

Analysis and Design of Wave Scattering by Weakly Non-uniform Waveguides.

by

Karl Peter Burr

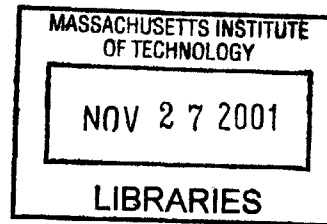
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in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Hydrodynamics

at the

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Abstract

When waves propagate through a medium with small irregularities of the size of the order of many wavelengths, a number of interesting phenomena may happen such as wave localization and large sensitivity of the wave field behavior with respect to the medium irregularity variation. These phenomena occur due to the interaction of the incident wave field with the medium small irregularities, and open the possibility of designing the system irregularity to achieve a desired vibratory response. This has a variety of engineering applications, such as the design of the sea bottom of coastal areas to provide protection against the incoming swells, or the design of the material and geometrical properties of the cross section of pipelines, risers and mooring lines such that vibration transmission is minimized.

The objective of this thesis is to understand how to tune the medium small irregularity such that the interaction of the incident wave field with the medium irregularity generates a desired reflected wave field. A particular design problem of interest is the prediction of the minimum amount of changes in the medium irregularity needed to minimize wave transmission to a desired level for a given range of frequencies of interest.

As a model problem, we considered disordered chains of repetitive systems with the size of the order of many wavelengths of the incident wave. We applied an asymptotic theory for wave propagation along the non-uniform chain. For weak coupling between subsystems, the asymptotic theory predicted new results, such as exponential small transmission due to wave tunneling and explained localization phenomena as a turning point problem. For strong coupling, the asymptotic theory provided fundamental understanding of the effects of the irregularity on wave propagation.

Pipelines and risers can be modeled as slender beams under tensile force. To describe well the effects of small irregularity in beams vibration, we derived asymptotically a simpler governing equation for the vibration problem. This new equation is asymptotic with respect to the beam irregularity steepness, but under the restriction of constant product of the flexural rigidity by the mass per unit length and constant tensile force, this new equation is an exact equation for the beam vibration and has

a Helmholtz-like form. Inverse scattering methods for the Helmholtz-like equations can be applied to design the beam non-uniformity such that desired wave scattering properties are achieved. We also constructed a high order asymptotic solution for the scattering of mono-chromatic waves by the irregularity in slender beams. The asymptotic method used is the WKB method, which is basically a wave refraction theory, but we improved it such that wave reflection and wave mode conversion were captured.

The asymptotic approach developed in the previous problems is extended and applied to the interaction of linear surface gravity waves with a bottom topography which varies slowly with respect to the length scale of the incident wave field. The asymptotic theory captured wave reflection and transmission and wave mode conversion, which leads to a more complete asymptotic representation of the wave field. This asymptotic theory also reveals the dependence of the scattering coefficients with respect to the bottom geometry, which is a fundamental step in the design of bottom topographies in coastal areas to provide protection against the incoming swells.

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Dedicated to my parents Jürgen and Inara.

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Chapter 1

Introduction

Waves generated by storms causing property damage in coastal areas is common. Harbours also suffer with incoming waves, specially when they are an open body of water linked to the ocean by a channel. In this case, one of the vibratory modes of the surface of the harbor water body can resonate with the incident wave, resulting in large wave motion inside the harbor. Rivers flowing to the sea also provide harbours that are subjected to wave effects. Incident waves can propagate upstream through the river, jeopardizing the harbor operation. Therefore, ways of minimizing the action of ocean waves on coastal areas and harbours is a subject of primary societal importance. Usual ways to protect coastal areas are very expensive and sometimes with non-desirable collateral effects, such as negative environment impact and erosion of adjacent areas. So, other means of coastal protection are on demand.

For coastal areas with gradually increasing water depth and channels, the bottom may be designed to work as a wave filter, minimizing wave transmission for a desired range of wave frequencies. This can be accomplished by tuning the bottom non-uniformity to maximize the wave localization phenomenon effects in the interaction of water waves with variable bottom topographies.

Localization phenomenon was initially discovered in the study of electric conductivity of disordered solids. Anderson [2] explained many of the transport properties of disordered solids. The theory of localization deals with linear waves, described by a Schrödinger equation, a Helmholtz equation or a similar wave equation, propagat-

ing in a static disordered medium. The wave undergo many partial reflections and transmissions from the scattering centers in the disordered medium and the various transmitted and reflected waves, which have a random de-phasing, interfere with each other. The theory of localization predicts the resulting behavior from all these interferences. The basic result is that, if the disorder is strong enough, the incident wave will be completely reflected as it tries to propagate through the medium. The amplitude of the wave disturbance along the medium shows an exponential decay, and the rate of decay is defined as the length scale of the localization phenomenon. Since the localization phenomenon is not specific to electrons but is a genuine wave interference effect, several applications of localization ideas have been proposed for classical waves propagating in one-dimensional disordered media: structural vibration (Hodges [29] and Hodges & Woodhouse [30]), electromagnetic waves in plasma (Escande & Souillard (1984)), light waves (Bouchaud & Daoud (1986) and Flesia, Johnston & Kunz (1984)), electrical waves in a chain of random impedances (Akkermans & Maynard (1984)).

In the context of water waves, localization phenomenon has been observed experimentally by Belzons et al [3]. They considered the interaction of linear gravity waves with a static one-dimensional random bottom. Parallel to their work, Devillard et al [19] performed theoretical calculations for the same kind of bottom used in the experiments in order to enable comparison between measured and predicted localization lengths (the exponential decay rate of the wave disturbance amplitude). The two principal experimental limitations for the observation of the localization phenomenon were the dissipation and finite size effects which were outside the scope of the theoretical calculations. The localization length depends on the wave frequency and disorder; localization could be unobservable if the localization length is too large, e. g. much larger than the exponential decay rate due to dissipation (length scale of the dissipation) or much larger than the size of the disordered bottom. Besides these limitation, theoretical calculations in Devillard et al (1987) for the random bottoms used in the experiment predicted a range of wave frequencies where the localization length should be smaller than the estimated dissipation and random bottom lengths and ob-

servable experimentally. For this range of wave frequencies, good agreement between the experiments and the theoretical prediction was observed. In these experiments, its size was of the same order as the localization length in the range of frequencies where localization was observable. This resulted in fluctuations in the transmission (reflection) coefficient which are not present in very large systems. These fluctuations are due to the occurrence of resonances. For very large systems (order of many wavelengths) the eigenmodes are localized, but for a finite system they are turned into resonances. Since these resonant modes are trapped along the non-uniform medium, the incident wave could pass by tunneling and thus transmission is enhanced for the corresponding resonant frequency. This implies a sample-to-sample dependence of the transmission (reflection). In other words, the system vibratory response shows a large sensitivity with respect to the non-uniformity variation, which opens the possibility of designing the non-uniformity to achieve a desired vibratory response, like minimizing wave transmission.

The objective of this thesis is to predict how to tune the medium non-uniformity such that the interaction of a given incident wave field with the medium non-uniformity generates a desired scattered wave field. A particular design problem of interest would be to predict the minimum amount possible of changes to the non-uniformity of the medium needed to minimize wave transmission to a desired level for a given range of wave frequencies. We have special interest in this design problem for water waves interacting with a variable bottom, as mentioned in the beginning of the introduction.

The key issue for the design problem related to localization phenomenon is the system vibratory response sensitivity to disorder variation. To discuss this issue we consider repetitive systems, which are chains of systems coupled with each other. A simple example is a chain of pendulums. Each pendulum is coupled through a spring with the pendulums after and before it in the chain. Thus, a chain of identical pendulums, one of which is forced to move, will respond with vibration extending to the whole chain. A repetitive system whose parameters deviate from a reference value randomly or in some prescribed manner is called a disordered repetitive system. Under certain conditions, such a disordered repetitive system exhibits localization of

the response, i.e., the forced response decays exponentially away from the driving point. In this situation, the normal modes are localized, i.e., the vibration is confined to a portion of the overall system, in contrast with the modes of a perfectly uniform system, which are extended, i.e., all parts of the system participate in the vibratory response.

Triantafyllou and Triantafyllou [65] showed that localization in disordered repetitive systems is associated with large sensitivity of both natural frequencies and natural modes with respect to small parametric change (disorder change). By allowing the disorder parameter to become complex, the eigenvalue equation for the system can be seen as a complex mapping from the complex parameter space to the complex frequency plane. They showed that frequency veering phenomena and high modal sensitivity, which characterize localization, is in fact caused by the presence of a branch point of this mapping in the complex parameter space. At the image of a branch point in the complex frequency plane, a saddle point, two or more natural frequencies coalesce. A real system, with parameters close to a branch point of the eigenvalue equation, should have to a large or small degree a large modal sensitivity. The closer the branch point lies to the real axis, the larger is the sensitivity of the vibratory response of the real system. We are going to illustrate these facts through a simple example, a two degree of freedom chain of coupled pendulums.

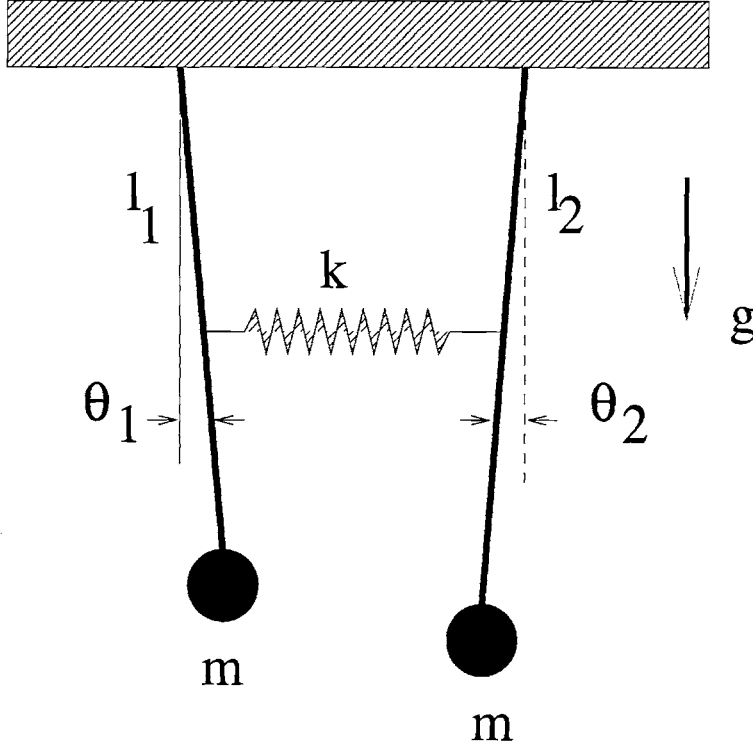


Figure 1-1: A two degree of freedom chain of coupled pendulums.

Both pendulums in figure 1-1 have the same mass m and the spring has stiffness k . g illustrated in figure 1-1 is the acceleration of gravity. The disorder in the system is characterized by the difference in length of the pendulums. l is the length of the first pendulum, and the length of the second pendulum is $l_2 = \frac{l}{1+\delta}$, where δ is the disorder parameter. θ_j is the angular displacement of the j -th pendulum. We consider the case of free vibration. The non-dimensional equations of motion in matrix form are

$$\begin{bmatrix} (1 + R - \omega^2) & -R \\ -R & (1 + \delta + R - \omega^2) \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = 0, \quad (1.1)$$

where $R = \frac{kl}{mg}$ is denoted as the coupling parameter, $z_1 = \theta_1$ and $z_2 = \frac{l_2}{l}\theta_2$. The non-dimensional frequency ω is the ratio of the system vibration frequency and the

frequency of oscillation ($\sqrt{g/l}$) for a single pendulum with length l . The natural frequencies are solutions of the eigenvalue equation for this system, given by the determinant of the system matrix (equation (1.1)). The natural frequencies are

$$\omega^2 = 1 + R + \frac{\delta}{2} \pm \frac{1}{2} \{\delta^2 + 4R^2\}^{1/2}. \quad (1.2)$$

The mapping between the disorder parameter δ and the non-dimensional frequency given by equation (1.2) has branch points in the complex parameter space at $\delta = \pm i2R$. At these values of δ the square root in equation (1.2) vanishes and we have a single value for the square of the natural frequency. This means that at $\delta = \pm i2R$ the natural frequencies coalesce. The sensitivity S_ω of the natural frequencies with respect to disorder is

$$S_\omega = \frac{d\omega^2}{d\delta} = \frac{1}{2} \pm \frac{1}{2} \frac{\delta}{\sqrt{\delta^2 + 4R^2}}. \quad (1.3)$$

For values of the disorder parameter close to the branch points $\pm i2R$, we can write $\delta = \pm i2R + \lambda$, and the natural frequency sensitivity is

$$S_\omega = \frac{1}{2} \pm \frac{1}{2} \frac{i2R + \lambda}{\sqrt{i4R + \lambda}} \lambda^{-1/2} \quad (1.4)$$

From equation (1.4) we already see large sensitivity of the natural frequency with respect to the disorder variation characterized now by the parameter λ . As $\lambda \rightarrow 0$, the natural frequency sensitivity becomes infinite. If $R \ll 1$ the coupling is weak, and the branch points $\pm i2R$ are close to the real axis in the δ complex plane. For this case the sensitivity of real systems (real disorder) is large since for real disorder we need $|\lambda| > |2R|$, but it is still $|\lambda| < 1$. As the coupling increases the branch points $\pm i2R$ move away from the real axis and the minimum value of λ which gives a real disorder increases. This implies in a decrease in the natural frequency sensitivity. The large

sensitivity with respect to disorder is not restricted to the natural frequencies. The natural modes

$$\{v_1 \ v_2\} = \left\{ \begin{array}{cc} 1 & 1 \\ -\frac{1}{R} \left(\frac{\delta}{2} + \frac{1}{2} \sqrt{\delta^2 + 4R^2} \right) & \frac{1}{R} \left(-\frac{\delta}{2} + \frac{1}{2} \sqrt{\delta^2 + 4R^2} \right) \end{array} \right\} \quad (1.5)$$

expressions have the same square root branch points at $\delta = \pm i2R$, as illustrated by equation (1.5). Notice that close to the branch points $\pm i2R$, we see a large variation in the amplitudes of the normal modes along the system. For the uniform system ($\delta = 0$), the normal modes are

$$\{v_1 \ v_2\} = \left\{ \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right\} \quad (1.6)$$

At the first mode ($\{v_1\}$), the two pendulums oscillate at the larger natural frequency with the same amplitude but out of phase, and at the second mode ($\{v_2\}$) the two pendulums oscillate at the smaller natural frequency with the same amplitude and in phase. Now, when $\delta \rightarrow \pm i2R$ the amplitudes of each mode changes drastically. From the equation (1.5) we see that for each mode, one of its amplitude becomes much smaller than the other, so the normal modes are localized. In this condition, if we excite, for example, the first pendulum, we observe the first pendulum oscillating at a larger amplitude than the second pendulum. So the system vibration stays localized in the first pendulum.

For more general systems, described by autonomous linear ordinary differential equations, or a set of linear partial differential equations with space-dependent coefficients subjected to a number of boundary conditions, the same qualitative results shown above apply, as discussed in Triantafyllou & Triantafyllou [65]. The natural frequencies of vibration (complex for damped or unstable systems) can be determined by the eigenvalue problem $\Delta(\omega, \delta_j) = 0$, where ω is the frequency and $\delta_j, j = 1, 2, \dots, N$ are N parameters of the system. Solving this relation for ω yields the natural fre-

quencies ω_n of the system as a function of the parameters δ_j . If we allow one of the parameters δ_j , let say δ , to become complex, the eigenvalue equation can be seen as a mapping between the complex δ plane and the complex ω frequency plane. This mapping has branch points δ_0 with saddle points ω_0 as images. For two natural frequencies to coalesce to ω_0 as we approach δ_0 in the complex δ plane, the points ω_0 and δ_0 have to satisfy the equations

$$\Delta(\omega_0, \delta_0) = 0 \text{ and } \frac{\partial \Delta}{\partial \omega}(\omega_0, \delta_0) = 0. \quad (1.7)$$

By virtue of equation (1.7), near the point ω_0, δ_0 , the eigenvalue equation can be expanded to lowest order as

$$\Delta(\omega, \delta) \approx A(\omega - \omega_0)^2 + B(\delta - \delta_0) = 0, \quad (1.8)$$

where A and B are complex constants. It is clear from equation (1.8) that the points ω_0 is a saddle point in the complex ω plane, and the point δ_0 is a branch point in the complex δ plane. The presence of a branch point requires the definition of a branch cut, the choice of which can be arbitrary, provided that the δ -real axis lies in the same Riemann sheet; otherwise we would have discontinuous variations of the natural frequencies with respect to the parameters. The approximate eigenvalue equation (1.8) can be written in the explicit form

$$\omega - \omega_0 = a\sqrt{\delta - \delta_0}, \quad (1.9)$$

where a is a complex constant. Hence the frequency variation is of order $\sqrt{\delta - \delta_0}$ for a variation in the governing equation of order $\delta - \delta_0$: i. e., it is infinitely more sensitive as $\delta \rightarrow \delta_0$.

Coalescence of more than two natural frequencies is possible. For an m -th order

frequency coalescence, the point (ω_0, δ_0) is defined as the solution of the simultaneous equations

$$\Delta(\omega_0, \delta_0) = 0 \text{ and } \frac{\partial^n \Delta}{\partial \omega^n}(\omega_0, \delta_0) = 0, n = 1, 2, \dots, m-1. \quad (1.10)$$

In the vicinity of the point (ω_0, δ_0) where the m -th order frequency coalescence happens, the eigenvalue equation $\Delta(\omega, \delta) = 0$ has the form

$$A(\omega - \omega_0)^m + B(\delta - \delta_0) = 0, \quad (1.11)$$

where A and B are complex constants, provided B is non-zero. It follows that we obtain an m -th order saddle point in the ω plane and an associated branch point in the δ plane. Consequently, close to ω_0, δ_0 we have

$$\omega - \omega_0 \approx a\{\delta - \delta_0\}^{1/m}, \quad (1.12)$$

demonstrating, again, infinite sensitivity to small parametric changes: i.e., as $\delta \rightarrow \delta_0$. In fact, the higher the order of the frequency coalescence, the higher the sensitivity. If the branch point δ_0 is close enough to the δ -real axis, the vibratory response of the real systems with $\delta \sim \Re\{\delta_0\}$ is also large.

If we allow a larger number of parameters to vary, the number of branch points of the system eigenvalue equation $\Delta(\omega, \delta) = 0$ (now δ is a vector of parameters which are allowed to be complex numbers) increases. For system with equations of motion given as a set of ordinary differential equations, the number of natural frequencies is equal to the number of degrees of freedom of the system. If the system equations of motion are linear partial differential equations, we have an infinite set of natural frequencies, enumerable or may be even continuous. The higher the number of natural frequencies, the higher the possibility of branch points where a large number of natural

frequencies coalesce, which implies in a larger system response sensitivity to variations in the system parameters. The number of normal modes that localized as we approach a branch point in the parameters space of the natural frequency eigenvalue equation is equal to the number of natural frequencies that coalesce.

From the design point of view of minimizing wave transmission to a desired degree (vibration propagation) for a given range of vibration frequencies, we would like to have the larger number possible of normal modes localized, and the localized modes should correspond to the natural frequencies laying inside the desired range of frequencies for which we want to minimize wave transmission. To succeed in this design problem we need to look for the branch point of the natural frequency eigenvalue equation $\Delta(\omega, \delta) = 0$ closest to the real part of the system parameters complex space (δ is a complex vector) where the larger number possible of natural frequencies coalesce. It is important also that the coalescing frequencies corresponds to the natural frequencies lying in the desired range of vibration frequencies. Once we find this branch point we project it into the real part of the system complex parameter space, and this should give the system with real parameters, or at least it should be close to the system with real parameter which gives the best vibration isolation possible.

As mentioned in the beginning of this section, we would like to be able to design the minimum amount of changes in a given sea bottom topography to minimize wave transmission to a desired range of wave frequencies. For finite size systems described by a set of differential equations or a set of linear partial differential equation under various boundary conditions, this design problem can be achieved by analyzing the system eigenvalue equation $\Delta(\omega, \delta) = 0$ as described above, but for water waves interacting with a non-uniform bottom we have a semi-infinite (a beach) or infinite domain (a channel, for example), and instead of only boundary conditions we have also radiation conditions. We have a more complex eigenvalue spectrum. For this class of problems (infinite and semi-infinite domains), the eigenvalue spectrum is a continuum and may have also a discrete set, which corresponds to trapped modes (localized standing wave) due to the bottom non-uniformity in the case of water

waves. So, the methodology previously described to handle the design problem of tuning the system parameters non-uniformity to minimizing wave transmission is very complex to use in the problem of surface waves propagating over non-uniform bottoms. Another approach for this design problem is welcome.

In chapter 2 we considered chains of repetitive systems with size of the order of many wavelengths of the incident wave. We applied an asymptotic theory for wave propagation along the non-uniform chain. For weak coupling between subsystems, the asymptotic theory predicted exponential small transmission due to wave tunneling and explained localization phenomena as a turning point problem. For not weak coupling, the asymptotic theory provided qualitative understanding of the effects of non-uniformity on wave propagation.

In chapter 3 we considered the vibration problem for a non-uniform beam under tensile force. To describe well the effects of the non-uniformity in the beam vibration, we derived asymptotically a simpler governing equation for the vibration problem. This equation is asymptotic with respect to the beam non-uniformity steepness. As the non-uniformity steepness decreases, this simple equation recovers the behavior of the original governing equation. Under the restriction of constant product of the flexural rigidity by the mass per unit length and constant tensile force, the simple equation is an exact governing equation for the beam vibration and has a Helmholtz-like form. These beams are not usually met in practice, but they can be designed and built. Inverse scattering method for the Helmholtz-like equation can be applied to design the beam non-uniformity such that desired wave scattering properties are achieved.

In chapter 4 we assume that the bottom varies slowly with respect to the wave disturbance length scale. We develop an asymptotic theory in the slowness of the bottom variation to predict the scattering coefficients for the wave interaction with a non-uniform bottom. This asymptotic theory reveals the dependence of the scattering coefficient with respect to the bottom geometry, what is a useful tool for the design problem in mind.

1.1 Background

The objective of this section is to outline the previous work done in wave propagation along non-uniform media, with special emphasis to wave localization phenomenon. The range of fields where wave localization has been studied is broad. We restrict the outline of the previous work on wave localization and wave propagation in non-uniform media to fields related to this thesis, namely, structural vibration and water waves. We also give a short account on the work related to wave localization done in solid state physics where this phenomenon was first discovered.

The phenomenon of localization has been known in the context of solid state physics for more than forty years. Anderson [2] explained many of the transport properties of disorder solids. The motion of electron in solids are governed by the Schrödinger wave equation. In the context of solid state physics, there is a large literature on the subject. Just to mention a few, we have Herbert & Jones (1971) who obtained improved conditions for localization in the Anderson's model, Thouless (1973) who related localization to backscattering on a continuously disordered one dimensional continuum, and the review by Erdos & Herdon (1982).

1.1.1 Structural Dynamics.

In the context of structural dynamics only recently localization phenomena had received attention. Hodges [29] was the first to recognize the relevance of localization theory to the context of structural dynamics, and in Hodges & Woodhouse [30] numerical and experimental evidence of localization were provided. Pierre [54] used an statistical approach to evaluate the localization factor. Kissel [34] used the transfer matrix method. The transfer matrices are considered random quantities with uniform distribution. He applied Furstenberg's theorem on the limiting behavior of products of random matrices to evaluate the localization factor. Castanier & Pierre [14] used perturbation technique to evaluate the Lyapunov exponent (localization factor) of the wave transfer matrix for multi-coupled disordered periodic linear systems. Bouzit & Pierre [8] used a wave transfer matrix approach and statistical perturbation methods

to evaluate the localization factor for a multi-span beam with slight randomness in spacing between supports.

Pierre [52] and [53] used an eigenvalue/mode perturbation approach to investigate normal mode localization for disordered structural systems consisting of weakly coupled subsystems. Triantafyllou & Triantafyllou [65] provide a geometric theory to explain frequency coalescence and mode localization. They consider the system eigenvalue problem as a complex mapping between the complex frequency plane and the complex parameter space. They showed that high modal sensitivity, which characterizes localization, is in fact caused by the presence of branch points of the mapping mentioned above in the complex parameter space near the real axis. Therefore, if we want to maximize the localization for a given system, we need to search for the branch points of the mapping mentioned where all natural frequencies coalesce. The projection of the branch point with smaller imaginary part on the real axis should give the parameter configuration for which all normal modes show some degree of localization. This configuration of non-uniformity should minimize transmission for the whole system passband. Therefore, Triantafyllou & Triantafyllou [65] geometric theory provides an approach to the design problem for minimum transmission for systems of small/medium size. For larger systems, order of hundreds of elements, the search for the branch points where all natural frequencies coalesce became too complex, so another approach is welcome. Results from the WKB method for second order difference equations, as discussed in chapter 2 of this work, may be suitable.

1.1.2 Water Waves.

The study of wave scattering by a variable bottom topography dates back to Kreisel [35] who proved the uniqueness of the two-dimensional problem in the presence of small bottom non-uniformity. For a general rough bottom, the uniqueness is still an open problem. For three-dimensional topographies, trapped modes are known to exist in general (Chamberlain & Porter [15]). When the trapped modes are excited by incident surface waves, large wave motions can be developed over the topography. The prediction of this general bottom topography is not available yet.

For wave scattering over a periodic bottom topography, much is understood, among which the most prominent phenomenon is the Bragg wave resonance due to sinusoidal bottom ripples. When the incident wave and bottom ripple wave numbers satisfy so-called Bragg conditions, the scattered wave becomes resonant and is greatly amplified. This was first predicted by a direct perturbation analysis which shows a singularity in the reflection coefficient when the periodic bottom undulation has a wavelength half that of the incident wave (Davies [18]). This has been confirmed in laboratory experiments (Davies & Heathershaw [18]). For mild incident wave and bottom slopes, reflection at or near this Bragg condition is well predicted by perturbation theory based on multiple scales and the assumption of linearized surface waves (Mei [43]). For larger waves and/or bottom steepness, higher-order Bragg resonances due to nonlinear interactions among the surface wave components are also expected. Guazzeli et al [25] showed experiments which demonstrate higher-order Bragg resonant interaction of linear surface gravity waves with doubly sinusoidal beds. Recently, Liu & Yue [40] addressed the problem of arbitrary generalized Bragg scattering involving multiple free-surface and bottom wave components. they obtained a generalized theory that places existing theoretical results in a systematic framework, explains the experimental observations and measurements, and predicts new nonlinear Bragg mechanisms which should have leading-order effects on near shore wave propagation.

When the bottom topography has a random variation, the phenomenon of wave localization was observed in experiments (Belzons et al. [3]). For monochromatic waves, Devilard et al. [19] developed a stochastic theory for wave localization using a step approximation for the bottom variation and obtained an estimate of the localization length. Nachbin & Papanicolaou [49] and Nachbin [48] considered a more general case involving multiple waves and random bottom profiles of large amplitude and developed an asymptotic theory, based on stochastic differential equations, for wave localization. For localization involving three-dimensional bottom topography, no theory is yet available. Furthermore, because of the requirement of extremely long bottom topography (relative to wavelength), the practical applicability of such stochastic approaches may be limited. In many cases, a deterministic prediction given

a (local) wave-bottom environment may be of much interest. In chapter 4 of this thesis we provide a deterministic, but asymptotic prediction of the wave scattering by a variable bottoms.

1.1.3 Design Problem.

The problem of designing the system disorder to minimize wave transmission for a given range of vibration frequencies was seldom addressed in the literature. The previous works that exists, to our knowledge, are Keane & Manohar [32], Gopalkrishna [56] and Langley [36]. Keane & Manohar [32] considered the optimum design, from an energy flow point of view, of two axially vibrating rods which are each composed of 20 piecewise uniform sections. Gopalkrishna [56] minimized wave transmission for system with a small number of degrees of freedom and weak coupling. He optimized (maximized) the exponential decay constants of the system normal modes in terms of the system parameters to achieve his design objectives. Langley [36] considered non-uniform one-dimensional repetitive systems embedded in infinite uniform systems. He provided tools to design structural filters which might block or allow wave transmission.

In the context of water waves, the design of the bottom non-uniformity to minimize wave transmission was never addressed in the literature.

1.2 Thesis Outline.

In this section we give an outline of the work done in the chapters of the thesis.

In chapter 2 we consider monochromatic wave propagation along finite, one-dimensional, slightly non-uniform repetitive system, whose ends are connected to uniform semi-infinite repetitive systems. A one-dimensional repetitive system is a chain of interconnected subsystems. An example is a chain of coupled pendulums. Each pendulum is coupled through springs to the pendulums before and after it in the chain. Each pendulum itself is a subsystem.

The non-uniformity in the system parameters, which is assumed slowly varying

and deterministic, can be tuned to produce desired scattered wave fields. With this objective in mind, we obtain an analytic solution for wave propagation along repetitive systems, asymptotic in the slowness of the variation of the system parameters. We consider systems governed by a second order finite difference equation and apply the WKB method allowing the index variable to be complex. This allows complex turning points to be considered. The coefficients of the difference equation are represented by their discrete Fourier modes. For complex turning points we obtain exponentially small reflection, a new result in the context of difference equations.

The asymptotic solution, besides revealing how the non-uniformity in the parameters affects wave propagation, furnishes an analytic expression for the system scattering matrix as a function of the system parameters. It also sheds light on the mechanism of localization phenomena for this class of repetitive systems. We also compare the asymptotic results with numerical experiments for large finite one-dimensional non-uniform chains of coupled pendulums.

In chapter 3 we consider the effects of geometrical and material non-uniformity on wave propagation along slender beams. Non-uniformity in beams arises either from manufacturing imperfections or by design, and can have a singular impact on the qualitative properties of the vibratory response of the beam. To describe the mechanism causing such large changes on the dynamics of the beam, we derived asymptotically in the first part of chapter 3 a simpler equation, in the form:

$$\chi_{ss} + Q(s)\chi(s) = 0.$$

The coefficient function $Q(s)$ is given by equation (3.52) herein in terms of the beam flexural rigidity, the mass per unit length and the tensile force applied to the beam. The equation is asymptotic to the non-uniformity of the beam, but under certain restrictions, viz. of having constant tension and a constant product of the beam mass per unit length and flexural rigidity, it is an exact governing equation for wave propagation along Bernoulli-Euler beams and it has a Helmholtz-like form. Then, inverse scattering methods for the Helmholtz equation can be used to design the

beam non-uniformity such that the scattered wave field has desired properties, like small or no wave transmission for a desired range of wave frequencies. The behavior of the equation is systematically explored and illustrated through numerical results. An example of a design problem is given. We design the beam non-uniformity such that functional relation between the reflection coefficient and the wave length of the incident wave is prescribed.

In the second part of chapter 3 we consider slender beams with slowly varying non-uniformity with respect to the length scale of the incident wave disturbance. We construct a high order asymptotic solution for the scattering of mono-chromatic waves by the beam non-uniformity. The asymptotic method used is the WKB method, which is basically a wave refraction theory, but wave reflection and wave mode conversion were captured by taking into account the effects of the turning points and singularities, like poles. At the turning points and singularities, the WKB solutions are not valid and a local analysis is needed to continue the asymptotic solution through the turning points, which are the branch points of the uniform system dispersion relation¹, assumed locally valid in the presence of slowly varying non-uniformity. At the branch points of the dispersion relation two or more wave numbers coalesce. The continuation of the asymptotic solutions through a turning point or singularity is called the connection problem.

The main objective of this part of chapter 3 was to understand the ideas proposed in Fuchs, Ko & Bers [22] and in Fuch, Bers & Harten [21] to perform the connection problem for high order differential equations using a local analysis, based on an approximate form of the original differential equation with order reduced to the number of wave numbers coalescing at the considered turning point. We apply these ideas to the asymptotic analysis performed of pseudo-differential equations in chapter 4. We did a local analysis based on second order differential equations at each turning points, but it did not provide a complete picture regarding wave mode conversion at some turning points. At these turning points we observed coalescence of four wave

¹The uniform system dispersion relation is now seen as a mapping between the space coordinate, now assumed complex, and the complex wavenumber space

numbers. So to handle this problem we decided to study first the connection problem for a Bernoulli-Euler beam with non-uniformity only in the flexural rigidity.

The Bernoulli-Euler beam model for monochromatic wave propagation is governed by a fourth order differential equation, what allowed us to obtain analytical expressions for the WKB waves and to perform the connection problem without using the ideas proposed in Fuchs, Ko & Bers [22] and in Fuch, Bers & Harten [21]. The experience obtained with this problem could be useful to extend the analysis performed in chapter 4, but we did not go further in this direction. We left it as future work.

In chapter 4 we consider the interaction of water waves with a slowly varying bottom topography with respect to the surface waves. We use the method of ordered operators to obtain pseudo-differential equations governing linear surface wave interaction with a slowly varying bottom. We consider one dimensional bottom topographies. We solve asymptotically the pseudo-differential equations. The asymptotic solution fails at points in complex coordinate variable plane. We take the effect of this points into account and obtain wave mode conversion (wave reflection is a particular case). We use the ideas proposed in Fuchs, Ko & Bers [22] and in Fuch, Bers & Harten [21] to perform the connection problem for the relevant turning points. Each turning point problem is modeled as a transfer matrix, relating the amplitude of the wave modes “before” and “after” the considered turning point. As a result, the wave interaction with the bottom non-uniform topography is modeled as a product of transfer matrices, and their product gives the whole system transfer matrix. We apply the asymptotic theory to analytic topographies and compare the asymptotic results for the scattering coefficients with results from direct numerical simulations.

1.3 Thesis contributions.

In this section we give an outline of the thesis contributions.

In chapter 2, we considered wave propagation along a non-uniform chain of repetitive systems of the size of many wavelengths of the incident wave, whose governing equation is a second order difference equation with variable coefficients. We used the

WKB method for second order difference equations to obtain an asymptotic solution for the wave propagation problem. We improved the WKB method for second order difference equations. Before this work, a procedure to continue (connection formula) the asymptotic solutions through complex turning points of the difference equation was not available. We derived connection formulas for pairs of complex conjugate first order turning points, for a cluster of a pair of complex or real first order turning points and for a complex or real second order turning point.

We also gave a systematic procedure to construct an asymptotic solution for a second order difference equation with a large number of turning points. This procedure is a blend between the WKB method and the method of transfer matrices.

For slowly varying non-uniformities relative to the wave, we were able to capture exponentially small reflection, a new result in the context of difference equations.

We also provided a qualitative understanding of localization phenomenon. For a uniform repetitive system, the dispersion relation provides the system passband. The larger the coupling, the larger the passband. Wave propagation happens only for frequencies inside the system passband. In the context of the WKB method, the uniform system dispersion relation is locally valid. Due to the non-uniformity along the chain, the size of the passband given by the uniform system dispersion relation varies along the chain. This leads to an effective passband for the whole chain. Waves with frequency outside the effective passband but inside the uniform system passband have exponentially small transmission and are strongly localized. Therefore, for whatever small disorder in the system, there will always exist a boundary layer close to both edges of the uniform system passband, and waves with frequency in these layers are strongly localized. For weak coupling, the effective passband may be very small or even cease to exist for a given disorder, and the strong wave localization observed can be explained by the WKB theory, which is a new result.

In chapter 3 we considered the effects of non-uniformity in the vibratory response of beams under tensile force. The contribution in this chapter is the derivation of a simple equation governing the beam vibratory response. This equation is asymptotic to the non-uniformity of the beam. As the steepness of the non-uniformity decreases,

the simple equation recovers the vibratory behavior predicted by the full governing equation. Under certain restrictions, viz. of having the constant tension and a constant product of the mass per unit length and flexural rigidity, the simple equation has a Helmholtz-like form and is an exact governing equation for the beam vibration response. These restrictions are not commonly met in practice, but such beams can be designed and built. Under these restrictions, inverse scattering methods for Helmholtz-like equations can be applied to design the beam non-uniformity such that wave transmission for a desired range of wave frequencies is minimized to a desired degree.

We also applied, in chapter 3, the WKB method to obtain an asymptotic solution of the vibration response on slender non-uniform beams. We consider non-uniformities with complex first turning points and poles. We obtained connection formulas for these turning points and poles. We use the standard technique (match near field solution with far field solution), but we obtained high order asymptotic approximations for all the scattering coefficients, which is the contribution of the second part of chapter 3.

In chapter 4 we applied a high order WKB for linear surface wave propagation over a non-uniform topography. The WKB method is a wave refraction theory and do not predict wave reflection or wave mode conversion, unless the effects of turning points are taking into account. We took the presence of turning points into account and obtained wave mode conversion. Wave reflection is a particular case of wave mode conversion. Wave reflection for surface waves in the context of the WKB method is not a new result in the literature (Meyer [46]), but wave mode conversion between the propagating modes and the evanescent modes, as we obtained in chapter 4, is a new result. The contribution of chapter 4 is a more complete asymptotic representation of the wave field for linear surface waves interacting with a non-uniform bottom topography.

Chapter 2

Asymptotic Analysis of Wave Propagation along Weakly Non-uniform Repetitive Systems.

A one-dimensional repetitive system is a chain of interconnected subsystems. The subsystems can be identical to each other (uniform repetitive system) or they may vary among each other (non-uniform repetitive system). Our aim is to study wave propagation along repetitive systems consisting of single degree of freedom subsystems, each interacting with its two nearest neighbors only. The systems are assumed to have a finite non-uniform part embedded in an infinite uniform part. The non-uniformity, i.e. disorder, is assumed slowly varying and deterministic. An example of this kind of repetitive system is a one-dimensional chain of coupled pendulums. The subsystem is a pendulum coupled with its two nearest neighbors through springs.

Wave propagation along uniform repetitive systems is possible only for continuous ranges of frequencies called passbands. For wave frequencies in the passband, energy can be transmitted through the system without attenuation, while no energy is transmitted at frequencies that lie outside the system passband. Therefore, we just consider wave frequencies in the system passband. We consider one-degree of freedom subsystems, and hence the system dispersion relation relates the wave frequency with two wave numbers of equal magnitude and opposite sign, which means a left and a

right propagating wave mode. A comprehensive discussion on wave propagation on more complex uniform repetitive systems is given in Brillouin [11].

When non-uniformity, i.e. deterministic disorder, is present along a repetitive system, the passband may become narrow, discrete or even disappear. If non-uniformity is present in a long enough part of a repetitive system and distributed in a random fashion, localization phenomena may show up as mentioned in chapter 1. We describe localization effects as follows. For wave frequencies inside the uniform system passband, the wave disturbance may show an exponential decay as it travels along the system. As a result, we have strong, or even complete reflection of the wave disturbance incident in the non-uniform part of the system, and the wave disturbance stays localized in space close to the beginning of the non-uniform part of the system. Another feature of the system behavior when localization phenomena are present, is the disproportionately large sensitivity of the system response (shape of natural modes and distribution of natural frequencies) with respect to small variation in the disorder of the system parameters, as pointed out by Pierre [53], Triantafyllou & Triantafyllou [65] and others. This large sensitivity with respect to the system parameters opens the possibility for the design of the non-uniformity of the system parameters to minimize wave transmission, or to allow perfect transmission at desired frequencies, depending upon the intended application. These design problems are one of the motivations for the work in this chapter, since an analytical expression for the system scattering matrix, even in an asymptotic sense, is a powerful tool to handle those design problems.

Slowly varying non-uniformity allows us to use the WKB method for second order finite difference equations. We focus on very large one-dimensional systems, consisting of hundreds of subsystems. Second order difference equations can be reduced to canonical forms. We chose the canonical form

$$z_{j+1} - 2z_j + z_{j-1} + Q_j z_j = 0, \quad (2.1)$$

where j is the independent variable and Q_j is a non-uniform sequence that varies

slowly with j . Since Q_j is finite, we assume that we can write it as a finite Fourier series. This allow us to extend Q_j for non-integer values of j . We allow j to be a complex variable, which opens the possibility to consider complex turning points, which, to the author's knowledge, were not considered in the literature before. We assume that the sequence Q_j does not have poles or any essential singularity, except at infinity. The WKB method for difference equations is applied similarly as for differential equations. The difference lays in the fact that for difference equations there are two turning point conditions, instead of only one, for differential equations. For pairs of complex conjugate turning points, the WKB method predicts an exponentially small reflection, which was known for differential equations, but is a new result for difference equations.

In the next section we give an outline of previous work on the WKB method for difference equations. Previous work on localization phenomena and on the design problems mentioned above was given, respectively, in the sections 1.1.1 and 1.1.3 of chapter 1. In section three, we consider an example of a repetitive system, namely a chain of coupled pendulums. The non-uniformity appears in the pendulums length and it is restricted to a finite part of the chain. We give the governing equation for wave propagation and reduce it to the canonical form (2.1). In section four, we discuss the WKB method for second order difference equations and give an asymptotic approximation for the system transfer matrix, that relates the amplitude of the incident and reflected waves with the amplitude of the transmitted wave for the whole non-uniform part of the system. We also discuss localization phenomena through the optics of the WKB method. In section five, we apply our asymptotic theory to simple non-uniformity configurations. We compare the asymptotic results with direct numeric simulation to access the performance of the asymptotic theory.

2.1 PREVIOUS WORK.

Previous work on the localization phenomena and on the design of the system disorder to minimize wave transmission were described, respectively, in sections 1.1.1 and 1.1.3

of chapter 1.

The application of the WKB method to second order finite difference equations dates back as early as 1949, according to [9], which gives an account of the literature about WKB method for difference equations before 1977. Early applications were usually in quantum mechanics. Braun [9] proposed a new method to obtain the Liouville-Green formulas and analyzed the matching condition at both simple and singular real turning points. Wilmott [67] considered also matching conditions for simple real turning points. Error bounds for the Liouville-Green formula for the discrete WKB method are discussed in [61] and [62]. The most recent applications of the discrete WKB method are asymptotic for orthogonal polynomials with slowly varying coefficients ([61] and [23]). Costin & Costin [17] considered recurrence relations of large finite-order. They use a matrix formulation and develop connection formulas for simple turning points. Instead of solving an approximate difference equation in the neighborhood of a turning point, they approximate the recurrence relation by a differential equation and obtained connection formulas using this approach.

2.2 ONE-DIMENSIONAL CHAIN OF COUPLED PENDULA.

We consider a one-dimensional chain of coupled pendulums where each pendulum is coupled to its two nearest neighbors by linear springs of constant k . We assume that the length of the pendulums is non-uniform for a finite part of the chain. The mass of the j -th pendulum m and its length is given as $(1 + \varepsilon_j)l$, where l is a reference length. The mass m and the spring constant k do not vary along the chain. The governing equation for the j -th pendulum is a result of the balance of the inertia force with gravity force and the forces applied by the two nearest neighbors. In non-dimensional form we obtain

$$\ddot{z}_j + \frac{1}{1 + \varepsilon_j} z_j - R [z_{j+1} - z_j] + R [z_j - z_{j-1}] = 0 \quad (2.2)$$

where $R = \frac{kl}{mg}$ is the coupling parameter and g is the acceleration of gravity. The first term in (2.2) corresponds to the inertia force, the second corresponds to the gravity force, the third term is the force applied by pendulum $j+1$, and the last term is the force applied by pendulum $j-1$. We consider a monochromatic wave of frequency ω incident on the non-uniform part of the chain. In this case the time dependence is given by

$$\exp(i\omega t) \quad (2.3)$$

and the governing equation (2.2) assume the form

$$-R z_{j+1} + \left(\frac{1}{1 + \varepsilon_j} + 2R - \omega^2\right) z_j - R z_{j-1} = 0 \quad (2.4)$$

which is the governing equation for mono-chromatic wave propagation along the chain of pendulums. Next, we illustrate how to reduce a general second order difference equation to the canonical form (2.1). We consider a general equation, as follows

$$a_j(x, y)z_{j+1} + b_j(x, y)z_j + c_j(x, y)z_{j-1} = 0, \text{ for } j_0 < j < j_0 + M, \quad (2.5)$$

and for $j < j_0$ and for $j > j_0 + M$, the coefficients in (2.5) are constants. For $j_0 \leq j \leq j_0 + M$, $a_j(x, y)$, $b_j(x, y)$ and $c_j(x, y)$ are the non-constant coefficients, which depend on the parameters x and y , and do not have any singularity, besides at infinity. The sequences in (2.5) have size M . To reduce equation (2.5) to the desired canonical form, we consider the change of variable

$$z_j = \left(\prod_{l=j_0-1}^j \beta_l \right) \alpha_j y_j, \text{ for } j_0 - 1 < j < j_0 + M, \quad (2.6)$$

where y_j is the new dependent variable. The sequences β_l and α_j are defined in such a way that the general recurrence relation (2.5) is reduced to the desired canonical form. The elements of the sequences β_l and α_j are given in terms of the sequences a_j

and c_j as follows.

$$\alpha_j = \frac{1}{c_{j+1}}, \text{ for } j \geq j_0 - 1 \quad (2.7)$$

$$\begin{aligned} \beta_j &= 1, \text{ for } j_0 - 1 \leq j \leq j_0 \\ \beta_{j+1} &= \frac{1}{a_j \beta_j \alpha_{j+1}}, \text{ for } j \geq j_0 \end{aligned} \quad (2.8)$$

After we apply the change of variable (2.6) to equation (2.5), we obtain the desired canonical form (2.1), where the sequence Q_j is defined in terms of the sequences a_j, b_j and c_j , as given below.

$$\begin{aligned} Q_j &= \frac{b_j}{c_{j+1}} + 2, \text{ for } j = j_0 \\ &= \frac{b_j}{a_{j-1}} + 2, \text{ for } j_0 + 1 \leq j < j_0 + M \end{aligned} \quad (2.9)$$

For the chain of coupled disordered pendulums the terms of the sequences a_j, b_j and c_j are given as

$$\begin{aligned} a_j &= -R \\ b_j &= \frac{1}{1 + \varepsilon_j} + 2R - \omega^2 \\ c_j &= -R \end{aligned} \quad (2.10)$$

In the canonical form (2.1), the term Q_j assumes the form

$$\begin{aligned} Q_j &= \frac{1}{-R} \left(\frac{1}{1 + \varepsilon_j} + 2R - \omega^2 \right) + 2, \text{ for } j = j_0 \\ &= \frac{1}{-R} \left(\frac{1}{1 + \varepsilon_j} + 2R - \omega^2 \right) + 2, \text{ for } j \geq j_0 + 1 \end{aligned} \quad (2.11)$$

Equation (2.11) is derived directly and simply from equations (2.1) and (2.4). The general analysis contained in equations (2.5) to (2.9) was carried out because the asymptotic theory below was designed to handle general second order difference equations.

In the next section we give an outline of the WKB method for second order difference equations.

2.3 THE WKB METHOD FOR SECOND ORDER DIFFERENCE EQUATIONS.

We give an outline of the WKB method for second order difference equations applied to our canonical form (2.1). As mentioned before, the WKB method for 2nd order difference and differential equations have many features in common, but the main difference lies in the fact that for difference equations we have two turning point conditions, instead of only one as in the case of differential equations. The first step is to obtain the Liouville-Green formulae. The asymptotic solution of (2.1) is a linear combination of the Liouville-Green functions.

2.3.1 LIOUVILLE-GREEN FUNCTIONS.

To obtain the Liouville-Green functions we proceed in a way similar to [9]. Through a change of variable, we transform the canonical form (2.1) into a non-linear difference equation. We solve it approximately through an iterative process. The solution of the first iteration gives the Liouville-Green functions. The change of variable to consider is

$$z_j = \prod_{l=l_0}^j u_l, \tag{2.12}$$

and if we substitute (2.12) in the canonical form (2.1), we obtain a discrete version of the Riccati equation for u_l , as follows

$$u_l^2 \left(\frac{u_{l+1}}{u_l} \right) + (Q_l - 2)u_l + 1 = 0 \quad (2.13)$$

We can solve (2.13) as a quadratic equation, and the natural logarithm of u_l follows

$$\ln u_l = -\frac{1}{2} \Delta \ln u_l + \ln \left(-D_l E_l \pm [(D_l E_l)^2 - 1]^{1/2} \right) \quad (2.14)$$

where the quantities D_l and E_l are defined as

$$D_l = \left(\frac{u_l}{u_{l+1}} \right)^{1/2} \quad (2.15)$$

$$E_l^2 = \frac{(Q_l - 2)^2}{4} \quad (2.16)$$

and Δ is the forward difference operator. The last term in (2.14) is the logarithm of a complex number with unitary modulus and phase θ_l , which can be written as

$$\theta_l = \arccos \left\{ -E_l \exp \left(-\frac{1}{2} \Delta \ln u_l \right) \right\} \text{ for } |D_l E_l| < 1 \quad (2.17)$$

$$\theta_l = -i \cosh^{-1} \left\{ -E_l \exp \left(-\frac{1}{2} \Delta \ln u_l \right) \right\} \text{ for } |D_l E_l| > 1 \quad (2.18)$$

and equation (2.14) assume the form

$$\ln u_l = -\frac{1}{2} \Delta \ln u_l \pm i \theta(l), \quad (2.19)$$

From (2.19) we develop an iterative scheme to compute approximations for u_l . We assume that the departure of $\frac{u_l}{u_{l+1}}$ from unity is small, so equation (2.19) simplifies by eliminating the forward difference of the logarithmic term. The expressions for θ_l , equations (2.17) and (2.18), also simplify by setting the forward difference of the

logarithmic term equal to zero. Then, the zero-th order approximation follows

$$\ln u_l^{(0)} = \pm i\theta_l \quad (2.20)$$

and here we redefine θ_l , as follows

$$\begin{aligned} \cos(\theta_l) &= E_l \text{ for } E_l \text{ real and } |E_l| < 1 \\ \cosh(i\theta_l) &= E_l \text{ for } E_l \text{ real and } |E_l| > 1 \end{aligned} \quad (2.21)$$

$$\theta_l = -i \ln(-E_l + \sqrt{E_l^2 - 1}) \text{ for } E_l \text{ complex} \quad (2.22)$$

where the last expression in (2.21) is the extension of the arccos function to the whole complex j plane, compatible with the second expression in (2.21). Higher order approximation of $\ln u_l$ are obtained by iterating equation (2.19), then $\ln u_l^{n+1}$ is given by

$$\ln u_l^{(n+1)} = -\frac{1}{2} \triangle \ln u_l^{(n)} \pm i \theta_l^{(n+1)} \quad (2.23)$$

where $\theta_l^{(n)}$ is the Taylor series expansion of equation (2.17), if $|E_l| < 1$, or (2.18) if $|E_l| > 1$, with respect to the forward difference of the logarithm of u_l , truncated at the n -th order term. To obtain the Liouville-Green functions, we need only to iterate to first order. To obtain $\ln u_l^{(1)}$ we need first to obtain $\theta_l^{(1)}$, which follows

$$\theta_l^{(1)} = \theta_l \mp \frac{i}{2} \cot \theta_l \triangle \theta_l \text{ for } |E_l| < 1 \quad (2.24)$$

$$= \theta_l \mp \frac{1}{2} \coth(i \theta_l) \triangle \theta_l \text{ for } |E_l| > 1 \quad (2.25)$$

Next, we substitute (2.24) and (2.25) with the zero-th order approximation (2.20) in the (2.23), and we obtain the first order approximation for $\ln u_l^{(1)}$, as follows

$$\ln u_l^{(1)} = \mp \frac{i}{2} \triangle \theta_l \pm i \left\{ \theta_l \pm \frac{i}{2} \triangle \theta_l \cot \theta_l \right\} \text{ for } |E_l| < 1 \quad (2.26)$$

and

$$\ln u_l^{(1)} = \mp \frac{i}{2} \Delta \theta_l \pm i \left\{ \theta_l \pm \frac{i}{2} \Delta \theta_l \coth(i \theta_l) \right\} \text{ for } |E_l| > 1 \quad (2.27)$$

Finally, to obtain the first order approximation for z_j , we substitute (2.26) and (2.27) in the change of variable (2.12), and we use the Euler-Maclaurin summation formula. We disregard terms of $O(\Delta)$ and higher. The final result is

$$z_j^{(1)} = D(\sin \theta_j)^{-1/2} \exp \left\{ \pm i \int_{j_0}^j \theta(l) dl \right\} \text{ for } |E_l| < 1 \quad (2.28)$$

and

$$z_j^{(1)} = D(\sinh(i \theta_j))^{-1/2} \exp \left\{ \pm i \int_{j_0}^j \theta(l) dl \right\} \text{ for } |E_l| > 1 \quad (2.29)$$

which are the Liouville-Green functions for the canonical equation (2.1). The quantity $\theta(l)$ in equations (2.27) and (2.28) is defined by (2.21). Notice that $\theta(l)$ given by (2.21) is the local wavenumber. In the context of the assumptions of the WKB method, the uniform system dispersion relation, actually given by (2.21), is locally valid. This implies that the passband given by the uniform system dispersion relation varies as we go along the chain. As a result, in the context of the WKB assumptions energy cannot propagate without attenuation along the chain for every wave frequency inside the uniform system passband. Now, energy can propagate without attenuation only for a subset of the uniform system passband, so we have an effective passband, which maybe an interval or/and some discrete frequencies, the frequencies of perfect transmission. To illustrate the concept of the effective passband, we assume that the non-uniformity varies along the chain in the interval $(-\varepsilon, \varepsilon)$. In the figure (2-1) below, we plot the two edges of the uniform system passband (wavenumber θ equals 0 and π) against values of the non-uniformity (variation of the pendulum length) for the chain of coupled pendulums. When these lines cross the line of a constant value of the non-uniformity, the interval so defined is the system passband for that specific value of non-uniformity. The intersection of the pass bands for non-uniformities $\pm \varepsilon$

defines the effective passband.

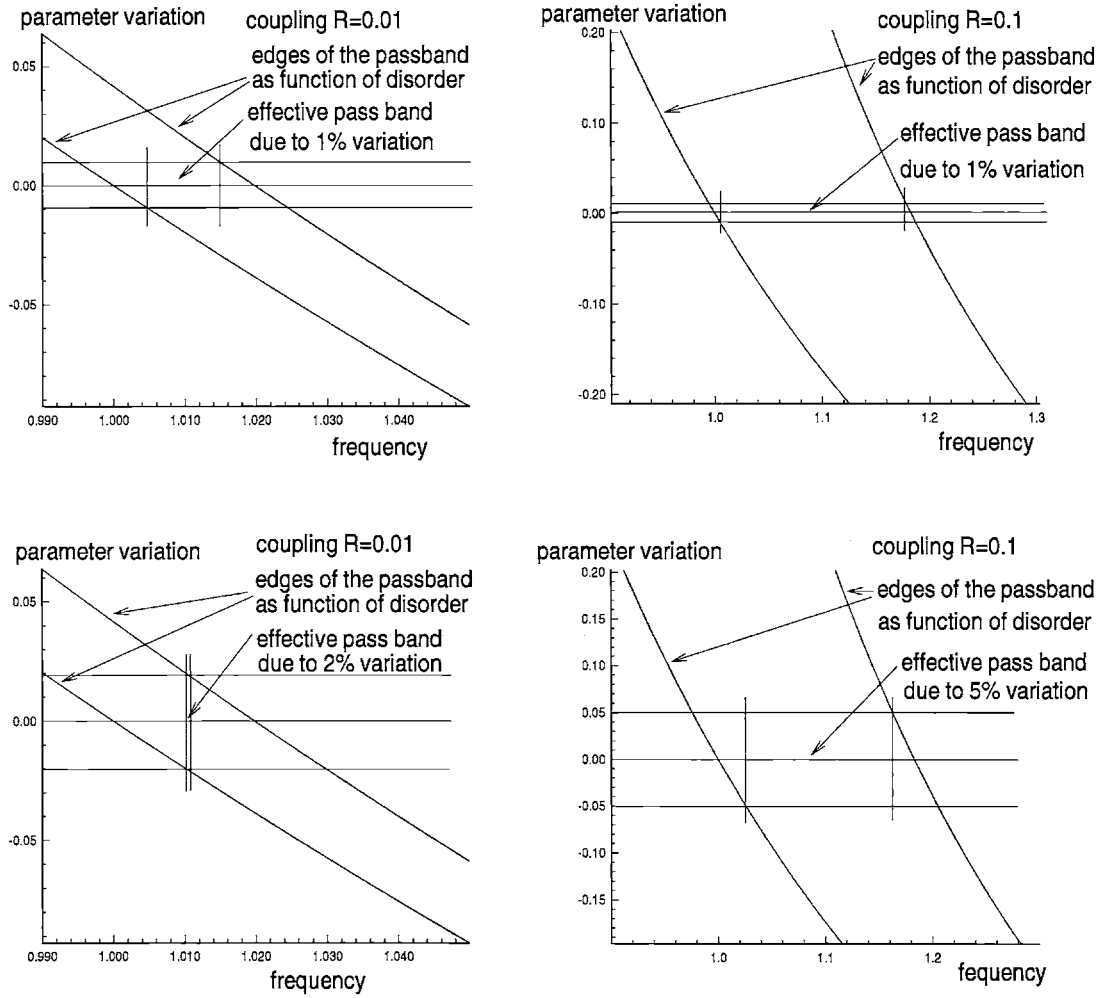


Figure 2-1: Examples of effective pass band for some values of the coupling parameter R and for some values of the range of the variation of the length of the pendula (%) along the chain.

For wave frequencies outside the effective passband, but inside the uniform system passband, the wave disturbance has an exponential decay along the chain length. The width of the effective passband depends on the strength of the coupling parameter and on the amplitude ε of the non-uniformity variation, but for any value of coupling, and no matter how small the non-uniformity is, there is always a boundary layer at both edges of the passband. For wave frequencies inside these boundary layers, the

asymptotic theory predicts that the wave disturbance always shows an exponential decay along the length of the chain, staying localized in space. Then for frequencies inside these boundary layers, we have strong localization. For frequencies inside the effective passband, we may have weak localization due to the multiple backscattering which takes place along the system. For long (many wavelengths), slowly varying non-uniform waveguides, the multiple backscattering along the many turning points of the system may be a good description for the multiple backscattering phenomenon which takes place along the system. Therefore, when this approximation is valid, the WKB approach described below should be able to predict weak localization effects.

As $E_l \rightarrow +1(-1)$, the wave number θ_l approaches $\pm 0(\pm\pi)$ and the Liouville-Green functions become unbounded, according to (2.28) and (2.29), and are no longer valid. Points in the complex j plane where $E_j = \pm 1$ are called turning points.

2.3.2 TURNING POINT CONDITIONS.

As pointed out in the last section, the first (second) turning point condition is $E_j = -1(E_j = 1)$. In terms of the sequence Q_j of the canonical equation (2.1), the turning point conditions are given as:

$$Q_j - 2 = \mp 2 \rightarrow \begin{cases} Q_j = 0 & \text{first condition} \\ Q_j = 4 & \text{second condition} \end{cases} \quad (2.30)$$

For a given sequence Q_j , we describe in the Appendix B a procedure on how to obtain the turning points in the complex j plane. Let us denote a turning point that satisfies the first (second) turning point condition (given in (2.30)) as $j_{*1}(j_{*2})$. In the following, the index k tagged to any constant or as a superscript denotes the kind of turning point, i.e. whether it satisfies the first ($k=1$) or second ($k=2$) turning point condition (2.30). We consider real and complex first order turning points and second order turning points, which result from the coalescence of a pair of real or complex first order turning points. Higher order and singular turning points may occur, but for the applications of the WKB method we have in mind, they seldom appear. We

impose some restrictions on the sequence Q_j , which follows:

- Q_j is real for real values of j .
- Q_j is a finite sequence.
- The extension of sequence Q_j to the complex j plane has no singularity besides at infinity, so for almost the whole complex j plane, the extension of Q_j is an analytic function.

As for differential equations, we can define the Stokes lines. For second order difference equations they are defined as the contours C in the complex j plane such that

$$\Im\left\{i \int_R^j \theta(l) dl\right\} = 0 \text{ with } j \in C, \quad (2.31)$$

and where R is a turning point that satisfies the first or second turning point condition. The Stokes lines given by (2.31) define regions in the complex j plane (see figure (2-2)).

The asymptotic solution of (2.1) given by the WKB method is a linear combination of the Liouville-Green functions. As we cross a Stokes line, the appropriate linear combination of the Liouville-Green functions changes, what is called in the literature the Stokes phenomena. Boundary conditions define the appropriate linear combination of the Liouville-Green functions in a region of the complex j plane bounded by Stokes lines, and if we want to know the asymptotic solution in another region of the complex j plane, we have to figure out how the linear combination of the Liouville-Green function changes as we cross Stokes lines.

To solve this question we develop connection formulae. If we want to connect the solution in two different regions, separated by a Stokes line, we need, first, to see which turning point is common to these two regions, and the necessary connection formula is obtained by solving an approximate form of (2.1), valid in the neighborhood of the turning point in question. Examples of the turning points configurations we are going to deal with are given in figure 2-2.

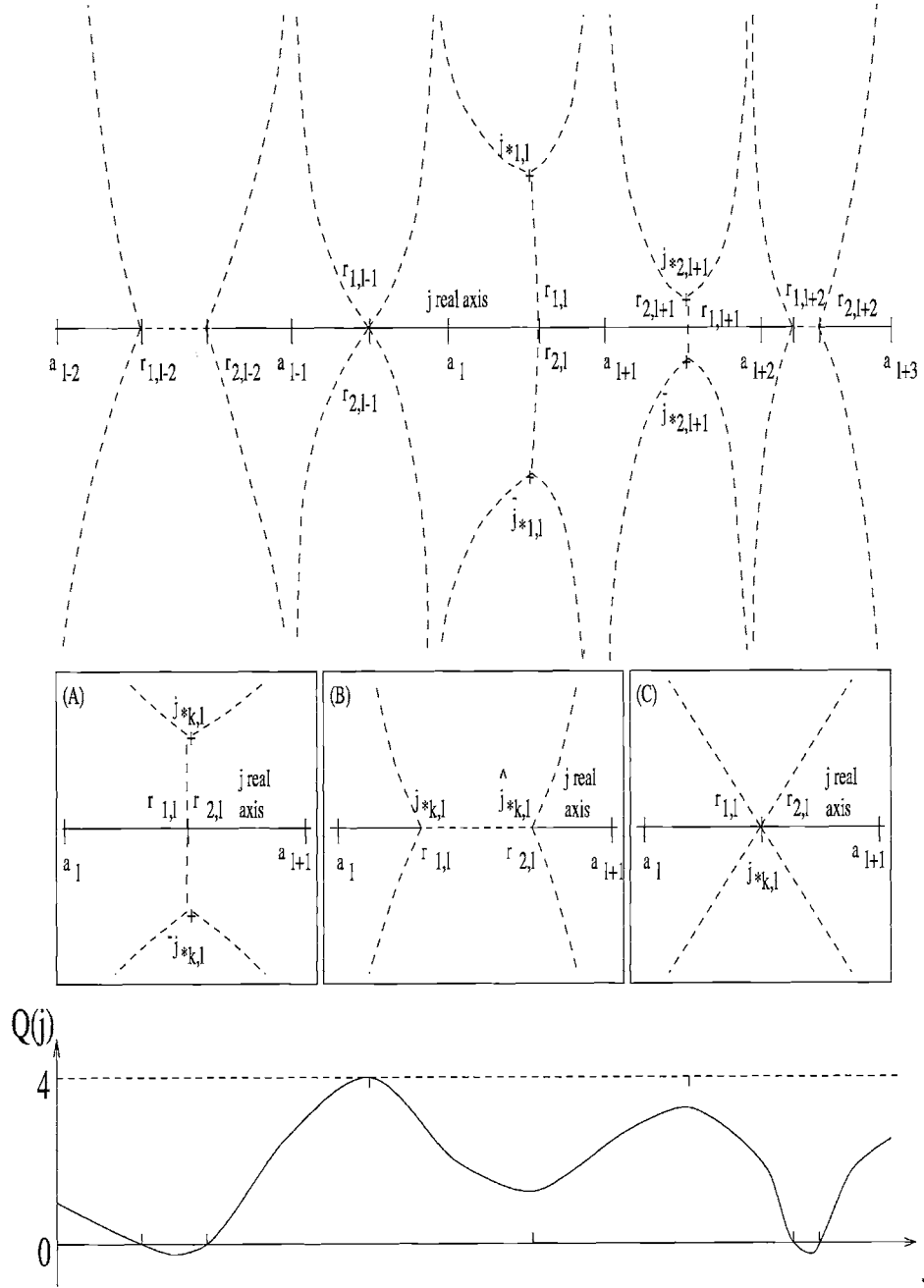


Figure 2-2: Examples of turning point problems (TPP j , $j=1, \dots, 5$) indicated by Stokes lines (---) and boundary points a_l . TPP1: $a_l < j < a_{l+1}$ (pair of complex conjugate turning points $j_{*1,l}$ and $\bar{j}_{*1,l}$, see details in box (A)). TPP2: $a_{l-2} < j < a_{l-1}$ (pair of real turning points $j_{*1,l-2}$ and $\hat{j}_{*1,l-2}$, details box (B)). TPP3: $a_{l+1} < j < a_{l+2}$ (pair of almost coalescent complex conjugate turning points $j_{*2,l+1}$ and $\bar{j}_{*2,l+1}$). TPP4: $a_{l+1} < j < a_{l+2}$ (pair of almost coalescent real turning points $j_{*1,l+2}$ and $\hat{j}_{*1,l+2}$). TPP5: $a_{l-1} < j < a_l$ (second order turning point $j_{2,l-1}$, details box (C)).

2.3.3 APPROXIMATE EQUATIONS.

In this section we give the approximate form of (2.1) in the neighborhood of a real or complex first order turning point, and in the neighborhood of a pair of almost coalescing real or complex first order turning points. The case of a second order turning point is the limiting case for almost coalescing turning points.

2.3.3.1 Approximate Equation for First Order Turning Points.

First, let us discuss the case of real or complex first order turning points. Let us consider j_{*k} a first order turning point. The approximate equation is obtained by a Taylor series expansion of Q_j in (2.1) with respect to j_{*k} to first order. Then (2.1) assumes the form

$$z_{j+1} + (-1)^k 2z_j + z_{j-1} + \alpha_k (j - j_{*k}) z_j = 0 \quad (2.32)$$

where the value of k (1 or 2) is related to which turning point condition j_{*k} satisfies, as discussed above. The constant α_k is defined as

$$\alpha_k = \frac{dQ(j_{*k})}{dj} \quad (2.33)$$

The solution of the approximate equation (2.32) is given in terms of a contour integral, what is discussed in Appendix C.1.

2.3.3.2 Approximate Equation for Almost Coalescing Pairs of Turning Points.

Next, let us consider the case of a pair of almost coalescing real or complex turning points. Let us label this pair of complex conjugate (real) turning points as j_{*k} and $\bar{j}_{*k}$ (j_{*k} and $\hat{j}_{*k}$). For a pair of almost coalescing complex conjugate (real) turning

points, we can assume that their real part is the same, so we can write

$$\begin{aligned}
j_{*k} &= a_k + ib_k & (\bar{j}_{*k} &= a_k - b_k) \\
\text{and} & & & \\
\bar{j}_{*k} &= a_k - ib_k & (\hat{j}_{*k} &= a_k + b_k)
\end{aligned} \tag{2.34}$$

and we assume that $b_k \ll 1$. For real turning points $a_k = |j_{*k} + \hat{j}_{*k}|/2$ and $b_k = |j_{*k} - \hat{j}_{*k}|/2$. To obtain the approximate equation in this case, we expand Q_j in Taylor series with respect to j_{*k} and $\bar{j}_{*k}$ (j_{*k} and $\hat{j}_{*k}$) and disregard terms of $O((j - j_{*k})$ and higher. The terms that were not disregarded are now expanded with respect to $\pm i b_k (\pm b_k)$. We keep only quadratic terms, and then we sum both expansions and divide by two. The approximate equation for this case follows

$$z_{j+1} + (-1)^k 2z_j + z_{j-1} + \psi_k [(j - a_k)^2 \pm (b_k)^2] z_j = 0 \tag{2.35}$$

with the constant ψ_k given as

$$\psi_k = \frac{1}{2} \frac{d^2 Q(a_k)}{dj^2} \tag{2.36}$$

and the $+$ ($-$) sign in the bracketed term of (2.35) is for the case of a pair of almost coalescing complex conjugate (real) turning points. The approximate difference equation (2.35) appears in a modified form as the recurrence relation of coefficients of the trigonometric series expansion of the solution of the Mathieu equation. This does not help us in finding a suitable closed form solution for these difference equations. If we try contour integral solutions, we end up trying to solve a second order differential equation that can be transformed to a Mathieu equation, which does not help us either.

