof of Theorem 10.

Definition 3. *We define the region A and the boundary regions B, B' as follow:*

$$A = \left\{z \in \mathbb{R}^2 | -\theta'_{\max} \leq \theta \leq \theta'_{\max}\right\}$$
$$B = \left\{z \in \mathbb{R}^2 | -\theta_{\max} \leq \theta < -\theta'_{\max}\right\}$$
$$B' = \left\{z \in \mathbb{R}^2 | \theta'_{\max} < \theta \leq \theta_{\max}\right\}$$

where $z(t) = [e(t) \ \theta(t)]^T$. *These regions are illustrated in Figure 5-1.*

Definition 4. *We divide the boundary region B into two regions as follow:*

$$B_L = \left\{z \in \mathbb{R}^2 | -\theta_{\max} \leq \theta \leq -(\theta'_{\max} + \varepsilon/2)\right\}$$
$$B_U = \left\{z \in \mathbb{R}^2 | -(\theta'_{\max} + \varepsilon/2) \leq \theta < -\theta'_{\max}\right\}.$$

Note that $B = B_L \cup B_U$. *These regions are illustrated in Figure 5-1.*

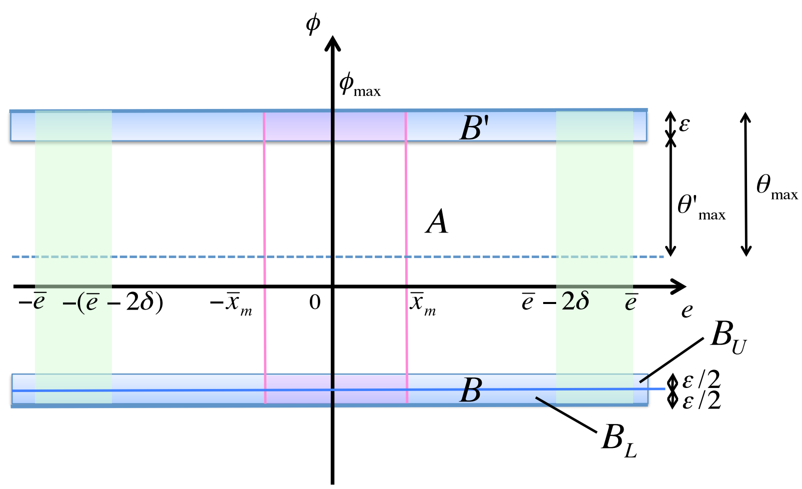


Figure 5-1: Definition of regions

Let positive constants δ, $\bar{e}$ defined by

$$\delta \in (0 \ 1] \tag{5.19}$$

and

$$\bar{e} = \max\left(\max_{t\in[t_0-\tau,t_0]} |\chi(t)| + \bar{x}_m + 2\delta, \ c_e, \ \beta\right) \tag{5.20}$$

where $\bar{x}_m \equiv \max_{t \geq t_0} |x_m(t)|$ and $c_e > 0$, $\beta > 0$ are given later in Proposition 2, Proposition 3, respectively. From the definitions of $\bar{e}$ and δ, it is immediate that $\bar{e} - \delta > 0$.

Condition 1. *$\pi(t)$ is said to satisfy Condition 1 at time t_a if*

$$|\pi(t)| \leq \bar{e} \quad \forall t \in [t_a - \tau,\ t_a], \tag{5.21}$$

$$|\pi(t_a)| = \bar{e} - \delta \tag{5.22}$$

where $t_a \geq t_0$ and $\bar{e} \in \mathbb{R}$, $\delta \in \mathbb{R}$ are positive constants with $\bar{e} - \delta > 0$.

5.3 Proof of Theorem 10

The closed-loop adaptive system given by (5.1), (5.3), (5.6), (5.12), (5.13) and (5.14) is equivalent to the error model described by

$$\dot{e}(t) = a_m e(t) + (\theta(t) - \theta^\star)(e(t) + x_m(t)) + \eta(t) \tag{5.23}$$

where η is given by (5.11), and the adaptive law described by

$$\begin{aligned} \dot{\theta}(t) &= -\frac{\theta_{\max}^2 - \theta^2}{\theta_{\max}^2 - \theta_{\max}'^2} \gamma e(t)\big(e(t) + x_m(t)\big) \\ &\text{if} \quad z \in (B \cup B') \text{ and } y\theta > 0 \end{aligned} \tag{5.24}$$

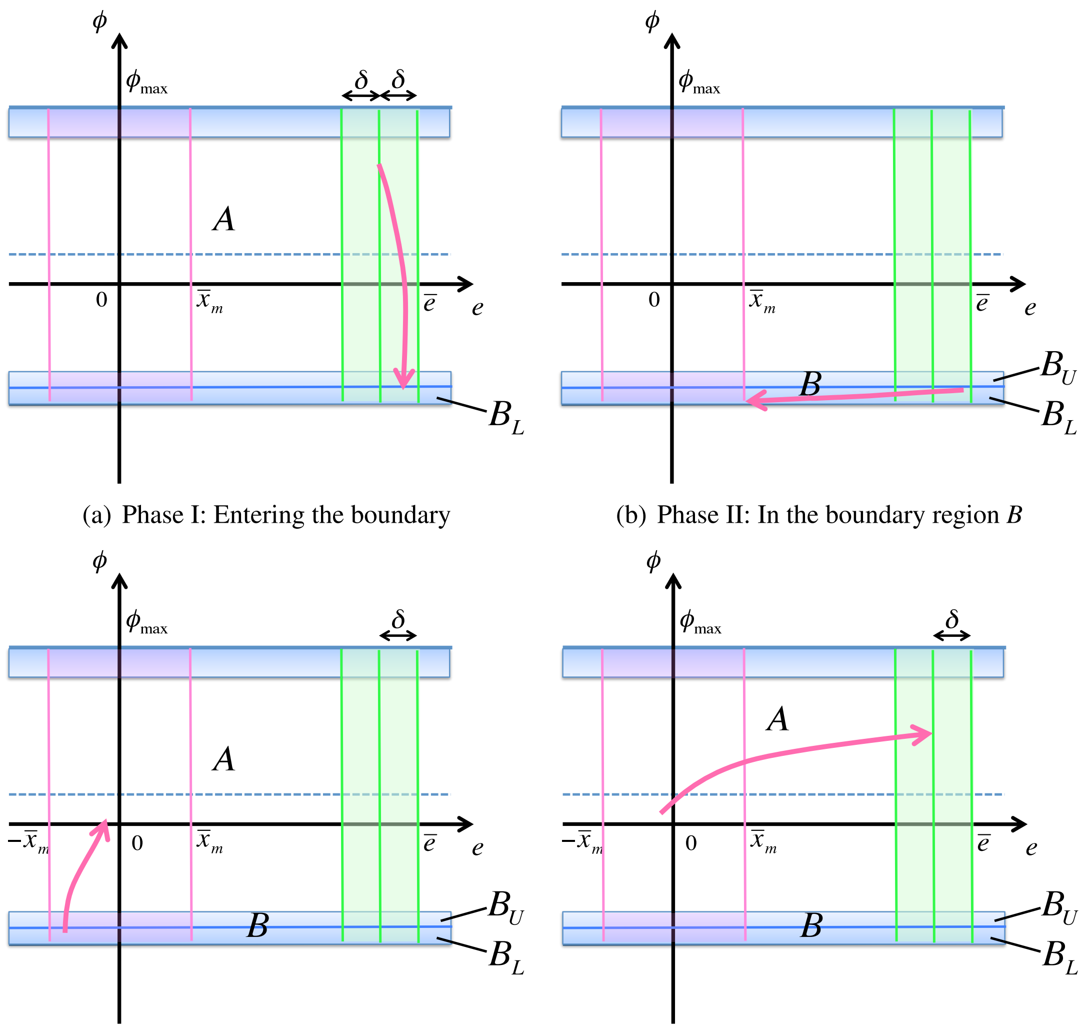
and

$$\dot{\theta}(t) = -\gamma e(t)\big(e(t) + x_m(t)\big) \quad \text{otherwise.} \tag{5.25}$$

We first note that since $|\chi_\theta(t)| \leq \theta_{\max}$, it follows from Lemma 4 that $|\theta(t)| \leq \theta_{\max}$ $\forall t \geq t_0$. Theorem 10 is therefore proven if the global boundedness of $e(t)$ is demonstrated.

The proof is completed using the following four phases.

Phase I: The error $e(t)$ satisfies Condition 1 for some $t = t_a$; this implies that the state z has to enter B at $t_b \in (t_a, t_a + \Delta T_{in,\max})$, where $\Delta T_{in,\max} > 0$ is a finite constant (see Figure 5-2(a)).



(a) Phase I: Entering the boundary

(b) Phase II: In the boundary region B

(c) Phase III: Exiting from the boundary

(d) Phase IV: Return to Condition 1

Figure 5-2: Phases I-IV of a trajectory

Phase II: When the trajectory enters B, the parameter enters the boundary of the projection algorithm; e is shown to be bounded by making use of the underlying linear time-varying system (see Figure 5-2(b)).

Phase III: There exists $\Delta T_{out,\min}$ such that the trajectory reenters A at $t_c > t_b + \Delta T_{out,\min}$ with $|e(t_c)| < \bar{x}_m$ (see Figure 5-2(c)).

Phase IV: The trajectory has only two alternatives: (IV-A): $|e(t)| < \bar{e} - \delta \ \forall t > t_c$ which proves Theorem 10; (IV-B): $e(t)$ satisfies Condition 1 for some $t_d > t_c$ (see Figure 5-2(d)). If the latter, we replace t_a by t_d and repeat Phases I to IV.

In the following subsections, we prove Phases I-IV rigorously.

5.3.1 Phase I: Entering the Boundary

From (5.20) and the definitions of $\bar{e}$ and δ, it can be shown that

$$|e(t)| < \bar{e} - \delta \quad \forall t \in [t_0 - \tau, t_0].$$

If $e(t)$ grows without bound, it implies that there exists $t_a > t_0$ such that

$$|e(t_a)| = \bar{e} - \delta. \tag{5.26}$$

That is, $e(t)$ satisfies Condition 1 at $t = t_a$. We note that if no t_a exists such that (5.26) holds, the global boundedness of $e(t)$ $\forall t \geq t_0$ is immediate. Without loss of generality, we assume that $z(t_a) \in A$.

Phase I is completed by proving the following proposition:

Proposition 2. *Let $e(t)$ satisfy Condition 1 at $t = t_a$ with δ, $\bar{e}$ given in* (5.19)*,* (5.20) *respectively and $z(t_a) \in A$. Then*

$$(i) \quad |e(t)| \leq \bar{e} \quad \forall t \in [t_a, t_a + \Delta T] \tag{5.27}$$

$$(ii) \quad \exists t_b' \in [t_a, t_a + \Delta T] \ \ s.t. \ \ z(t_b') \in B_L, \tag{5.28}$$

where

$$\Delta T = \frac{\delta}{b_0\bar{e}+b_1},$$
$$b_0 = 3\theta_{\max} + |a_m| + |\theta^\star|, \qquad b_1 = (3\theta_{\max} + |\theta^\star|)\bar{x}_m + 2\bar{r}, \tag{5.29}$$

and $\bar{r} \equiv \max_{t\geq t_0} |r(t)|$.

Proof of Proposition 2 (i):

We note from (5.23) that

$$|\dot{e}(t)| \leq |a_m + \theta(t) - \theta^\star||e(t)| + |\theta(t) - \theta^\star||x_m(t)| + |\eta(t)| \tag{5.30}$$

where $\eta(t)$ is given by (5.11). Since

$$|\eta(t)| \leq 2\theta_{\max}\left(\max_{[t-\tau,\, t]} |e(t)| + \bar{x}_m\right) + 2\bar{r}, \tag{5.31}$$

it follows that

$$|\dot{e}(t)| \leq b_0\bar{e}' + b_1 \qquad \forall t \in [t_a, t_a + \Delta T] \tag{5.32}$$

where

$$\bar{e}' = \max_{t\in[t_a-\tau, t_a+\Delta T]} |e(t)|. \tag{5.33}$$

We therefore have that $\forall \Delta t \in [0, \Delta T]$,

$$\begin{aligned} |e(t_a+\Delta t)| &\leq |e(t_a)| + \max_{t\in[t_a,t_a+\Delta T]} |\dot{e}(t)|\Delta T \\ &\leq (\bar{e}-\delta) + \delta\left(1 + \frac{b_0(\bar{e}'-\bar{e})}{b_0\bar{e}+b_1}\right) \\ &= (1-b_0\Delta T)\bar{e} + b_0\Delta T\bar{e}' \end{aligned} \tag{5.34}$$

from (5.32) and (5.22) since Condition 1 is satisfied for $t = t_a$. Therefore

$$\max_{t\in[t_a,t_a+\Delta T]} |e(t)| \leq (1-b_0\Delta T)\bar{e} + b_0\Delta T\bar{e}'. \tag{5.35}$$

From (5.33) and since $e(t)$ satisfies (5.21), there are only two possible cases, (a) $\bar{e}' = \bar{e}$ and (b) $\bar{e}' > \bar{e}$. If case (a) holds, it immediately implies from (5.33) that Proposition 2 (i) is true. If we suppose case (b) holds, it implies $\bar{e}' = \max_{t\in[t_a,t_a+\Delta T]} |e(t)|$ and from (5.35) it follows that

$$(1-b_0\Delta T)\bar{e}' \leq (1-b_0\Delta T)\bar{e}.$$

Since $1-b_0\Delta T > 0$ this in turn implies that $\bar{e}' \leq \bar{e}$, which contradicts with the condition of the case and therefore we obtain $\bar{e}' = \bar{e}$. □

Proof of Proposition 2 (ii).

We note from (5.32) that

$$|e(t)| \geq |e(t_a)| - (b_0\bar{e}' + b_1)\Delta T \qquad \forall\, t \in [t_a, t_a+\Delta T]$$

which can be simplified, using Proposition 2 (i) and the fact that $e(t)$ satisfies (5.26), as

$$|e(t)| \geq \bar{e} - 2\delta \qquad \forall\, t \in [t_a, t_a+\Delta T]. \tag{5.36}$$

From (5.36) and the choices of δ and $\bar{e}$ in (5.19) and (5.20), it can be shown that

$$\bar{e} - 2\delta > \bar{x}_m.$$

This in turn implies that $\dot{\theta}(t)$ is negative and

$$\begin{aligned} -\dot{\theta}(t) &\geq \gamma|e(t)|(|e(t)| - |x_m(t)|) \\ &\geq \gamma(\bar{e}-2\delta)((\bar{e}-2\delta) - \bar{x}_m) \qquad \forall\, t \in T_A \end{aligned} \tag{5.37}$$

where T_A is defined as

$$T_A : \Big\{ t | z(t) \in A \;\text{ and }\; t \in [t_a, t_a + \Delta T] \Big\}.$$

From (5.37) and noting that $\dot{\theta}(t) < 0 \;\forall t \in [t_a, t_a + \Delta T]$ since $|e(t)| > \bar{x}_m \;\forall t \in [t_a, t_a + \Delta T]$, it follows that

$$\theta(t_a) - \theta(t_a + \Delta t) \geq \gamma(\bar{e} - 2\delta)(\bar{e} - 2\delta - \bar{x}_m)\Delta t \qquad \forall \Delta t \in [0, \Delta T]. \tag{5.38}$$

Hence defining

$$\Delta T_{in,\max} = \frac{2\theta_{\max}}{\gamma(\bar{e} - 2\delta)(\bar{e} - 2\delta - \bar{x}_m)} \tag{5.39}$$

and if $\Delta T_{in,\max} \leq \Delta T$, from (5.38), $|\theta(t)| \leq \theta_{\max} \;\forall t \geq t_0$ and definition of region B, it follows that $z(t)$ enters B at $t_b \in (t_a, t_a + \Delta T_{in,\max})$.

We now show that $z(t)$ enters B_L at $t'_b \in (t_a, t_a + \Delta T'_{in,\max})$ for some $\Delta T'_{in,\max} > \Delta T_{in,\max}$. First, it can be proven that

$$|\text{Proj}(\theta, y)| > \frac{1}{2}|y| \qquad \forall z \in B_U. \tag{5.40}$$

Using similar arguments as above, then it can be shown that

$$-\dot{\theta}(t) > \frac{\gamma}{2}(\bar{e} - 2\delta)(\bar{e} - 2\delta - \bar{x}_m) \qquad \forall t \in T_{BU} \tag{5.41}$$

where T_{BU} is defined as

$$T_{BU} : \Big\{ t | z(t) \in B_U \;\text{ and }\; t \in [t_a, t_a + \Delta T] \Big\}.$$

Noting the definitions of B_U and B_L given by Definition 4, the maximum time that $z(t)$ can spend in B_U can be derived, using (5.41), to be $\{\varepsilon/2\}/\{\frac{\gamma}{2}(\bar{e} - 2\delta)(\bar{e} - 2\delta - \bar{x}_m)\}$. This implies that $z(t)$ enters region B_L at $t'_b \in (t_a, t_a + \Delta T'_{in,\max})$ where

$$\begin{aligned}\Delta T'_{in,\max} &= \Delta T_{in,\max} + \frac{\varepsilon/2}{\gamma(\bar{e}-2\delta)(\bar{e}-2\delta-\bar{x}_m)/2} \\ &= \frac{2\theta_{\max}+\varepsilon}{\gamma(\bar{e}-2\delta)(\bar{e}-2\delta-\bar{x}_m)}\end{aligned} \tag{5.42}$$

if $\Delta T'_{in,\max} \leq \Delta T$, since then the inequality in (5.41) is satisfied for all $t \in (t_b, t'_b]$.