We use a different approach. We approximate (2.35) by a second order differential equation. The second order finite difference operator can be approximated by the second order differential operator in the context of the WKB method assumptions. For the first turning point condition ($k=1$) the approximate differential equation of

(2.35) follows:

$$\frac{d^2 y(j)}{dj^2} + |\psi_1|[(j - a_1)^2 \pm (b_1)^2]y(j) = 0 \quad (2.37)$$

For the second turning point condition, we make the change of the dependent variable $y_j = \exp(i\pi j)w_j$. For the dependent variable w_j the approximate differential equation of (2.35) with the minus sign is

$$\frac{d^2 w(j)}{dj^2} + |\psi_2|[(j - a_2)^2 \pm (b_2)^2]w(j) = 0 \quad (2.38)$$

Let us consider the change of the independent variable

$$\exp(-i\frac{\pi}{4})(4|\psi_k|)^{1/4}(j - a_k) = x \quad (2.39)$$

which allows us to write equations (2.37) and (2.38), respectively, as

$$\frac{d^2 y}{dx^2}(x) = (d_1 - \frac{1}{4}x^2)y(x) \quad (2.40)$$

$$\frac{d^2 w}{dx^2}(x) = (d_2 - \frac{1}{4}x^2)w(x) \quad (2.41)$$

with $d_k = \pm i\sqrt{\mu_k \frac{(b_k)^2}{4}}$ (plus sign for complex conjugate turning points and the minus sign for pairs of real turning points), and we define $\mu_k = 4|\psi_k|$. Both equations are related to the parabolic cylinder equation. The functions $D_\nu(x)$ and $D_\nu(-x)$ give a satisfactory solution for the parabolic cylinder equation, when $\nu \notin N$, in the form given in pages 531 and 532 of Bender & Orszag [4]. The solution of equations (2.37) and (2.38) in terms of parabolic cylinder functions are

$$\begin{aligned} y(j) = & A_1 D_{\pm i\frac{\sqrt{\mu_1}}{4}(b_1)^2 - \frac{1}{2}}((\mu_1)^{1/4}e^{-i\pi/4}(j - a_1)) \\ & + B_1 D_{\pm i\frac{\sqrt{\mu_1}}{4}(b_1)^2 - \frac{1}{2}}(-(\mu_1)^{1/4}e^{-i\pi/4}(j - a_1)) \end{aligned} \quad (2.42)$$

$$\begin{aligned}
w(j) = & A_2 D_{\pm i \frac{\sqrt{\mu_2}}{4} (b_2)^2 - \frac{1}{2}} ((\mu_2)^{1/4} e^{-i\pi/4} (j - a_2)) \\
& + B_2 D_{\pm i \frac{\sqrt{\mu_2}}{4} (b_2)^2 - \frac{1}{2}} (-(\mu_2)^{1/4} e^{-i\pi/4} (j - a_2))
\end{aligned} \tag{2.43}$$

For the case of a second order turning point, we proceed the same way as we did for almost coalescing pairs of first order turning points. We obtain an approximate difference equation of the form (2.35), but with $b_k = 0$ and a_k as the second order turning point. We again approximate the second order difference operator in our approximate difference equation by a second order differential operator, which leads to differential equations of the form (2.40) and (2.41), but with $d_k = 0$. Therefore, for a second order turning point, the solution of the approximate equation is given by (2.42) or (2.43) with b_k set to zero. In Appendix C.2 we give the asymptotic expansions of the solutions of the approximate equations for pairs of almost coalescing real or complex turning points. These asymptotic expansions are used to obtain the connection formulae for these turning point problems.

2.3.4 CONNECTION FORMULAE.

Our objective is to study wave propagation along a repetitive system with slowly varying non-uniformity. Waves propagate only on the real axis of the j complex plane. Therefore, our concern is to see how a given combination of the Liouville-Green functions valid in an interval of the real axis changes as we cross a Stokes line or a real turning point. With this point in mind, we consider the following turning points problems

- Pair of complex conjugate first order turning points (TPP1).
- Pair of real first order turning points (TPP2).
- Pair of almost coalescing complex conjugate first order turning points (TPP3).
- Pair of almost coalescing real first order turning points (TPP4).
- Second order real turning point (TPP5), which is a particular case of the two cases above.

Our strategy is the following. First, we solve each of these turning point problems alone. We obtain connection formulas for each of them. Second, we use these connection formulas to build the asymptotic solution for repetitive systems with the turning point problems listed above. This will be postponed to the section where we discuss on how to build the asymptotic approximation of whole system transfer matrix. An example picture of a sequence of turning point problems with the turning point problems listed above is given in figure 2-2. In the following, if we refer to a specific turning point problem contained in the list of turning points problems above, we use the symbol TTPj ($j = 1, \dots, 5$).

Here, let us concentrate in obtaining the connection formulae for the turning point problems listed above. We consider two boundary conditions. The first one is a wave incident from left of the the turning point. Let us consider points $a_l < a_{l+1}$ along the real axis of the j plane. At points a_l and a_{l+1} the boundary conditions are applied. We consider also the reference points $r_{1,l}$ and $r_{2,l}$, such that $a_l < r_{1,l} \leq r_{2,l} < a_{l+1}$ (see boxes A, B and C on figure 2-2). The index l used below specifies that we are dealing with the l -th turning point problem in a sequence of turning point problems, as illustrated by an example in figure 2-2. The other boundary condition to consider is an incident wave from the right of the turning point. Let us formulate these boundary conditions in terms of the Liouville-Green functions. For left incidence, we have

- For $a_l < j < r_{1,l}$, we have an incident plus a reflected wave.

$$\sqrt{\sin(\theta_j)} z_j \sim A \exp(-iI(a_l, r_{1,l}) - i \int_{r_{1,l}}^j \theta(n) dn) + R^- A \exp(iI(a_l, r_{1,l}) + i \int_{r_{1,l}}^j \theta(n) dn) \quad (2.44)$$

- For $r_{2,l} < j < a_{l+1}$, we have a transmitted wave.

$$\sqrt{\sin(\theta_j)} z_j \sim T^- A \exp(-i \int_{r_{2,l}}^j \theta(n) dn) \quad (2.45)$$

where $I(a,b)$ is defined as

$$I(a,b) = \int_a^b \theta(n) dn, \quad (2.46)$$

and the coefficients R^- and T^- are, respectively, the reflection and transmission coefficients for left incidence.

For right incidence we have

- For $a_l < j < r_{1,l}$, we have a transmitted wave.

$$\sqrt{\sin(\theta_j)} z_j \sim T^+ A \exp(i \int_{r_{2,l}}^j \theta(n) dn) \quad (2.47)$$

- For $r_{2,l} < j < a_{l+1}$, we have an incident plus a reflected wave.

$$\sqrt{\sin(\theta_j)} z_j \sim A \exp(i I(a_{l+1}, r_{2,l}) + i \int_{r_{2,l}}^j \theta(n) dn) + R^+ A \exp(-i I(a_l, r_{2,l}) - i \int_{r_{2,l}}^j \theta(n) dn) \quad (2.48)$$

where the coefficients R^+ and T^+ are, respectively, the reflection and transmission coefficients for right incidence. For each of the turning point problems, the connection formulae are the expressions for the coefficients $R^\pm$ and $T^\pm$. In Appendix C.1, we obtain the expressions for the coefficients $R^\pm$ and $T^\pm$ for TPP1, and in Appendix C.2 we obtain the expressions for the coefficients $R^\pm$ and $T^\pm$ for cases TPP3, TPP4 and TPP5. For TPP2, we just refer the reader to the literature ([9] and [67]). In what follows, we just list and discuss the results obtained in Appendices C.1 and C.2.

2.3.4.1 Pair of Complex Conjugate First Order Turning Points.

For TPP1 (see box A in figure 2-2) and the boundary conditions given above, we describe the connection problem in Appendix C.1. Let us define the constant $\gamma = \frac{(-1)^k i}{\alpha_{k,l}}$. For this case we have that the reference points $r_{1,l}$ and $r_{2,l}$ are such that $r_{1,l} = r_{2,l} = r_l = \Re\{j_{*k,l}\}$ (real part of the pair of complex conjugate turning points). The connection formula follows.

- TPP1 with turning points that satisfies the first turning point condition.
 - Left incidence boundary condition.

$$R^- = -\exp(-i2I(r_l, \bar{j}_{*1,l}) - i2I(a_l, r_l) - i\frac{\pi}{2}), \text{ for } -\frac{\pi}{2} < \arg\{\gamma\} < \frac{\pi}{2} \quad (2.49)$$

$$T^- = \exp(-iI(a_l, r_l)), \text{ for } -\frac{\pi}{2} < \arg\{\gamma\} < \frac{\pi}{2} \quad (2.50)$$

where $I(a,b)$ is defined by (2.46). The integral $I(r_l, \bar{j}_{*1,l})$ is imaginary and negative, which implies that the reflection coefficient R^- is exponentially small. The integral $I(a_l, r_l)$ is real, so it gives a phase change for the reflection and transmission coefficients. The transmission coefficient has modulus equal unity. The incident wave suffers an exponentially small reflection, and the transmitted wave has a phase shift with respect to the incident wave.

- Right incidence boundary condition.

$$R^+ = -\exp(i2I(r_l, j_{*1,l}) - i2I(r_l, a_{l+1}) + i\frac{\pi}{2}), \text{ for } -\pi < \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } \frac{\pi}{2} < \arg\{\gamma\} \leq \pi \quad (2.51)$$

$$T^+ = \exp(-iI(r_l, a_{l+1})), \text{ for } -\pi < \arg\{\gamma_{1,l}\} < -\frac{\pi}{2} \text{ and } \frac{\pi}{2} < \arg\{\gamma\} \leq \pi \quad (2.52)$$

The reflection coefficient is exponentially small due to the integral $I(r_l, j_{*1,l})$ which is imaginary and positive.

- TPP1 with turning points that satisfy the second turning point condition.

– Left incidence boundary condition.

$$R^- = -\exp(-i2I(r_l, j_{*2,l}) - i2I(a_l, r_l) + i2\pi j_{*2,l} + i\frac{\pi}{2}), \text{ for } -\frac{\pi}{2} < \arg\{\gamma\} < \frac{\pi}{2} \quad (2.53)$$

$$T^- = \exp(-iI(a_l, r_l)), \text{ for } -\frac{\pi}{2} < \arg\{\gamma\} < \frac{\pi}{2} \quad (2.54)$$

The coefficient R^- is again exponentially small due to the difference between the integral $I(r_l, j_{*2,l})$ (imaginary and positive) and the term $2\pi j_{*2,l}$ (imaginary part positive). The transmission coefficient T^- has modulus equal unity.

– Right incidence boundary condition.

$$R^+ = \exp(i2I(r_l, \bar{j}_{*2,l}) - i2I(r_l, a_{l+1}) - i2\pi \bar{j}_{*2,l} - i\frac{\pi}{2}), \text{ for } -\pi < \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } \frac{\pi}{2} < \arg\{\gamma\} \leq \pi \quad (2.55)$$

$$T^+ = \exp(-iI(r_l, a_{l+1})), \text{ for } -\pi \leq \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } \frac{\pi}{2} < \arg\{\gamma\} < \pi \quad (2.56)$$

2.3.4.2 Pair of Real First Order Turning Points.

For TPP2 with the pair of real turning points denoted as $j_{*k,l}$ and $\hat{j}_{*k,l}$ (see box (B) of figure 2-2) and the boundary conditions given above, we just list results presented in the literature using our notation. The connection formula follows.

• Left incidence boundary condition.

$$R^- = \exp(-i2I(a_l, j_{*k,l}) + (-1)^k i\frac{\pi}{2} + i2\pi j_{*k,l} \delta_{2,k}) \quad (2.57)$$

$$T^- = \exp(-iI(a_l, j_{*k,l}) - iI(j_{*k,l}, \hat{j}_{*k,l})) \quad (2.58)$$

- Right incidence boundary condition.

$$R^+ = \exp(-i2I(\hat{j}_{*k,l}, a_{l+1}) - (-1)^k i \frac{\pi}{2} - i2\pi \hat{j}_{*k,l} \delta_{2,k}) \quad (2.59)$$

$$T^+ = \exp(-iI(\hat{j}_{*k,l}, a_{l+1}) - iI(j_{*k,l}, \hat{j}_{*k,l})) \quad (2.60)$$

2.3.4.3 Pair of Almost Coalescing First Order Complex Conjugate or Real Turning Points.

In this section we give the connection formula for the cases TPP3 and TPP4. For TPP3 the reference points $r_{1,l}$ and $r_{2,l}$ are such that $r_{1,l} = r_{2,l} = \Re\{j_{*k,l}\}$, and for TPP4 we have $r_{1,l} = j_{*k,l} < r_{2,l} = \hat{j}_{*k,l}$, with $j_{*k,l}$ and $\hat{j}_{*k,l}$ representing the pair of real turning points. The quantities b_k and μ_k were defined in section 4.3.2, and here we relabel them, respectively, as $b_{k,l}$ and $\mu_{k,l}$. The connection formulas for a given boundary condition does not change if the turning points satisfies the first or the second turning point condition. Therefore, we distinguish the connection formula with respect to the boundary condition used. In Appendix C.2, we discuss in detail how these connection formulae were derived. In the expressions below, the upper (lower) sign refers TPP3 (TPP4). The connection formula follows

- Left incidence boundary condition.

$$R^- = - \frac{\Gamma(\mp \frac{i}{4} \sqrt{\mu_{k,l}} (b_{k,l})^2 + \frac{1}{2})}{\sqrt{2\pi}} 2^{\pm i \frac{3}{2} \sqrt{\mu_{k,l}} (b_{k,l})^2} \mu_{k,l}^{\mp \frac{i}{8} \sqrt{\mu_{k,l}} (b_{k,l})^2} \exp(\mp \frac{\pi}{8} \sqrt{\mu_{k,l}} (b_{k,l})^2 + i \frac{\pi}{2} - i2I(a_l, r_{1,l}) + i2\pi r_{1,l} \delta_{k,2}) \quad (2.61)$$

$$T^- = \frac{\Gamma(\mp \frac{i}{4} \sqrt{\mu_{k,l}} (b_{k,l})^2 + \frac{1}{2})}{\sqrt{2\pi}} 2^{\pm i \frac{3}{2} \sqrt{\mu_{k,l}} (b_{k,l})^2} \mu_{k,l}^{\mp \frac{i}{8} \sqrt{\mu_{k,l}} (b_{k,l})^2} \exp(\pm \frac{\pi}{8} \sqrt{\mu_{k,l}} (b_{k,l})^2 - iI(a_l, r_{1,l})) \quad (2.62)$$

For the case TPP3 (TPP4) the reflection (transmission) coefficient is an exponentially small quantity which has $1/\sqrt{2}$ as its maximum value. The transmission (reflection) coefficient varies from $1/\sqrt{2}$ to one. These formulas are useful only for small values of the parameters $b_{k,l}$ and $\mu_{k,l}$. Small values of $b_{k,l}$ implies that the turning points are very close to each other, almost coalescing, and

small values of $\mu_{k,l}$ implies that the sequence Q_j varies very slowly.

- Right incidence boundary condition.

$$R^+ = - \frac{\Gamma(\frac{\mp i}{4} \sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2})}{\sqrt{2\pi}} 2^{\pm i \frac{3}{2} \sqrt{\mu_{k,l}}(b_{k,l})^2} \mu_{k,l}^{\mp \frac{i}{8} \sqrt{\mu_{k,l}}(b_{k,l})^2} \exp(\mp \frac{\pi}{8} \sqrt{\mu_{k,l}}(b_{k,l})^2 + i \frac{\pi}{2} - i 2I(r_{1,l}, a_{l+1}) - i 2\pi r_{2,l} \delta_{k,2}) \quad (2.63)$$

$$T^+ = \frac{\Gamma(\frac{\mp i}{4} \sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2})}{\sqrt{2\pi}} 2^{\pm i \frac{3}{2} \sqrt{\mu_{k,l}}(b_{k,l})^2} \mu_{k,l}^{\mp \frac{i}{8} \sqrt{\mu_{k,l}}(b_{k,l})^2} \exp(\pm \frac{\pi}{8} \sqrt{\mu_{k,l}}(b_{k,l})^2 - i I(r_{1,l}, a_{l+1})) \quad (2.64)$$

The expressions for the reflection and transmission coefficients are almost the same as in the previous case. We have just a change in the phase of the reflection and transmission coefficients.

For TPP5, we need to set $b_{k,l}$ equal to zero in the expressions for TPP3 or TPP4, and the desired reflection and transmission coefficients follows.

2.3.4.4 Improvement of the Asymptotic Results for Reflection and Transmission Coefficients.

For TPP1, the WKB method gave the transmission coefficient of modulus equal to unity and an exponentially small reflection coefficient. For the case TPP2, we have the opposite situation, i.e. exponentially small transmission and reflection coefficient of modulus equal to unity. We can improve the estimate for the transmission coefficient in the case of TPP1, and for the reflection coefficient in the case of TPP2, by making use of some quantity that depends on the solutions of (2.1) and is constant (does not depend on the independent variable j). If the reflection (transmission) coefficient is exponentially small, we can use the quantity mentioned above to estimate the modulus of the transmission (reflection) coefficient. An example of this quantity follows.

$$\bar{z}_j z_{j+1} - z_j \bar{z}_{j+1} = \text{constant with } z_j \text{ solution of (2.1)} \quad (2.65)$$

where $\bar{z}_j$ is the complex conjugate of z_j . We want to use (2.65) to obtain a relation between the reflection and transmission coefficients. We consider the boundary condition of left incidence given by (2.44) and (2.45). If we substitute (2.44) into (2.65), we have

- For j real and $a_l < j < r_{1,l}$,

$$\frac{2A^2}{\sqrt{\sin(\theta_j) \sin(\theta_{j+1})}} \{-\sin(I(j, j+1)) + |R^-| \sin(I(j, j+1))\} = \text{constant} \quad (2.66)$$

- For j real and $r_{2,l} < j < a_{l+1}$,

$$\frac{2A^2|T^-|^2}{\sqrt{\sin(\theta_j) \sin(\theta_{j+1})}} \{-\sin(I(j, j+1))\} = \text{constant} \quad (2.67)$$

We can subtract these two relations, but first we set $j+1 = r_{1,l}$ in (2.66) and we set $j = r_{2,l}$ in (2.67). We obtain

$$|R^-|^2 + |T^-|^2 \frac{\sqrt{\sin(\theta_{r_{1,l}-1}) \sin(\theta_{r_{1,l}}) \sin(I(r_{2,l}, r_{2,l}+1))}}{\sqrt{\sin(\theta_{r_{2,l}}) \sin(\theta_{r_{2,l}+1}) \sin(I(r_{1,l}-1, r_{1,l}))}} = 1 \quad (2.68)$$

We have that $\sin(\theta(r_{1,l})) = \sin(\theta(r_{2,l}))$ and we can assume that $\sin(\theta(r_{1,l}-1)) = \sin(\theta(r_{2,l}+1))$ and $\sin(I(r_{1,l}-1, r_{1,l})) = \sin(I(r_{2,l}, r_{2,l}+1))$. As a result we can rewrite (2.68) as

$$|R^-|^2 + |T^-|^2 = 1 \quad (2.69)$$

For the boundary condition of right incidence we can proceed in the same way as we did for the left incidence case and obtain, basically, the same result, as follows:

$$|R^+|^2 + |T^+|^2 = 1 \quad (2.70)$$

Finally, we use (2.69) and (2.70) to improve the WKB predictions for the reflection and transmission coefficient, as described bellow.

- Exponentially small reflection. For this case we improve the estimate for the transmission coefficient. From (2.69) and (2.70) the modulus of the transmission coefficient is now given by

$$|T^\pm| = \sqrt{1 - |R^\pm|^2} \quad (2.71)$$

and we keep the phase given by the WKB method.

- Exponentially small transmission coefficient. For this case we improve the estimate for the reflection coefficient. From (2.69) and (2.70) the modulus of the reflection coefficient is now given by

$$|R^\pm| = \sqrt{1 - |T^\pm|^2} \quad (2.72)$$

and we keep the phase given by the WKB method.

2.3.5 ASYMPTOTIC FORM OF THE SYSTEM TRANSFER MATRIX.

The objective of this section is to describe how to obtain the transfer matrix that relates the wave disturbance amplitude at the ends of the non-uniform part of a repetitive system. Wave scattering, in the context of the WKB method occurs at the turning points. Therefore, first we describe how to obtain the transfer matrix for the turning point problems listed in section 4.4, and then we give the whole system transfer matrix as a product of the transfer matrices of the turning point problems that may appear along the non-uniform part of the chain. Let us consider points $a_l < r_{1,l} \leq r_{2,l} < a_{l+1}$ along the real axis of the j complex plane, as defined at the beginning of section 4.4. For TPP1, TPP3 and TPP5 $r_{1,l} = r_{2,l}$ (see boxes (A) and (C) of figure 2-2). If we have TPP2 or TPP4, $r_{1,l} < r_{2,l}$ and they actually are the

turning points (see box (B) of figure 2-2). In terms of the Liouville-Green functions, the wave disturbance can be written as follows:

- For $a_l < j < r_{1,l}$ we have

$$\sqrt{\sin(\theta_j)} z_j \sim C_l \exp(-i \int_{a_l}^j \theta_n dn) + D_l \exp(i \int_{a_l}^j \theta_n dn) \quad (2.73)$$

- For $r_{2,l} < j < a_{l+1}$ we have

$$\sqrt{\sin(\theta_j)} z_j \sim C_{l+1} \exp(-i \int_{r_{2,l}}^j \theta_n dn) + D_{l+1} \exp(i \int_{r_{2,l}}^j \theta_n dn) \quad (2.74)$$

The transfer matrix for the l -th turning point problem relates the wave amplitudes (C_l, D_l) at the boundary point a_l to the wave amplitudes (C_{l+1}, D_{l+1}) at the boundary point a_{l+1} . To obtain the desired transfer matrix, labeled $\mathbf{M}_l$, we consider the boundary conditions of left and right incidence. For left incidence we have

$$(C_l, D_l) = (A, R^- A) \quad \text{and} \quad (2.75)$$

$$(C_{l+1}, D_{l+1}) = (T^- A \exp(-iI(r_{2,l}, a_{l+1})), 0),$$

and for right incidence we have

$$(C_l, D_l) = (0, T^+ A) \quad \text{and} \quad (2.76)$$

$$(C_{l+1}, D_{l+1}) = (R^+ A \exp(-iI(r_{2,l}, a_{l+1})), A \exp(iI(r_{2,l}, a_{l+1}))),$$

The reflection and transmission coefficients for both boundary conditions were obtained in sections 4.1, 4.2, 4.3 and 4.4 according to the turning point problem in consideration. Therefore, we can assume the reflection and transmission coefficients above as known. We apply the boundary conditions (2.75) and (2.76) to the matrix

equation

$$\begin{pmatrix} C_{l+1} \\ D_{l+1} \end{pmatrix} = \begin{bmatrix} M_{l,11} & M_{l,12} \\ M_{l,21} & M_{l,22} \end{bmatrix} \begin{pmatrix} C_l \\ D_l \end{pmatrix} \quad (2.77)$$

and obtain the matrix elements in terms of the reflection $R^\pm$ and transmission $T^\pm$ coefficients. The elements of the matrix $\mathbf{M}_l$ and its inverse $\mathbf{M}_l^{-1}$ follows

$$M_{l,11} = \left(T^- - \frac{R^- R^+}{T^+} \right) \exp(-iI(r_{2,l}, a_{l+1})) \quad (2.78)$$

$$M_{l,12} = \frac{R^+}{T^+} \exp(-iI(r_{2,l}, a_{l+1})) \quad (2.79)$$

$$M_{l,21} = -\frac{R^-}{T^+} \exp(iI(r_{2,l}, a_{l+1})) \quad (2.80)$$

$$M_{l,22} = \frac{1}{T^+} \exp(iI(r_{2,l}, a_{l+1})) \quad (2.81)$$

and for the inverse matrix

$$M_{l,11}^{-1} = \frac{1}{T^-} \exp(iI(r_{2,l}, a_{l+1})) \quad (2.82)$$

$$M_{l,12}^{-1} = -\frac{R^+}{T^-} \exp(-iI(r_{2,l}, a_{l+1})) \quad (2.83)$$

$$M_{l,21}^{-1} = \frac{R^-}{T^-} \exp(iI(r_{2,l}, a_{l+1})) \quad (2.84)$$

$$M_{l,22}^{-1} = \left(T^+ - \frac{R^- R^+}{T^-} \right) \exp(-iI(r_{2,l}, a_{l+1})) \quad (2.85)$$

We have the transfer matrix for each of the turning point problems listed in section 4, since for each of these problems, we obtained the reflection and transmission coefficients $R^\pm$ and $T^\pm$. In the context of the WKB method the wave diffraction problem due to the system non-uniformity is replaced by the wave diffraction problem due to the presence of turning points in the complex j plane. For a large system we have a sequence of turning point problems as we travel along the real axis of the j plane. The transfer matrix that relates the wave amplitudes before and after the non-uniform section of the system is given as the product of the transfer matrices for each of the turning point problems encountered as we travel along the real axis

of the j plane. Suppose we want to relate the wave amplitudes at the beginning of the non-uniform section a_0 to the wave amplitudes at the end of the non-uniform section a_n . We assume that between a_0 and a_n we have n turning point problems. The transfer matrix relating these two positions follows as the product of the transfer matrices of the turning point problems found between these two positions:

$$\begin{pmatrix} C_n \\ D_n \end{pmatrix} = \prod_{l=1}^n [\mathbf{M}_l] \begin{pmatrix} C_0 \\ D_0 \end{pmatrix} \quad (2.86)$$

or

$$\begin{pmatrix} C_0 \\ D_0 \end{pmatrix} = \prod_{l=n}^1 [\mathbf{M}_l^{-1}] \begin{pmatrix} C_n \\ D_n \end{pmatrix} \quad (2.87)$$

The global transmission and reflection coefficients can be obtained from the product of matrices in (2.86) and (2.87). For left incidence boundary condition on the non-uniform part of the chain we use the product of matrices (2.86). The global reflection and transmission coefficients R_G^- and T_G^- for left incidence follows

$$R_G^- = - \frac{(\prod_{j=1}^N [\mathbf{M}_j])_{21}}{(\prod_{j=1}^N [\mathbf{M}_j])_{22}} \exp(-i2\theta(a_0)a_0) \quad (2.88)$$

$$T_G^- = \left(\left(\prod_{j=1}^N [\mathbf{M}_j] \right)_{11} \exp(-i\theta(a_0)a_0) - \frac{(\prod_{j=1}^N [\mathbf{M}_j])_{12} (\prod_{j=1}^N [\mathbf{M}_j])_{21}}{(\prod_{j=1}^N [\mathbf{M}_j])_{22}} \exp(i\theta(a_0)a_0) \right) \exp(i\theta(a_N)a_N) \quad (2.89)$$

For the boundary condition of right incidence, we derived expressions for the global transmission and reflection coefficients R_G^+ and T_G^+ from (2.87), as follows

$$R_G^+ = - \frac{(\prod_{j=N}^1 [\mathbf{M}_j^{-1}])_{12}}{(\prod_{j=N}^1 [\mathbf{M}_j^{-1}])_{11}} \exp(i2\theta(a_N)a_N) \quad (2.90)$$

$$T_G^+ = \left((\prod_{j=1}^N [\mathbf{M}_j^{-1}])_{22} - \frac{(\prod_{j=1}^N [\mathbf{M}_j^{-1}])_{21} (\prod_{j=1}^N [\mathbf{M}_j^{-1}])_{12}}{(\prod_{j=1}^N [\mathbf{M}_j^{-1}])_{11}} \right) \exp(i\theta(a_N)a_N - i\theta(a_0)a_0) \quad (2.91)$$

We use these expressions to evaluate the global transmission and reflection coefficients, since they are more stable when these asymptotic results (reflection and transmission coefficients) are evaluated numerically.

2.4 APPLICATION.

We compare the asymptotic results obtained in section 4 with numerical simulation. The physical system used to illustrate the performance of the asymptotic theory is a chain of coupled pendulums. We gave the governing equation for wave propagation on a chain of coupled pendulums in section 3. We consider only the variation of the pendulums length as the source of non-uniformity. At first, we give results for systems with only one pair of turning points. Second, we give results for systems with two pairs of turning points. We give asymptotic and numerical results for the reflection and transmission coefficients as a function of the wave frequencies and size of the non-uniform part of the system.

2.4.1 SYSTEM WITH ONE PAIR OF TURNING POINTS.

The shape of the non-uniformity is assumed to be a flat Gaussian curve, described by

$$\varepsilon_j = \pm A \exp\left(-\frac{(j - M/2 + j_0)^2}{2\sigma^2}\right), \text{ for } j_0 < j < j_0 + M \quad (2.92)$$

where M is the size of the non-uniform part of the chain and j_0 is where the non-uniform part starts. The amplitude A is chosen such that we have a second order

turning point for a specific wave frequency value that we chose. The value of σ is chosen such that we have a smooth transition to the uniform part of the chain. As M increases, the Gaussian become flatter, so the matching between numerical and asymptotic results increases. The $+$ ($-$) sign in equation (2.92) gives turning points that satisfy the first (second) turning point condition. For frequencies at the left (right) side of the uniform system passband (see (2-1)), we have a pair of real turning points (TPP2) that coalesce and become a pair of complex conjugate turning points (TPP1). In the following figures we present the asymptotic and numerical results for the modulus of the reflection and transmission coefficients as function of the incident wave frequency and size of the non-uniform part of the system. The wave frequency ranges over the uniform system passband.

In figures 2-3 and 2-4 we considered left incidence boundary condition. For right incidence, we have the same results as for left incidence due to the symmetry of the shape considered. According to the results in figures 2-3 and 2-4, as the steepness of the non-uniform part decreases, the match between asymptotic and numerical results increases, except for frequencies where the turning points are close to coalesce. This fact lead us to use a different approach for almost coalescing turning points (TPP3 and TPP4), outlined in sections 4.4.3 and 4.4.4 and in Appendix C.2. We combined the corrected WKB approximation (see section 4.4.5) with the results for two almost coalescing turning points (TPP3 and TPP4) to obtain a better approximation for the modulus of the reflection and transmission coefficients. In the figure 2-5 we illustrate the asymptotic results for a pair of almost coalescing real (TPP4) and complex turning points (TPP3). We present results for the modulus of the reflection coefficient only for the left incidence boundary condition, as function of wave frequency and steepness of the Gaussian shape. We consider that the pair of turning points satisfies the first turning point condition only. We just show the part of the passband where the modulus of the reflection/transmission coefficients goes from almost one to exponentially small values.

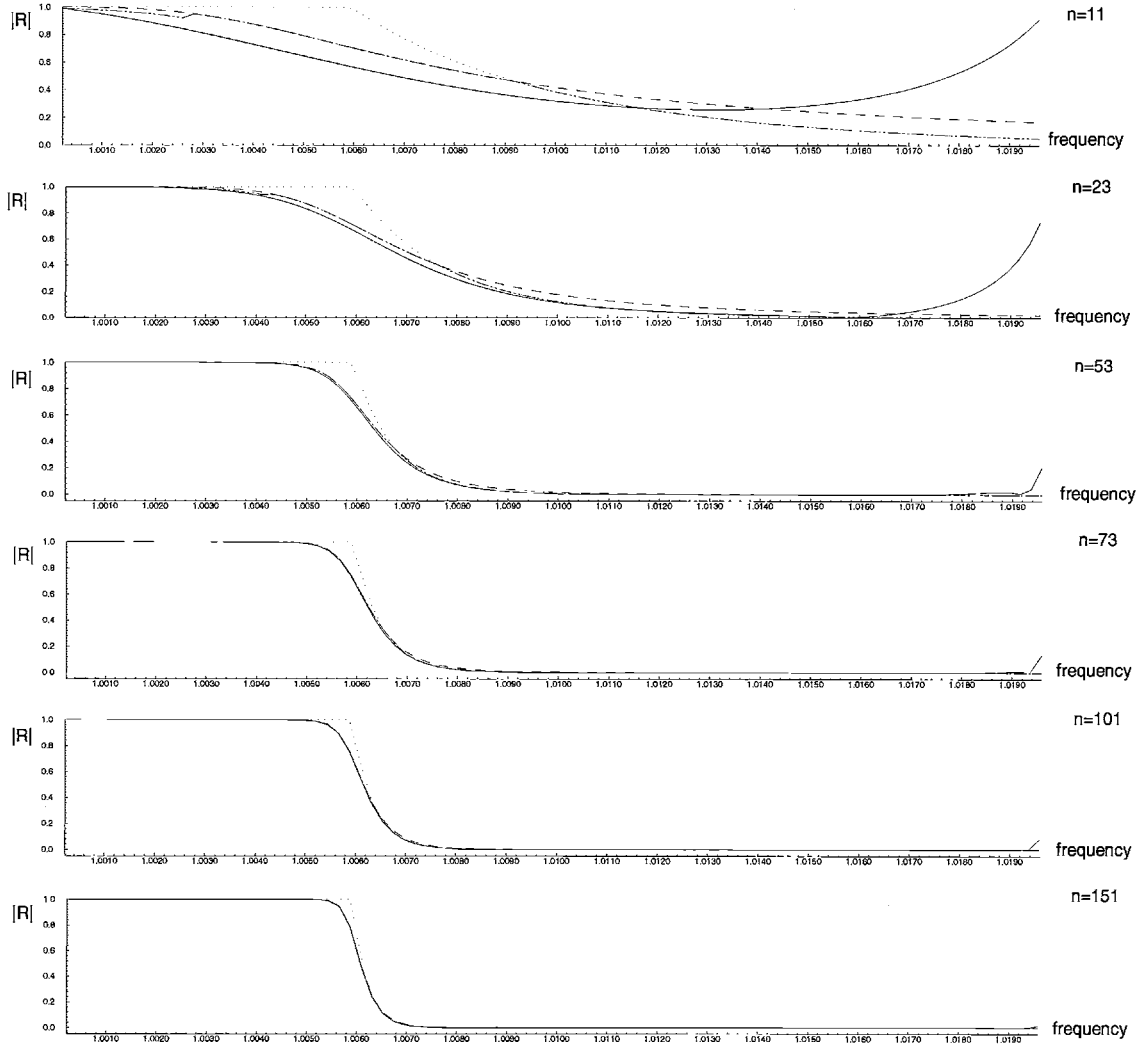


Figure 2-3: Modulus of the Reflection coefficient for left incidence. We have a second order turning point at 30% of the uniform system passband. The turning points satisfy the first turning point condition. Amplitude $A = 0.012$. Coupling parameter is $R = 0.01$. The full line denotes the numerical results. The $\cdots$ line denotes the usual WKB approximation. The $---$ line denotes the reflection coefficient given by (2.61) and (2.63). The $-\cdots-$ line denotes a composite approximation. For frequencies close to the frequency for which we have a second order turning point, the reflection coefficient is given by (2.61) and (2.63). Otherwise, it is given by (2.49) and (2.57).

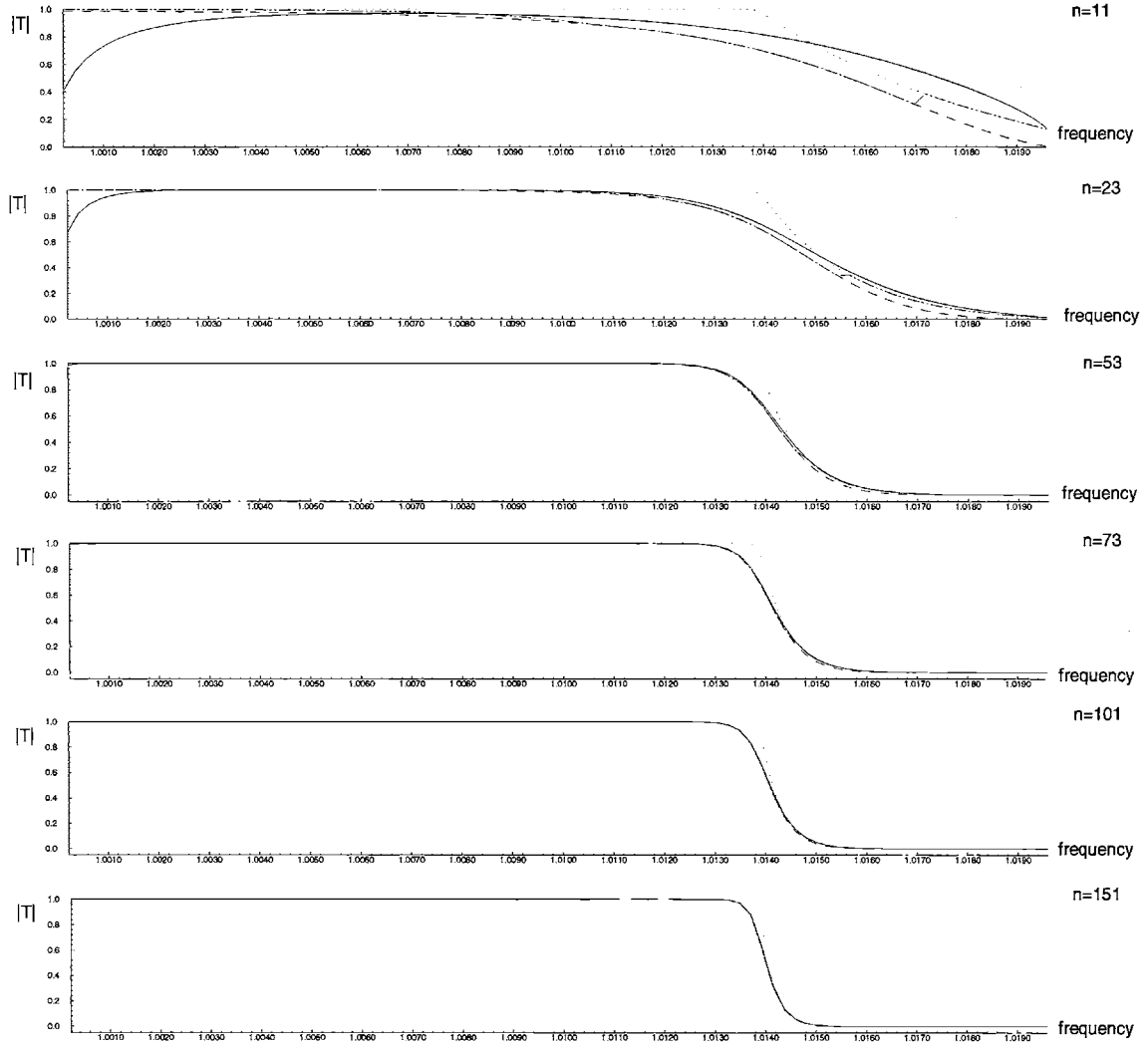


Figure 2-4: Modulus of the transmission coefficient for left incidence. We have a second order turning point at 70% of the uniform system passband. The turning points satisfy the second turning point condition. Amplitude $A = 0.012$. Coupling parameter is $R = 0.01$. The full line denotes the numerical results. The $\cdots$ line denotes the usual WKB approximation. The $---$ line denotes the transmission coefficient given by (2.64). The $- \cdots - \cdots -$ line denotes a composite approximation. For frequencies close to the frequency for which we have a second order turning point, the transmission coefficient is given by (2.64). Otherwise, it is given by (2.52) and (2.60).

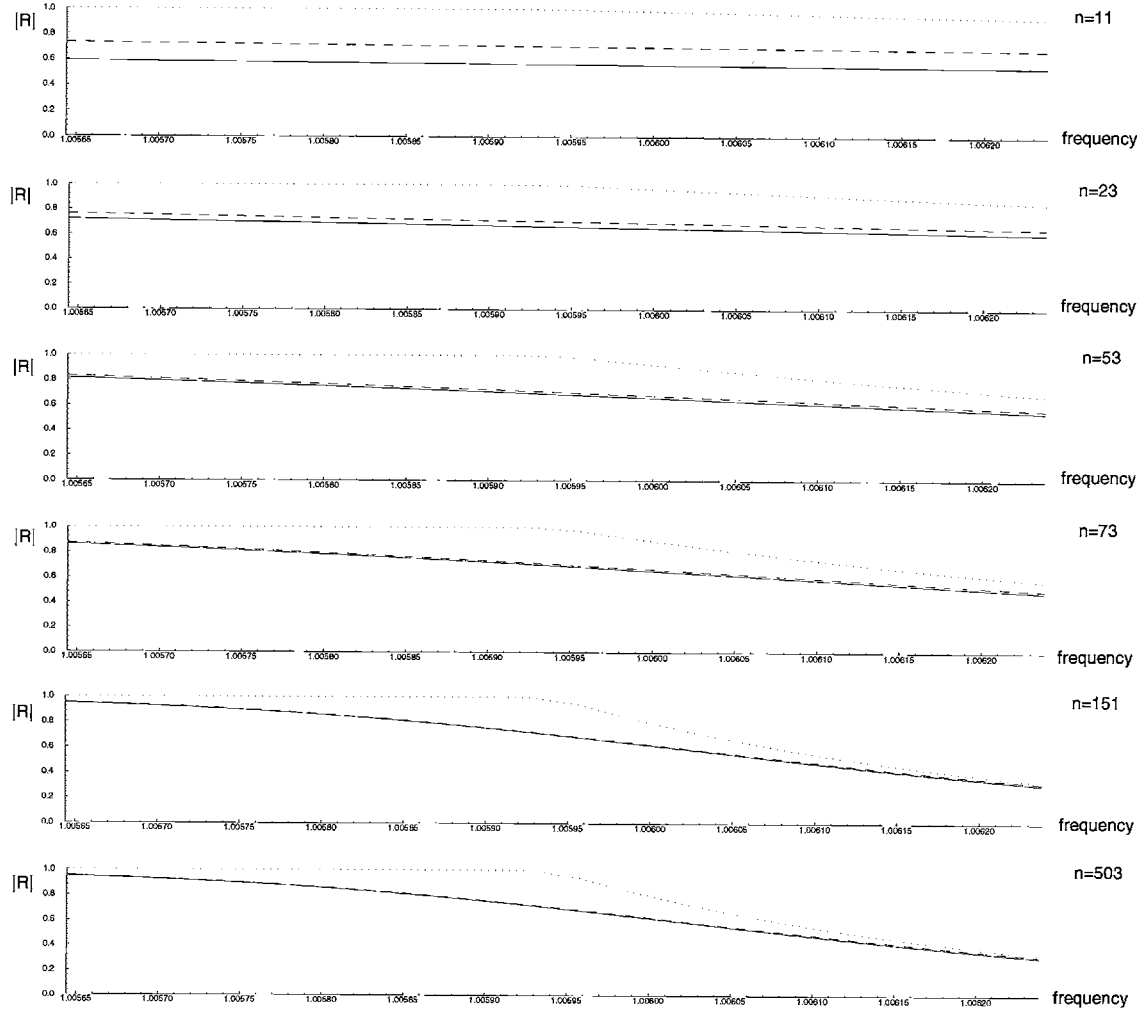


Figure 2-5: Modulus of the Reflection coefficient for left incidence. We have a second order turning point at 30% of the uniform system passband. The turning points satisfy the first turning point condition. Amplitude $A = 0.012$. Coupling parameter is $R = 0.01$. The full line denotes the numerical results. The $---$ line denotes the reflection coefficient given by (2.61) and (2.63). The $\cdots$ line denotes given by (2.49) and (2.57).

2.4.2 SYSTEMS WITH TWO PAIRS OF TURNING POINTS.

Again, we use a simple shape to illustrate the asymptotic form of the transfer matrix. We consider a shape that has two TPPjs. One of the TPPj is related to the first

turning point condition and the other is related to the second turning point condition. The shape is a sum of two flat Gaussian curves with the same amplitude, described by

$$\varepsilon_j = \pm A \exp\left(-\frac{(j - M/4 + j_0)^2}{2\sigma^2}\right) \mp A \exp\left(-\frac{(j - 3M/4 + j_0)^2}{2\sigma^2}\right), \text{ for } j_0 < j < j_0 + M \quad (2.93)$$

where M is the size of the non-uniform part of the system and j_0 is where the non-uniform part starts. There is a maximum/minimum at $M/4$ and a minimum/maximum at $3M/4$. The amplitude A and σ are chosen as in the previous section. The plus (minus) sign in equation (2.93) gives a shape that has a first TPPj associated with the first (second) turning point condition and a second TPPj associated with the second (first) turning point condition. In figures 2-6 and 2-7 we give the modulus of the global reflection and transmission coefficients as function of wave frequency and size of the non-uniform part of the system. The wave frequency ranges over the uniform system passband. We show results for the left incidence boundary condition only. The results for the right incidence boundary condition are similar to the left incidence case due to the symmetry of the shapes considered. For shape (2.93) we give results for the modulus of the global reflection and transmission in figures 2-6 and 2-7.

The asymptotic results given in figures 2-6 and 2-7 for the reflection and transmission coefficients are given by expressions (2.88) to (2.91).

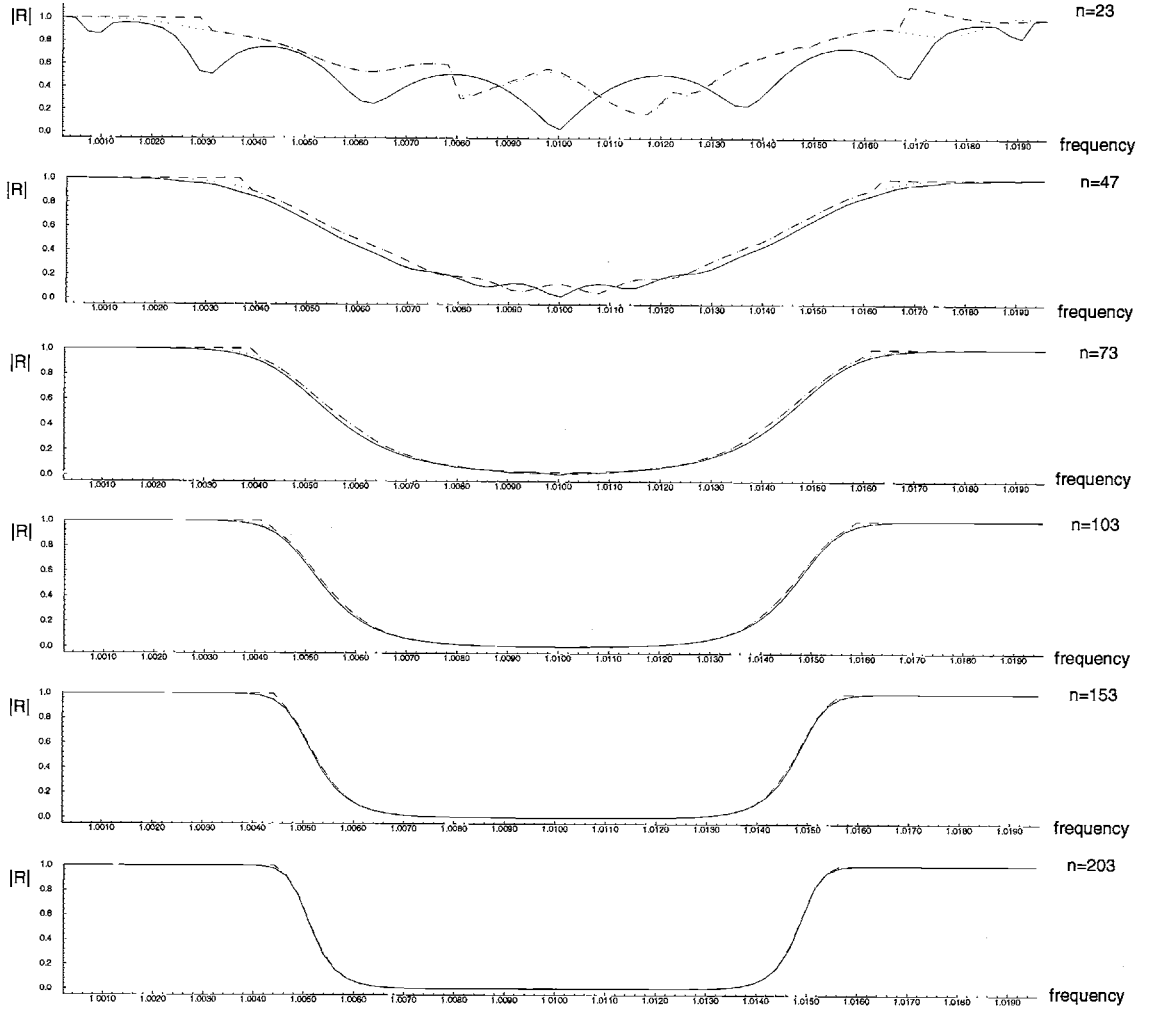


Figure 2-6: Modulus of the Reflection coefficient for left incidence. We have a second order turning point at 25% and 75% of the uniform system passband. Amplitude $A = 0.0098$. Coupling parameter is $R = 0.01$. The full line denotes the numerical results. The $---$ line denotes the reflection coefficient given by the transfer matrix method. We use the local reflection/transmission coefficients given by the usual WKB approximation. For the $\cdots$ line we use the local reflection and transmission coefficients corrected by the approximation for almost coalescing pairs of turning points problem. We use this correction when the wave frequency is close to the frequency for which the turning point problem is a second order real turning point.

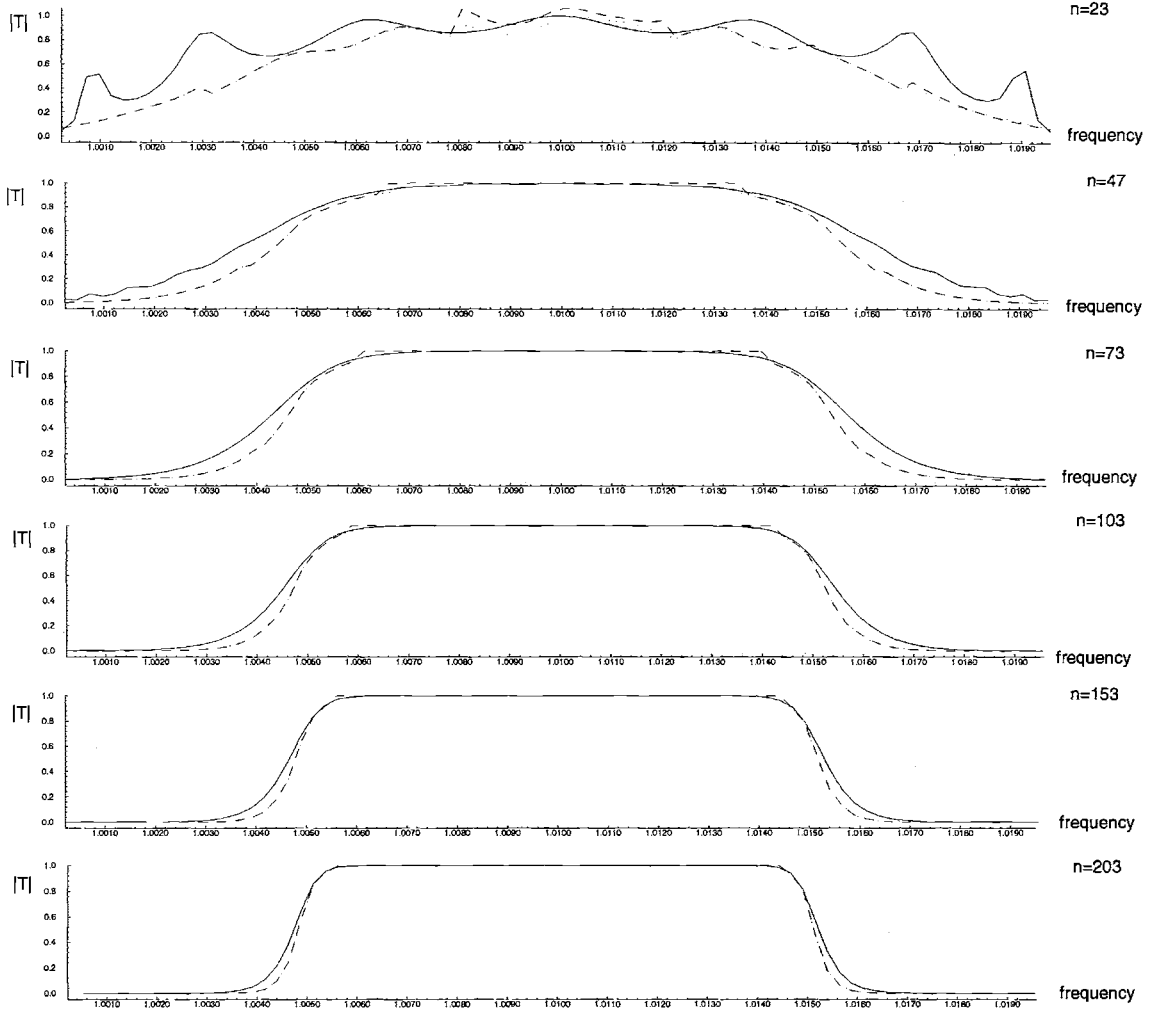


Figure 2-7: Modulus of the Transmission coefficient for left incidence. We have a second order turning point at 25% and 75% of the uniform system passband. Amplitude $A = 0.0098$. Coupling parameter is $R = 0.01$. The full line denotes the numerical results. The $- - -$ line denotes the transmission coefficient given by the transfer matrix method. We use the local reflection/transmission coefficients given by the usual WKB approximation. For the $\cdots$ line we use the local reflection and transmission coefficients corrected by the approximation for almost coalescing pairs of turning points. We use this correction when the wave frequency is close to the frequency for which the turning point problem is of second order.

2.5 DISCUSSION AND CONCLUSIONS.

We outline how the asymptotic results can be used to design the slowly varying non-uniformity in a very long chain of coupled pendulums to minimize transmission for the whole system passband. We also comment on some possibilities for future work.

The asymptotic approximations for the reflection and transmission coefficients obtained in section 4 are meaningful only for frequencies inside the uniform system passband, since for other values of wave frequency there is no wave propagation. These asymptotic approximations for the reflection and transmission coefficients are not able to “see” both edges of the passband. Figure 2-3 [figure 2-4] illustrates this fact for a non-uniform chain with only one turning point problem with turning points that satisfy the first [second] turning point condition (2.30). As the wave frequency approaches the upper [lower] edge of the uniform system passband, the modulus of the reflection (transmission) coefficient goes to one (zero), but the asymptotic result stays exponentially small (equal to one). In other words, the asymptotic approximation does not model the transition of the modulus of the reflection (transmission) coefficient from exponentially small values to one (one to zero). An example of this kind of non-uniformity is given by (2.92) in section 5.1. For a chain with many turning point problems, we find that some turning points satisfy the first turning point condition (2.30), and other turning points satisfy the second turning point condition (2.30). Therefore, for chains with many turning point problems, both uniform system passband edges are modeled properly. The upper (lower) edge is taken into account by turning points that satisfy the first (second) turning point condition (2.30). We add that turning points that satisfy the first (second) turning point condition (2.30), model correctly the behavior at the lower (upper) edge of the system effective passband as illustrated in figures 2-3 and 2-4.

The asymptotic approximation for the reflection and transmission coefficients given by the WKB method does not give good results when two turning points start to coalesce. For this situation we considered in a neighborhood of the pair of turning points an approximate form of equation (2.1) that takes both turning points into

account, as described in section 4.3.2 and in Appendix C.2. This approach seems to be successful as shown in figure 2-5. We use these asymptotic results to obtain a better asymptotic approximation for the reflection and transmission coefficients. We combine the usual WKB approximation with the asymptotic results valid when the turning points are almost coalescing, to obtain an asymptotic approximation that gives better results for the whole system passband. Results for this combined approximation are illustrated in figures 2-3 and 2-4.

Regarding localization phenomena, the asymptotic results obtained through the WKB method give some qualitative understanding, as discussed in the end of section 4.1. If the non-uniformity is periodic in space, we have perfect transmission at a discrete set of wave frequencies, with some of them even outside the effective passband. If no periodicity is present in the non-uniformity, frequencies of perfect transmission are not likely to occur, as pointed out in [36]. The problem of perfect transmission was not addressed here. We could have used the asymptotic approximations for the reflection and transmission coefficients to obtain estimates for frequencies of perfect transmission, but due to the complexity of the expressions for the global transmission and reflection coefficients we would have to use a numerical approach to search for the desired frequencies.

Regarding the design problem for minimum transmission, we can use the asymptotic results to design a chain with slowly varying non-uniformity in space. For a maximum allowed amount of non-uniformity at each element of the chain, the idea is to search for the non-uniformity configuration which provides turning points as close as possible to the real axis and has no periodicity. For the problem of a chain of coupled pendulums, we have a simple dependence of the sequence Q_j with respect to the wave frequency ω , as illustrated by equation (2.11). Therefore, by minimizing transmission for a single frequency, located in the middle of the effective passband, it may be sufficient to obtain a non-uniformity configuration that minimizes transmission for the whole effective passband.

We can extend this work in at least two directions. One direction is to extend the WKB method to high order difference equations that arise in the modeling of

repetitive system where the interaction of the subsystem is not restricted to the two nearest neighbors. Some work have been done in this direction in [17] and in the Appendix A. Another direction is to extend the analysis done here for second order difference equations to two dimensional systems of first order difference equations. More interesting systems can be modeled by two dimensional systems of first order difference equations, like, for example, wave propagation along a one dimensional channel with the bottom represented by shelves; or wave propagation along coupled uniform beams with different cross section.

In the Appendix A we discuss how to apply the WKB method for system of first order difference equations. We also outline how to obtain a high order asymptotic solution for a n -th order difference equation, but we do not discuss how to continue the high order asymptotic solution through the turning points this difference equation may have. In section A.4 of Appendix A we also illustrate how to write a difference equation as a pseudo-differential equation. The asymptotic analysis of pseudo-differential equations is very similar to the asymptotic analysis of differential equations, and all the machinery for differential equations can be applied to pseudo-differential equations almost directly. This simplifies the asymptotic analysis of difference equations, specially if high order approximations are desired.

Chapter 3

Asymptotic Analysis and Design of Wave Propagation Along a Non-Uniform Euler-Bernoulli Beam.

In this chapter we consider the vibratory response of a non-uniform Euler-Bernoulli beam. Non-uniformity in beams arise either from manufacturing imperfections or by design, and can have a singular importance on the qualitative properties of the beam vibration response.

To describe the mechanism causing such a large changes in the dynamics of the beam, in the first part of this chapter, we derived asymptotically a simple equation governing the beam motion. This simple equation is a second order differential equation, asymptotic to the non-uniformity of the beam. Under certain restrictions, viz. of having constant tension and a constant product of the beam mass per unit length and flexural rigidity, it has a Helmholtz-like form and is an exact governing equation. So, we can use this simple equation to design the non-uniformity of the beam to minimize wave transmission for a desired range of wave frequencies. We can use inverse-scattering methods for Helmholtz-like equations to accomplish the design

problem mentioned above.

In the second part of chapter 3 we consider slender beams with slowly varying non-uniformity with respect to the wave length of the incident wave. We construct a high order asymptotic solution for the beam vibration. The asymptotic method used is the WKB method, which is basically a wave refraction theory. To incorporate wave reflection and wave mode conversion we take into account the effects of turning points and singularities, like poles. At the turning points and singularities the asymptotic theory breaks down, and a local analysis is needed to continue the asymptotic solution through the turning points and singularities. The result of the local analysis are the connection formulas. For second order differential equations, connection formulas for turning points and singularities are well known, but for high order differential equations not much is known, so further progress is needed. The objective of the second part of this chapter was to understand the difficulties involving in obtaining such connection formulas for turning point problems where four wave modes couple. Coupling of four wave modes also appeared in the asymptotic analysis performed in chapter 4 for pseudo-differential equations. The ideas proposed in Fuchs, Ko & Bers (1981) and in Fuchs, Bers & Harte (1985) based on only second order differential equations did not give a complete picture, so fourth order equations maybe necessary to extend the results obtained in chapter 4, what is left as future work.

The Euler-Bernoulli beam model is governed by a fourth-order differential equation in the frequency domain, and non-uniformity in the flexural rigidity only leads to turning point problems where four wave modes couple. We consider non-uniformity in the flexural rigidity which gives only first order turning points to make possible the approximate form of the governing equation, valid in a neighborhood of a turning point, to have an analytic solution. This is fundamental for the connection problem to be solved in closed form. The analysis in the second part of chapter 3 is a first step towards the extension of the ideas discussed in Fuchs, Ko & Bers (1981) and in Fuchs, Bers & Harte (1985) for the case where four wave modes couple at a turning point or group of turning points.

3.1 Asymptotic Governing Equation for Wave Propagation along Weakly Non-uniform Euler-Bernoulli Beams.

The vibration of non-uniform beams, whether smooth or stepped, has been studied extensively, and is still receiving attention in the literature. The non-uniformity may arise in the material properties and in the geometry of the beam. Usually, the effect of the non-uniformity on the natural frequencies and natural modes of finite beams is investigated, under various boundary conditions. Wave propagation along long non-uniform beams has received less attention, especially in the case of non-periodic, smooth and continuous non-uniformity.

For periodic non-uniformity, analytical techniques can be applied, such as Floquet theory and perturbation methods, such as the method of multiple scales. Wave interaction with periodic non-uniformity can give rise to strong resonance effects. For incident waves with wavelength of the order of two times the wavelength scale of the non-uniformity, the incident wave resonates with the non-uniformity, leading to almost complete reflection if the non-uniform part of the beam is long enough. This strong interaction is known as Bragg reflection (see Hawwa [27]).

The case of non-periodic non-uniformity is even more interesting. For beams with a long enough non-uniform part, vibration with a fairly broad spectrum may stay localized in a finite region close to its source. The normal modes are not extended anymore, and they become localized in space. In terms of wave propagation, we may have almost complete reflection and exponentially small transmission. These are manifestations of localization phenomena in the vibration of mechanical systems. Another feature of the behavior of such localized systems is the disproportionately large sensitivity of the system response with respect to small variation in the non-uniformity of the system parameters, as pointed out by Pierre [53] and Triantafyllou and Tryantafyllou [65].

In summary, small non-uniformity in the material and geometrical properties in

beams can have a singular importance, causing large effects on vibration propagation relative to its own magnitude. To help describe these mechanisms well, and given the significance these effects may have to a large number of applications, we derived asymptotically a simpler equation, which captures the essence of the localization phenomena.

We derived asymptotically a second order differential equation governing wave propagation along non-uniform Bernoulli-Euler beams under the action of non-uniform tensile force along their length. This second order governing equation is asymptotic with respect to the steepness of the non-uniformity. In other words, as the non-uniformity steepness decreases, the agreement between the behavior predicted by the second order governing equation and the full governing equation increases.

When the tensile force and the product of the beam flexural rigidity by its mass per unit length are constants, the second order governing equation is an exact governing equation for wave propagation along non-uniform Bernoulli-Euler beams. These restrictions are not commonly met in practice, but such beams can be designed and built.

The second order governing equation under the restrictions mentioned above can be used to design the beam non-uniformity to achieve, for example, passive vibration isolation. This can be done in two ways. The first approach consists of using a second order differential equation with non-constant coefficient which has a closed form solution and models well the wave propagation problem. The analytical solution furnishes a functional relation between the wave frequency and the scattering coefficients, such as the reflection coefficient, using the non-uniformity in the coefficient function as a parameter. Then, if we specify the form of the functional relation between the reflection coefficient and the wave frequency, we end up designing the non-uniformity in the coefficient function of the second order differential equation. Once the non-uniformity in the coefficient function is specified, the non-uniformity in the beam flexural rigidity and mass per unit length is also determined. We give an example of this approach in section 3.1.5.

The second approach consists of using inverse scattering techniques for Helmholtz-

like equations, since the second order governing equation under the restrictions mentioned above has a Helmholtz-like form.

In the next section, we give an outline of previous work on wave propagation along smooth non-uniform beams, on analytical solutions of second order differential equations and on inverse-scattering techniques for the Helmholtz equation. In section 3.1.2, we give the governing equation for the Bernoulli-Euler beam in terms of the chosen non-dimensional variables. In section 3.1.3, we derive asymptotically the second order differential equation governing wave propagation along non-uniform Bernoulli-Euler beams. We also discuss the qualitative behavior predicted by this governing equation for general non-uniformities, and we describe under which restriction over the non-uniformity this equation is an exact governing equation. In section 3.1.4, we use the second order governing equation to predict wave reflection by the beam non-uniformity. We compare results for the modulus of the reflection coefficient from the numerical simulation of both governing equations. In section 3.1.5, we illustrate the first approach for the design problem mentioned above.

3.1.1 Previous Work.

Wave propagation along smooth non-uniform periodic Bernoulli-Euler beams can be studied through analytical techniques and perturbation methods. Lee & Ke [37] applied Floquet theory to study flexural wave propagation along Bernoulli-Euler beams with periodic non-uniformity. They show that flexural waves in a periodic beam can be interpreted as a superposition of two pairs of waves propagating in opposite directions, of which one pair behaves as an attenuated wave. Hawwa [27] considered beams with a periodically varying cross section, and used a straightforward asymptotic expansion to show that resonance between the beam periodicity and the wave field occur when the wavenumber of the flexural wave is half the wavenumber of that of the beam periodicity. A uniform expansion near the resonance condition was obtained through the method of multiple scales. An account of the literature regarding wave propagation along periodic beams is given in the two references mentioned above.

To study vibration of non-periodic non-uniform Bernoulli-Euler beams, numerical methods, like the finite element method, the finite difference method and the transfer matrix method are usually applied. The exception are geometrical and material non-uniformities which are polynomial functions of the space parameterization. Naguleswaran [50] considered Bernoulli-Euler beams with a variation in breadth proportional to x^s ($s \leq 0$, and x is the distance from the “sharp” end). Abrate [1] considered beams with cross-sectional area and its second moment as arbitrary polynomial functions of the space parameterization. Approximations for the natural frequencies were obtained using the Rayleigh-Ritz method.

Heading [28] gives a list of indices of refraction such that the resulting one-dimensional Helmholtz equation has closed form solutions in terms of transcendental functions. He also discusses how to generate more complex indices of refraction from simple ones. He also gives an account of the literature about one-dimensional Helmholtz and Schrödinger like equations with non-constant coefficient functions, which have a closed form solution.

An account of the literature about inverse scattering methods for the one-dimensional Helmholtz-like equations up to 1987, with emphasis in the context of seismology, is given in Merab [45]. An exact inverse method for the one-dimensional Helmholtz equation in the half line is described in Sylvester et al [64] and Sylvester & Winebrenner [63]. They developed a new layer stripping technique for the inverse scattering problem of the Helmholtz equation on the half line. In Sylvester et al [64], they proved convergence of the algorithm and wellposedness of the forward and inverse scattering problems, and in Sylvester & Winebrenner [63], they constructed a numerical inverse algorithm based on the non-linear Riesz transform.

3.1.2 Bernoulli-Euler Beam Governing Equation.

Here we describe the choice of non-dimensional variables and we give the non-dimensional form of the Bernoulli-Euler beam governing equation.

We consider a one-dimensional continuous model for an elastic beam under small transverse motions, which are represented by the transversal displacement $\eta(x, t)$ of

the neutral line of the beam cross section. The variables x and t represent, respectively, the space and time coordinates. The Bernoulli-Euler beam model assigns only transverse inertia (i.e., ignores rotational inertia) and bending elasticity (i.e., ignores shear deformation) to the continuum. We also consider the elastic restoring force due to the tensile force applied to the beam. For the beam model considered, the governing equation for the transversal displacement results from the balance between the cross-sectional inertia force and the gradient of the shear force due to the bending moment and tensile force, as follows:

$$(I(x)E(x)\eta_{xx})_{xx} - (P(x)\eta_x)_x + \rho(x)A(x)\eta_{tt} = 0. \quad (3.1)$$

$A(x)$ and $I(x)$ are, respectively, the cross-sectional area and its second moment. $E(x)$ and $\rho(x)$ are, respectively, the modulus of elasticity and density of the material. $P(x)$ represents the tensile load applied to the beam. The Bernoulli-Euler beam is considered non-uniform for $0 \leq x \leq L$. For $x \leq 0$ and $x \geq L$, the beam is uniform, but not necessarily with the same geometrical and material properties. We consider the following set of non-dimensional variables:

$$\begin{aligned}
s &= \frac{x}{\lambda}, \\
\tau &= f_0 t \text{ with } f_0 = \sqrt{\frac{E_0 I_0}{\rho_0 A_0 \lambda^4}}, \\
\omega &= f_0 \Omega, \\
y &= \frac{\eta}{h_0}, \\
\bar{L} &= \frac{L}{\lambda}, \\
\bar{P}(s) &= \frac{\lambda^2}{E_0 I_0} P(x), \\
m(s) &= \frac{\rho(x) A(x)}{\rho_0 A_0}, \\
ei(s) &= \frac{I(x) E(x)}{I_0 E_0},
\end{aligned} \tag{3.2}$$

where λ is the wavelength of the wave disturbance on the uniform part of the beam at $x \leq 0$. The quantities $\rho_0 A_0$ and $E_0 I_0$ are, respectively, the reference values for the mass per unit length and for the flexural rigidity. The quantity h_0 is the half beam-cross section reference height, and f_0 is the time non-dimensionalization factor. All reference quantities are taken from the uniform part located at $x \leq 0$. In terms of the non-dimensional variables, the governing equation (3.1) assumes the form:

$$(ei(s)y_{ss})_{ss} - (\bar{P}(s)y_s)_s + m(s)y_{\tau\tau} = 0. \tag{3.3}$$

The non-dimensional mass per unit length $m(s)$ and the non-dimensional flexural rigidity $ei(s)$ are defined as follows:

$$m(s) = \begin{cases} 1 & \text{for } s < 0 \\ \frac{\rho A(\lambda s)}{\rho_0 A_0} & \text{for } 0 \leq s \leq \bar{L} \\ \frac{\rho_1 A_1}{\rho_0 A_0} & \text{for } s > \bar{L} \end{cases}$$

and

$$ei(s) = \begin{cases} 1 & \text{for } s < 0 \\ \frac{EI(\lambda s)}{E_0 I_0} & \text{for } 0 \leq s \leq \bar{L} \\ \frac{E_1 I_1}{E_0 I_0} & \text{for } s > \bar{L} \end{cases} \quad (3.4)$$

Since we are interested in the interaction of mono-chromatic waves with the beam non-uniformity, we assume the time dependence:

$$\exp(-i\omega\tau), \quad (3.5)$$

with ω as the non-dimensional wave frequency. The non-dimensional governing equation (3.3) assumes the form

$$(ei(s)y_{ss})_{ss} - (\bar{P}(s)y_s)_s - \omega^2 m(s)y(s) = 0, \quad (3.6)$$

which is a fourth order differential equation with variable coefficients.

3.1.3 Second Order Governing Equation for the Bernoulli-Euler Beam.

In this section we derive asymptotically a second order differential equation governing wave propagation along a non-uniform Bernoulli-Euler beam.

First, we consider a change of the dependent variable to transform the governing equation (3.6) into a four-dimensional system of first order differential equations.

Second, we discuss the restrictions on the beam non-uniformity which allows wave propagation to be governed by a two-dimensional system of first order differential equations, and from this system of equations we obtain a second order differential equation through a sequence of changes of the dependent variable. We also discuss the qualitative insight on the effects of the non-uniformity on wave propagation given by the second order governing equation. Third, we pay special attention to beams where the tensile force and the product of the beam flexural rigidity by its mass per unit length are constants. When these restrictions are satisfied, the second order governing equation is an exact governing equation, and has a Helmholtz-like form.

3.1.3.1 Governing Equation for the Wave Modes Amplitude.

Here we consider a change of the dependent variable which transforms equation (3.6) in a four-dimensional system of first order differential equations.

Along the uniform part of the beam, the general solution of the governing equation (3.6) is given as a superposition of four wave modes, as follows:

$$y(s) = A \exp(i k_1 s) + B \exp(-i k_1 s) + C \exp(k_2 s) + D \exp(-k_2 s) \quad (3.7)$$

In the equation (3.7), the first two wave modes are propagating modes, and the last two are evanescent modes. Each wave mode has an associated wavenumber. For the propagating modes, the wave numbers are pure imaginary numbers with the same modulus, but opposite phase. For the evanescent modes, the wave numbers are real numbers with the same modulus, but opposite sign. The wave numbers $i k_1$ and k_2 are solutions of the dispersion relation

$$e i k^4 - \bar{P} k^2 - \omega^2 m = 0. \quad (3.8)$$

The wave mode amplitudes A, B, C and D are specified by boundary conditions

(finite beam) or by radiation conditions.

The rotation of the beam cross-section, the bending moment and the shear force can also be written in terms of the wave modes.

We assume that the transverse displacement along the non-uniform part of the beam can be written in the same form as in the case of the uniform beam, but now with the wave mode amplitudes and wave numbers as functions of the space coordinate, as follows:

$$y(s) = \underbrace{A(s) \exp(i k_1(s)s)}_{\tilde{A}(s)} + \underbrace{B(s) \exp(-i k_1(s)s)}_{\tilde{B}(s)} + \underbrace{C(s) \exp(k_2(s)s)}_{\tilde{C}(s)} + \underbrace{D(s) \exp(-k_2(s)s)}_{\tilde{D}(s)} \quad (3.9)$$

The new dependent variables are defined by incorporating the phase factor to the wave modes amplitude. These new dependent variables are denoted as $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$, according to equation (3.9). The wave numbers are still given by the uniform system dispersion relation (3.8), now assumed locally valid and with the non-dimensional flexural rigidity ei , the non-dimensional mass per unit length m and the non-dimensional tensile force $\bar{P}$ as functions of the space coordinate. The wave numbers are now given by the equations

$$i k_1(s) = \frac{i}{\sqrt{2ei(s)}} \left\{ -\bar{P}(s) + \sqrt{\bar{P}^2(s) + 4\omega^2 ei(s)m(s)} \right\}^{1/2}, \quad (3.10)$$

$$k_2(s) = \frac{1}{\sqrt{2ei(s)}} \left\{ \bar{P}(s) + \sqrt{\bar{P}^2(s) + 4\omega^2 ei(s)m(s)} \right\}^{1/2}. \quad (3.11)$$

For the non-uniform beam, we assume that the rotation of the beam cross-section, the bending moment and the shear force along the non-uniform beam are given in the same form as in the case of the uniform beam, but now with wave modes amplitudes and wave numbers as functions of the space coordinate. The functions $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$ will be sought such that:

$$\frac{dy}{ds} = i k_1(s)(\tilde{A}(s) - \tilde{B}(s)) + k_2(s)(\tilde{C}(s) - \tilde{D}(s)) \quad (3.12)$$

$$ei(s)\frac{d^2y}{ds^2} = -k_1^2(s)(\tilde{A}(s) + \tilde{B}(s)) + k_2^2(s)(\tilde{C}(s) + \tilde{D}(s)) \quad (3.13)$$

$$\begin{aligned} \frac{d}{ds} \left(ei(s)\frac{d^2y}{ds^2} \right) - \bar{P}(s)\frac{dy}{ds} = & -ik_1(s)(k_1^2(s) + ei(s)[k_2^2(s) - k_1^2(s)]) \left(\tilde{A}(s) - \tilde{B}(s) \right) \\ & + k_2(s)(k_2^2(s) - ei(s)[k_2^2(s) - k_1^2(s)]) \left(\tilde{C}(s) - \tilde{D}(s) \right) \end{aligned} \quad (3.14)$$

These representation for the rotation of the beam cross-section, for the bending moment and for the shear force can be justified if and only if equations (D.2), (D.3) and (D.4) are satisfied (see Appendix D). This set of equations furnishes the three first order differential equations for the quantities $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$. If we substitute the equations for the transverse displacement (3.9), for the bending moment (3.13) and for the shear stress (3.14) on the governing equation (3.6), we obtain one more equation involving the quantities $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$ and their first order derivatives, which is equation (D.5) in Appendix D.

The equation (D.5) plus the equations (D.2), (D.3) and (D.4) furnish a four dimensional system of first order differential equations for the new dependent variables $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$, as follows:

$$\begin{Bmatrix} \frac{d\tilde{A}(s)}{ds} \\ \frac{d\tilde{B}(s)}{ds} \\ \frac{d\tilde{C}(s)}{ds} \\ \frac{d\tilde{D}(s)}{ds} \end{Bmatrix} = [\mathbf{M}(s)] \begin{Bmatrix} \tilde{A}(s) \\ \tilde{B}(s) \\ \tilde{C}(s) \\ \tilde{D}(s) \end{Bmatrix} \quad (3.15)$$

The equations describing the elements of the system matrix, denoted $\mathbf{M}(s)$, in terms of the system parameters are given in Appendix D.

The linear combination of the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$ accounts for the propagating component of the wave disturbance along the non-uniform beam, and the linear

combination of the quantities $\tilde{C}(s)$ and $\tilde{D}(s)$ accounts for the evanescent component of the wave field along the non-uniform beam. Radiation conditions for left or right incidence can be written entirely in terms of the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$. Therefore, under conditions of weak or no coupling between the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$ and the quantities $\tilde{C}(s)$ and $\tilde{D}(s)$, wave propagation along the non-uniform beam should be entirely described by the evolution of the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$, as is shown in the next section.

3.1.3.2 Asymptotic Second Order Governing Equation.

We define quantities that are useful in discussing the restrictions necessary for the evolution of the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$ to decouple asymptotically from the evolution of the quantities $\tilde{C}(s)$ and $\tilde{D}(s)$.

The ratio $\nu(s)$ between the inertia force per unit length and the flexural rigidity and the ratio $\theta(s)$ between the tensile load and the flexural rigidity are defined, respectively, as:

$$\nu(s) = \frac{\omega^2 m(s)}{ei(s)}, \quad (3.16)$$

$$\theta(s) = \frac{\bar{P}(s)}{ei(s)} \quad (3.17)$$

We can express the wave numbers $k_1(s)$ and $k_2(s)$ and their derivative in terms of the quantities defined above, according to equations (E.1) to (E.4) of Appendix E. The derivatives of the quantities $\nu(s)$ and $\theta(s)$, present in equations (E.3) and (E.4) for the derivatives of the wave numbers, are given in terms of the beam parameters, as follows:

$$\frac{d\nu}{ds} = \frac{\nu(s)}{m(s)} \frac{dm}{ds} - \frac{\nu(s)}{ei(s)} \frac{d ei}{ds}, \quad (3.18)$$

$$\frac{d\theta}{ds} = \frac{1}{ei(s)} \frac{d\bar{P}}{ds} - \frac{\theta(s)}{ei(s)} \frac{d ei}{ds}. \quad (3.19)$$

The equations for elements of matrix $\mathbf{M}(s)$ are given in Appendix D in terms of the beam parameters, their derivatives, and in terms of the wave numbers $k_1(s)$ and $k_2(s)$ and their derivatives. Therefore, we need to estimate the order of magnitude of these quantities to be able to study the order of magnitude of the matrix elements $\mathbf{M}_{jk}$. The non-dimensional mass per unit length and the non-dimensional flexural rigidity are considered as quantities of order $O(1)$. To estimate the order of magnitude of the derivatives of these quantities, we define the ratio between the length scale λ of the incident wave and the length scale λ' of the non-uniformity variation as:

$$\Lambda = \frac{\lambda}{\lambda'}. \quad (3.20)$$

The magnitude of the non-uniformity in the non-dimensional flexural rigidity, in the non-dimensional mass per unit length and in the non-dimensional tensile force per one wavelength of the incident wave are defined as:

$$\max_{0 < s < \bar{L}} \{|ei(s+1) - ei(s)|\} \sim O(\varepsilon'), \quad (3.21)$$

$$\max_{0 < s < \bar{L}} \{|m(s+1) - m(s)|\} \sim O(\varepsilon'), \quad (3.22)$$

$$\max_{0 < s < \bar{L}} \{|\bar{P}(s+1) - \bar{P}(s)|\} \sim O(\delta). \quad (3.23)$$

Now, we can write the magnitude of the derivatives of the non-dimensional tensile force, of the non-dimensional mass per unit length and of the non-dimensional flexural rigidity in terms of the quantities defined above, as follows:

$$\frac{d ei}{ds} \sim O(\varepsilon' \Lambda), \quad (3.24)$$

$$\frac{dm}{ds} \sim O(\varepsilon \Lambda), \quad (3.25)$$

$$\frac{d\bar{P}}{ds} \sim O(\delta \Lambda). \quad (3.26)$$

The ratio between the non-dimensional inertia force per unit length and the non-dimensional flexural rigidity is a quantity of order $O(1)$ for large values of the frequency of the wave disturbance. The exception is the long wave limit, where $\nu(s)$ is a quantity of order of magnitude larger than one. In general, we can say that

$$\nu(s) \sim O(1) \quad (3.27)$$

Based on the estimates above for the order of magnitude of the non-dimensional mass per unit length, non-dimensional flexural rigidity, non-dimensional tensile force and their derivatives, we are ready to estimate the order of magnitude of the elements of matrix $\mathbf{M}(s)$. Since they can all be expressed, basically, in terms of the elements $M_{11}(s)$, $M_{21}(s)$, $M_{31}(s)$, $M_{13}(s)$, $M_{33}(s)$, $M_{43}(s)$ and M_{44} , these are the only ones which need to be analyzed. We denote them as the basic elements of matrix $\mathbf{M}(s)$.

We consider two regimes. First, the non-dimensional tensile force is very small compared to the non-dimensional flexural rigidity ($\theta \rightarrow 0$). Second, the non-dimensional tensile force is of the same or larger order than the order of magnitude of the non-dimensional flexural rigidity ($\theta \geq 1$).

For each of the two regimes considered, we express in the appendix E the order of magnitude of the wave numbers and their derivatives. We also describe in detail the order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$.

3.1.3.2.1 Regime $\theta \rightarrow 0$.

The order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$ is given in the Appendix E.1 in terms of the order of magnitude of the beam parameters and their derivatives, according to the equations (E.10) to (E.16). We define ε'' as

$$\varepsilon'' = \max\{\varepsilon, \varepsilon', \delta\}. \quad (3.28)$$

Now, the order of magnitude of the basic elements of matrix $\mathbf{M}(s)$ is given as:

$$M_{11}(s) \sim O(\nu^{1/4}), \quad (3.29)$$

$$M_{21}(s) \sim O(\varepsilon''\Lambda), \quad (3.30)$$

$$M_{31}(s) \sim O(\varepsilon''\Lambda), \quad (3.31)$$

$$M_{13}(s) \sim O(\varepsilon''\Lambda), \quad (3.32)$$

$$M_{33}(s) \sim O(\nu^{1/4}), \quad (3.33)$$

$$M_{43}(s) \sim O(\varepsilon''\Lambda), \quad (3.34)$$

$$M_{44}(s) \sim O(\varepsilon''\Lambda). \quad (3.35)$$

We assume the non-uniformity steepness of the beam parameters to be small. In other words,

$$\varepsilon''\Lambda \ll 1. \quad (3.36)$$

According to equations (3.29) to (3.35) and under the restriction given by equation (3.36), the evolution described by the four-dimensional system of first order differential equations (3.15) can be asymptotically described by the evolution of a diagonal system of first order differential equations with an error of $O(\varepsilon''\Lambda)$, which is small. This diagonal matrix is just the main diagonal of matrix $\mathbf{M}(s)$. Therefore, under this regime and the restriction given above, the evolution of the components describing the propagating and evanescent parts of the solution can be considered decoupled with an error of the order of $O(\varepsilon''\Lambda)$. Since, we want to study resonance effects between the non-uniformity and the wave field, we keep the coupling between the components describing the propagating part of the wave field. Therefore, the propagating part of the solution is described with an error of order of $O(\varepsilon''\Lambda)$ by a two dimensional system of first order differential equations with matrix given by the 2×2 upper left block of matrix $\mathbf{M}(s)$.

3.1.3.2.2 Regime $\theta \geq 1$.

The order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$ is given in the Appendix E.2 in terms of the order of magnitude of the beam parameters and their derivatives according to the equations (E.22) to (E.28).

We consider the same assumption as in the previous regime regarding the order of magnitude of the steepness of the non-uniformity. We give the order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$, as follows:

$$M_{11}(s) \sim O(\nu^{1/4}) \quad (3.37)$$

$$M_{21}(s) \sim O(\nu^{-1/2}\theta\epsilon''\Lambda) \quad (3.38)$$

$$M_{31}(s) \sim O(\epsilon''\Lambda) \quad (3.39)$$

$$M_{13}(s) \sim O(\nu^{1/4}\theta^{1/2}\epsilon''\Lambda) \quad (3.40)$$

$$M_{33}(s) \sim O(\theta^{1/2}) \quad (3.41)$$

$$M_{43}(s) \sim O(\epsilon''\Lambda) \quad (3.42)$$

$$M_{44}(s) \sim O(\theta^{1/2}) \quad (3.43)$$

According to equations (3.37) to (3.43), even with the assumption of small steepness, the coupling between the propagation part and the evanescent part of the solution is not necessarily small. For the propagating part of the solution to be described by a two dimensional system of first order differential equations, we need little energy to be transferred from the propagating part of the solution to its evanescent part. The transfer of energy from the propagating modes to the evanescent modes is governed by the elements $M_{31}(s)$, $M_{32}(s)$, $M_{41}(s)$ and $M_{42}(s)$, which have an order of magnitude of the order of $O(\epsilon''\Lambda)$. Therefore, under the restriction given by equation (3.36), the transfer of energy between the propagating and evanescent components of the wave field is small, and the evolution of the propagating part of the wave field can be described by the two-dimensional system of first order differential equation with matrix given by the 2×2 upper left block of matrix $\mathbf{M}(s)$. The error in this approximation

is of the order of $O(\varepsilon''\Lambda)$, which is small.

In both regimes, if the restriction of small steepness in the non-uniformity is satisfied, the propagating part of the wave field can be described asymptotically by a two dimensional system of first order differential equations with matrix given by the 2×2 upper left block of matrix $\mathbf{M}(s)$, as follows:

$$\begin{Bmatrix} \frac{d\tilde{A}(s)}{ds} \\ \frac{d\tilde{B}(s)}{ds} \end{Bmatrix} = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \begin{Bmatrix} \tilde{A}(s) \\ \tilde{B}(s) \end{Bmatrix} \quad (3.44)$$

This two-dimensional system of equations can be reduced through a change of the dependent variables to a second order differential equation, as discussed in the next section.

3.1.3.2.3 Second order differential equation.

We describe the change of the dependent variables used to obtain the second order governing equation from the system of equation (3.44), which governs the evolution of the propagating part of the wave field.

The new dependent variables are $\psi(s)$, the transverse displacement, and $\phi(s)$, the rotation of the beam cross-section due to the propagating component of the wave field, normalized by $-k_1(s)$. These new dependent variables are related to the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$ through the matrix equation:

$$\begin{Bmatrix} \psi(s) \\ \phi(s) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{Bmatrix} \tilde{A}(s) \\ \tilde{B}(s) \end{Bmatrix} \quad (3.45)$$

We substitute the matrix equation (3.45) into the system of equations (3.44). This leads to a new system of equations in terms of the new dependents variables ψ and ϕ .

The first equation of the system of equations for the dependent variables ψ and ϕ allow us to write the variable ϕ in terms of the transverse displacement ψ and its

derivative $\frac{d\psi}{ds}$. If we substitute this expression for ϕ into the second equation of the system of equations for the dependent variables ψ and ϕ , we obtain the second order differential equation

$$\psi_{ss} + p(s)\psi_s + q(s)\psi(s) = 0. \quad (3.46)$$

The quantities $p(s)$ and $q(s)$ are defined in terms of the beam parameters and in terms of the wavenumber $k_1(s)$ and $k_2(s)$ as:

$$p(s) = \frac{1}{ei(s)} \frac{dei(s)}{ds} + \frac{1}{k_1^2(s) + k_2^2(s)} \frac{d}{ds} (k_1^2(s) + k_2^2(s)), \quad (3.47)$$

$$q(s) = \frac{1}{4} \left(\frac{1}{ei(s)} \frac{dei(s)}{ds} \right)^2 - \frac{1}{[k_1^2(s) + k_2^2(s)]^2} \left[2 k_1(s) \frac{d k_1}{ds} - \frac{\bar{P}}{2(ei(s))^2} \frac{dei}{ds} \right]^2 \\ + \frac{1}{k_1^2(s) + k_2^2(s)} \left\{ \frac{d}{ds} \left[2k_1(s) \frac{dk_1}{ds} + \frac{k_1^2(s)}{ei(s)} \frac{dei}{ds} \right] \right\} + k_1^2(s). \quad (3.48)$$

Let us write this second order equation in a more standard form through a change of the dependent variable, which is given below by the equation

$$\psi(s) = \chi(s) \exp\left(-\frac{1}{2} \int^s p(t) dt\right) = \chi(s) \left\{ \bar{P}^2(s) + 4\omega^2 m(s) ei(s) \right\}^{-1/4}. \quad (3.49)$$

Equation (3.46) reduces to the form

$$\chi_{ss} + Q(s)\chi(s) = 0. \quad (3.50)$$

The coefficient function $Q(s)$ is given in terms of $p(s)$ and $q(s)$ by the equation

$$Q(s) = -\frac{1}{2} \frac{dp}{ds} - \frac{1}{4} (p(s))^2 + q(s). \quad (3.51)$$

In terms of the beam non-dimensional flexural rigidity, non-dimensional tensile force and non-dimensional mass per unit length, the coefficient function $Q(s)$ assumes the form:

$$Q(s) = -\frac{1}{2}F(s)^{-1/2}\frac{d^2\bar{P}}{ds^2} - \frac{1}{4F(s)}\left(\frac{d\bar{P}}{ds}\right)^2 + \frac{1}{4}F(s)^{-3/2}\frac{dF}{ds}\frac{d\bar{P}}{ds} + k_1^2(s), \quad (3.52)$$

with

$$F(s) = \bar{P}(s)^2 + 4\omega^2 m(s)ei(s). \quad (3.53)$$

Before discussing the aspects of the qualitative behavior of the second order differential equation (3.49), we discuss the radiation conditions for left and right wave incidence. The radiation conditions are imposed on the uniform parts of the beam ($s \rightarrow \pm\infty$). In terms of the variable $\psi(s)$, the radiation conditions for left and right wave incidence follows:

- For left wave incidence we have the radiation conditions:

$$\psi(s) = \begin{cases} A \exp(i k_1^- s) + R^- A \exp(-i k_1^- s) & \text{as } s \rightarrow -\infty, \\ T^- A \exp(i k_1^+ s) & \text{as } s \rightarrow +\infty. \end{cases} \quad (3.54)$$

- For right wave incidence we have the radiation conditions:

$$\psi(s) = \begin{cases} T^+ A \exp(-i k_1^- s) & \text{as } s \rightarrow -\infty, \\ A \exp(-i k_1^+ s) + R^+ A \exp(i k_1^+ s) & \text{as } s \rightarrow +\infty. \end{cases} \quad (3.55)$$

A is the amplitude of the incident wave, and R^- and T^- (R^+ and T^+) are the reflection and transmission coefficients for left (right) wave incidence radiation condition. The wave numbers k_1^- and k_1^+ are the value of the wavenumber $k_1(s)$ for $s \leq 0$ and $s \geq \bar{L}$ respectively.

Along the uniform part of the beam $\chi(s)$ is proportional to $\psi(s)$. Therefore, the radiation conditions in terms of the dependent variable $\chi(s)$ has the same form as

the radiation condition in terms of the dependent variable $\psi(s)$. If the parameters of the beam are the same at both semi-infinite uniform parts, the radiation conditions in terms of the variables ψ and χ are exactly the same. The radiation condition in terms of the variable $\chi(s)$ follows:

- For left wave incidence we have the radiation conditions:

$$\chi(s) = \begin{cases} A \exp(i k_1^- s) + \tilde{R}^- A \exp(-i k_1^- s) & \text{as } s \rightarrow -\infty, \\ \tilde{T}^- A \exp(i k_1^+ s) & \text{as } s \rightarrow +\infty. \end{cases} \quad (3.56)$$

- For right wave incidence we have the radiation conditions:

$$\chi(s) = \begin{cases} \tilde{T}^+ A \exp(-i k_1^- s) & \text{as } s \rightarrow -\infty, \\ A \exp(-i k_1^+ s) + \tilde{R}^+ A \exp(i k_1^+ s) & \text{as } s \rightarrow +\infty. \end{cases} \quad (3.57)$$

$\tilde{R}^-$ and $\tilde{T}^-$ ($\tilde{R}^+$ and $\tilde{T}^+$) are the reflection and transmission coefficients for left (right) wave incidence in terms of the variable $\chi(s)$. Between the coefficients $T^\pm$ and $\tilde{T}^\pm$ we have the following relation:

$$\tilde{T}^- = \left\{ \frac{(P_0)^2 + 4\omega^2 m_0 e i_0}{(P_1)^2 + 4\omega^2 m_1 e i_1} \right\}^{1/4} T^-, \quad (3.58)$$

$$\tilde{T}^+ = \left\{ \frac{(P_1)^2 + 4\omega^2 m_1 e i_1}{(P_0)^2 + 4\omega^2 m_0 e i_0} \right\}^{1/4} T^+. \quad (3.59)$$

In the equations (3.58) and (3.59), the index 0 and 1 in the beam parameters refer, respectively, to the parameter values for $s \leq 0$ and $s \geq \bar{L}$. The reflection coefficients $R^\pm$ are equal to the reflection coefficients $\tilde{R}^\pm$.

In the next section, we discuss the qualitative behavior of the second order differential equation in terms of the Bernoulli-Euler beam parameters.

3.1.3.2.4 Properties of the second order differential equation.

The second order differential equation (3.49) in terms of the variable $\chi(s)$ satisfies energy flux conservation. In other words:

$$|\tilde{R}^\pm|^2 + |\tilde{T}^\pm|^2 = 1. \quad (3.60)$$

This is not true for the second order differential equation (3.46). Its reflection and transmission coefficients satisfy the following relation:

$$\pm i 2 k_1 A^2 (\pm |T^\mp|^2 \pm |R^\mp|^2 \mp 1) = \exp(-\int_0^{\bar{L}} p(t) dt). \quad (3.61)$$

The coefficient function $Q(s)$ can be simplified if we make assumptions regarding the Bernoulli-Euler beam parameters. If these parameters are constants, the coefficient function $Q(s)$ is a constant, which is equal to the square of the wavenumber k_1 . The second order equation (3.50) in this case is a one-dimensional Helmholtz equation with constant wavenumber k_1 , which is the wavenumber associated with the propagating modes for the Bernoulli-Euler beam with constant material and geometrical properties. Therefore, along the uniform part of the beam, the coefficient function $Q(s)$ gives the right wavenumber for the propagating modes.

If we assume only the tensile force to be a constant, equation (3.52) simplifies to

$$Q(s) = k_1^2(s). \quad (3.62)$$

The second order differential equation under the restriction of constant tensile force looks like a Helmholtz equation. This is not true, since we can not write the coefficient function $Q(s)$ as a “wave frequency” times an index of refraction. The dependence on the frequency for the coefficient function $Q(s)$ in the form above is more complicated than the dependence encountered in Helmholtz-like equations, as revealed by equation (3.10). If we impose special restrictions on the non-dimensional mass per unit length and on the non-dimensional flexural rigidity, we end up with a Helmholtz-like equation, which is discussed in the next section.

Since the asymptotic governing equation is a second order differential equation, its qualitative behavior with respect to the non-dimensional tensile force, non-dimensional mass per unit length and non-dimensional flexural rigidity is revealed through the study of the sign of the coefficient function $Q(s)$. When the non-dimensional tensile force is a constant, the coefficient function $Q(s)$ is always positive, and the governing equation predicts propagating wave-like solutions. If the tensile force is not a constant, the coefficient function $Q(s)$ (see equation (3.52)) may assume negative values, which implies exponentially decaying or growing solutions. If this happens for some interval of s along the beam, we have exponentially small wave transmission and almost complete wave reflection. If $Q(s)$ happens to be negative for a range of wave frequencies at disjoint intervals of the space coordinate s , trapped modes for some wave frequency values may exist.

The coefficient function $Q(s)$ may be negative for some range of s , according to equation (3.52), if and only if

$$\frac{d^2 \bar{P}}{ds^2} > 0, \quad (3.63)$$

$$\frac{dF}{ds} \frac{d\bar{P}(s)}{ds} < 0. \quad (3.64)$$

The full governing equation should behave in the way predicted by the second order governing equation (3.49), specially if the restriction of small non-uniformity steepness is satisfied. Therefore, a Bernoulli-Euler beam under variable tensile force satisfying the equations (3.63) and (3.64) may have trapped modes, and allow only exponentially small transmission. In section 3.1.4, we give an example of a non-uniformity which satisfies equations (3.63) and (3.64) for intervals of the space coordinate s .

3.1.3.3 Second Order Governing Equations as an Exact Governing Equation.

According to the system of equations (3.15), the coupling between the propagating and evanescent part of the wave field is governed by the elements $M_{jk}(s)$ and

$M_{kj}(s)$ ($j=1,2$ and $k=3,4$). If the non-dimensional tensile force and the product of the non-dimensional mass per unit length by the non-dimensional flexural rigidity are constants, we show in Appendix F that the matrix elements $M_{jk}(s)$ and $M_{kj}(s)$ are zero, which implies that the second order differential equation (3.49) is an exact equation governing mono-chromatic wave propagation along the non-uniform beam, and assumes the Helmholtz-like form

$$\chi_{ss} + \hat{\Omega}^2(\omega, \bar{P}, C) \tilde{n}^2(s) \chi(s) = 0. \quad (3.65)$$

$\tilde{n}(s)$ is the index of refraction, which is given in terms of the non-dimensional flexural rigidity or non-dimensional mass per unit length, according to the equation

$$\tilde{n}(s) = \frac{1}{\sqrt{ei(s)}} = \sqrt{\frac{m(s)}{C}}. \quad (3.66)$$

The “wave frequency” $\hat{\Omega}(\omega, \bar{P}, C)$ is defined by the equation

$$\hat{\Omega}(\omega, \bar{P}, C) = \left\{ -\frac{\bar{P}}{2} + \frac{1}{2} \sqrt{\bar{P}^2 + 4\omega^2 C} \right\}^{1/2}. \quad (3.67)$$

We define the dimensional counter-part of $\chi(s)$ as $w(x)$. The relation between these two dependent variables is given by the equation

$$w(x) = \frac{h_0 \sqrt{E_0 I_0}}{\lambda} \chi(s). \quad (3.68)$$

The square of the variable $w(x)$ has the dimension of force times length squared. The dimensional form of the second order governing equation (3.65) follows:

$$w_{xx} + \tilde{\Omega}^2(P, \Omega, C') n(x)^2 w(x) = 0, \quad (3.69)$$

where the index of refraction $n(x)$ is given by the equation

$$n(x) = \frac{1}{\sqrt{EI(x)}} = \sqrt{\frac{\rho A(x)}{C'}}. \quad (3.70)$$

The “wave frequency” $\tilde{\Omega}(P, \Omega, C')$ has the dimension of force and its expression in terms of the constant tensile force P and the constant C' is given by the equation

$$\tilde{\Omega}(P, \Omega, C') = \left\{ -\frac{P}{2} + \frac{1}{2} \sqrt{P^2(x) + 4\Omega C'} \right\}^{1/2}. \quad (3.71)$$

The constant C' has the dimension of mass times energy, and it is the value of the product of the mass per unit length by the flexural rigidity along the entire beam.

3.1.4 Applications - The Analysis Problem.

In this section, we consider wave interaction with five examples of non-uniformities. For the first three examples, the beam material properties are assumed to be constant. We allow variations only in the cross-sectional geometry and in the tensile force. For the last two examples, we prescribe the flexural rigidity and the mass per unit length such that their product is a constant. The tensile force is assumed constant.

For each example of non-uniformity, we give results for the modulus of the reflection coefficient for both governing equations. Results from the second and full governing equations are obtained through numerical simulation. The exception is the last example of non-uniformity, where the second order governing equation has an analytic solution.

We use the finite difference method to simulate numerically the governing equations. The radiation conditions are incorporated in the finite difference method.

A detailed description of each example considered is given in table 3.1.

3.1.4.1 Non-uniform Geometry.

For the first three examples, we considered a beam with rectangular cross-section. The only non-uniformity is in the changing dimensions of the cross-section. The

material properties are considered constant along the entire beam, which is assumed to be made of aluminum with density $\rho_0 = 2.7 \times 10^3 \text{ kg/m}^3$ and with modulus of elasticity $E_0 = 7.1 \times 10^{10} \text{ N/m}^2$. Along the non-uniform part of the beam, the height of the cross-section varies in a prescribed way. The height below and above the mean line are prescribed by different given functions. Along the uniform parts of the beam ($s \leq 0$ and $s \geq \bar{L}$), the height h_0 of the cross-section is constant and its value is 0.01 m . The width of the cross-section is constant along the entire beam, and it is denoted as b . The value of the width b is 0.05 m .

Table 3.1: Description of the Examples.

Example	First	Second	Third	Fourth	Fifth
Associated figure	Figure 3-1	Figure 3-2	Figure 3-3	Figure 3-4	Figure 3-5
Non-dimensional Flexural rigidity	Equation (3.73)			Eq. (3.77)	Eq. (3.78)
Non-dim. mass per unit length	Equation (3.72)			$m(s)ei(s) = C$	
Non-dimensional Tensile force	constant		Eq. (3.76)	constant	
Function $f(s)$	Equation (3.74)		Eq. (3.75)		
Function $g(s)$	$= f(s)$	$= -f(s)$	$= -f(s)$		
Constant C .				1	1
Non-uniformity parameters.	A/h_0	A/h_0	$\frac{A}{h_0}, A_p, k_g$ and k_p .	A, k_g, s_0, γ and κ .	S, N, M and s_0 .
Parameter A/h_0	0.2, 0.1 and 0.05	0.1, 0.05 and 0.01	0.05 and 0.01		
Parameter A .				0.2, 0.1 and 0.05	
Parameter A_p			0N and 10000N		
For other Parameters			Caption of Figure 3-3	Caption of Figure 3-4	Table 3.3
Function $Q(s)$	Equation (3.62)		Eq. (3.52)	$\hat{\Omega}^2(\omega, \bar{P}, C)\tilde{n}^2(s)$ and eqs. (3.66) and (3.67)	
Purpose	Asymptotic behavior of 2-nd order governing equation.		Both gov. eqs. with same qualitative behavior	2-nd order equation as an exact governing equation.	

The non-dimensionalization constants for the mass per unit length and flexural rigidity in the first three examples are $\rho_0 A_0 = \rho_0 b h_0$ and $E_0 I_0 = E_0 \frac{b h_0^3}{12}$, and the non-dimensional expressions for the mass per unit length and flexural rigidity in terms of the functions $f(s)$ and $g(s)$, describing the height non-uniformity, are

$$m(s) = \begin{cases} 1 & \text{for } s \leq 0 \text{ and } s \geq \bar{L}, \\ 1 + \frac{A}{h_0}(f(s) - g(s)) & \text{for } 0 < s < \bar{L}, \end{cases} \quad (3.72)$$

$$ei(s) = \begin{cases} 1 & \text{for } s \leq 0 \text{ and } s \geq \bar{L}, \\ (1 + \frac{A}{h}(f(s) - g(s)))^3 + 3(1 + \frac{A}{h}(f(s) - g(s)))(\frac{A}{h}(f(s) + g(s)))^2 & \text{for } 0 < s < \bar{L}. \end{cases} \quad (3.73)$$

A is the amplitude of the height non-uniformity.

3.1.4.1.1 Constant Tensile Force.

For the first two examples the tensile force is assumed constant. If we substitute the equations (3.72) and (3.73) into equation (3.62) for the coefficient function $Q(s)$, we obtain its expression in terms of the functions $f(s)$ and $g(s)$.

The function $f(s)$ for the first two examples is

$$f(s) = \cos(k_g \lambda s + \gamma). \quad (3.74)$$

The parameter $k_g \lambda$ is the non-dimensional wavenumber of the non-uniformity and here γ is a phase factor. We consider that $k_g = 2\pi$, which implies a non-uniformity with wavelength equal to $1m$. The size of the non-uniformity is equal to 22 times its wavelength. The function $g(s)$ is specified in terms of the function $f(s)$ according to table 3.1.

For the first example, the non-uniformity comes from the flexural rigidity, and in its expression we have the square of the parameter A/h_0 . Therefore, the actual non-uniformity magnitude is $(A/h_0)^2$. The actual wavelength of the non-uniformity is equal to half the wavelength of the function $f(s)$.

For the second example, the non-dimensional mass per unit length is a linear function of $f(s)$, and the magnitude of the uniformity is given by the parameter

A/h_0 . For the non-dimensional flexural rigidity, the magnitude in the non-uniformity is now given by the cube of the parameter A/h_0 . The non-dimensional mass per unit length and the non-dimensional flexural rigidity have the same periodicity as the function $f(s)$. Values of the non-uniformity parameter A/h_0 are given in table 3.1.

Results regarding the first and second examples are illustrated, respectively, in figures 3-1 and 3-2. A detailed description of figures 3-1 and 3-2 is given in the second and third column of table 3.2. According to parts (b) and (d) of figure 3-1 and 3-2, the agreement between the results for the modulus of the reflection coefficient from the numerical simulation of both governing equations increases as the magnitude (parameter A/h_0) of the geometry non-uniformity decreases. This illustrates the asymptotic nature of the second order governing equation with respect to the steepness in the beam non-uniformity.

The peak in the reflection coefficient in part (a) of figure 3-1 and in parts (a) and (c) of figure 3-2 is due to Bragg scattering phenomenon. Bragg scattering peaks in the reflection coefficient happens when the ratio between the wavenumber of the incident wave multiplied by 2 and the non-uniformity wavenumber is a natural number. The Bragg scattering peak (arrow) illustrated in figures 3-1 and 3-2 are first Bragg scattering peaks.

Table 3.2: Description of the Figures.

Figure	3-1	3-2	3-3	3-4	3-5
Results	Modulus of the reflection coefficient for left wave incidence as a function of the wave frequency.				
Part (a) description	Modulus of the reflection coeff. from the numerical simulation of equations (3.6) and (3.50) for 3 values of A/h_0 and with $P = 0N$.	Modulus of the reflection coeff. from the numerical simulation of eqs. (3.6) and (3.50) with $A/h_0 = 0.05$ and 0.01 , $A_P = 0N$, $k_P = 0$ and $k_g = 2$.	Modulus of the reflection coeff. from the numerical simulation of eqs. (3.6) and (3.50) with $A = 0.2, 0.1$ and 0.05 and $P = 0N$.	Modulus of the reflection coeff. from the numerical simulation of eq. (3.6) and from eq. (3.79) with $k_g = 10$ and 5 .	
Part (b) description	Difference between the modulus of the reflection coeff. from the numerical simulation of eq. (3.6) and eq. (3.50) for 3 values of A/h_0 with $P = 0N$.	Same as Part (a), but with $A_P = 10000N$, $k_P = 10$ and $k_g = 2$.	Same as Part (a), but with $P = 10000N$.	Same as part (a).	
Part (c) description	Same as part (a), but with $P = 10000N$.	Intervals of s where $Q(s) < 0$ as a function of wave frequency.		Same as part (a).	
Part (d) description	Same as part (b), but with $P = 10000N$.			Same as part (a).	
Part (e) description				Non-dim. Flexural rigidity distributions.	
Resonance	1st Bragg scattering peak in part (a)	1st Bragg scattering peak in parts (a) and (c).		1st and 2nd Bragg scattering peaks in parts (a) and 1st peak in part (b).	

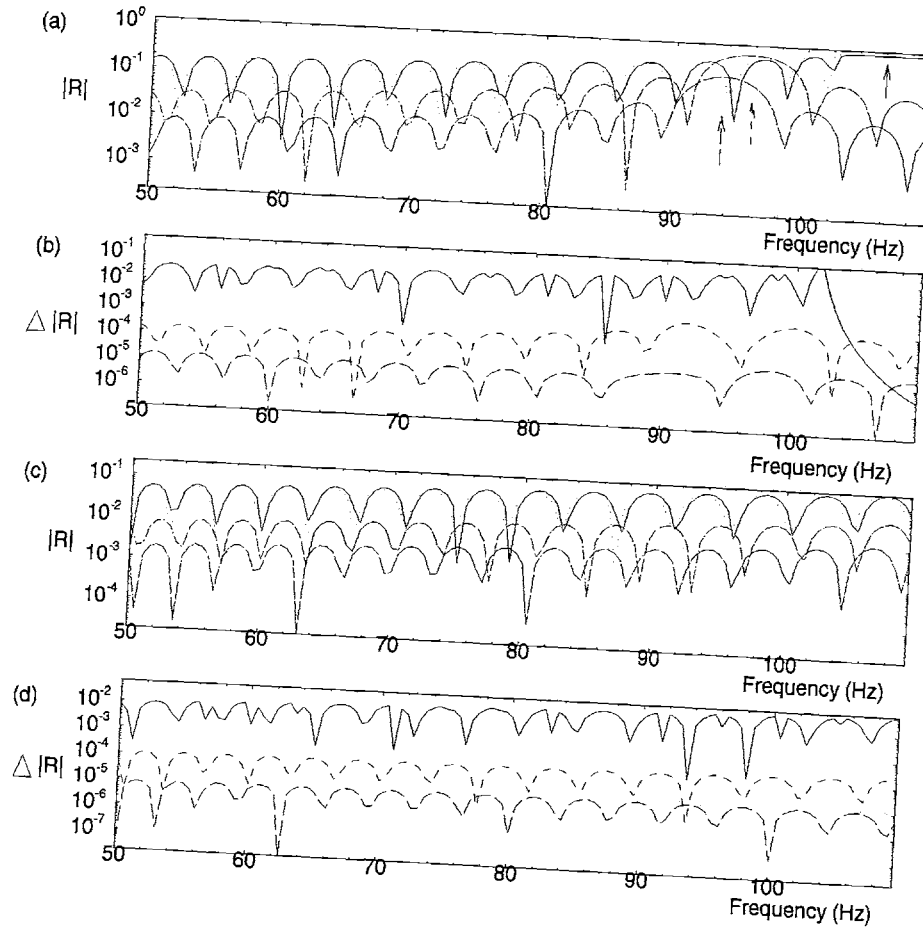


Figure 3-1: Reflection coefficient as a function of wave frequency of the incident wave. $|R|$: modulus of the reflection coefficient. $\Delta|R|$: difference between modulus of the reflection coefficient. Vertical arrows: Bragg scattering peaks. In parts (a) and (c), lines ———, - - - - - and - - - - - : numerical simulation of the full governing equation (3.6), respectively, for $A/h_0 = 0.2, 0.1$ and 0.05 , and lines ·····, - - - - - and - - - - - : numerical simulation of the second order governing equation (3.49) for $A/h_0 = 0.2, 0.1$ and 0.05 . In parts (b) and (d), lines ———, - - - - - and - - - - - : difference between modulus of the reflection coefficient, respectively, for $A/h_0 = 0.2, 0.1$ and 0.05 .

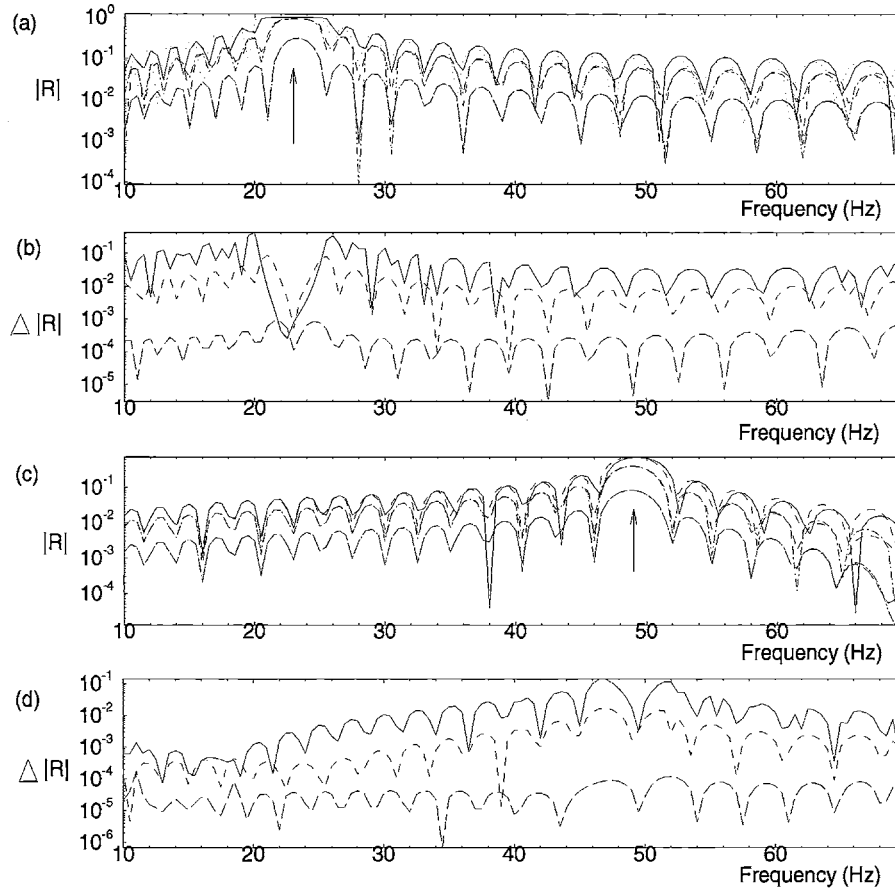


Figure 3-2: Reflection coefficients as a function of wave frequency of the incident wave. $|R|$: modulus of the reflection coefficient. $\Delta|R|$: difference between modulus of the reflection coefficient. Vertical arrows: Bragg scattering peaks. In parts (a) and (c), lines ———, - - - - - and - - - - - : numerical simulation of the full governing equation (3.6), respectively, for $A/h_0 = 0.1, 0.05$ and 0.01 , and lines ······, - - - - - and - - - - - : numerical simulation of the second order governing equation (3.49), respectively, for $A/h_0 = 0.1, 0.05$ and 0.01 . In parts (b) and (d), lines ———, - - - - - and - - - - - : difference between modulus of the reflection coefficient, respectively, for $A/h_0 = 0.1, 0.05$ and 0.01 .

3.1.4.1.2 Variable Tensile Force.

Results in figure 3-3 for the third example illustrate the fact that the qualitative behavior predicted by the second order governing equation is in agreement with the

results obtained using the full governing equation. Results presented in figure 3-3 are described in the fourth column of table 3.2.

For the third example, the function $f(s)$ is

$$f(s) = -\text{sech}(k_g \lambda (s - s_0)). \quad (3.75)$$

The product $A k_g \lambda / h_0$ gives the magnitude of this non-uniformity steepness. The function $g(s)$ is related to the function $f(s)$ according to table 3.1. The non-dimensional tensile force is given as:

$$\bar{P}(s) = -A_p \text{sech}(k_p \lambda (s - s_0)), \quad (3.76)$$

where A_p gives the magnitude of the non-dimensional non-uniformity in the tensile load, and the product $A_p k_p \lambda$ gives its steepness.

The non-dimensional mass per unit length and the non-dimensional flexural rigidity reaches a maximum or minimum at $s = s_0$, and the non-dimensional tensile force reaches a maximum negative value at $s = s_0$. We may choose the values of the non-uniformity parameters A/h_0 , A_p , k_g and k_p such that the coefficient function $Q(s)$ reaches negative values for intervals of the space coordinate. When this happens, the second order equation (3.50) predicts exponentially small transmission and almost complete reflection.

For part (b) of figure 3-3, we chose the parameters A/h_0 , A_p , k_g and k_p such that the coefficient function $Q(s)$ assumes negative values for two intervals of the space coordinate. In this case, large wave reflection is observed. The size of the intervals where $Q(s)$ is negative is illustrated in part (c) of figure 3-3 as a function of the frequency of the incident wave.

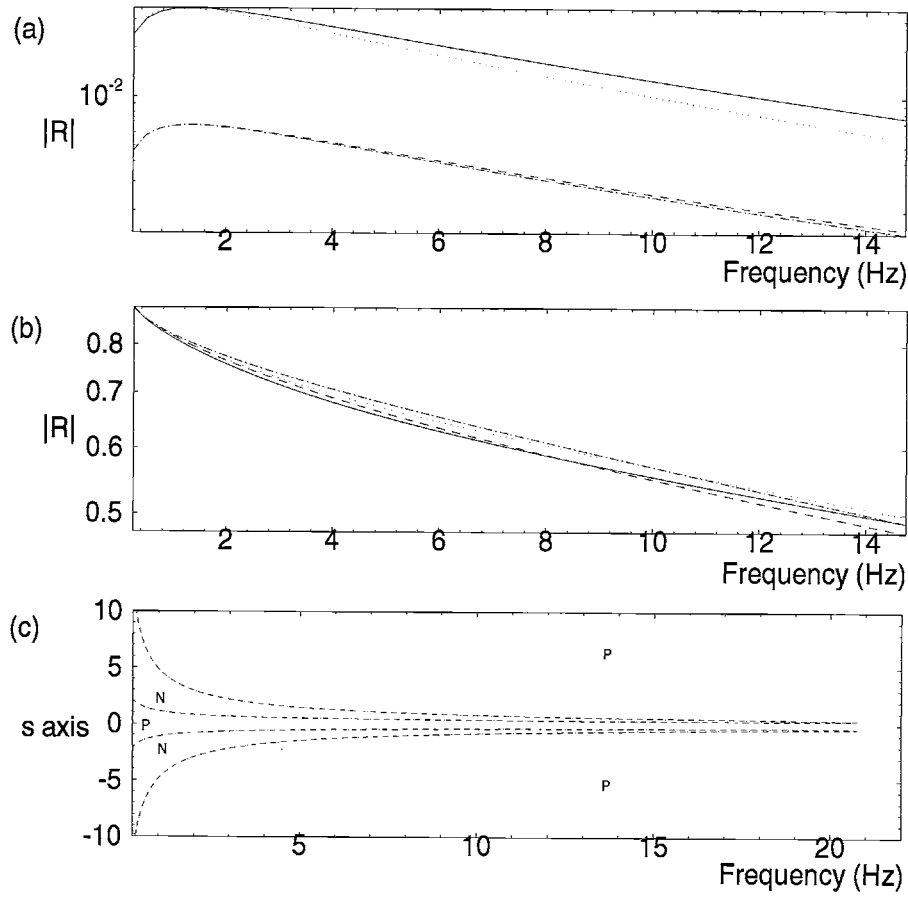


Figure 3-3: Modulus of the reflection coefficient as a function of wave frequency of the incident wave. In parts (a) and (b), lines ——— and - - - - - : numerical simulation of the full governing equation (3.6) with A/h_0 , respectively, 0.05 and 0.01, and lines and - · - · - · : numerical simulation of the second order governing equation (3.49) with A/h_0 , respectively, 0.05 and 0.01. Part (c) displays contour plots for $Q(s) = 0$. The letters P and N in part (c) stands for positive and negative values of $Q(s)$. Line - - - - - : $A_P = 10000N$, $k_p = 10$, $k_g = 2$ and $A/h_0 = 0.05$, and line - · - · - · : $A_P = 10000N$, $k_p = 10$, $k_g = 2$ and $A/h_0 = 0.01$.

3.1.4.2 Tensile force and the mass per unit length times flexural rigidity are constant.

For the fourth and fifth examples, the tensile force and the product of the flexural rigidity by the mass per unit length are constants. Under such conditions, the results predicted by both governing equations are the same, as illustrated in figures 3-4 and 3-5. These figures are described, respectively, in the fifth and sixth columns of table 3.2.

The fourth example is described by the fifth column of table 3.1. The non-dimensional flexural rigidity for this example is prescribed according to the equation

$$ei(s) = 1 + \frac{A}{2} \cos(k_g \lambda s + \gamma) (\tanh(\kappa(s + s_0)) + \tanh(\kappa(s - s_0))), \quad (3.77)$$

which is a periodic function along a finite part of the beam and otherwise zero.

The fifth example is described in the sixth column of table 3.1. The non-dimensional flexural rigidity for this example is

$$ei(s) = S - N \frac{\exp(k_g \lambda(s + s_0))}{1 + \exp(k_g \lambda(s + s_0))} - 4M \frac{\exp(k_g \lambda(s + s_0))}{(1 + \exp(k_g \lambda(s + s_0)))^2}. \quad (3.78)$$

For $N = 0$, equation (3.78) is basically the square of the hyperbolic secant function, and for $M = 0$, equation (3.78) gives basically the hyperbolic tangent function. The second order equation for this non-uniformity has an analytic solution in terms of Hyper-geometric functions, as discussed in Appendix G. The reflection and transmission coefficients are given in terms of Gamma functions by the equations

$$R^- = \frac{\Gamma(\beta)\Gamma(1-\gamma)\Gamma(\gamma-\delta)}{\Gamma(\beta-\gamma+1)\Gamma(\gamma-1)\Gamma(1-\delta)}, \quad (3.79)$$

$$T^- = \frac{\Gamma(\beta)\Gamma(\gamma-\delta)}{\Gamma(\beta-\delta+1)\Gamma(\gamma-1)}. \quad (3.80)$$

The quantities β , γ and δ are given in terms of the parameters $\tilde{\Omega}(\omega, \bar{P}, C)$, k_g , λ , S , N

and M according to the equations

$$\gamma = 1 + i2 \frac{\tilde{\Omega}(\omega, \bar{P}, C)}{|k_g \lambda|} \sqrt{S}, \quad (3.81)$$

$$\delta = \frac{1}{2} + \frac{1}{2} \left[1 - \frac{16\tilde{\Omega}(\omega, \bar{P}, C)^2 M}{(k_g \lambda)^2} \right] + i \frac{\tilde{\Omega}(\omega, \bar{P}, C)}{|k_g \lambda|} \left(\sqrt{S} - \sqrt{S - N} \right), \quad (3.82)$$

$$\beta = \frac{1}{2} + \frac{1}{2} \left[1 - \frac{16\tilde{\Omega}(\omega, \bar{P}, C)^2 M}{(k_g \lambda)^2} \right] + i \frac{\tilde{\Omega}(\omega, \bar{P}, C)}{|k_g \lambda|} \left(\sqrt{S} + \sqrt{S - N} \right), \quad (3.83)$$

as suggested by Brekhovskikh [10], page 55. The parameter $\tilde{\Omega}(\omega, \bar{P}, C)$ is defined by equation (3.67).

According to figure 3-5, the modulus of the reflection coefficient from the numerical simulation of the full governing equation (3.6) agrees with the modulus of the reflection coefficient given by equation (3.79). The non-uniformity parameters used in parts (a), (b), (c) and (d) of figure 3-5 are displayed in table 3.3. Part (e) of figure 3-5 illustrates the distributions of the non-dimensional flexural rigidity along the beam used to generate the results in parts (a), (b), (c) and (d) of figure 3-5.

Table 3.3: Parameters for the non-dimensional flexural rigidity and tensile force values.

Figure	Parameter S	Parameter N	Parameter M	Parameter s_0	Tensile force P
3-5 (a)	1.0	0.5	0.0	0.0	0 N
3-5 (b)	1.0	0.5	0.0	0.0	10000 N
3-5 (c)	1.0	0.0	-0.5	0.0	0 N
3-5 (d)	1.0	0.0	-0.5	0.0	10000 N

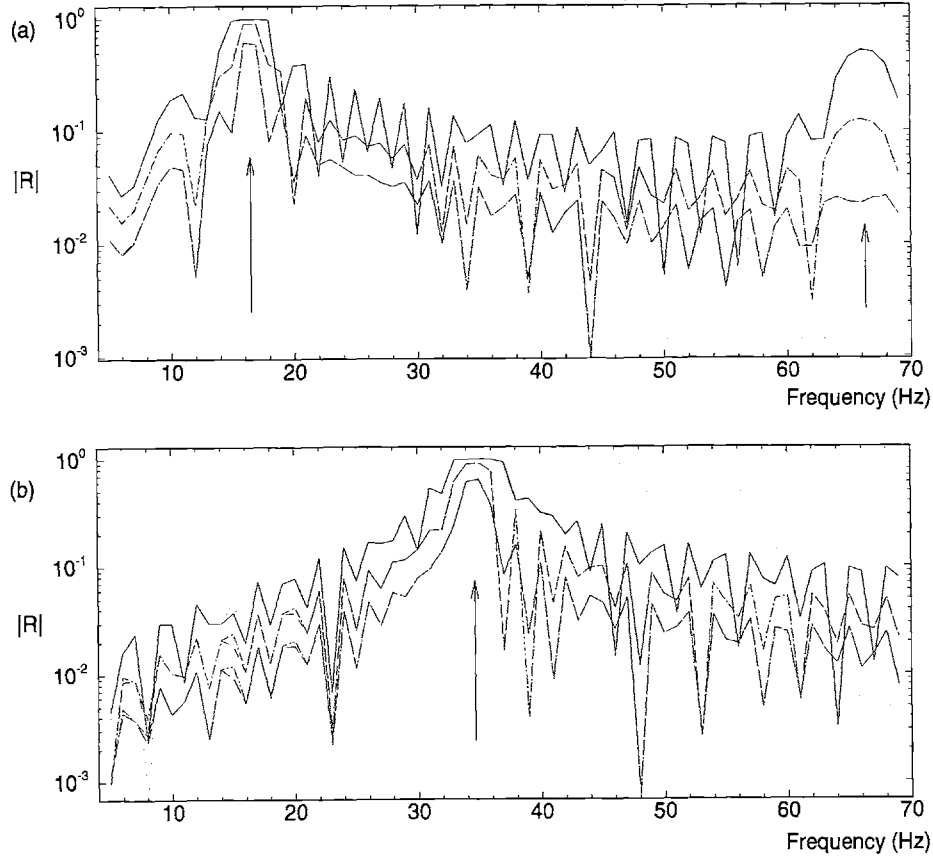


Figure 3-4: Modulus of the reflection coefficient as a function of wave frequency of the incident wave. Non-dimensional flexural rigidity and mass per unit length defined by equations (3.77) and (F.3), with $C = 1$, $\kappa = 20$, $k_g = 2\pi$, $s_0 = 10.25$ and $\gamma = 0$. Vertical arrow: Bragg scattering peak. Lines ———, - - - - - and - - - - : numerical simulation of the full governing equation (3.6) with A , respectively, equals to 0.2, 0.1 and 0.05. Lines ······, -·-·-·- and -·-·-·- : numerical simulation of the second order governing equation (3.49) with A , respectively, equals to 0.2, 0.1 and 0.05.

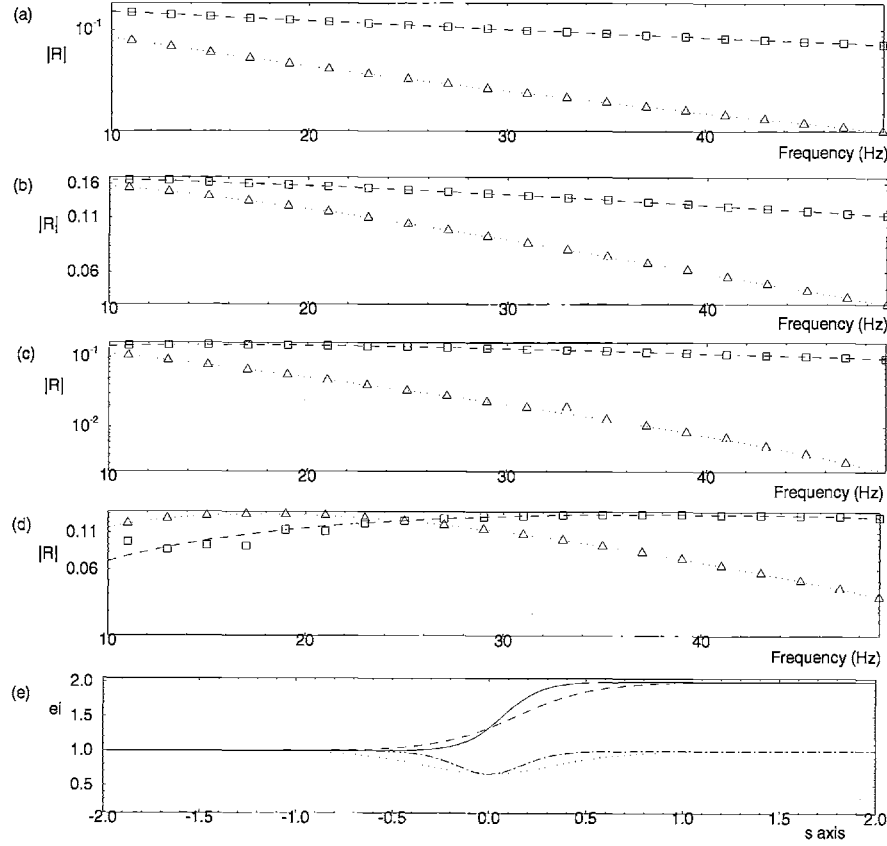


Figure 3-5: Modulus of the reflection coefficient as a function of wave frequency of the incident wave. Non-dimensional flexural rigidity and mass per unit length defined by equations (3.78) and (F.3) with $C = 1$. Lines - - - - - and : numerical simulation of the full governing equation (3.6). Symbols $\square$ and $\triangle$: Closed form solution given by equations (3.78) and (3.81). Line - - - - - and symbol $\square$: $k_g = 10$. Line and symbol $\triangle$: $k_g = 5$. Values of the parameters S, N, M and s_0 and the tensile force P for parts (a), (b), (c) and (d) are listed in table 3.3. In part (e), lines ——— and - - - - - : non-dimensional flexural rigidity distribution used in parts (a) and (b) with k_g , respectively, equals to 10 and 5 , and lines - - - - - and : non-dimensional flexural rigidity distribution used in parts (c) and (d) with k_g , respectively, equals to 10 and 5.

3.1.5 Applications - The Design Problem.

We are going to design the non-uniformity in the mass per unit length and in the flexural rigidity of a Bernoulli-Euler beam such that the relationship between the reflection coefficient and the wavelength of the incident wave has a prescribed form.

We assume the tensile force applied along the beam and the product of the mass per unit length by the flexural rigidity of the beam to be constants. Under these restrictions, the second order governing equation is an exact governing equation for wave propagation along the non-uniform Bernoulli-Euler beam. We consider a non-uniformity shape such that the second order governing equation for the Bernoulli-Euler beam has a known analytical solution. In this situation, the relation between the reflection coefficient and the wavelength of the incident wave can be derived in closed form. The parameters describing the non-uniformity shape, i.e. its steepness, size and amplitude, are also the parameters of the relation between the reflection coefficient and the wavelength of the incident wave. Therefore, if we prescribe the form of the relationship between the reflection coefficient and the wavelength of the incident wave, we can determine the desired shape of the non-uniformity in the mass per unit length and in the flexural rigidity.

We consider a Bernoulli-Euler beam consisting of three uniform pieces with a constant tensile force P applied to it. The distribution of mass per unit length and flexural rigidity for each piece is defined by the equations

$$\rho A(x) = \begin{cases} \rho A_0 & \text{for } x < 0 \text{ and } x > L, \\ \rho A_1 & \text{for } 0 \leq x \leq L, \end{cases} \quad (3.84)$$

$$EI(x) = \begin{cases} EI_0 & \text{for } x < 0 \text{ and } x > L, \\ EI_1 & \text{for } 0 \leq x \leq L. \end{cases} \quad (3.85)$$

The restriction that the product of the mass per unit length by the flexural rigidity is a constant is expressed by the equation

$$\rho A_0 EI_0 = \rho A_1 EI_1 = C'' . \quad (3.86)$$

The value of the constant C'' is fixed by the equation (3.86), since the quantities ρA_0 and EI_0 are assumed known.

The second order governing equation for the Euler-Bernoulli beam under such restrictions assumes the form (3.69), and its coefficient function in terms of the beam parameters is

$$\tilde{\Omega}^2(\Omega, P, C'')n(x)^2 = \begin{cases} k_0^2 = \frac{1}{EI_0} \left\{ -\frac{P}{2} + \frac{1}{2}\sqrt{P^2 + 4\Omega^2 C''} \right\} & \text{for } x < 0 \text{ and } x > L, \\ k_1^2 = \frac{1}{EI_1} \left\{ -\frac{P}{2} + \frac{1}{2}\sqrt{P^2 + 4\Omega^2 C''} \right\} & \text{for } 0 \leq x \leq L. \end{cases} \quad (3.87)$$

The quantity Ω is the dimensional frequency of the incident wave. k_0 and k_1 are wave numbers.

According to equation (3.87), the wavenumber k_1 can be written in terms of the wavelength λ of the incident wave, as follows:

$$k_1 = \frac{EI_0}{EI_1} \frac{2\pi}{\lambda}. \quad (3.88)$$

Equation (3.69) has an analytical solution for the function $\tilde{\Omega}^2(\Omega, P, C'')n(x)^2$ defined by equation (3.87). We consider the radiation condition of left wave incidence given by equation (3.56) with $k_1^\pm = k_0$.

The reflection coefficient for left wave incidence can be written in terms of the wave numbers k_0 and k_1 and in terms of the size L of the part of the beam with unknown properties, according to the equation

$$\tilde{R}^- = \frac{\alpha(1-\beta)}{1+\alpha^2\beta}, \quad (3.89)$$

where the quantities α, β and Γ are defined according to the equations

$$\alpha = \frac{k_0^2 - k_1^2}{k_0 + k_1} = \frac{EI_1 - EI_0}{EI_1 + EI_0}, \quad (3.90)$$

$$\Gamma = 4k_1 L = \frac{8\pi}{\lambda} \frac{EI_0}{EI_1} L, \quad (3.91)$$

$$\beta = \exp(i\Gamma), \quad (3.92)$$

The modulus of the reflection coefficient $\tilde{R}^-$ is

$$|\tilde{R}^-| = \left\{ \left(\frac{\alpha (1 - \cos(\Gamma)) (1 - \alpha^2 \cos(\Gamma))}{(1 - \alpha^2 \cos(\Gamma))^2 + \alpha^4 (\sin(\Gamma))^2} + \frac{\alpha^3 (\sin(\Gamma))^2}{(1 - \alpha^2 \cos(\Gamma))^2 + \alpha^4 (\sin(\Gamma))^2} \right)^2 + \left(-\frac{\alpha \sin(\Gamma) (1 - \alpha^2 \cos(\Gamma))}{(1 - \alpha^2 \cos(\Gamma))^2 + \alpha^4 (\sin(\Gamma))^2} + \frac{\alpha^3 (1 - \cos(\Gamma)) \sin(\Gamma)}{(1 - \alpha^2 \cos(\Gamma))^2 + \alpha^4 (\sin(\Gamma))^2} \right)^2 \right\}^{1/2}. \quad (3.93)$$

The reflection coefficient $\tilde{R}^-$ is an oscillatory function of the wavelength λ of the incident wave, according to equations (3.91) and (3.89). According to equation (3.89), the reflection coefficient attains zero value at a discrete set of wavelengths of the incident wave. These are the wavelengths for perfect wave transmission. They satisfy the condition for perfect transmission

$$1 - \exp(i\Gamma) = 0. \quad (3.94)$$

We design the quantities $\rho A_1, EI_1$ and L to have perfect transmission at chosen wavelengths λ_a and λ_b ($\lambda_b > \lambda_a$). We also prescribe the maximum value that the

modulus of the reflection coefficient assumes for wavelengths between λ_a and λ_b . Once we prescribe a wavelength of perfect transmission, i.e. λ_a , the next wavelength of perfect transmission, i.e. λ_b , could not be prescribed in an arbitrary way.

First, we chose two desired wavelengths of perfect transmission, i.e. λ_a and λ_b , but with λ_b as a function of λ_a such that the condition for perfect transmission, as given in the third line of table 3.4, is satisfied. Second, we chose the maximum value of the modulus of the reflection coefficient for $\lambda_a < \lambda < \lambda_b$. With these three quantities chosen, we follow the steps given in table 3.4.

Table 3.4: Design Problem Steps.

Step	Description
Constant C''	Given by eq. (3.86) in terms of the known quantities ρA_0 and EI_0 .
Chose λ_a	We chose a desired wavelength of perfect transmission. No restriction posed on this choice.
Chose λ_b	Next wavelength of perfect transmission. It has to satisfy the condition of perfect transmission (3.94), which can be written as $\lambda_b = \frac{n}{n-1}\lambda_a$ for n a natural number. This relation fixes the value of n .
Chose $ R_{max} $	We chose the maximum value of the modulus of reflection coefficient for $\lambda_a < \lambda < \lambda_b$. According to eqs. (3.92) and (3.93), $\frac{d}{d\lambda} R = 0 \rightarrow \frac{d}{d\Gamma} R = 0$ implies $\Gamma = m\pi$ with m an integer number. $\Gamma = \pi + 2m\pi$, with m integer, gives $ R_{max} = \frac{2\alpha}{\alpha^2+1}$
Obtain α	This quantity is defined by eq. (3.90). In terms of $ R_{max} $, we chose $\alpha = \frac{1}{ R_{max} }(1 - \sqrt{1 - R_{max} ^2})$.
Obtain EI_1	Given in terms of α as $EI_1 = \frac{\alpha+1}{1-\alpha}EI_0$.
Obtain L	The size of the part of the beam to be designed, denoted as L , is given by the equation $L = \frac{n\lambda_a EI_1}{4EI_0} = \frac{n\lambda_a}{4} \frac{\alpha+1}{1-\alpha}$
Obtain ρA_1	The mass per unit length of the part to be designed is given by the equation $\rho A_1 = \frac{C''}{EI_1}$

In figure 3-6, we present plots of equation (3.93) as a function of the wavelength of the incident wave, and we compare them with the values of the reflection coefficient obtained from the exact solution of the full governing equation. These values of the modulus of the reflection coefficient have been obtained in the Appendix H.

We also present table 3.5 with the values of the designed mass per unit length ρA_1 and flexural rigidity EI_1 , respectively, as a fraction of the known values of the mass per unit length ρA_0 and flexural rigidity EI_0 . We also give the chosen wavelengths of perfect transmission λ_a and λ_b , the size L and the maximum value of the modulus of the reflection coefficient $|\tilde{R}^-|_{max}$. These values presented in table 3.5 are associated with figures 3-6 (a), 3-6 (b), 3-6 (c) and 3-6 (d).

Table 3.5: Parameters of the designed Bernoulli-Euler beam.

Figure	Ratio $\frac{\rho A_1}{\rho A_0}$	Ratio $\frac{EI_1}{EI_0}$	λ_a	λ_b	$ \tilde{R}^- _{max}$	Length L
3-6 (a)	0.11	9.0	1.0 m	2.0 m	0.8	2.736 m
3-6 (b)	0.33	3.0	1.0 m	2.0 m	0.5	1.914 m
3-6 (c)	0.33	3.0	0.75 m	1.5 m	0.5	1.436 m
3-6 (d)	0.0526315789	19.0	0.75 m	1.5 m	0.9	2.747 m

3.1.6 Discussion and Conclusions.

In the previous section, we illustrated how to use a Helmholtz-like second order differential equation with known solution to design a non-uniform Bernoulli-Euler beam with constants tensile force and product of the flexural rigidity by the mass per unit length such that the functional relation between the reflection coefficient and the wavelength of the incident wave had a prescribed form. This approach to the design problem is limited, since there are few Helmholtz-like second order differential equations with variable coefficient that have analytical solutions. To handle general design problems, a more general approach is necessary.