From (5.20)

$$\bar{e} \geq c_e. \tag{5.43}$$

From (5.29) and (5.42), if we let the positive constant c_e defined by

$$\begin{aligned} c_e &= \frac{-a_2 + \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}, \\ a_1 &= \delta\gamma \\ a_2 &= -\delta\gamma(4\delta + \bar{x}_m) - (2\theta_{\max} + \varepsilon)\, b_0 \\ a_3 &= 2\delta^2\gamma(2\delta + \bar{x}_m) - (2\theta_{\max} + \varepsilon)\, b_1, \end{aligned} \tag{5.44}$$

then $\Delta T'_{in,\max} \leq \Delta T$ is implied from (5.43). This proves Proposition 2 (ii). □

Remark 6. *We note that since* $\Delta T'_{in,\max} \sim O\left(\bar{e}^{-2}\right)$ *from* (5.42) *and* $\Delta T \sim O\left(\bar{e}^{-1}\right)$ *from* (5.29)*, the sufficient condition of* $\Delta T'_{in,\max} \leq \Delta T$ *can be obtained in a form of* $\bar{e} \geq c_e$*, where* c_e *is given as a solution of quadratic equation in* $\bar{e}$*. Since the exact solution takes a messy expression as given by* (5.44)*, we derive an upper-bound here. From* (5.19) *and relative sizes among the constants* $\theta_{\max}$*,* $\theta^\star$ *and* ε*, it can be shown using algebraic manipulations that*

$$c_e < \frac{16}{\delta\gamma}(\theta_{\max}^2 + \gamma)(1 + \bar{x}_m). \tag{5.45}$$

Therefore if $\bar{e} \geq \frac{16}{\delta\gamma}(\theta_{\max}^2 + \gamma)(1 + \bar{x}_m)$*, then* $\Delta T'_{in,\max} < \Delta T$ *is implied. The right hand side of* (5.45) *is just one example of an upper-bound on* c_e *and may be too conservative, however the dependencies of the guaranteed bound* $\bar{e}$ *on each parameter* γ*,* $\theta_{\max}$ *and* $\bar{x}_m$ *are more* *transparent.*

5.3.2 Phase II: In the Boundary Region B

Let the trajectory stays in B for $t \in (t_b, t_c)$ for some $t_c > t_b$. From the definition of B, it follows that

$$\theta(t) = -\theta_{\max} + \varepsilon(t) \quad \text{for} \quad t \in (t_b, t_c) \tag{5.46}$$

where

$$\varepsilon(t) \in [0 \;\; \varepsilon).$$

Hence, from (5.1), (5.3), and (5.6), the error dynamics can be shown to be of the form

$$\dot{e}(t) = m_0 e(t) + m_1 e(t-\tau) + r_B(t) \quad \forall t \in (t_b, t_c) \tag{5.47}$$

where

$$\begin{aligned} m_0 &\equiv a_p \\ m_1 &\equiv -\theta_{\max} + \varepsilon(t-\tau) \\ r_B(t) &\equiv -\theta^\star x_m(t) + m_1 x_m(t-\tau) + (r(t-\tau) - r(t)) . \end{aligned} \tag{5.48}$$

Note that the boundedness of r_B is immediate since r and x_m are bounded.

Proposition 3 contains the main result of this section.

Proposition 3. *There exists $\beta > 0$ such that for any $\tau \leq \bar{\tau}$,*

$$|e(t)| \leq \max\{|e(t_b)|, \beta\} \quad \forall t \in (t_b, t_c)$$

where

$$\bar{\tau} = \frac{-(\bar{a} - \theta_{\max} + \varepsilon)}{4\theta_{\max}^2}. \tag{5.49}$$

Proof. The proof is built upon Proposition 6.7 in [23], model transformation, and Razumikhin Theorem. The key idea is that m_0, m_1 in (5.47) given by (5.48) satisfy $m_0 + m_1 < 0$ for $t \in (t_b, t_c)$, or when the trajectory stays in B, from (5.16) and (5.17).

Using

$$e(t-\tau) = e(t) - \int_{-\tau}^{0} \dot{e}(t+\zeta)d\zeta,$$

with $\dot{e}(t+\zeta)$ replaced by the right hand side of the system equation (5.47) with appropriate time shift, we obtain the following transformed system:

$$\begin{aligned}
\dot{e}(t) &= (m_0 + m_1(t))e(t) + r_B(t) \\
&\quad - m_1(t)\int_{-\tau}^{0}\big(m_0 e(t+\zeta) + m_1(t+\zeta)e(t+\zeta-\tau) \\
&\qquad\qquad + r_B(t+\zeta)\big)d\zeta \qquad (5.50)\\
&= \bar{m}_0 e(t) + \int_{-2\tau}^{0} \bar{m}(t,\zeta)e(t+\zeta)d\zeta + \bar{r}_B(t), \\
&\Big(\bar{m}_0(t), \bar{m}(t,\cdot)\Big) \in \bar{\Omega},
\end{aligned}$$

where

$$\bar{\Omega} = \left\{ \Big(\bar{m}_0, \bar{m}(\cdot)\Big) \;\middle|\; \begin{array}{l} \bar{m}_0 = m_0 + m_1 \\ \bar{m}(\zeta) = -m_1 m_{0\zeta}, \\ \bar{m}(-\tau+\zeta) = -m_1 m_{1\zeta} \\ \quad -\tau \le \zeta < 0 \end{array} \right\} \qquad (5.51)$$

$$m_{i\zeta}(t) = m_i(t+\zeta)$$

and

$$\bar{r}_B \equiv r_B(t) - m_1(t)\int_{-\tau}^{0} r_B(t+\zeta)d\zeta.$$

$\bar{r}_B(t)$ is bounded since $r_B(t)$ and $m_1(t)$ are bounded. That is, there exists a scalar $r_{\max}$ such $|\bar{r}_B(t)| \le r_{\max}$ $\forall t \ge t_0$. Equation (5.50) can be seen to be a system with distributed delays, whose stability can be shown using the Razumikhin method, as shown below.

Define

$$V(e) = e^2 \tag{5.52}$$

$$e_t(t) = \max_{\zeta\in[-2\tau,0]} |e(t+\zeta)|, \tag{5.53}$$

$$\bar{V}(e_t) = \max_{\zeta\in[-2\tau,0]} V(e(t+\zeta)) \tag{5.54}$$

and a set Ω_t

$$\Omega_t \equiv \left\{ t \middle| t\in(t_b,t_c),\ \ V(e(t)) = \bar{V}(e_t) \right\}. \tag{5.55}$$

It follows that for all $t\in(t_b,t_c)$, there are two cases, (a) $t\in\Omega_t$, (b) $t\in(t_b,t_c)\backslash\Omega_t$. We provide the proof for each case separately.

(a) $\underline{t\in\Omega_t}$: From the definitions in (5.54) and (5.55), it follows that in this case,

$$V(e(t+\zeta)) \leq V(e(t)) \quad \text{for all} \ \ -2\tau\leq\zeta\leq 0. \tag{5.56}$$

Hence we obtain that

$$\begin{aligned}\dot{V}(e) \leq{}& 2\bar{m}_0(t)e^2(t) + \int_{-2\tau}^{0} \bar{m}(t,\zeta)e(t+\zeta)e(t)d\zeta + 2e(t)\bar{r}_B(t) \\ &+ \int_{-2\tau}^{0} \alpha(\zeta)\left[e^2(t) - e^2(t+\zeta)\right]d\zeta\end{aligned}$$

with any scalar positive function $\alpha(\zeta)$, since the last term then becomes positive due to (5.56). We therefore obtain that

$$\dot{V}(e) \leq \int_{-2\tau}^{0} E_\zeta^T(t)\Psi(t,\zeta)E_\zeta(t)d\zeta + 2r_{\max}|e(t)| \tag{5.57}$$

where

$$\Psi(t,\zeta) \equiv \begin{bmatrix} n_p(t,\zeta) & \bar{m}(t,\zeta) \\ \bar{m}(t,\zeta) & -\alpha(\zeta) \end{bmatrix}, \tag{5.58}$$

$$n_p(t,\zeta) = \frac{1}{\tau}(m_0 + m_1) + \alpha(\zeta), \tag{5.59}$$

and $E_\zeta(t) = [e(t)\ e(t+\zeta)]^T$.

With the selection

$$\alpha(\zeta) = \begin{cases} \bar{a}^2 & -\tau < \zeta \leq 0 \\ \theta_{\max}^2 & -2\tau \leq \zeta \leq -\tau \end{cases},$$

noting that $\theta_{\max} > \bar{a}$, from (5.59) and (5.48) it can be seen that if

$$\tau < \frac{-(a_p - \theta_{\max} + \varepsilon)}{\theta_{\max}^2}, \tag{5.60}$$

then

$$n_p(t,\zeta) < 0 \quad \forall t, \zeta. \tag{5.61}$$

Furthermore, together with (5.48) and (5.51), the determinant of the matrix $\Psi(t,\zeta)$ given by (5.58) can be computed and bounded as

$$\begin{aligned} &\det(\Psi(t,\zeta)) \\ &\quad = \begin{cases} -(\frac{1}{\tau}(m_0 + m_1) + \bar{a}^2)\bar{a}^2 - m_1^2 m_{0\zeta}^2 \\ -(\frac{1}{\tau}(m_0 + m_1) + \theta_{\max}^2)\theta_{\max}^2 - m_1^2 m_{1\zeta}^2 \end{cases} \\ &\geq \begin{cases} -\frac{1}{\tau}(a_p - \theta_{\max} + \varepsilon)\bar{a}^2 - \bar{a}^4 - \theta_{\max}^2 \bar{a}^2 & -\tau < \zeta \leq 0 \\ -\frac{1}{\tau}(a_p - \theta_{\max} + \varepsilon)\theta_{\max}^2 - \theta_{\max}^4 - \theta_{\max}^4 & -2\tau \leq \zeta \leq -\tau. \end{cases} \end{aligned} \tag{5.62}$$

Again noting that $\theta_{\max} > \bar{a}$, it can therefore be seen that if

$$\tau < \frac{-(a_p - \theta_{\max} + \varepsilon)}{2\theta_{\max}^2} \tag{5.63}$$

then

$$\det(\Psi(t,\zeta)) > 0 \quad \forall t, \zeta. \tag{5.64}$$

From (5.60) and (5.63), we see that if

$$\tau \leq \bar{\tau} \tag{5.65}$$

where

$$\bar{\tau} \equiv \frac{-(\bar{a} - \theta_{\max} + \varepsilon)}{4\theta_{\max}^2}, \tag{5.66}$$

then (5.61) and (5.64) are both satisfied, proving that

$$\Psi(t,\zeta) < 0 \quad \forall t, \zeta \tag{5.67}$$

for all $a_p \in [-\bar{a}, \bar{a}]$.

Defining

$$\varepsilon_v \equiv \min_{t,\zeta,\tau \in [0,\bar{\tau}]} (-\text{eig}(\Psi(t,\zeta))), \tag{5.68}$$

(5.57) can therefore be simplified as

$$\dot{V}(e(t)) \leq -\varepsilon_v |e(t)|^2 + 2r_{\max}|e(t)|. \tag{5.69}$$

From (5.69),

$$\dot{V}(e(t)) < 0 \quad \forall t \in \Omega_t \backslash \{t \,|\, |e(t)| > \beta\} \tag{5.70}$$

where $\beta = 2r_{\max}/\varepsilon_v$. Since $\bar{V}(e_t(t)) = V(e(t))$ as we defined Ω_t in (5.55), it can be concluded that

$$\dot{\bar{V}}(e_t(t)) < 0 \quad \forall t \in \Omega_t \backslash \{t||e(t)| > \beta\}. \tag{5.71}$$

(b):$\underline{t \in (t_b, t_c) \backslash \Omega_t}$: From the definitions in (5.54) and (5.55), it follows that for any t in Case (b),

$$\bar{V}(e_t(t)) > V(e(t)). \tag{5.72}$$

Suppose there exists a $t = t_s \in (t_b, t_c) \backslash \Omega_t$ such that

$$\dot{\bar{V}}(e_t(t_s)) > 0.$$

Then it follows that

$$V(e(t_s^+)) > \bar{V}(e_t(t_s))$$

from the definition of $\bar{V}(e_t)$ in (5.54). This contradicts (5.72), and therefore we can conclude that

$$\dot{\bar{V}}(e_t(t)) \leq 0 \quad \forall t \in (t_b, t_c) \backslash \Omega_t.$$

From Case (a) and (b),

$$\dot{\bar{V}}(e_t(t)) \leq 0 \quad \forall t \in (t_b, t_c) \backslash \{t||e(t)| > \beta\}. \tag{5.73}$$

From (5.53), we have that $e_t(t)$ is always positive. Therefore (5.73) implies that

$$\dot{e}_t(t) \leq 0 \quad \forall t \in (t_b, t_c) \backslash \{t||e(t)| > \beta\}$$

and therefore

$$e_t(t) \leq \max\{e_t(t_b), \beta\}. \tag{5.74}$$

Since $|e(t)| \leq e_t(t)$ from (5.53), (5.74) implies that

$$|e(t)| \leq \max\{|e(t_b)|, \beta\} \quad \forall t \in (t_b, t_c),$$

completing the proof.

□

From Proposition 2, $|e(t_b)| \leq \bar{e}$ and since $\bar{e} \geq \beta$ from (5.20), it can be concluded that $|e(t)| \leq \bar{e}\ \forall t \in (t_b, t_c)$.

5.3.3 Phase III: Exiting from the Boundary

Proposition 4. *Let $z(t_b') \in B_L$. Then either*

(I) $z(t) \in B\ \forall t \geq t_b'$, or

(II) there exists $t_c > t_b'$ such that $z(t_c) \in A$ and $z(t) \in B\ \forall t \in [t_b',\ t_c)$.

In addition, in case (II),

$$t_c - t_b' \geq \Delta T_{exit,\min} \tag{5.75}$$

where

$$\Delta T_{exit,\min} = \frac{2\varepsilon}{\gamma \bar{x}_m^2}, \tag{5.76}$$

and

$$|e(t_c)| < \bar{x}_m. \tag{5.77}$$

Proof. It is straightforward to see that cases (I) and (II) are mutually and collectively exclusive.

From the definition of regions A and B_L, it follows that

$$\theta(t_b') \leq -(\theta'_{\max} + \varepsilon/2), \quad \theta(t_c) \geq -\theta'_{\max}.$$

In addition, from (5.24) and (5.25)

$$\dot{\theta}(t) \leq \frac{1}{4}\gamma \bar{x}_m^2 \qquad \forall\, t.$$

Hence

$$t_c - t_b' \geq \frac{2\varepsilon}{\gamma \bar{x}_m^2},$$

completing the proof for (5.75).

We now prove (5.77) as follows. From the conditions in case (II), it is seen that

$$\theta(t_c - \Delta t_c) < -\theta'_{\max}, \quad \theta(t_c) \geq -\theta'_{\max}$$

for any $\Delta t_c \in (0, t_c - t_b']$. Letting Δt_c tend to zero from the right hand side, it follows that $\dot{\theta}(t_c) > 0$. From (5.24) and (5.25), this in turn implies that $|e(t)| < |x_m(t)|$, proving (5.77).

□

5.3.4 Phase IV: Return to Condition 1

So far, we have shown on Phases I through III that if at $t = t_a$, $e(t)$ satisfies Condition 1,

I. $z(t_b) \in B$ for $t_b < t_a + \Delta T_{in,\max}$, with $|e(t)| \leq \bar{e}\ \forall t \in [t_a, t_b]$

II. Defining t_c such that $z(t) \in B\ \forall t \in (t_b, t_c)$, if $\tau \leq \bar{\tau}$, then $|e(t)| \leq \bar{e}\ \forall t \in (t_b, t_c)$.

III. Either (a) $t_c = \infty$, or (b) $t_c \geq t_b' + \Delta T_{exit,\min}$

where $z(t_c) \in A$ and $|e(t_c)| < \bar{x}_m$.

We also infer from (5.19) and (5.20) that $\bar{x}_m < \bar{e} - 2\delta$.

I to **III** above imply therefore that there are only two possibilities:

(A) $|e(t)| < \bar{e} - \delta$ for all $t \geq t_c$, or

(B) there exists $t_d > t_c$ s.t. $|e(t_d)| = \bar{e} - \delta$ and $|e(t)| < \bar{e} - \delta\ \forall t \in [t_c, t_d)$.

Global boundedness of $z(t)$ is immediate in case **(A)**. If case **(B)** holds, then from the condition of the case it immediately implies that $e(t)$ satisfies Condition 1 (5.22) for $t = t_d$. We note that $\forall t \in (t_b, t_c)$, $z(t) \in B$ with $|e(t)| \leq \bar{e}$. This together with the condition of the case implies that

$$|e(t)| \leq \bar{e} \quad \forall t \in [t_b, t_d].$$

Hence if $\tau \leq \Delta T_{exit,\min}$, it follows that $e(t)$ satisfies Condition 1 (5.21) for $t = t_d$. This proves Phase IV.

5.3.5 Final Part of the Proof

The above phases imply that starting $t = t_a$, there are only one of three possibilities: The trajectory stays on Phase II for all $t > t_1$ for some finite $t_1 \geq t_b$; (ii) The trajectory stays on Phase IV-(A) for all $t \geq t_2$ for some $t_2 \geq t_c$; (iii) The trajectory visits all four phases infinitely often. The discussions in sections 5.3.1 through 5.3.4 imply that in all three cases (i)-(iii), $e(t)$ always remains bounded, which proves Theorem 10. In particular, it follows from (5.27), Proposition 3, and (5.77) that in all cases (i)-(iii), if $\tau \leq \tau_l^\star$ defined as

$$\tau_l^\star = \min\left[\Delta T_{exit,\min}, \bar{\tau}\right], \tag{5.78}$$

then,

$$|e(t)| \leq \bar{e} \quad \forall t \geq t_0$$

and hence

$$|z(t)| \leq M \quad \forall t \geq t_0,$$

where $M \equiv \sqrt{\bar{e}^2 + \theta_{\max}^2}$, proving global boundedness.

5.3.6 Delay Margin of the Adaptive System

From (5.66), (5.76), and (5.78), we obtain that the solutions of the overall adaptive system is bounded for all $\tau \leq \tau_l^\star$. Hence, the lower bound of the delay margin $\tau^\star$ is given by $\tau_l^\star$, with

$$\tau_l^\star = \min\left[\frac{2\varepsilon}{\gamma \bar{x}_m^2}, \frac{\theta_{\max} - \bar{a} - \varepsilon}{4\theta_{\max}^2}\right]. \tag{5.79}$$

Noting (5.16), if we choose

$$\theta_{\max} - \varepsilon = \bar{a} + |a_m|,$$

(5.79) can be rewritten as

$$\tau_l^\star = \min\left[\frac{2\varepsilon}{\gamma \bar{x}_m^2}, \frac{|a_m|}{4\left(\bar{a} + |a_m| + \varepsilon\right)^2}\right]. \tag{5.80}$$

The delay margin obtained as given by (5.79) or (5.80) is intuitive. As adaptation speed γ is set to be larger, it is seen that the delay margin becomes smaller. As reference input $r(t)$ is more aggressive, which usually leads to larger $\bar{x}_m$, the guaranteed delay margin is reduced. As there is larger size of uncertainty $\bar{a}$ in the system, it requires larger $\theta_{\max}$ to ensure the ideal gain to lie in the inner projection set, leading to smaller delay margin.