We discuss how to apply inverse scattering techniques for the one-dimensional

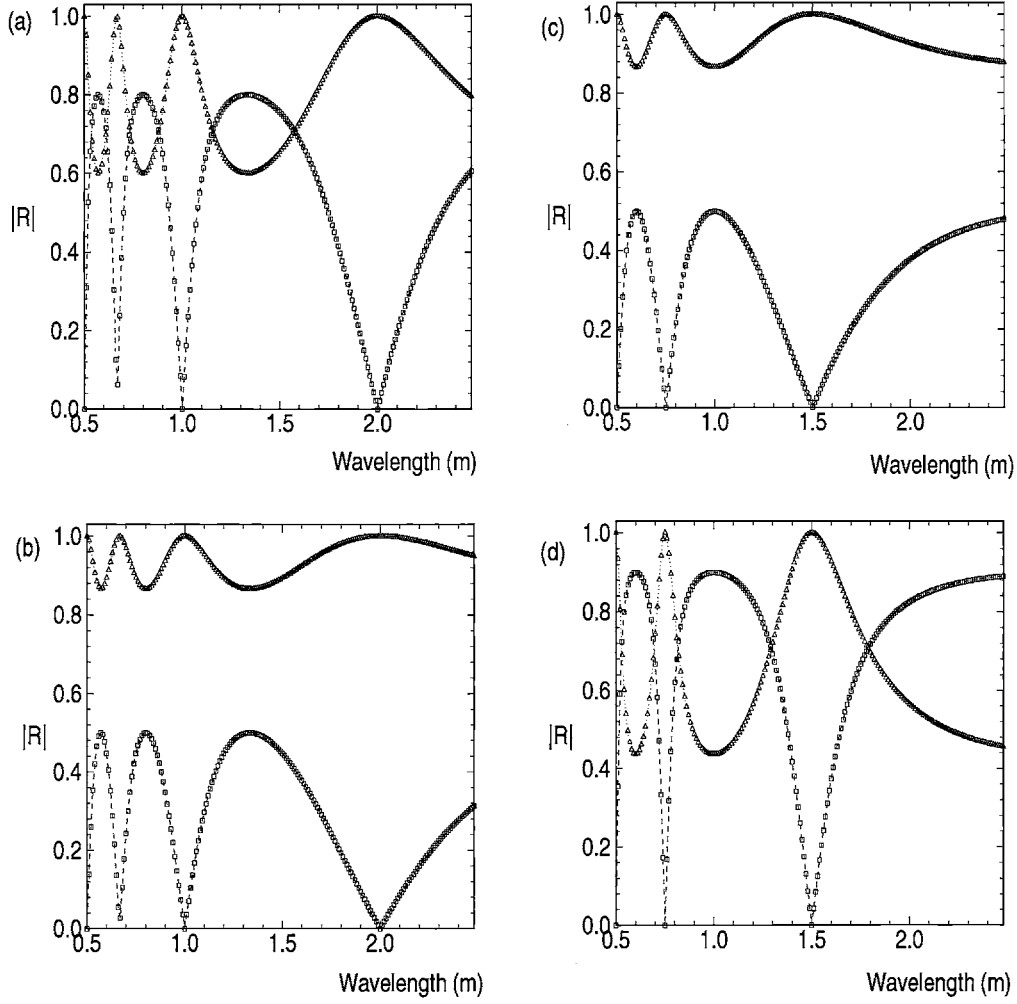


Figure 3-6: Modulus of the reflection and transmission coefficients as a function of wave length of the incident wave for the designed Euler-Bernoulli beam. $|\tilde{R}^-|$ from equation (3.93): - - - - - . $|\tilde{T}^-| = \sqrt{1 - |\tilde{R}^-|^2}$: Numerical simulation for the reflection coefficient (H.13): $\square$. Numerical results for the transmission coefficient (H.13): $\triangle$. Properties of the designed beams for (a), (b), (c) and (d) are given in table (3.5).

Helmholtz equation in the half line to the design problem of a Bernoulli-Euler beam under constant tensile force and with the product of the mass per unit length by the flexural rigidity being a constant.

Under the restriction mentioned above, wave propagation along a non-uniform Bernoulli-Euler beam is governed by the second order differential equation (3.69). The relationship between the frequency of the incident wave Ω and the “wave frequency” $\tilde{\Omega}$ is given by the equation (3.71). The constant tensile force P enters as a parameter in the equation (3.71) and the constant C' is the value of the product of the mass per unit length times the flexural rigidity along the entire beam.

Once the value of the tensile force is prescribed, the relationship between Ω and $\tilde{\Omega}$ is one-to-one onto for $\Omega \geq 0$. Then, the knowledge of the functional relation between the reflection coefficient and frequency of the incident wave Ω implies that we know the functional relation between the reflection coefficient and the “wave frequency” $\tilde{\Omega}$. Therefore, we can apply inverse scattering techniques developed for the Helmholtz equation on the half line to design the index of refraction $n(x)$ of equation (3.69) for $x > 0$, given a prescribed functional relation between the reflection coefficient at $x = 0$ and the “wave frequency” $\tilde{\Omega}$. Designing the index of refraction of equation (3.69) implies designing the inverse of the flexural rigidity of the Bernoulli-Euler beam. Once the flexural rigidity is obtained from the design of the index of refraction $n(x)$, the mass per unit length follows from the restriction given by the equation (3.70).

Regarding techniques to solve the inverse problem for the Helmholtz equation on the half line, we can mention the nonlinear approximate method described in Jaggard & Kim [31] and the exact method based on layer stripping technique described in Silvester, Winebrenner & Gyls-Colwell [64] and in Silvester & Winebrenner [63]. These techniques to handle the inverse scattering problem can be applied to design the non-uniformity in a Bernoulli-Euler beam under the restrictions of the tensile force and the product of the mass per unit length by the flexural rigidity being constants, as discussed above. Bernoulli-Euler beams satisfying the restrictions described above can be built, and they could be useful in engineering applications, where passive vibration isolation is desired.

According to the numerical results displayed in section 3.1.4, the second order governing equation (3.49) recovers the behavior predicted by the full governing equation (3.6) for general non-uniformities when its steepness is small. The second order governing equation (3.49) is useful in studying wave scattering phenomenon along weakly non-uniform Bernoulli-Euler beams of many wavelengths. Under the restriction of the tensile force and the product of the mass per unit length by the flexural rigidity being constants, the second order governing equation (3.49) is able to predict wave propagation along the Bernoulli-Euler beam even for large deviation from uniformity, since it is an exact governing equation.

Higher order WKB or phase integral methods can be applied to the new governing equation to obtain analytic approximations for the beam scattering quantities, since for second order equations these methods are nowadays well developed. For an account on the WKB method and phase integral methods see Fröman and Fröman [20].

3.2 High Order WKB Method for the Euler-Bernoulli Beam.

We consider an Euler-Bernoulli beam of infinite length with a finite non-uniform part under tensile force. The cross-sectional geometry of the non-uniform part of the beam is assumed to vary slowly along its length.

One objective of this work is to construct a high order asymptotic solution for the scattering problem of monochromatic waves incident in the non-uniform part of the beam. We obtain high order asymptotic expressions for the reflection, transmission and mode conversion coefficients due to the interaction of the monochromatic incident wave with the beam non-uniformity.

The asymptotic method considered is the WKB method. Scattering in the context of the WKB method happens due to the presence of turning points and singularities like poles. Connection formulae for the WKB waves across turning points, singular-

ities and regions in the complex plane of the independent variable bounded by the analogous of the Stokes lines for second order differential equations are necessary to capture wave reflection and wave mode conversion in the context of the WKB method.

The main objective of this work is to study the connection problem for turning points or group of turning point where four wave mode couple. We chose the Euler-Bernoulli beam model since its governing equation in the frequency domain¹ is a fourth order differential equation, so coupling of four wave modes at a turning point is possible, and we can still obtain analytical expressions for the WKB waves.

In the context of the WKB method, the dispersion relation for the uniform system is locally valid. For a given wave frequency, the locally valid dispersion relation gives the wave numbers as a function of the system parameters, which are now functions of the space coordinate. So the wave numbers are functions of the space coordinate. We now assume that the space coordinate and the wavenumber are complex variable, so the locally valid uniform system dispersion relation can be considered as a mapping from the complex space coordinate plane into the wavenumber complex plane. This mapping has branch points in the space coordinate plane, and their images in the wavenumber plane are the saddle points. When we approach a branch point in the space coordinate plane, two or more wave numbers coalesce to a saddle point, which implies that we have wave mode conversion. If at a branch point we have the coalescence of two wave numbers, and they are the wave numbers associated with the right and left propagating waves, the wave mode conversion that happens at this branch point is wave reflection. The main reason to chose to study wave propagation along a non-uniform beam, besides its simplicity, is the way, in the context of the WKB method, the wave modes couple at some turning points, which are the branch points of the dispersion relation. There are two possibilities for wave mode coupling at turning points. In the first possibility, the two propagating modes couple and in the second possibility, each propagating mode couples with one evanescent mode. These two forms of mode coupling are observed in the WKB analysis of the problem of linear surface waves interacting with a one-dimensional channel with non-uniform

¹Time dependence of the form $\exp(-i\omega t)$, where t is the time coordinate.

bottom. This problem is discussed in chapter 4. Therefore, the WKB analysis of the Euler-Bernoulli beam is a model problem for the WKB analysis of linear surface wave scattering by a one-dimensional non-uniform bottom.

The approach adopted is to match the WKB waves (far field asymptotic solution) with the asymptotic expansion of the solution of the approximate form of the Euler-Bernoulli beam model governing equation valid in a neighborhood of the considered turning point. This gives the Stokes constants necessary to obtain an asymptotic approximation for the scattering coefficients. We basically consider the connection problem for first order turning points only.

The outline of this work follows. The Euler-Bernoulli beam governing equation is given in section 3.1.2. In the next section we perform a rescaling of the independent variable, since now the coefficients of the governing equation are slowly varying functions. In section 3.2.2 we obtain the high order form of the Liouville-Green functions (WKB waves). We assume a suitable form for the WKB expansion. We use the ideas of the calculus of non-commutative operators, discussed in Maslov [42], to derive the transport equation for the WKB wave amplitude. The space derivative of the phase of the WKB waves is a solution of the principal part of the symbol of the Euler-Bernoulli beam non-dimensional rescaled governing equation. The final result of this section is a recurrence relation for the terms of the series expansion assumed for the amplitude of the WKB waves. In section 3.2.3 we analyze the locally valid uniform system dispersion relation, which is the principal part of the symbol of the beam governing equation. We obtain the branch points of the dispersion relation and we describe how the wave numbers coalesce. The branch points of the locally valid uniform system dispersion relation are the turning points. We also consider simple analytical forms of non-uniformity. We obtain the turning points configuration for these non-uniformities as functions of the system parameters. We give emphasis for non-uniformity in the beam flexural rigidity only. We carry out the connection problem for this particular non-uniformity. In section 3.2.4 we consider non-uniformity in the flexural rigidity only. This non-uniformity gives first order poles close to first order turning points. We obtain the approximate form of the governing equation, valid

in a neighborhood of a turning point plus pole. We solve this approximate equation in terms of contour integrals and we obtain a high order asymptotic expansion of this closed form solutions for a wide range of the argument of the independent variable. We use the steepest descent method method to extract the asymptotic expansion of the contour integral solutions. In section 3.2.5 we consider the matching of the WKB waves with the asymptotic expansion of the contour integral solutions to high order. The matching gives the connection formulae for the WKB waves for a first order turning point plus pole. As a result, reflection, transmission and mode conversion coefficients are obtained.

3.2.1 Euler-Bernoulli Beam Governing Equation Revisited.

The governing equation for mono-chromatic wave propagation along a non-uniform Euler-Bernoulli beam under tensile force, in terms of non-dimensional variables, is given in section 3.1.2 by equation (3.6). Here we consider a rescale of the independent variable since the coefficient functions of the governing equation are assumed slowly varying functions of the space coordinate.

In section 3.1.2, the space coordinate x was non-dimensionalized by the wave length of the incident wave, so in terms of the non-dimensional space coordinate s , the non-uniformity always “sees” an incident wave of unitary wave length. As mentioned before, the coefficient functions of equation (3.6) are slowly varying functions of the independent variable s . Let us consider a small parameter ε , which is the ratio between the incident wavelength and the length scale of the non-uniformity variation. To stress the fact that the coefficient functions are slowly varying quantities, we write equation (3.6) in the form

$$(ei(\varepsilon s)y_{ss})_{ss} - (\bar{P}(\varepsilon s)y_s)_s - \omega^2 m(\varepsilon s)y = 0 \quad (3.95)$$

We consider the rescaled independent variable z , related to s through the trans-

formation

$$z = \varepsilon s. \quad (3.96)$$

In terms of the space coordinate z , the non-uniformity “sees” an incident wave with wavelength ε . Now equation (3.95) can be written as

$$\varepsilon^4 (ei(z)y_{zz})_{zz} - \varepsilon^2 (P(z)y_z)_z - \omega^2 m(z)y(z) = 0. \quad (3.97)$$

Equation (3.97) is the rescaled equation used to derive the asymptotic results that are described in the next sections.

3.2.2 High Order Liouville-Green Functions.

In this section we derive the first step necessary to built an asymptotic solutions for equation (3.97) along the real axis of the z complex plane. We obtain the Liouville-Green functions for equation (3.97), which are valid asymptotic solutions around the complex z plane, besides at the turning points, where they break down (became unbounded).

The WKB waves (Liouville-Green functions) are constructed in this section according to the steps outlined as follows. We consider ideas from the calculus of non-commutative operators described in Maslov [42]. First, we write equation (3.97) in terms of the operator $\hat{L}$ as

$$\hat{L}(z, \varepsilon \frac{d}{dz})y(z) = 0. \quad (3.98)$$

Second, we assume a specific form for the WKB waves, which is a phase function times an amplitude, as follows:

$$y(z) = \exp(\frac{1}{\varepsilon}S(z))w(z). \quad (3.99)$$

We also assume that the amplitude function $w(z)$ is given as a series in the parameter

ε . We substitute the functional form (3.99) in equation (3.98). This furnish an equation for the phase function $S(z)$ in terms of the symbol of the operator $\hat{L}$, and a transport equation for the amplitude function $w(z)$. Since we assumed that the amplitude function $w(z)$ can be written as a series in the parameter ε , the transport equation reduces to a sequence of first order ordinary differential equations for the terms in the series in ε representing the amplitude function.

Third, we solve the equation for the phase function $S(z)$. We obtain four phase functions, each representing the phase of a wave like solution. For each phase, we have an amplitude function $w(z)$, solution of the transport equation. Since each of these four amplitude functions, as mentioned above, are given as series in ε , the transport equations for each amplitude function reduces to a sequence of first order in-homogeneous differential equations. The solution of these sequences of first order differential equations gives a sequence of recurrence relations for the terms in the series expansions of the amplitude functions. The recurrence relation for the n -th term in the series for the amplitude function depends on the previous four terms and their derivatives up to fourth order. We solve the recurrence relation up to first order in ε only. Higher order terms can be obtained numerically.

The symbol of the operator $\hat{L}$ is

$$L(\overset{2}{z}, \overset{1}{p}) = R_0(\overset{2}{z}, \overset{1}{p}) + \varepsilon R_1(\overset{2}{z}, \overset{1}{p}) + \varepsilon^2 R_2(\overset{2}{z}, \overset{1}{p}), \quad (3.100)$$

where the symbol $R_0(\overset{2}{z}, \overset{1}{p})$ is the principal part of operator $\hat{L}$. The expression for the symbols $R_j(\overset{2}{z}, \overset{1}{p})$ is

$$R_0(\overset{2}{z}, \overset{1}{p}) = ei(\overset{2}{z})(\overset{1}{p})^4 - \bar{P}(\overset{2}{z})(\overset{1}{p})^2 - \omega^2 m(\overset{2}{z}), \quad (3.101)$$

$$R_1(\overset{2}{z}, \overset{1}{p}) = 2 \frac{d}{dz} ei(\overset{2}{z})(\overset{1}{p})^3 + \frac{d}{dz} \bar{P}(\overset{2}{z}) \overset{1}{p}, \quad (3.102)$$

$$R_2(\overset{2}{z}, \overset{1}{p}) = \frac{d^2}{dz^2} ei(\overset{2}{z})(\overset{1}{p})^2. \quad (3.103)$$

Next, we substitute $y(z)$ in equation (3.98) by the expression (3.99). This result in a

equation for the phase function $S(z)$ and amplitude function $w(z)$, as follows:

$$L(z, \varepsilon \frac{d}{dz} + \frac{dS}{dz})w(z) = 0. \quad (3.104)$$

If we collect terms of order $O(\varepsilon^0)$, we obtain for the phase function $S(z)$ the equation

$$R_0(z, \frac{dS}{dz}) = 0. \quad (3.105)$$

In terms of the coefficient functions $ei(z)$, $\bar{P}(z)$ and $m(z)$, the equation (3.105) has the form

$$ei(z)(\frac{dS}{dz})^4 - \bar{P}(z)(\frac{dS}{dz})^2 - \omega^2 m(z) = 0. \quad (3.106)$$

It can be solved, giving four phase functions

$$S_1(z) = \int_r^z \sqrt{\frac{1}{2}D(z) - \frac{1}{ei(z)}\sqrt{F(z)}}, \quad (3.107)$$

$$S_2(z) = -S_1(z), \quad (3.108)$$

$$S_3(z) = \int_r^z \sqrt{\frac{1}{2}D(z) + \frac{1}{2ei(z)}\sqrt{F(z)}}, \quad (3.109)$$

$$S_4(z) = -S_3(z) \quad (3.110)$$

where r is a reference point which can be arbitrarily chosen. The functional form of $D(z)$ and $F(z)$ in terms of the coefficient functions $ei(z)$, $\bar{P}(z)$ and $m(z)$ follows:

$$D(z) = \frac{\bar{P}(z)}{ei(z)}, \quad (3.111)$$

$$F(z) = \bar{P}(z)^2 + 4\omega^2 m(z)ei(z). \quad (3.112)$$

The transport equation for the amplitude function $w(z)$ in terms of the symbols $R_j(z, p)$ (see equations (3.101) to (3.103)) and their derivatives with respect to p has

the form

$$\begin{aligned}
& \left\{ \varepsilon^4 \left[\frac{1}{24} \frac{\partial^4 R_0}{\partial p^4} \left(z, \frac{dS}{dz} \right) \frac{d^4}{dz^4} + \frac{1}{6} \frac{\partial^3 R_1}{\partial p^3} \left(z, \frac{dS}{dz} \right) \frac{d^3}{dz^3} + \frac{1}{2} \frac{\partial^2 R_2}{\partial p^2} \left(z, \frac{dS}{dz} \right) \frac{d^2}{dz^2} \right] \right. \\
& + \varepsilon^3 \left[\frac{1}{6} \frac{\partial^3 R_0}{\partial p^3} \left(z, \frac{dS}{dz} \right) \frac{d^3}{dz^3} + \frac{1}{24} \frac{\partial^4 R_0}{\partial p^4} \left(z, \frac{dS}{dz} \right) \frac{d^4 S}{dz^4} + \frac{1}{6} \frac{\partial^4 R_0}{\partial p^4} \left(z, \frac{dS}{dz} \right) \frac{d^3 S}{dz^3} \frac{d}{dz} \right. \\
& + \frac{1}{4} \frac{\partial^4 R_0}{\partial p^4} \left(z, \frac{dS}{dz} \right) \frac{d^2 S}{dz^2} \frac{d^2}{dz^2} + \frac{1}{2} \frac{\partial^2 R_1}{\partial p^2} \left(z, \frac{dS}{dz} \right) \frac{d^2}{dz^2} + \frac{1}{6} \frac{\partial^3 R_1}{\partial p^3} \left(z, \frac{dS}{dz} \right) \frac{d^3 S}{dz^3} \\
& + \frac{1}{2} \frac{\partial^3 R_1}{\partial p^3} \left(z, \frac{dS}{dz} \right) \frac{d^2 S}{dz^2} \frac{d}{dz} + \frac{\partial R_2}{\partial p} \left(z, \frac{dS}{dz} \right) \frac{d}{dz} + \left. \frac{1}{2} \frac{\partial^2 R_2}{\partial p^2} \left(z, \frac{dS}{dz} \right) \frac{d^2 S}{dz^2} \right] \\
& + \varepsilon^2 \left[\frac{1}{2} \frac{\partial^2 R_0}{\partial p^2} \left(z, \frac{dS}{dz} \right) \frac{d^2}{dz^2} + \left(\frac{1}{6} \frac{\partial^3 R_0}{\partial p^3} \left(z, \frac{dS}{dz} \right) \frac{d^3 S}{dz^3} + \frac{1}{8} \frac{\partial^4 R_0}{\partial p^4} \left(z, \frac{dS}{dz} \right) \left(\frac{d^2 S}{dz^2} \right)^2 \right) \right. \\
& + \frac{1}{2} \frac{\partial^3 R_0}{\partial p^3} \left(z, \frac{dS}{dz} \right) \frac{d^2 S}{dz^2} \frac{d}{dz} + \frac{\partial R_1}{\partial p} \left(z, \frac{dS}{dz} \right) \frac{d}{dz} + \frac{1}{2} \frac{\partial^2 R_1}{\partial p^2} \left(z, \frac{dS}{dz} \right) \frac{d^2 S}{dz^2} + R_2 \left(z, \frac{dS}{dz} \right) \left. \right] \\
& + \varepsilon \left[\frac{\partial}{\partial p} R_0 \left(z, \frac{dS}{dz} \right) \frac{d}{dz} + \frac{1}{2} \frac{\partial^2}{\partial p^2} R_0 \left(z, \frac{dS}{dz} \right) \frac{d^2 S}{dz^2} + R_1 \left(z, \frac{dS}{dz} \right) \right] \Big\} w(z) = 0
\end{aligned} \tag{3.113}$$

Notice that the coefficient of the transport equation (3.113) are functions of the wavenumber dS/dz . Since we have four wave numbers, solutions of equation (3.105), we have a set of four transport equations. For each $dS_j/dz, j = 1, \dots, 4$ we have a transport equation. Next, we obtain asymptotic solutions for the transport equation (3.113).

As mentioned before, we assume that the amplitude function $w(z)$ is given as series expansion in the parameter ε . The form of the series expansion in ε is

$$w(z) = w_0(z) \left\{ 1 + \sum_{n=1}^{\infty} \varepsilon^n w_n(z) \right\} \tag{3.114}$$

We substitute the expansion (3.114) for $w(z)$ in the transport equation (3.113). We collect terms of the same order in ε . This gives a sequence of first order ordinary differential equations for the terms $w_n(z)$ of the series expansion for the function $w(z)$. Below we list the sequence of first order differential equations for the amplitude corrections $w_n(z)$.

- To first order in ε we have

$$\frac{1}{w_0(z)} \frac{dw_0}{dz}(z) = -\frac{1}{2} \frac{1}{\frac{\partial R_0}{\partial p}(z, \frac{dS}{dz})} \frac{d}{dz} \left(\frac{\partial R_0}{\partial p}(z, \frac{dS}{dz}) \right), \quad (3.115)$$

which has solution in the form

$$w_0(z) = \frac{1}{\sqrt{\frac{\partial R}{\partial p}(z, \frac{dS}{dz})}}. \quad (3.116)$$

Since the derivative of the phase function appears in the expression for $w_0(z)$ and we have a total of four phase functions $S_j(z)$, we make explicit the dependence of the function $w_0(z)$ with respect to the wave numbers $dS_j/dz, j = 1, \dots, 4$, by appending to it an extra index. Now we write $w_{0j}(z)$ to specify that its functional form is given in terms of the wavenumber dS_j/dz . The expressions for the functions $w_{0j}(z)$ in terms of the functions $ei(z)$, $D(z)$ and $F(z)$ are

$$w_{01}(z) = \frac{1}{\sqrt{-\sqrt{F(z) \left(\frac{1}{2}D(z) - \frac{1}{2ei(z)}\sqrt{F(z)} \right)}}} \quad (3.117)$$

$$w_{02}(z) = \frac{1}{\left[F(z) \left(\frac{1}{2}D(z) - \frac{1}{2ei(z)}\sqrt{F(z)} \right) \right]^{1/4}} \quad (3.118)$$

$$w_{03}(z) = \frac{1}{\left[F(z) \left(\frac{1}{2}D(z) + \frac{1}{2ei(z)}\sqrt{F(z)} \right) \right]^{1/4}} \quad (3.119)$$

$$w_{04}(z) = \frac{1}{\sqrt{-\sqrt{F(z) \left(\frac{1}{2}D(z) + \frac{1}{2ei(z)}\sqrt{F(z)} \right)}}} \quad (3.120)$$

The higher order wave amplitude corrections $w_n(z)$ are also functions of the phase functions $S_j(z)$, according to the higher order transport equations given below. So, for each phase function $S_j(z)$ we have an nth order amplitude correction $w_n(z)$, and to make explicit the dependence of the amplitude correction with respect to the phase $S_j(z)$, we append the extra index j ($j = 1, \dots, 4$) to them.

- To second order in ε we have

$$\begin{aligned}
w_{1j}(z) = \int_r^z \left\{ \left(-\frac{1}{2} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{\partial R_1}{\partial p} \left(t, \frac{dS}{dt} \right) \right) w_{0j}(t) \frac{d}{dt} w_{0j}(t) \right. \\
+ \left(-\frac{1}{6} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^3 S}{dt^3} - \frac{1}{8} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \left(\frac{d^2 S}{dt^2} \right)^2 - \frac{1}{2} \frac{\partial^2 R_1}{\partial^2} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} \right. \\
\left. \left. - R_2 \left(t, \frac{dS}{dt} \right) \right) (w_{0j}(t))^2 - \frac{1}{2} \frac{\partial^2 R_0}{\partial p^2} \left(t, \frac{dS}{dt} \right) \frac{d^2 w_{0j}}{dt^2} w_{0j}(t) \right\} dt
\end{aligned} \tag{3.121}$$

- To third and higher order ($N \geq 3$) in parameter ε we have

$$\begin{aligned}
w_{(N-1)j}(z) = - \int_r^z w_{0j}(t) \left\{ \sum_{l=2}^{N-1} \left\{ \sum_{n=0}^{\min\{l-1, M\}} \left(\sum_{k=0}^{l-1-n} \left[\sum_{q=0}^k \sum_{m=0}^{\min\{l-k-n, |j_n+n-l\}} \left(\sum_{\substack{j_m+\dots+j_1=l-k+m-n \\ j_m, \dots, j_1 \geq 2}} \frac{1}{(k-j)! j_m! \dots j_1!} \right) \frac{d^{k-q} w_{0j}}{dt^{k-q}} \frac{d^q w_{(N-l)j}}{dt^q} \right] \right. \right. \right. \\
\left. \left. + \delta_{(l-n), \min\{l-n, j_n\}} \sum_{q=0}^{l-n} \left[\frac{1}{(l-n-q)! q!} \frac{\partial^{(l-n)} R_n}{\partial p^{(l-n)}} (t, p) \Big|_{p=\frac{dS}{dt}} \frac{d^{(l-n-q)} w_{0j}}{dt^{(l-n-q)}} \frac{d^q w_{(N-l)j}}{dt^q} \right] \right) \right\} \right. \\
+ \sum_{n=0}^{\min\{N-1, M\}} \left(\sum_{k=0}^{N-1-n} \left[\sum_{m=0}^{\min\{N-k-n, |j_n+n-N\}} \left(\sum_{\substack{j_m+\dots+j_1=N-k+m-n \\ j_m, \dots, j_1 \geq 2}} \frac{1}{(k-j)! j_m! \dots j_1!} \right) \frac{\partial^{(N-m-n)} R_n}{\partial p^{(N-m-n)}} (t, p) \Big|_{p=\frac{dS}{dt}} \frac{d^{j_m} S}{dt^{j_m}} \times \dots \times \frac{d^{j_1} S}{dt^{j_1}} \right] \frac{d^k w_{0j}}{dt^k} + \frac{\delta_{(N-n), \min\{N-n, j_n\}}}{(N-n)!} \right. \\
\left. \left. \frac{\partial^{N-n} R_n}{\partial p^{N-n}} (t, p) \Big|_{p=\frac{dS}{dt}} \frac{d^{N-n} w_{0j}}{dt^{N-n}} \right) + \sum_{l=\max\{N-n, 1\}}^{N-2} R_{N-l} \left(t, \frac{dS}{dt} \right) \right\} dt
\end{aligned} \tag{3.122}$$

where the value of M in the equation above is the maximum order of the parameter ε ($M = 2$) in the expansion for the symbol of the operator $\hat{L}$, given by equation (3.100). The index j_n gives the maximum power of the variable p appearing in the expression for the symbol $R_n(x, p)$, given by equations (3.101) to (3.103). The symbol δ with two indexes represents the Kronecker delta function, which assumes value one when

the indexes are equal and zero otherwise.

The procedure to derive the equation (3.122) is outlined in the Appendix L. We also provide the script of a Maple procedure which computes the integrand of equation (3.100). This script is listed in Appendix L. Below we give the transport equation for the amplitude corrections w_{2j} and w_{3j} , which were computed with the script given in Appendix L.

- Transport equation for the amplitude correction $w_{2j}(z)$.

$$\begin{aligned}
w_{2j}(z) = \int_{\tau}^z & \left[\left(-\frac{1}{2} \frac{\partial^2 R_0}{\partial p^2} \left(t, \frac{dS}{dt} \right) w_{1j}(t) - \frac{1}{2} \frac{\partial^2 R_1}{\partial p^2} \left(t, \frac{dS}{dt} \right) - \frac{1}{4} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} \right) w_{0j}(t) \frac{d^2 w_{0j}}{dt^2} \right. \\
& + \left(-\frac{\partial R_2}{\partial p} \left(t, \frac{dS}{dt} \right) - \frac{\partial^2 R_0}{\partial^2} \left(t, \frac{dS}{dt} \right) \frac{dw_{1j}}{dt} - \frac{1}{6} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^3 S}{dt^3} \right. \\
& + \left(-\frac{1}{2} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{\partial R_1}{\partial p} \left(t, \frac{dS}{dt} \right) \right) w_{1j}(t) - \frac{1}{2} \frac{\partial^3 R_1}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} \Big) w_{0j}(t) \frac{dw_{0j}}{dt} \\
& + \left(\left(-\frac{1}{6} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^3 S}{dt^3} - \frac{1}{8} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \left(\frac{d^2 S}{dt^2} \right)^2 - \frac{1}{2} \frac{\partial^2 R_1}{\partial p^2} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} \right. \right. \\
& \left. \left. - R_2 \left(t, \frac{dS}{dt} \right) \right) w_{1j}(t) + \left(-\frac{1}{2} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{\partial R_1}{\partial p} \left(t, \frac{dS}{dt} \right) \right) \frac{dw_{1j}}{dt} \right. \\
& - \frac{1}{2} \frac{\partial^2 R_0}{\partial p^2} \left(t, \frac{dS}{dt} \right) \frac{d^2 w_{1j}}{dt^2} - \frac{1}{6} \frac{\partial^3 R_1}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^3 S}{dt^3} - \frac{1}{2} \frac{\partial^2 R_2}{\partial p^2} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} \\
& \left. \left. - \frac{1}{24} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^4 S}{dt^4} \right) (w_{0j}(t))^2 - \frac{1}{6} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^3 w_{0j}}{dt^3} w_{0j}(t) \right] dt
\end{aligned} \tag{3.123}$$

- Transport equation for the amplitude correction $w_{3j}(z)$.

$$\begin{aligned}
w_{3j}(z) = \int_{\tau}^z & \left\{ \left(-\frac{1}{6} \frac{\partial^3 R_1}{\partial p^3} \left(t, \frac{dS}{dt} \right) - \frac{1}{6} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) w_{1j}(t) \right) w_{0j}(t) \frac{d^3 w_{0j}}{dt^3} \right. \\
& + \left(-\frac{1}{2} \frac{\partial^2 R_2}{\partial p^2} \left(t, \frac{dS}{dt} \right) + \left(-\frac{1}{4} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{1}{2} \frac{\partial^2 R_1}{\partial p^2} \left(t, \frac{dS}{dt} \right) \right) w_{1j}(t) \\
& \left. - \frac{1}{2} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{dw_{1j}}{dt} - \frac{1}{2} \frac{\partial^2 R_0}{\partial p^2} \left(t, \frac{dS}{dt} \right) w_{2j}(t) \right) w_{0j}(t) \frac{d^2 w_{0j}}{dt^2}
\end{aligned}$$

$$\begin{aligned}
& + \left(\left(-\frac{1}{2} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{\partial R_1}{\partial p} \left(t, \frac{dS}{dt} \right) \right) w_{2j}(t) - \frac{1}{2} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 w_{1j}}{dt^2} \right. \\
& + \left(-\frac{\partial^2 R_1}{\partial p^2} \left(t, \frac{dS}{dt} \right) - \frac{1}{2} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} \right) \frac{dw_{1j}}{dt} - \frac{\partial^2 R_0}{\partial^2} \left(t, \frac{dS}{dt} \right) \frac{dw_{2j}}{dt} \\
& + \left(-\frac{\partial R_2}{\partial p} \left(t, \frac{dS}{dt} \right) - \frac{1}{2} \frac{\partial^3 R_1}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{1}{6} \frac{\partial^4 R_0}{\partial^4} \left(t, \frac{dS}{dt} \right) \frac{d^3 S}{dt^3} \right) w_{1j}(t) \Bigg) w_{0j}(t) \frac{dw_{0j}}{dt} \\
& + \left(\left(-\frac{1}{2} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{\partial R_1}{\partial p} \left(t, \frac{dS}{dt} \right) \right) \frac{dw_{2j}}{dt} + \left(-\frac{1}{6} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^3 S}{dt^3} \right. \right. \\
& \left. \left. - \frac{1}{8} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \left(\frac{d^2 S}{dt^2} \right)^2 - \frac{1}{2} \frac{\partial^2 R_1}{\partial p^2} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - R_2 \left(t, \frac{dS}{dt} \right) \right) w_{2j}(t) \right. \\
& + \left(-\frac{1}{4} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{1}{2} \frac{\partial^2 R_1}{\partial p^2} \left(t, \frac{dS}{dt} \right) \right) \frac{d^2 w_{1j}}{dt^2} + \left(-\frac{\partial R_2}{\partial p} \left(t, \frac{dS}{dt} \right) \right. \\
& \left. - \frac{1}{2} \frac{\partial^3 R_1}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} - \frac{1}{6} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^3 S}{dt^3} \right) \frac{dw_{1j}}{dt} - \frac{1}{6} \frac{\partial^3 R_0}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^3 w_{1j}}{dt^3} \\
& + \left(-\frac{1}{6} \frac{\partial^3 R_1}{\partial p^3} \left(t, \frac{dS}{dt} \right) \frac{d^3 S}{dt^3} - \frac{1}{24} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^4 S}{dt^4} - \frac{1}{2} \frac{\partial^2 R_2}{\partial p^2} \left(t, \frac{dS}{dt} \right) \frac{d^2 S}{dt^2} \right) w_{1j}(t) \\
& \left. - \frac{1}{2} \frac{\partial^2 R_0}{\partial p^2} \left(t, \frac{dS}{dt} \right) \frac{d^2 w_{2j}}{dt^2} \right) (w_{0j}(t))^2 - \frac{1}{24} \frac{\partial^4 R_0}{\partial p^4} \left(t, \frac{dS}{dt} \right) \frac{d^4 w_{0j}}{dt^4} w_{0j}(t) \Bigg\} dt
\end{aligned} \tag{3.124}$$

The functions $w_{nj}(z)$ with $n \geq 4$ are given by equation (3.122) in terms of the functions $w_{(n-1)j}$, $w_{(n-2)j}$ and $w_{(n-3)j}$ only. Therefore, what equation (3.122) gives is a recurrence relation, which gives $w_{nj}(z)$ in terms of the functions $w_{(n-1)j}$, $w_{(n-2)j}$ and $w_{(n-3)j}$ for $n \geq 4$. Since $w_{1j}(z)$ is a function of $S_j(z)$, w_{0j} and their first and second order derivatives, it is easy to see through equation (3.123) that $w_{2j}(z)$ is also a function of $S_j(z)$, w_{0j} and their first, second and third order derivatives. If we continue to iterate the recurrence relation, we conclude that the function $w_{nj}(z)$ is a function of $S_j(z)$, w_{0j} and their derivatives up to $n+1$ -th order. Therefore, to be able to obtain the n -th order term in the series expansion for the amplitude we need the coefficients of the governing equation to be continuous functions up to $n+2$ -th order. Otherwise, we can go only to an order equal to $n-2$, where n is the degree of smoothness of the coefficient functions of the governing equation (3.97).

From equation (3.123), (3.124) and (3.122), we notice that the amplitude correc-

tions $w_{nj}(z)$ for $n \geq 2$ have a term proportional to $(w_{1j}(z))^n$. Therefore, the transport equation (3.122) can be written in the form

$$w_{(N-1)j}(z) = \frac{1}{(N-1)!} (w_{1j}(z))^{N-1} + G(w_{j,N-2}, \dots, w_{j,0}, \frac{dS_j}{dx}, x). \quad (3.125)$$

The amplitude correction $w_{1j}(z)$ is actually a phase correction, since the first term in the right side of equation (3.125) corresponds to the $(n-1)$ th term of the series expansion of

$$\exp(\varepsilon w_{1j}(z))$$

with respect to ε . We can write the wave amplitude expansion (3.114) in terms of the parameter ε in the form

$$\begin{aligned} w_j(z) &= w_{0j} \left\{ 1 + \varepsilon w_{1j}(z) + \sum_{n=2}^{\infty} \varepsilon^n \left(\frac{1}{n!} (w_{1j}(z))^n + \tilde{w}_{nj}(z) \right) \right\} \\ &= w_{0j}(z) \left\{ \exp(\varepsilon w_{1j}(z)) + \sum_{n=2}^{\infty} \varepsilon^n \tilde{w}_{nj}(z) \right\} \end{aligned} \quad (3.126)$$

To second order in ε the Liouville-Green functions have actually the form

$$y(z) = w_{0j}(z) \exp\left(\frac{1}{\varepsilon} S_j(z) + \varepsilon w_{1j}(z)\right) + O(\varepsilon^2) \quad (3.127)$$

We could have assumed another form for the solution of equation (3.98). Instead of the form (3.99), we could have chosen the form

$$y(z) = \exp\left(\frac{1}{\varepsilon} S(z) + \sum_{n=1}^{\infty} \varepsilon^n v_n(z)\right) v_0(z) \quad (3.128)$$

where we have phase corrections (the terms $v_n(z)$) instead of amplitude corrections. The phase function $S(z)$ in the form (3.128) also satisfies the Eikonal equation (3.105). If we expand the exponential of the sum in the equation (3.128) in terms of the parameter ε and compare with equation (3.114), we realize the relations

$$\begin{aligned}
v_{0j}(z) &= w_{0j}(z), \\
v_{1j}(z) &= w_{1j}(z), \\
v_{2j}(z) &= w_{2j}(z) - \frac{1}{2}v_{1j}(z), \\
v_{3j}(z) &= w_{3j}(z) - \frac{1}{6}v_{1j}(z) - v_{1j}(z)v_{2j}(z), \\
&\dots\dots\dots \\
v_{nj}(z) &= w_{nj}(z) - \frac{1}{n!} \sum_{l=2}^n \left(\sum_{j_1+\dots+j_l=n} (j_1! \dots j_l!) v_{j_1j}(z) \dots v_{j_lj}(z) \right). \quad (3.129)
\end{aligned}$$

According to the intended use, the solution form (3.128) may be more convenient than the the form (3.99), but once we have computed the amplitude corrections $w_{nj}(z)$ by solving its transport equation (3.122), we can compute the phase corrections v_{nj} using the relation (3.129). The solution form given by equation (3.99) is convenient when we perform the matching of the Liouville-Green functions with the asymptotic expansion of the solution of the approximate form of the governing equation (3.97) at a turning point of the operator $\hat{L}$, as we are going to see in section 3.2.5.

If the wave numbers dS_j/dz are different from each other we have four Liouville-Green functions, which can be seen as a base to represent any asymptotic solution for the governing equation (3.97). If at some point in the complex z plane two or more wave numbers dS_j/dz turn out to be equal, we do not have four independent Liouville-Green functions to represent our asymptotic solution at this particular point, so our asymptotic solution is not valid at those particular points, which are called the turning points of the operator $\hat{L}$. They are studied in the next section.

In summary, in this section we constructed recurrence relations for the WKB waves (Liouville-Green functions) up to any order. To first order we were able to

solve the recurrence relation in closed form (functions $w_{0j}(z)$), but for high orders, the recurrence relations have to be solved numerically. At first, we assumed that the solution of the governing equation (3.97) has the form of an amplitude times the exponential of a phase, functions of the space coordinate z (see equation (3.99)). Then, we substituted equation (3.99) into the governing equation (3.97) and obtained an equation for the phase $S_j(z)$ and a transport equation for the wave amplitudes $w_j(z)$. The equation for the wave numbers $\frac{dS_j}{dz}$ (equation (3.105)) is the uniform system dispersion relation, assumed locally valid in the context of the WKB method. Second, we assumed an expansion in ε for the wave amplitude $w_j(z)$, and substituted this expansion in the transport equation, which furnished a set of first order differential equations for the wave amplitude corrections $w_{nj}(z)$. We were able to solve only to first order, which gave $w_{0j}(z)$. The high order amplitude corrections are solutions of the recurrence relation (3.122). We also realized that the wave amplitude correction w_{1j} is actually a phase correction (see equation (3.125)). We proposed another form for the solution of equation (3.98), where we have phase corrections instead of amplitude corrections. At the end we show how to compute the phase corrections in the solution form given by (3.128) from the amplitude corrections solutions of the transport equation (3.122).

3.2.3 Turning Points of the Operator $\hat{L}$.

The asymptotic solution for the governing equation (3.97) is given as a linear combination of four WKB waves (Liouville-Green functions). Each Liouville-Green function (WKB wave) is a function of one of the wave numbers $\frac{dS_j}{dz}$. Four different wave numbers $\frac{dS_j}{dz}$ give four linearly independent Liouville-Green functions. If at some point of the complex z plane one or more of the wave numbers $\frac{dS_j}{dz}$ coalesce, two or more Liouville-Green functions became linearly dependent and we do not have an appropriate base to represent the asymptotic solutions. Points in the complex z plane where this happens are called turning points.

In this section we discuss how to obtain the turning points of the operator $\hat{L}$ for a given beam non-uniformity. We also discuss which wave numbers $\frac{dS_j}{dz}$ coalesce at the

various types of turning points we may have. We consider the case of non-uniformity in the beam flexural rigidity only. We chose such uniformity such that we have first order turning points and simple poles as singularities. We discuss in detail how the wavenumber couple at the first order turning points and poles.

3.2.3.1 Dispersion Relation Analysis.

As mentioned before, the wave numbers dS_j/dz are solutions of equation (3.106). So, the symbol $R_0(\overset{2}{z}, \overset{1}{p})$ of the operator $\hat{R}$ is basically the uniform system dispersion relation

$$R_0(z, p) = ei(z)p^4 - \bar{P}(z)p^2 + 4\omega^2 m(z) = 0 \quad (3.130)$$

assumed locally valid. For a given wave frequency ω , the dispersion relation, given by equation (3.130), relates the space coordinate z with the local wave number p through the non-dimensional beam flexural rigidity $ei(z)$, the non-dimensional tensile force $\bar{P}(z)$ and non-dimensional mass per unit length $m(z)$. If we assume z and the wavenumber p as complex variables, we can see this dispersion relation as an implicit mapping from the z complex plane into the complex wavenumber p plane.

$$z \mapsto p$$

$$(z, p) \rightarrow R_0(z, p) = 0 \quad (3.131)$$

This mapping has branch points in the complex z plane, and their images in the complex wavenumber p plane are called the saddle points. The branch points of the implicit mapping (3.131) are solutions of the set of equations

$$R_0(z, p) = 0 \quad (3.132)$$

$$\frac{\partial^n R_0}{\partial p^n}(z, p) = 0 \text{ for } n = 1, \dots, 3. \quad (3.133)$$

At the branch points of the dispersion relation, given by equation (3.130), we have that two or more wave numbers coalesce. So, at the branch points of the dispersion relation two or more Liouville-Green functions are linearly dependent. Then, the branch points of the dispersion relation, given by equation (3.130), are the turning points of the governing equation (3.97). The solutions of the equations (3.132)-(3.133) give rise to three turning point conditions, which follows

$$F(z) = 0, \quad (3.134)$$

$$m(z) = 0, \quad (3.135)$$

$$ei(z) = 0. \quad (3.136)$$

The function $F(z)$ is given in terms of the function $ei(z)$, $\bar{P}(z)$ and $m(z)$ through equation (3.112). The solutions of the turning point condition (3.136) are actually poles of the governing equation (3.97).

We denoted the solutions of the locally valid uniform system dispersion relation (see equation (3.130) or (3.106)) as the wave numbers dS_j/dz . Next, we are going to study how the wave numbers dS_j/dz coalesce as we approach a turning point in the z complex plane. First, we consider a turning point solution of the equation (3.134). According to expressions (3.107) to (3.110), the wave numbers at a turning point solution of equation (3.134) assume the form

$$\frac{dS_1}{dz} = \sqrt{\frac{1}{2}D(z)}, \quad (3.137)$$

$$\frac{dS_2}{dz} = -\sqrt{\frac{1}{2}D(z)}, \quad (3.138)$$

$$\frac{dS_3}{dz} = \sqrt{\frac{1}{2}D(z)}, \quad (3.139)$$

$$\frac{dS_4}{dz} = -\sqrt{\frac{1}{2}D(z)}. \quad (3.140)$$

According to equations (3.137) to (3.140), we realize that the wave numbers coalesce in pairs at turning points solution of equation (3.134). More specifically, wave numbers dS_1/dz and dS_3/dz coalesce to $\sqrt{D(z)/2}$ and wave numbers dS_2/dz and dS_4/dz

coalesce to $-\sqrt{D(z)}/2$. Second, we look at what happens with the wave numbers dS_j/dz when we approach a turning point solution of the equation (3.135). The function $F(z)$ at this type of turning point is just given as

$$F(z) = \bar{P}(z)^2. \quad (3.141)$$

This implies, according to expressions (3.107) to (3.110), that the wave numbers at a this type of turning point assume the form

$$\frac{dS_1}{dz} = 0, \quad (3.142)$$

$$\frac{dS_2}{dz} = 0, \quad (3.143)$$

$$\frac{dS_3}{dz} = \sqrt{D(z)}, \quad (3.144)$$

$$\frac{dS_4}{dz} = -\sqrt{D(z)}, \quad (3.145)$$

According to equations (3.142) to (3.145), we have only one pairwise coalescence of wave numbers. The wave numbers dS_1/dz and dS_2/dz coalesce to 0, but the wave numbers dS_3/dz and dS_4/dz do not coalesce, they approach, respectively, $\sqrt{D(z)}$ and $-\sqrt{D(z)}$. Third, we discuss what happens with the wave numbers dS_j/dz when we approach a point in the complex z plane that satisfies equation (3.136). These points are actually poles of the governing equation (3.97), but also branch points of the locally valid uniform system dispersion relation, given by equation (3.130). As we approach these points in the complex z plane, the wave numbers dS_j/dz may assume any complex value, it is just a matter of how the function $ei(z) \rightarrow 0$. So, to make a proper discussion on how the wave numbers coalesce at the turning points solutions of equation (3.136), we need first to specify the expression for the non-dimensional flexural rigidity $ei(z)$. We assume that for a neighborhood close to the turning point the flexural rigidity has the form

$$ei(z) \sim \beta(z - z^*)^\alpha \text{ with } \alpha \geq 1, \quad (3.146)$$

where z^* is a turning point given by equation (3.136) and β is a complex constant. To simplify our analysis, we assume that the non-dimensional tensile force and the non-dimensional mass per unit length are constants. Under such conditions, the wave numbers dS_j/dz have the form

$$\frac{dS_1}{dz} = \sqrt{\frac{\bar{P}}{2\beta(z - z^*)^\alpha}} \sqrt{1 - \sqrt{1 + \frac{4\omega^2 m}{\bar{P}^2} \beta(z - z^*)^\alpha}}, \quad (3.147)$$

$$\frac{dS_2}{dz} = - \sqrt{\frac{\bar{P}}{2\beta(z - z^*)^\alpha}} \sqrt{1 - \sqrt{1 + \frac{4\omega^2 m}{\bar{P}^2} \beta(z - z^*)^\alpha}}, \quad (3.148)$$

$$\frac{dS_3}{dz} = \sqrt{\frac{\bar{P}}{2\beta(z - z^*)^\alpha}} \sqrt{1 + \sqrt{1 + \frac{4\omega^2 m}{\bar{P}^2} \beta(z - z^*)^\alpha}}, \quad (3.149)$$

$$\frac{dS_1}{dz} = - \sqrt{\frac{\bar{P}}{2\beta(z - z^*)^\alpha}} \sqrt{1 + \sqrt{1 + \frac{4\omega^2 m \beta}{\bar{P}^2} (z - z^*)^\alpha}}. \quad (3.150)$$

Now, we assume that $|(z - z^*)^\alpha| \ll \frac{\bar{P}^2}{4\omega^2 m \beta}$. Under such assumption, we can write

$$\sqrt{1 + \frac{4\omega^2 m \beta}{\bar{P}^2} (z - z^*)^\alpha} \sim 1 + \frac{2\omega^2 m \beta}{\bar{P}^2} (z - z^*)^\alpha + O((z - z^*)^{2\alpha}) \quad (3.151)$$

and equations (3.147) to (3.150) simplifies to

$$\frac{dS_1}{dz} \sim \omega \sqrt{\frac{m\beta}{\bar{P}}}, \quad (3.152)$$

$$\frac{dS_2}{dz} \sim -\omega \sqrt{\frac{m\beta}{\bar{P}}}, \quad (3.153)$$

$$\frac{dS_3}{dz} \sim \sqrt{\frac{\bar{P}}{2\beta(z-z^*)^\alpha}} \sqrt{2 + \frac{2\omega^2 m\beta}{\bar{P}^2}(z-z^*)^\alpha}, \quad (3.154)$$

$$\frac{dS_4}{dz} \sim -\sqrt{\frac{\bar{P}}{2\beta(z-z^*)^\alpha}} \sqrt{2 + \frac{2\omega^2 m\beta}{\bar{P}^2}(z-z^*)^\alpha}. \quad (3.155)$$

From the equations (3.152) and (3.153) we realize that the wave numbers dS_1/dz and dS_2/dz do not coalesce at the turning points given by the equation (3.136). From equations (3.154) and (3.155), as we approach the turning point z^* , the wave number $dS_3/dz \rightarrow \infty \exp(\frac{i}{2} \arg\{(z-z^*)\})$ and the wave number $dS_4/dz \rightarrow -\infty \exp(\frac{i}{2} \arg\{(z-z^*)\})$. Therefore, the wave numbers dS_3/dz and dS_4/dz coalesce to the infinite as we approach a turning point z^* , solution of equation (3.136).

In this section we obtained the equations whose solutions are the turning points of the governing equation (3.97). These equations are written in terms of the beam non-dimensional mass per unit length and non-dimensional flexural rigidity, and in terms of the non-dimensional tensile force. We also discussed how the wave numbers dS_j/dz coalesce as we approach the different types of turning points discussed above. So far, we did not specify the functional form of the non-dimensional mass per unit length $m(z)$, of the non-dimensional flexural rigidity $ei(z)$ and of the non-dimensional tensile force $\bar{P}(z)$. We are going to do this in the next section. After we specify the functional form of the parameters $m(z)$, $ei(z)$ and $\bar{P}(z)$, we give the new form of the turning point conditions (3.134) to (3.136).

3.2.3.2 Example of Non-uniformities and Their Turning Points.

The non-uniformity in the non-dimensional mass per unit length and in the non-dimensional flexural rigidity have a finite support. We assume that the uniform part of the beam before and after the non-uniform section have the same flexural rigidity

and mass per unit length. So we can write

$$m(z) = 1 + F_1(z), \quad (3.156)$$

$$ei(z) = 1 + F_2(z). \quad (3.157)$$

The non-dimensional tensile force $\bar{P}(z)$ is assumed to be non-uniform only along a finite part of the beam. We write it as a superposition of a constant tensile load plus a localized variation

$$\bar{P}(z) = \bar{P} + F_3(z). \quad (3.158)$$

The support of the function $F_3(z)$ is the interval $-\varepsilon L \leq z \leq \varepsilon L$. The functions $F_1(z)$ and $F_2(z)$ are assumed to be non-zero in the interval $-\varepsilon L \leq z \leq \varepsilon L$. Next, we write the turning point conditions (3.134) to (3.136) in terms of the functions $F_j(z)$. The first turning point condition, given by equation (3.134), assumes the form

$$2PF_3(z) + (F_3(z))^2 + 4\omega^2[F_1(z) + F_2(z) + F_1(z)F_2(z)] = -(P^2 + 4\omega^2). \quad (3.159)$$

The second and third turning point conditions, given, respectively, by equations (3.135) and (3.136), assume, respectively, the forms

$$F_1(z) = -1, \quad (3.160)$$

$$F_2(z) = -1. \quad (3.161)$$

According to the definition of functions $m(z)$ and $ei(z)$, for real values of z we have that

$$|F_1(z)| < 1 \text{ for } z \text{ real}, \quad (3.162)$$

$$|F_2(z)| < 1 \text{ for } z \text{ real}. \quad (3.163)$$

Restrictions (3.162) and (3.163) imply that the turning points, solutions of equations (3.135) and (3.136), could not assume real values for the kind of beam considered. These turning point conditions give real turning points at points where the cross sectional area of the beam goes to zero. If turning points, solutions of equations (3.135) and (3.136), exist for a given functions $F_1(z)$ and $F_2(z)$, they are complex numbers. For the first turning point condition (3.159), the question regarding if the turning points given by this condition are real or not has not an so easy answer. If there is no tensile force applied to the waveguide, then the turning points solutions of equation (3.159) are complex numbers due to restrictions (3.162) and (3.163). If the tensile force assume negative values along the real axis, then we may have real turning points.

The first turning point condition (3.159) is not an easy to solve equation, since in general the functions $F_j(z)$ differs from each other. We consider some simplifications. In the following we assume that only one of the functions $F_j(z)$ is non-constant.

- **Non-constant non-dimensional mass per unit length.**

The function $F_1(z)$ is considered not constant. The non-dimensional flexural rigidity and non-dimensional tensile force are assumed constants. Then

$$F_1(z) = A_1 t(z), \quad (3.164)$$

$$F_2(z) = 0, \quad (3.165)$$

$$F_3(z) = 0. \quad (3.166)$$

The first turning point condition, given by equation (3.159), assumes the form

$$t(z) = -\frac{(P^2 + 4\omega^2)}{4\omega^2 A_1}, \quad (3.167)$$

and the second turning point condition, given by equation (3.160), assumes the form

$$t(z) = -\frac{1}{A_1}. \quad (3.168)$$

The third turning point condition does not apply for this configuration of the parameters of the Euler-Bernoulli beam.

- **Non-constant non-dimensional flexural rigidity.**

The function $F_2(z)$ is considered non-constant. The non-dimensional mass per unit length and the non-dimensional tensile load are assumed constants. Then

$$F_1(z) = 0 \quad (3.169)$$

$$F_2(z) = A_2 t(z) \quad (3.170)$$

$$F_3(z) = 0 \quad (3.171)$$

The first turning point condition, given by equation (3.159), assumes the form

$$t(z) = -\frac{(P^2 + 4\omega^2)}{4\omega^2 A_2}, \quad (3.172)$$

and the third turning point condition, given by equation (3.161), assumes the form

$$t(z) = -\frac{1}{A_2}. \quad (3.173)$$

The second turning point condition does not apply for this case since the function $m(z)$ is assumed constant.

- **Non-constant non-dimensional tensile load.** The function $F_3(z)$ is assumed non-constant. The non-dimensional mass per unit length and the non-dimensional flexural rigidity are assumed constants. Then

$$F_1(z) = 0, \quad (3.174)$$

$$F_2(z) = 0, \quad (3.175)$$

$$F_3(z) = A_3 t(z). \quad (3.176)$$

The first turning point condition, given by equation (3.159), assumes the form

$$t(z) = \begin{cases} -\frac{P}{A_3} + \frac{2\omega}{A_3} \\ -\frac{P}{A_3} - \frac{2\omega}{A_3} \end{cases} \quad (3.177)$$

The second and third turning point conditions do not apply for this case since the functions $m(z)$ and $ei(z)$ are assumed constant.

Next, we consider a simple analytical function to represent the function $t(z)$, which is free of poles. We consider

$$t(z) = \exp(-z^2). \quad (3.178)$$

We assume that the turning point conditions, represented by equations (3.159), (3.160) and (3.161), give turning points which satisfy the equation $t(z) = t_l$. The index l assumes values 1, 2 and 3. $l = 1$ refers to the first turning point condition given by equation (3.159). $l = 2$ and 3 refers, respectively, to the turning point condition given by equations (3.160) and (3.161). Next, we find the turning points in the z plane that corresponds to the quantity t_l . Examples of the quantity t_l are given by the right side of equations (3.167), (3.168), (3.172), (3.173) and (3.177). Now we have to solve the equation

$$t_l = \exp(-z^2). \quad (3.179)$$

If we write t_l in the form

$$t_l = \rho_l \exp(i\theta_l), \quad (3.180)$$

where ρ_l is the modulus of t_l and θ_l is the phase of t_l . The solution of equation (3.179)

gives the following set of turning points:

$$z_{2n-1,l} = \{\ln^2(\rho_l) + (\theta_l + 2n\pi)^2\}^{1/4} \exp \left\{ \frac{i}{2} \arctan \left(\frac{\theta_l + 2n\pi}{\ln(\rho)} \right) \right\} \text{ with } n \text{ as an integer} \quad (3.181)$$

$$z_{2n,l} = - \{\ln^2(\rho_l) + (\theta_l + 2n\pi)^2\}^{1/4} \exp \left\{ \frac{i}{2} \arctan \left(\frac{\theta_l + 2n\pi}{\ln(\rho)} \right) \right\} \text{ with } n \text{ as an integer} \quad (3.182)$$

We have an infinite set of turning points, but only four pairs of complex conjugate turning points are relevant. The four pairs closest to the real axis.

For the study of the WKB method we have in mind in this work we consider non-uniformity in the flexural rigidity given by the function (3.178). The other coefficient functions $m(z)$ and $\bar{P}(z)$ are assumed constants. Figures (3-7) and (3-8) illustrate the topology of the turning points closest to the real axis for the non-uniformity in the flexural rigidity given by the function (3.178). In these figures, for each set of eight turning points, the four turning points closest to the real axis are given by the third turning point condition (3.173). The other four turning points are given by the turning point condition (3.172).

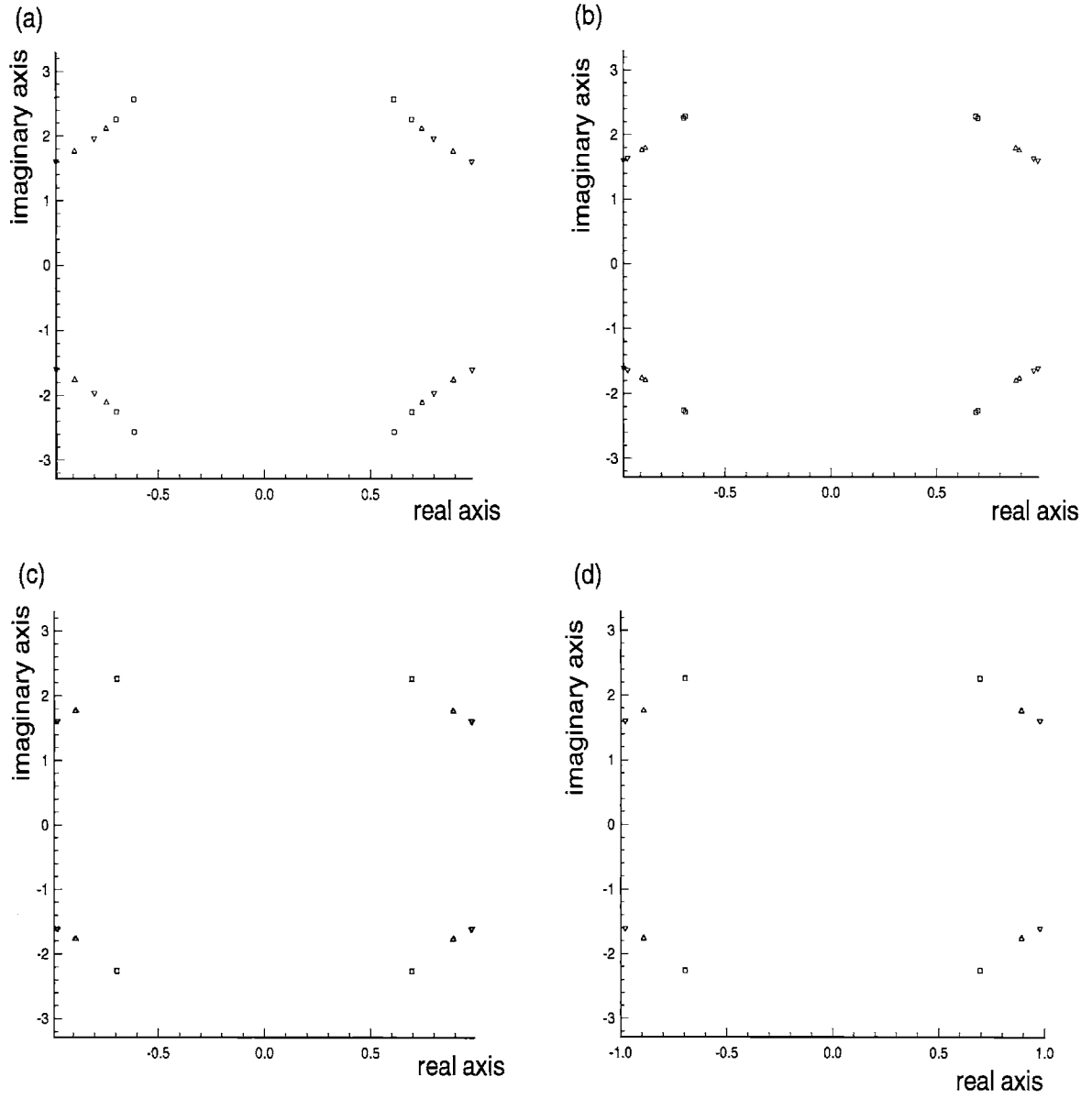


Figure 3-7: Turning points for non-uniformity in the flexural rigidity. Tensile load equals to 1000 N. Wave frequency: (a) - 2 Hz, (b) - 10 Hz, (c) - 20 Hz and (d) - 30 Hz. The symbols representing the position of the turning points are associated with the non-uniformity non-dimensional amplitude. Symbols: $\square$ - $A = 0.01$, $\triangle$ - $A = 0.1$, ∇ - $A = 0.2$.

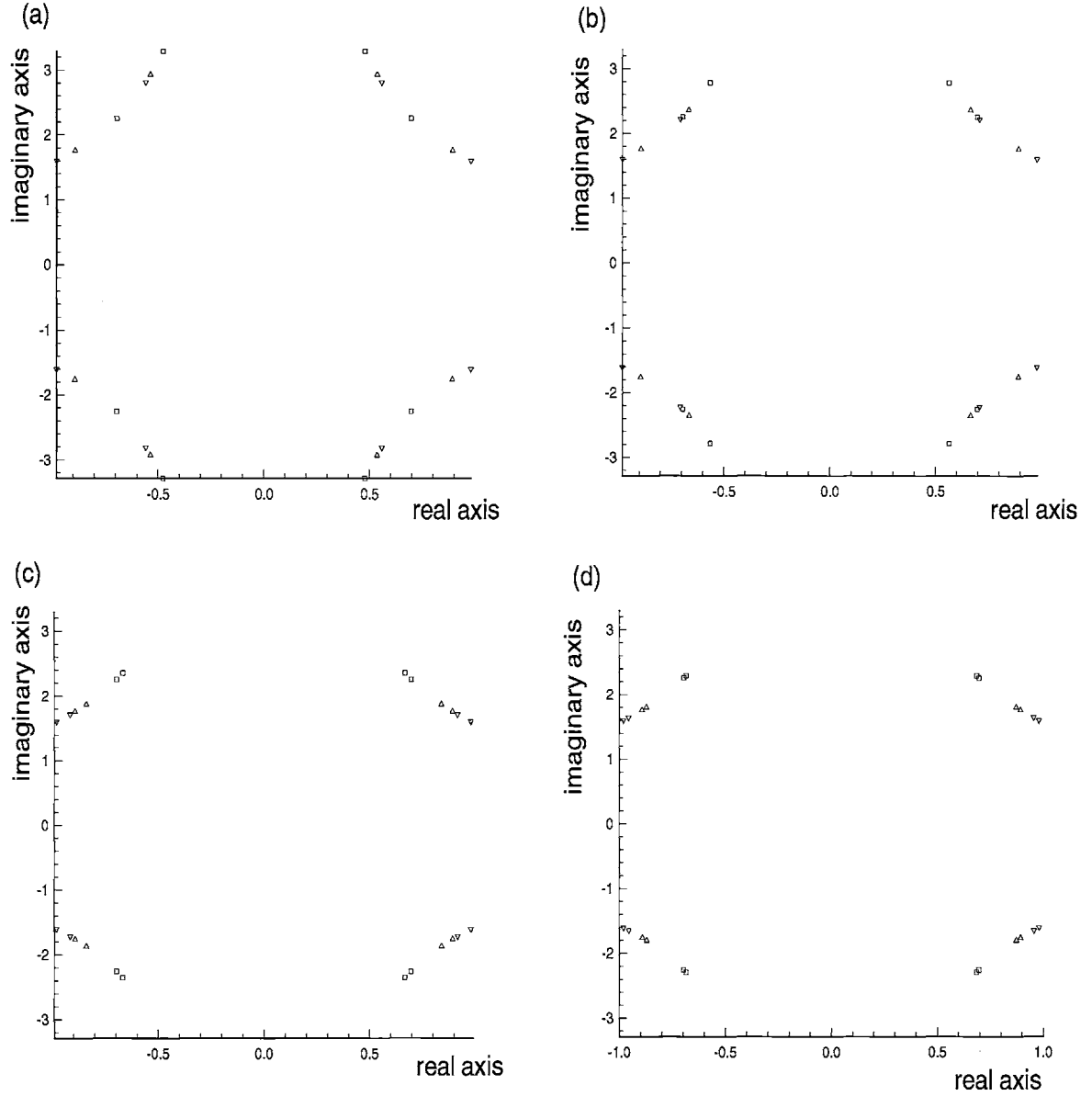


Figure 3-8: Turning points for non-uniformity in the flexural rigidity. Tensile load equals to 10,000 N. Wave frequency: (a) - 2 Hz, (b) - 10 Hz, (c) - 50 Hz and (d) - 90 Hz. The symbols representing the position of the turning points are associated with the non-uniformity non-dimensional amplitude. Symbols: $\square$ - $A = 0.01$, $\triangle$ - $A = 0.1$, ∇ - $A = 0.2$.

In the next section we discuss the evolution of the wave numbers dS_j/dz as we deform the real axis to encircle chosen turning points or groups of turning points.

The equations (3.107) to (3.110) that define the wave numbers dS_j/dz have branch points at the turning points of the governing equation (3.97). Then, for the wave numbers dS_j/dz to be well defined, we need to prescribe branch cuts in the complex z plane. We avoided to do so until now, because the topology of the branch points (turning points) depends on the form of the non-uniformity in the coefficient functions $m(z)$, $ei(z)$ and $\bar{P}(z)$. For the chosen non-uniformity in the flexural rigidity, we have an infinite set of branch points. We have a set of four pairs of branch points closest to the real axis. From these eight branch points, four of them lay in the upper complex plane and the other fours are their complex conjugates. So, we define four branch cuts, each of them connecting one of the four branch points in the upper complex plane to its complex conjugate pair in the lower complex plane. For the rest of the branch points in the upper complex plane, we define the branch cuts connecting each branch point to infinity, without crossing the real axis. We do the same for the rest of the branch points laying in the lower complex plane. Each branch point has a branch cut connecting it to infinity, without crossing the real axis.

We consider radiation or boundary conditions before and after the non-uniform part of the beam imposed at a specific Riemann sheet. So, we need to deform the real axis to a contour that do not cross the branch cuts that cross the real axis. If we cross one of the branch cuts, we end up in another Riemann sheet where we do not have radiation or boundary conditions imposed. Candidates for the deformed contours are illustrated in figure 3-9.

As a result of the deformation of the real axis into a contour around the branch cuts, we need to follow along this contour what happens with the wave numbers dS_j/dz . As we go around a branch cut and return to the real axis, the numerical values of the wave numbers dS_j/dz change. For example, we consider the contour illustrated in the top part of figure 3-9. The wavenumber dS_3/dz assumes a positive real number along the real axis before we reach the first branch cut that crosses the real axis. As we go around this branch cut (first detour from the real axis, as we travel from left to right in the top of figure 3-9), the wavenumber dS_3/dz assumes negative real values (dash dotted line of part (a) of figure 3-10), and as we go around the

second branch cut (second detour from the real axis, as we travel from left to right in the top of figure 3-9), the wavenumber dS_3/dz assumes pure imaginary values (dash dotted line in part (b) of figure 3-10).

3.2.3.3 Evolution of The Wave numbers dS_j/dz .

In this section, we study how the numerical value of the wave numbers dS_j/dz vary as we follow the contours illustrated in figure 3-9. This procedure allow us to verify which wave modes couple (which wave numbers coalesce) at chosen turning points. To illustrate how encircling a chosen turning point reveals how the wave numbers coalesce at this turning point, we first remind that a turning point is a branch point of the dispersion relation. So, close to a turning point z^* , the locally valid uniform system dispersion relation can be represented as

$$(k - k^*)^n \sim A(z - z^*), \quad (3.183)$$

where k^* is the saddle point in the complex wavenumber plane, image of the branch point z^* through the complex mapping given by the locally valid uniform system dispersion relation. We assume coalescence of only two wave numbers ($n = 2$ in equation (3.183)), which are denoted $k^\pm$. They are solutions of the approximate dispersion relation (3.183), and are given by the equation

$$k^\pm - k^* \sim \pm \sqrt{A(z - z^*)}. \quad (3.184)$$

In polar coordinates, we can write $A(z - z^*) = \rho \exp(i\theta)$, such that

$$k^\pm - k^* \sim \pm \rho^{1/2} \exp(i\frac{\theta}{2}). \quad (3.185)$$

Before encircling the turning point z^* , we assume that $\theta = \theta_0$ and the wave numbers are given by the equation

$$k^\pm - k^* \sim \pm \rho^{1/2} \exp(i \frac{\theta_0}{2}). \quad (3.186)$$

As we vary the argument of $z - z^*$ by 2π , θ assume the value $\theta_0 + 2\pi$. So the wave numbers are now given by the equation

$$k^\pm - k^* \sim \mp \rho^{1/2} \exp(i \frac{\theta_0}{2}). \quad (3.187)$$

So, as we encircle the turning point z^* in the complex z plane, the wave numbers describe a half circle in the wavenumber plane, and the wave numbers $k^\pm$ exchange their numerical values ($k^+(\theta_0 + 2\pi) = k^-(\theta_0)$ and $k^-(\theta_0 + 2\pi) = k^+(\theta_0)$).

If a wavenumber do not coalesce to another wavenumber as we approach a chosen turning point in the complex z plane, it just perform a closed contour in the complex wavenumber plane as we encircle the chosen turning point, but if the wave numbers coalesce at a chosen turning point, they perform arcs in the complex wavenumber plane as the example above illustrates. If n wavenumbers coalesce at a chosen turning point, as we encircle the chosen turning point, each of the coalescing wavenumbers perform an arc in the complex wavenumber plane with a total argument variation of $2\pi/n$, and the sum of these arcs perform a closed contour. For the example above, $n = 2$ and the total argument variation of the arc is π (half circle).

Next, we illustrate the continuum of numerical values the wave numbers dS_j/dz assume as we travel along a contour in the complex z plane. This contour is basically the real axis plus deformations in the complex z plane to avoid crossing branch cuts and changes of Riemann sheets, and to encircle chosen turning points. The topology of the turning points is illustrated in Figures (3-7) and (3-8). We consider basically two types of contours. The first type circles each turning point separately and return to the real axis before encircling the next chosen turning point. The second type

treats the turning points as groups. It goes around two turning points at once. In Figures (3-7) and (3-8), we see that as the values of the wave frequency increases, there are pairs of turning points that became closer and closer. They became identical only if the tensile force goes to zero, otherwise, they may be very close, but are not the same. The group of turning points to be encircled by the contours in the z plane are two of the four pairs of merging turning points closest to the real axis.

For real values of the z independent variable before the beam non-uniform part, dS_1/dz and dS_2/dz are pure imaginary numbers with opposite sign, and dS_3/dz and dS_4/dz are real numbers with opposite sign. As we go in the complex z plane around the turning points and branch cuts, some or all of the wave numbers dS_j/dz exchange their numerical values among themselves. How they exchange, and if they exchange, their numerical value depends on which turning point condition the encircled turning point satisfies, or if the contour encircles a pair of turning points.

Next, we illustrate the two shapes of contours considered in the complex z plane. We also give the images of these two contours in the complex wavenumber plane. These images are the continuum of numerical values that the wave numbers dS_j/dz assume along these contours. An example of the two types of contours considered in the z plane is given in Figure 3-9.

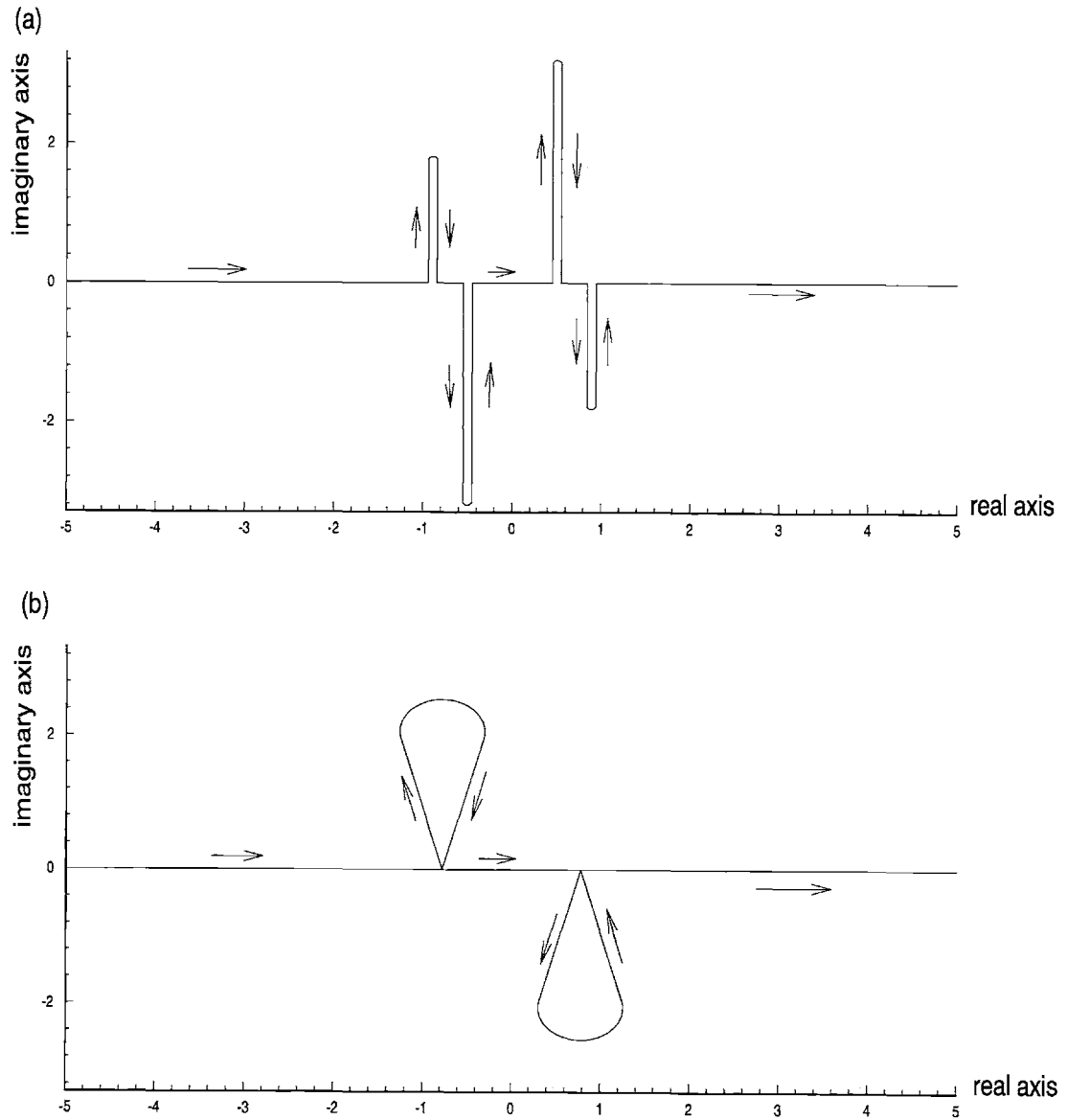


Figure 3-9: Examples of the two types of contours in the z plane. Arrows indicates in which sense we travel along the contours. (a) - contour of the first type. (b) - contour of the second kind. Tensile load $P = 10,000$ N. The wave frequency for (a) is 2 Hz, which gives turning points further apart of each other. In (b) the wave frequency is 10 Hz.

As we travel along the real axis in the z plane, the contour encircles turning points or group in turning points in an alternate fashion. If the contour encircled

a turning point (group of turning points) in the the upper plane, the next encircled turning point (group of turning points) is in the lower plane and vice-versa. For this particular non-uniformity in the flexural rigidity, this way of encircling turning points or groups of turning points allows the wave numbers dS_j/dz to return to the same numerical values they had at the starting point (before the non-uniform part of the beam). Therefore, they are associated with the same kind of wave mode at both ends of the non-uniform part of the waveguide. This fact makes more easy to write the radiation conditions for the wave scattering problem due to the non-uniformity along a finite part of the beam.

Other ways to deform the real axis to avoid crossing branch cuts and to encircle turning points are fine, but we need to keep track of the change in the numerical value of the wavenumbers, since the same wavenumber in different parts of the real axis is associated with different wave modes (evanescent or propagating wave mode).

In figures 3-10 and 3-12, we illustrate the evolution of the wave numbers as we travel along the contour illustrated in part (a) of Figure 3-9. We illustrate only the sequence of numerical values that the wave numbers assume as we encircle the chosen turning points. As we encircle the turning point, we start and return to the real axis. The first encircled turning point satisfies the third turning point condition, given by equation (3.136). According to part (a) of Figure 3-10, we see that dS_3/dz and dS_4/dz exchange numerical values. The other two wavenumbers keep their values, as is illustrated in parts (a) and (b) of figure 3-12. This means that the two evanescent modes interact with each other. The second encircled turning is given by the first turning point condition (equation (3.134)). According to part (b) of Figure 3-10, the wave numbers dS_1/dz and dS_4/dz (dS_2/dz and dS_3/dz) exchange numerical values. This implies that a right (left) propagating wave mode interacts with a left (right) evanescent wave mode. The next encircled turning point satisfies the same turning point condition as the previous turning point. According to part (c) of Figure (3-10), the evolution of the wave numbers for this case is exactly the opposite of the previous case. The last encircled turning point satisfies the third turning point condition. According to part (d) of figure 3-10, wave numbers dS_3/dz and dS_4/dz exchange

numerical values. The wave numbers dS_1/dz and dS_2/dz perform a closed contour in the wavenumber plane, similar to figure 3-12.

In figure 3-11, we illustrate the evolution of the wave numbers dS_j/dz as we encircle pairs of coalescing turning points according to the contour illustrated in part (b) of Figure 3-9. According to part (a) of Figure 3-11, as we encircle the first pair of turning points, all the wave numbers dS_j/dz exchange numerical values among themselves. Before encircling the first pair of turning points, the wavenumber $dS_1/dz(dS_2/dz)$ is a positive (negative) imaginary number, and $dS_3/dz(dS_4/dz)$ is a positive (negative) real number, but after we encircle the first pair of turning points, the wavenumber $dS_1/dz(dS_2/dz)$ becomes a negative (positive) real number and the wavenumber $dS_3/dz(dS_4/dz)$ becomes a positive (negative) imaginary number. In part (b) of Figure 3-11, the exchange of numerical values among the wave numbers dS_j/dz is the opposite of what was verified in part (a).

As we encircled a pair of merging turning points, all four wave modes interact among themselves. This suggests that we need at least a fourth order differential equation to describe the wave mode coupling at a pair of almost coalescing turning points. This situation is similar to what happens in the WKB analysis of the problem of linear surface gravity waves interacting with a one-dimensional channel with non-uniform bottom, where we can have coalescence of four wavenumbers at the turning points. The analysis performed in the second part of this chapter can be seen as an assessment of the challenges we need to face to perform the study of the coupling of four wave modes in the context of the WKB method.

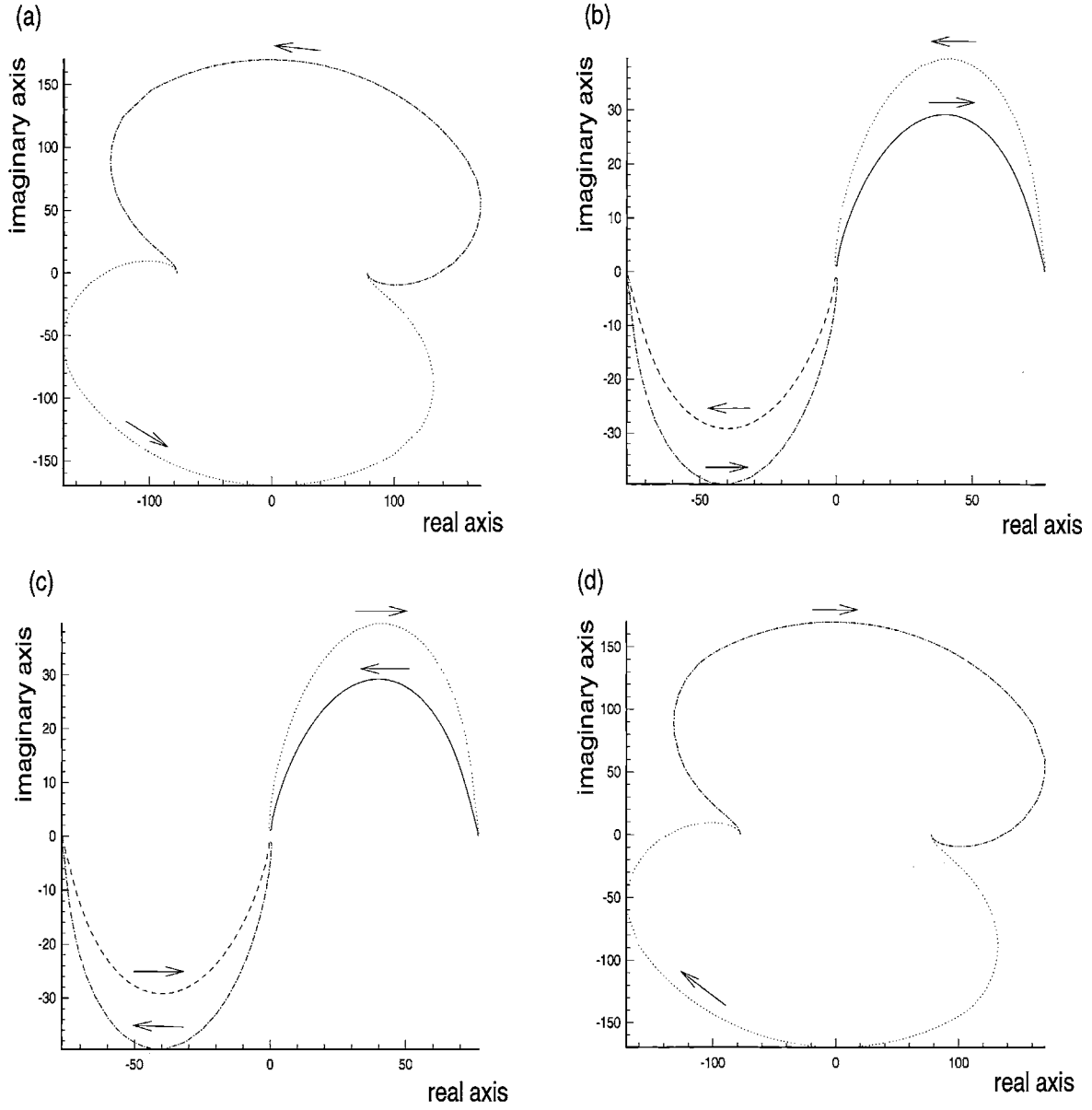


Figure 3-10: Continuum of numerical values that the wave numbers dS_j/dz assume as the contour in part (a) of Figure (3-9) encircles chosen turning points. Arrows indicate in which sense the wave numbers evolve as we encircle a turning point. Tensile load $P = 10,000$ N and wave frequency is 2 Hz. (a) - 1 st chosen turning point . (b) - 2 nd chosen turning point. (c) - 3 rd chosen turning point. (d) - 4 th chosen turning point. Line ——— - numerical value of dS_1/dz . Line - - - - - - numerical value of dS_2/dz . Line - · - · - · - numerical value of dS_3/dz . Line · · · · · - numerical value of dS_4/dz .

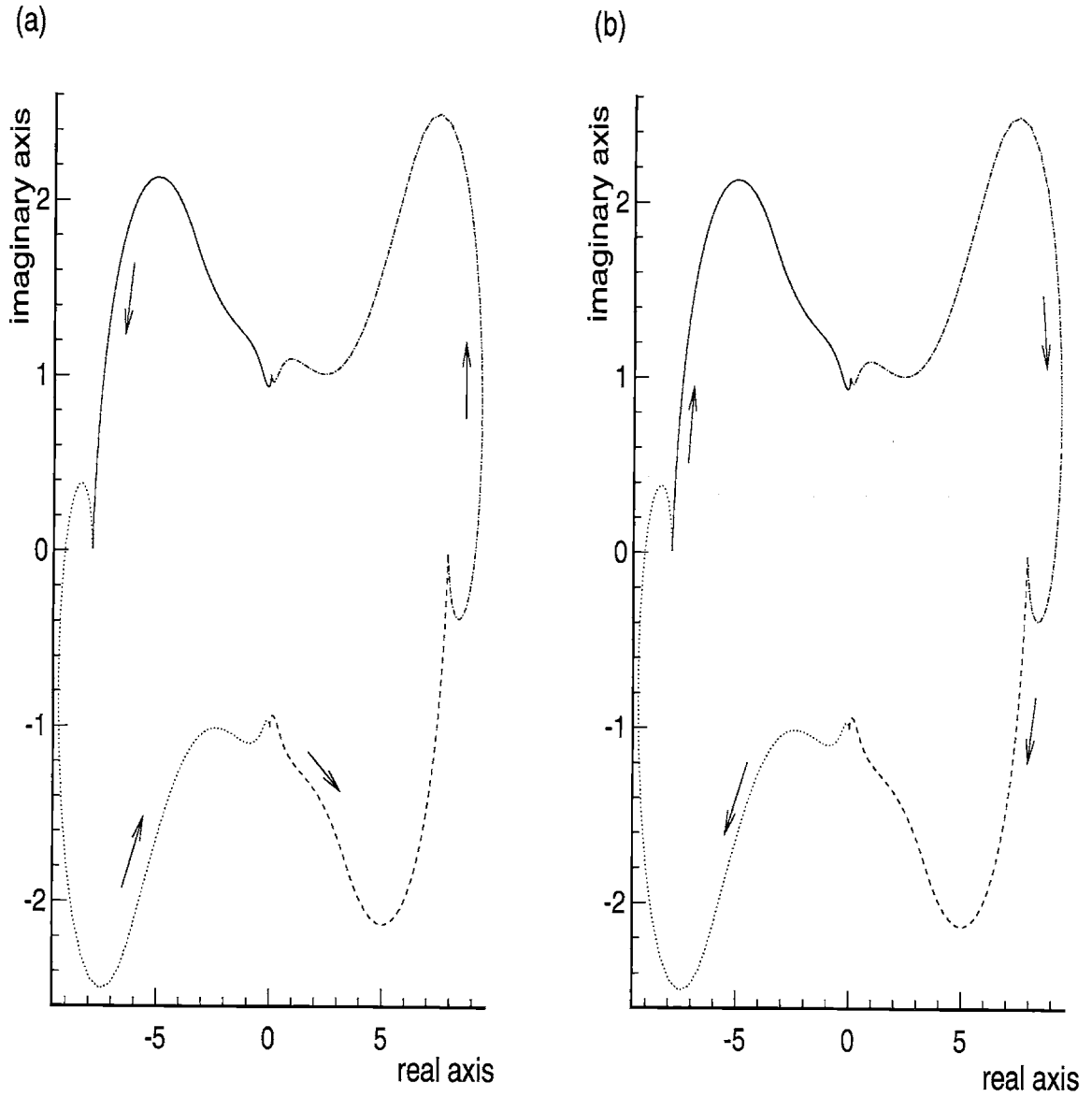


Figure 3-11: Continuum of numerical values the wave numbers dS_j/dz assume as the contour in part (b) of Figure (3-9) encircles chosen pairs of turning points. Arrows indicates in which sense the wave numbers evolve as we encircle a pair of turning points. Tensile load $P = 10,000$ N and wave frequency is 10 Hz. (a) - 1 st chosen pair of turning points . (b) - 2 nd chosen pair of turning points. Line ——— - numerical value of dS_1/dz . Line - - - - - - numerical value of dS_2/dz . Line - · - · - · - numerical value of dS_3/dz . Line · · · · · - numerical value of dS_4/dz .

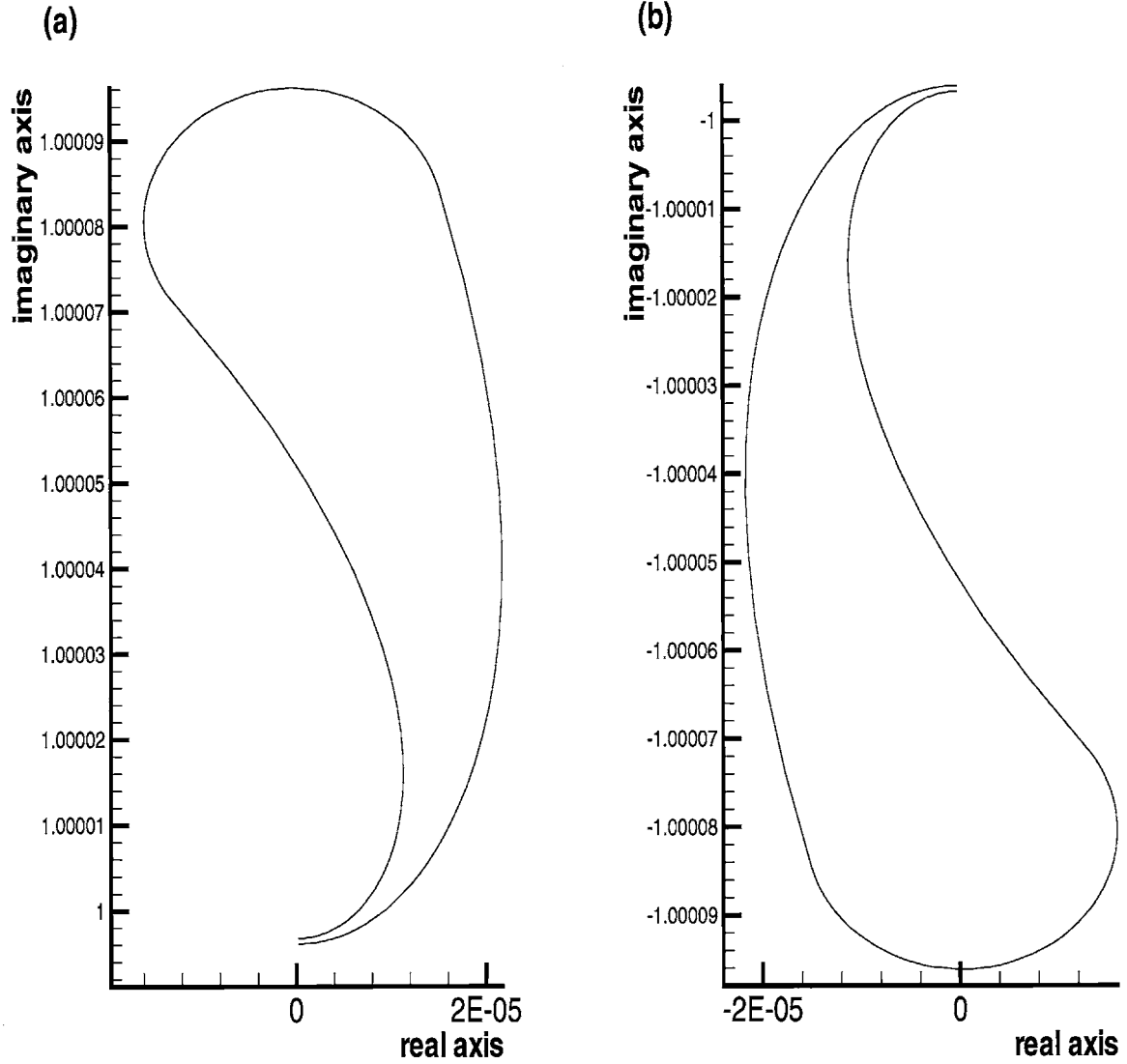


Figure 3-12: Continuum of numerical values assumed by the wave numbers dS_1/dz (part (a)) and dS_2/dz (part (b)). Tensile load $P = 10,000$ N and wave frequency is 2 Hz.

In section 3.2.3 we obtained the equations whose solutions are the turning points of the governing equation (3.97), we also discussed and illustrated how the wave numbers dS_j/dz coalesce as we approach turning points in the complex z plane (section 3.2.3.1). We also considered an example of non-uniformity (section 3.2.2). We consider the

case of non-uniformity in the flexural rigidity only. We obtained the turning points for this non-uniformity and we defined the necessary branch cuts in the complex z plane, so that the equations for the wave numbers dS_j/dz became well defined (section 3.2.3.3). We gave two possible contours, deformations of the real axis, to continue the wave numbers around the branch cuts, such that we always stay in the same Riemann sheet, where the boundary or radiation conditions are given (section 3.2.3.3).

To perform the connection problem (to continue the asymptotic solution across turning points and Stokes-line) we need to solve in closed form an approximate form of the governing equation (3.97), valid in a neighborhood of a chosen turning point or pair of turning points. This analysis is discussed in the next section.

3.2.4 Approximate form of the Beam Equation for a Neighborhood of a First Order Turning Point.

We consider the non-uniformity in the flexural rigidity only, as in the example in the previous section. The functions $m(z)$ and $\bar{P}(z)$ are assumed constants. We consider the function $t(z)$ given by equation (3.179). There are two types of first order turning points for this particular non-uniformity. One type satisfies the first turning point condition and the other type satisfies the third turning point condition. This last type of turning points are actually a first order pole of the governing equation (3.97) for the type of non-uniformity chosen.

For the non-uniformity in the flexural rigidity, the topology of the turning points and poles in the complex plane is illustrated by figures 3-9 and 3-10.

Through the figures 3-9 and 3-10, we see that we have a pole and a turning point close together. Therefore, we need to consider a turning point and its neighbor pole together regarding the connection problem. We obtain an approximate form of the governing equation valid in a neighborhood of a turning point plus its neighbor pole. We expand the flexural rigidity in Taylor series with respect to one of its pole z^* and keep the first non-zero term in the expansion. We consider the change of variable $z = (z - z^*)$, so we end up with an approximate equation of the form:

$$\varepsilon^4 \chi z \frac{d^4 y}{dz^4} + 2\varepsilon^4 \chi \frac{d^3 y}{dz^3} - \varepsilon^2 \bar{P} \frac{d^2 y}{dz^2} - \omega^2 y(z) = 0 \quad (3.188)$$

It has a singular point at $z = 0$ and a turning point at $z = \frac{\bar{P}}{\omega^2 \chi}$. The point $z = 0$ corresponds to the zeros of the non-dimensional flexural rigidity. Close to each zero of the flexural rigidity we have a turning point, and this turning point corresponds to one of the turning points of the full governing equation. The solution of this equation is given in terms of contour integrals, as follows:

$$y(z) = \int_C \frac{1}{t^2} \exp\left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha)\right) dt, \quad (3.189)$$

with

$$\lambda = \frac{|\chi| \varepsilon^4}{\omega^2}, \quad (3.190)$$

$$\beta = \arg\{\chi\}, \quad (3.191)$$

$$\alpha = \frac{\bar{P}}{\omega^2}, \quad (3.192)$$

$$\delta = \frac{\chi}{\omega^2} z, \quad (3.193)$$

and

$$f(t, \delta, \varepsilon, \alpha) = \exp(-i\beta) \left(\frac{1}{3t^2} + \frac{\varepsilon^2 \alpha}{t} + \varepsilon^4 \delta t \right). \quad (3.194)$$

The end points of the integration contour C has to satisfy the restriction

$$\varepsilon^4 \chi t^2 \exp\left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha)\right)|_C = 0. \quad (3.195)$$

To define well solutions of the differential equation (3.188) given by the contour integral (3.189), we need to specify the integration contour C . The end points of each possible contour C has to satisfy the condition given by equation (3.195). We need

at least four linearly independent solutions, since equation (3.188) is a fourth order differential equation. So we need to specify at least four contours C such that we obtain four linearly independent solutions. In the limit $|t| \rightarrow 0$, equation (3.193) is satisfied along the rays

$$\arg\{t\} = \frac{\pi}{3} - \frac{\beta}{3} + \frac{2n\pi}{3} \text{ with } n \text{ an integer.} \quad (3.196)$$

This gives three rays emanating from the origin. Their direction depends on the parameter β , which has its value restricted to the range $-\pi < \beta < \pi$. We give the argument of these rays modulo 2π , as follows:

$$\arg\{t\} = -\frac{\beta}{3} - \frac{\pi}{3}, \quad (3.197)$$

$$\arg\{t\} = -\frac{\beta}{3} - \frac{\pi}{3} + \frac{2\pi}{3}, \quad (3.198)$$

$$\arg\{t\} = -\frac{\beta}{3} - \frac{\pi}{3} - \frac{2\pi}{3}, \quad (3.199)$$

We label the rays given by equations (3.197), (3.198) and (3.199), respectively, as r_3 , r_1 and r_2 . So, for integration contours C which approach the origin along one of these three rays, the contour end point condition (3.193) is satisfied. We can define three integration contours by connecting two consecutive rays (we do not encircle the origin of the t complex plane) by a loop contour. Each of these contours gives a linearly independent solution of equation (3.188). We label each of these three contour as A_1 , A_2 and A_3 . They are specified as follows:

- Contour A_1 approaches the origin along the rays r_1 and r_3 . This integration contour do not cross the ray r_2 .
- Contour A_2 approaches the origin along the rays r_1 and r_2 . This integration contour do not cross the ray r_3 .
- Contour A_3 approaches the origin along the rays r_2 and r_3 . This integration contour do not cross the ray r_1 .

These contours are illustrated in the figure 3-13 below.

The contour integral solutions defined by the contours A_j illustrated in figure 3-13 define three solutions, denoted as $A_j(\delta, \lambda, \varepsilon, \alpha)$ and given by the equation

$$A_j(\delta, \lambda, \varepsilon, \alpha) = \int_{A_j} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 1, 2, 3. \quad (3.200)$$

We need one more linearly independent solution. We may consider a closed contour which encircles the origin, but the contour integral solution given by this contour can be written as a linear combination of the contour integrals based on the contours A_j , so we do not obtain one more linearly independent solution. We may consider contours emanating from the rays r_j towards the infinite in the complex t plane. For the contour end point condition to be satisfied for $|t| \rightarrow \infty$, we need that

$$\cos(\arg\{t\} + \gamma) < 0, \quad (3.201)$$

where $\gamma = \arg\{z\}$. The restriction on the $\arg\{t\}$ can be rewritten as

$$\frac{\pi}{2} - \gamma + 2n\pi < \arg\{t\} < \frac{3\pi}{2} - \gamma + 2n\pi. \quad (3.202)$$

This restriction implies that the end point condition (3.189) cannot be satisfied by any ray going to infinity in the t complex plane for all possible values of $\gamma = \arg\{z\}$. Our strategy to define contour integrals solutions with one contour end point at the infinite and the other at the origin is outlined below.

- First, we define rays going towards to the infinite in the complex t plane. These rays are labeled R_1, R_2 and R_3 , and their arguments are given as follows:

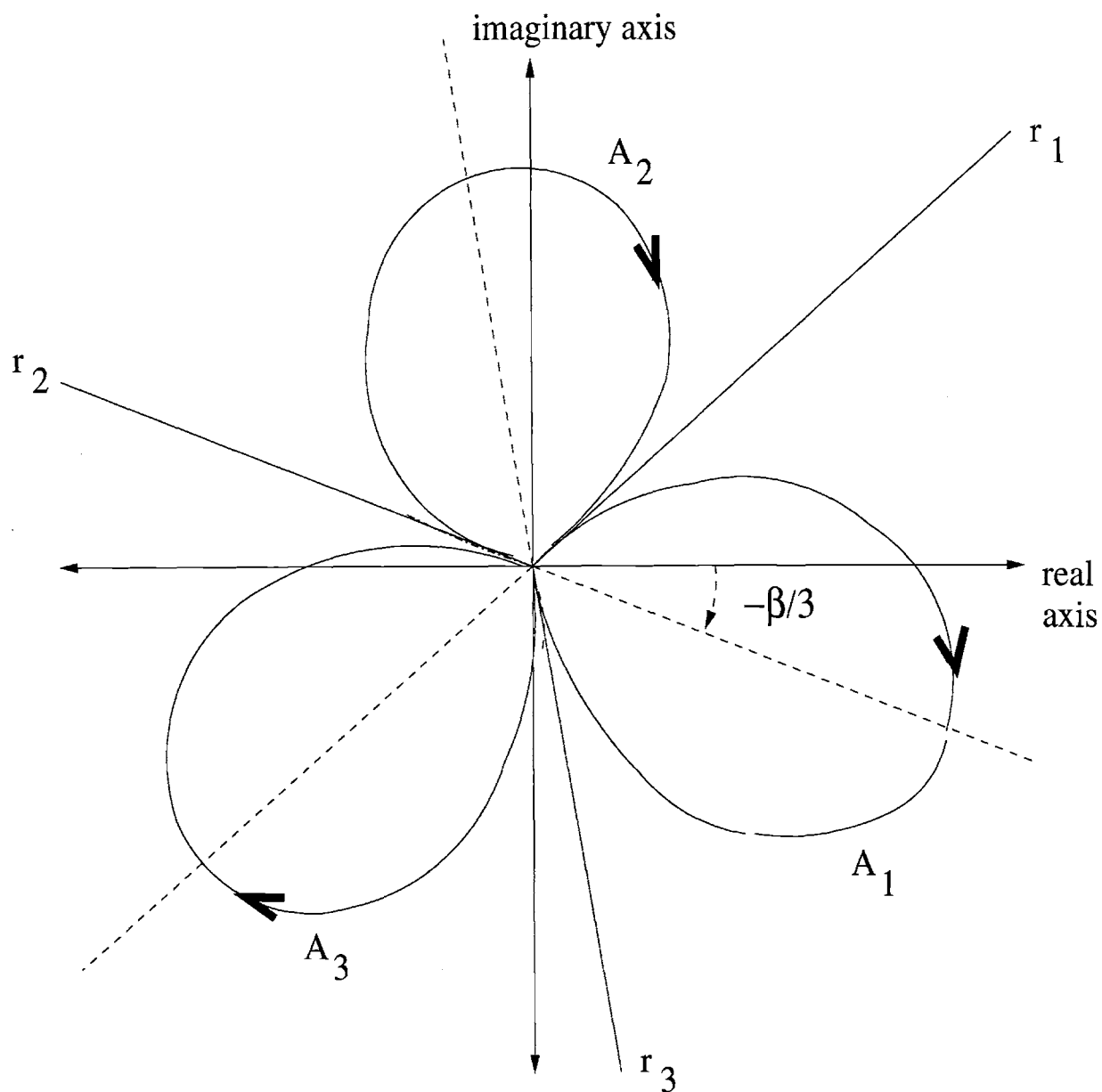


Figure 3-13: A sketch in the complex t plane of the integration contours $A_j, j = 1, \dots, 3$ and the rays r_1, r_2 and r_3 specified, respectively, by equations (3.198), (3.199) and (3.197). The arrows in the integration contours A_j indicate the way we travel along the contours as the integration is performed.

$$\arg\{R_1\} = -\frac{\beta}{3}, \quad (3.203)$$

$$\arg\{R_2\} = -\frac{\beta}{3} + \frac{2\pi}{3}, \quad (3.204)$$

$$\arg\{R_3\} = -\frac{\beta}{3} - \frac{2\pi}{3}. \quad (3.205)$$

The end point contour condition is satisfied along each ray R_j for different ranges of $\gamma = \arg\{z\}$. We define sectors in the z complex plane where the contour end point condition is satisfied as $|t| \rightarrow \infty$ along the rays R_j . To each ray R_j we associate the sector SR_j , which are given below:

$$SR_1 : \frac{\beta}{3} + \frac{\pi}{2} + 2n\pi < \gamma < \frac{3\pi}{2} + \frac{\beta}{3} + 2n\pi, \text{ with } n \text{ an integer}, \quad (3.206)$$

$$SR_2 : \frac{\beta}{3} - \frac{\pi}{6} + 2n\pi < \gamma < \frac{5\pi}{6} + \frac{\beta}{3} + 2n\pi, \text{ with } n \text{ an integer}, \quad (3.207)$$

$$SR_3 : \frac{\beta}{3} - \frac{5\pi}{6} + 2n\pi < \gamma < \frac{\pi}{6} + \frac{\beta}{3} + 2n\pi, \text{ with } n \text{ an integer}. \quad (3.208)$$

The layout of the regions SR_j is illustrated in the figure 3-14 given below.

- Second, we consider the contours A_{jl} , with $j = 4, 5, 6$ and $l = 1, 2, 3$ illustrated in figure 3-15 below, connecting the origin and the infinite of the complex t plane. The position of the shadow with respect to the line crossing the origin in the t complex plane in figure 3-15 indicates in which half plane the contour end point condition is satisfied as $|t| \rightarrow \infty$. If the shadow is above (below) the line crossing the origin, the mentioned half plane is above (below) this line. The argument of the line crossing the origin is a function of the parameter β and of the argument of the independent z variable. As we travel in the z complex plane, the half plane where the contour end point condition is satisfied for $|t| \rightarrow \infty$ rotates around the origin of the complex t plane.
- Third, we define contour integrals in terms of the contours A_{jk} , $j = 4, 5, 6$ and

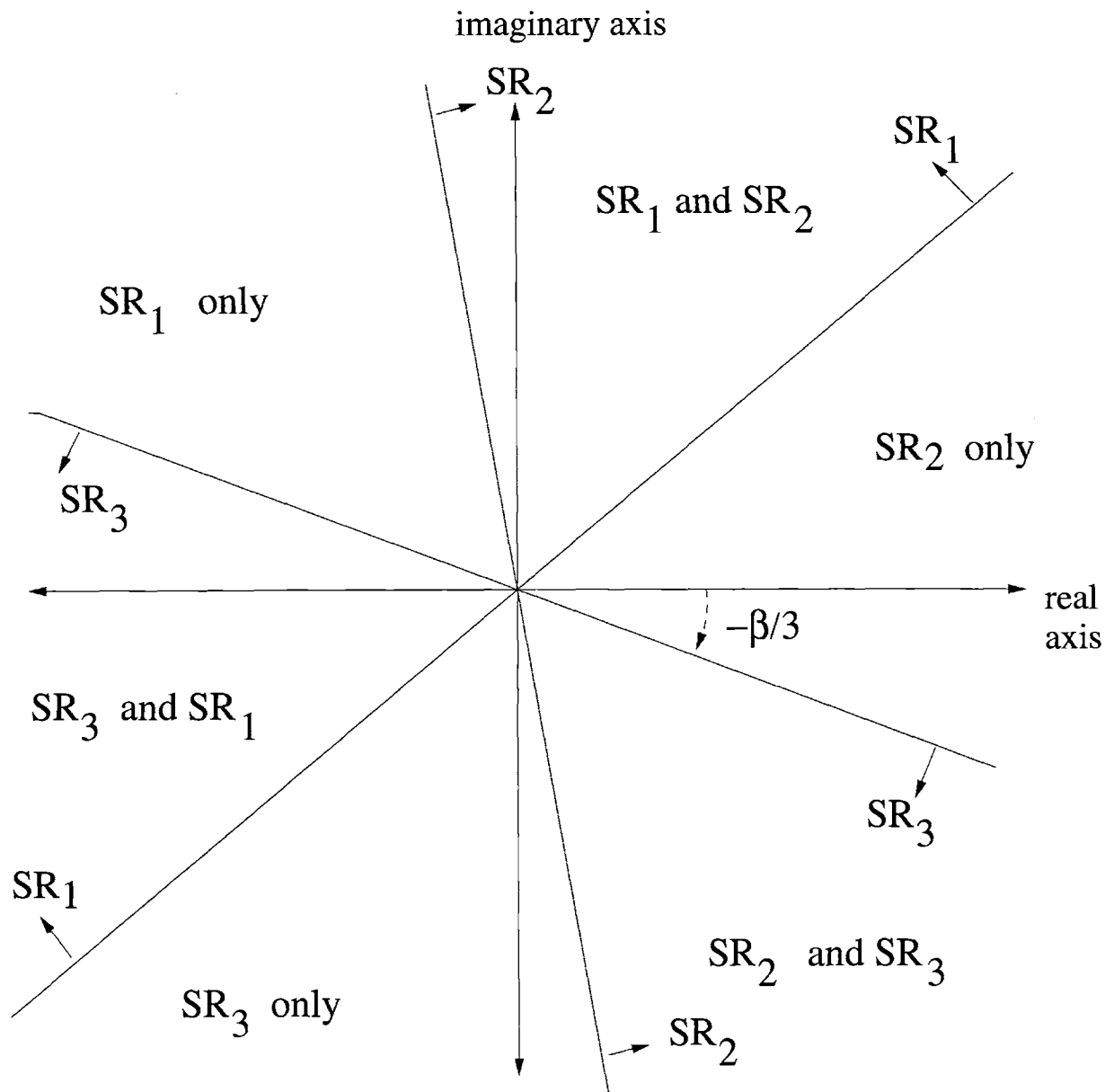


Figure 3-14: Sketch of the sectors SR_j in the complex z plane. The range of $\arg\{z\}$ for these sectors is given by equations (3.206) to (3.208) with $n = 0$.

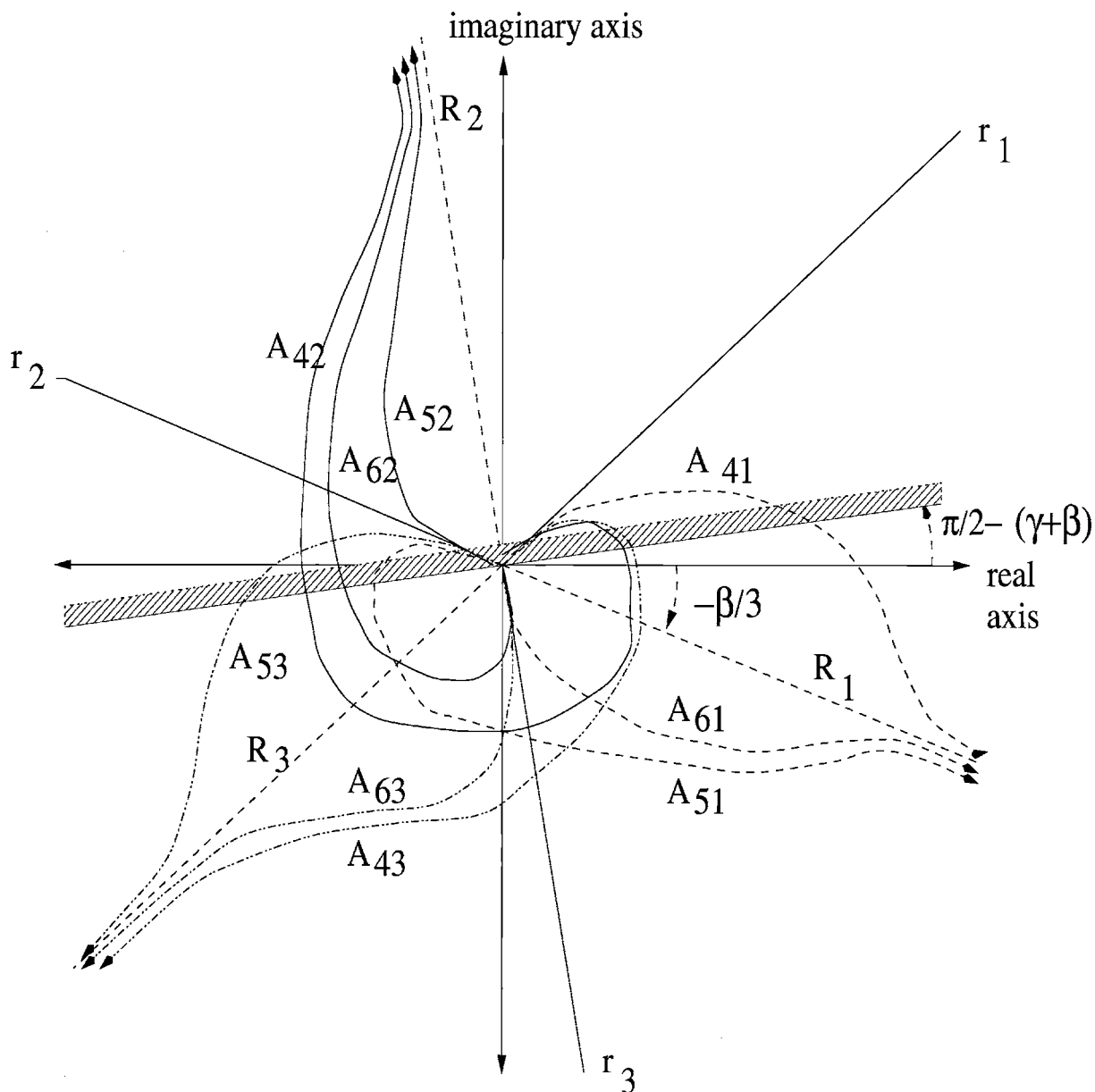


Figure 3-15: A sketch in the complex t plane of the integration contours A_{jl} , $j = 4, 5, 6$ and $l = 1, 2, 3$, of the rays r_1 , r_2 and r_3 and of the rays R_1 , R_2 and R_3 specified, respectively, by equations (3.203), (3.204) and (3.205). The arrows in the integration contours A_{jl} indicate the way we travel along the contours as the integration is performed. The half plane where the contour end point condition is satisfied as $|t| \rightarrow \infty$ is given by the position of the shadow with respect to the line crossing the origin. If the shadow is above (below) the line, the contour end point condition is satisfied for $|t| \ll 1$ with argument above (below) this line.

$k = 2, 3$, which are denoted, respectively, as $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$, $j = 4, 5, 6$. The index n is related with the definition of the sectors SR_l , $l = 1, 2, 3$. The value of n defines the range of $\gamma = \arg\{z\}$ in the sectors SR_l . For β in the range $-\pi < \beta < 0$, the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ are given by the equation

$$A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) = \int_{A_{j2}} \frac{1}{t^2} \exp\left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha)\right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_2. \quad (3.209)$$

As an example, the functions $A_{j-3,0}(\delta, \lambda, \varepsilon, \alpha)$ are defined according to the equations (3.209) for $-\pi < \beta < 0$ and $\beta/3 - \pi/6 < \gamma < 5\pi/6 + \beta/3$. For the parameter β in the range $0 < \beta < \pi$, the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ are given by the equation

$$A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) = \int_{A_{j3}} \frac{1}{t^2} \exp\left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha)\right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_3. \quad (3.210)$$

These contour integrals are well defined only for $\arg\{z\}$ given by equation (3.207) (for $-\pi < \beta < 0$) and by equation (3.208) (for $0 < \beta < \pi$) with the appropriate value of n . As we vary the value of n we define new functions. This will be clear when we discuss how to continue the contour integral solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ to a wider range of the $\arg\{z\}$. This continuation process is necessary since we are going to use the solutions of the equation (3.188) to continue the Liouville-Green functions through the turning points.

- Fourth, we continue the solutions defined by the contour integrals along the contours A_{jk} , $k = 2, 3$ to other ranges of the $\arg\{z\}$ besides the sectors SR_2 and SR_3 with the appropriate value of n . Here we discuss the continuation of the contour integrals defining the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$, $j = 4, 5, 6$ for the

parameter β in the range $-\pi < \beta < 0$. The continuation of the contour integral solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ for $0 < \beta < \pi$ is postponed to the next item.

The sectors SR_j overlap, and on these overlapping sectors the contour end point condition is satisfied along two rays R_j and R_l as $|t| \rightarrow \infty$. So, on these overlapping sectors we can define contour integral solutions of equation (3.188) using the rays R_j and R_l . We first continue these contour integral solutions into the anti-clockwise sense. To continue the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ to the sector SR_1 , we deform the integration contour. For the overlapping region of the sectors SR_2 and SR_1 in the complex z plane, we apply Cauchy's integral theorem to the closed contour in the complex t plane defined by the union of the contours A_{j2}, C_{21} and A_{j1} . This contour encircles the origin of the complex t plane (figure 3-16 illustrates the case for $j = 4$), which is an essential singularity of the integrand, as we can see by looking at the equation (3.194). So, this contour integral is equal to the residue of the singularity of the integrand at the origin of the complex t plane. We avoid dealing with it. We consider a modified closed contour. We add the contours A_1, A_2 and A_3 illustrated in the figure 3-13 to the previous contour. We end up with a closed contour that does not encircle the origin and does not contain any singularity of the integrand. Since the integral along the contour C_{21} vanishes we obtain that

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j1}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&\quad + \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_1.
\end{aligned}
\tag{3.211}$$

When we continued the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ from the sector SR_2 to the sector SR_1 , we moved in the complex z plane in the anti-clockwise sense. To continue the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ from the sector SR_1 to the sector SR_3 , we keep moving in the complex z plane in the anti clockwise sense. The so-

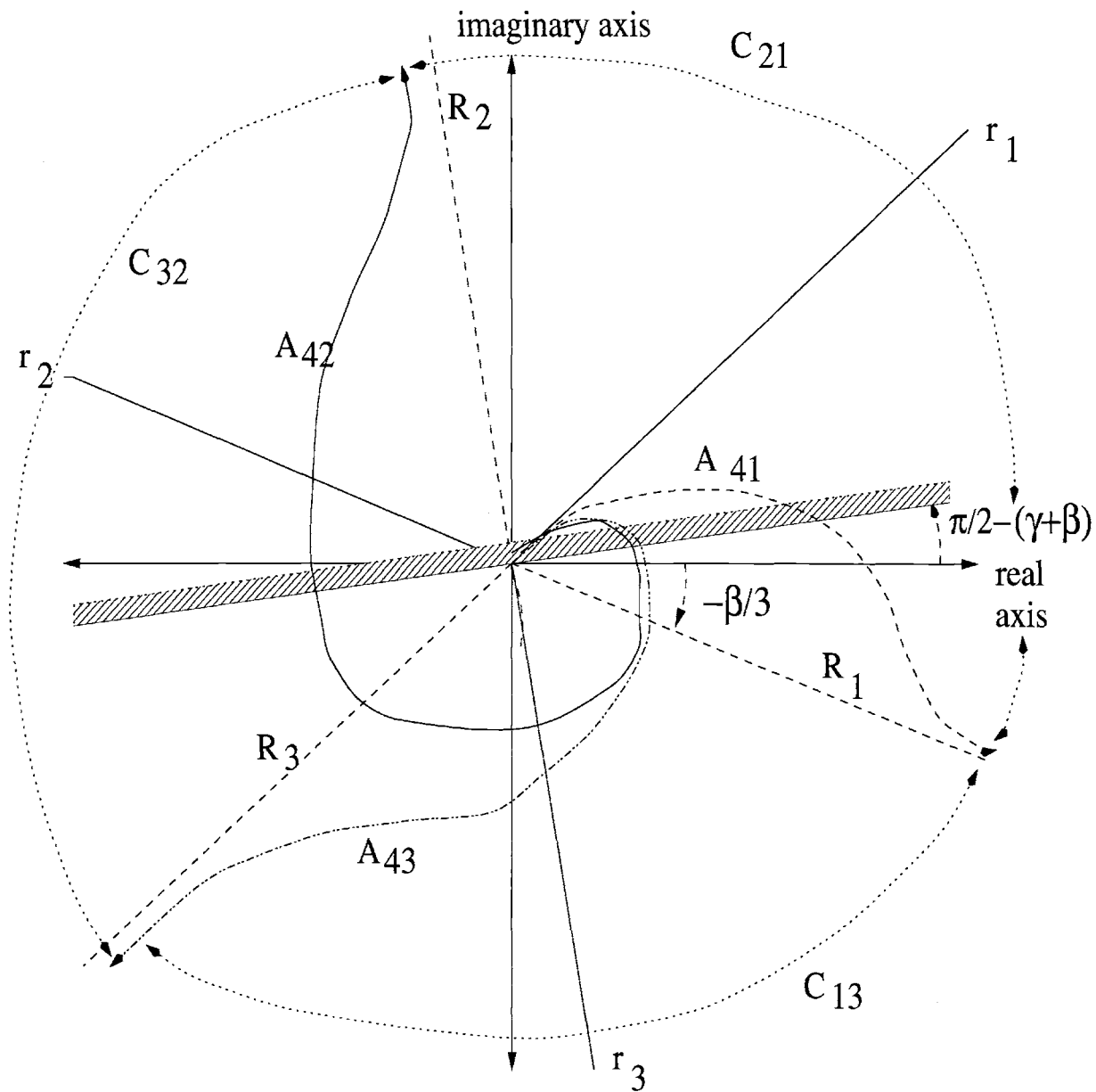


Figure 3-16: A sketch in the complex t plane of the closed integration contours used to continue the solution $A_{1,n}$, defined in the sector SR_2 (SR_3) for $-\pi < \beta < 0$ ($0 < \beta < \pi$) by the integration contour A_{422} (A_{43}) to the sectors SR_3 (SR_2) and SR_1 .

lutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ in the sector SR_1 are given as contour integrals along the contours A_{j1} , so to continue these solutions we need to deform the contours A_{j1} in terms of the contours A_{j3} . In the sector where SR_1 and SR_3 overlap, we define a closed contour in the complex t plane given by the union of the contours A_{j1} , A_{j3} and C_{13} (see figure 3-16 for the case $j = 4$). Since this closed contour does not contain any singularity of the integrand, the integral over this closed contour is zero according to Cauchy's integral theorem. The integral over the contour C_{13} vanishes, so we end up with

$$\left(\int_{A_{j1}} - \int_{A_{j3}} \right) \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt = 0,$$

which implies that

$$\begin{aligned} A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j3}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\ &+ \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_3, \end{aligned} \quad (3.212)$$

where the argument of z is given by equation (3.208) with $n + 1$ instead of just n .

Next, we continue from the sector SR_3 to the sector SR_2 . We consider the closed contour in the complex t plane defined by the union of the contours A_{j3} , C_{32} and A_{j2} (the case for $j = 4$ is illustrated in figure 3-16). There are no singularities inside this closed contour in the complex t plane and the contour integral along the contour C_{32} vanishes, so the final results is

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j2}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&+ \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_2.
\end{aligned}
\tag{3.213}$$

Notice that the argument of z in the region SR_2 now differs from the previous range. To obtain the argument of z now in the region SR_2 , we need to add 1 to the value of n used to define the solution $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$. The expression of the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ is also different, as we can see by comparing the equations (3.213) and (3.210). This suggests that the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ are multivalued functions of the complex z variable. To make these functions well defined we need to define a branch cut connecting the origin to the infinite in the complex z plane. We postpone the branch cut definition to the next section, where we give the asymptotic expansions of the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ for $|\lambda| \ll 1$.

We can keep continuing the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ along the complex z plane in the anti-clockwise sense. If we continue from the sector SR_2 to the sector SR_1 , we end up encircling the origin one more time. As a result we obtain

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j1}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&+ \int_{2(A_1 \cup A_2 \cup A_3)} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_1,
\end{aligned}
\tag{3.214}$$

where the argument of z is now given by equation (3.206) with $n + 1$ instead of n . Every time we continue the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ from the region SR_2 to the region SR_1 of the complex z plane, the integration contour in the complex t plane end up encircling the origin of the complex t plane one more

time. As a result we can write

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j1}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&+ \int_{k(A_1 \cup A_2 \cup A_3)} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_l,
\end{aligned}
\tag{3.215}$$

where k is the number of times the origin of the complex t plane is encircled as we travel along the complex z plane. The argument of z for $z \in SR_2$ is given by equation (3.207) with $n + k$ instead of n . The argument of z for $z \in SR_1$ is given by equation (3.206) with $n + k - 1$ instead of n , and the argument of z for $z \in SR_3$ is given by equation (3.208) with $n + k$ instead of n .

We can also continue the contour integral solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ along the complex z plane in the clockwise sense. We continue from the sector SR_2 to the sector SR_3 . In the region where the sectors SR_2 and SR_3 overlap, we consider the closed contour in the complex t plane given by the union of the contours A_{j3}, C_{23} and A_{j2} . This contour do not encircle any singularity of the integrand, so according to Cauchy's theorem we have that

$$\left(\int_{A_{j2}} - \int_{A_{j3}} \right) \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt = 0,$$

which implies that

$$A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) = \int_{A_{j3}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_3.
\tag{3.216}$$

If we keep moving in the clockwise sense in the complex z plane, we need to

continue the contour integral solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ from the sector SR_3 to the sector SR_1 . In the region where the sectors SR_3 and SR_1 overlap, we consider the closed contour in the complex t plane given by the union of the contours A_{j3}, C_{31} and A_{j1} . This contour do not encircle any singularity of the integrand and the integral around C_{31} vanishes, so the final result is

$$A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) = \int_{A_{j3}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_1, \quad (3.217)$$

The argument of z in the region SR_1 is now given by the equation (3.206) with $n - 1$ instead of n . Next we continue from the region SR_1 to the region SR_2 in the clockwise sense. In the overlapping between the regions SR_1 and SR_2 , we consider a closed integration contour in the complex t plane given by the union of the contours A_{j1}, C_{21} and A_{j2} . This contour encircles the origin of the complex t plane. To avoid dealing with the integrand singularity, we add the contour A_1, A_2 and A_3 illustrated in figure 3-13.

$$A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) = \int_{A_{j2}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\ - \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_2. \quad (3.218)$$

The argument of z in SR_2 is given by equation (3.207), but with $n - 1$ instead of n . So, the argument of z in the region SR_2 changed, and the expression for the solution $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ also changed, as we can see by comparing equations (3.218) and (3.209). We can keep continuing the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ along the complex z plane in the clockwise sense. If we continue from the sector SR_2 to the sector SR_3 , and we end up with

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j3}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&\quad - \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_3.
\end{aligned}
\tag{3.219}$$

The argument of z in the region SR_3 is now given by equation (3.208), but with $n - 1$ instead of n . If we continue from the region SR_3 to the region SR_1 , we obtain the same expression for the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ as the one given by equation (3.219), but with the first integral along the contour A_{j1} instead of the contour A_{j3} and with the argument of z in region SR_1 given by equation (3.206) but with $n - 2$ instead of n . If we continue from region SR_1 to the region SR_2 , the integration contour in the complex t plane end up encircling the origin of the complex t plane one more time. As a result we obtain

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j2}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&\quad - \int_{2(A_1 \cup A_2 \cup A_3)} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_2,
\end{aligned}
\tag{3.220}$$

where the argument of z in SR_2 is given by equation (3.207) with $n - 2$ instead of n . Every time we continue the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ from the region SR_1 to the region SR_2 (clockwise sense), the integration contour in the complex t plane encircles the origin of the complex t plane one more time. As a result we can write

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{ji}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&\quad - \int_{k(A_1 \cup A_2 \cup A_3)} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_l.
\end{aligned}
\tag{3.221}$$

The argument of z for $z \in SR_2$ is given by equation (3.207) with $n - k$ instead of n . The argument of z for $z \in SR_1$ is given by equation (3.206) with $n - k - 1$ instead of n , and the argument of z for $z \in SR_3$ is given by equation (3.208) with $n - k$ instead of n .

- Fifth, we continue the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$, $j = 4, 5, 6$ defined by contour integrals along the contours A_{j3} to other ranges of the $\arg z$. Here we discuss the continuation of the contour integrals defining the function $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ for the parameter β in the range $0 < \beta < \pi$.

We continue from the sector SR_3 to the sector SR_2 in the anti-clockwise sense. In the overlapping region between these two sectors we deform the integration contour in the complex t plane in the same way we did in the previous item. We consider a closed contour given as the union of the contours A_{j3} , A_{j2} and C_{32} . This contour do not encircle any singularity of the integrand and the integral along C_{32} vanishes, so the final result is that we can write the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ given by equation (3.210) in the form

$$A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) = \int_{A_{j2}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_2,
\tag{3.222}$$

and the argument of z is given by equation (3.207).

Next, we continue from the sector SR_2 to the sector SR_1 in the anti-clockwise sense. In the overlapping region between these two sectors we deform the in-

tegration contour. The deformed contour given as the union of the contours A_{j2} , A_{j1} and C_{21} encloses the origin. So, we add the contours A_1 , A_2 and A_3 to obtain a closed contour that does not encircle the origin. We apply Cauchy's integral theorem to the resulting contour integral and we end up with

$$\begin{aligned} A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j1}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\ &+ \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_1. \end{aligned} \quad (3.223)$$

Next, we continue from the sector SR_1 to the sector SR_3 (anti-clockwise sense). We proceed in the same way as before. In the overlapping region between sectors SR_1 and SR_3 we define a closed contour as the union of the contours A_{j1} , A_{j3} and C_{13} . After applying Cauchy's contour integral theorem, we end up with

$$\begin{aligned} A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j3}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\ &+ \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_3, \end{aligned} \quad (3.224)$$

but now the argument of z is given by equation (3.208) with $n + 1$ instead of n . Notice that the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ in region SR_3 is now different, as we can see by comparing equations (3.210) and (3.224).

We continue the contour integrals one more time. We continue from the sector SR_3 to the sector SR_2 . In the overlapping region of sectors SR_3 and SR_2 we define a closed contour given as the union of the contours A_{j3} , A_{j2} and C_{32} . After applying Cauchy's contour integral theorem, we obtain

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j2}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&+ \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_2,
\end{aligned}
\tag{3.225}$$

but now with the argument of z given by the equation (3.207) with $n + 1$ instead of n . Next, we continue from the sector SR_2 to the sector SR_1 . In the overlapping regions between these two sectors, we define a closed contour in the complex t plane given as the union of the contours A_{j2}, A_{j1} and C_{21} . This contour encloses the origin of the complex t plane one more time. As a result we obtain

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j1}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&+ \int_{2(A_1 \cup A_2 \cup A_3)} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_1,
\end{aligned}
\tag{3.226}$$

where the argument of z is now given by equation (3.207) with $n + 2$ instead of n . every time we continue the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ from the region SR_2 to the region SR_1 of the complex z plane, the integration contour in the complex t plane end up encircling the origin of the complex t plane one more time. As a result, we can write

$$\begin{aligned}
A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{jl}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\
&+ \int_{k(A_1 \cup A_2 \cup A_3)} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_l.
\end{aligned}
\tag{3.227}$$

The argument of z for $z \in SR_1$ is given by the equation (3.206) with $n + k - 1$ instead of n . The argument of z for $z \in SR_2$ is given by the equation (3.207) with $n + k$ instead of n , and the argument of z for $z \in SR_3$ is given by the equation (3.208) with $n + k$ instead of n .

Next, we can continue the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ in the clockwise sense in the complex z plane. We continue from the sector SR_3 to the sector SR_1 . As before, in the overlapping region between sectors SR_3 and SR_1 , we define a closed integration contour given as the union of contours A_{j3}, A_{j1} and C_{31} (see figure 3-16). Then, we apply Cauchy's contour integral theorem to the resulting contour integral to obtain that

$$A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) = \int_{A_{j1}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_1, \quad (3.228)$$

but with the argument of z given by equation (3.206) with $n - 1$ instead on n . Next, we continue the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ to the sector SR_2 . In the overlapping region between sectors SR_1 and SR_2 , we define a closed integration contour as the union of the contours $A_{j1}, A_{j2}, C_{21}, A_1, A_2$ and A_3 . We use contours A_1, A_2 and A_3 to avoid encircling the origin of the complex t plane. We apply Cauchy's integral theorem to the resulting contour integral to obtain

$$\begin{aligned} A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j2}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\ &\quad - \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_2, \end{aligned} \quad (3.229)$$

but with the argument of z given by equation (3.207) with $n - 1$ instead of n .

Next, we continue the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ from the sector SR_2 to the

sector SR_3 (clockwise sense). In the overlapping region between these two sectors we consider a closed contour given as the union of the contours A_{j1}, A_{j3} and C_{13} . After applying Cauchy's contour integral theorem to the resulting contour integral, we obtain

$$\begin{aligned} A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j3}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\ &\quad - \int_{A_1 \cup A_2 \cup A_3} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_3, \end{aligned} \quad (3.230)$$

but with the argument of z given by equation (3.208) with $n - 1$ instead of n . Notice that the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ are given now by a different contour integral in the sector SR_3 , as we can see from the equations (3.230) and (3.210). We can keep continuing the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ along the complex z plane. If we continue in the clockwise sense from the sector SR_3 to the sector SR_1 , we obtain an expression for the functions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$ similar to the one given by equation (3.230), but with the contour A_{j1} instead of the contour A_{j3} , and the argument of z is given by the equation (3.206), but with $n - 2$ instead of n . To continue from the sector SR_1 to the sector SR_2 , we deform the integration contour in the complex t plane and we end up encircling the origin of the complex t plane one more time. So we obtain

$$\begin{aligned} A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) &= \int_{A_{j3}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \\ &\quad - \int_{2(A_1 \cup A_2 \cup A_3)} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_2, \end{aligned} \quad (3.231)$$

and the argument of z is given by the equation (3.207), but with $n - 2$ instead of n . Every time we continue from the sector SR_1 to the sector SR_2 in the

clockwise sense the integration contour in the complex t plane end up encircling the origin of the complex t plane. As a result we can write

$$A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha) = \int_{A_{ji}} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt - \int_{k(A_1 \cup A_2 \cup A_3)} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 4, 5, 6 \text{ and } z \in SR_l. \quad (3.232)$$

The argument of z for $z \in SR_1$ is given by equation (3.206), but with $n - k - 1$ instead of n . The argument of z for $z \in SR_2$ is given by equation (3.207), but with $n - k$ instead of n , and the argument of z for $z \in SR_3$ is given by equation (3.208), but with $n - k$ instead of n .

For our intended application, the connection of the Liouville-Green functions across a combination of a turning point and pole and related Stokes-lines, we need solutions of the equation (3.188) defined for ranges of the argument of the independent z variable of at least 2π . Once the necessary range of the argument of the z variable is given, we choose the appropriate contour integral solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$, $j = 4, \dots, 6$ defined above in the sector SR_2 ($-\pi < \beta < 0$) or in the sector SR_3 ($0 < \beta < \pi$). Then, we analytically continue the solutions $A_{j-3,n}(\delta, \lambda, \varepsilon, \alpha)$, $j = 4, \dots, 6$ to the overlap of the given range of the argument of the z variable with the appropriate sectors SR_1 , SR_2 and SR_3 by using the deformation contour procedures outlined above.