5.3.7 Remarks

Theorem 10 establishes global boundedness in the presence of time delay and a computable lower bound of the delay margin is obtained as in (5.79) or (5.80). Instead of utilizing any Lyapunov function, which is a fixture in most adaptive control proofs, a first principles approach was used to ensure the global boundedness of the errors.

It should be noted that while Theorem 10 ensures global boundedness for a range of time delays, if $\tau = 0$ (no time delay in the system) convergence of the state error $e(t)$ can be shown for the adaptive system as follows, utilizing the Lyapunov function. If $\tau = 0$, with a V as in (5.7), we obtain

$$\dot{V} = a_m e^2 + \phi\left(ex_p + \text{Proj}(\theta, -ex_p)\right) \tag{5.81}$$

since $\eta(t) = 0\ \forall t$ from (5.11). Equation (3.22) in Lemma 2 together with $y = -ex_p$ implies that the term within the parentheses in Eq. (5.81) is non-positive. This in turn implies that

$$\dot{V} \leq a_m e^2,$$

where a_m is negative as given in (5.6).

5.4 Numerical Example

In this section we demonstrate using a simple example as to how the main result in this chapter can be used to obtain delay margin of adaptive systems. We consider the roll dynamics of a conventional aircraft which can be approximated by a scalar plant.

Consider the aircraft roll dynamics in the form of

$$\dot{p} = L_p p + L_{\delta_a} \delta_a$$

where p is the aircraft roll rate in stability axis (radians/s), δ_a is the total differential aileron-spoiler deflection (radians), L_p is the roll damping derivative, and L_{δ_a} is the dimensional rolling moment derivative with respect to differential aileron-spoiler deflection, (the aileron-to-roll control effectiveness). Given $L_p = -0.8(s^{-1})$ and $L_{\delta_a} = 1.6(s^{-1})$, we design a nominal controller

$$\delta_a = k_p p + k_{cmd} p_{cmd}$$

where $k_p = -0.75$ and $k_{cmd} = 1.25$. Then the ideal closed-loop dynamics is given as

$$\dot{p}_{ideal} = -2p_{ideal} + 2p_{cmd}.$$

Now we assume that the constant roll damping derivative L_p is unknown, but known to be $|L_p| \leq 2$. An adaptation can be introduced into k_p as

$$\dot{k}_p = -\gamma p e_p$$

where γ is a positive constant and $e_p = p - p_{ideal}$. A projection algorithm as described above is introduced, modifying the adaptive law as

$$\dot{k}_p = \text{proj}\left(k_p, -\gamma p e_p\right).$$

Noting the upper bound on L_p, we choose the projection parameters $\theta'_{\max} = 2.7$ and $\varepsilon = 0.01$.

Also, we set $\gamma = 1$ and assume that $p_{cmd}(t)$ is such that

$$|p_{ideal}(t)| \leq 0.2(radians/s),$$

which specifies $\bar{x}_m$ in (5.79). We can therefore calculate the delay margin using (5.79) as

$$\tau_l^\star = 0.024(s).$$

According to some numerical simulation studies[1], it was seen that the actual delay margin of the uncertain adaptive system is around $0.38(s)$. It can be therefore concluded that the analytical lower bound of a delay margin obtained in this chapter is not overly conservative.

5.5 Summary

In this chapter, robust adaptive control of scalar plants in the presence of time delay is established. It is shown that a standard MRAC adaptive law modified only with a projection algorithm ensures global boundedness of the overall adaptive system for a range of non-zero delays. The upper bound of such delays, i.e. the delay margin, is explicitly computed

[1]Extensive time simulations were conducted over the parameter space specified.

and demonstrated using the aircraft roll dynamics. By taking a close look at how the trajectory behaves and relying on first principle analysis, not overly conservative results are obtained.

An extension to higher dimensional plants where states are accessible is presented in the next chapter.

Chapter 6

Guaranteed Delay Margins for Adaptive Systems with State Variables Accessible

In Chapter 5, it was shown that global boundedness can be derived with a computable delay margin using a modified adaptive law based on projection for a first-order uncertain plant. In contrast to the results in Chapter 4, not overly conservative results were obtained, by virtue of taking a close look at how the trajectory behaves with a scalar plant and relying on first principle analysis.

In this chapter, we extend this result to higher dimensional plants with a scalar input, where states are accessible. Although some complications arise when we depart from the first-order plant case, similar arguments can be still applied to obtain global results and computable delay margins.

The problem is stated in Section 6.1. The main result is stated in Section 6.2 and proved in Section 6.3.

6.1 Problem Statement

An nth order plant with a scalar input and a parameter uncertainty is given by

$$\dot{x_p}(t) = A_p x_p(t) + b_p u(t-\tau) \tag{6.1}$$

where A_p is an unknown parameter, b_p is known, and $\tau \geq 0$ is an unknown time delay.

A control law is chosen as

$$u(t) = \theta^T(t)x_p(t) + r(t) \tag{6.2}$$

where $\theta(t)$ is time varying due to adaptation and $r(t)$ is a reference input. The problem is to ensure bounded solutions with the control law as in (6.2) using a suitable adaptive law for adjusting $\theta(t)$.

(6.1) can be rewritten into the form of

$$\dot{x_p}(t) = A_p x_p(t) + b_p\big(u(t) + \eta(t)\big) \tag{6.3}$$

where

$$\eta(t) = u(t-\tau) - u(t). \tag{6.4}$$

Therefore the system subject to the input time delay can be interpreted as a perturbed system by the unmodeled dynamics $\eta(t)$.

A reference model is chosen as

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t) \tag{6.5}$$

where A_m is Hurwitz. Therefore with a reference input $r(t)$ bounded, boundedness of x_m is straightforward. We define $\bar{r} \equiv \max_{t \geq t_0} |r(t)|$. Also we take $b_m = b_p$.

Assuming that $\theta^\star$ exists such that

$$A_p + b_p\theta^{\star T} = A_m, \tag{6.6}$$

we define the parameter and state errors

$$\phi(t) = \theta(t) - \theta^{\star} \tag{6.7}$$

$$e(t) = x_p(t) - x_m(t). \tag{6.8}$$

The closed-loop adaptive system is given by

$$\dot{x}_p(t) = A_m x_p(t) + b_p\big(\phi^T(t)x_p(t) + r(t) + \eta(t)\big) \tag{6.9}$$

and the error equation

$$\dot{e}(t) = A_m e(t) + b_m\big(\phi^T(t)x_p(t) + \eta(t)\big). \tag{6.10}$$

It is known that since A_m is Hurwitz, for any positive definite symmetric matrix Q, there exists a positive definite symmetric matrix P which satisfies the Lyapunov equation

$$A_m^T P + P A_m = -Q. \tag{6.11}$$

As in the first-order plant case discussed in Section 5.1, it can be seen that a standard adaptive law based on MRAC [44]

$$\dot{\theta}(t) = -\Gamma x_p(t) b_m^T P e(t) \tag{6.12}$$

where Γ is a positive definite symmetric matrix and P is given by (6.11), with the quadratic function

$$V = \frac{1}{2}e^T P e + \frac{1}{2}\phi^T \Gamma^{-1}\phi \tag{6.13}$$

which truly serves as a Lyapunov function to ensure stability if there is no delay, only yields

$$\dot{V} = -e^T Q e + e^T P b_m \eta \tag{6.14}$$

with a state-dependent disturbance η. Therefore, in the presence of a delay, stability can no longer be assured.

In fact, as shown in Theorem 1 and 2 in Chapter 3, the adaptive stabilizer with the standard adaptive law (6.12) has unbounded solutions.

The question is if a different adaptive law than (6.12) can ensure a delay margin. This is the problem addressed in Section 6.2 for general nth order plants. In particular, we demonstrate that a modified adaptive law based on projection algorithm guarantees the determination of a $\tau_l^\star$ such that the adaptive system consisting of the plant in (6.1) and the control law in (6.2) has globally bounded solutions for all $\tau \in [0,\ \tau_l^\star]$.

6.2 Boundedness in the Presence of Time Delay

We now establish a robust adaptive controller for higher dimensional plants with a scalar input, which ensures global boundedness in the presence of a certain range of finite time delays, using projection algorithm.

Before we proceed with the main theorem, we first describe the specific adaptive law based on projection algorithm that we will use to adjust the parameter θ in (6.2). In order to ensure robustness, we apply projection to a set of transformed error states. As will be seen in this section, the nonsingular transformation helps in focusing on two key scalars, one each in e and θ which are central to the proof. This transformation is described in Section 6.2.1. The adaptive law based on projection is discussed in Section 6.2.2. Certain features of the reference model parameters are discussed in Section 6.2.3 using the transformation. A key property of the projection algorithm is revisited in Section 6.2.4. Selections of the projection algorithm parameters are discussed in Section 6.2.5. Following these preliminaries, the main result is stated in Section 6.2.6 and proved in Section 6.3.

6.2.1 A Nonsingular Transformation

In this section, we will derive transformed state error $\mathscr{E}(t)$ and parameter $\vartheta(t)$ using transfer matrices C and M so that

$$\mathscr{E}(t) \equiv Ce(t), \tag{6.15}$$

$$\vartheta(t) \equiv M\theta(t). \tag{6.16}$$

We refer to ith component of these states by $\mathscr{E}_i(t)$, $\vartheta_i(t)$ respectively, $i = 0, 1, \cdots, n-1$. The introduction of C and M are needed in order to identify crucial scalars that capture the dominant effect of the perturbation η. We now describe the construction of C and M.

The matrix M in (6.16) is chosen as follows. First we define

$$c_0 = \frac{Pb_m}{p_{bb}} \tag{6.17}$$

where P is given in (6.11) and $p_{bb} \equiv \sqrt{b_m^T P b_m}$. We note that

$$c_0^T b_m = \frac{b_m^T P}{p_{bb}} b_m = p_{bb}. \tag{6.18}$$

Then we pick $n-1$ vectors c_i for $i = 1, 2, \cdots, n-1$ which satisfy

$$c_i^T P^{-1} c_j = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j \end{cases} \tag{6.19}$$

where $j = 0, 1, \cdots, n-1$. We therefore note that

$$c_i^T b_m = c_i^T P^{-1} c_0 p_{bb} = 0 \quad \text{for} \quad i = 1, 2, \cdots, n-1. \tag{6.20}$$

We obtain an invertible matrix C by defining

$$C = \begin{bmatrix} c_0^T \\ c_1^T \\ \vdots \\ c_{n-1}^T \end{bmatrix}. \tag{6.21}$$

From (6.17), (6.19) and (6.21), it can be seen that

$$CP^{-1}C^T = I \tag{6.22}$$

$$\sum_{j=0}^{n-1} c_j c_j^T = P. \tag{6.23}$$

Using P and C, we choose M as

$$M = p_{bb} C P^{-1}. \tag{6.24}$$

6.2.2 A Modified Adaptive Law with the Projection Algorithm

Several approaches have been taken in robust adaptive control to establish boundedness, which includes the modification of the standard MRAC adaptive law. One such example is to utilize projection algorithm as demonstrated in Chapter 4, Chapter 5, and [41, 51, 55].

The adaptive law we propose in this chapter is of the form

$$\dot{\theta}(t) = M^{-1} w \tag{6.25}$$

where $w = [w_1 \; w_2 \; \cdots \; w_n]^T$ and

$$w_i = \mathrm{Proj}\Big(\{M\theta(t)\}_i, -\{M\Gamma x_p(t) b_m^T P e(t)\}_i\Big). \tag{6.26}$$

The projection $\mathrm{Proj}(,)$ in (6.26) is a scalar function with scalar arguments and is defined by[1]

$$\mathrm{Proj}(\Theta, y) = \begin{cases} \frac{\theta_{\max}^2 - \Theta^2}{\theta_{\max}^2 - \theta_{\max}'^2} y & \text{if } [\Theta \in \Omega_1 \backslash \Omega_0 \wedge y\Theta > 0] \\ y & \text{otherwise} \end{cases} \tag{6.27}$$

where $\theta_{\max} > \theta'_{\max}$ are any positive constants,

$$\varepsilon = \theta_{\max} - \theta'_{\max}, \tag{6.28}$$

and

$$\begin{aligned} \Omega_0 &= \left\{ \Theta \in \mathbb{R}^1 | -\theta'_{\max} \leq \Theta \leq \theta'_{\max} \right\} \\ \Omega_1 &= \left\{ \Theta \in \mathbb{R}^1 | -\theta_{\max} \leq \Theta \leq \theta_{\max} \right\}. \end{aligned} \tag{6.29}$$

When the projection is not activated ($\mathrm{Proj}(\Theta, y) = y$), the adaptive law given by (6.25) and (6.26) is reduced to the standard adaptive law (6.12). For the sake of simplicity, we will assume that $\Gamma = \gamma P$.

6.2.3 Properties Regarding the Reference Model

We define scalars

$$\alpha_{ij} \equiv c_i^T A_m P^{-1} c_j, \quad i, j = 0, \cdots, n-1 \tag{6.30}$$

and an $(n \times n)$ matrix

$$\mathscr{A}_m = C A_m P^{-1} C^T. \tag{6.31}$$

[1]The projection function is identical with the one given by (5.13) and (5.14) in Chapter 5. For the sake of completeness we repeat them here.

We partition $\mathscr{A}_m$ as

$$\mathscr{A}_m = \begin{bmatrix} \alpha_{00} & a_1^T \\ a_0 & \mathscr{A}_m' \end{bmatrix} \tag{6.32}$$

where $\mathscr{A}_m'$ is an $(n-1)\times(n-1)$ matrix. We show below that $\mathscr{A}_m'$ is Hurwitz.

Lemma 6. *$\mathscr{A}_m$ is Hurwitz.*

Proof. From (6.22),

$$P^{-1}C^T = C^{-1} \tag{6.33}$$

and therefore (6.31) can be rewritten as

$$\mathscr{A}_m = CA_mC^{-1}. \tag{6.34}$$

Then we obtain

$$\det(sI - \mathscr{A}_m) = \det\left(C(sI - A_m)C^{-1}\right) = \det(C)\det(sI - A_m)\det(C^{-1}),$$

which becomes zero only and if

$$\det(sI - A_m) = 0$$

since $\det(C) \neq 0$. Therefore the eigenvalues of $\mathscr{A}_m$ and those of A_m are identical. Since A_m is Hurwitz, this implies that $\mathscr{A}_m$ is also Hurwitz. □

We note that the eigenvectors of $\mathscr{A}_m$ and those of A_m are not necessary identical.

Lemma 7. *$\mathscr{A}_m'$ is Hurwitz.*

Proof. From (6.34) and (6.33),

$$A_m^TP + PA_m = C^T\left(\mathscr{A}_m + \mathscr{A}_m^T\right)C = PC^{-1}\left(\mathscr{A}_m + \mathscr{A}_m^T\right)C. \tag{6.35}$$

Noting $P > 0$, and $A_m^T P + PA_m < 0$ from (6.11), we obtain

$$C^{-1}\left(\mathscr{A}_m + \mathscr{A}_m^T\right)C < 0.$$

Therefore

$$\mathscr{A}_m + \mathscr{A}_m^T < 0. \tag{6.36}$$

Considering the principal $(n-1)\times(n-1)$ submatrix, it can be then concluded that

$$\mathscr{A}_m' + \mathscr{A}_m'^T < 0. \tag{6.37}$$

Since the symmetric part is negative definite, $\mathscr{A}_m'$ is Hurwitz. □

Remark 7. *$\mathscr{A}_m$, as seen in* (6.31)*, has a special structure with C chosen using* (6.17)*,* (6.19) *and* (6.21)*. While in general a Hurwitz matrix X need not have its submatrix X' to be Hurwitz, because of the special structure of $\mathscr{A}_m$, it is true that $\mathscr{A}_m'$ is Hurwitz, as shown in Lemma 7. Two examples are shown below to demonstrate that this does hold. Since one of our main interests is flight applications, this includes one with flight dynamics.*

Example 1. *Consider a 3×3 matrix in a control canonical form*

$$A_m = \begin{bmatrix} 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \\ -1.4142 & -3.4545 & -4.5726 \end{bmatrix}, \quad b_m = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

P is obtained as a unique solution of the Lyapunov equation $A_m^T P + PA_m = -Q$ with $Q = I$,

$$P = \begin{bmatrix} 2.2426 & 1.9947 & 0.3536 \\ 1.9947 & 3.8719 & 0.7222 \\ 0.3536 & 0.7222 & 0.2673 \end{bmatrix}.$$

From b_m and P, we construct C as

$$C = \begin{bmatrix} 0.6839 & 1.3968 & 0.5170 \\ 1.1474 & 0.0898 & 0 \\ 0.6771 & 1.3830 & 0 \end{bmatrix}.$$

Therefore we obtain

$$\mathscr{A}_m = \begin{bmatrix} -1.8707 & -0.1736 & 1.1039 \\ 0.1736 & -0.5091 & 0.6873 \\ 2.6751 & -0.3004 & -2.1928 \end{bmatrix}, \quad \mathscr{A}_m' = \begin{bmatrix} -0.5091 & 0.6873 \\ -0.3004 & -2.1928 \end{bmatrix}$$

whose eigenvalues are $\{-3.7525, -0.4101 \pm 0.4569i\}$ and $\{-0.6422, -2.0597\}$, respectively. This does not contradict with Lemma 7.