3.2.4.1 Asymptotic Expansion.

Here we obtain the asymptotic expansion of the contour integral solutions $A_l(\delta, \lambda, \varepsilon, \alpha)$ and $A_{l,n}(\delta, \lambda, \varepsilon, \alpha)$, $l = 1, 2, 3$ of the approximate equation (3.188). We use the steepest descent method to obtain the asymptotic expansion of the contour integrals defining the solutions $A_l(\delta, \lambda, \varepsilon, \alpha)$ and $A_{l,n}(\delta, \lambda, \varepsilon, \alpha)$ when $\lambda \ll 1$. The application of the steepest descent method to the contour integrals $A_l(\delta, \lambda, \varepsilon, \alpha)$ and $A_{l,n}(\delta, \lambda, \varepsilon, \alpha)$ is discussed in detail in the Appendix I. Here give the results of the application of the

steepest descent method to the contour integrals $A_l(\delta, \lambda, \varepsilon, \alpha)$ and $A_{l,n}(\delta, \lambda, \varepsilon, \alpha)$. The first step in applying the steepest descent method is to deform the integration contours in terms of the steepest descent paths, along which $\Im\{f(t, \delta, \varepsilon, \alpha)\} = 0$. Therefore, the original contour integrals are written as a linear combination of contour integrals along the steepest descent paths, denoted as $C_j, j = 1, \dots, 4$. Each of the paths C_j in the complex t plane cross a saddle point of the function $f(t, \delta, \varepsilon, \alpha)$. The saddle points t_j are given by the equations

$$t_1 = \frac{1}{\varepsilon} \left(\frac{\rho''}{\rho} \right)^{1/2} \exp(-i\frac{\pi}{2} - \frac{i}{2}\theta + \frac{i}{2}\theta''), \quad (3.233)$$

$$t_2 = -t_1, \quad (3.234)$$

$$t_3 = \frac{1}{\varepsilon} \left(\frac{\rho'''}{\rho} \right)^{1/2} \exp(-\frac{i}{2}\theta + \frac{i}{2}\theta'''), \quad (3.235)$$

$$t_4 = -t_3, \quad (3.236)$$

where the polar coordinate systems $(\rho, \theta), (\rho', \theta'), (\rho'', \theta'')$ and (ρ''', θ''') are given in terms of δ according to

$$\delta = \rho \exp(i\theta), \quad (3.237)$$

$$\left(\frac{\alpha}{2} \right)^2 + \delta = \rho' \exp(i\theta'), \quad (3.238)$$

$$-\sqrt{\rho'} \exp(\frac{i}{2}\theta') + \frac{\alpha}{2} = \rho'' \exp(i\theta''), \quad (3.239)$$

$$\sqrt{\rho'} \exp(\frac{i}{2}\theta') + \frac{\alpha}{2} = \rho''' \exp(i\theta'''). \quad (3.240)$$

The existence of four saddle points implies that we have four steepest descent paths C_j over the complex t plane. We define four contour integrals over the steepest descent paths C_j , which are denoted as

$$H_j(\delta, \lambda, \varepsilon, \alpha) = \int_{C_j} \frac{1}{t^2} \exp \left(\frac{1}{\lambda} f(t, \delta, \varepsilon, \alpha) \right) dt \text{ with } j = 1, \dots, 4 \quad (3.241)$$

So the contour integrals $A_l(\delta, \lambda, \varepsilon, \alpha)$ and $A_{l,n}(\delta, \lambda, \varepsilon, \alpha)$ are expressed as linear combinations of the contour integrals $H_j(\delta, \lambda, \varepsilon, \alpha)$ over the steepest descent paths. Therefore, to obtain the asymptotic expansion of the contour integrals $A_l(\delta, \lambda, \varepsilon, \alpha)$ and $A_{l,n}(\delta, \lambda, \varepsilon, \alpha)$, we need first to obtain the deformation of their integration contour in terms of the steepest descent paths (their linear combination in terms of the contour integrals $H_j(\delta, \lambda, \varepsilon, \alpha)$), and then to obtain the asymptotic expansion of the contour integrals $H_j(\delta, \lambda, \varepsilon, \alpha)$. The analysis outlined above is performed in the appendix I. The asymptotic expansion of the contour integrals $H_j(\delta, \lambda, \varepsilon, \alpha)$ is also given in the appendix I, and are proportional to the exponential

$$\exp\left(\frac{1}{\lambda}f(t_j, \delta, \varepsilon, \alpha)\right),$$

with the functions $f(t_j, \delta, \varepsilon, \alpha)$ given by the equations

$$\begin{aligned} f(t_1, \delta, \varepsilon, \alpha) = & \frac{4}{3}\varepsilon^3 \exp(-i\beta) \left\{ \left(\frac{\rho}{\rho''}\right)^{3/2} \exp(-i\frac{\pi}{2} + i\frac{3}{2}(\theta - \theta'')) \right. \\ & \left. + \frac{3}{2}\alpha \left(\frac{\rho}{\rho''}\right)^{1/2} \exp(i\frac{\pi}{2} + i\frac{1}{2}(\theta - \theta'')) \right\}, \end{aligned} \quad (3.242)$$

$$f(t_2, \delta, \varepsilon, \alpha) = -f(t_1, \delta, \varepsilon, \alpha), \quad (3.243)$$

$$\begin{aligned} f(t_3, \delta, \varepsilon, \alpha) = & \frac{4}{3}\varepsilon^3 \exp(-i\beta) \left\{ \left(\frac{\rho}{\rho'''}\right)^{3/2} \exp(i\frac{3}{2}(\theta - \theta''')) \right. \\ & \left. + \frac{3}{2}\alpha \left(\frac{\rho}{\rho'''}\right)^{1/2} \exp(i\frac{1}{2}(\theta - \theta''')) \right\}, \end{aligned} \quad (3.244)$$

$$f(t_4, \delta, \varepsilon, \alpha) = -f(t_3, \delta, \varepsilon, \alpha). \quad (3.245)$$

For a given contour integral solution $A_l(\delta, \lambda, \varepsilon, \alpha)$ or $A_{l,n}(\delta, \lambda, \varepsilon, \alpha)$, its linear combination in terms of the contour integrals $H_j(\delta, \lambda, \varepsilon, \alpha)$ along the steepest descent paths changes as we cross the Stokes lines defined by the differences

$$\Im\{f(t_j, \delta, \varepsilon, \alpha) - f(t_k, \delta, \varepsilon, \alpha)\} = 0 \text{ for } j < k = 1, \dots, 4. \quad (3.246)$$

For the asymptotic analysis of the contour integral solutions, it really matters only the Stokes lines generated by the cases $(j, k) = (1, 3), (1, 4)$. These lines are labeled DS_{jk} . For the asymptotic expansions to be properly constructed we need to define Riemann sheets labeled RS_l , since the functions $f(t_j, \delta, \varepsilon, \alpha)$ are multivalued functions over the δ complex plane. We define four Riemann sheets RS_l according to the equations

$$RS_1 - \text{1st Riemann Sheet:} \left\{ \begin{array}{ll} \frac{-3\pi}{2} < \underbrace{\theta - \beta}_{\gamma} < \frac{\pi}{2} & \frac{-3\pi}{2} < \theta' - \beta < \frac{\pi}{2} \quad \text{for } -\pi < \beta < 0 \\ \frac{-\pi}{2} < \underbrace{\theta - \beta}_{\gamma} < \frac{3\pi}{2} & \frac{-\pi}{2} < \theta' - \beta < \frac{3\pi}{2} \quad \text{for } 0 < \beta < \pi \end{array} \right. \quad (3.247)$$

$$RS_2 - \text{2nd Riemann Sheet:} \left\{ \begin{array}{ll} \frac{\pi}{2} < \underbrace{\theta - \beta}_{\gamma} < \frac{5\pi}{2} & \frac{\pi}{2} < \theta' - \beta < \frac{5\pi}{2} \quad \text{for } -\pi < \beta < 0 \\ \frac{3\pi}{2} < \underbrace{\theta - \beta}_{\gamma} < \frac{7\pi}{2} & \frac{3\pi}{2} < \theta' - \beta < \frac{7\pi}{2} \quad \text{for } 0 < \beta < \pi \end{array} \right. \quad (3.248)$$

$$RS_3 - \text{3rd Riemann Sheet:} \left\{ \begin{array}{ll} \frac{5\pi}{2} < \underbrace{\theta - \beta}_{\gamma} < \frac{9\pi}{2} & \frac{5\pi}{2} < \theta' - \beta < \frac{9\pi}{2} \quad \text{for } -\pi < \beta < 0 \\ \frac{7\pi}{2} < \underbrace{\theta - \beta}_{\gamma} < \frac{11\pi}{2} & \frac{7\pi}{2} < \theta' - \beta < \frac{11\pi}{2} \quad \text{for } 0 < \beta < \pi \end{array} \right. \quad (3.249)$$

$$RS_4 - \text{4th Riemann Sheet:} \left\{ \begin{array}{ll} \frac{9\pi}{2} < \underbrace{\theta - \beta}_{\gamma} < \frac{13\pi}{2} & \frac{9\pi}{2} < \theta' - \beta < \frac{13\pi}{2} \quad \text{for } -\pi < \beta < 0 \\ \frac{11\pi}{2} < \underbrace{\theta - \beta}_{\gamma} < \frac{15\pi}{2} & \frac{11\pi}{2} < \theta' - \beta < \frac{15\pi}{2} \quad \text{for } 0 < \beta < \pi \end{array} \right. \quad (3.250)$$

These four Riemann sheets define a Riemann surface. Over this Riemann surface, the variables θ'' and θ''' assume, respectively, values in the range $0 < \theta'' < 4\pi$ and $-\pi < \theta''' < 3\pi$. We give now which functions $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ are going to be considered (which are the values of n). Below we give two tables (the first for $-\pi < \beta < 0$, which

is table 3.6, and the second for $0 < \beta < \pi$, which is table 3.7) stating which functions $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ we are going to consider over each Riemann sheet RS_l . We also give the integration contours used to define the considered functions $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$.

The Stokes lines over the four Riemann sheets are given in figures 3-17, 3-18, 3-19 and 3-20, respectively, for values of the parameter $\beta = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$. The Riemann surface over the complex δ plane is divided in regions labeled $S_l, l = 0, \dots, 11$. At each region S_l , we have a different asymptotic expansion of the contour integral solutions.

Next, we provide the representation of the contour integrals $A_j(\delta, \lambda, \varepsilon, \alpha)$ and $A_{j,n}(\delta, \lambda, \varepsilon, \alpha), j = 1, 2, 3$ (see tables 3.6 and 3.7) in terms of the steepest descent paths integrals $H_l(\delta, \lambda, \varepsilon, \alpha), l = 1, \dots, 4$ over the Riemann surface defined above. The asymptotic expansion of the contour integrals $A_j(\delta, \lambda, \varepsilon, \alpha)$ and $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ is given in terms of the asymptotic expansion of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$. The representation of the functions $A_j(\delta, \lambda, \varepsilon, \alpha)$ and $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ changes as we move in the complex z (or δ) plane from one region S_n to S_{n+1} or to S_{n-1} (see figures 3-16 to 3-20). Since we are going to use the solutions $A_j(\delta, \lambda, \varepsilon, \alpha)$ and $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ to continue the Liouville-Green functions specified by the boundary conditions across the Stokes lines in the complex z plane, we give the representation of these functions only in the regions S_n which are used in the continuation process. For $-\pi < \beta < 0$ we consider the regions S_0, S_3, S_6 and S_9 , and for $0 < \beta < \pi$ we consider the regions S_2, S_5, S_8 and S_{11} .

First we consider $-\pi < \beta < 0$.

- For region S_0 we have:

Table 3.6: Functions $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)(-\pi < \beta < 0)$ to be considered over the Riemann sheets RS_l . The contour A_{123} stands for the contour $A_1 \cup A_2 \cup A_3$. The index j assume values from 4 to 6.

Riemann sheet	RS_1			
Sector	SR_2	SR_1	SR_3	SR_2
Function	Integration Contour			
$A_{j-3,-1}$	A_{j2}	$A_{j1} \cup A_{123}$	$A_{j3} \cup A_{123}$	$A_{j2} \cup A_{123}$
$A_{j-3,0}$	$A_{j2} \cup -A_{123}$	A_{j1}	A_{j3}	A_{j2}
$A_{j-3,1}$	$A_{j2} \cup -2A_{123}$	$A_{j1} \cup -A_{123}$	$A_{j3} \cup -A_{123}$	$A_{j2} \cup -A_{123}$
$A_{j-3,2}$	$A_{j2} \cup -3A_{123}$	$A_{j1} \cup -2A_{123}$	$A_{j3} \cup -2A_{123}$	$A_{j2} \cup -2A_{123}$
Riemann sheet	RS_2			
Sector	SR_2	SR_1	SR_3	SR_2
Function	Integration Contour			
$A_{j-3,0}$	A_{j2}	$A_{j1} \cup A_{123}$	$A_{j3} \cup A_{123}$	$A_{j2} \cup A_{123}$
$A_{j-3,1}$	$A_{j2} \cup -A_{123}$	A_{j1}	A_{j3}	A_{j2}
$A_{j-3,2}$	$A_{j2} \cup -2A_{123}$	$A_{j1} \cup -A_{123}$	$A_{j3} \cup -A_{123}$	$A_{j2} \cup -A_{123}$
$A_{j-3,3}$	$A_{j2} \cup -3A_{123}$	$A_{j1} \cup -2A_{123}$	$A_{j3} \cup -2A_{123}$	$A_{j2} \cup -2A_{123}$
Riemann sheet	RS_3			
Sector	SR_2	SR_1	SR_3	SR_2
Function	Integration Contour			
$A_{j-3,1}$	A_{j2}	$A_{j1} \cup A_{123}$	$A_{j3} \cup A_{123}$	$A_{j2} \cup A_{123}$
$A_{j-3,2}$	$A_{j2} \cup -A_{123}$	A_{j1}	A_{j3}	A_{j2}
$A_{j-3,3}$	$A_{j2} \cup -2A_{123}$	$A_{j1} \cup -A_{123}$	$A_{j3} \cup -A_{123}$	$A_{j2} \cup -A_{123}$
$A_{j-3,4}$	$A_{j2} \cup -3A_{123}$	$A_{j1} \cup -2A_{123}$	$A_{j3} \cup -2A_{123}$	$A_{j2} \cup -2A_{123}$
Riemann sheet	RS_4			
Sector	SR_2	SR_1	SR_3	SR_2
Function	Integration Contour			
$A_{j-3,2}$	A_{j2}	$A_{j1} \cup A_{123}$	$A_{j3} \cup A_{123}$	$A_{j2} \cup A_{123}$
$A_{j-3,3}$	$A_{j2} \cup -A_{123}$	A_{j1}	A_{j3}	A_{j2}
$A_{j-3,4}$	$A_{j2} \cup -2A_{123}$	$A_{j1} \cup -A_{123}$	$A_{j3} \cup -A_{123}$	$A_{j2} \cup -A_{123}$
$A_{j-3,5}$	$A_{j2} \cup -3A_{123}$	$A_{j1} \cup -2A_{123}$	$A_{j3} \cup -2A_{123}$	$A_{j2} \cup -2A_{123}$

Table 3.7: Functions $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ ($0 < \beta < \pi$) to be considered over the Riemann sheets RS_l . The contour A_{123} stands for the contour $A_1 \cup A_2 \cup A_3$. The index j assume values from 4 to 6.

Riemann sheet	RS_1			
Sector	SR_3	SR_2	SR_1	SR_3
Function	Integration Contour			
$A_{j-3,-1}$	$A_{j3} \cup A_{123}$	$A_{j2} \cup A_{123}$	$A_{j1} \cup 2A_{123}$	$A_{j3} \cup 2A_{123}$
$A_{j-3,0}$	A_{j3}	A_{j2}	$A_{j1} \cup A_{123}$	$A_{j3} \cup A_{123}$
$A_{j-3,1}$	$A_{j3} \cup -A_{123}$	$A_{j2} \cup -A_{123}$	A_{j1}	A_{j3}
$A_{j-3,2}$	$A_{j3} \cup -2A_{123}$	$A_{j2} \cup -2A_{123}$	$A_{j1} \cup -A_{123}$	$A_{j3} \cup -A_{123}$
Riemann sheet	RS_2			
Sector	SR_3	SR_2	SR_1	SR_3
Function	Integration Contour			
$A_{j-3,0}$	$A_{j3} \cup A_{123}$	$A_{j2} \cup A_{123}$	$A_{j1} \cup 2A_{123}$	$A_{j3} \cup 2A_{123}$
$A_{j-3,1}$	A_{j3}	A_{j2}	$A_{j1} \cup A_{123}$	$A_{j3} \cup A_{123}$
$A_{j-3,2}$	$A_{j3} \cup -A_{123}$	$A_{j2} \cup -A_{123}$	A_{j1}	A_{j3}
$A_{j-3,3}$	$A_{j3} \cup -2A_{123}$	$A_{j2} \cup -2A_{123}$	$A_{j1} \cup -A_{123}$	$A_{j3} \cup -A_{123}$
Riemann sheet	RS_3			
Sector	SR_3	SR_2	SR_1	SR_3
Function	Integration Contour			
$A_{j-3,1}$	$A_{j3} \cup A_{123}$	$A_{j2} \cup A_{123}$	$A_{j1} \cup 2A_{123}$	$A_{j3} \cup 2A_{123}$
$A_{j-3,2}$	A_{j3}	A_{j2}	$A_{j1} \cup A_{123}$	$A_{j3} \cup A_{123}$
$A_{j-3,3}$	$A_{j3} \cup -A_{123}$	$A_{j2} \cup -A_{123}$	A_{j1}	A_{j3}
$A_{j-3,4}$	$A_{j3} \cup -2A_{123}$	$A_{j2} \cup -2A_{123}$	$A_{j1} \cup -A_{123}$	$A_{j3} \cup -A_{123}$
Riemann sheet	RS_4			
Sector	SR_3	SR_2	SR_1	SR_3
Function	Integration Contour			
$A_{j-3,2}$	$A_{j3} \cup A_{123}$	$A_{j2} \cup A_{123}$	$A_{j1} \cup 2A_{123}$	$A_{j3} \cup 2A_{123}$
$A_{j-3,3}$	A_{j3}	A_{j2}	$A_{j1} \cup A_{123}$	$A_{j3} \cup A_{123}$
$A_{j-3,4}$	$A_{j3} \cup -A_{123}$	$A_{j2} \cup -A_{123}$	A_{j1}	A_{j3}
$A_{j-3,5}$	$A_{j3} \cup -2A_{123}$	$A_{j2} \cup -2A_{123}$	$A_{j1} \cup -A_{123}$	$A_{j3} \cup -A_{123}$

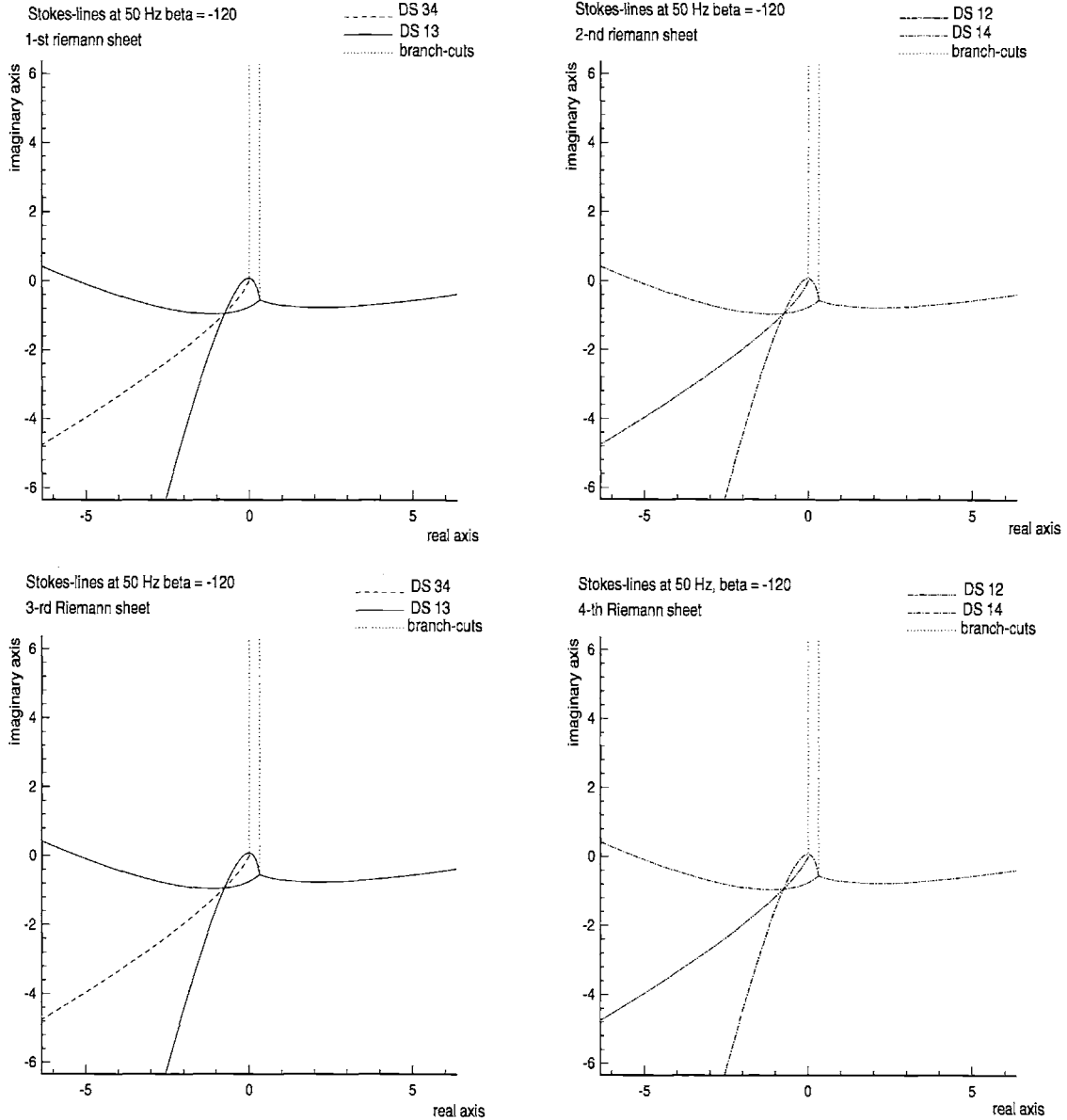


Figure 3-17: Stokes lines for the approximate equation (3.188) with $\beta = -\frac{2\pi}{3}$. These lines are defined according to equation (3.246). At the 1st (3rd) Riemann sheet (top (bottom) left figure): Region S_0 (S_6) - 1st quadrant (bounded by branch-cut and right line DS_{13}), region S_{11} (S_5) - 4th quadrant (bounded by the bottom and right going lines DS_{13}), region S_{10} (S_4) - 3rd quadrant (bounded by the bottom and left going lines DS_{13}) and region S_9 (S_3) - 2nd quadrant (bounded by the left going line DS_{13} and the branch-cut). At the 2nd (4th) Riemann sheet (top (bottom) right figure): Region S_3 (S_9) - 1st quadrant, region S_2 (S_8) - 4th quadrant, region S_1 (S_7) - 3rd quadrant and region S_0 (S_6) - 2nd quadrant. Notice that for this Riemann sheet, regions are bounded by lines DS_{14} (DS_{14}) and branch-cuts.

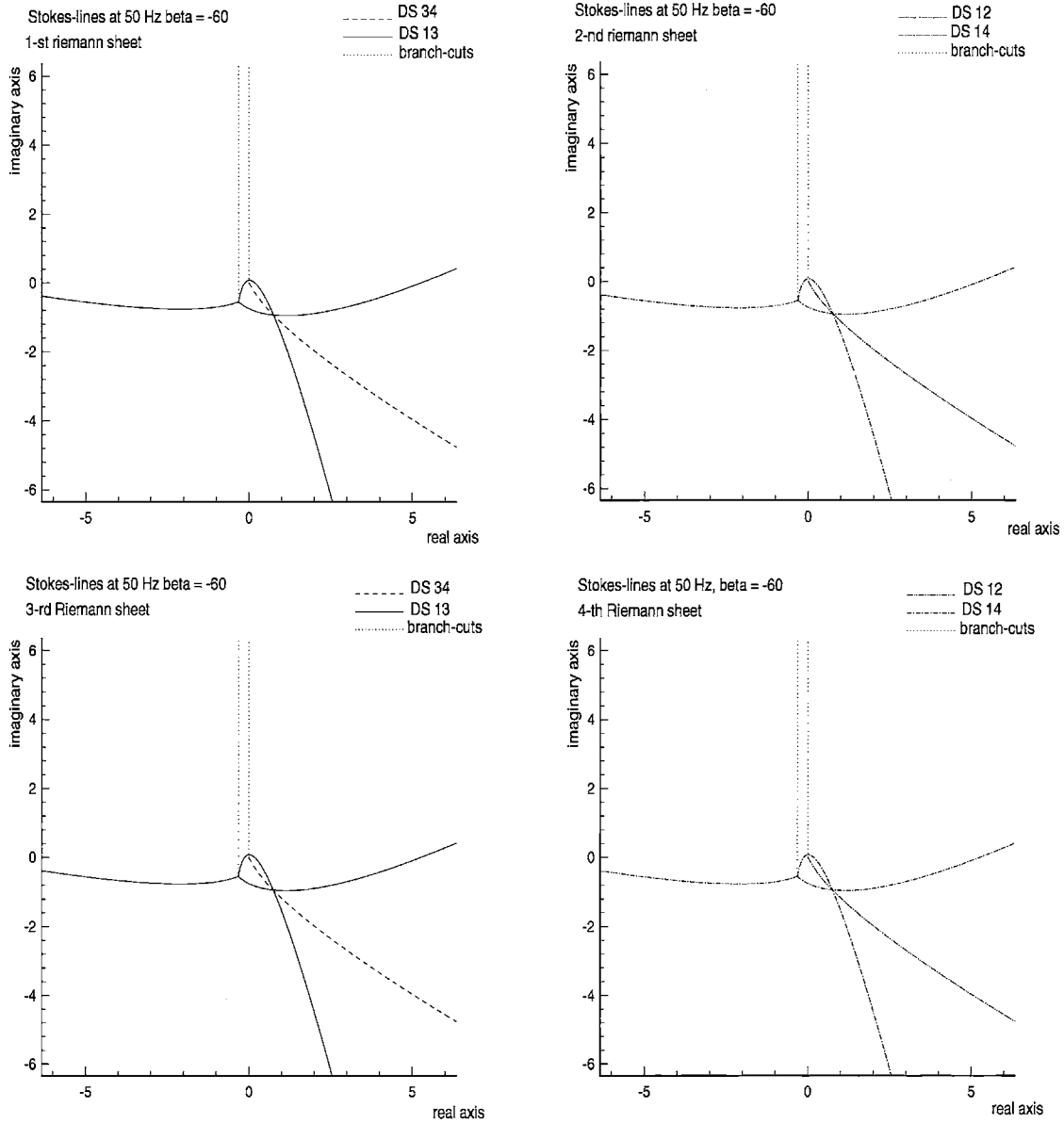


Figure 3-18: Stokes lines for the approximate equation (3.188) with $\beta = -\frac{\pi}{3}$. These lines are defined according to equation (3.246). At the 1st (3rd) Riemann sheet (top (bottom) left figure): Region S_0 (S_6) - 1st quadrant (bounded by branch-cut and right line DS_{13}), region S_{11} (S_5) - 4th quadrant (bounded by the bottom and right going lines DS_{13}), region S_{10} (S_4) - 3rd quadrant (bounded by the bottom and left going lines DS_{13}) and region S_9 (S_3) - 2nd quadrant (bounded by the left going line DS_{13} and the branch-cut). At the 2nd (4th) Riemann sheet (top (bottom) right figure): Region S_3 (S_9) - 1st quadrant, region S_2 (S_8) - 4th quadrant, region S_1 (S_7) - 3rd quadrant and region S_0 (S_6) - 2nd quadrant. Notice that for this Riemann sheet, regions are bounded by lines DS_{14} (DS_{14}) and branch-cuts.

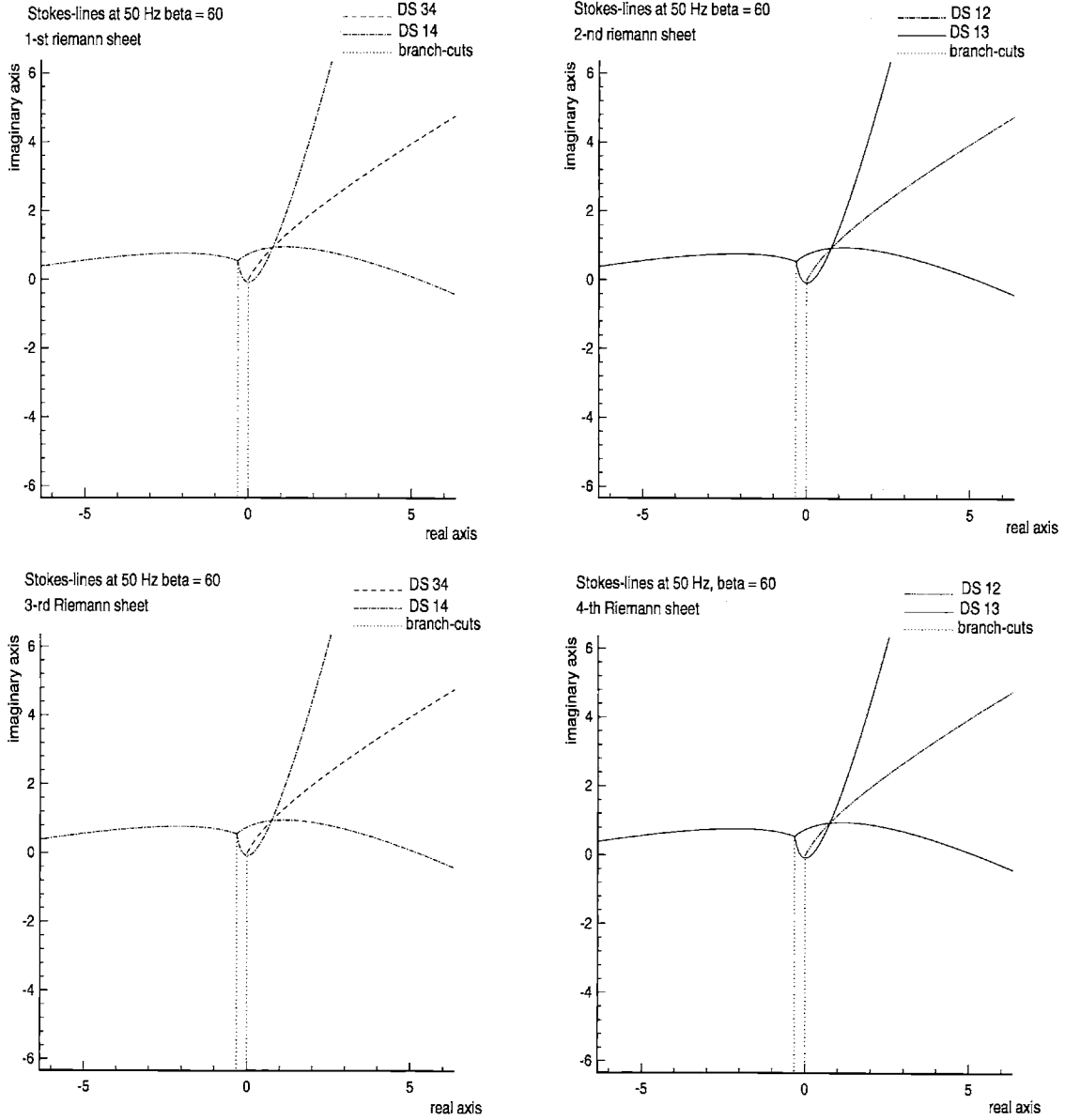


Figure 3-19: Stokes lines for the approximate equation (3.188) with $\beta = \frac{\pi}{3}$. These lines are defined according to equation (3.246). At the 1st (3rd) Riemann sheet (top (bottom) left figure): Region S_0 (S_6) - 1st quadrant (bounded by the top and right going lines DS_{14}), region S_{11} (S_5) - 4th quadrant (bounded by the branch-cut and the right going line DS_{14}), region S_2 (S_8) - 3rd quadrant (bounded by the branch-cut and the left going lines DS_{14}) and region S_1 (S_7) - 2nd quadrant (bounded by the left and top going lines DS_{14}). At the 2nd (4th) Riemann sheet (top (bottom) right figure): Region S_3 (S_9) - 1st quadrant, region S_2 (S_8) - 4th quadrant, region S_5 (S_{11}) - 3rd quadrant and region S_4 (S_{10}) - 2nd quadrant. Notice that for this Riemann sheet, regions are bounded by lines DS_{13} (DS_{13}) and branch-cuts.

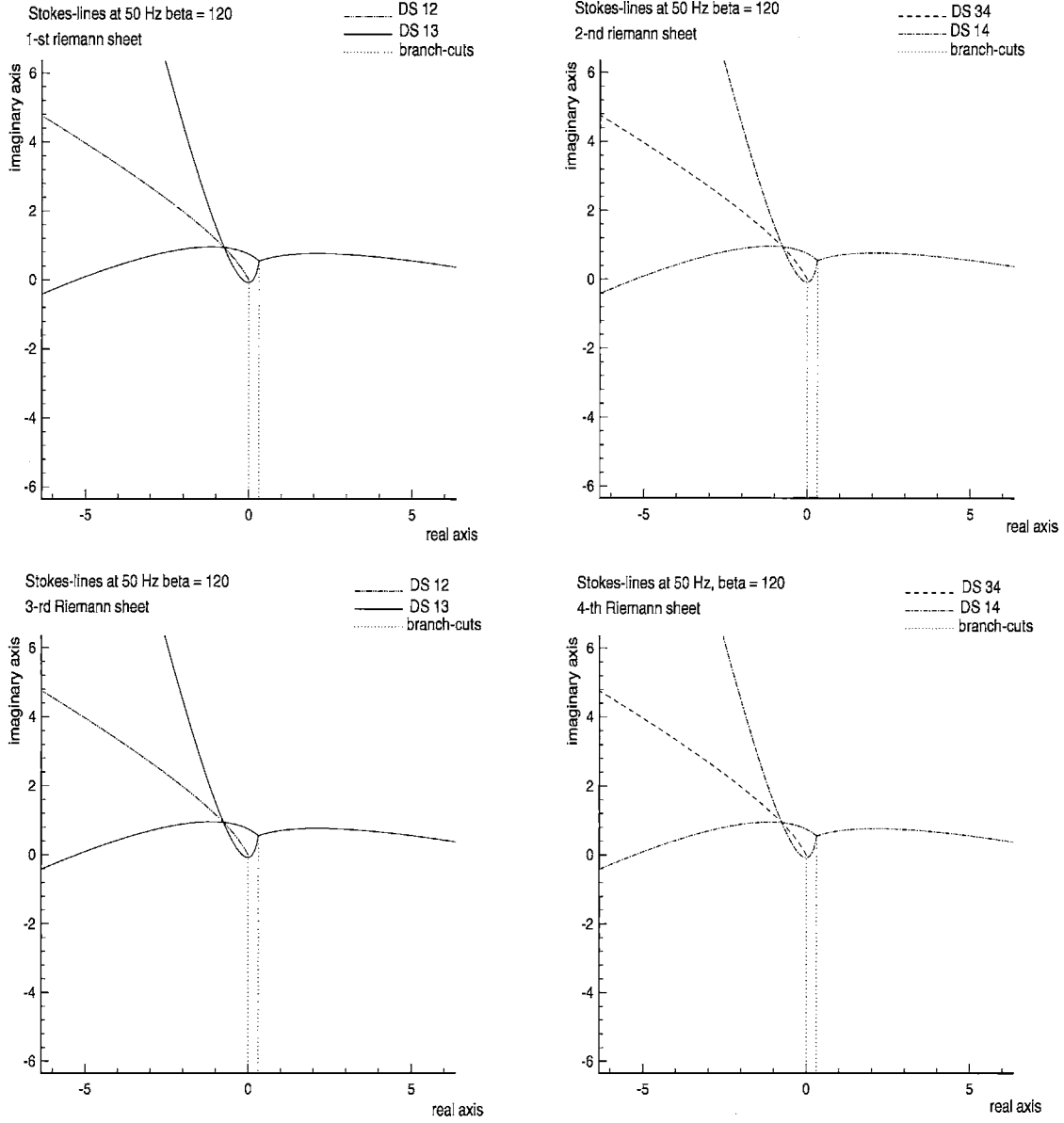


Figure 3-20: Stokes lines for the approximate equation (3.188) with $\beta = \frac{2\pi}{3}$. These lines are defined according to equation (3.246). At the 1st (3rd) Riemann sheet (top (bottom) left figure): Region S_0 (S_6) - 1st quadrant (bounded by the top and right going lines DS_{13}), region S_{11} (S_5) - 4th quadrant (bounded by the branch-cut and the right going line DS_{13}), region S_2 (S_8) - 3rd quadrant (bounded by the branch-cut and the left going lines DS_{13}) and region S_1 (S_7) - 2nd quadrant (bounded by the left and top going lines DS_{13}). At the 2nd (4th) Riemann sheet (top (bottom) right figure): Region S_3 (S_9) - 1st quadrant, region S_2 (S_8) - 4th quadrant, region S_5 (S_{11}) - 3rd quadrant and region S_4 (S_{10}) - 2nd quadrant. Notice that for this Riemann sheet, regions are bounded by lines DS_{14} (DS_{14}) and branch-cuts.

$$A_1(\delta, \lambda, \varepsilon, \alpha) = H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.251)$$

$$A_2(\delta, \lambda, \varepsilon, \alpha) = H_4(\delta, \lambda, \varepsilon, \alpha) + H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.252)$$

$$A_3(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.253)$$

$$\sum_{j=1}^3 A_j(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.254)$$

$$A_{1,-1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.255)$$

$$A_{2,-1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.256)$$

$$A_{3,-1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.257)$$

$$A_{1,0}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.258)$$

$$A_{2,0}(\delta, \lambda, \varepsilon, \alpha) = H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.259)$$

$$A_{3,0}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.260)$$

$$A_{1,1}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.261)$$

$$A_{2,1}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.262)$$

$$A_{3,1}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.263)$$

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.264)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.265)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.266)$$

- For the region S_3 we have:

$$A_1(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.267)$$

$$A_2(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.268)$$

$$A_3(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.269)$$

$$\sum_{j=1}^3 A_j(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.270)$$

$$A_{1,0}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.271)$$

$$A_{2,0}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.272)$$

$$A_{3,0}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.273)$$

$$A_{1,1}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.274)$$

$$A_{2,1}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.275)$$

$$A_{3,1}(\delta, \lambda, \varepsilon, \alpha) = H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.276)$$

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = -H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.277)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.278)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.279)$$

$$A_{1,3}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.280)$$

$$A_{2,3}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.281)$$

$$A_{3,3}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.282)$$

- For the region S_6 we have:

$$A_1(\delta, \lambda, \varepsilon, \alpha) = -H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.283)$$

$$A_2(\delta, \lambda, \varepsilon, \alpha) = -H_3(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.284)$$

$$A_3(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.285)$$

$$\sum_{j=1}^3 A_j(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.286)$$

$$A_{1,1}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.287)$$

$$A_{2,1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.288)$$

$$A_{3,1}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.289)$$

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.290)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = -H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.291)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.292)$$

$$A_{1,3}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.293)$$

$$A_{2,3}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.294)$$

$$A_{3,3}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.295)$$

$$A_{1,4}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.296)$$

$$A_{2,4}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.297)$$

$$A_{3,4}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.298)$$

- For the sector S_9 we have:

$$A_1(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.299)$$

$$A_2(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.300)$$

$$A_3(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.301)$$

$$\sum_{j=1}^3 A_j(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.302)$$

$$(3.303)$$

Regarding the considered functions $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ we to consider the Riemann sheets SR_1 and SR_4 separately. For $S_9 \cap RS_1$ we have:

$$A_{1,-1}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.304)$$

$$A_{2,-1}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.305)$$

$$A_{3,-1}(\delta, \lambda, \varepsilon, \alpha) = H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.306)$$

$$A_{1,0}(\delta, \lambda, \varepsilon, \alpha) = H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.307)$$

$$A_{2,0}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.308)$$

$$A_{3,0}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.309)$$

$$A_{1,1}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.310)$$

$$A_{2,1}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.311)$$

$$A_{3,1}(\delta, \lambda, \varepsilon, \alpha) = 2H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.312)$$

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.313)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.314)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.315)$$

For $S_9 \cap RS_4$ we have:

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = 2H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.316)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.317)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.318)$$

$$A_{1,3}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.319)$$

$$A_{2,3}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.320)$$

$$A_{3,3}(\delta, \lambda, \varepsilon, \alpha) = -H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.321)$$

$$A_{1,4}(\delta, \lambda, \varepsilon, \alpha) = H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.322)$$

$$A_{2,4}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.323)$$

$$A_{3,4}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.324)$$

Next, we consider the regions S_n for the range $0 < \beta < \pi$.

- For the region S_2 we have:

$$A_1(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.325)$$

$$A_2(\delta, \lambda, \varepsilon, \alpha) = H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.326)$$

$$A_3(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.327)$$

$$\sum_{j=1}^3 A_j(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.328)$$

$$A_{1,-1}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) - 3H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.329)$$

$$A_{2,-1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.330)$$

$$A_{3,-1}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.331)$$

$$A_{1,0}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.332)$$

$$A_{2,0}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.333)$$

$$A_{3,0}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.334)$$

$$A_{1,1}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.335)$$

$$A_{2,1}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.336)$$

$$A_{3,1}(\delta, \lambda, \varepsilon, \alpha) = H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.337)$$

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.338)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.339)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.340)$$

$$A_{1,3}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.341)$$

$$A_{2,3}(\delta, \lambda, \varepsilon, \alpha) = -3H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.342)$$

$$A_{3,3}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.343)$$

- For the region S_5 we have:

$$A_1(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.344)$$

$$A_2(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.345)$$

$$A_3(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.346)$$

$$\sum_{j=1}^3 A_j(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.347)$$

$$A_{1,0}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) - 3H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.348)$$

$$A_{2,0}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.349)$$

$$A_{3,0}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.350)$$

$$A_{1,1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.351)$$

$$A_{2,1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.352)$$

$$A_{3,1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.353)$$

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = -H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.354)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.355)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.356)$$

$$A_{1,3}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.357)$$

$$A_{2,3}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.358)$$

$$A_{3,3}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.359)$$

$$A_{1,4}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.360)$$

$$A_{2,4}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) + 3H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.361)$$

$$A_{3,4}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.362)$$

- For the region S_8 we have:

$$A_1(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.363)$$

$$A_2(\delta, \lambda, \varepsilon, \alpha) = H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.364)$$

$$A_3(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.365)$$

$$\sum_{j=1}^3 A_j(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.366)$$

$$A_{1,1}(\delta, \lambda, \varepsilon, \alpha) = 3H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.367)$$

$$A_{2,1}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.368)$$

$$A_{3,1}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.369)$$

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = 2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.370)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.371)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.372)$$

$$A_{1,3}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.373)$$

$$A_{2,3}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.374)$$

$$A_{3,3}(\delta, \lambda, \varepsilon, \alpha) = H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.375)$$

$$A_{1,4}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.376)$$

$$A_{2,4}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.377)$$

$$A_{3,4}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.378)$$

$$A_{1,5}(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.379)$$

$$A_{2,5}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) - 3H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.380)$$

$$A_{3,5}(\delta, \lambda, \varepsilon, \alpha) = -2H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.381)$$

- For the region S_{11} we have:

$$A_1(\delta, \lambda, \varepsilon, \alpha) = -H_1(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.382)$$

$$A_2(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) \quad (3.383)$$

$$A_3(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.384)$$

$$\sum_{j=1}^3 A_j(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.385)$$

For the functions $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ we consider separately the intersection of the region S_{11} with the Riemann sheets RS_1 and RS_4 . For the region $S_{11} \cap RS_1$ we have:

$$A_{1,-1}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.386)$$

$$A_{2,-1}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.387)$$

$$A_{3,-1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.388)$$

$$A_{1,0}(\delta, \lambda, \varepsilon, \alpha) = H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.389)$$

$$A_{2,0}(\delta, \lambda, \varepsilon, \alpha) = H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.390)$$

$$A_{3,0}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.391)$$

$$A_{1,1}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.392)$$

$$A_{2,1}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.393)$$

$$A_{3,1}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.394)$$

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.395)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = -2H_2(\delta, \lambda, \varepsilon, \alpha) + 3H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.396)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.397)$$

For the region $S_{11} \cap RS_4$ we have:

$$A_{1,2}(\delta, \lambda, \varepsilon, \alpha) = 2H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) + 3H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.398)$$

$$A_{2,2}(\delta, \lambda, \varepsilon, \alpha) = 2H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.399)$$

$$A_{3,2}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.400)$$

$$A_{1,3}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.401)$$

$$A_{2,3}(\delta, \lambda, \varepsilon, \alpha) = H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.402)$$

$$A_{3,3}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.403)$$

$$A_{1,4}(\delta, \lambda, \varepsilon, \alpha) = H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.404)$$

$$A_{2,4}(\delta, \lambda, \varepsilon, \alpha) = H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.405)$$

$$A_{3,4}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) \quad (3.406)$$

$$A_{1,5}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) \quad (3.407)$$

$$A_{2,5}(\delta, \lambda, \varepsilon, \alpha) = -H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) - 2H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.408)$$

$$A_{3,5}(\delta, \lambda, \varepsilon, \alpha) = H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha) \quad (3.409)$$

The asymptotic expansions of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$ (integrals along the steepest descent paths) are given in the Appendix I. With these asymptotic expansions and the deformation of the contour integrals $A_j(\delta, \lambda, \varepsilon, \alpha)$ and $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$ in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$ we can construct the asymptotic expansion of the functions $A_j(\delta, \lambda, \varepsilon, \alpha)$ and $A_{j,n}(\delta, \lambda, \varepsilon, \alpha)$, which are solutions of the approximate equation (3.188). The asymptotic expansion of these solutions are going to be used in the next section to continue the Liouville-Green functions across the turning points and Stokes-lines.

3.2.5 Matching Process.

We discuss the matching of the Liouville-Green functions with the asymptotic expansion of the solution of the approximate equation for the particular case of non-uniformity in the beam flexural rigidity only. This results in connection formulas for first order turning point problems. We have to consider two turning point problems (pair of a first order turning point plus a pole). We first consider the boundary conditions for each turning point problem. Second we obtain the form of the Liouville-Green functions valid in a neighborhood of the turning point problems. Third, we match the local form of the Liouville-Green functions with the asymptotic expansion of the appropriate solution of the approximate equation (3.188). This gives the local reflection, transmission and wave mode conversion coefficients. Fourth, we use these local scattering coefficients to build a local transfer matrix which represents the wave scattering at each turning point problem. The product of the local transfer matrices gives the global transfer matrix, from which we obtain the global reflection and transmission coefficients. Next, we discuss the boundary conditions for the turning point problems.

3.2.5.1 Boundary Conditions for the Turning Point Problems.

We consider four boundary conditions for the turning point problems, namely, right and left incidence of propagating and evanescent wave modes. We divide the real axis of the complex z plane in intervals (A_n, A_{n+1}) . We impose the boundary conditions at the points A_n and A_{n+1} . The points $B_n \in (A_n, A_{n+1})$ and $C_n \in (A_n, A_{n+1})$ are defined as the point where the branch cuts cross the real axis of the complex z plane. A sketch of the complex z plane with the turning point problems is given in figure 3-21.

For non-uniformity in the flexural rigidity only specified by equation (3.178), we have two turning point problems, each of which consists of a first order turning point close to coalesce with a first order pole. Below we give the boundary conditions for both turning point problems.

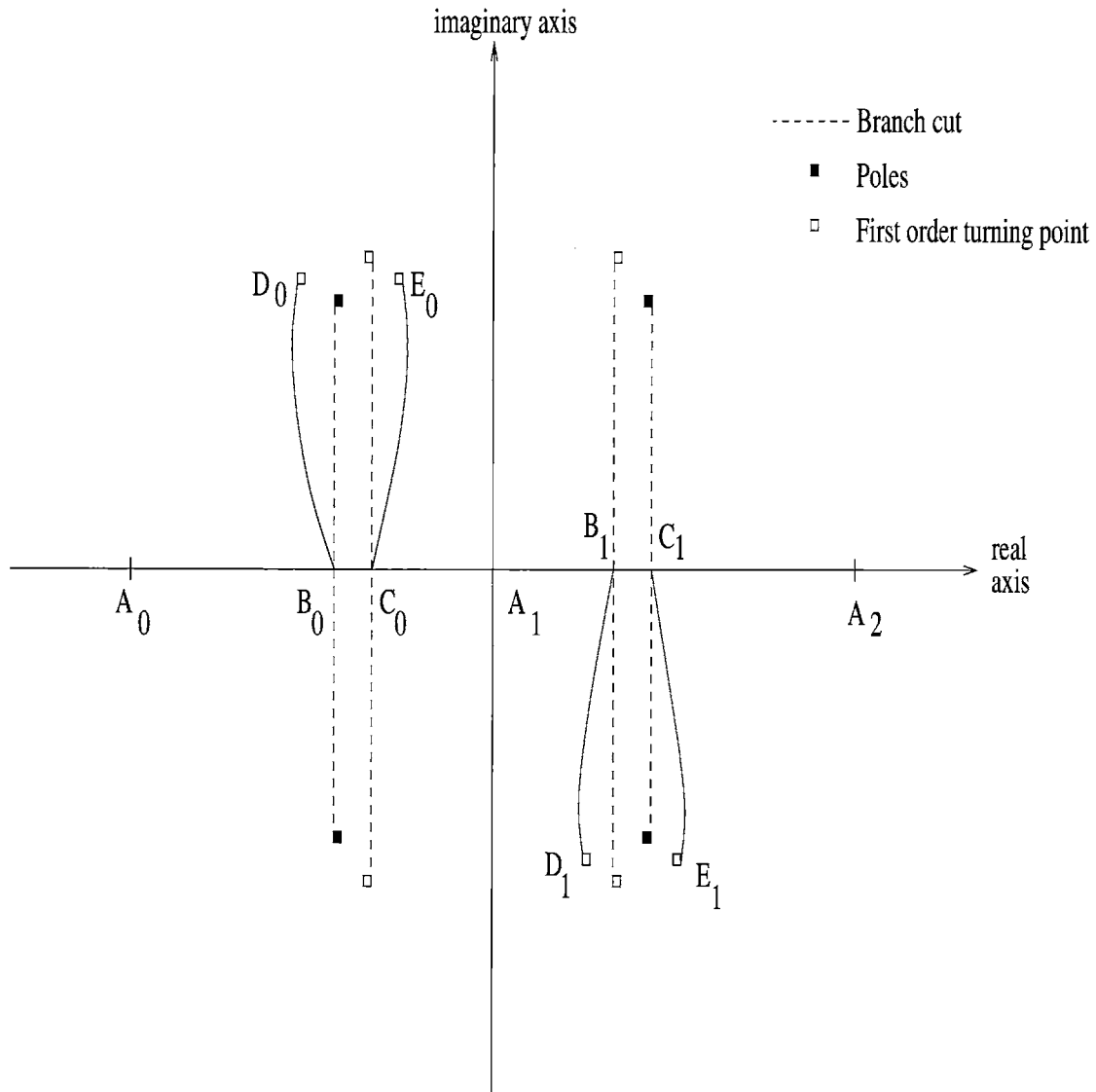


Figure 3-21: Sketch of turning points problems and points A_n , B_n and C_n in the complex z plane for the non-uniformity in the flexural rigidity only, and specified by equation (3.178).

- Case of left wave incidence.

– For $A_0 < z < B_0$ we have that (first turning point problem):

$$\begin{aligned}
y(z) \sim & Av_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, A_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, A_0) \right) \\
& + R_0^- Av_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, A_0) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, A_0) \right) \\
& + C_0^- Av_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, B_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, B_0) \right),
\end{aligned} \tag{3.410}$$

where the points A_0 and B_0 appearing in the equation above are reference points for the integrals giving the quantities $S_j(z, r)$ and $v_{nj}(z, r)$ defined, respectively, by equations (3.107)-(3.110) and (3.129). We make this dependence on the reference point r explicit since it is necessary to define with respect to which point we are obtaining the scattering coefficients. In the summations appearing in the equation above and below, N is the order of the approximation considered. The scattering coefficients R_0^- and C_0^- are, respectively, the reflection coefficient at point A_0 and left wave mode conversion coefficient at point B_0 .

– For $C_0 < z < A_1$ we have that (first turning point problem):

$$\begin{aligned}
y(z) \sim & T_0^- Av_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, C_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, C_0) \right) \\
& + D_0^- Av_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, C_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, C_0) \right).
\end{aligned} \tag{3.411}$$

The scattering coefficients T_0^- and D_0^- are, respectively, the transmission coefficient at point C_0 and the right wave mode conversion coefficient at point C_0 . Next, we consider the second turning point problem ($A_1 < z < A_2$).

– For $A_1 < z < B_1$ we have that (second turning point problem):

$$\begin{aligned}
y(z) \sim & Av_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, A_1) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, A_1) \right) \\
& + R_1^- Av_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, A_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, A_1) \right) \\
& + C_1^- Av_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, B_1) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, B_1) \right),
\end{aligned} \tag{3.412}$$

The scattering coefficients R_1^- and C_1^- are, respectively, the reflection coefficient at point A_1 and the left wave mode conversion coefficient at point B_1 .

– For $C_1 < z < A_2$ we have that (second turning point problem):

$$\begin{aligned}
y(z) \sim & T_1^- Av_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, C_1) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, C_1) \right) \\
& + D_1^- Av_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, C_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, C_1) \right).
\end{aligned} \tag{3.413}$$

The scattering coefficients T_1^- and D_1^- are, respectively, the transmission coefficient at point C_1 and the right wave mode conversion coefficient at point C_1 . Next, we consider right wave incidence for the first turning point problem.

- Case of right wave incidence.

– For $A_0 < z < B_0$ we have that (first turning point problem):

$$\begin{aligned}
y(z) \sim & T_0^+ Av_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, B_0) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, B_0) \right) \\
& + C_0^+ Av_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, B_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, B_0) \right).
\end{aligned} \tag{3.414}$$

The scattering coefficients T_0^+ and C_0^+ are, respectively, the transmission coefficient at point B_0 and the left wave mode conversion coefficient at point B_0 .

– For $C_0 < z < A_1$ we have that (first turning point problem):

$$\begin{aligned}
y(z) \sim & Av_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, A_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, A_1) \right) \\
& + R_0^+ Av_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, A_1) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, A_1) \right) \\
& + D_0^+ Av_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, C_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, C_0) \right).
\end{aligned} \tag{3.415}$$

The scattering coefficients R_0^+ and D_0^+ are, respectively, the reflection coefficient at point A_1 and the right wave mode conversion coefficient at point C_0 . Next, we consider right wave incidence for the second turning point problem.

– For $A_1 < z < B_1$ we have that (second turning point problem):

$$\begin{aligned}
y(z) \sim & T_1^+ Av_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, B_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, B_1) \right) \\
& + C_1^+ Av_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, B_1) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, B_1) \right),
\end{aligned} \tag{3.416}$$

The scattering coefficients T_1^+ and C_1^+ are, respectively, the transmission

coefficient at point B_1 and the left wave mode conversion coefficient at point B_1 .

- For $C_1 < z < A_2$ we have that (second turning point problem):

$$\begin{aligned}
y(z) \sim & Av_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, A_2) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, A_2) \right) \\
& + R_1^+ Av_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, A_2) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, A_2) \right) \\
& + D_1^+ Av_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, C_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, C_1) \right).
\end{aligned} \tag{3.417}$$

The scattering coefficients R_1^+ and D_1^+ are, respectively, the reflection coefficient at point A_2 and the right wave mode conversion coefficient at point C_1 . Next, we consider left incidence of evanescent wave mode at the first turning point problem.

- Case of left incidence of evanescent wave mode.

- For $A_0 < z < B_0$ we have that (first turning point problem):

$$\begin{aligned}
y(z) \sim & Av_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, A_0) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, A_0) \right) \\
& + \tilde{R}_0^- Av_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, A_0) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, A_0) \right) \\
& + \tilde{C}_0^- Av_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, B_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, B_0) \right),
\end{aligned} \tag{3.418}$$

The scattering coefficients $\tilde{R}_0^-$ and $\tilde{C}_0^-$ are, respectively, the left propagating wave mode conversion coefficient at point A_0 and the reflection coefficient at point C_0 .

- For $C_0 < z < A_1$ we have that (first turning point problem):

$$\begin{aligned}
y(z) \sim & \tilde{T}_0^- Av_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, C_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, C_0) \right) \\
& + \tilde{D}_0^- Av_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, C_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, C_0) \right).
\end{aligned} \tag{3.419}$$

The scattering coefficients $\tilde{T}_0^-$ and $\tilde{D}_0^-$ are, respectively, the right propagating wave mode conversion coefficient at point C_0 and the transmission coefficient at point C_0 . Next, we consider left incidence of evanescent wave mode at the second turning point problem.

– For $A_1 < z < B_1$ we have that (second turning point problem):

$$\begin{aligned}
y(z) \sim & Av_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, A_1) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, A_1) \right) \\
& + \tilde{R}_1^- Av_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, A_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, A_1) \right) \\
& + \tilde{C}_1^- Av_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, B_1) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, B_1) \right),
\end{aligned} \tag{3.420}$$

The scattering coefficients $\tilde{R}_1^-$ and $\tilde{C}_1^-$ are, respectively, the reflection coefficient at point A_1 and the left wave mode coefficient at point B_1 .

– For $C_1 < z < A_2$ we have that (second turning point problem):

$$\begin{aligned}
y(z) \sim & \tilde{T}_1^- Av_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, C_1) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, C_1) \right) \\
& + \tilde{D}_1^- Av_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, C_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, C_1) \right).
\end{aligned} \tag{3.421}$$

The scattering coefficients $\tilde{T}_1^-$ and $\tilde{D}_1^-$ are, respectively, the right propagating wave mode conversion coefficient at point C_1 and the transmission

coefficient at point C_1 . Next, we consider the case of right incidence of evanescent wave mode at the second turning point problem.

- Case of right incidence of evanescent wave mode.

- For $A_0 < z < B_0$ we have that (first turning point problem):

$$\begin{aligned} y(z) \sim & \tilde{T}_0^+ A v_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, B_0) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, B_0) \right) \\ & + \tilde{C}_0^+ A v_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, B_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, B_0) \right). \end{aligned} \quad (3.422)$$

The scattering coefficients $\tilde{T}_0^+$ and $\tilde{C}_0^+$ are, respectively, left propagating wave mode conversion coefficient at point B_0 and the transmission coefficient at point B_0 .

- For $C_0 < z < A_1$ we have that (first turning point problem):

$$\begin{aligned} y(z) \sim & A v_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, A_1) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, A_1) \right) \\ & + \tilde{R}_0^+ A v_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, A_1) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, A_1) \right) \\ & + \tilde{D}_0^+ A v_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, C_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, C_0) \right). \end{aligned} \quad (3.423)$$

The scattering coefficients $\tilde{R}_0^+$ and $\tilde{D}_0^+$ are, respectively, the right propagating wave mode conversion coefficient at point A_1 and reflection coefficient at point C_0 . Next, we consider right incidence of evanescent wave mode at the second turning point problem.

- For $A_1 < z < B_1$ we have that (second turning point problem):

$$\begin{aligned}
y(z) \sim & \tilde{T}_1^+ A v_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, B_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, B_1) \right) \\
& + \tilde{C}_1^+ A v_{02}(z) \exp \left(\frac{1}{\varepsilon} S_2(z, B_1) + \sum_{n=1}^N \varepsilon^n v_{n2}(z, B_1) \right), \tag{3.424}
\end{aligned}$$

The scattering coefficients $\tilde{T}_1^+$ and $\tilde{C}_1^+$ are, respectively, the left propagating wave mode conversion coefficient at point B_1 and the transmission coefficient at point B_1 .

– For $C_1 < z < A_2$ we have that (second turning point problem):

$$\begin{aligned}
y(z) \sim & A v_{03}(z) \exp \left(\frac{1}{\varepsilon} S_3(z, A_2) + \sum_{n=1}^N \varepsilon^n v_{n3}(z, A_2) \right) \\
& + \tilde{R}_1^+ A v_{01}(z) \exp \left(\frac{1}{\varepsilon} S_1(z, A_2) + \sum_{n=1}^N \varepsilon^n v_{n1}(z, A_2) \right) \\
& + \tilde{D}_1^+ A v_{04}(z) \exp \left(\frac{1}{\varepsilon} S_4(z, C_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(z, C_1) \right). \tag{3.425}
\end{aligned}$$

The scattering coefficients $\tilde{R}_1^+$ and $\tilde{D}_1^+$ are, respectively, the right propagating wave mode conversion coefficient at point A_2 and the reflection coefficient at point C_1 .

To be able to match the combinations of the Liouville-Green functions given by the boundary conditions above with the asymptotic expansion of the solution of the approximate equation (3.188) we need to continue the Liouville-Green functions up to a neighborhood (points D_n and E_n of figure 3-21 of the chosen pair of first order turning point plus pole. So, in a neighborhood of a chosen turning point plus pole we can write the Liouville-Green functions in the form given below.

- For $A_n < z < B_n$ and left incidence of a propagating wave mode or evanescent wave mode we have that:

$$\begin{aligned}
v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, A_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, A_n) \right) &\sim v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(B_n, A_n) \right. \\
&\quad \left. + \sum_{m=1}^N \varepsilon^m v_{mj}(B_n, A_n) + \frac{1}{\varepsilon} S_j(D_n, B_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(D_n, B_n) \right) \\
&\quad \exp \left(\frac{1}{\varepsilon} S_j(z, D_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, D_n) \right) \text{ with } j = 1, 2 \text{ if } n = 0 \\
&\quad \text{or } j = 3, 4 \text{ if } n = 1,
\end{aligned} \tag{3.426}$$

$$\begin{aligned}
v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, A_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, A_n) \right) &\sim v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(D_n, B_n) \right. \\
&\quad \left. + \sum_{m=1}^N \varepsilon^m v_{mj}(D_n, B_n) \right) \exp \left(\frac{1}{\varepsilon} S_j(z, D_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, D_n) \right) \\
&\quad \text{with } j = 3, 4 \text{ if } n = 0 \text{ or } j = 1, 2 \text{ if } n = 1.
\end{aligned} \tag{3.427}$$

- For $C_n < z < A_{n+1}$ and left incidence of a propagating wave mode or evanescent wave mode we have that:

$$\begin{aligned}
v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, C_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, C_n) \right) &\sim v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(E_n, C_n) \right. \\
&\quad \left. + \sum_{m=1}^N \varepsilon^m v_{mj}(E_n, C_n) \right) \exp \left(\frac{1}{\varepsilon} S_j(z, E_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, E_n) \right) \\
&\quad \text{for } j = 1, 2, 3, 4
\end{aligned} \tag{3.428}$$

- For $A_n < z < B_n$ and right incidence of a propagating wave mode or evanescent wave mode we have that:

$$\begin{aligned}
v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, B_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, B_n) \right) &\sim v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(D_n, B_n) \right. \\
&\quad \left. + \sum_{m=1}^N \varepsilon^m v_{mj}(D_n, B_n) \right) \exp \left(\frac{1}{\varepsilon} S_j(z, D_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, D_n) \right) \\
&\text{for } j = 1, 2, 3, 4
\end{aligned} \tag{3.429}$$

- For $C_n < z < A_{n+1}$ and right incidence of a propagating wave mode or evanescent wave mode we have that:

$$\begin{aligned}
v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, A_{n+1}) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, A_{n+1}) \right) &\sim v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(C_n, A_{n+1}) \right. \\
&\quad \left. + \sum_{m=1}^N \varepsilon^m v_{mj}(C_n, A_{n+1}) + \frac{1}{\varepsilon} S_j(E_n, C_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(E_n, C_n) \right) \\
&\exp \left(\frac{1}{\varepsilon} S_j(z, E_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, E_n) \right) \text{ with } j = 3, 4 \text{ if } n = 0 \\
&\text{or } j = 1, 2 \text{ if } n = 1.
\end{aligned} \tag{3.430}$$

$$\begin{aligned}
v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, C_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, C_n) \right) &\sim v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(E_n, C_n) \right. \\
&\quad \left. + \sum_{m=1}^N \varepsilon^m v_{mj}(E_n, C_n) \right) \exp \left(\frac{1}{\varepsilon} S_j(z, E_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, E_n) \right) \\
&\text{with } j = 1, 2 \text{ if } n = 0 \text{ or } j = 3, 4 \text{ if } n = 1.
\end{aligned} \tag{3.431}$$

3.2.5.2 Liouville-Green Functions in Terms of the Inner Variables

To be able to match the Liouville-Green functions with the asymptotic expansion of the appropriate solution of the approximate equation (3.188), we need to write expressions for the phase $S_j(z)$, the wave amplitude v_{0j} and the exponential of the phase correction $\sum_{m=1}^N \varepsilon^m v_{mj}(z)$ in terms of the inner variables. This is the objective of this section. For the non-uniformity considered we have

$$m(z) = 1, \quad (3.432)$$

$$ei(z) = \chi(z - z^*), \quad (3.433)$$

$$\bar{P}(z) = \bar{P}, \quad (3.434)$$

where z^* represents the first order pole of the turning point problem considered and

$$\chi = \frac{dt}{dz}(z^*), \quad (3.435)$$

with $t(z)$ given by the equation (3.178). The independent inner variable δ is related to the far field variable z according to the equation

$$z = z^* + \frac{\omega^2}{\chi} \delta. \quad (3.436)$$

If we substitute equations (3.432)-(3.434) into the equations (3.107)-(3.110) for the phases $S_j(z)$ and if we consider the change of variable given by equation (3.436), we obtain the expressions for the phases $S_j(z)$ in terms of the inner variable δ as given by the equation

$$\frac{1}{\varepsilon} S_j(\delta, r) = \frac{1}{|\lambda|} \{f(t_j, \delta, \varepsilon, \alpha) - f(t_j, r, \varepsilon, \alpha)\} \text{ for } j = 1, \dots, 4, \quad (3.437)$$

where r is one of the points D_n or E_n . The functions $f(t_j, \delta, \varepsilon, \alpha)$ are given by the equations (3.242)-(3.245). To obtain the wave amplitude $v_{0j}(z)$ in terms of the inner variable δ , we substitute equations (3.432)-(3.434) into the transport equation (3.115), and we also use the change of variable given by equation (3.436). We solve the resulting transport equation in terms of the inner variable δ . We obtain:

$$w_{01}(\delta) = \sqrt{\frac{1}{2\varepsilon^9}} \left(\frac{\rho}{\rho''\rho'} \right)^{1/4} \exp \left(i\frac{\theta}{4} - i\frac{\theta''}{4} - i\frac{\theta'}{4} + i\frac{3}{2}\beta \right), \quad (3.438)$$

$$w_{02}(\delta) = \sqrt{\frac{1}{2\varepsilon^9}} \left(\frac{\rho}{\rho''\rho'} \right)^{1/4} \exp \left(i\frac{\theta}{4} - i\frac{\theta''}{4} - i\frac{\theta'}{4} - i\frac{\pi}{2} + i\frac{3}{2}\beta \right), \quad (3.439)$$

$$w_{03}(\delta) = \sqrt{\frac{1}{2\varepsilon^9}} \left(\frac{\rho}{\rho'''\rho'} \right)^{1/4} \exp \left(i\frac{\theta}{4} - i\frac{\theta'''}{4} - i\frac{\theta'}{4} - i\frac{\pi}{2} + i\frac{3}{2}\beta \right), \quad (3.440)$$

$$w_{04}(\delta) = \sqrt{\frac{1}{2\varepsilon^9}} \left(\frac{\rho}{\rho''\rho'} \right)^{1/4} \exp \left(i\frac{\theta}{4} - i\frac{\theta'''}{4} - i\frac{\theta'}{4} + i\frac{3}{2}\beta \right). \quad (3.441)$$

To obtain the high order phase corrections $v_{nj}(z)$, we first substitute equations (3.432)-(3.434) into the transport equation (3.121) and (3.122) and use the change of variable given by equation (3.436). We solve the resulting transport equations to obtain the amplitude corrections $w_{nj}(z)$, and from them and relations given by equation (3.129) we obtain the phase corrections $v_{nj}(z)$. After that we expand the exponential of the phase correction $\sum_{m=1}^N \varepsilon^m v_{mj}(z)$ up to the N th order in ε , we realize that

$$\exp \left(\sum_{m=1}^N \varepsilon^m v_{mj}(\delta, r) \right) \sim \exp \left(\sum_{m=1}^N \lambda^m \tilde{D}_{mj}(r) \right) \left\{ 1 + \sum_{m=1}^N \lambda^m D_{mj}(\delta) \right\} \text{ for } j = 1, \dots, 4, \quad (3.442)$$

where the functions $D_{mj}(\delta)$ are given in the Appendix I by equations (I.5)-(I.12) up to $m = 4$. The parameter λ is the small parameter in terms of the inner variable and it is defined by equation (3.190) and r is one of the reference points D_n or E_n . The quantities $\tilde{D}_{mj}(r)$ are given in terms of the functions $D_{mj}(\delta)$ according to the relations

$$\begin{aligned}
\tilde{D}_{1j}(r) &= D_{1j}(r), \\
\tilde{D}_{2j}(r) &= D_{2j}(r) - \frac{1}{2}\tilde{D}_{1j}(r), \\
\tilde{D}_{3j}(r) &= D_{3j}(r) - \frac{1}{6}\tilde{D}_{1j}(r) - \tilde{D}_{1j}(r)\tilde{D}_{2j}(r), \\
&\dots\dots\dots
\end{aligned} \tag{3.443}$$

$$\tilde{D}_{nj}(r) = D_{nj}(r) - \frac{1}{n!} \sum_{l=2}^n \left(\sum_{j_1+\dots+j_l=n} (j_1! \dots j_l!) \tilde{D}_{j_1j} \dots \tilde{D}_{j_lj} \right). \tag{3.444}$$

From the results of this section, we can write

$$\begin{aligned}
v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, r) + \sum_{n=1}^N \varepsilon^n v_{nj}(z, r) \right) &\sim C \exp \left(\frac{1}{|\lambda|} [f(t_j, \delta, \varepsilon, \alpha) - f(t_j, r, \varepsilon, \alpha)] \right. \\
&\quad \left. + \sum_{m=1}^N \lambda^m D_{mj}(r) \right) v_{0j}(\delta) \left\{ 1 + \sum_{m=1}^N \lambda^m D_{mj}(\delta) \right\} \\
&\text{for } j = 1, \dots, 4 \text{ and } N = 4,
\end{aligned} \tag{3.445}$$

where C is a constant. If we compare this expression with the asymptotic expansion for the functions $H_j(\delta, \lambda, \varepsilon, \alpha)$, we realize that they match perfectly for $N = 4$. We evaluated the functions $D_{nj}(\delta)$ up to $n = 4$, but we conjecture that we could go to higher orders and the matching would still be perfect.

3.2.5.3 Solutions of the Approximate Equation.

Here we give the solutions of the approximate equation (3.188), valid in a neighborhood of a turning point plus pole, for the various boundary conditions considered in section 3.2.5.1. We use these solutions to continue the linear combination of Liouville-Green functions (far field asymptotic solution) across the turning points and Stokes-lines. These solutions of the approximate equation (3.188) are going to be matched with the local form of the Liouville-Green functions valid in a neighborhood

of the pairs of turning points plus poles. This matching process gives the scattering coefficients at the turning point problems. For the non-uniformity considered we have only two turning point problems. Below we are going to list the solutions of the approximate equation (3.188) for each turning point problem and for each boundary conditions considered for the turning point problems.

- Solutions of the approximate equation (3.188) for the first turning point problem ($A_0 < \Re\{z\} < A_1$).
 - Boundary condition of left wave incidence. We consider regions S_{11} and S_8 illustrated in figure 3-20.

$$y(\delta) = B \{A_{1,3}(\delta, \lambda, \varepsilon, \alpha) - 2A_{3,3}(\delta, \lambda, \varepsilon, \alpha)\}, \quad (3.446)$$

where B is a constant to be obtained through the matching of the asymptotic expansion of this solution with the local form of linear combination of Liouville-Green functions specified by the boundary condition given by equations (3.410) and (3.411). In the regions S_{11} and S_8 of the complex δ plane, the solution above can be written in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, as follows:

$$y(\delta) = B \{2H_1(\delta, \lambda, \varepsilon, \alpha) - H_2(\delta, \lambda, \varepsilon, \alpha) + H_3(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_{11} \quad (3.447)$$

$$y(\delta) = B \{H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_8 \quad (3.448)$$

- Boundary condition of right wave incidence. We consider regions S_8 and S_5 illustrated in figure 3-20.

$$y(\delta) = B \{A_{2,3}(\delta, \lambda, \varepsilon, \alpha)\}, \quad (3.449)$$

where B is a constant to be obtained through the matching of the asymptotic expansion of this solution with the local form of the linear combination of Liouville-Green functions specified by the boundary condition given by equations (3.414) and (3.415). In the regions S_8 and S_5 of the complex δ plane, the solution above can be written in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, as follows:

$$y(\delta) = B \{H_2(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_8 \quad (3.450)$$

$$y(\delta) = B \{-2H_4(\delta, \lambda, \varepsilon, \alpha) + H_1(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_5 \quad (3.451)$$

- Boundary condition of left incidence of evanescent wave mode. We consider regions S_{11} and S_8 illustrated in figure 3-20.

$$y(\delta) = B \{A_1(\delta, \lambda, \varepsilon, \alpha) + A_{3,3}(\delta, \lambda, \varepsilon, \alpha)\}, \quad (3.452)$$

where B is a constant to be obtained through the matching of the asymptotic expansion of this solution with the local form of linear combination of Liouville-Green functions specified by the boundary condition given by equations (3.418) and (3.419). In the regions S_{11} and S_8 of the complex δ plane, the solution above can be written in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, as follows:

$$y(\delta) = B \{ 2H_4(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \} \text{ for } \delta \in S_{11} \quad (3.453)$$

$$y(\delta) = B \{ H_1(\delta, \lambda, \varepsilon, \alpha) \} \text{ for } \delta \in S_8 \quad (3.454)$$

- Boundary condition of right incidence of evanescent wave mode. We consider regions S_8 and S_5 illustrated in figure 3-20.

$$y(\delta) = B \{ A_{2,3}(\delta, \lambda, \varepsilon, \alpha) - 2A_{3,3}(\delta, \lambda, \varepsilon, \alpha) \}, \quad (3.455)$$

where B is a constant to be obtained through the matching of the asymptotic expansion of this solution with the local form of the linear combination of Liouville-Green functions specified by the boundary condition given by equations (3.422) and (3.423). In the regions S_8 and S_5 of the complex δ plane, the solution above can be written in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, as follows:

$$y(\delta) = B \{ -H_2(\delta, \lambda, \varepsilon, \alpha) - 2H_3(\delta, \lambda, \varepsilon, \alpha) \} \text{ for } \delta \in S_8 \quad (3.456)$$

$$y(\delta) = B \{ H_1(\delta, \lambda, \varepsilon, \alpha) - 2H_2(\delta, \lambda, \varepsilon, \alpha) - H_3(\delta, \lambda, \varepsilon, \alpha) \} \text{ for } \delta \in S_5 \quad (3.457)$$

- Solutions of the approximate equation (3.188) for the second turning point problem ($A_1 < \Re\{z\} < A_2$).

- Boundary condition of left wave incidence. We consider regions S_0 and S_9 illustrated in figure 3-18.

$$y(\delta) = B \{A_{2,0}(\delta, \lambda, \varepsilon, \alpha) - A_2(\delta, \lambda, \varepsilon, \alpha)\}, \quad (3.458)$$

where B is a constant to be obtained through the matching of the asymptotic expansion of this solution with the local form of linear combination of Liouville-Green functions specified by the boundary condition given by equations (3.412) and (3.413). In the regions S_0 and S_9 of the complex δ plane, the solution above can be written in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, as follows:

$$y(\delta) = B \{-H_1(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_0 \quad (3.459)$$

$$y(\delta) = B \{H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_3 \quad (3.460)$$

– Boundary condition of right wave incidence. We consider regions S_0 and S_3 illustrated in figure 3-18.

$$y(\delta) = B \{A_{3,0}(\delta, \lambda, \varepsilon, \alpha) - 2A_3(\delta, \lambda, \varepsilon, \alpha)\}, \quad (3.461)$$

where B is a constant to be obtained through the matching of the asymptotic expansion of this solution with the local form of the linear combination of Liouville-Green functions specified by the boundary condition given by equations (3.416) and (3.417). In the regions S_0 and S_3 of the complex δ plane, the solution above can be written in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, as follows:

$$y(\delta) = B \{H_2(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_9 \quad (3.462)$$

$$y(\delta) = B \{-H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_2(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_0 \quad (3.463)$$

- Boundary condition of left incidence of evanescent wave mode. We consider regions S_0 and S_9 illustrated in figure 3-18.

$$y(\delta) = B \{A_2(\delta, \lambda, \varepsilon, \alpha) + A_{2,0}(\delta, \lambda, \varepsilon, \alpha)\}, \quad (3.464)$$

where B is a constant to be obtained through the matching of the asymptotic expansion of this solution with the local form of linear combination of Liouville-Green functions specified by the boundary condition given by equations (3.420) and (3.421). In the regions S_0 and S_9 of the complex δ plane, the solution above can be written in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, as follows:

$$y(\delta) = B \{H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_4(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_0 \quad (3.465)$$

$$y(\delta) = B \{-2H_1(\delta, \lambda, \varepsilon, \alpha) + H_2(\delta, \lambda, \varepsilon, \alpha) - H_4(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_9 \quad (3.466)$$

- Boundary condition of right incidence of evanescent wave mode. We consider regions S_0 and S_3 illustrated in figure 3-17.

$$y(\delta) = B \{A_{1,0}(\delta, \lambda, \varepsilon, \alpha) - A_1(\delta, \lambda, \varepsilon, \alpha)\}, \quad (3.467)$$

where B is a constant to be obtained through the matching of the asymp-

otic expansion of this solution with the local form of the linear combination of Liouville-Green functions specified by the boundary condition given by equations (3.424) and (3.425). In the regions S_0 and S_3 of the complex δ plane, the solution above can be written in terms of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, as follows:

$$y(\delta) = B \{-H_2(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_0 \quad (3.468)$$

$$y(\delta) = B \{-H_1(\delta, \lambda, \varepsilon, \alpha) + 2H_3(\delta, \lambda, \varepsilon, \alpha) + H_4(\delta, \lambda, \varepsilon, \alpha)\} \text{ for } \delta \in S_3 \quad (3.469)$$

Now we can use the solutions of the approximate equation (3.188) given above to continue the combinations of Liouville-Green functions specified by the various boundary conditions across the turning points and Stokes-lines. To do so, we match the local form of the Liouville-Green functions with the asymptotic expansion of the appropriate solution of the approximate equation (3.188). This matching process gives the scattering coefficients at each turning point problem. In the next section we illustrate the matching process and we give the scattering coefficients at each turning point problem for the various boundary conditions.

3.2.5.4 Local Reflection, Transmission and Wave Mode Conversion Coefficients

We illustrate the matching process only for the case of left wave incidence at the first turning point problem ($A_0 < \Re\{z\} < A_1$). For the other boundary conditions specified for this and the other turning point problem, the matching process is basically the same. For the problem of left wave incidence at the first turning point problem, we first match the local form of the linear combination of the Liouville-Green functions specified by the boundary conditions with the solution of the approximate equation (3.188) in the region S_{11} illustrated in figure 3-20. If we match equation (3.447) with equations (3.426) and (3.427), we obtain that

$$2B = A \exp \left(\frac{1}{\varepsilon} S_1(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_0, A_0) + \frac{1}{\varepsilon} S_1(D_0, B_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(D_0, B_0) - \frac{1}{\lambda} f(t_1, D_0, \varepsilon, \alpha) - \sum_{m=1}^N \lambda^m \tilde{D}_{m1}(D_0) \right) \quad (3.470)$$

$$-B = R_0^- A \exp \left(\frac{1}{\varepsilon} S_2(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n2}(B_0, A_0) + \frac{1}{\varepsilon} S_2(D_0, B_0) + \sum_{n=1}^N \varepsilon^n v_{n2}(D_0, B_0) - \frac{1}{\lambda} f(t_2, D_0, \varepsilon, \alpha) - \sum_{m=1}^N \lambda^m \tilde{D}_{m2}(D_0) \right) \quad (3.471)$$

$$B = C_0^- A \exp \left(\frac{1}{\varepsilon} S_3(D_0, B_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(D_0, B_0) - \frac{1}{\lambda} f(t_3, D_0, \varepsilon, \alpha) - \sum_{m=1}^N \lambda^m \tilde{D}_{m3}(D_0) \right) \quad (3.472)$$

If we solve the system of equations above, we obtain the reflection and left evanescent wave mode conversion coefficient.

$$R_0^- = -\frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_1(B_0, A_0) - \frac{1}{\varepsilon} S_2(B_0, A_0) + \sum_{n=1}^N \varepsilon^n [v_{n1}(B_0, A_0) - v_{n2}(B_0, A_0)] + \frac{1}{\varepsilon} S_1(D_0, B_0) - \frac{1}{\varepsilon} S_2(D_0, B_0) + \sum_{n=1}^N \varepsilon^n [v_{n1}(D_0, B_0) - v_{n2}(D_0, B_0)] - \frac{1}{\lambda} [f(t_1, D_0, \varepsilon, \alpha) - f(t_2, D_0, \varepsilon, \alpha)] - \sum_{m=1}^N \lambda^m [\tilde{D}_{m1}(D_0) - \tilde{D}_{m2}(D_0)] \right) \quad (3.473)$$

$$C_0^- = \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_1(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_0, A_0) + \frac{1}{\varepsilon} S_1(D_0, B_0) - \frac{1}{\varepsilon} S_3(D_0, B_0) + \sum_{n=1}^N \varepsilon^n [v_{n1}(D_0, B_0) - v_{n3}(D_0, B_0)] - \frac{1}{\lambda} [f(t_1, D_0, \varepsilon, \alpha) - f(t_3, D_0, \varepsilon, \alpha)] - \sum_{m=1}^N \lambda^m [\tilde{D}_{m1}(D_0) - \tilde{D}_{m3}(D_0)] \right) \quad (3.474)$$

Second, we match the local form of the linear combination of the Liouville-Green functions specified by the boundary conditions with the solution of the approximate

equation (3.188) in the region S_8 illustrated in figure 3-20. If we match equation (3.448) with equations (3.428), we obtain that

$$B = D_0^- A \exp \left(\frac{1}{\varepsilon} S_1(E_0, C_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(E_0, C_0) - \frac{1}{\lambda} f(t_1, E_0, \varepsilon, \alpha) - \sum_{m=1}^N \lambda^m \tilde{D}_{m1}(E_0) \right) \quad (3.475)$$

$$-2B = T_0^- A \exp \left(\frac{1}{\varepsilon} S_3(E_0, C_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(E_0, C_0) - \frac{1}{\lambda} f(t_3, E_0, \varepsilon, \alpha) - \sum_{m=1}^N \lambda^m \tilde{D}_{m3}(E_0) \right) \quad (3.476)$$

These two equations gives the transmission and right evanescent wave mode conversion coefficients, which are given below.

$$\begin{aligned} T_0^- = - \exp & \left(\frac{1}{\varepsilon} S_1(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_0, A_0) + \frac{1}{\varepsilon} S_1(D_0, B_0) - \frac{1}{\varepsilon} S_3(E_0, C_0) \right. \\ & + \sum_{m=1}^N \varepsilon^m [v_{m1}(D_0, B_0) - v_{m3}(E_0, C_0)] - \frac{1}{\lambda} [f(t_1, D_0, \varepsilon, \alpha) - f(t_3, E_0, \varepsilon, \alpha)] \\ & \left. - \sum_{m=1}^N \lambda^m [\tilde{D}_{m1}(D_0) - \tilde{D}_{m3}(E_0)] \right) \end{aligned} \quad (3.477)$$

$$\begin{aligned} D_0^- = \frac{1}{2} \exp & \left(\frac{1}{\varepsilon} S_1(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_0, A_0) + \frac{1}{\varepsilon} S_1(D_0, B_0) - \frac{1}{\varepsilon} S_1(E_0, C_0) \right. \\ & - \sum_{n=1}^N \varepsilon^n v_{n1}(E_0, C_0) - \frac{1}{\lambda} [f(t_1, D_0, \varepsilon, \alpha) - f(t_1, E_0, \varepsilon, \alpha)] \\ & \left. - \sum_{m=1}^N \lambda^m [\tilde{D}_{m1}(D_0) - \tilde{D}_{m1}(E_0)] \right) \end{aligned} \quad (3.478)$$

Now we need to write properly the integrals in the expressions above for the scattering coefficients R_0^- , T_0^- , C_0^- and D_0^- . For the integrals along the branch cuts appearing in the expression for R_0^- and C_0^- we consider the contour Γ_0 illustrated in figure 3-22. We propagate the point D_0 until the other side of the branch cuts, as illustrated in figure 3-22. We also make E_0 to coincide with D_0 . Note that point D_0 lies close to

the branch points such that the local approximation is still valid. For the integrals appearing in the scattering coefficients T_0^- and D_0^- we use also the contours Γ_{01} and Γ_{02} illustrated in figure 3-22 below.

Now we give the expression of the scattering coefficients R_0^- , T_0^- , C_0^- and D_0^+ in terms of contour integrals Γ_0 , Γ_{01} and Γ_{02} , as follows:

$$R_0^- = -\frac{1}{2} \exp \left(\frac{2}{\varepsilon} S_1(B_0, A_0) + 2 \sum_{n=1}^N \varepsilon^n v_{n1}(B_0, A_0) + \frac{2}{\varepsilon} S_1(\Gamma_0) + 2 \sum_{n=1}^N \varepsilon^n v_{n1}(\Gamma_0) - \frac{2}{\lambda} [f(t_1, D_0, \varepsilon, \alpha) - 2 \sum_{m=1}^N \lambda^m \tilde{D}_{m1}(D_0)] \right) \quad (3.479)$$

$$C_0^- = \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_1(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_0, A_0) + \frac{1}{\varepsilon} S_1(\Gamma_0) - \frac{1}{\varepsilon} S_3(\Gamma_0) + \sum_{n=1}^N \varepsilon^n [v_{n1}(\Gamma_0) - v_{n3}(\Gamma_0)] - \frac{1}{\lambda} [f(t_1, D_0, \varepsilon, \alpha) - f(t_3, D_0, \varepsilon, \alpha)] - \sum_{m=1}^N \lambda^m [\tilde{D}_{m1}(D_0) - \tilde{D}_{m3}(D_0)] \right) \quad (3.480)$$

$$T_0^- = -\exp \left(\frac{1}{\varepsilon} S_1(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_0, A_0) + \frac{1}{\varepsilon} S_1(\Gamma_0) - \frac{1}{\varepsilon} S_3(\Gamma_{01}) + \sum_{m=1}^N \varepsilon^m [v_{m1}(\Gamma_0) - v_{m3}(\Gamma_{01})] - \frac{1}{\lambda} [f(t_1, D_0, \varepsilon, \alpha) - f(t_3, D_0, \varepsilon, \alpha)] - \sum_{m=1}^N \lambda^m [\tilde{D}_{m1}(D_0) - \tilde{D}_{m3}(D_0)] \right) \quad (3.481)$$

$$D_0^- = \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_1(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_0, A_0) + \frac{1}{\varepsilon} S_1(\Gamma_{02}) + \sum_{n=1}^N \varepsilon^n v_{n1}(\Gamma_{02}) \right) \quad (3.482)$$

For the other boundary conditions specified for the first turning point problem, we just give the final result from the matching process, which are the scattering coefficients. We use the contours Γ_0 , Γ_{01} and Γ_{02} illustrated in figure 3-22.

- Right wave incidence for the first turning point problem ($A_0 < \Re\{z\} < A_1$).

The matching process is done in regions S_8 and S_5 .

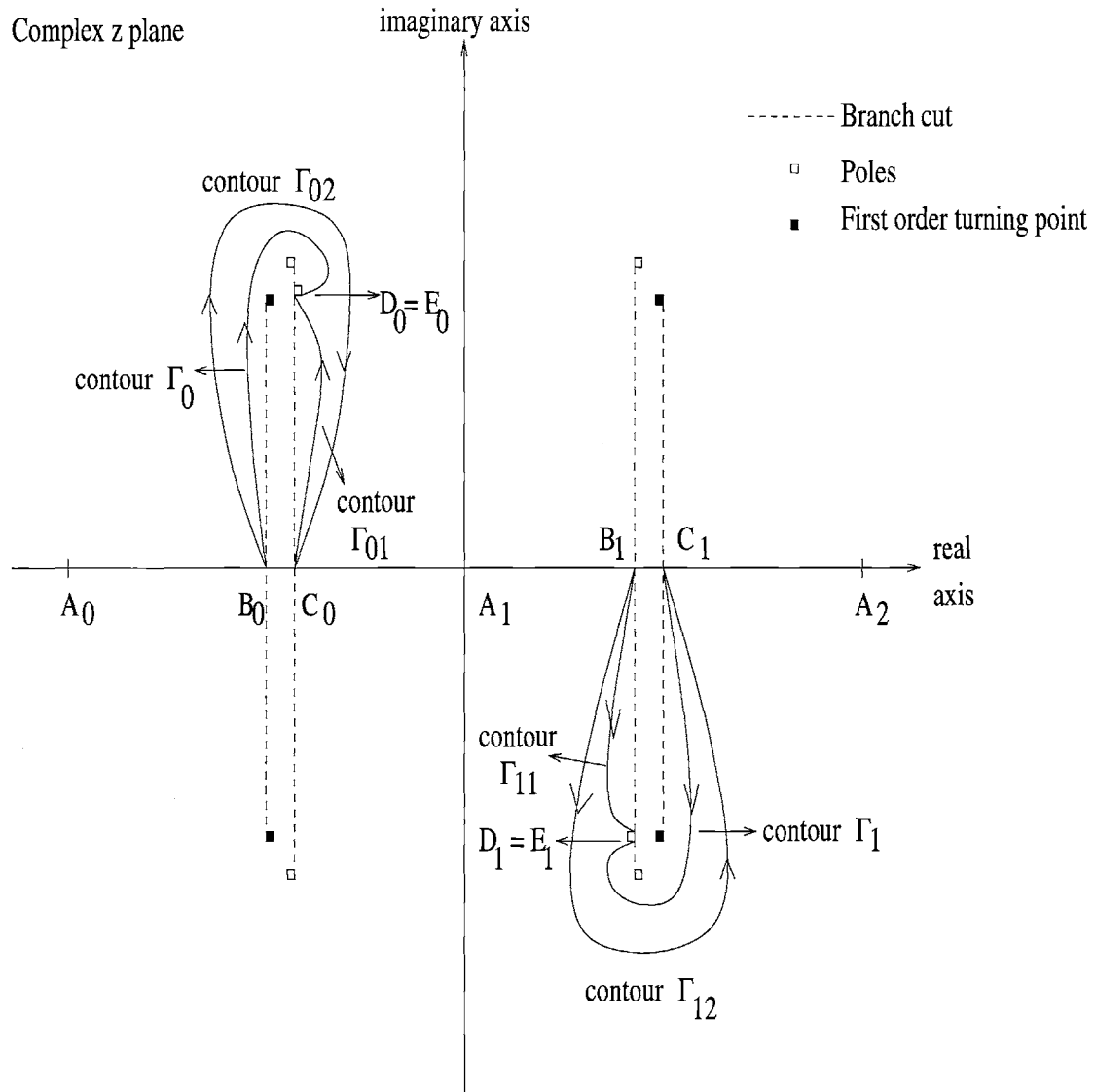


Figure 3-22: Sketch of the integration contours $\Gamma_0, \Gamma_{01}, \Gamma_{02}, \Gamma_1, \Gamma_{11}$ and Γ_{12} in the complex z plane for the non-uniformity in the flexural rigidity only, and specified by equation (3.178).

$$R_0^+ = \frac{1}{2} \exp \left(\frac{2}{\varepsilon} S_4(C_0, A_1) + 2 \sum_{n=1}^N \varepsilon^n v_{n4}(C_0, A_1) + \frac{2}{\varepsilon} S_4(\Gamma_{01}) + 2 \sum_{n=1}^N \varepsilon^n v_{n4}(\Gamma_{01}) - \frac{2}{\lambda} f(t_4, D_0, \varepsilon, \alpha) - 2 \sum_{n=1}^N \lambda^n \tilde{D}_{n4}(D_0) \right) \quad (3.483)$$

$$D_0^+ = -\frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_4(C_0, A_1) - \frac{1}{\varepsilon} S_1(C_0, A_1) + \sum_{n=1}^N \varepsilon^n [v_{n4}(C_0, A_1) - v_{n1}(C_0, A_1)] + \frac{1}{\varepsilon} S_4(\Gamma_{01}) - \frac{1}{\varepsilon} S_1(\Gamma_{01}) + \sum_{n=1}^N \varepsilon^n [v_{n4}(\Gamma_{01}) - v_{n1}(\Gamma_{01})] - \frac{1}{\lambda} [f(t_4, D_0, \varepsilon, \alpha) - f(t_1, D_0, \varepsilon, \alpha)] - \sum_{n=1}^N \lambda^n [\tilde{D}_{n4}(D_0) - \tilde{D}_{n1}(D_0)] \right) \quad (3.484)$$

$$T_0^+ = -\exp \left(\frac{1}{\varepsilon} S_4(C_0, A_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(C_0, A_1) + \frac{1}{\varepsilon} S_4(\Gamma_{01}) - \frac{1}{\varepsilon} S_2(\Gamma_0) + \sum_{n=1}^N \varepsilon^n [v_{n4}(\Gamma_{01}) - v_{n2}(\Gamma_0)] - \frac{1}{\lambda} [f(t_4, D_0, \varepsilon, \alpha) - f(t_2, D_0, \varepsilon, \alpha)] - \sum_{n=1}^N [\tilde{D}_{n4}(D_0) - \tilde{D}_{n2}(D_0)] \right) \quad (3.485)$$

$$C_0^+ = 0 \quad (3.486)$$

- Left incidence of evanescent wave mode for the first turning point problem ($A_0 < \Re\{z\} < A_1$). The matching process is done in regions S_{11} and S_8 .

$$\tilde{R}_0^- = \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_4(B_0, A_0) - \frac{1}{\varepsilon} S_2(B_0, A_0) + \sum_{n=1}^N \varepsilon^n [v_{n4}(B_0, A_0) - v_{n2}(B_0, A_0)] + \frac{1}{\varepsilon} S_4(\Gamma_0) - \frac{1}{\varepsilon} S_2(\Gamma_0) + \sum_{n=1}^N \varepsilon^n [v_{n4}(\Gamma_0) - v_{n2}(\Gamma_0)] - \frac{1}{\lambda} [f(t_4, D_0, \varepsilon, \alpha) - f(t_2, D_0, \varepsilon, \alpha)] - \sum_{n=1}^N \lambda^n [\tilde{D}_{n4}(D_0) - \tilde{D}_{n2}(D_0)] \right) \quad (3.487)$$

$$\begin{aligned}\tilde{C}_0^- = & -\frac{1}{2} \exp \left(\frac{2}{\varepsilon} S_4(B_0, A_0) + 2 \sum_{n=1}^N \varepsilon^n v_{n4}(B_0, A_0) + \frac{2}{\varepsilon} S_4(\Gamma_0) + 2 \sum_{n=1}^N \varepsilon^n v_{n4}(\Gamma_0) \right. \\ & \left. - \frac{2}{\lambda} f(t_4, D_0, \varepsilon, \alpha) - 2 \sum_{n=1}^N \lambda^n \tilde{D}_{n4}(D_0) \right) \end{aligned} \quad (3.488)$$

$$\tilde{T}_0^- = 0 \quad (3.489)$$

$$\begin{aligned}\tilde{D}_0^- = & \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_4(B_0, A_0) + \sum_{n=1}^N \varepsilon^n v_{n4}(B_0, A_0) + \frac{1}{\varepsilon} S_4(\Gamma_0) - \frac{1}{\varepsilon} S_1(\Gamma_{01}) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n4}(\Gamma_0) - v_{n1}(\Gamma_0)] - \frac{1}{\lambda} [f(t_4, D_0, \varepsilon, \alpha) - f(t_1, D_0, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N [\tilde{D}_{n4}(D_0) - \tilde{D}_{n1}(D_0)] \right) \end{aligned} \quad (3.490)$$

- Right incidence of evanescent wave mode for the first turning point problem ($A_0 < \Re\{z\} < A_1$). The matching process is done in regions S_8 and S_5 .

$$\begin{aligned}\tilde{R}_0^+ = & \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_2(C_0, A_1) - \frac{1}{\varepsilon} S_3(C_0, A_1) + \sum_{n=1}^N \varepsilon^n [v_{n2}(C_0, A_1) - v_{n3}(C_0, A_1)] \right. \\ & + \frac{1}{\varepsilon} S_2(\Gamma_{01}) - \frac{1}{\varepsilon} S_3(\Gamma_{01}) + \sum_{n=1}^N \varepsilon^n [v_{n2}(\Gamma_{01}) - v_{n3}(\Gamma_{01})] - \frac{1}{\lambda} [f(t_2, D_0, \varepsilon, \alpha) \\ & \left. - f(t_3, D_0, \varepsilon, \alpha)] - \sum_{n=1}^N \lambda^n [\tilde{D}_{n2}(D_0) - \tilde{D}_{n3}(D_0)] \right) \end{aligned} \quad (3.491)$$

$$\begin{aligned}\tilde{D}_0^+ = & -\frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_2(C_0, A_1) + \sum_{n=1}^N \varepsilon^n v_{n2}(C_0, A_1) + \frac{1}{\varepsilon} S_2(\Gamma_{01}) - \frac{1}{\varepsilon} S_1(\Gamma_{01}) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n2}(\Gamma_{01}) - v_{n1}(\Gamma_{01})] - \frac{1}{\lambda} [f(t_2, D_0, \varepsilon, \alpha) - f(t_3, D_0, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N [\tilde{D}_{n2}(D_0) - \tilde{D}_{n3}(D_0)] \right) \end{aligned} \quad (3.492)$$

$$\tilde{T}_0^+ = \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_2(C_0, A_1) + \sum_{n=1}^N \varepsilon^n v_{n2}(C_0, A_1) - \frac{1}{\varepsilon} S_2(\Gamma_{02}) - \sum_{n=1}^N v_{n2}(\Gamma_{02}) \right) \quad (3.493)$$

$$\begin{aligned} \tilde{C}_0^+ = & - \exp \left(\frac{1}{\varepsilon} S_2(C_0, A_1) + \sum_{n=1}^N \varepsilon^n v_{n2}(C_0, A_1) - \frac{1}{\varepsilon} S_2(\Gamma_{01}) - \frac{1}{\varepsilon} S_3(\Gamma_0) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n2}(\Gamma_{01}) - v_{n3}(\Gamma_0)] - \frac{1}{\lambda} [f(t_2, D_0, \varepsilon, \alpha) - f(t_3, D_0, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N \lambda^n [\tilde{D}_{n2}(D_0) - \tilde{D}_{n3}(D_0)] \right) \end{aligned} \quad (3.494)$$

Next, we consider the second turning point problem ($A_1 < \Re\{z\} < A_2$) under various boundary conditions. We use the contours Γ_1, Γ_{11} and Γ_{12} illustrated in figure 3-22. We give only the final result of the matching process for this turning point problem, which is the scattering coefficients.

- Left wave incidence for the second turning point problem ($A_1 < \Re\{z\} < A_2$).

The matching process is done in regions S_9 and S_0 illustrated in figure 3-17.

$$\begin{aligned} R_1^- = & - \frac{1}{2} \exp \left(\frac{2}{\varepsilon} S_3(B_1, A_1) + 2 \sum_{n=1}^N \varepsilon^n v_{n3}(B_1, A_1) + \frac{2}{\varepsilon} S_3(\Gamma_{11}) \right. \\ & \left. + 2 \sum_{n=1}^N \varepsilon^n v_{n3}(\Gamma_{11}) - \frac{2}{\lambda} f(t_3, D_1, \varepsilon, \alpha) - 2 \sum_{n=1}^N \lambda^n \tilde{D}_{n3}(D_1) \right) \end{aligned} \quad (3.495)$$

$$\begin{aligned} C_1^- = & \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_3(B_1, A_1) + \sum_{n=1}^N \varepsilon^n v_{n3}(B_1, A_1) + \frac{1}{\varepsilon} S_3(\Gamma_{11}) - \frac{1}{\varepsilon} S_2(\Gamma_{11}) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n3}(\Gamma_{11}) - v_{n2}(\Gamma_{11})] - \frac{1}{\lambda} [f(t_3, D_1, \varepsilon, \alpha) - f(t_2, D_1, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N \lambda^n [\tilde{D}_{n3}(D_1) - \tilde{D}_{n2}(D_1)] \right) \end{aligned} \quad (3.496)$$

$$D_1^- = 0 \quad (3.497)$$

$$\begin{aligned} T_1^- = & -\frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_3(C_1, A_1) + \sum_{n=1}^N \varepsilon^n v_{n3}(C_1, A_1) + \frac{1}{2\varepsilon} S_3(\Gamma_1) - \frac{1}{\varepsilon} S_1(\Gamma_{11}) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n3}(\Gamma_1) - v_{n1}(\Gamma_{11})] - \frac{1}{\lambda} [f(t_3, D_1, \varepsilon, \alpha) - f(t_1, D_1, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N \lambda^n [\tilde{D}_{n3}(D_1) - \tilde{D}_{n1}(D_1)] \right) \end{aligned} \quad (3.498)$$

- Right wave incidence for the second turning point problem ($A_1 < \Re\{z\} < A_2$).
The matching process is done in regions S_9 and S_0 illustrated in figure 3-17.

$$\begin{aligned} R_1^+ = & -\frac{1}{2} \exp \left(\frac{2}{\varepsilon} S_2(C_1, A_2) + 2 \sum_{n=1}^N \varepsilon^n v_{n2}(C_1, A_2) + \frac{2}{\varepsilon} S_2(\Gamma_1) + 2 \sum_{n=1}^N \varepsilon^n v_{n2}(\Gamma_1) \right. \\ & \left. - \frac{2}{\lambda} f(t_2, E_1, \varepsilon, \alpha) - 2 \sum_{n=1}^N \lambda^n \tilde{D}_{n2}(E_1) \right) \end{aligned} \quad (3.499)$$

$$\begin{aligned} D_1^+ = & \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_2(C_1, A_2) + \sum_{n=1}^N \varepsilon^n v_{n2}(C_1, A_2) + \frac{1}{\varepsilon} S_2(\Gamma_1) - \frac{1}{\varepsilon} S_4(\Gamma_1) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n2}(\Gamma_1) - v_{n4}(\Gamma_1)] - \frac{1}{\lambda} [f(t_2, E_1, \varepsilon, \alpha) - f(t_4, E_1, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N \lambda^n [\tilde{D}_{n2}(E_1) - \tilde{D}_{n4}(E_1)] \right) \end{aligned} \quad (3.500)$$

$$C_1^+ = \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_2(C_1, A_2) + \sum_{n=1}^N \varepsilon^n v_{n2}(C_1, A_2) - \frac{1}{\varepsilon} S_2(\Gamma_{12}) - \sum_{n=1}^N \varepsilon^n v_{n2}(\Gamma_{12}) \right) \quad (3.501)$$

$$\begin{aligned} T_1^+ = \exp & \left(\frac{1}{\varepsilon} S_2(C_1, A_2) + \sum_{n=1}^N \varepsilon^n v_{n2}(C_1, A_2) - \frac{1}{\varepsilon} S_2(\Gamma_1) - \frac{1}{\varepsilon} S_4(\Gamma_{11}) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n2}(\Gamma_1) - v_{n4}(\Gamma_{11})] - \frac{1}{\lambda} [f(t_2, E_1, \varepsilon, \alpha) - f(t_4, E_1, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N \lambda^n [\tilde{D}_{n2}(E_1) - \tilde{D}_{n4}(E_1)] \right) \end{aligned} \quad (3.502)$$

- Left incidence of evanescent wave mode for the second turning point problem ($A_1 < \Re\{z\} < A_2$). The matching process is done in regions S_3 and S_0 illustrated in figure 3-17.

$$\begin{aligned} \tilde{R}_1^- = \frac{1}{2} \exp & \left(\frac{1}{\varepsilon} S_1(B_1, A_1) - \frac{1}{\varepsilon} S_4(B_1, A_1) + \sum_{n=1}^N \varepsilon^n [v_{n1}(B_1, A_1) - v_{n4}(B_1, A_1)] \right. \\ & + \frac{1}{\varepsilon} S_1(\Gamma_{11}) - \frac{1}{\varepsilon} S_4(\Gamma_{11}) + \sum_{n=1}^N \varepsilon^n [v_{n1}(\Gamma_{11}) - v_{n4}(\Gamma_{11})] - \frac{1}{\lambda} [f(t_1, D_1, \varepsilon, \alpha) \\ & \left. - f(t_4, D_1, \varepsilon, \alpha)] - \sum_{n=1}^N \lambda^n [\tilde{D}_{n1}(D_1) - \tilde{D}_{n4}(D_1)] \right) \end{aligned} \quad (3.503)$$

$$\begin{aligned} \tilde{C}_1^- = -\frac{1}{2} \exp & \left(\frac{2}{\varepsilon} S_1(B_1, A_1) + 2 \sum_{n=1}^N \varepsilon^n v_{n1}(B_1, A_1) + \frac{2}{\varepsilon} S_1(\Gamma_{11}) + 2 \sum_{n=1}^N \varepsilon^n v_{n1}(\Gamma_{11}) \right. \\ & \left. - \frac{2}{\lambda} f(t_1, D_1, \varepsilon, \alpha) - 2 \sum_{n=1}^N \lambda^n \tilde{D}_{n1}(D_1) \right) \end{aligned} \quad (3.504)$$

$$\tilde{T}_1^- = -\frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_1(B_1, A_1) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_1, A_1) + \frac{1}{\varepsilon} S_1(\Gamma_{12}) + \sum_{n=1}^N \varepsilon^n v_{n1}(\Gamma_{12}) \right) \quad (3.505)$$

$$\begin{aligned} \tilde{D}_1^- = & -\exp \left(\frac{1}{\varepsilon} S_1(B_1, A_1) + \sum_{n=1}^N \varepsilon^n v_{n1}(B_1, A_1) + \frac{1}{\varepsilon} S_1(\Gamma_{11}) - \frac{1}{\varepsilon} S_4(\Gamma_1) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n1}(\Gamma_{11}) - v_{n4}(\Gamma_1)] - \frac{1}{\lambda} [f(t_1, D_1, \varepsilon, \alpha) - f(t_4, D_1, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N \lambda^n [\tilde{D}_{n1}(D_1) - \tilde{D}_{n4}(D_1)] \right) \end{aligned} \quad (3.506)$$

- Right incidence of evanescent wave mode for the second turning point problem ($A_1 < \Re\{z\} < A_2$). The matching process is done in regions S_3 and S_0 illustrated in figure 3-17.

$$\begin{aligned} \tilde{R}_1^+ = & -\frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_3(C_1, A_2) - \frac{1}{\varepsilon} S_1(C_1, A_2) + \sum_{n=1}^N \varepsilon^n [v_{n3}(C_1, A_2) - v_{n1}(C_1, A_2)] \right. \\ & + \frac{1}{\varepsilon} S_3(\Gamma_1) - \frac{1}{\varepsilon} S_1(\Gamma_1) + \sum_{n=1}^N \varepsilon^n [v_{n3}(\Gamma_1) - v_{n1}(\Gamma_1)] - \frac{1}{\lambda} [f(t_3, E_1, \varepsilon, \alpha) \\ & \left. - f(t_1, E_1, \varepsilon, \alpha)] - \sum_{n=1}^N \lambda^n [\tilde{D}_{n3}(E_1) - \tilde{D}_{n1}(E_1)] \right) \end{aligned} \quad (3.507)$$

$$\begin{aligned} \tilde{D}_1^+ = & \frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_3(C_1, A_2) + \sum_{n=1}^N \varepsilon^n v_{n3}(C_1, A_2) + \frac{2}{\varepsilon} S_3(\Gamma_1) \right. \\ & \left. + 2 \sum_{n=1}^N \varepsilon^n v_{n3}(\Gamma_1) - \frac{2}{\lambda} f(t_3, E_1, \varepsilon, \alpha) - 2 \sum_{n=1}^N \lambda^n \tilde{D}_{n3}(E_1) \right) \end{aligned} \quad (3.508)$$

$$\tilde{T}_1^+ = 0 \quad (3.509)$$

$$\begin{aligned} \tilde{C}_1^+ = & -\frac{1}{2} \exp \left(\frac{1}{\varepsilon} S_3(C_1, A_2) + \sum_{n=1}^N \varepsilon^n v_{n3}(C_1, A_2) + \frac{1}{\varepsilon} S_3(\Gamma_1) - \frac{1}{\varepsilon} S_2(\Gamma_{11}) \right. \\ & + \sum_{n=1}^N \varepsilon^n [v_{n3}(\Gamma_1) - v_{n2}(\Gamma_{11})] - \frac{1}{\lambda} [f(t_3, E_1, \varepsilon, \alpha) - f(t_2, E_1, \varepsilon, \alpha)] \\ & \left. - \sum_{n=1}^N \lambda^n [\tilde{D}_{n3}(E_1) - \tilde{D}_{n2}(E_1)] \right) \end{aligned} \quad (3.510)$$

Here we obtained the scattering coefficients at each turning point problem for various boundary conditions. The scattering coefficients obtained for the turning point problems are going to be used to build a transfer matrix which models the wave scattering at the considered turning point problem. The interaction of the incident wave with the beam non-uniformity is modeled as a product of the local transfer matrices, which gives the system global transfer matrix. The global reflection and transmission coefficients are obtained from the global transfer matrix. In the next section we built the local transfer matrix for each turning point problem.

3.2.5.5 Local Transfer Matrices

Along the real axis of the z complex plane we can represent the wave field as in asymptotic form as a superposition of the Liouville-Green functions according to the equations below

- For $A_n < z < B_n$ we have

$$y(z) = \sum_{j=1}^4 F_{j,n} v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, A_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, A_n) \right) \quad (3.511)$$

- For $C_n < z < A_{n+1}$ we have

$$y(z) = \sum_{j=1}^4 F_{j,n+1} v_{0j}(z) \exp \left(\frac{1}{\varepsilon} S_j(z, A_n) + \sum_{m=1}^N \varepsilon^m v_{mj}(z, A_n) \right) \quad (3.512)$$

We can connect the wave modes amplitude at A_n with the wave modes amplitude at A_{n+1} through a transfer matrix. So, the wave scattering at a turning point problem located at $A_n < \Re\{z\} < A_{n+1}$ can be represented by a transfer matrix. In other words, we can write

$$\begin{Bmatrix} F_{1,n+1} \\ F_{2,n+1} \\ F_{3,n+1} \\ F_{4,n+1} \end{Bmatrix} = [M_n] \begin{Bmatrix} F_{1,n} \\ F_{2,n} \\ F_{3,n} \\ F_{4,n} \end{Bmatrix} \quad \text{for } n = 0, 1 \quad (3.513)$$

The elements of the matrix $[M_n]$ can be obtained from the scattering coefficients at the n th turning point problem for the boundary conditions of left and right wave incidence and the boundary condition of left and right evanescent wave mode incidence. These scattering coefficients for the two turning points problems ($n = 0, 1$) are given in the previous section. If we apply the four boundary conditions mentioned above, the matrix equation (3.513) for $n = 0$ assumes the form

$$\begin{bmatrix} (D_0^-)^* & (D_0^+)^* & (\tilde{D}_0^-)^* & (\tilde{D}_0^+)^* \\ 0 & 0 & 0 & 1 \\ (T_0^-)^* & R_0^+ & (\tilde{T}_0^-)^* & \tilde{R}_0^+ \\ 0 & 1 & 0 & 0 \end{bmatrix} = [M_0] \begin{bmatrix} 1 & 0 & 0 & 0 \\ R_0^- & (T_0^+)^* & \tilde{R}_0^- & (\tilde{T}_0^+)^* \\ (C_0^-)^* & (C_0^+)^* & (\tilde{C}_0^-)^* & (\tilde{C}_0^+)^* \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (3.514)$$

The first column in the first and last matrices of the equation (3.514) has the scattering coefficients for the boundary condition of left wave incidence for the first turning point

problem. The second column in the first and last matrices of the equation (3.514) has the scattering coefficients for the boundary condition of right wave incidence for the first turning point problem. The third column in the first and last matrices of the equation (3.514) has the scattering coefficients for the boundary condition of incidence of left evanescent wave mode at the first turning point, and the fourth column of the first and last matrices has the scattering coefficients for the boundary condition of incidence of right evanescent wave mode.

The elements of the matrix $[M_0]$ are obtained by solving the equation (3.514). All the scattering coefficients in the first (last) matrix of the equation (3.514) are obtained at the point A_0 (A_1), but the coefficients $D_0^\pm, \tilde{D}_0^\pm, T_0^\pm$ and $\tilde{T}_0^\pm$ are obtained with respect to the points B_0 or C_0 (see figure 3-21). So, we define the equivalent to these scattering coefficients, but now with respect to the points A_0 or A_1 . These new scattering coefficients are denoted as $(D_0^\pm)^*, (\tilde{D}_0^\pm)^*, (T_0^\pm)^*$ and $(\tilde{T}_0^\pm)^*$ and they are related to the scattering coefficients $D_0^\pm, \tilde{D}_0^\pm, T_0^\pm$ and $\tilde{T}_0^\pm$ according to the equations

$$(C_0^\pm)^*[(\tilde{C}_0^\pm)^*] = C_0^\pm[\tilde{C}_0^\pm] \exp \left(\frac{1}{\varepsilon} S_3(A_0, B_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(A_0, B_0) \right) \quad (3.515)$$

$$(D_0^\pm)^*[(\tilde{D}_0^\pm)^*] = D_0^\pm[\tilde{D}_0^\pm] \exp \left(\frac{1}{\varepsilon} S_1(A_1, C_0) + \sum_{n=1}^N \varepsilon^n v_{n1}(A_1, C_0) \right) \quad (3.516)$$

$$(T_0^-)^*[(\tilde{T}_0^-)^*] = T_0^-[\tilde{T}_0^-] \exp \left(\frac{1}{\varepsilon} S_3(A_1, C_0) + \sum_{n=1}^N \varepsilon^n v_{n3}(A_1, C_0) \right) \quad (3.517)$$

$$(T_0^+)^*[(\tilde{T}_0^+)^*] = T_0^+[\tilde{T}_0^+] \exp \left(\frac{1}{\varepsilon} S_2(A_0, B_0) + \sum_{n=1}^N \varepsilon^n v_{n2}(A_0, B_0) \right) \quad (3.518)$$

Next, we apply the boundary conditions of left and right wave incidence and left and right incidence of evanescent mode to the second turning point problem ($n = 1$), so we can write the matrix equation (3.513) in the form

$$\begin{bmatrix} (T_1^-)^* & R_1^+ & (\tilde{T}_1^-)^* & \tilde{R}_1^+ \\ 0 & 1 & 0 & 0 \\ (D_1^-)^* & (D_1^+)^* & (\tilde{D}_1^-)^* & (\tilde{D}_1^+)^* \\ 0 & 0 & 0 & 1 \end{bmatrix} = [M_1] \begin{bmatrix} 0 & 0 & 1 & 0 \\ (C_1^-)^* & (C_1^+)^* & (\tilde{C}_1^-)^* & (\tilde{C}_1^+)^* \\ 1 & 0 & 0 & 0 \\ R_1^- & (T_1^+)^* & \tilde{R}_1^- & (\tilde{T}_1^+)^* \end{bmatrix} \quad (3.519)$$

We solve this matrix equation to obtain the elements of the matrix $[M_1]$. All the scattering coefficients in the first (last) matrix of the equation (3.519) are obtained at the point A_1 (A_2), but the coefficients $D_1^\pm, \tilde{D}_1^\pm, T_1^\pm$ and $\tilde{T}_1^\pm$ are obtained with respect to the points B_1 or C_1 (see figure 3-21). So, we define the equivalent to these scattering coefficients, but now with respect to the points A_1 or A_2 . These new scattering coefficients are denoted as $(D_1^\pm)^*, (\tilde{D}_1^\pm)^*, (T_1^\pm)^*$ and $(\tilde{T}_1^\pm)^*$ and they are related to the scattering coefficients $D_1^\pm, \tilde{D}_1^\pm, T_1^\pm$ and $\tilde{T}_1^\pm$ according to the equations

$$(C_1^\pm)^*[(\tilde{C}_1^\pm)^*] = C_1^\pm[\tilde{C}_1^\pm] \exp \left(\frac{1}{\varepsilon} S_3(A_1, B_1) + \sum_{n=1}^N \varepsilon^n v_{n3}(A_1, B_1) \right) \quad (3.520)$$

$$(D_1^\pm)^*[(\tilde{D}_1^\pm)^*] = D_1^\pm[\tilde{D}_1^\pm] \exp \left(\frac{1}{\varepsilon} S_1(A_2, C_1) + \sum_{n=1}^N \varepsilon^n v_{n1}(A_2, C_1) \right) \quad (3.521)$$

$$(T_1^+)^*[(\tilde{T}_1^+)^*] = T_1^+[\tilde{T}_1^+] \exp \left(\frac{1}{\varepsilon} S_4(A_1, B_1) + \sum_{n=1}^N \varepsilon^n v_{n4}(A_1, B_1) \right) \quad (3.522)$$

$$(T_1^-)^*[(\tilde{T}_1^-)^*] = T_1^-[\tilde{T}_1^-] \exp \left(\frac{1}{\varepsilon} S_1(A_2, C_1) + \sum_{n=1}^N \varepsilon^n v_{n1}(A_2, C_1) \right) \quad (3.523)$$

In this section we obtained the local transfer matrix related to each turning point problem. The wave scattering by the beam non-uniformity can be modeled as the product of the local transfer matrices, which are matrices $[M_0]$ and $[M_1]$. The product of the local transfer matrices gives the system global transfer matrix. It relates the Liouville-Green functions amplitude at point A_0 (see figure 3-21) with the amplitude of the Liouville-Green functions at point A_2 for the particular non-uniformity consid-

ered. From the global transfer matrix we can extract the global wave reflection and wave transmission coefficients. This is addressed in the next section.

3.2.6 Global Reflection and Transmission Coefficients

We define the system global transfer matrix $[M]$ as the product of the local transfer matrices. For the particular beam non-uniformity, we have only two local transfer matrices, each of which is related to a turning point problem. The global transfer matrix is given in terms of the local transfer matrices according to the equation

$$[M] = \prod_{l=1}^2 [M_l] \quad (3.524)$$

Next, we discuss how to use the system global transfer matrix $[M]$ to obtain the global wave reflection and transmission coefficients for the boundary conditions of left and right wave incidence on the non-uniform part of the beam. The global transfer matrix relates the Liouville-Green functions amplitude at the point A_0 to the Liouville-Green functions amplitude at the point A_2 (see figure 3-21). In other words, we have that

$$\begin{pmatrix} F_{1,2} \\ F_{2,2} \\ F_{3,2} \\ F_{4,2} \end{pmatrix} = [M] \begin{pmatrix} F_{1,0} \\ F_{2,0} \\ F_{3,0} \\ F_{4,0} \end{pmatrix}, \quad (3.525)$$

where $F_{j,2}$ and $F_{j,0}$ are the Liouville Green functions amplitudes, respectively, at points A_2 and A_0 . For the boundary condition of left wave incidence the equation (3.525) assumes the form

$$\begin{Bmatrix} T_G^- \\ 0 \\ 0 \\ D_G^- \end{Bmatrix} = [M] \begin{Bmatrix} 1 \\ R_G^- \\ C_G^- \\ 0 \end{Bmatrix}, \quad (3.526)$$

where R_G^- , T_G^- , C_G^- and D_G^- are, respectively, the wave reflection coefficient, the wave transmission coefficient, the left evanescent wave mode conversion coefficient and the right evanescent wave mode conversion coefficient. Their expression in terms of the elements of the global transfer matrix $[M]$ is given by the equations

$$R_G^- = \frac{M_{33}M_{21} - M_{31}M_{23}}{M_{32}M_{23} - M_{33}M_{22}} \quad (3.527)$$

$$T_G^- = M_{11} + \frac{M_{12}(M_{33}M_{21} - M_{31}M_{23}) + M_{13}(M_{22}M_{31} - M_{21}M_{32})}{M_{32}M_{23} - M_{33}M_{22}} \quad (3.528)$$

$$C_g^- = \frac{M_{22}M_{31} - M_{21}M_{32}}{M_{32}M_{23} - M_{33}M_{22}} \quad (3.529)$$

$$D_G^- = M_{41} + \frac{M_{42}(M_{33}M_{21} - M_{31}M_{23}) + M_{43}(M_{22}M_{31} - M_{21}M_{32})}{M_{32}M_{23} - M_{33}M_{22}} \quad (3.530)$$

For the boundary condition of right wave incidence the equation (3.525) assumes the form

$$\begin{Bmatrix} R_G^+ \\ 1 \\ 0 \\ D_G^+ \end{Bmatrix} = [M] \begin{Bmatrix} 0 \\ T_G^+ \\ C_G^+ \\ 0 \end{Bmatrix}, \quad (3.531)$$

where R_G^+ , T_G^+ , C_G^+ and D_G^+ are, respectively, the wave reflection coefficient, the wave transmission coefficient, the left evanescent wave mode conversion coefficient and the right evanescent wave mode conversion coefficient. Their expression in terms of the elements of the global transfer matrix $[M]$ is given by the equations

$$R_G^+ = \frac{M_{33}M_{12} - M_{13}M_{32}}{M_{22}M_{33} - M_{32}M_{23}} \quad (3.532)$$

$$T_G^+ = \frac{M_{33}}{M_{22}M_{33} - M_{32}M_{23}} \quad (3.533)$$

$$C_g^+ = \frac{M_{32}}{M_{32}M_{23} - M_{33}M_{22}} \quad (3.534)$$

$$D_G^+ = \frac{M_{42}M_{33} - M_{43}M_{32}}{M_{22}M_{33} - M_{32}M_{23}} \quad (3.535)$$

Comparison between the asymptotic results for the scattering coefficients $R_G^\pm, T_G^\pm, C_G^\pm$ and $D_G^\pm$ and the results from direct numerical simulation of equation (3.97) are left as future work.

3.2.7 Discussions and Conclusions

In chapter 4 we develop an asymptotic theory to study the interaction of linear surface gravity waves with a non-uniform bottom topography. In the context of this asymptotic theory, the asymptotic solution is given as a superposition of plane waves with variable amplitude and phase and the uniform bottom dispersion relation is assumed locally valid. The superposition of plane waves is composed by two propagating plane waves and an enumerable set of evanescent plane waves. This is basically a refraction theory, no wave scattering is predicted, unless we take into account the effect of the points where this asymptotic representation for the solution breaks down. These points are the branch points of the uniform bottom dispersion relation. At these points two or more wave modes couple (their associated wavenumbers coalesce). To be able to describe this wave mode coupling, we obtained from the uniform system dispersion relation an ordinary differential equation of the order equal to the number of interacting wave modes. For the water waves problem, we have two types of branch points. For the first type, we observe coupling of the two propagating wave modes, and for the second type of branch points we observe coupling of each propagating wave mode with an evanescent wave mode, which gives a coupling of four wave modes. For the first type of branch point, a second order equation describes well the wave mode

coupling, as is shown in chapter 4, but for the second type of branch points we need a fourth order differential equation to describe well the coupling among four wave modes.

In the second part of 3 we applied the asymptotic ideas used in chapter 4 to the fourth order governing equation for the transversal vibration of the Euler-Bernoulli beam model. For this governing equation, the asymptotic representation of the solution is given as a superposition of two propagating plane waves and two evanescent plane waves with variable phase and amplitude. Due to the non-uniformity, these wave modes couple at the the branch points of the uniform system dispersion relation for the Euler-Bernoulli beam model. For the non-uniformity in the beam flexural rigidity only we have coupling among the four wave modes at the branch points of the uniform system dispersion relation, similar to the wave mode coupling observed at the second type of branch point for the problem of interaction of linear surface gravity waves with a non-uniform bottom topography. We use the governing equation for the Euler-Bernoulli beam model with non-uniformity in the beam flexural rigidity only as a model problem to understand how to extend the ideas used to describe the coupling of two wave modes to the case of coupling among four wave modes.

The asymptotic theory used in the second part of chapter 3 is the WKB method. To take the effects of the turning points (branch points of the uniform system dispersion relation) into account, we consider an approximate form of the governing equation (3.97) valid in a neighborhood of the turning points. For the type of non-uniformity in the flexural rigidity only, the branch points are first order poles and first order turning points (the variable coefficient function of the governing equation is a linear function close to the branch point). For these type of turning points the approximate equation can be solved in closed form. We match the asymptotic form of the solution of the approximate equation with the WKB solution (WKB waves), which gives the reflection and wave mode conversion coefficients due to the presence of turning points. This approach is limited, since we need to be able to solve the approximate form of the governing equation close to the turning points, and for high order turning points this may not be possible. More general techniques are needed

and an interesting technique is the Borel-Laplace transform.

Despite the lack of generality in the non-uniformity considered, many aspect of the use of the asymptotic theory for high order ordinary differential equations became evident. For second order differential equations, the form of the asymptotic expansion changes (exponentially small terms appear) as we cross Stokes-lines emanating from the turning points. For high order differential equations we have a set of Stokes-lines. The number of Stokes-lines is equal to $\frac{n!}{2!(n-2)!}$, where n is the number of branches of the uniform system dispersion relation (for example, the functions $\frac{dS_j}{dz}$ in this chapter). For the problem in this chapter the Stokes-lines are solutions of

$$\Im\{S_j(z) - S_k(z)\} = 0 \text{ for } k > j \text{ and } j = 1, \dots, 4. \quad (3.536)$$

If some symmetry is present (for example, for the functions $S_j(z)$ we have that $S_l = -S_{l+1}, l = 1, 3$), the number of Stokes-lines to consider may be reduced. For high order differential equations, not all Stokes-lines are important. The asymptotic representation of the solution does not change as we cross some Stokes-lines, as is illustrated in figures 3-17 to 3-20. The asymptotic representation of the solution of the approximate equation (3.188) does not change as we cross the Stokes-lines labeled DS_{12} and DS_{34} , and they do not define any boundary for the sectors S_l . Why this is so? As we approach the turning points (branch points of the uniform system dispersion relation) from where the Stokes-lines emanate, we realize that the wavenumbers $\frac{dS_j}{dz}, j = 1, 2$ coalesce, respectively, with the wavenumbers $\frac{dS_j}{dz}, j = 3, 4$. As a results, only as we cross the Stokes-lines labeled DS_{13} and $DS_{24} = DS_{14}$ the asymptotic representation of the solution changes. The wavenumbers $\frac{dS_j}{dz}, j = 3, 4$ coalesce to infinite as we approach the turning points (first order poles) given by equation (3.136), but they approach the infinite in opposite directions, so the Stokes-line given by

$$\Im\{S_3(z) - S_4(z)\} = 0 \quad (3.537)$$

is not relevant. In summary, for Stokes-lines emanating from a given turning point or cluster of turning points, only the Stokes-lines associated with wavenumbers coalescing at this particular turning point or cluster of turning points are important for the asymptotic analysis.

The scattering coefficients obtained from the asymptotic analysis of second order differential equations are given in terms of integrals connecting the real axis to the turning points of the equation that were taken into account. If high order expansion are considered, we need to regularize those integrals. This is discussed in pages 85 to 90 of Slavyanov [59] or in Fröman and Fröman [20]. The wavenumbers coalescing at the turning points of a second order equation have the form

$$\pm\sqrt{Q(z)} \tag{3.538}$$

where $Q(z)$ is the coefficient function of a second order differential equation when it is reduced to the form

$$\frac{d^2y}{dz^2} + Q(z)y(z) = 0.$$

To regularize those integrals we can substitute them by loop integrals around the considered turning points. The integration contour encloses only the considered turning point. It may also enclose the complex conjugate of the considered turning point. For high order differential equations, the solutions of the uniform system dispersion relation do not have in general the form given by equation (3.538), so closed integration contours enclosing the considered turning points cannot, in general, be used to regularize the phase integrals appearing in the scattering coefficients. In section 3.2.5.4, we proposed as integration contours the contours $\Gamma_0, \Gamma_{0j}, j = 1, 2$ and the contours $\Gamma_1, \Gamma_{1j}, j = 1, 2$ to regularize the phase integrals appearing in the scattering coefficients. We also added terms from the asymptotic expansion of the solution of the approximate governing equation (3.188). This was done to eliminate the depen-

dence of the phase integrals from the chosen end point when the contours $\Gamma_0, \Gamma_{01}, \Gamma_1$ and Γ_{11} were used.

At last, for second order differential equations we always have the coalescence of the wavenumbers given by equation (3.538) at the turning points of the equation. For high order differential equations, more complex patterns of wave mode coupling (wavenumbers coalescence) are possible, what makes the asymptotic analysis more intricate, and a flavor of it was given in the second part of this chapter.

Chapter 4

Interaction of Linear Surface Gravity Waves with a Weakly Non-uniform Bottom Topography.

The work on this chapter addresses the problem of wave propagation from deep ocean towards the shore, with special focus on the interactions with non-uniform bottom topographies. Non-uniform bottoms with a mild slope are a common topography in near-shore areas of oceans and seas around the world (e.g. Gierloff-Emden [24]; Snead [60]; Wijnberg [66]). In these cases, the observed mean bottom slope typically varies from 0.001 to 0.01 and a typical wavelength of the bottom variation range from $O(1000)$ meters to $O(100)$ meters. The latter can be comparable to that of the dominant ocean wave and strong interactions between the waves and bottom may happen and extend to $O(10 - 100)$ wave lengths.

This work addresses such scenario and specifically considers wave and bottom parameters given by: $\lambda/\Lambda < O(1)$, $kh = O(1)$, and $L_b/\lambda = O(10 - 100)$. Here λ and Λ are the characteristics length scales, respectively, of the surface wave and of the bottom variation, $k = \lambda/2\pi$ is the wavenumber of the surface waves, h is the water depth, and L_b is the dimension of the bottom topography.

For these range of physical parameters, stochastic theories (valid for much larger L_b/λ) are of limited applicability, while simulations based on direct discretization

(such as boundary-integral or finite-difference methods) are practically prohibitive for $L_b/\lambda \gg O(1)$. When the variation of the bottom is slow relative to surface waves ($\lambda/\Lambda \ll 1$), a number of powerful approximations are available. These include methods based on geometric optics, mild-slope equations and the parabolic approximation.

Methods based on geometrical optics are wave refraction theories and are not able to predict wave reflection due to the bottom non-uniformity, unless a local analysis is performed at the regions where the geometrical optics approximation breaks down. For two dimensional bottom topographies, the regions where the geometrical optics approximations breaks down are called caustics. Considerable wave reflection is obtained through a local analysis of the wave propagation along the caustics. For one-dimensional bottom topographies, the analogous to caustics would be points in the space coordinate, denoted as real turning points, where the geometrical optics approximation breaks down, but this do not happens in the interaction of linear surface waves with a one-dimensional bottom topography. If we consider the space coordinate as a complex variable, for one-dimensional bottom topographies the points where the geometrical optics approximation breaks down lay in the complex plane, away from the real axis.

In this chapter we apply geometrical optics ideas to the interaction of linear surface waves with a non-uniform bottom. We construct a high order asymptotic solution for the wave interaction problem. This asymptotic theory breaks down at the turning points in the complex plane. To continue the asymptotic solution through the turning points, we perform a local analysis of the wave propagation problem. This allow us to incorporate wave reflection and wave mode conversion to the refraction model. By wave mode conversion, we meant the coupling between the propagating wave modes and evanescent waves that appear due to the bottom variation.

We represent the wave scattering at each relevant turning point by a transfer matrix, which results from the local analysis of the wave propagation problem performed in a neighborhood of the considered turning point. The relevant turning points are those closest to the real axis, where the wave propagates. This approach models the interaction of linear surface gravity waves with a variable bottom as a product of

transfer matrices. We expect that the blend of the high order asymptotic with the transfer matrix method would result in an asymptotic theory capable to predict wave interaction with a variable topography not only for $\lambda/\Lambda \ll 1$, but also when $\lambda/\Lambda \rightarrow 1$.

The asymptotic theory developed in this chapter, besides predicting wave reflection and wave mode conversion when $\lambda/\Lambda \ll 1$, also allow a qualitative understanding on how the bottom non-uniformity variation affects wave reflection and wave mode conversion when $\lambda/\Lambda \sim O(1)$. So the asymptotic theory developed in this chapter is a very useful tool to access the minimum change in the bottom non-uniformity such that wave transmission is minimized to a desired level for a given range of wave frequencies of the incident wave.

4.1 Previous Work.

As mentioned before, the interest of linear water waves scattering by a variable bottom dates back from the late forties. The earliest contribution was by Kreisel [35] on the understanding of the uniqueness problem for the diffraction of linear surface gravity waves by a one-dimensional non-uniform bottom. Kreisel proved uniqueness of the reflection and transmission coefficients, but for the velocity potential he proved uniqueness only when the non-uniformity is a small perturbation of the uniform bottom. The uniqueness for the diffraction of linear surface waves by a general rough bottom in a one-dimensional channel is still an open problem, and Simon & Ursell [58] and Chamberlain & Porter [15] gave a review of the progress done so far.

A review of ocean waves in random media is given by Mysak [47]. In particular Hasslemann [26] used the formalism of quantum field theory to describe the scattering of internal and surface gravity waves by topographic irregularities in a stratified fluid. Both wave amplitudes and bottom displacement are considered small so that perturbation techniques are applicable. Long [41] used Hasslemann's theory in homogeneous fluid case for surface waves alone and observed that the presence of a random topography could be sufficient to account for swell decay.

Previous work on periodic bottom topographies and random topographies, specif-

ically localization phenomenon, were outlined in the Introduction chapter of this thesis.

For slow varying bottom topographies relative to the surface waves there are approximate theories, which are the geometrical optics approximation for progressive waves over a gradually varying bottom, the mild slope equation and the parabolic approximations of the water waves equations. A derivation of the equations of geometric optics for linear surface gravity waves through a perturbation analysis is given in Mei [44]. The merit of the ray approximation (Geometrical optics) is to reduce a three dimensional problem to a continuum of one-dimensional problems. Yet, near a caustic (regions where the geometrical optics theory breaks down), effects need to be taken into account transverse to a ray. Hence, the problem is locally two-dimensional. The mild-slope equation is an approximation that accounts for slow depth variations and allows rapid horizontal variations associated with diffraction, and was first obtained by Berkhoff [5]. References about other derivations of the same result are given in Mei [44]. Parabolic approximations were first used in the context of three dimensional water waves by Liu & Mei [39] and Rader [55]. The effects of current and variable bathymetry were explored by Booij [7], Liu [38] and Kirby [33]. Stochastic variations in the ocean depth were considered by Reeve [57].

Regarding applications of the concept of pseudo-differential operators and the method of ordered operators to water waves, there is an extensive literature. Dobrokhotov (1983) applied the method of order operators (see Maslov [42] and Maslov & Nazaikinskii (1988)) to find approximate (asymptotic) solutions of linear equations describing the propagation of gravitational waves on the surface of an ideal fluid (irrotational flow and incompressible inviscid fluid) in a basin with a variable, slowly changing bottom. He considered the evolution of a wave fronts and provided conditions (“quantization conditions”) to determine trapped modes for two-dimensional bottom topographies. Dobrokhotov & Zhevandrov (1985) is similar to Dobrokhotov (1983), but surface tension effects are included in the linearized free surface boundary conditions, and the Cherenkov radiation of gravity capillary waves is considered for a non-uniform slowly varying basin. In Dobrokhotov (1987), the method of ordered

operators is used to derive nonlocal analogous of the nonlinear Boussinesq equation for surface waves with allowance of all dispersive terms when the pool is no longer shallow, but the amplitudes of the wave field are still assumed small. Nonlocal pseudo-differential equations are given for the surface waves, including quadratic and cubic terms.

In this chapter we use the method of ordered operators to obtain pseudo-differential equations governing the interaction of linear surface waves with a slowly varying bottom. We consider one dimensional bottom topographies. We solve asymptotically the pseudo-differential equations. The asymptotic solution fails at discrete points (turning points) in complex coordinate variable plane. We take the effect of this points into account and obtain wave mode conversion (wave reflection is a particular case). For wave reflection this is not a new result. Meyer [46] obtained wave reflection for slowly varying one-dimensional analytical topographies by taking the effects of turning points of the governing equation into account. Our approach is different from his approach, and we recover his results and obtained wave mode conversion between the propagating wave modes and evanescent wave modes, which is a new result. So, we obtain an improved asymptotic description of the wave field.

4.2 Chapter Overview.

In section 4.3 we formulate the general boundary value problem for surface gravity waves interacting with a non-uniform bottom topography in the context of potential theory, which implies an irrotational flow and an inviscid fluid. We present the choice of the appropriate length and time scales to make the boundary value problem non-dimensional. We assume the wave steepness small to be able to linearize the boundary value problem, and we state our assumptions regarding the non-dimensional parameters.

In section 4.4 we apply the concept of pseudo-differential operator and the operator method approach (see Maslov [42] and Maslov and Nazaikinskii (1988)) to the non-dimensional boundary value problem for linear surface gravity waves interacting with

a variable bottom. In the first part of section 4.4 we follow closely the first section of Dobrokhotov (1983) and the second section of Dobrokhotov & Zhevandrov (1985). We assume the velocity field potential as the action of an unknown operator over the free surface potential (function only of the horizontal coordinate and time). We substitute this representation for the potential function into the boundary value problem for the linear surface gravity waves to obtain a boundary value problem in the vertical coordinate for the symbol of the unknown operator and a pseudo partial differential equation for the free surface potential. The boundary value problem for the symbol of the unknown operator is independent of the free surface potential, but the governing equation for the later is a function of the unknown operator. So, we solve the depth dependence (obtain the unknown operator) to obtain the governing equation for the free surface potential, which is the variable of interest.

In section 4.5, we construct an asymptotic solution for the free surface potential. We assume that the free surface potential is given as a modulated plane wave, with variable phase and amplitude. We assume incidence of mono-chromatic waves and one-dimensional bottom topographies (wave propagation only in the direction of the bottom variation), such that the governing equation for the free surface potential is an ordinary pseudo-differential equation in the space coordinate. We substitute the assumed form of the solution into the ordinary pseudo-differential equation for the free surface potential. This result in the Eikonal equation for the phase and in the transport equation for the wave amplitude. To solve the transport equation, we assume a series expansion in the parameter β for the wave amplitude, which result in a sequence of first order non-homogeneous differential equations for the terms in the series expansion of the wave amplitude. In section 4.5.1 we solve these equations in closed form up to first order in β . Solutions of the equations for the high order corrections of the wave amplitude can be obtained by numerical quadrature.

The Eikonal equation is actually the uniform bottom dispersion relation assumed locally valid. This is an implicit equation for the wave numbers which has an infinite enumerable set of solutions. The wave numbers are now functions of the space coordinate. To each solution of the Eikonal equation we have a transport equation.

Therefore, we have an enumerable base for the asymptotic solution. Which functions of these enumerable base are used to represent the asymptotic solution depend on the radiation conditions for the problem. We denote the functions of the enumerable base of the asymptotic solution as the Liouville-Green functions. If we assume the space coordinate and the wave numbers as complex variables, the dispersion relation can be considered as a mapping from the space coordinate complex plane into the wavenumber complex plane. This mapping has branch points in the space coordinate complex plane, and their image in the wavenumber complex plane are the saddle points. In section 4.5.3 we obtain the branch points of the uniform bottom dispersion relation assumed locally valid. We also give the saddle points associated with these branch points. We show that the set of branch points is infinite enumerable. We actually evaluate a finite number of branch points. At the branch points, groups of wavenumbers coalesce to one or more branch points (different groups of wave numbers coalesce to different saddle points). This is discussed in section 4.5.3.1. So, at the branch points groups of Liouville-Green functions became linearly dependent and we lose part of the base for our asymptotic solution. To continue the asymptotic solution through the turning points (branch point of the dispersion relation), we need to develop connection formulas, the same way we did in the second part of chapter 3.

For the linear surface waves interacting with a variable bottom, the wave numbers coalesce in pairs at the branch points of the dispersion relation (see section 4.5.3.1), so we use ideas similar to the ideas proposed in Fuchs, Ko & Bers [22] and in Fuchs, Bers & Harten [21] to perform the connection problems we may have. This is discussed in sections 4.5.3 and 4.5.4. After we solved the connection problem for each relevant turning point (the turning points closest to the real axis), we model the wave mode conversion (wave reflection is a special case) as a transfer matrix, relating the amplitude of the wave modes “before” and “after” the considered turning point.

The result of this asymptotic approach is a sequence of transfer matrices, relating the amplitude of the various wave modes participating in the wave interaction problem. We give the global system transfer matrix, which is the product of the local transfer matrices. We consider applications of the asymptotic theory to simple

analytic one-dimensional bottom topographies. We compare the asymptotic results with direct numerical simulations.

4.3 Linear Surface Gravity Waves Boundary Value Problem.

We consider surface gravity waves propagating over a non-uniform bottom topography. We assume the flow irrotational and the fluid incompressible and inviscid, such that the velocity field can be described as the gradient of a potential function $\phi(x, y, t)$. In the vertical direction, the fluid domain is bounded by the free surface which has its position given by the equation $y = \eta(x, t)$, and by the bottom which has its position given by the equation $y = -h(x)$. In the horizontal direction the fluid domain is of infinite size. The potential function $\phi(x, y, t)$ satisfies Laplace equation (4.1) in the fluid domain. At the free surface we have the kinematic boundary condition (4.2), which expresses the fact that the free surface is a material contour. A fluid particle at the free surface can move only tangentially to it. On the free surface (air-water interface), both η and ϕ are unknown, and it is necessary to add a dynamical boundary condition concerning forces, given by equation (4.3). The bottom is also a material contour, so only tangential motion to it is allowed, which is expressed by the boundary condition (4.4). The equations describing the boundary value problem are

$$\nabla^2 \phi(x, y, t) = 0 \text{ for } -\infty < x < \infty \text{ and } -h(x) < y < \eta(x, t), \quad (4.1)$$

$$\phi_y - \eta_t - \phi_x \eta_x = 0 \text{ at } y = \eta(x, t), \quad (4.2)$$

$$\phi_t + \frac{1}{2}(\phi_x)^2 + g\eta(x, t) = 0 \text{ at } y = \eta(x, t), \quad (4.3)$$

$$\phi_y - H_x \phi_x = 0 \text{ at } y = -h_0 + H(x), \quad (4.4)$$

where h_0 is the average depth of the fluid domain and $H(x)$ accounts for the depth variation. We shall study surface waves with characteristic wavelength λ and char-

acteristic amplitude η_0 at the free surface in a domain with characteristic horizontal length scale Λ (wavelength of the bottom non-uniformity variation). The depth variation has a length scale A . All the length scales mentioned above are illustrated in figure 4-1 below.

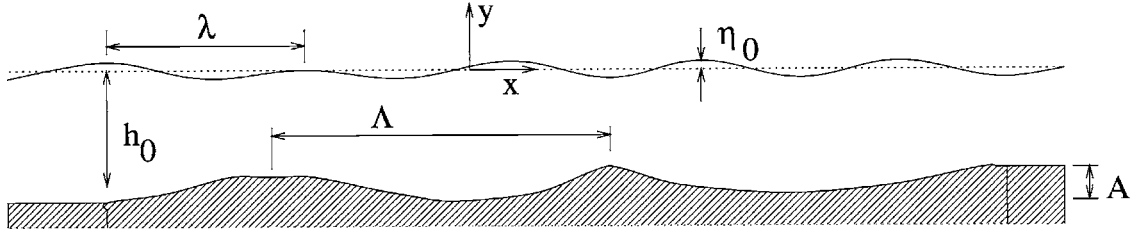


Figure 4-1: Example of a one-dimensional non-uniform bottom topography.