Example 2. *The linearized short period dynamics is of the form*

$$A_p = \begin{bmatrix} -0.2950 & 1.0000 \\ -13.0798\lambda_\alpha & -0.2084\lambda_q \end{bmatrix}, \quad b_p = \begin{bmatrix} 0 \\ -9.4725 \end{bmatrix}$$

where λ_α, λ_q are parametric uncertainties with a nominal value of one. A reference model is chosen as

$$A_m = \begin{bmatrix} -0.2950 & 1.0000 \\ -12.9121 & -6.6762 \end{bmatrix},$$

whose eigenvalues are $-3.4856 \pm 1.6529i$. With $Q = I$, we obtain the unique solution of the Lyapunov equation

$$P = \begin{bmatrix} 1.0901 & 0.0138 \\ 0.0138 & 0.0770 \end{bmatrix}.$$

Choosing $b_m = b_p$, we compute

$$C = \begin{bmatrix} -0.0498 & -0.2774 \\ 1.0429 & 0 \end{bmatrix}.$$

Therefore $\mathscr{A}_m$, $\mathscr{A}_m'$ can be calculated as

$$\mathscr{A}_m = \begin{bmatrix} -6.4967 & 3.1386 \\ -3.7592 & -0.4746 \end{bmatrix}, \quad \mathscr{A}_m' = \begin{bmatrix} -0.4746 \end{bmatrix}$$

whose eigenvalues are $-3.4856 \pm 1.6529i$ and -0.4746, respectively.

6.2.4 Properties of the Lipschitz Continuous Projection Algorithm

The most interesting property of the projection algorithm is its ability to ensure the boundedness of the parameter estimate independent of the system dynamics. We note Lemma 2-Lemma 4, which are the important lemmas regarding the projection algorithm. For the sake of completeness, we state a lemma[2] below.

Lemma 8. *Consider the dynamics of $\Theta \in \mathbb{R}^1$*

$$\dot{\Theta} = Proj(\Theta, y) \tag{6.38}$$

with Projection algorithm in (6.27), (6.29). Then,

$$|\Theta(t_0)| \leq \theta_{\max} \implies |\Theta(t)| \leq \theta_{\max} \forall t \geq t_0. \tag{6.39}$$

The proof of Lemma 8 is straightforward from Lemma 3, or Lemma 4.

The implications of Lemma 8 on boundedness of the control parameter θ are obvious. If the adaptive law is chosen as in (6.17)-(6.27) with $\Theta = \{M\theta\}_i$ and $y = -\{M\Gamma x_p b_m^T Pe\}_i$ for each $i = 1, 2, \cdots, n$, then irrespective of the boundedness of y and e, it follows that $\{M\theta(t)\}_i$ is bounded, i.e. $|\{M\theta(t_0)\}_i| \leq \theta_{i,\max} \implies |\{M\theta(t)\}_i| \leq \theta_{i,\max} \forall t \geq t_0$.

[2]Lemma 8 is a scalar version of Lemma 4.

We let sets Ω_{0R} and Ω_{1R} be defined as

$$\Omega_{0R} = \left\{\theta \in \mathbb{R}^n \middle| -\theta'_{i,\max} \leq \{M\theta\}_i \leq \theta'_{i,\max} \ \ \forall i\right\},$$
$$\Omega_{1R} = \left\{\theta \in \mathbb{R}^n \middle| -\theta_{i,\max} \leq \{M\theta\}_i \leq \theta_{i,\max} \ \ \forall i\right\}.$$

6.2.5 Choice of Projection Algorithm Parameters

The projection algorithm (6.26) requires $\theta_{i,\max}$ and ε_i to be specified, whose selections are discussed below.

We assume that upper bounds $\theta^\star_{i,\max} \in \mathbb{R}$ on the uncertain parameter $\theta^\star$ are known and are defined as

$$\theta^\star_{i,\max} = \max_{\theta^\star} \left|\vartheta^\star_i\right| \quad i = 0, 1, \cdots, n-1$$

where $\vartheta^\star = M\theta^\star$ and $\vartheta^\star_i$ refers to its ith component. We choose the control parameters $\theta_{i,\max}$ and ε_i for $i = 1, 2, \cdots, n-1$ such that

$$\theta_{i,\max} - \varepsilon_i \geq \theta^\star_{i,\max}. \tag{6.40}$$

Letting p_φ be a positive number, we then choose $\theta_{0,\max}$ and ε_0 such that

$$-(\theta_{0,\max} - \varepsilon_0) + \theta^\star_{0,\max} < -\alpha_{00} + \frac{1}{2p_\varphi s_{Q'}} \left(\|P'a_0\| + (\|a_1\| + \phi'_{\max})p_\varphi\right)^2 \tag{6.41}$$

where α_{00}, a_0, and a_1 are defined as in (6.32), P' is the solution of

$$\mathscr{A}'^T_m P' + P'\mathscr{A}'_m = -Q'$$

with a positive definite symmetric matrix Q' and

$$\phi'_{\max} \equiv \sqrt{\sum_1^{n-1} \left(\theta_{i,\max} + \theta^\star_{i,\max}\right)^2}.$$

It should be noted that choosing control parameters which satisfy (6.41) is always possible by taking $\theta_{0,\max}$ to be large enough. We also note that (6.40) implies that the condition

$\Theta^\star \in \Omega_0$ in Lemma 2 is satisfied with $\Theta = \vartheta_i$ for $i = 0, 1, \cdots, n-1$. The choice of $\theta_{0,\max}$ in (6.41) will become clear in Section 6.3.6, where the boundedness of e and θ are addressed.

Lastly, we define

$$\Theta_{\max} \equiv \sqrt{\sum_{i=0}^{n-1} \theta_{i,\max}^2} \tag{6.42}$$

and

$$\phi_{\max} \equiv \sqrt{\sum_{0}^{n-1} \left(\theta_{i,\max} + \theta_{i,\max}^\star\right)^2}. \tag{6.43}$$

6.2.6 Main Result

Theorem 11. *There exists a* $\tau^\star$ *such that the closed-loop adaptive system with the plant in* (6.1)*, reference model in* (6.5)*, control law in* (6.2)*, and adaptive law in* (6.25)*,* (6.27)*,* (6.29) *together with the projection parameters as in* (6.40)*,* (6.41) *has globally bounded solutions for any initial conditions*

$$x_p(t) = \chi(t), \quad \theta(t) = \chi_\theta(t) \qquad t \in [t_0 - \tau, t_0] \tag{6.44}$$

where $\chi(t) : \mathbb{R} \to \mathbb{R}^n$*,* $\chi_\theta(t) : \mathbb{R} \to \Omega_{1R}$*, and* $\forall \tau \in [0, \tau^\star)$.

Theorem 11 implies that the adaptive system consisting of MRAC with the projection algorithm has a nonzero time delay margin $\tau^\star$.

6.2.7 Preliminaries

Before we proceed to the proof of Theorem 11, a few important constants and a specific condition are first defined. This condition will be shown to be satisfied by the trajectory in the proof.

Notations

Throughout the chapter, we again use the following notations. Let

$$\begin{aligned} \underline{s}_A &= \min_i \left|\Re\big(\lambda_i(A)\big)\right| \\ \bar{s}_A &= \max_i \left|\Re\big(\lambda_i(A)\big)\right| \end{aligned} \tag{6.45}$$

where λ_i is ith eigenvalue of a square matrix A and $\Re(\lambda_i)$ is its real part.

Definitions

Definition 5. *We define regions A, B, and B' as follows (See Figure 6-1): Let* $z(t) = [\mathscr{E}^T(t)\ \ \vartheta^T(t)]^T$.

$$\begin{aligned} A &= \left\{z \in \mathbb{R}^{2n} \middle| -\theta'_{0,\max} \le \vartheta_0 \le \theta'_{0,\max}\right\} \\ B &= \left\{z \in \mathbb{R}^{2n} \middle| -\theta_{0,\max} \le \vartheta_0 < -\theta'_{0,\max}\right\} \\ B' &= \left\{z \in \mathbb{R}^{2n} \middle| \theta'_{0,\max} < \vartheta_0 \le \theta_{0,\max}\right\}. \end{aligned}$$

Definition 6. *We divide the boundary region B into two regions as follows (See Figure 6-1):*

$$\begin{aligned} B_L &= \left\{z \in \mathbb{R}^{2n} \middle| -\theta_{0,\max} \le \vartheta_0 \le -(\theta'_{0,\max} + \varepsilon_0/2)\right\} \\ B_U &= \left\{z \in \mathbb{R}^{2n} \middle| -(\theta'_{0,\max} + \varepsilon_0/2) \le \vartheta_0 < -\theta'_{0,\max}\right\}. \end{aligned}$$

We note that $B = B_L \cup B_U$, and that A, B_L, B_U, and B' are all regions in $\mathbb{R}^{2n}$ that lie between two hyperplanes. All of these hyperplanes are specified using only one scalar state variable, ϑ_0.

Constants

Let positive constants δ, E_0 defined by

$$\delta \in (0\ 1] \tag{6.46}$$

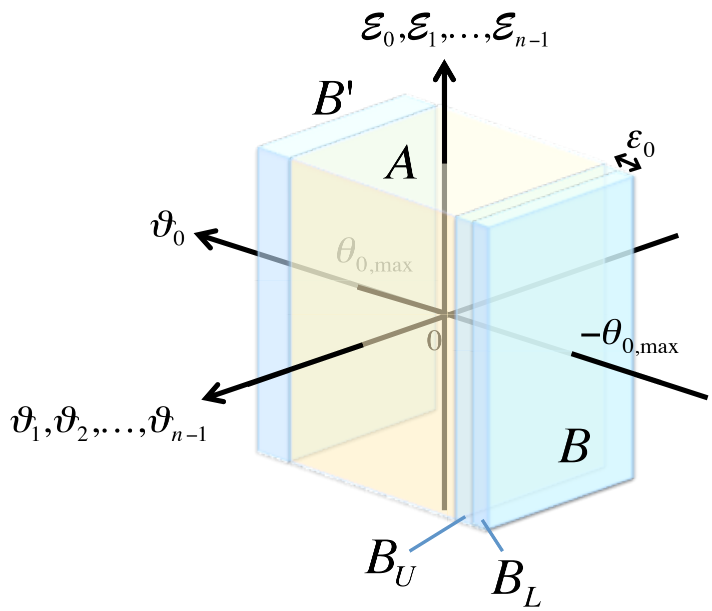


Figure 6-1: Definition of regions

and

$$E_0 = \max\left(\max_{t\in[t_0-\tau,t_0]} |\mathscr{E}_0(t)| + \overline{m}_0 + 2\delta,\ c_e,\ \beta\right) \tag{6.47}$$

where $\overline{m}_0 \equiv \max_{t\geq t_0} \left|c_0^T x_m(t)\right|$ and $c_e > 0$, $\beta > 0$ are specified later in Proposition 6, Lemma 10, respectively. From the definitions of E_0 and δ, it can be shown that $E_0 - 2\delta > \overline{m}_0$.

We also define a positive constant E' by

$$E' = \max\left(\sqrt{\frac{\overline{s}_{P'}}{\underline{s}_{P'}}} \max_{t\in[t_0-\tau,t_0]} \|\mathscr{E}(t)\|, \sqrt{\frac{l^2 r_p}{1-l^2 r_p}} E_0\right) \tag{6.48}$$

where positive constants l, r_p are such that

$$r_p > 1, \tag{6.49}$$

$$\sqrt{\frac{l^2 r_p}{1-l^2 r_p}} < 1. \tag{6.50}$$

From the definition of E', it is seen that

$$E' < E_0. \tag{6.51}$$

Using r_p, E_0 and E', we further define

$$E = \sqrt{r_p}\sqrt{E_0^2 + E'^2}. \tag{6.52}$$

Since $r_p > 1$, it is seen that

$$E > E_0. \tag{6.53}$$

Also from the definitions of E' and E, it can be proven that

$$lE \leq E'. \tag{6.54}$$

Condition

Condition 2. *$\pi(t) \in \mathbb{R}^n$ is said to satisfy Condition 2 at time t_a if*

$$|\pi_0(t)| \leq E \quad \forall t \in [t_a - \tau,\ t_a], \tag{6.55}$$

$$|\pi_0(t_a)| = E_0 - \delta, \tag{6.56}$$

$$\pi'^T(t_a - \tau)P'\pi'(t_a - \tau) \leq \underline{s}_{P'}E'^2 \tag{6.57}$$

where π_i is the ith component of π $i = 0, 1, \cdots, n-1$ and $\pi'(t) = \begin{bmatrix}\pi_1 & \pi_2 & \cdots & \pi_{n-1}\end{bmatrix}^T \in \mathbb{R}^{n-1}$. $t_a \geq t_0$ and $E_0 \in \mathbb{R}$, $\delta \in \mathbb{R}$, $E' \in \mathbb{R}$ are positive constants with $E_0 - \delta > 0$. P' is a positive definite matrix.

6.3 Proof of the Main Result

The closed-loop adaptive system is equivalent to the error model described by

$$\dot{e}(t) = A_m e(t) + b_m \Big\{ \big(\theta^T(t) - \theta^{\star T}\big)\big(e(t) + x_m(t)\big) + \eta(t) \Big\} \tag{6.58}$$

where the state-dependent disturbance η due to input delay is given by (6.4), and the adaptive law in (6.25) and (6.26) which can be rewritten as

$$\{M\dot{\theta}(t)\}_i = \mathrm{Proj}\Big(\{M\theta(t)\}_i, -\big\{M\Gamma(e(t)+x_m(t))b_m^T Pe(t)\big\}_i\Big). \tag{6.59}$$

We first note that since $\chi_\theta(t_0) \in \Omega_{1R}$, it follows from Lemma 8 that $\theta(t) \in \Omega_{1R}$ $\forall t \geq t_0$, or $|\vartheta_i(t)| \leq \theta_{i,\max}$ $\forall i$ $\forall t \geq t_0$. Theorem 11 is therefore proved if the global boundedness of $e(t)$ is demonstrated. In sections 6.3.1 through 6.3.3, the transformed error dynamics are discussed. An outline of the proof, with four phases, is provided in Section 6.3.4. The details of the four phases of the proof are provided in sections 6.3.5 through 6.3.8.

6.3.1 Transformed State Error Dynamics

In order to prove boundedness of $e(t)$, we will utilize the transformed error $\mathscr{E}(t)$ introduced in (6.15). The global boundedness of $e(t)$ is demonstrated if the global boundedness of $\mathscr{E}(t)$ is shown. In this section, we will derive the dynamics of $\mathscr{E}$. In what follows, $y \in \mathbb{R}^{n-1}$ is said to be a subvector of a $x \in \mathbb{R}^n$ if its jth element

$$y_j = x_{j+1}, \quad j = 1, \cdots, n-1.$$

Noting that $\mathscr{E}_i$ refers to the ith component of $\mathscr{E}$ and c_i^T is the ith row vector of C, it follows from (6.15) that for $i = 0, \cdots, n-1$

$$\dot{\mathscr{E}}_i(t) = c_i^T \dot{e}(t). \tag{6.60}$$

For $i = 1, \cdots, n-1$, using the property in (6.20) and the fact (6.23), it can be shown from (6.58) that

$$\begin{aligned}\dot{\mathscr{E}}_i(t) &= c_i^T A_m I e(t) \\ &= c_i^T A_m P^{-1} \left(\sum_{j=0}^{n-1} c_j c_j^T \right) e(t).\end{aligned} \tag{6.61}$$

Noting the definition of α_{ij} in (6.30), (6.61) can be rewritten as

$$\dot{\mathscr{E}}_i(t) = \sum_{j=1}^{n-1} \alpha_{ij} \mathscr{E}_j(t) + \alpha_{i0} \mathscr{E}_0(t). \tag{6.62}$$

The definition of $\mathscr{A}'_m$ in (6.31) and a_0 in (6.32) implies that the subvector $\mathscr{E}'$ of $\mathscr{E}$ given by

$$\mathscr{E}'(t) \equiv [\mathscr{E}_1 \ \mathscr{E}_2 \ \cdots \ \mathscr{E}_{n-1}] \tag{6.63}$$

satisfies the error dynamics

$$\dot{\mathscr{E}}'(t) = \mathscr{A}'_m \mathscr{E}'(t) + a_0 \mathscr{E}_0(t). \tag{6.64}$$

We now return to (6.60) and consider the special case when $i = 0$. Using the property in (6.18) and the definition of α_{ij} in (6.30), the dynamics of the critical state error $\mathscr{E}_0$ can be obtained from (6.58) as

$$\begin{aligned}\dot{\mathscr{E}}_0(t) &= c_0^T A_m e(t) + p_{bb}\big(\theta^T(t) - \theta^{\star T}\big)\big(e(t) + x_m(t)\big) + p_{bb}\eta(t) \\ &= \sum_{j=0}^{n-1} \alpha_{0j} \mathscr{E}_j(t) + p_{bb}\big(\theta^T(t) - \theta^{\star T}\big)\big(e(t) + x_m(t)\big) + p_{bb}\eta(t).\end{aligned} \tag{6.65}$$

Defining

$$m_i(t) \equiv c_i^T x_m(t) \tag{6.66}$$

and noting (6.23) and (6.16), the error equation (6.65) can be rewritten as

$$
\begin{aligned}
\dot{\mathscr{E}}_0(t) &= \sum_{j=0}^{n-1} \alpha_{0j}\mathscr{E}_j(t) + p_{bb}\big(\theta^T(t) - \theta^{\star T}\big)P^{-1}\Big(\sum_{j=0}^{n-1} c_j c_j^T\Big)\big(e(t) + x_m(t)\big) + p_{bb}\eta(t) \\
&= \sum_{j=0}^{n-1} \alpha_{0j}\mathscr{E}_j(t) + \sum_{j=0}^{n-1} \big(\vartheta_j(t) - \vartheta_j^\star\big)\big(\mathscr{E}_j(t) + m_j(t)\big) + p_{bb}\eta(t) \\
&= \big(\alpha_{00} + \vartheta_0(t) - \vartheta_0^\star\big)\mathscr{E}_0(t) + \big(\vartheta_0(t) - \vartheta_0^\star\big)m_0(t) + p_{bb}\eta(t) \\
&\qquad + \sum_{j=1}^{n-1} \Big\{\big(\alpha_{0j} + \vartheta_j(t) - \vartheta_j^\star\big)\mathscr{E}_j(t) + \big(\vartheta_j(t) - \vartheta_j^\star\big)m_j(t)\Big\} \\
&= \big(\alpha_{00} + \vartheta_0(t) - \vartheta_0^\star\big)\mathscr{E}_0(t) + \big(\vartheta_0(t) - \vartheta_0^\star\big)m_0(t) + p_{bb}\eta(t) \\
&\qquad + \big(a_1 + \vartheta'(t) - \vartheta'^\star\big)\mathscr{E}'(t) + \big(\vartheta'(t) - \vartheta'^\star\big)m'(t)
\end{aligned}
\tag{6.67}
$$

where $\vartheta^\star = M\theta^\star$ and $\vartheta'^\star$ is its subvector. $m(t) = [m_0\ m_1\ \cdots\ m_{n-1}]$ and $m'(t)$ is its subvector. Since $x_m(t)$ is known to be bounded, boundedness of $m_i(t)$ is straightforward from (6.66) and we define $\overline{m} \equiv \max_{t\geq t_0}\|m(t)\|$. We also note that the definition of $\overline{m}_0$ in Section 6.2.7 can be rewritten as $\overline{m}_0 \equiv \max_{t\geq t_0}|m_0(t)|$.

Equations (6.64) and (6.67) represent the transformed tracking error dynamics $\mathscr{E}$. These equations show that the perturbation η due to the time delay τ appears directly only in the dynamics of $\mathscr{E}_0$ and not in $\mathscr{E}_i$, $i = 1, \cdots, n-1$.

In what follows, we will relate the boundedness of $\mathscr{E}'$ to that of $\mathscr{E}_0$ using Lemma 7.

Proposition 5. *Suppose*

$$
|\mathscr{E}_0(t)| \leq W \quad t \in \ \mathscr{T}_s = [t_s, t_{ss}] \tag{6.68}
$$

where $t_{ss} > t_s \geq t_0$. Then

$$
V'(t) \leq \max\left(V'(t_s),\ \frac{1}{2}s_{P'}\,(lW)^2\right) \quad \forall t \in \mathscr{T}_s, \tag{6.69}
$$

where $V'(.)$ is defined as

$$
V'(t) = \frac{1}{2}\mathscr{E}'^T(t)P'\mathscr{E}'(t), \tag{6.70}
$$

$P' = P'^T > 0$ is the solution of $\mathscr{A}_m'^T P' + P'\mathscr{A}_m' = -Q'$ for some $Q' = Q'^T > 0$, and a positive constant l is defined as

$$l = \frac{2\bar{s}_{P'}^2 \|a_0\|}{\underline{s}_{P'}\underline{s}_{Q'}}. \tag{6.71}$$

Proof. Since $\mathscr{A}_m'$ is Hurwitz, for any positive definite symmetric matrix Q' there exists a positive definite symmetric matrix P' which satisfies the Lyapunov equation

$$\mathscr{A}_m'^T P' + P'\mathscr{A}_m' = -Q'. \tag{6.72}$$

Considering a Lyapunov-like function (6.70), we obtain its time derivative as

$$\begin{aligned} \dot{V}' &= -\frac{1}{2}\mathscr{E}'^T Q'\mathscr{E}' + \mathscr{E}'^T P' a_0 \mathscr{E}_0 \\ &\leq -\frac{1}{2}\min_i \left(\Re\left(\lambda_i(Q')\right)\right) \|\mathscr{E}'\|^2 + \|P'a_0\| W \|\mathscr{E}'\|. \end{aligned} \tag{6.73}$$

Noting that

$$\frac{1}{2}\underline{s}_{P'}\|\mathscr{E}'(t)\|^2 \leq V'(t) \leq \frac{1}{2}\bar{s}_{P'}\|\mathscr{E}'(t)\|^2, \tag{6.74}$$

(6.73) can be simplified as

$$\dot{V}' \leq -k_1 V' + k_2\sqrt{V'} \tag{6.75}$$

where

$$k_1 = \frac{\underline{s}_{Q'}}{\bar{s}_{P'}}, \quad k_2 = \frac{\sqrt{2}\bar{s}_{P'}\|a_0\| W}{\sqrt{\underline{s}_{P'}}}. \tag{6.76}$$

For positive constants Δ_1, Δ_2 such that $\Delta_1 < k_1$ and $4\Delta_1\Delta_2 \geq k_2^2$, it can be shown that for any V',

$$\Delta_1 V' + \Delta_2 \geq k_2\sqrt{V'} \tag{6.77}$$

through a straightforward completion of squares. Inequalities (6.75) and (6.77) imply that

$$\dot{V}' \leq -(k_1 - \Delta_1)V' + \Delta_2.$$

Defining $\Delta_1 = k_1/2$ and $\Delta_2 = k_2^2/(4\Delta_1) = k_2^2/(2k_1)$, we therefore obtain

$$\dot{V}' \leq -\frac{k_1}{2}V' + \frac{k_2^2}{2k_1}. \tag{6.78}$$

(6.78) implies that

$$\dot{V}'(t) \leq 0 \quad \text{if} \quad V'(t) \geq K_1 \tag{6.79}$$

where

$$K_1 = \left(\frac{k_2}{k_1}\right)^2 = \frac{1}{2}\underline{s}_{P'}(lW)^2. \tag{6.80}$$

This proves Proposition 5. □

Corollary 1. *Suppose* (6.68) *is satisfied where $t_{ss} > t_s \geq t_0$. Then*

$$\underline{s}_{P'}\|\mathscr{E}'(t)\|^2 \leq \max\left(\mathscr{E}'^T(t_s)P'\mathscr{E}'(t_s),\ \underline{s}_{P'}(lW)^2\right) \quad \forall t \in \mathscr{T}_s. \tag{6.81}$$

Proof. From Proposition 5 and noting (6.74), (6.81) follows. □

6.3.2 Transformed Parameter Dynamics

Similar to Section 6.3.1, we now focus on the transformed parameter $\vartheta(t)$ in (6.16). From (6.59) and noting that $\{M\theta(t)\}_i = \vartheta_i$ and (6.24), we obtain, for $i = 0, \cdots, n-1$,

$$\begin{aligned}\dot{\vartheta}_i(t) &= \text{Proj}\Big(\vartheta_i, -\gamma p_{bb} c_i^T (e(t) + x_m(t)) b_m^T P e(t)\Big) \\ &= \gamma p_{bb} \text{Proj}\Big(\vartheta_i, -(\mathscr{E}_i(t) + m_i(t)) b_m^T P e(t)\Big).\end{aligned}$$

We also note that $b_m^T Pe(t) = p_{bb} c_0^T e(t) = p_{bb} \mathscr{E}_0(t)$ from (6.17) and (6.15). Therefore we obtain

$$\dot{\vartheta}_i(t) = \gamma' \text{Proj}\Big(\vartheta_i, -(\mathscr{E}_i(t) + m_i(t))\mathscr{E}_0(t)\Big), \quad i = 0, \cdots, n-1 \tag{6.82}$$

where $\gamma' = \gamma p_{bb}^2$. As will be seen later, $\mathscr{E}_0$ is the main component of interest. We therefore examine (6.82) for $i = 0$ in more detail. Returning to the definition of $\text{Proj}(\cdot,\cdot)$ in (6.27), it follows that

$$\begin{aligned} \dot{\vartheta}_0(t) &= -\gamma' \big(\mathscr{E}_0(t) + m_0(t)\big)\mathscr{E}_0(t) \\ &\quad \text{if} \quad [z \in A] \vee \big[(z \in (B \cup B')) \wedge (\mathscr{E}_0(t) + m_0(t))\mathscr{E}_0(t)\vartheta_0 \geq 0\big] \end{aligned} \tag{6.83}$$

and

$$\begin{aligned} \dot{\vartheta}_0(t) = -&\left(\frac{\theta_{0,\max}^2 - \vartheta_0^2}{\theta_{0,\max}^2 - \theta_{0,\max}'^2}\right)\gamma' \big(\mathscr{E}_0(t) + m_0(t)\big)\mathscr{E}_0(t) \\ &\qquad \text{if} \quad \big[(z \in (B \cup B')) \wedge (\mathscr{E}_0(t) + m_0(t))\mathscr{E}_0(t)\vartheta_0 < 0\big]. \end{aligned} \tag{6.84}$$

It is seen that $\dot{\vartheta}_0 < 0$ when $\mathscr{E}_0 < -\overline{m}_0$ or $\overline{m}_0 < \mathscr{E}_0$.

Equations (6.82) for $i = 1, \cdots, n-1$ and (6.83), (6.84) constitute the complete adaptive law.

6.3.3 Complete Transformed State Error and Parameter Dynamics

The two states in the adaptive system are the state error $\mathscr{E}$ and the parameter ϑ. The former is given by (6.65) and (6.64), and the latter by (6.82) for $i = 1, \cdots, n-1$ and (6.83), (6.84). Of the $2n$ states $\mathscr{E}$ and ϑ, two scalar variables $\mathscr{E}_0(t)$ and $\vartheta_0(t)$ will be seen to be more crucial. We note that while η explicitly appears in the dynamics of $\mathscr{E}_0$, it does not in the dynamics of $\mathscr{E}_i$, $i \geq 1$. Among the parameter states, only ϑ_0 is affected by a nonlinear function of $\mathscr{E}_0$ whereas ϑ_i, $i \geq 1$ includes only linear function of $\mathscr{E}_0$. The effects of such features in $\mathscr{E}$ and ϑ, $\mathscr{E}_0$ and ϑ_0 in particular, will become clear in the following sections.

6.3.4 Outline of the Proof

The proof is completed using the following four phases.

Phase I: The transformed error $\mathscr{E}(t)$ satisfies Condition 2 for some $t = t_a$; this implies that the state $z(t)$ has to enter B at $t_b \in (t_a, t_a + \Delta T_{in,\max})$, where $\Delta T_{in,\max} > 0$ is a finite constant (see Figure 6-2(a)).

Phase II: While the trajectory stays in B, the parameter $\vartheta_0(t)$ stays in the boundary of the projection algorithm; $\mathscr{E}(t)$ is shown to be bounded by making use of the underlying linear time-varying system (see Figure 6-2(b)).

Phase III: There exists $\Delta T_{out,\min}$ such that the trajectory reenters A at $t_c > t_b + \Delta T_{out,\min}$ with $|\mathscr{E}_0(t_c)| < \overline{m}_0$ (see Figure 6-2(c)).

Phase IV: The trajectory has only two alternatives: (IV-A): $|\mathscr{E}_0(t)| < E_0 - \delta \ \forall t > t_c$ which proves Theorem 1; (IV-B): $\mathscr{E}_0(t)$ satisfies Condition 2 (6.56) for some $t_d > t_c$. If the latter, we replace t_a by t_d and repeat Phases I to IV.

In the following subsections, we prove Phases I-IV in detail. Lemmas and propositions are introduced as needed in order to prove these phases.

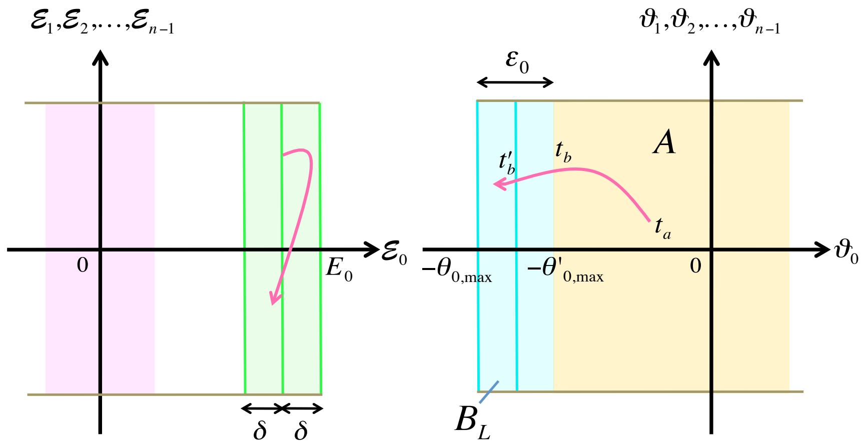
6.3.5 Phase I: Entering the Boundary

The goal of this section is to prove the following proposition.

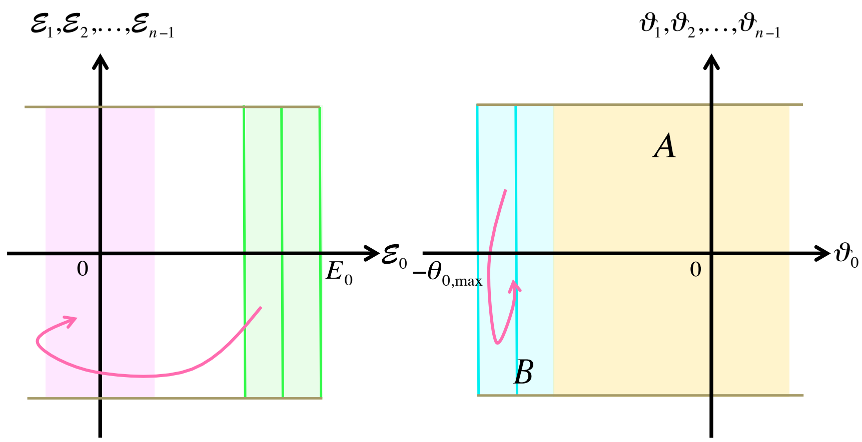
Proposition 6. *Let $\mathscr{E}(t)$ satisfy Condition 2 at $t = t_a$ with δ, E_0, E' given in* (6.46)*,* (6.47)*,* (6.48) *respectively and $z(t_a) \in A$ where $z = \begin{bmatrix} \mathscr{E}^T & \vartheta^T \end{bmatrix}^T$. Then*

$$(i) \quad |\mathscr{E}_0(t)| < E_0 \quad \forall t \in [t_a,\ t_a + \Delta T] \tag{6.85}$$

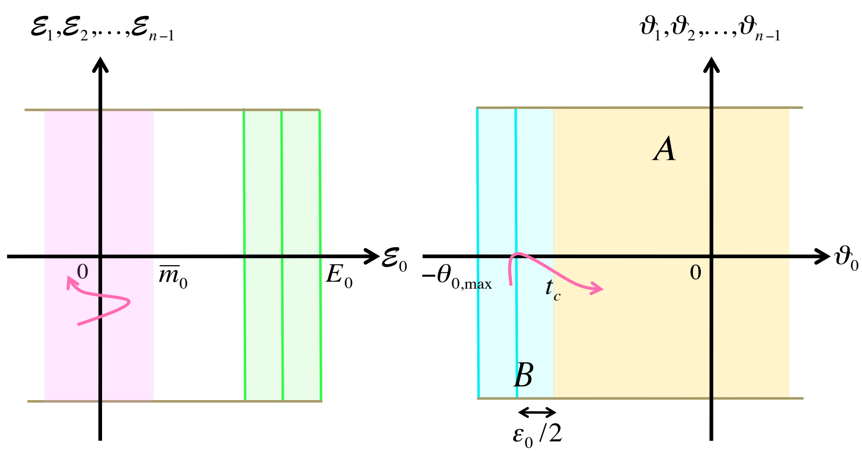
$$(ii) \quad \exists t_b' \in (t_a,\ t_a + \Delta T] \ \ s.t. \ \ z(t_b') \in B_L, \tag{6.86}$$



(a) Phase I: Entering the boundary



(b) Phase II: In the boundary region B



(c) Phase III: Exiting from the boundary

Figure 6-2: Phases I-III of a trajectory *where*

$$\Delta T = \frac{\delta}{b_0 E + b_1}, \tag{6.87}$$

$$b_0 = B + B', \quad b_1 = \left(\phi_{\max} + 2\frac{\bar{s}_c}{\underline{s}_c}\Theta_{\max}\right)\overline{m} + 2p_{bb}\bar{r},$$

$$B = |a_{00}| + |\vartheta_0^\star| + \left(1 + 2\frac{\bar{s}_c}{\underline{s}_c}\right)\Theta_{\max}, \quad B' = \|a_1\| + \|\vartheta'^\star\| + \left(1 + 2\frac{\bar{s}_c}{\underline{s}_c}\right)\Theta_{\max}.$$

Proof of Proposition 6 (i):

We note from (6.67) that

$$\begin{aligned} |\dot{\mathscr{E}}_0(t)| \le{}& |a_{00} + \vartheta_0(t) - \vartheta_0^\star||\mathscr{E}_0(t)| + |\vartheta_0(t) - \vartheta_0^\star||m_0(t)| + p_{bb}|\eta(t)| \\ &+ \|a_1 + \vartheta'(t) - \vartheta'^\star\|\|\mathscr{E}'(t)\| + \|\vartheta'(t) - \vartheta'^\star\|\|m'(t)\|. \end{aligned} \tag{6.88}$$

From (6.4) and (6.2) it can be also seen that

$$|\eta(t)| \le \frac{2}{p_{bb}}\frac{\bar{s}_c}{\underline{s}_c}\Theta_{\max}\left(\max_{[t-\tau,\ t]}\|\mathscr{E}(t)\| + \overline{m}\right) + 2\bar{r}. \tag{6.89}$$

From (6.88) together with (6.89), it follows after elaborate algebraic manipulations that

$$|\dot{\mathscr{E}}_0(t)| \le B\hat{\mathscr{E}}_0 + B'\hat{\mathscr{E}}' + b_1 \qquad \forall t \in [t_a,\ t_a + \Delta T] \tag{6.90}$$

where

$$\hat{\mathscr{E}}_0 = \max_{t\in[t_a-\tau, t_a+\Delta T]}|\mathscr{E}_0(t)|, \quad \hat{\mathscr{E}}' = \max_{t\in[t_a-\tau, t_a+\Delta T]}\|\mathscr{E}'(t)\|. \tag{6.91}$$

By applying Proposition 5, with $t_a - \tau$ replacing t_s, $t_a + \Delta T$ replacing t_{ss}, and $\hat{\mathscr{E}}_0$ replacing W, we obtain that

$$\mathscr{E}'^T(t)P'\mathscr{E}'(t) \le \max\left(\mathscr{E}'^T(t_a - \tau)P'\mathscr{E}'(t_a - \tau),\ \underline{s}_{P'}(l\hat{\mathscr{E}}_0)^2\right) \quad \forall t \in [t_a - \tau, t_a + \Delta T].$$

Since $\mathscr{E}(t)$ satisfies Condition 2 (6.57) at $t = t_a$, the right hand side can be simplified to obtain

$$\mathscr{E}'^T(t)P'\mathscr{E}'(t) \le \max\left(\underline{s}_{P'}E'^2,\ \underline{s}_{P'}(l\hat{\mathscr{E}}_0)^2\right) \quad \forall t \in [t_a-\tau, t_a+\Delta T].$$

Noting the definition of $\hat{\mathscr{E}}'$ in (6.91), we therefore obtain

$$\hat{\mathscr{E}}' \le \sqrt{\frac{1}{\underline{s}_{P'}}\max\left(\underline{s}_{P'}E'^2,\ \underline{s}_{P'}(l\hat{\mathscr{E}}_0)^2\right)}.$$

Since $l < 1$ and $E' < E_0$ (6.51), it follows that

$$\hat{\mathscr{E}}' \le \max\left(E_0, \hat{\mathscr{E}}_0\right). \tag{6.92}$$

From (6.92), it can be seen that there are two possible cases, (A) $E_0 \le \hat{\mathscr{E}}_0$ and (B) $E_0 > \hat{\mathscr{E}}_0$.

Case (A) $E_0 \le \hat{\mathscr{E}}_0$

Condition of Case (A) and (6.92) imply that $\hat{\mathscr{E}}' \le \hat{\mathscr{E}}_0$. This allows us to simplify (6.90) as

$$|\dot{\mathscr{E}}_0(t)| \le b_0\hat{\mathscr{E}}_0 + b_1 \qquad \forall t \in [t_a, t_a+\Delta T] \tag{6.93}$$

where $b_0 \equiv B + B'$. Note that $\forall \Delta t \in [0, \Delta T]$

$$|\mathscr{E}_0(t_a+\Delta t)| \le |\mathscr{E}_0(t_a)| + \max_{t\in[t_a,t_a+\Delta T]} |\dot{\mathscr{E}}_0(t)|\Delta T. \tag{6.94}$$

From (6.93), the definition of ΔT in (6.87), and (6.56) in Condition 2 which is satisfied for $t = t_a$, it follows that

$$|\mathscr{E}_0(t_a+\Delta t)| \le (E_0-\delta) + \delta\left(1 + \frac{b_0(\hat{\mathscr{E}}_0 - E)}{b_0E + b_1}\right) \quad \forall \Delta t \in [0, \Delta T]. \tag{6.95}$$

Therefore

$$\max_{t\in[t_a,t_a+\Delta T]} |\mathscr{E}_0(t)| \le E_0 + b_0\Delta T\left(\hat{\mathscr{E}}_0 - E\right). \tag{6.96}$$

Noting the definition of $\hat{\mathscr{E}}_0$ in (6.91) and since $\mathscr{E}_0(t)$ satisfies (6.55),

$$\hat{\mathscr{E}}_0 = \max\left\{E, \max_{t\in[t_a,t_a+\Delta T]} |\mathscr{E}_0(t)|\right\}$$

and therefore there are only two possible cases, (A-a) $\hat{\mathscr{E}}_0 = E$ and (A-b) $\hat{\mathscr{E}}_0 > E$. If case (A-a) holds, it immediately implies from (6.96) that Proposition 6 (i) is true. If we suppose case (A-b) holds, it implies $\hat{\mathscr{E}}_0 = \max_{t\in[t_a,t_a+\Delta T]} |\mathscr{E}_0(t)|$ and from (6.96) it follows that

$$(1-b_0\Delta T)\hat{\mathscr{E}}_0 \le E_0 - b_0\Delta T E.$$

Noting $E > E_0$ and $1 - b_0\Delta T > 0$, we can therefore obtain

$$\hat{\mathscr{E}}_0 < \frac{E_0 - b_0\Delta T E_0}{1-b_0\Delta T} = E_0 < E.$$

This contradicts with the condition of the case and therefore we obtain $\hat{\mathscr{E}}_0 = E$.

Case (B) $E_0 > \hat{\mathscr{E}}_0$

Condition of Case (B) and (6.92) imply that $\hat{\mathscr{E}}' \le E_0$. This together with Condition of Case (B) allows us to simplify (6.90) as

$$|\dot{\mathscr{E}}_0(t)| \le b_0E_0 + b_1 \qquad \forall t \in [t_a, t_a+\Delta T]. \tag{6.97}$$

Noting that (6.94) $\forall \Delta t \in [0, \Delta T]$, we therefore obtain using (6.87) and (6.56) that

$$\begin{aligned} |\mathscr{E}_0(t_a+\Delta t)| &\le (E_0-\delta) + \delta\frac{b_0E_0+b_1}{b_0E+b_1} \\ &< (E_0-\delta) + \delta\frac{b_0E+b_1}{b_0E+b_1} \\ &= E_0, \end{aligned} \tag{6.98}$$

which again implies that Proposition 6 (i) is true.

Proof of Proposition 6 (ii).

Equation (6.90) together with (6.92) gives that

$$|\dot{\mathscr{E}}_0(t)| \leq b_0 \max\left(E_0, \hat{\mathscr{E}}_0\right) + b_1 \qquad \forall t \in [t_a, t_a + \Delta T].$$

Thus, since $E \geq \max\left(E_0, \hat{\mathscr{E}}_0\right)$ from the proof of Proposition 6 (i),

$$|\mathscr{E}_0(t)| \geq |\mathscr{E}_0(t_a)| - (b_0 E + b_1)\Delta T \qquad \forall\, t \in [t_a, t_a + \Delta T]$$

which can be simplified, using the fact that $\mathscr{E}_0(t)$ satisfies (6.56), as

$$|\mathscr{E}_0(t)| \geq E_0 - 2\delta \qquad \forall\, t \in [t_a, t_a + \Delta T]. \tag{6.99}$$

From the choices of δ and E_0 in (6.46) and (6.47), it can be shown that $E_0 - 2\delta \geq \overline{m}_0$ and therefore

$$|\mathscr{E}_0(t)| \geq \overline{m}_0 \qquad \forall\, t \in [t_a, t_a + \Delta T].$$

From (6.83), this in turn implies that $\dot{\vartheta}_0(t)$ is non-positive and

$$\begin{aligned} -\dot{\vartheta}_0(t) &\geq \gamma' |\mathscr{E}_0(t)| \left(|\mathscr{E}_0(t)| - |m_0(t)|\right) \\ &\geq \gamma' (E_0 - 2\delta)((E_0 - 2\delta) - \overline{m}_0) \qquad \forall\, t \in T_A \end{aligned} \tag{6.100}$$

where T_A is defined as

$$T_A : \left\{ t | z(t) \in A \ \text{ and } \ t \in [t_a, t_a + \Delta T] \right\}.$$

From (6.100), it follows that

$$\vartheta_0(t_a) - \vartheta_0(t_a + \Delta t) \geq \gamma' (E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)\Delta t \tag{6.101}$$

for all $\Delta t \in [0, \Delta T]$ which satisfy $[t_a, t_a + \Delta t] \subset T_A$. Hence defining

$$\Delta T_{in,\max} = \frac{2\theta_{0,\max}}{\gamma'(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)} \tag{6.102}$$

and if $\Delta T_{in,\max} \leq \Delta T$, from (6.101), (6.39) and definition of regions A and B, it follows that $z(t)$ enters B at $t_b \in (t_a \; t_a + \Delta T_{in,\max})$.

We now show that $z(t)$ enters B_L at $t'_b \in (t_a \; t_a + \Delta T'_{in,\max})$ for some $\Delta T'_{in,\max} > \Delta T_{in,\max}$. First, it can be proven that

$$|\text{Proj}(\theta, y)| > \frac{1}{2}|y| \qquad \forall z \in B_U. \tag{6.103}$$

Using similar arguments as above, then it can be shown that

$$-\dot{\vartheta}_0(t) > \frac{\gamma'}{2}(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0) \qquad \forall t \in T_{BU} \tag{6.104}$$

where T_{BU} is defined as

$$T_{BU} : \Big\{ t | z(t) \in B_U \;\text{ and }\; t \in [t_a, t_a + \Delta T] \Big\}.$$

Noting Definition 6, the maximum time that $z(t)$ can spend in B_U can be derived, using (6.104), to be $\{\varepsilon_0/2\}/\{\frac{\gamma'}{2}(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)\}$. This implies that $z(t)$ enters region B_L at $t'_b \in (t_a, t_a + \Delta T'_{in,\max})$ where

$$\begin{aligned} \Delta T'_{in,\max} &= \Delta T_{in,\max} + \frac{\varepsilon_0/2}{\gamma'(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)/2} \\ &= \frac{2\theta_{0,\max} + \varepsilon_0}{\gamma'(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)} \end{aligned} \tag{6.105}$$

if $\Delta T'_{in,\max} \leq \Delta T$, since then the inequality in (6.104) is satisfied for all $t \in (t_b, t'_b]$.

From (6.47)

$$E_0 \geq c_e. \tag{6.106}$$

Noting that $E < \frac{E_0}{l}$ from (6.54) and (6.51), if we let the positive constant c_e defined by

$$\begin{aligned} c_e &= \frac{-l_2 + \sqrt{l_2^2 - 4 l_1 l_3}}{2 l_1}, \\ l_1 &= \delta \gamma' \\ l_2 &= -\delta \gamma' \left(4\delta + \overline{m}_0\right) - \left(2\theta_{0,\max} + \varepsilon_0\right) \frac{b_0}{l} \\ l_3 &= 2\delta^2 \gamma' \left(2\delta + \overline{m}_0\right) - \left(2\theta_{0,\max} + \varepsilon_0\right) b_1, \end{aligned} \tag{6.107}$$

then (6.106) implies $\Delta T'_{in,\max} < \Delta T$ from (6.87) and (6.105). This proves Proposition 6 (ii).

Remark 8. *As in Chapter 5, an upper-bound of c_e can be derived. It can be shown using algebraic manipulations that*

$$c_e < \frac{16}{\delta \gamma'} (\theta_{0,\max}^2 + \gamma')(1 + \overline{m}_0).$$

6.3.6 Phase II: In the Boundary Region B

We return to the overall adaptive system, which can be written using (6.1), (6.2), and (6.6) as

$$\dot{x_p}(t) = \left\{A_m - b_m \theta^{\star T}\right\} x_p(t) + b_m \left\{\theta^T(t-\tau) x_p(t-\tau) + r(t-\tau)\right\} \tag{6.108}$$

which leads to the error dynamics

$$\dot{e}(t) = A_m e(t) - b_m \theta^{\star T} x_p(t) + b_m \theta^T(t-\tau) x_p(t-\tau) + b_m \big(r(t-\tau) - r(t)\big). \tag{6.109}$$

Noting that $\mathscr{E} = Ce$, we then obtain

$$\begin{aligned} \dot{\mathscr{E}}(t) &= C A_m (P^{-1} C^T C) e(t) - C b_m \theta^{\star T} (P^{-1} C^T C) \big(e(t) + x_m(t)\big) \\ &\quad + C b_m \theta^T(t-\tau) (P^{-1} C^T C) \big(e(t-\tau) + x_m(t-\tau)\big) + C b_m \big(r(t-\tau) - r(t)\big) \\ &= M_0 \mathscr{E}(t) + M_1 \mathscr{E}(t-\tau) + R(t) \end{aligned} \tag{6.110}$$

where the matrices M_0, M_1 and the vector R are defined as

$$\begin{aligned} M_0 &\equiv \mathscr{A}_m - c_I \vartheta^{\star T} \\ M_1 &\equiv c_I \vartheta(t-\tau) \\ R(t) &\equiv -p_{bb} c_I \theta^{\star T} x_m(t) + p_{bb} c_I \theta^T(t-\tau) x_m(t-\tau) + p_{bb} c_I \big(r(t-\tau) - r(t)\big) \\ c_I &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T. \end{aligned} \tag{6.111}$$

Let the trajectory stay in B for $t \in (t_b, t_c)$ for some $t_c > t_b$. From the definition of B, it follows that

$$\vartheta_0(t) = -\theta_{0,\max} + \varepsilon(t) \quad \text{for} \quad t \in (t_b, t_c) \tag{6.112}$$

where

$$\varepsilon(t) \in [0 \;\; \varepsilon_0).$$

We show below that $\mathscr{E}(t)$ is guaranteed to converge to a bounded set if the trajectory remains in B. Before we proceed to this result, we study the properties of $M_0 + M_1$ while in B. Let us define a set as follows:

$$\Omega_B = \{(M_0, M_1) | z \in B\}.$$

Lemma 9. *There exists* $\underline{q} > 0$ *such that*

$$(M_0 + M_1)^T \mathscr{P} + \mathscr{P}(M_0 + M_1) < -\underline{q} I \tag{6.113}$$

is satisfied for all $(M_0, M_1) \in \Omega_B$*, where* $\mathscr{P}$ *is a constant matrix defined as*

$$\begin{aligned} \mathscr{P} &= \mathring{I}^T \mathring{\mathscr{P}} \mathring{I}, \\ \mathring{\mathscr{P}} &= \begin{bmatrix} P' & 0 \\ 0 & p_\varphi \end{bmatrix}, \quad \mathring{I} = \begin{bmatrix} 0_{1\times(n-1)} & 1 \\ I_{(n-1)\times(n-1)} & 0_{(n-1)\times 1} \end{bmatrix}. \end{aligned} \tag{6.114}$$

Proof. From (6.111), it is seen that

$$M_0 + M_1 = \mathscr{A}_m + \begin{bmatrix} \vartheta^T(t-\tau) - \vartheta^{\star T} \\ 0 \end{bmatrix}. \tag{6.115}$$

From (6.114), (6.115) and (6.32), we obtain that

$$S(t) \equiv \mathring{I}(M_0 + M_1)\mathring{I}^T = \begin{bmatrix} \mathscr{A}'_m & a_0 \\ a_1^T + \varphi'^T(t-\tau) & \alpha_{00} + \varphi_0(t-\tau) \end{bmatrix} \tag{6.116}$$

where $\varphi_0 \in \Re$, $\varphi' \in \Re^{n-1}$ are given by

$$\begin{bmatrix} \varphi_0(t) & \varphi'^T(t) \end{bmatrix}^T = \vartheta(t) - \vartheta^\star. \tag{6.117}$$

Defining a symmetric matrix function $\mathring{\mathscr{Q}}(.)$ as

$$\begin{aligned} \mathring{\mathscr{Q}}(S) &= -\left(\mathring{\mathscr{P}}S + S^T\mathring{\mathscr{P}}\right) \\ &= \begin{bmatrix} Q' & -q_d(\varphi') \\ -q_d^T(\varphi') & -2p_\varphi(\alpha_{00} + \varphi_0(t-\tau)) \end{bmatrix} \end{aligned} \tag{6.118}$$

where $q_d(\varphi') \equiv P'a_0 + (a_1 + \varphi'(t-\tau))p_\varphi$, we can show that $\mathring{\mathscr{Q}}(S)$ is positive definite for all $S(t)$ if $z(t) \in B$ as follows.

From (6.72), we have that $Q' > 0$. Therefore all k leading principal minors of $\mathring{\mathscr{Q}}$ are positive for $k = 1, 2, \cdots, n-1$. Also, noting from (6.118) that

$$\det\{\mathring{\mathscr{Q}}\} = \det\{Q'\}\Big(-2p_\varphi(\alpha_{00} + \varphi_0(t-\tau)) - q_d^T(\varphi')Q'^{-1}q_d(\varphi')\Big) \tag{6.119}$$

and the design of the projection algorithm (6.41) which implies

$$\varphi_0(t-\tau) < -\alpha_{00} - \frac{1}{2p_\varphi}q_d^T(\varphi')Q'^{-1}q_d(\varphi') \quad \text{if} \quad z \in B,$$

we obtain

$$\det\left\{\mathring{\mathscr{Q}}\right\} > 0 \quad \text{if} \quad z \in B.$$

Since all the leading principal minors of $\mathring{\mathscr{Q}}$ are positive, we obtain that $\mathring{\mathscr{Q}}$ is positive definite while $z \in B$.

Noting the definition of S in (6.116) and from the fact that $\mathring{I}^T = (\mathring{I})^{-1}$, we obtain

$$\begin{aligned} -\mathring{I}^T \mathring{\mathscr{Q}} \mathring{I} &= \mathring{I}^T \mathring{\mathscr{P}} S \mathring{I} + \mathring{I}^T S^T \mathring{\mathscr{P}} \mathring{I} \\ &= \mathring{I}^T \mathring{\mathscr{P}} \left(\mathring{I}(M_0+M_1)\mathring{I}^T\right)\mathring{I} + \mathring{I}^T \left(\mathring{I}(M_0+M_1)^T \mathring{I}^T\right) \mathring{\mathscr{P}} \mathring{I} \\ &= (\mathring{I}^T \mathring{\mathscr{P}} \mathring{I})(M_0+M_1) + (M_0+M_1)^T (\mathring{I}^T \mathring{\mathscr{P}} \mathring{I}). \end{aligned} \tag{6.120}$$

Equation (6.120) serves as a Lyapunov equation for M_0+M_1, since it can be rewritten into the form of

$$-\mathscr{Q} = \mathscr{P}(M_0+M_1) + (M_0+M_1)^T \mathscr{P} \tag{6.121}$$

with $\mathscr{P} \equiv \mathring{I}^T \mathring{\mathscr{P}} \mathring{I}$ and $\mathscr{Q} \equiv \mathring{I}^T \mathring{\mathscr{Q}} \mathring{I}$. From the definition, $\mathscr{P}$ is symmetric and positive definite since $\mathring{\mathscr{P}}$ is a symmetric positive definite matrix. In the same manner, it can be seen that $\mathscr{Q}$ is symmetric and positive definite for all $(M_0, M_1) \in \Omega_B$ since the symmetric matrix function $\mathring{\mathscr{Q}}$ is positive definite while $z \in B$. This proves Lemma 9. □

Lemma 10. *Consider the uncertain time-varying system* (6.110) *with the selection of the projection parameters which satisfies* (6.41). *Let the solutions of the system lie in B for $t \in (t_b, t_c)$. Then there exist $\bar{\tau}$ and $\beta > 0$ such that for any $\tau \leq \bar{\tau}$,*

$$V(\mathscr{E}(t)) \leq \max\left\{V(\mathscr{E}(t_b)),\ \bar{s}_{\mathscr{P}}\beta^2\right\} \quad \forall t \in (t_b, t_c) \tag{6.122}$$

where

$$V(\mathscr{E}) = \mathscr{E}^T \mathscr{P} \mathscr{E}. \tag{6.123}$$

Proof. Lemma 10 is a vector version of Proposition 3 in Chapter 5 and its proof is built upon Proposition 6.7 in [23] utilizing Lemma 9, model transformation, and Razumikhin Theorem.

Using

$$\mathscr{E}(t-\tau) = \mathscr{E}(t) - \int_{-\tau}^{0} \dot{\mathscr{E}}(t+\zeta)d\zeta, \tag{6.124}$$

with $\dot{\mathscr{E}}(t+\zeta)$ replaced by the right hand side of the system equation (6.110) with appropriate time shift, we obtain the following transformed system:

$$\begin{aligned}
\dot{\mathscr{E}}(t) &= (M_0 + M_1(t))\mathscr{E}(t) + R(t) \\
&\quad - M_1(t)\int_{-\tau}^{0} \big(M_0\mathscr{E}(t+\zeta) + M_1(t+\zeta)\mathscr{E}(t+\zeta-\tau) + R(t+\zeta)\big)d\zeta \\
&= \bar{M}_0\mathscr{E}(t) + \int_{-2\tau}^{0} \bar{M}(t,\zeta)\mathscr{E}(t+\zeta)d\zeta + \bar{R}(t), \\
&\Big(\bar{M}_0(t), \bar{M}(t,\cdot)\Big) \in \bar{\Omega},
\end{aligned} \tag{6.125}$$

where

$$\begin{aligned}
\bar{\Omega} &= \left\{ \Big(\bar{M}_0, \bar{M}(\cdot)\Big) \left| \begin{array}{c} \bar{M}_0 = M_0 + M_1 \\ \bar{M}(\zeta) = -M_1 M_{0\zeta},\ \bar{M}(-\tau+\zeta) = -M_1 M_{1\zeta},\ -\tau \le \zeta < 0 \\ \text{for } (M_0, M_1) \in \Omega_B \text{ and } (M_{0\zeta}, M_{1\zeta}) \in \Omega_B \end{array} \right. \right\}, \\
M_{k\zeta}(t) &= M_k(t+\zeta)
\end{aligned} \tag{6.126}$$

and

$$\bar{R}(t) \equiv R(t) - M_1(t)\int_{-\tau}^{0} R(t+\zeta)d\zeta. \tag{6.127}$$

$\bar{R}(t)$ is bounded since $R(t)$ and $M_1(t)$ are bounded. That is, there exists a scalar $R_{\max}$ such that $\|\mathscr{P}\bar{R}(t)\| \le R_{\max}\ \forall t \ge t_0$. Equation (6.125) can be seen to be a system with distributed delays, whose stability can be shown using the Razumikhin method, as shown below.

Define

$$\bar{V}(\mathscr{E}_t) = \max_{\zeta \in [-2\tau, 0]} V(\mathscr{E}(t+\zeta)) \tag{6.128}$$

and a set Ω_t

$$\Omega_t \equiv \left\{ t \middle| t \in (t_b, t_c),\ V(\mathscr{E}(t)) = \bar{V}(\mathscr{E}_t) \right\}. \tag{6.129}$$

It follows that for all $t \in (t_b, t_c)$, there are two cases, (a) $t \in \Omega_t$, (b) $t \in (t_b, t_c) \backslash \Omega_t$. We provide the proof for each case separately.

(a) $\underline{t \in \Omega_t}$: From the definitions in (6.128) and (6.129), it follows that in this case,

$$V(\mathscr{E}(t+\zeta)) \leq V(\mathscr{E}(t)) \quad \text{for all} \quad -2\tau \leq \zeta \leq 0. \tag{6.130}$$

Hence we obtain from (6.123) and (6.125) that

$$\begin{aligned} \dot{V}(\mathscr{E}) \leq\ & 2\mathscr{E}^T(t)\mathscr{P}\bar{M}_0(t)\mathscr{E}(t) + 2\int_{-2\tau}^{0} \mathscr{E}^T(t)\mathscr{P}\bar{M}(t,\zeta)\mathscr{E}(t+\zeta)d\zeta + 2\mathscr{E}^T(t)\mathscr{P}\bar{R}(t) \\ & + \int_{-2\tau}^{0} \alpha(\zeta)\left[\mathscr{E}^T(t)\mathscr{P}\mathscr{E}(t) - \mathscr{E}^T(t+\zeta)\mathscr{P}\mathscr{E}(t+\zeta)\right]d\zeta \end{aligned} \tag{6.131}$$

with any scalar positive function $\alpha(\zeta)$, since the last term then becomes non negative due to (6.130). Equation (6.131) can be simplified as

$$\dot{V}(\mathscr{E}) \leq \int_{-2\tau}^{0} E_\zeta^T(t)\Psi(t,\zeta)E_\zeta(t)d\zeta + 2R_{\max}\|\mathscr{E}(t)\| \tag{6.132}$$

where

$$\Psi(t,\zeta) \equiv \begin{bmatrix} N_p(t,\zeta) & \mathscr{P}\bar{M}(t,\zeta) \\ (\mathscr{P}\bar{M}(t,\zeta))^T & -\alpha(\zeta)\mathscr{P} \end{bmatrix}, \tag{6.133}$$

$$N_p(t,\zeta) = \frac{1}{2\tau}[\mathscr{P}(M_0+M_1)+(M_0+M_1)^T\mathscr{P}]+\alpha(\zeta)\mathscr{P}, \tag{6.134}$$

and $E_\zeta(t) = [\mathscr{E}^T(t)\ \mathscr{E}^T(t+\zeta)]^T$.

We take

$$\alpha(\zeta) = \Theta_{\max}\sqrt{\frac{\bar{s}_{\mathscr{P}}}{\underline{s}_{\mathscr{P}}}}\cdot\begin{cases} \|M_{0\zeta}\| & -\tau<\zeta\leq 0 \\ \|M_{1\zeta}\| & -2\tau\leq\zeta\leq-\tau \end{cases}. \tag{6.135}$$

We now state and prove a sublemma:

Sublemma 1. *There exist ε_v, $\bar{\tau}$ such that $\Psi(t,\zeta)\leq-\varepsilon_v I$ if $\tau\leq\bar{\tau}$.*

Proof. From (6.134), (6.135) and Lemma 9 (6.113), it can be seen that if

$$\tau < \frac{1}{2\Theta_{\max}\|M_{k\zeta}\|\bar{s}_{\mathscr{P}}}\sqrt{\frac{\underline{s}_{\mathscr{P}}}{\bar{s}_{\mathscr{P}}}}\,\underline{q} \qquad k=0,1 \tag{6.136}$$

then

$$N_p(t,\zeta)<0 \quad \forall t,\zeta. \tag{6.137}$$

Using (6.137), it can be then shown that for any vectors v_1, $v_2\in\Re^n$

$$\begin{aligned}&\begin{bmatrix} v_1^T & v_2^T\end{bmatrix}\Psi(t,\zeta)\begin{bmatrix} v_1 \\ v_2\end{bmatrix}\\ &\leq -\underline{s}_{N_p(t,\zeta)}\left(\|v_1\|-\frac{\|PM_1M_{k\zeta}\|\|v_2\|}{\underline{s}_{N_p(t,\zeta)}}\right)^2+\left(\frac{\|PM_1M_{k\zeta}\|^2}{\underline{s}_{N_p(t,\zeta)}}-\alpha(\zeta)\underline{s}_{\mathscr{P}}\right)\|v_2\|^2,\end{aligned} \tag{6.138}$$

and also noting (6.134) and (6.113),

$$\underline{s}_{N_p(t,\zeta)} \geq \frac{1}{2\tau}\underline{q}-\alpha(\zeta)\bar{s}_{\mathscr{P}} \tag{6.139}$$

holds. From the definition of M_k, $k=0,1$ given in (6.111) and noting (6.6), it can be obtained that

$$\|M_k\| \leq \Theta_{\max}. \tag{6.140}$$

Therefore noting that

$$\|\mathscr{P}M_1M_{k\zeta}\| \leq \bar{s}_{\mathscr{P}}\Theta_{\max}\|M_{k\zeta}\|,$$

and integrating (6.139) into (6.138), we can further simplify the inequality as

$$\begin{bmatrix} v_1^T & v_2^T \end{bmatrix} \Psi(t,\zeta) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \leq \left(\frac{(\bar{s}_{\mathscr{P}}\, \Theta_{\max}\|M_{k\zeta}\|)^2}{\frac{1}{2\tau}\underline{q} - \alpha(\zeta)\bar{s}_{\mathscr{P}}} - \alpha(\zeta)\underline{s}_{\mathscr{P}} \right) \|v_2\|^2 \tag{6.141}$$

where $k=0$ if $-\tau < \zeta \leq 0$ and $k=1$ if $-2\tau \leq \zeta \leq -\tau$. With α substituted by (6.135), it is seen that the parenthesis in (6.141) becomes negative which in turn implies that

$$\Psi(t,\zeta) < 0$$

for all t,ζ if

$$\tau < \frac{1}{4\Theta_{\max}\|M_{k\zeta}\|\bar{s}_{\mathscr{P}}} \sqrt{\frac{\underline{s}_{\mathscr{P}}}{\bar{s}_{\mathscr{P}}}}\, \underline{q} \quad \text{for} \quad k = 0, 1. \tag{6.142}$$

Noting (6.140) again, it can be seen that (6.136) and (6.142) are satisfied if

$$\tau < \frac{1}{4\Theta_{\max}^2\bar{s}_{\mathscr{P}}} \sqrt{\frac{\underline{s}_{\mathscr{P}}}{\bar{s}_{\mathscr{P}}}}\, \underline{q}\,. \tag{6.143}$$

We let

$$\bar{\tau} \equiv \frac{1}{(4+\varsigma)\Theta_{\max}^2\bar{s}_{\mathscr{P}}} \sqrt{\frac{\underline{s}_{\mathscr{P}}}{\bar{s}_{\mathscr{P}}}}\, \underline{q}\,, \quad \varsigma > 0. \tag{6.144}$$

Then defining

$$\varepsilon_v \equiv \min_{t,\zeta,\tau\in[0,\bar{\tau}]} (-\mathrm{eig}(\Psi(t,\zeta))), \tag{6.145}$$

$$\Psi(t,\zeta) \le -\varepsilon_v I$$

is satisfied. This proves Sublemma 1. □

(6.132) can therefore be simplified as

$$\dot{V}(\mathscr{E}(t)) \le -\varepsilon_v \|\mathscr{E}(t)\|^2 + 2R_{\max}\|\mathscr{E}(t)\|. \tag{6.146}$$

From (6.146),

$$\dot{V}(\mathscr{E}(t)) < 0 \quad \forall t \in \Omega_t \setminus \{t|\ \|\mathscr{E}(t)\| > \beta\} \tag{6.147}$$

where

$$\beta = 2R_{\max}/\varepsilon_v. \tag{6.148}$$

Since $\bar{V}(\mathscr{E}_t(t)) = V(\mathscr{E}(t))$ as we defined Ω_t in (6.129), it can be concluded that

$$\dot{\bar{V}}(\mathscr{E}_t(t)) < 0 \quad \forall t \in \Omega_t \setminus \{t|\ \|\mathscr{E}(t)\| > \beta\}. \tag{6.149}$$

(b):$\underline{t \in (t_b,t_c)\backslash\Omega_t}$: From the definitions in (6.128) and (6.129), it follows that for any t in Case (b),

$$\bar{V}(\mathscr{E}_t(t)) > V(\mathscr{E}(t)). \tag{6.150}$$

Suppose there exists a $t = t_s \in (t_b,t_c)\backslash\Omega_t$ such that

$$\dot{\bar{V}}(\mathscr{E}_t(t_s)) > 0.$$

Then it follows that

$$V(\mathscr{E}(t_s^+)) > \bar{V}(\mathscr{E}_t(t_s)) \tag{6.151}$$

from the definition of $\bar{V}(\mathscr{E}_t)$ in (6.128). This contradicts (6.150), and therefore we can conclude that

$$\dot{\bar{V}}(\mathscr{E}_t(t)) \leq 0 \quad \forall t \in (t_b, t_c) \backslash \Omega_t. \tag{6.152}$$

From Case (a) and (b) ((6.149) and (6.152)), with β as in (6.148),

$$\dot{\bar{V}}(\mathscr{E}_t(t)) \leq 0 \quad \forall t \in (t_b, t_c) \backslash \{t | \, \|\mathscr{E}(t)\| > \beta\}. \tag{6.153}$$

Therefore

$$\bar{V}(\mathscr{E}_t(t)) \leq \max\left\{\bar{V}(\mathscr{E}_t(t_b)),\ \bar{s}_{\mathscr{P}}\beta^2\right\}. \tag{6.154}$$

Since $V(\mathscr{E}(t)) \leq \bar{V}(\mathscr{E}_t(t))$ from the definition given by (6.128), (6.154) implies that

$$V(\mathscr{E}(t)) \leq \max\left\{V(\mathscr{E}(t_b)),\ \bar{s}_{\mathscr{P}}\beta^2\right\} \quad \forall t \in (t_b, t_c),$$

completing the proof.

□

Proposition 7 contains the main result of this section.

Proposition 7. *If* $\tau \leq \bar{\tau}$, *then* $\|\mathscr{E}(t)\| < E \ \forall t \in [t_b, t_c)$.

Proof. From Lemma 10, $\forall t \in [t_b, t_c)$

$$\begin{aligned} V(\mathscr{E}(t)) &\leq \max\left\{V(\mathscr{E}(t_b)),\ \bar{s}_{\mathscr{P}}\beta^2\right\} \\ &\leq \max\left\{\bar{s}_{\mathscr{P}}\left(\mathscr{E}_0(t_b)^2 + \|\mathscr{E}'(t_b)\|^2\right),\ \bar{s}_{\mathscr{P}}\beta^2\right\}. \end{aligned} \tag{6.155}$$

We note from Proposition 6 that $|\mathscr{E}_0(t_b)| < E_0$. Also applying Corollary 1 (6.81) with

$t_s = t_a - \tau$, $t_{ss} = t_b$, $W = E_0$ and noting that Condition 2 (6.57) is satisfied at $t = t_a$, it can be shown that $\|\mathscr{E}'(t_b)\| \le \max(E', lE_0)$. Therefore (6.155) can be simplified as

$$V(\mathscr{E}(t)) \le \bar{s}_{\mathscr{P}} \max\left\{\left(E_0^2 + \max(E'^2,\ l^2E_0^2)\right),\ \beta^2\right\}.$$

Furthermore, from the definition of E_0 (6.47), $E_0 \ge \beta$. Also from (6.53) and (6.54), $E' > lE_0$. Therefore we obtain

$$V(t) \le \bar{s}_{\mathscr{P}} \left(E_0^2 + E'^2\right) \quad \forall t \in [t_b, t_c). \tag{6.156}$$

Noting that

$$\underline{s}_{\mathscr{P}} \|\mathscr{E}(t)\|^2 \le V(t) \le \bar{s}_{\mathscr{P}} \|\mathscr{E}(t)\|^2,$$

(6.156) implies that

$$\|\mathscr{E}(t)\| \le \sqrt{\frac{\bar{s}_{\mathscr{P}}\left(E_0^2 + E'^2\right)}{\underline{s}_{\mathscr{P}}}} \quad \forall t \in [t_b, t_c).$$

By taking

$$r_p \equiv \frac{\bar{s}_{\mathscr{P}}}{\underline{s}_{\mathscr{P}}},$$

it can be therefore concluded that

$$\|\mathscr{E}(t)\| \le E \quad \forall t \in [t_b, t_c).$$

□

6.3.7 Phase III: Exiting from the Boundary

Proposition 8. *Let $z(t_b') \in B_L$. Then either*

(I) $z(t) \in B\ \forall t \ge t_b'$, or

(II) there exists $t_c > t'_b$ such that $z(t_c) \in A$ and $z(t) \in B\ \forall t \in [t'_b,\ t_c)$.

In addition, in case (II),

$$t_c - t'_b \geq \Delta T_{exit,\min} \tag{6.157}$$

where

$$\Delta T_{exit,\min} = \frac{2\varepsilon_0}{\gamma' \overline{m}_0^2}, \tag{6.158}$$

and

$$|\mathscr{E}_0(t_c)| < \overline{m}_0. \tag{6.159}$$

Proof. It is straightforward to see that cases (I) and (II) are mutually and collectively exclusive.

From the definition of regions A and B_L, it follows that

$$\vartheta_0(t'_b) \leq -(\theta'_{0,\max} + \varepsilon_0/2), \quad \vartheta_0(t_c) \geq -\theta'_{0,\max}.$$

In addition, from (6.83)

$$\dot{\vartheta}_0(t) \leq \frac{1}{4}\gamma' \overline{m}_0^2 \qquad \forall\, t.$$

Hence

$$t_c - t'_b \geq \frac{2\varepsilon_0}{\gamma' \overline{m}_0^2},$$

completing the proof for (6.157).

We now prove (6.159) as follows. From the conditions in case (II), it is seen that

$$\vartheta_0(t_c - \Delta t_c) < -\theta'_{0,\max}, \quad \vartheta_0(t_c) \geq -\theta'_{0,\max} \tag{6.160}$$

for any $\Delta t_c \in (0, t_c - t'_b]$. Letting Δt_c tend to zero from the right hand side, it follows that $\dot{\vartheta}_0(t_c) > 0$. From (6.83), this in turn implies that $|\mathscr{E}_0(t_c)| < |m_0(t)|$, proving (6.159). □

6.3.8 Phase IV: Return to Condition 2

So far, we have shown on phases I through III the following:

Phase I. At $t = t_a$, $\mathscr{E}(t)$ satisfies Condition 2. Then $z(t'_b) \in B_L$ for $t'_b < t_a + \Delta T'_{in,\max}$, with $|\mathscr{E}_0(t)| < E_0 \; \forall t \in [t_a, t_a + \Delta T]$.

Phase II. Defining t_c such that $z(t) \in B \; \forall t \in (t_b, t_c)$, if $\tau \leq \bar{\tau}$, then $\|\mathscr{E}(t)\| < E \; \forall t \in [t_b, t_c)$.

Phase III. Either (a) $t_c = \infty$, or (b) $t_c \geq t'_b + \Delta T_{exit,\min}$ where $z(t_c) \in A$ and $|\mathscr{E}_0(t_c)| < \overline{m}_0$.

The following proposition contains the main result of this section:

Proposition 9. *Either $\mathscr{E}(t)$ returns to Condition 2 for some $t = t_d$ or the boundedness of $\mathscr{E}(t)$ is immediate.*

Proof. In case (a) in Phase III, the boundedness of $\mathscr{E}(t)$ is guaranteed since Phase II implies that $\|\mathscr{E}(t)\| < E \; \forall t \geq t_b$. In Phase III case (b), noting (6.159) and that $E_0 - \delta > \overline{m}_0$ from (6.47), there are only two possibilities:

(A) $|\mathscr{E}_0(t)| < E_0 - \delta$ for all $t \geq t_c$, or

(B) there exists $t_d > t_c$ s.t. $|\mathscr{E}_0(t_d)| = E_0 - \delta$ and $|\mathscr{E}_0(t)| < E_0 - \delta \; \forall t \in [t_c, t_d)$.

In case **(A)**, applying Corollary 1 with $t_s = t_c$, $t_{ss} = \infty$, and $W = E_0 - \delta$, it can be shown from (6.81) that

$$\|\mathscr{E}'(t)\| \leq \max\left(\sqrt{\frac{\bar{s}_{P'}}{\underline{s}_{P'}}}\|\mathscr{E}'(t_c)\|,\; l(E_0 - \delta)\right) \quad \forall t \geq t_c.$$

This implies that $\mathscr{E}(t)$ and therefore $z(t)$ is bounded.

If case **(B)** holds, then from the condition of the case it immediately implies that $\mathscr{E}(t)$ satisfies (6.56) in Condition 2 for $t = t_d$. We note that $\forall t \in (t_b, t_c)$, $z(t) \in B$ with $\|\mathscr{E}(t)\| < E$. This together with the condition of the case $|\mathscr{E}_0(t)| \leq E_0 - \delta \; \forall t \in [t_c, t_d]$ implies that

$$|\mathscr{E}_0(t)| < E \quad \forall t \in (t_b, t_d] \tag{6.161}$$

since $|\mathscr{E}_0(t)| \leq \|\mathscr{E}(t)\|$ and $E > E_0$. Hence if $\tau \leq \Delta T_{exit,\min}$, it follows that $\mathscr{E}_0(t)$ satisfies (6.55) in Condition 2 for $t = t_d$. Furthermore, since $\mathscr{E}_0(t)$ satisfies (6.55) in Condition 2 at $t = t_a$, and from Phase I $|\mathscr{E}_0(t)| < E_0 \ \forall t \in [t_a, t_a + \Delta T]$, we obtain

$$|\mathscr{E}_0(t)| < E \quad \forall t \in [t_a - \tau, t_d]. \tag{6.162}$$

Then applying Proposition 5 with $t_s = t_a - \tau$, $t_{ss} = t_d - \tau$ and $W = E$, it follows that

$$V'(t_d - \tau) \leq \max\left(V'(t_a - \tau),\ \frac{1}{2}\underline{s}_{P'}\,(lE)^2\right). \tag{6.163}$$

Noting that (6.57) in Condition 2 is satisfied by $\mathscr{E}'(t)$ for $t = t_a$, and using (6.54), we obtain

$$\begin{aligned} V'(t_d - \tau) &\leq \max\left(\frac{1}{2}\underline{s}_{P'}E'^2,\ \frac{1}{2}\underline{s}_{P'}E'^2\right) \\ &= \frac{1}{2}\underline{s}_{P'}E'^2. \end{aligned} \tag{6.164}$$

Hence $\|\mathscr{E}'(t)\|$ satisfies Condition 2 (6.57) for $t = t_d$. This implies that $\mathscr{E}(t)$ satisfies Condition 2 for $t = t_d$, proving Proposition 9. □

6.3.9 Final Part of the Proof

The above phases imply that starting $t = t_a$, there are only one of three possibilities: (i) The trajectory stays in Phase II for all $t \geq t_1$ for some finite $t_1 \geq t_b$; (ii) The trajectory stays on Phase IV-A for all $t \geq t_2$ for some $t_2 \geq t_c$; (iii) The trajectory visits all four phases infinitely often. The discussions in sections 6.3.5 through 6.3.8 imply that in all three cases (i)-(iii), $\mathscr{E}(t)$ always remains bounded, which proves Theorem 1. In particular, it follows from (6.85), Lemma 10, and (6.159) that in all cases (i)-(iii), if $\tau \leq \tau_l^\star$ defined as

$$\tau_l^\star = \min\left[\Delta T_{exit,\min}, \bar{\tau}\right], \tag{6.165}$$

then

$$|\mathscr{E}_0(t)| \leq E \quad \forall t \geq t_0. \tag{6.166}$$

Again applying Proposition 5 with $t_s = t_a - \tau$ and $W = E$, we obtain

$$V'(t) \leq \max\left(\frac{1}{2}\underline{s}_{P'}E'^2, \frac{1}{2}\underline{s}_{P'}(lE)^2\right) \quad \forall t \geq t_a - \tau.$$

Noting (6.53) and (6.54), it follows that

$$\|\mathscr{E}'(t)\| \leq E' \quad \forall t \geq t_a - \tau. \tag{6.167}$$

Hence

$$|z(t)| \leq M \quad \forall t \geq t_0, \tag{6.168}$$

where

$$M \equiv \sqrt{E^2 + \max\left(E', \max_{[t_0,\ t_a-\tau]} \|\mathscr{E}'(t)\|\right)^2 + \Theta_{\max}^2},$$

proving global boundedness.

6.3.10 Delay Margin of the Adaptive System

From (6.144), (6.158), and (6.165), we obtain that the solutions of the overall adaptive system is bounded for all $\tau \leq \tau_l^\star$. Hence, the lower bound of the delay margin $\tau^\star$ is given by $\tau_l^\star$, with

$$\tau_l^\star = \min\left[\frac{2\varepsilon_0}{\gamma'\overline{m}_0^2}, \frac{1}{(4+\varsigma)\Theta_{\max}^2\bar{s}_{\mathscr{P}}}\sqrt{\frac{\underline{s}_{\mathscr{P}}}{\bar{s}_{\mathscr{P}}}\,\underline{q}}\right]. \tag{6.169}$$

6.3.11 Differences from Chapter 5

As we establish guaranteed delay margins for adaptive systems with first-order plants in Chapter 5 and extend the result to higher-order plants in this chapter, it can be seen that the main theorems (Theorem 10 and Theorem 11) and their proofs share many similarities. However more complexities had to be dealt with in the vector case due to the higher dimensions of the state errors and parameters. Here we summarize the key differences.

The most significant difference is the non-singular transformations we take to extract the crucial scalar states $\mathscr{E}(t)$ and $\vartheta(t)$. The transformation involves matrices C and M which are constructed utilizing the direction of input vector b_p. Consequently, the proposed adaptive control law applies the projection algorithm on the transformed parameter state $\vartheta(t)$.

Another difference is the extra condition (6.41) in choosing the projection algorithm parameters $\theta_{i,\max}$ and ε_i. Given the size of parametric uncertainties in A_p, the condition (6.40) is necessary in order to guarantee the existence of $\vartheta^\star \in \Omega_{0R}$ which satisfies the matching condition (6.6). This is identical with the case of first-order plants where the parameter estimate is a scalar. However in the vector case, in addition to the condition (6.40), $\theta_{0,\max}$ needs to be sufficiently large so that the condition (6.41) is also satisfied. This is necessary to prove Lemma 9 and eventually Lemma 10, where the boundedness of the adaptive system while staying on the boundary region B is discussed.

The last difference we note is Proposition 5. Treating the crucial scalar state $\mathscr{E}_0$ as an input, the boundedness of the other states $\mathscr{E} = [\mathscr{E}_1 \; \mathscr{E}_2 \; \cdots \; \mathscr{E}_{n-1}]$ for a finite period of time is discussed. Lemma 7, which is proven by utilizing the special structure of $\mathscr{A}_m$ (the transformed reference model dynamics matrix), plays the critical role in showing this.

6.3.12 Remarks

The results of Theorem 11 together with Theorem 10 represent an important step in robust adaptive control. From establishing global boundedness in the presence of disturbances and unmodeled dynamics, this thesis takes the next step in robust adaptive control and extends it to time delays for a class of adaptive systems. A computable delay margin is demonstrated to exist, thereby providing a theoretical framework for verification of adaptive control systems in flight as well in other applications. The most important point to note is the absence of any Lyapunov function, a fixture in most adaptive control proofs. A first principles approach was used instead in this chapter as well as in Chapter 5 to ensure the global boundedness of the tracking errors, which is a distinctly different type of proof than those employed in robust adaptive control to-date. As can be seen in the proof of Theorem 11 as well as in the proof of Theorem 10, the two most crucial pieces of the proof involve the boundary of the projection algorithm in the adaptive law - the first says that the trajectory will hit the boundary region in a finite time (Phase I). The second is that once it hits the boundary region, it cannot become unbounded while remaining on the boundary region. These two were central points that helped establish global boundedness in this challenging problem.

In this chapter, for the sake of simplicity we assumed that b_p is known and let $b_m = b_p$. However it is expected that the result can be extended straightforwardly for the case $b_p = \lambda b_m$, where $\lambda > 0$ is an unknown parameter.

6.4 Numerical Example

In this section we demonstrate using a simple example as to how the main result in this chapter can be used to obtain delay margin of adaptive systems. We consider the short period dynamics of a conventional aircraft which can be approximated by a second-order plant with a scalar input.

From [57], short period dynamics of a fixed-wind aircraft with zero bank angle can be expressed as

$$\underbrace{\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix}}_{\dot{x}_p} = \underbrace{\begin{bmatrix} -L_\alpha & -L_q \\ \lambda_\alpha M_\alpha & \lambda_q M_q \end{bmatrix}}_{A_p} \underbrace{\begin{bmatrix} \alpha \\ q \end{bmatrix}}_{x_p} + \lambda_\delta \underbrace{\begin{bmatrix} 0 \\ M_\delta \end{bmatrix}}_{b_p} \Big(\underbrace{\delta}_{u} + d_{\mathrm{trm}} \Big) \tag{6.170}$$

where α is the aircraft angle of attack (radians), q is the body pitch rate in stability axis (radians/s), and δ is the total differential elevator deflection (radians). The scalars $\lambda_\delta > 0$,

λ_α, and λ_q represent uncertainties in the parameter values, and d_{trm} denotes an unknown trim input component. The nominal values are given as $\lambda_\alpha = \lambda_q = \lambda_\delta = 1$ and $d_{\text{trm}} = 0$. In the following, we assume that there are no uncertainties in the control effectiveness and the trim input, i.e. $\lambda_\delta = 1$ and $d_{\text{trm}} = 0$. Also we assume that the size of uncertainties are known and $\lambda_\alpha \in [.6\ 1]$ and $\lambda_q \in [.7\ 1]$. The rest of parameters represent the so-called aircraft stability and control derivatives. The values of the stability and control derivatives used in this example are

$$L_\alpha = 0.6582, \quad L_q = -0.9705, \quad M_\alpha = -3.3105, \quad M_q = -1.4741, \text{ and } \quad M_\delta = -3.6764.$$

These values can be found from a numerical linearization of a nonlinear aircraft model.

A state-feedback controller architecture is used for the controller so that

$$\delta = \theta_x^T(t) x_p + k_\delta \delta_{cmd}$$

where $\theta_x = \begin{bmatrix} \theta_\alpha & \theta_q \end{bmatrix}^T$ and δ_{cmd} is the pilot command. Let the dynamics without uncertainty be denoted (i.e. (6.170) where $\lambda_\alpha = \lambda_q = 1$)

$$\dot{x}_p = A_{p,nom} x_p + b_p \delta. \tag{6.171}$$

The Linear Quadratic (LQ) optimal control design techniques [61] is straightforwardly applied to the dynamics in (6.171) to obtain a nominal controller. In this example, the values minimizing the cost function

$$J = \frac{1}{2} \int_0^\infty x_p^T \left(Q_J + k_x R_J k_x^T \right) x_p dt$$

with $Q_J = diag([2\ \ 1])$ and $R_J = 1$ are calculated as $k_x = \begin{bmatrix} -0.2816 & -0.7434 \end{bmatrix}^T$. This nominal gain is used as an initial condition for the parameters $\theta_x(t)$ to be adapted, i.e. $\theta_x(t_0) = k_x$. The feed forward gain k_δ is designed to produce the angle of attack following so that $k_\delta = 1/g_\alpha$, where $\begin{bmatrix} g_\alpha & g_q \end{bmatrix}^T = -(A_{p,nom} + b_p k_x^T)^{-1} b_p$ is the steady state gain of (6.171) with the feedback corresponding to k_x. Then the closed loop dynamics of (6.171) with the nominal controller can be written as

$$\dot{x}_m = A_m x_m + b_m \delta_{cmd} \tag{6.172}$$

where $A_m = A_{p,nom} + b_p k_x^T$ and $b_m = b_p$. (6.172) will serve as a reference model. An adaptation can be then introduced into $\theta_x(t)$ as

$$\dot{\vartheta}_i = \gamma' \text{Proj}\,(\vartheta_i, -(\mathscr{E}_i + m_i)\mathscr{E}_0) \quad i = 0, 1$$

where $\mathscr{E}$, ϑ, and m are the transformed state error, parameter, and reference state, as introduced in (6.15), (6.16), and (6.66), respectively. A Transfer matrix C is constructed utilizing b_m and A_m (which gives $P > 0$, a solution of Lyapunov equation (6.11)), and given as

$$C = \begin{bmatrix} 0.1247 & 0.4572 \\ -0.4572 & 0.1247 \end{bmatrix}.$$

Similarly, M and $\mathscr{A}_m$ are constructed from (6.24) and (6.31). We then choose the projection parameters from equations (6.40) and (6.41) and the size of uncertainties in λ_α, λ_q as $\theta_{0,\max} = 6.0$, $\theta_{1,\max} = 1.5$ and $\varepsilon_1 = \varepsilon_2 = 0.01$. Also, we set the adaptation gain $\gamma' = 10.83$ based on ad-hoc tuning and assume that δ_{cmd} is such that $|\alpha_m(t)| \leq 0.1745(radians)$, $|q_m(t)| \leq 0.6109(radians/s)$ $\forall t \geq t_0$ which leads to $\overline{m}_0 = 0.3010$ from (6.66). $\underline{q}$ is set to 1 and $\bar{s}_{\mathscr{P}}$, $\underline{s}_{\mathscr{P}}$ are calculated from $\mathscr{A}_m$, $\theta_{i,\max}$, ε_i, and $\vartheta^\star$. We can therefore calculate the delay margin using (6.169) as

$$\tau_l^\star = 6.8(ms).$$

According to numerical simulation studies, it was seen that the actual delay margin of the adaptive system is around $0.070(s)$. It can be therefore again argued that the analytically computable bound of the delay margin established in this chapter is not overly conservative.

6.5 Summary

In this chapter, the result in Chapter 5 is extended and robust adaptive control of general *n*th order plants with a scalar input in the presence of time delay is established. The proposed adaptive control law applies the projection algorithm to the transformed parameter state component-wise. The transformation involves a matrix M which is constructed utilizing the direction of input vector b_p. Together with Chapter 5, these results clearly demonstrate that adaptive systems with state variables accessible have a guaranteed delay margin, providing one solution to a non-trivial open problem in this field.

Chapter 7

Concluding Remarks and Future Works

In this thesis, we have focused on robust adaptive control technology for safe flight. The developed adaptive controller in Chapter 2 consists of a fixed controller that provides satisfactory performance under nominal flying conditions, and a direct Model Reference Adaptive Controller (MRAC) that provides stability in the presence of failures or damages. Using a NASA Generic Transport Model (GTM), which is a model of a transport aircraft, the behavior of the adaptive control system is simulated in the presence of various uncertainties in this chapter. While a significant improvement in flight safety was observed in several failure and damage scenarios, an undesirable flight performance and robustness concerns also become apparent. One such case was in the presence of time delays, illustrating that robustness of adaptive control systems in the presence of delays has to be addressed. Chapter 3 presents fundamental theoretical results related to adaptive control of scalar plants in the presence of time delays. The main instability results are summarized as Theorem 1 and 2 for the case when the standard adaptive laws without any modification except for σ-modification. Projection algorithm can be then introduced as a tool to avoid instability. The properties of the Lipschitz continuous projection algorithm, which are utilized throughout the rest of the thesis, are formulated in this chapter together with its key definitions and lemmas. We show that with this projection algorithm, a robust adaptive stabilizer that guarantees global boundedness in the presence of time delay can be established.

In Chapter 4, a class of adaptive systems is examined in the presence of unmodeled dynamics and robustness results are derived using a Lyapunov function approach. Even though the result in the chapter is quite restrictive, it sheds light on the potential of the projection algorithm as a tool to achieve desirable global results.

The main results of this thesis are presented in Chapter 5 and 6 in Theorem 10 and 11, where robust adaptive control of general plants with a single input and states accessible in the presence of time delay is established. In contrast to Theorem 9 in Chapter 4, we show that global boundedness can be derived without requiring any approximation. One of the main goals of the thesis, an analytically computable delay margin, is also achieved. In Chapter 5, global boundedness and a delay margin are derived for the adaptive control of scalar plants with the adaptive law straightforwardly modified based on the projection algorithm. In the adaptive law proposed in Chapter 6, the projection algorithm is applied to a transformed parameter state component-wise, and this transformation enables the use of two crucial scalar states, and allows the results of Chapter 5 to be applied. The results demonstrate that adaptive systems with state variables accessible have a guaranteed delay margin, providing a solution to a long standing open problem in the field of adaptive control.

The results of this thesis, while solving a highly difficult problem in robust adaptive control, need to be generalized much further. The following are some examples:

- Developing robust adaptive control for plants with multiple inputs and states accessible, which ensures global boundedness in the presence of time delays (Multivariable control),
- Developing robust adaptive control where state variables are not accessible, which ensures global boundedness in the presence of time delays (Output feedback),
- Developing robust adaptive control for the above two cases in the presence of unmodeled dynamics

One important aspect of the main result in this thesis is that the robustness properties of the adaptive systems with projection are partially determined by the properties of the LTV system exhibited while the trajectory stays on the projection boundary. This notion is promising, since utilizing the similar analysis, it may be possible to study the performance of general robust adaptive systems.

Lastly, it should be noted that while the modification based on the projection algorithm proposed in this thesis ensures global boundedness and therefore a nonzero delay margin, the performance of the overall adaptive system may be far from satisfactory in the presence of time delay. In the worst case, the system may exhibits an oscillating behavior by visiting all four phases infinitely often. In most of the actual systems this is not acceptable. Filling the gap between boundedness and satisfactory performance is not an easy task, but that would be also an important future work of the thesis.

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