The non-dimensional variables are:

$$x' = x/\Lambda,$$

$$y' = y/\lambda,$$

$$\eta' = \eta/\eta_0,$$

$$H' = H/A,$$

$$t' = t/(\Lambda/\sqrt{g\lambda}),$$

$$\phi' = \phi/(\eta_0\sqrt{g\lambda}).$$

The non-dimensional form of the boundary value problem (4.1)-(4.4) is given by the set of equations below. For simplicity the symbol (') over the non-dimensional variables was omitted.

$$\beta^2 \phi_{xx} + \phi_{yy} = 0 \text{ for } -\infty < x < \infty \text{ and } -\gamma + \frac{\epsilon}{\beta} H(x) < y < \delta \eta(x, t), \quad (4.12)$$

$$\phi_y - \beta \eta_t - \delta \beta \phi_x \eta_x = 0 \text{ at } y = \delta \eta(x, t), \quad (4.13)$$

$$\phi_t + \frac{1}{2} \delta \beta (\phi_x)^2 + \frac{1}{\beta} \eta(x, t) = 0 \text{ at } y = \delta \eta(x, t), \quad (4.14)$$

$$\phi_y - \epsilon \beta H_x \phi_x = 0 \text{ at } y = -\gamma + \frac{\epsilon}{\beta} H(x). \quad (4.15)$$

We define the non-dimensional parameters β , ϵ , δ and γ according to the equations

$$\beta = \frac{\lambda}{\Lambda}, \quad (4.16)$$

$$\epsilon = \frac{A}{\Lambda}, \quad (4.17)$$

$$\delta = \frac{\eta_0}{\lambda}, \quad (4.18)$$

$$\gamma = \frac{h_0}{\lambda}. \quad (4.19)$$

The parameter β is the ratio between the length scale of the wave disturbance and the length scale of the bottom variation. The parameter ϵ gives the steepness of the bottom topography, the parameter δ gives the magnitude of the steepness of the free surface and γ is the ratio between the average depth and the length scale of the surface waves. We assume that

$$\delta \ll 1, \quad (4.20)$$

$$\epsilon < 1, \quad (4.21)$$

$$\beta < 1, \quad (4.22)$$

$$\gamma < 1/2. \quad (4.23)$$

Assumption (4.23) (finite depth regime) implies that the surface wave feels the pres-

ence of the bottom, and assumptions (4.21) and (4.22) implies that the surface waves feels the bottom variation. We consider waves with small steepness ($\delta \ll 1$), such that we can linearize the free surface boundary conditions. We expand the quantity $\phi(x, y, t)$ at the free surface boundary conditions (4.13) and (4.14) with respect to the y variable around $y = 0$ and we disregard terms of $O(\delta^2)$. The boundary value problem (4.12)-(4.15) assumes the form

$$\beta^2 \phi_{xx} + \phi_{yy} = 0 \text{ for } -\infty < x < \infty \text{ and } -\gamma + \frac{\epsilon}{\beta} H(x) < y < 0, \quad (4.24)$$

$$\phi_y - \beta \eta_t = 0 \text{ at } y = 0, \quad (4.25)$$

$$\beta \phi_t + \eta = 0 \text{ at } y = 0, \quad (4.26)$$

$$\phi_y - \epsilon \beta H_x \phi_x = 0 \text{ at } y = -\gamma + \frac{\epsilon}{\beta} H(x). \quad (4.27)$$

The boundary value problem (4.24)-(4.27) is composed of linear equations. The difficulty in solving this boundary value problem lays in the boundary condition at the bottom, which do not allow us to solve this boundary value problem in closed form since the bottom geometry is a non-trivial known surface. To make further progress we assume that the flow potential can be written as the action of an unknown operator $\hat{R}$ over the free surface potential $\psi(x, t)$.

$$\phi(x, y, t) = R(\overset{2}{x}, y, -i\beta \frac{\partial}{\partial x}, \overset{1}{p}, \beta) \psi(x, t), \quad (4.28)$$

where $R(\overset{2}{x}, y, \overset{1}{p}, \beta)$ is the assumed form for the symbol of the operator $\hat{R}$ and $p = -i\beta \frac{\partial}{\partial x}$.

We substitute this representation for the potential function into the boundary value problem (4.24)-(4.27). This leads to a boundary value problem in terms of the vertical coordinate y only for the symbol of the unknown operator $\hat{R}$. The horizontal coordinate x and time t appear only as parameters. This boundary value problem for the symbol of the unknown operator $\hat{R}$ is independent of the free surface potential

$\psi(x, t)$, but the governing equation for the later is a function of the unknown operator. So, in the next section, we solve the depth dependence through an asymptotic expansion of the symbol of the unknown operator $\hat{R}$ in terms of the parameter β to obtain the governing equation of the free surface potential $\psi(x, t)$, which is the variable of interest.

4.4 Operator Method Approach for Linear Surface Gravity Waves.

In this section we use the ordered operator method approach (see Maslov [42] and Maslov and Nazaikinskii (1988)) to obtain pseudo-differential equations governing the interaction of linear surface gravity waves with a variable bottom. We follow closely the first section of Dobrokhotov (1983) and the second section of Dobrokhotov & Zhevandrov (1985). We assume the velocity field potential in the form (4.28), and we substitute this representation for the potential function into the boundary value problem (4.24)-(4.27), which result, as mentioned before, in a boundary value problem for the symbol of the unknown operator $\hat{R}$ and in a pseudo-partial differential equation for the free surface potential. The boundary value problem (4.24)-(4.27) assumes the form

$$\left[- \left(\overset{1}{p} - i\beta \overset{3}{\frac{\partial}{\partial x}} \right)^2 R(\overset{2}{x}, y, \overset{1}{p}, \beta) + \frac{\partial^2}{\partial y^2} R(\overset{2}{x}, y, \overset{1}{p}, \beta) \right] \psi(x, t) = 0 \text{ for } -\infty < x < +\infty$$

and $-\gamma + \frac{\epsilon}{\beta} H(x) < y < 0,$

(4.29)

$$\beta^2 R(\overset{2}{x}, y, \overset{1}{p}, \beta) \frac{\partial^2 \psi}{\partial t^2}(x, t) + \frac{\partial R}{\partial y}(\overset{2}{x}, y, \overset{1}{p}, \beta) \psi(x, t) = 0 \text{ at } y = 0, \quad (4.30)$$

$$\left[\frac{\partial R}{\partial y}(\overset{2}{x}, y, \overset{1}{p}, \beta) - i\beta \epsilon \frac{dH}{dx} (\overset{1}{p} - i\beta \overset{3}{\frac{\partial}{\partial x}}) R(\overset{2}{x}, y, \overset{1}{p}, \beta) \right] \psi(x, t) = 0 \text{ at } y = -\gamma + \frac{\epsilon}{\beta} H(x).$$

(4.31)

We assume that the free surface potential $\psi(x, t)$ satisfies a pseudo-partial differential equation of the form

$$\beta^2 \psi_{tt} + L(\overset{2}{x}, -i\beta \overset{1}{\frac{\partial}{\partial x}}, \beta) \psi = 0, \quad (4.32)$$

where $\hat{L}$ is an unknown operator with symbol $L(\overset{2}{x}, \overset{1}{p}, \beta)$. We also assume that the symbol of the operator $\hat{L}$ is related to the symbol of the operator $\hat{R}$ according to the equation

$$L(\overset{2}{x}, \overset{1}{p}, \beta) = \frac{\partial}{\partial y} R(\overset{2}{x}, 0, \overset{1}{p}, \beta). \quad (4.33)$$

Now we substitute equations (4.32) and (4.33) into the boundary value problem (4.29)-(4.31) involving the symbol of the operator $\hat{R}$ and the free surface potential $\psi(x, t)$. This results in a boundary value problem for the symbol of the operator $\hat{R}$ in the vertical coordinate y . The boundary value problem has the form

$$-p^2 R + i2\beta p \frac{\partial}{\partial x} R + \beta^2 \frac{\partial^2}{\partial x^2} R + \frac{\partial^2 R}{\partial y^2} = 0 \text{ for } -\gamma + \frac{\epsilon}{\beta} H(x) < y < 0, \quad (4.34)$$

$$R = 1 \text{ at } y = 0, \quad (4.35)$$

$$R_y - i\epsilon(H_x \cdot p)R - \epsilon\beta H_x \cdot \frac{\partial}{\partial x} R = 0 \text{ at } y = -\gamma + \frac{\epsilon}{\beta} H(x). \quad (4.36)$$

This is a boundary value problem in the y variable with the variables x and t as parameters. We assume a series expansion for the symbol of the operators $\hat{L}$ and $\hat{R}$ in the parameter β . We have

$$R(\overset{2}{x}, y, \overset{1}{p}, \beta) = \sum_{n=0}^{\infty} (-i\beta)^n R_n(\overset{2}{x}, y, \overset{1}{p}), \quad (4.37)$$

$$L(\overset{2}{x}, \overset{1}{p}, \beta) = \sum_{n=0}^{\infty} (-i\beta)^n L_n(\overset{2}{x}, \overset{1}{p}). \quad (4.38)$$

We also make the assumption that the bottom non-uniformity variation with respect to the surface wave is of the same order of magnitude of the bottom steepness. In other words, we assume that

$$\epsilon \sim \beta. \quad (4.39)$$

We substitute the series expansion (4.37) of the symbol of the operator $\hat{R}$ in the boundary value problem (4.34)-(4.36) with the assumption (4.39) in mind. This results in a sequence of boundary value problems for the symbols $R_n(\overset{2}{x}, y, \overset{1}{p})$.

- To order $O(1)$ in the parameter β we have the boundary value problem

$$(R_0)_{yy} - p^2 R_0(x, y, p) = 0, \quad (4.40)$$

$$R_0 = 1 \text{ at } y = 0, \quad (4.41)$$

$$(R_0)_y = 0 \text{ at } y = -\gamma + H(x). \quad (4.42)$$

- To order $O(\beta)$ we have the boundary value problem

$$(R_1)_{yy} - p^2 R_1(x, y, p) = 2p \frac{\partial}{\partial x} R_0, \quad (4.43)$$

$$R_1 = 0 \text{ at } y = 0, \quad (4.44)$$

$$(R_1)_y = -H_x p R_0 \text{ at } y = -\gamma + H(x). \quad (4.45)$$

- To order $O(\beta^n); n = 2, 3, \dots$ we have the boundary value problem

$$(R_n)_{yy} - p^2 R_n(x, y, p) = 2p \frac{\partial}{\partial x} R_{n-1} + \frac{\partial^2}{\partial x^2} R_{n-2}, \quad (4.46)$$

$$R_n = 0 \text{ at } y = 0, \quad (4.47)$$

$$(R_n)_y = -H_x p R_{n-1} - H_x \frac{\partial}{\partial x} R_{n-2} \text{ at } y = -\gamma + H(x). \quad (4.48)$$

The boundary value problem of $O(1)$ in the parameter β , given by equations (4.40)-(4.42), has the solution

$$R_0(x, y, p) = A_0^0(x, p) \cosh(p(y + h(x))), \quad (4.49)$$

where

$$A_0^0(x, p) = \frac{1}{\cosh(ph(x))}. \quad (4.50)$$

The boundary value problem for the symbol $R_n(x, y, p)$ depends on the boundary value problems for $R_{n-1}(x, y, p)$ and $R_{n-2}(x, y, p)$. We assume a solution for the

symbol $R_n(x, y, p)$ in the form

$$R_n(x, y, p) = \sum_{m=0}^n \{A_m^n(x, p)y^m \cosh(p(y + h(x))) + B_m^n(x, p)y^m \sinh(p(y + h(x)))\}, \quad (4.51)$$

where the recurrence relation for the symbols A_m^n and B_m^n with $m = 0, \dots, n$ and their expressions are given in the appendix J. We also give maple scripts in the Appendix J to evaluate the expressions for the symbols A_m^n and B_m^n . The symbol of the operator $\hat{L}$ is given in terms of the symbol of the operator $\hat{R}$ according to the equation (4.33). This equation plus equation (4.51) implies that the symbols $L_n(x, p)$ are given by the equation

$$\begin{aligned} L_n(x, p) &= \frac{\partial}{\partial y} R_n(x, y, p) \\ &= A_0^n(x, p)p \sinh(ph(x)) + B_0^n(x, p)p \cosh(ph(x)) \\ &\quad + A_1^n(x, p) \cosh(ph(x)) + B_1^n(x, p) \sinh(ph(x)). \end{aligned} \quad (4.52)$$

The asymptotic expansion of the symbol $L(x, p, \beta)$ in terms of the parameter β is now given in the form

$$\begin{aligned} L(x, p, \beta) &= p \tanh(ph(x)) + \sum_{n=1}^{\infty} (-i\beta)^n \{A_1^n(x, p) \cosh(ph(x)) + B_1^n(x, p) \sinh(ph(x)) \\ &\quad + p [A_0^n(x, p) \sinh(ph(x)) + B_1^n(x, p) \cosh(ph(x))]\}. \end{aligned} \quad (4.53)$$

We assume time dependence of the form $\exp(-i\frac{\alpha}{\beta}t)$, where $\alpha = \omega \frac{\Lambda}{\sqrt{g\lambda}}\beta$ and ω is the wave frequency in radians per second. The governing equation (4.32) for the free surface potential $\psi(x, t)$ now assumes the form

$$\left\{ -\alpha^2 + L(\overset{2}{x}, -i\beta \frac{\partial}{\partial x}, \beta) \right\} \psi(x) = 0. \quad (4.54)$$

This equation is an ordinary pseudo-differential equation for the free surface potential $\psi(x)$. In the next section we construct an asymptotic solution for this equation.

4.5 Asymptotic Solution Through a High Order WKB Approach.

We assume that the free surface potential $\psi(x, t)$ is given as a modulated plane wave, with variable phase and amplitude. We substitute the assumed form of the solution into the ordinary pseudo-differential equation for the free surface potential. This result in the Eikonal equation for the phase and in the transport equation for the wave amplitude. To solve the transport equation, we assume a series expansion in the parameter β for the wave amplitude, which result in a sequence of first order non-homogeneous differential equations for the terms in the series expansion of the wave amplitude. In section 4.5.1 we solve these equations in closed form up to first order in β . Solutions of the high order transport equations can be obtained by numerical quadrature. The assumed form of the asymptotic solution for the pseudo-differential equation (4.54) is

$$\psi(x) = \exp(\frac{i}{\beta} S(x)) w(x). \quad (4.55)$$

If we substitute equation (4.55) into the pseudo-differential operator (4.54), we obtain

$$\left\{ -\alpha^2 + L(\overset{3}{x}, \frac{d}{dx} \overset{2}{S}(x) - i\beta \frac{d}{dx} \overset{1}{S}(x), \beta) \right\} w(x) = 0. \quad (4.56)$$

The term $\frac{dS}{dx}$ in the expression for the symbol of the operator $\hat{L}$ is due to the action of the operator $\hat{L}$ over the exponential in the assumed form for the free surface potential $\psi(x, t)$, given by equation (4.55). We expand equation (4.56) in terms of the parameter β . Notice that the symbol of the operator $\hat{L}$ has a series expansion in the parameter β , given by equation (4.53).

4.5.1 The Eikonal Equation and the Transport Equation.

The term of order $O(1)$ in the expansion mentioned above gives the Eikonal equation for the phase $S(x)$, which is given by the equation

$$-\alpha^2 + L_0(x, \frac{\partial}{\partial x} S(x)) = 0, \quad (4.57)$$

or

$$-\alpha^2 + k(x) \tanh(k(x)h(x)) = 0 \text{ with } k(x) = \frac{\partial}{\partial x} S(x). \quad (4.58)$$

The transport equation for the wave amplitude $w(x)$ is the sum of the terms of order $O(\beta^1)$ and higher in the expansion mentioned above. The transport equation has the form

$$\begin{aligned} \sum_{l=1}^{\infty} (-i\beta)^n & \left(\sum_{n=0}^{l-1} \left\{ \sum_{k=0}^{(l-n)-1} \left(\sum_{m=1}^{(l-k)-n} \left[\sum_{\substack{j_m + \dots + j_1 = (l-k)-n+m \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k! j_m! \dots j_1! m!} \right. \right. \right. \right. \\ & \left. \left. \left. \frac{\partial^{l+m-n}}{\partial p^{l+m-n}} L_n(x, p) \right|_{p=DS} D^{j_l} S \times \dots \times D^{j_1} S \times D^p \right] \right) \frac{d^k}{dx} w(x) \right. \\ & \left. + \frac{1}{(l-n)!} \frac{\partial^{(l-n)}}{\partial p^{(l-n)}} L_n(x, p) \right|_{p=DS} \frac{d^{(l-n)}}{dx} w(x) \Big\} + L_l(x, DS) w(x) \Big) = 0. \end{aligned} \quad (4.59)$$

The derivation of the transport equation for the amplitude $w(x)$ is discussed in detail in the Appendix L. We also provide a maple procedure to compute symbolically any

term of the series in the parameter β of the operator $\hat{L}$.

To solve the transport equation we assume a series expansion of the wave amplitude $w(x)$ in terms of the parameter β . We give the amplitude $w(x)$ in the form

$$w(x) = w_0(x) \left\{ 1 + \sum_{n=1}^{\infty} (-i\beta)^n w_n(x) \right\}. \quad (4.60)$$

We substitute the expansion for the wave amplitude $w(x)$ into the transport equation (4.59). This leads to a sequence of first order transport equations for the wave amplitude corrections $w_n(x)$, $n = 0, 1, 2, \dots$. In the Appendix L we discuss in detail how to derive the transport equations for the amplitude corrections $w_n(x)$. We also provide a maple procedure to compute symbolically the transport equation for any $w_n(x)$. Here we give only the final result. The transport equation for $w_0(x)$ is

$$\frac{\partial}{\partial p} L_0(x, p)|_{p=DS} \frac{d}{dx} w_0(x) + \left(\frac{1}{2!} \frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=DS} D^2 S + L_1(x, DS) \right) w_0(x) = 0. \quad (4.61)$$

The transport equations of order $O(\beta^N)$ can be written as non-homogeneous first order differential equations of the form

$$\begin{aligned}
\frac{d}{dx}w_{N-1}(x) = & -\frac{1}{\frac{\partial}{\partial p}L_0(x,p)|_{p=DS}w_0(x)} \left[\sum_{l=2}^{N-1} \left\{ \sum_{n=0}^{l-1} \left(\sum_{k=0}^{l-1-n} \left[\sum_{j=0}^k \left\{ \sum_{m=1}^{l-k-n} \left(\sum_{\substack{j_m+\dots+j_1=l-k+m-n \\ j_m,\dots,j_1 \geq 2}} \right. \right. \right. \right. \right. \right. \\
& \left. \left. \left. \frac{1}{(k-j)!j!j_m!\dots j_1!m!} \frac{\partial^{l+m-n}}{\partial p^{l+m-n}} L_n(x,p)|_{p=DS} D^{j_m}S \times \dots \times D^{j_1}S \right) \right\} \right. \\
& \left. \frac{d^{k-j}}{dx^{k-j}}w_0(x) \frac{d^j}{dx^j}w_{N-l}(x) \right] + \sum_{j=0}^{l-n} \left[\frac{1}{(l-n-j)!j!} \frac{\partial^{l-n}}{\partial p^{l-n}} L_n(x,p)|_{p=DS} \right. \\
& \left. \frac{d^{l-n-j}}{dx^{l-n-j}}w_0(x) \frac{d^j}{dx^j}w_{N-l}(x) \right] \left. \right\} + \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1-n} \left[\sum_{m=1}^{N-k-n} \left(\sum_{\substack{j_m+\dots+j_1=N-k+m-n \\ j_m,\dots,j_1 \geq 2}} \right. \right. \right. \right. \\
& \left. \left. \left. \frac{1}{k!j_m!\dots j_1!m!} \frac{\partial^{N+m-n}}{\partial p^{N+m-n}} L_n(x,p)|_{p=DS} D^{j_m}S \times \dots \times D^{j_1}S \right) \right] \frac{d^k}{dx^k}w_0(x) \right. \\
& \left. + \frac{1}{(N-n)!} \frac{\partial^{N-n}}{\partial p^{N-n}} L_n(x,p)|_{p=DS} \frac{d^{N-n}}{dx^{N-n}}w_0(x) \right) + L_N(x,DS) \Big].
\end{aligned} \tag{4.62}$$

For this transport equation to be useful, we need the expression for the symbols $L_n(x, p)$. The expression for these symbols is given in the appendix K. The transport equation (4.61) can be solved in closed form. The wave amplitude $w_0(x)$ is given by the equation

$$w_0(x) = \left\{ \left(\frac{\partial}{\partial p} L_0(x, p) \right) \Big|_{p=DS(x)} \right\}^{-1/2}, \tag{4.63}$$

where

$$\left(\frac{\partial}{\partial p} L_0(x, p) \right) \Big|_{p=DS(x)} = \frac{1}{2} \frac{\sinh(2h(x)DS) + 2DS h(x)}{\cosh^2(h(x)DS)}. \tag{4.64}$$

Notice that the transport equation (4.63) is a function of the phase function $S(x)$ and their derivatives. The Eikonal equation (4.57), which gives the phase function

$S(x)$, has an enumerable set of solutions. So, we attach to the phase function an index j to denote the different solutions $S_j(x)$ of the Eikonal equation. Once we obtain a solutions $S_j(x)$ of the Eikonal equation, we can compute its derivatives in an iterative way, illustrated in the Appendix L. For each solution $S_j(x)$, we have an amplitude function $w(x)$, now denoted $w_j(x)$, and the amplitude corrections are now $w_{j,n}(x)$. Therefore, the asymptotic solution of the equation (4.54) has an enumerable basis. The boundary or radiation conditions imposed and the wave mode coupling which occur along the medium are going to define which elements of this enumerable base are going to form the asymptotic solution of the equation (4.54). The wave mode coupling is due to the presence of turning points of the asymptotic solution in the complex x plane. At these points two or more wave numbers dS_j/dx assume the same value, so we lose two or more elements from the base used to construct the asymptotic expansion of the equation (4.54). The turning points of the asymptotic solution of the equation (4.54) are located in the complex x plane, at some distance from the real axis where the wave propagates. If we do not take these points into account, the asymptotic expansion discussed in this section gives only wave refraction. To incorporate wave reflection and wave mode conversion we need to take the effects of the turning points into account. Since the asymptotic expansion breaks down at these points, we find an approximate form for the symbol of the operator $\hat{L}$ in equation (4.54) valid in a neighborhood of the considered turning point. We reduce the symbol of this operator to the symbol of an ordinary differential equation of the order equal to the number of the wave modes that couple at the considered turning point. The solution of this locally valid differential equation allow us to continue the asymptotic solution through the considered turning point and to capture wave reflection and wave mode conversion.

This approach is closely related to the ideas described in Fuchs, Ko & Bers [22] and in Fuchs, Bers & Harten [21]. They extracted from the uniform system dispersion relation (Eikonal equation, where dS/dx is the wavenumber), assumed locally valid in the context of the WKB method, an ordinary differential equation describing the wave mode interaction in a neighborhood of a turning point. Their local equation

is basically the same local equation we obtained, since the symbol of a differential operator is intimately related to the dispersion relation of the ordinary or partial differential equation the operator is associated with. For example, the principal part of the symbol of the operator $\hat{L}$, denoted as $L_0(x, p)$, plus $-\alpha^2$ is the uniform system dispersion relation, and the way they approximate the dispersion relation close to one of its branch points to extract the local equation is basically the same way we approximate the symbol $L(x, p, \beta)$ to obtain the local equation.

The first step in handling the turning points is to obtain them. The Eikonal equation is basically the uniform bottom dispersion relation, assumed locally valid. It relates the wavenumber dS/dx to the space coordinate x through the depth function $h(x)$, so we can see this dispersion relation as a mapping between the complex x plane and the wavenumber complex plane. The turning points are the branch points of this dispersion relation, and their image in the wavenumber complex plane are the saddle points to which the wave numbers dS_j/dx coalesce as we approach a turning point. We elaborate further this topic in the next section.

Similar to what we observed in chapter 3, the transport equation (4.62) for the amplitude corrections $w_{j,n}(x)$ for $n \geq 2$ can be written in the form

$$\frac{dw_{j,N-1}}{dx} = \frac{1}{(N-2)!} (w_{j,1}(x))^{(N-2)} \frac{dw_{j,1}}{dx} + G(w_{j,n-2}, \dots, w_{j,0}, \frac{dS_j}{dx}, x), \quad (4.65)$$

which implies that the amplitude correction $w_{j,1}(x)$ is actually a phase correction. Instead of a solution of the form (4.55) for the equation (4.54), we may assume another form, which may be more convenient according to the application in mind. Instead of amplitude corrections, we consider phase corrections, so we have

$$y(x) = v_0(x) \exp \left(\frac{i}{\beta} S(x) + \sum_{n=1}^{\infty} (-i\beta)^n v_n(x) \right). \quad (4.66)$$

If we expand the expression

$$\exp \left(\sum_{n=1}^{\infty} (-i\beta)^n v_n(x) \right)$$

in Taylor series with respect to the parameter β , and compare with equation (4.60) we realize the relations

$$\begin{aligned} v_0(x) &= w_0(x), \\ v_1(x) &= w_1(x), \\ v_2(x) &= w_2(x) - \frac{1}{2}(v_1(x))^2, \\ &\dots\dots \\ v_n(x) &= w_n(x) - \frac{1}{n!} \left\{ \sum_{l=2}^{n-1} \left(\sum_{j_l+\dots+j_1=n} (j_1! \dots j_l!) v_{j_l} \dots v_{j_1} \right) \right\}. \end{aligned} \quad (4.67)$$

Therefore, if we obtain the amplitude corrections $w_{j,n}(x)$ by integrating the transport equation (4.62), we can use the relations (4.67) to compute the phase corrections $v_{j,n}(x)$. The phase function $S(x)$ in both representations is a solution of the Eikonal equation (4.57).

4.5.2 Analysis of the Uniform Bottom Dispersion Relation.

As we mentioned before, the asymptotic expansion built in the previous section breaks down at the turning points, which are important since we are able to capture wave mode conversion in the context of the WKB method by taking these points into account in our analysis. The first step in taking the effect of the turning points into account is to find these points in the complex x plane. Then for each turning point we analyze how the wave numbers dS/dx coalesce at the turning points. This analysis tell us how the wave mode couple at the turning points, but it does not reveal how the wave mode amplitudes change due to the wave mode coupling. To answer this question we find an approximate form for the symbol of the operator $\hat{L}$, present in

equation (4.54), which is valid in a neighborhood of the considered turning point. We reduce the symbol of this operator to the symbol of an ordinary differential equation of the order equal to the number of wave modes that couple at the considered turning point. This ordinary differential equation describes the wave mode coupling in a neighborhood of the considered turning. These points are addressed in this section.

Before starting our analysis, we consider a change of notation. We use just k_j instead of dS_j/dx to label the wave numbers of the uniform system dispersion relation, now written in the form

$$-\alpha^2 + L_0(x, k) = 0 \text{ or } -\alpha^2 + k \tanh(kh(x)) = 0 \quad (4.68)$$

We assume x and k as complex variables. The wave numbers k_j are implicit solutions of the equation (4.68), and are functions of the x variable. So, the uniform system dispersion relation can be seen as a mapping from the complex x plane into the complex k plane, and the solutions $k_j(x)$ are the many branches of this implicit mapping. The branches $k_j(x)$ may coalesce at the branch points of the implicit mapping given by the uniform depth dispersion relation. Then the branch points of the dispersion relation are the turning points of the asymptotic solution built in the previous section, and they are solutions of the set of equations

$$\begin{aligned} -\alpha^2 + L_0(x, k) = 0 \text{ or } & -\alpha^2 + k \tanh(kh(x)) = 0, \\ \frac{\partial L_0}{\partial k}(x, k) = 0 \text{ or } & -\tanh(kh(x)) - kh(x)(1 - \tanh^2(kh(x))) = 0. \end{aligned} \quad (4.69)$$

These two equations can be reduced to a single equation through the change of variable

$$\sigma = \frac{\alpha^2}{k}. \quad (4.70)$$

If substitute (4.70) in equation (4.68), we obtain the identity

$$\sigma = \tanh(kh(x)). \quad (4.71)$$

With this identity we can write the equation (4.69) in the form

$$kh(x) = \frac{\sigma}{\sigma^2 - 1}. \quad (4.72)$$

Now, if we substitute (4.72) into the identity (4.71) we finally obtain a transcendental equation for σ , which is the equation

$$\sigma + \tanh\left(\frac{\sigma}{\sigma^2 - 1}\right) = 0. \quad (4.73)$$

The strategy to obtain the branch points in the x plane and their images, the saddle points in the complex k plane, is described as follows: First, we solve equation (4.73) in the complex σ plane. Second, we use the change of variable (4.70) to obtain the saddle points in the complex k plane. Third, with equations (4.70) and (4.72), we obtain

$$h(x) = \frac{1}{\alpha^2} \frac{\sigma^2}{\sigma^2 - 1}, \quad (4.74)$$

which give the branch point in the complex h plane. Once the depth function $h(x)$ is given, the branch points are given as the solutions in the complex x plane of the implicit equation (4.74).

Equation (4.73) has an enumerable set of solutions in the complex σ plane. This translates into an enumerable set of saddle points in the complex k plane and an enumerable set of branch points in the complex h plane. We use the depth h as an intermediate variable, since the depth function is not yet chosen. The most simple solution of equation (4.73) is $\sigma = 0$. This solution corresponds to a branch point at the origin of the complex h plane, and its image in the complex k plane lies in the infinite. In figure 4-2 we display part of the branch points of the locally valid uniform

depth dispersion relation (4.68). The branch points are pairs of complex conjugate numbers. We numerate the branch points by their closeness to the real axis and the sign of their imaginary part. For positive (negative) imaginary part, we assign an odd (even) integer. This results in the enumeration displayed in figure 4-2.

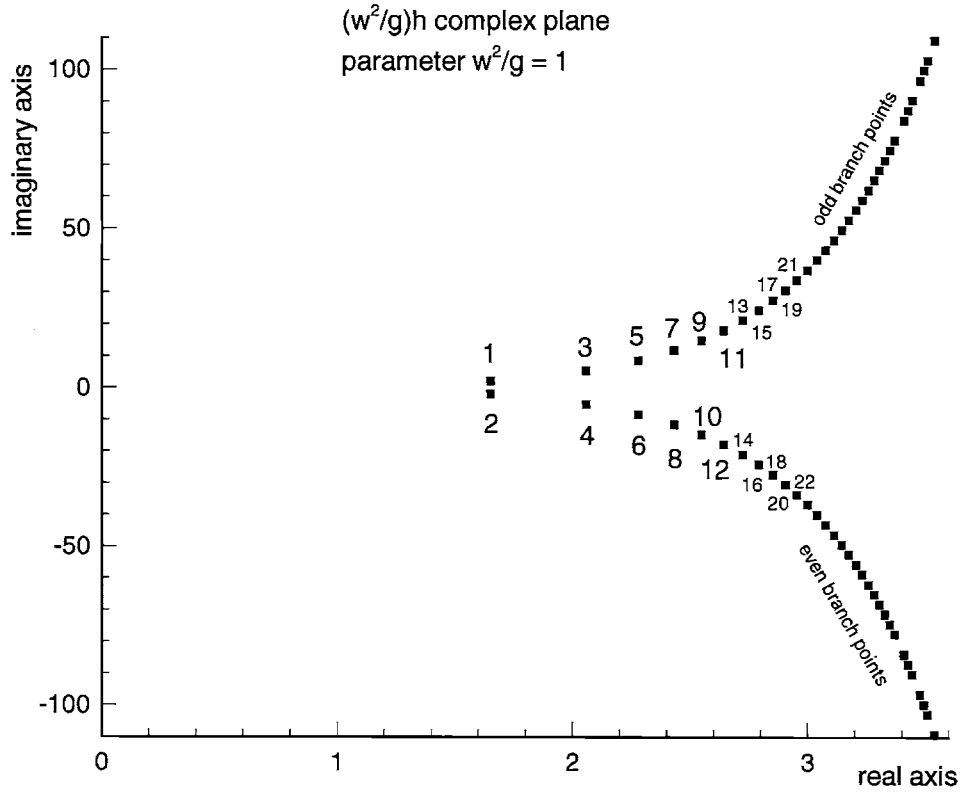


Figure 4-2: Enumerated branch points in the complex h plane. Branch point at the origin not displayed.

The image of the branch points through (4.68) in the complex k plane are the saddle points displayed in figure 4-3. The saddle points carry the number of the branch points they are image of. We see that to each branch point corresponds a pair of saddle points, as shown in figure 4-3.

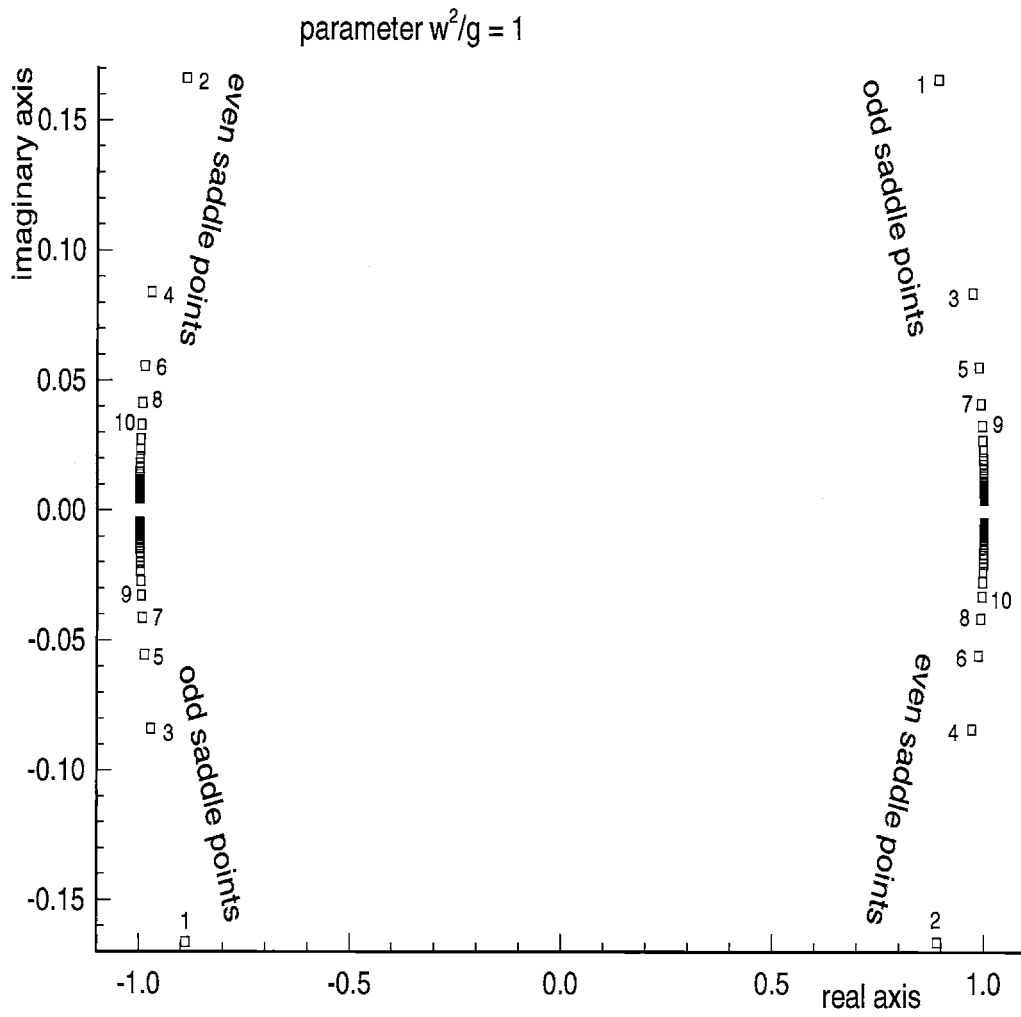


Figure 4-3: Saddle points in the complex wavenumber k plane, which are images of the enumerated branch points in the complex h plane.

4.5.2.1 Wave Mode Interaction Pattern.

Here we discuss how to obtain the interaction pattern of the wave modes which couple at each branch point of the dispersion relation. First, we see how many wave modes couple at a given branch point. We take a look on the higher order derivatives of (4.68) with respect to the wavenumber k . We have

$$\frac{\partial^2 L_0}{\partial p^2}(x, k) = -2(1 - \tanh^2(kh))h + 2k \tanh(kh)(1 - \tanh^2(kh))h^2, \quad (4.75)$$

$$\begin{aligned} \frac{\partial^3 L_0}{\partial p^3}(x, k) &= 6 \tanh(kh)(1 - \tanh^2(kh))h^2 + 2k(1 - \tanh^2(kh))^2 h^3 \\ &\quad - 4k \tanh^2(kh)(1 - \tanh^2(kh))h^3, \end{aligned} \quad (4.76)$$

$$\begin{aligned} \frac{\partial^4 L_0}{\partial p^4}(x, k) &= 8(1 - \tanh^2(kh))^2 h^3 - 16 \tanh^2(kh)(1 - \tanh^2(kh))h^3 \\ &\quad - 16k \tanh(kh)(1 - \tanh^2(kh))^2 h^4 \\ &\quad + 8k \tanh^3(kh)(1 - \tanh^2(kh))h^4, \end{aligned} \quad (4.77)$$

and if the pair (h, k) is a branch point in the complex h plane plus one of its image, these equations assume the form:

$$\frac{\partial^2 L_0}{\partial p^2}(x_b, k_s) = -2 \frac{g}{\alpha^2} \frac{\sigma_s^2}{\sigma_s^2 - 1}, \quad (4.78)$$

$$\frac{\partial^3 L_0}{\partial p^3}(x_b, k_s) = 4\sigma_s \left[\frac{g}{\alpha^2} \frac{\sigma_s^2}{\sigma_s^2 - 1} \right]^2, \quad (4.79)$$

$$\frac{\partial^4 L_0}{\partial p^4}(x_b, k_s) = -8(2\sigma_s^2 - 1) \left[\frac{g}{\alpha^2} \frac{\sigma_s^2}{\sigma_s^2 - 1} \right]^3, \quad (4.80)$$

where (k_s, h_b) is a pair of a saddle and branch point of the uniform depth dispersion relation. We denote σ_s as the solution of (4.73) which corresponds to the pair (k_s, h_b) . High order derivatives of (4.68) with respect to the wavenumber k at a pair (k_s, h_b) are non-zero. Next, we check the derivatives of dispersion relation (4.68) with respect

to x , as follows:

$$\frac{\partial L_0}{\partial x}(x, k) = k^2(1 - \tanh^2(kh)) \frac{dH}{dx}, \quad (4.81)$$

$$\frac{\partial^2 L_0}{\partial x^2}(x, k) = 2k^3 \tanh(kh)(1 - \tanh^2(kh)) \left(\frac{dH}{dx} \right)^2 + k^2(1 - \tanh^2(kh)) \frac{d^2 H}{dx^2}. \quad (4.82)$$

At a pair (k_s, x_b) the equations above assume the form

$$\frac{\partial L_0}{\partial x}(k_s, x_b) = -(\sigma_s^2 - 1) \left[\frac{\alpha^2}{g} \frac{1}{\sigma_s} \right]^2 \frac{dH}{dx}(x_b), \quad (4.83)$$

$$\frac{\partial^2 L_0}{\partial x^2}(k_s, x_b) = -2 \frac{\sigma_s^2 - 1}{\sigma_s^2} \left[\frac{\alpha^2}{g} \right]^3 \left(\frac{dH}{dx}(x_b) \right)^2 - (\sigma_s^2 - 1) \left[\frac{\alpha^2}{g} \frac{1}{\sigma_s} \right]^2 \frac{d^2 H}{dx^2}(x_b). \quad (4.84)$$

As a consequence of equations (4.78) and (4.83), we can write the first terms of the Taylor series expansion of the dispersion relation (4.68), valid in a neighborhood of (k_s, h_b) , as

$$\begin{aligned} -\alpha^2 + L_0(x, k) &\sim \frac{1}{2} \frac{\partial^2 L_0}{\partial k^2}(x_b, k_s) (k - k_s)^2 + \frac{\partial L_0}{\partial x}(x_b, k_s) (x - x_b) \\ &\quad + O[(x - x_b)^2, (x - x_b)(k - k_s)^2, (k - k_s)^3]. \end{aligned} \quad (4.85)$$

Therefore, according to equation (4.85), as we approach a branch point x_b in the complex x plane, two wave numbers coalesce at a image k_s (saddle point) of x_b . We have assumed that the first derivative of the function $H(x)$ (depth variation) is non-zero at x_b . Therefore, the wave modes interact in pairs. Since, for each branch point we have as its image a pair of saddle points, we can say that four wave modes interact as we approach a branch point. Next, we obtain the wavenumber coalescence pattern. We work on the complex h plane instead of the complex x plane to simplify our analysis.

To obtain the interaction pattern between wave numbers we use the procedure:

- First, we chose a branch point h_b in the complex h plane.
- Second, we evaluate the wave numbers for $h = \Re\{h_b\}$.
- Third, we consider a contour in the complex h plane that goes from $\Re\{h_b\}$ to

h_b . The contour can be parameterized as follows:

$$\begin{aligned} [0, \Im\{h_b\}] &\rightarrow C(h) \quad (\text{complex } h \text{ plane}) \\ t &\mapsto \Re\{h_b\} + i t \end{aligned} \quad (4.86)$$

- Fourth, as we go from the real axis of the complex h plane to the chosen branch point h_b , we trace the trajectory of each wavenumber in the complex k plane. The second item gives the initial condition for the wave numbers. As we approach the branch point h_b , two pairs of wave numbers should coalesce at the two saddle points, which are images of the branch point h_b .

An illustration of the contours (4.86) in the complex h plane for some branch points is given in the figure 4-4.

The result of this procedure is the following wavenumber coalescence pattern.

- For the odd branch points: At the $(2n - 1)$ -th branch point, the wavenumber $+k_0(-k_0)$ coalesce with the wavenumber $+k_n(-k_n)$, $n = 1, 2, \dots$. This implies that at the $(2n - 1)$ -th branch point, the propagating wave mode associated with the wavenumber $+k_0(-k_0)$ interact with the evanescent mode associated with the wavenumber $+k_n(-k_n)$, $n = 1, 2, \dots$
- For the even branch points: At the $2n$ -th branch point, the wavenumber $+k_0(-k_0)$ coalesce with the wavenumber $-k_n(+k_n)$, $n = 1, 2, \dots$. This implies that at the $2n$ -th branch point, the propagating wave mode associated with the wavenumber $+k_0(-k_0)$ interact with the evanescent mode associated with the wavenumber $-k_n(+k_n)$, $n = 1, 2, \dots$
- At the branch point $h = 0$, the wave numbers $\pm k_0$ coalesce. So, at the images of this branch point in the complex x plane, the two propagating modes interact, which implies in wave reflection.

We illustrate the coalescence pattern of the wave numbers. As we travel in the complex h plane close to the $2m - 1$ -th or the $2m$ -th branch point, the wave numbers

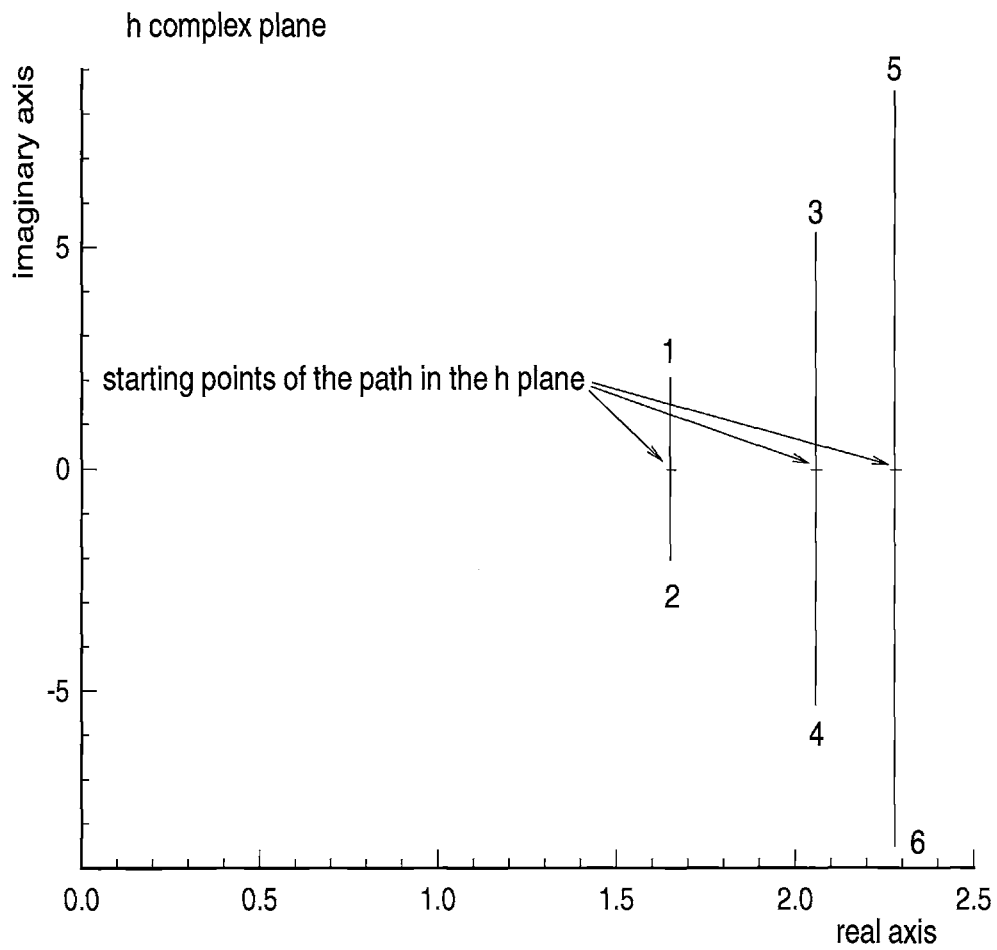


Figure 4-4: Some contour given by equation (4.86) in the complex h plane.

$\pm k_0$ coalesce with the wave numbers $\pm k_m$. Figures 4-5 to 4-10 show the wavenumber coalescence pattern that happens as we approach each of the first six branch points. In each of these figures, the large left picture illustrates the trajectory of the coalescing wave numbers $\pm k_0$ and $\pm k_m$ plus trajectories of wave numbers $\pm k_n$, with $n = 1, \dots, m - 1$. The right top picture shows the trajectory of the wave numbers $\pm k_0$, and the bottom right picture show the same for the wave numbers $\pm k_m$.

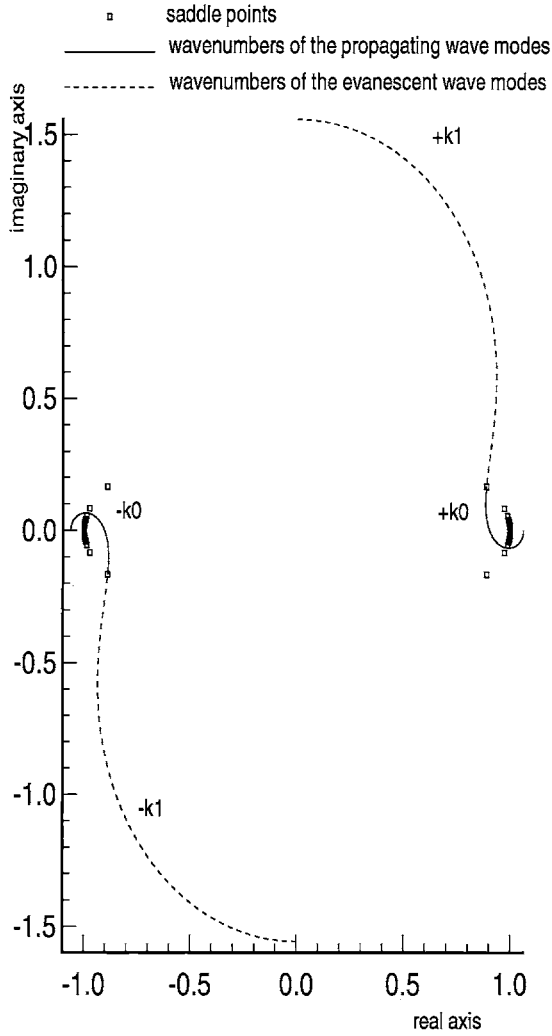
For the branch point at the origin, we perform a different exercise to study the wavenumber coalescence. We travel around a path in the complex h which encircles the origin as illustrated in figure 4-11 below. The image in the complex k plane for the wave numbers $\pm k_0$ is a half circle (see figure 4-13), which implies that this two wave numbers coalesce to the infinite (saddle point, image of this branch points, lies at the infinite of the complex k plane) as we approach the branch points in the x complex plane solution of $h(x) = 0$. The other wave numbers perform a full circle as we encircle the origin of the complex h plane, as illustrated in figure 4-13. So, the other wave numbers do not coalesce at the branch point $h = 0$.

4.5.2.2 Wave Modes Boundary and Approximate Forms of the Dispersion Relation.

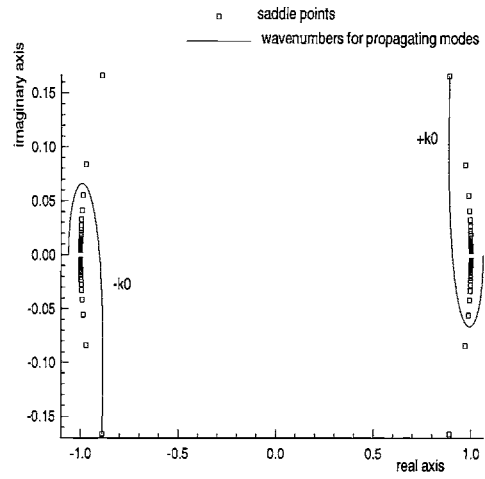
The analysis of the dispersion relation (4.68) reveals how the wave modes couple at the branch points of the dispersion relation, but this analysis does not tell us anything about the amount of energy exchanged among the coupling wave modes. To obtain this information we approximate the operator $\hat{L}$ in equation (4.54) such that equation (4.54) for the free surface potential is approximated as a locally valid ordinary differential equation, which describes the wave mode coupling that happens at the considered branch point. According to the analysis performed in the previous section, at each branch point we have two pairs of wave modes that couple, and a second order differential equation seems enough to describe the coupling between two wave modes.

The first step to obtain the second order differential equation describing the coupling of two wave modes is to realize a suitable way to reduce the operator $\hat{L}$ in an

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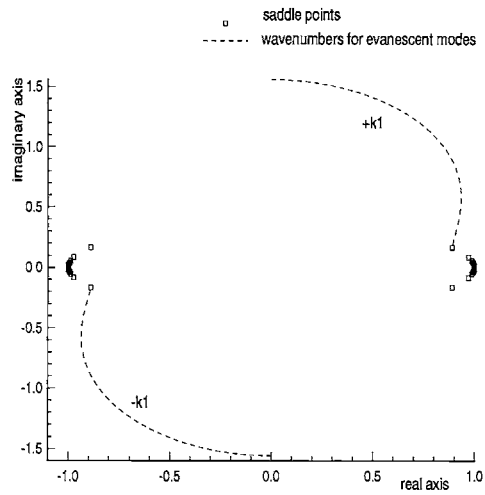
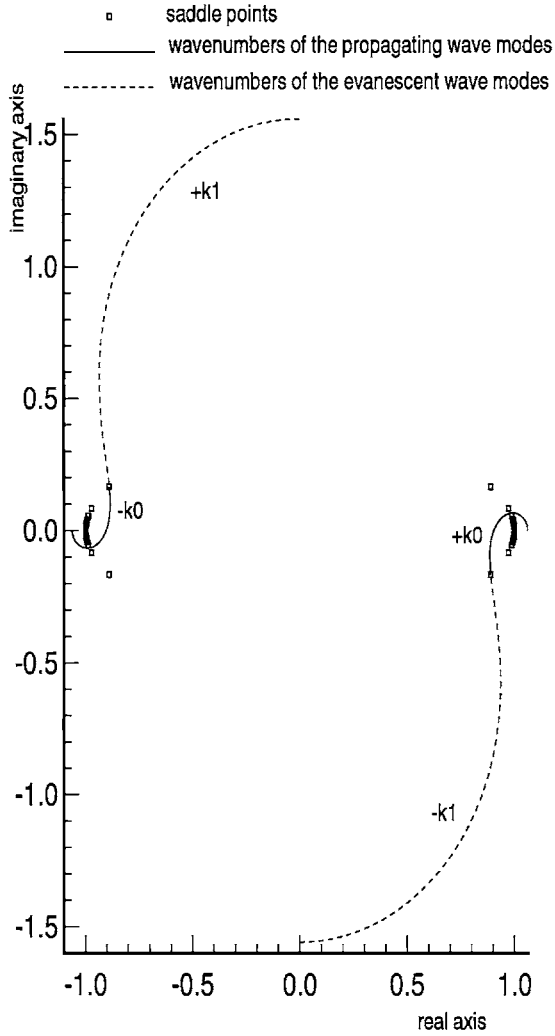
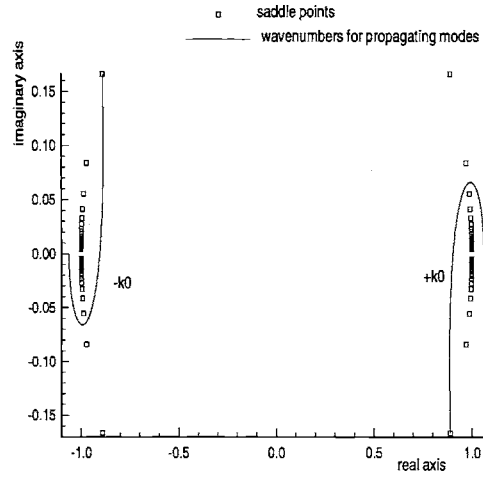


Figure 4-5: Trajectory of the wave numbers that coalesce at the two saddles, image of the first branch point, as we go along contour (4.86) with h_b as the first branch point ($m = 1$).

trajectory of wavenumbers in the complex k plane



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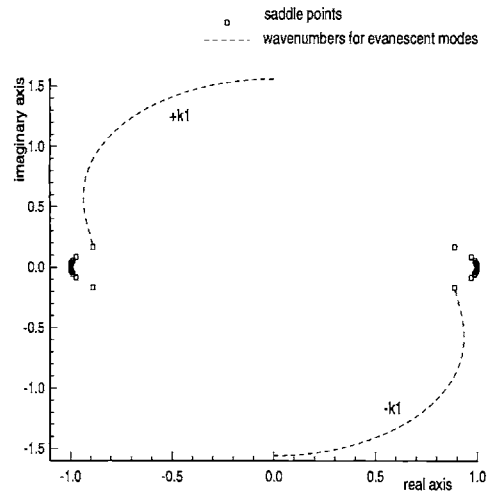
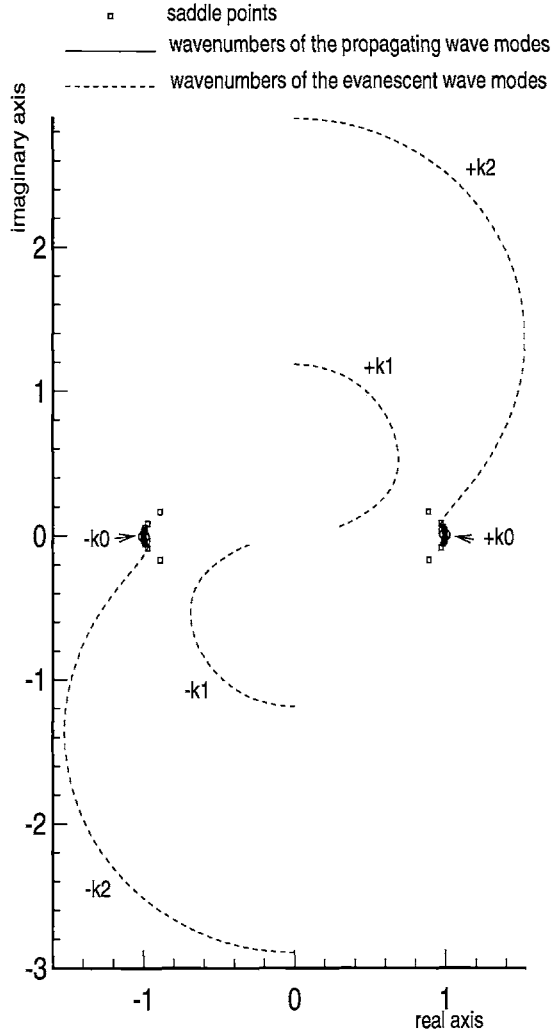
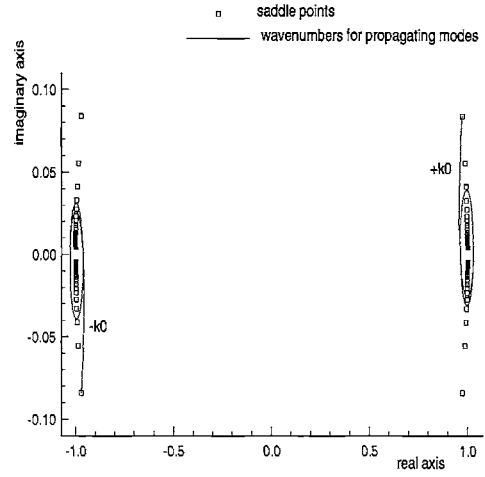


Figure 4-6: Trajectory of the wave numbers that coalesce at the two saddles, image of the second branch point, as we go along contour (4.86) with h_b as the second branch point ($m = 1$).

trajectory of wavenumbers in the complex k plane



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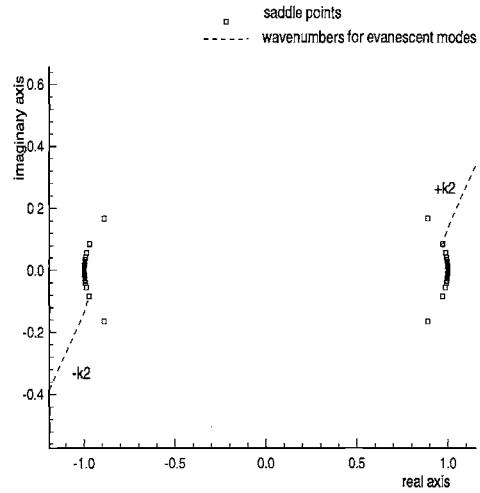
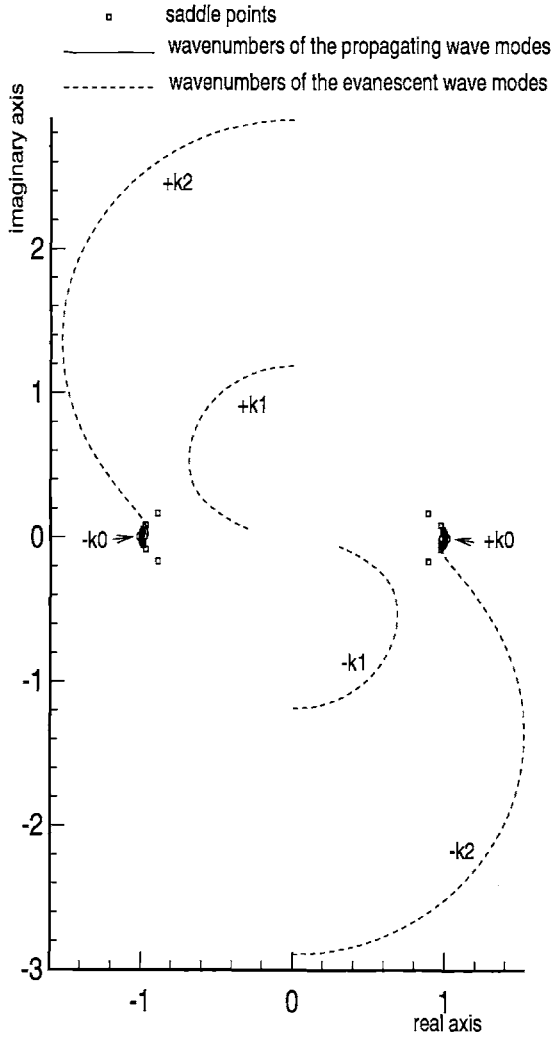
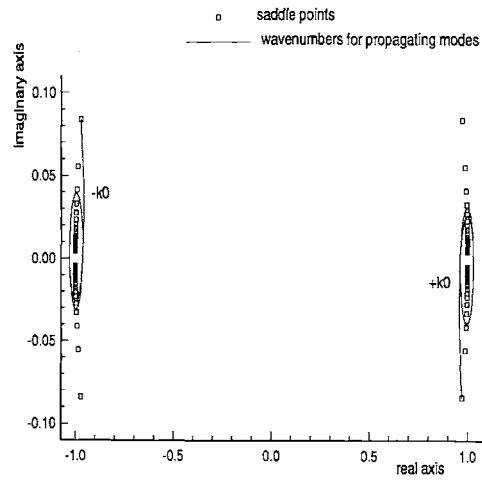


Figure 4-7: Trajectory of the wave numbers that coalesce at the two saddles, image of the third branch point, as we go along contour (4.86) with h_b as the third branch point ($m = 2$).

trajectory of wavenumbers in the complex k plane



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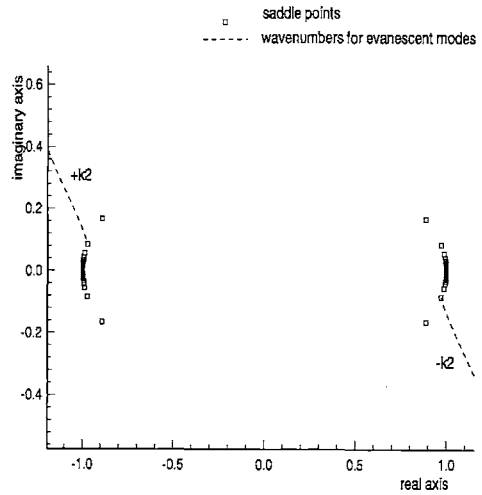
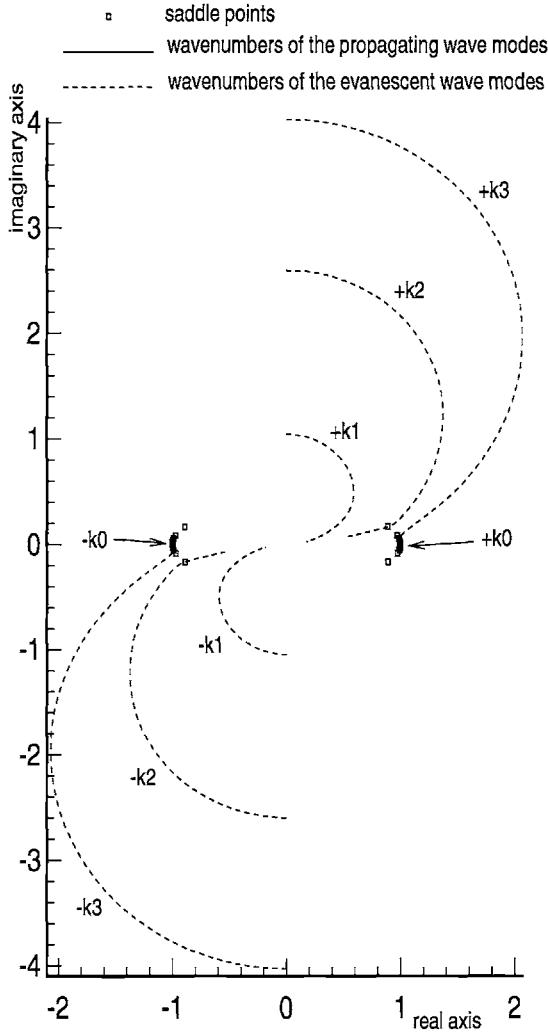
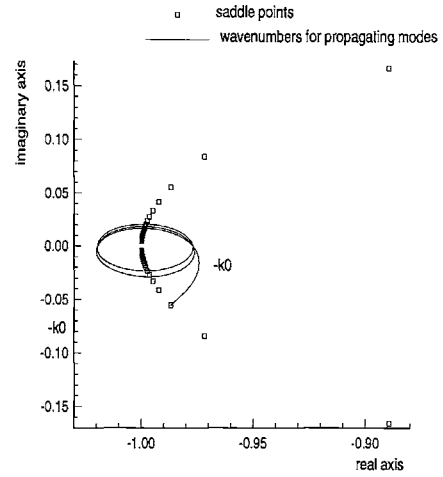


Figure 4-8: Trajectory of the wave numbers that coalesce at the two saddles, image of the fourth branch point, as we go along contour (4.86) with h_b as the fourth branch point ($m = 2$).

trajectory of wavenumbers in the complex k plane



trajectory of wavenumbers in the complex k plane



trajectory of wavenumbers in the complex k plane

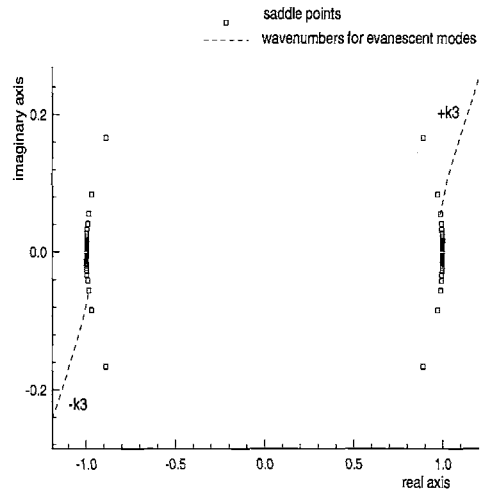
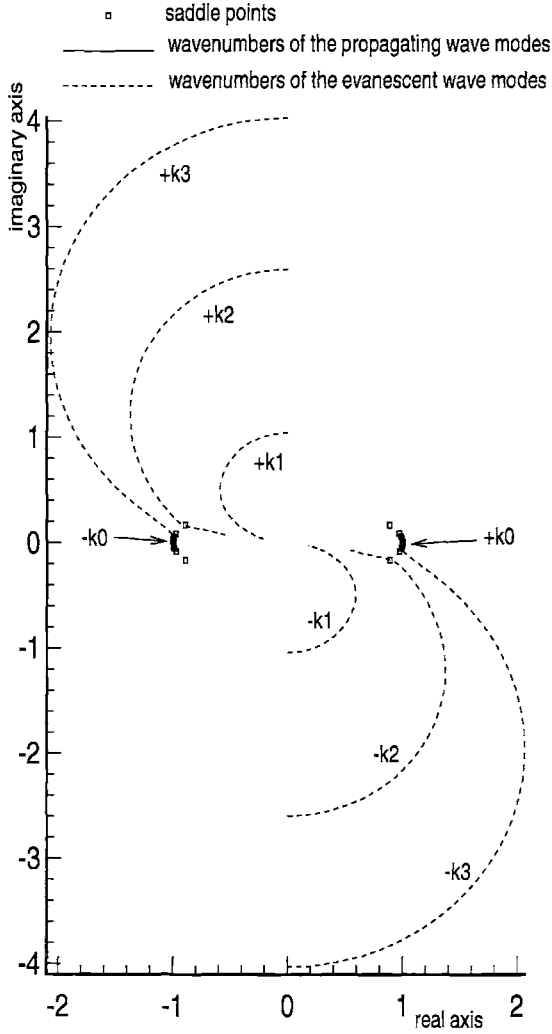
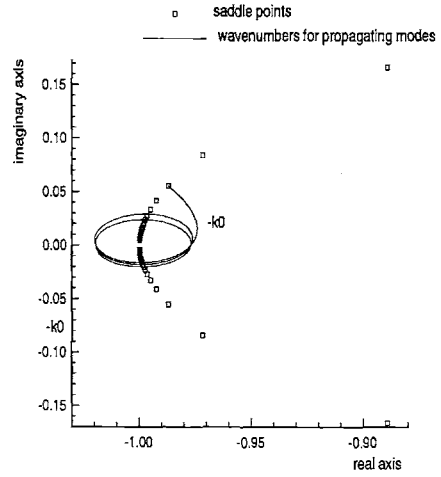


Figure 4-9: Trajectory of the wave numbers that coalesce at the two saddles, image of the fifth branch point, as we go along contour (4.86) with h_b the fifth branch point ($m = 3$).

trajectory of wavenumbers in the complex k plane



trajectory of wavenumbers in the complex k plane



trajectory of wavenumbers in the complex k plane

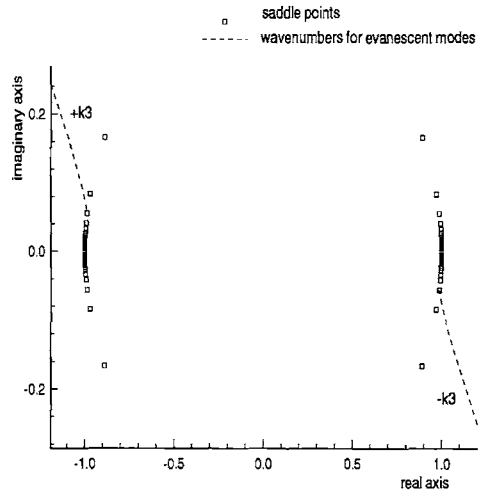


Figure 4-10: Trajectory of the wave numbers that coalesce at the two saddles, image of the sixth branch point, as we go along contour (4.86) with h_b the sixth branch point ($m = 3$).

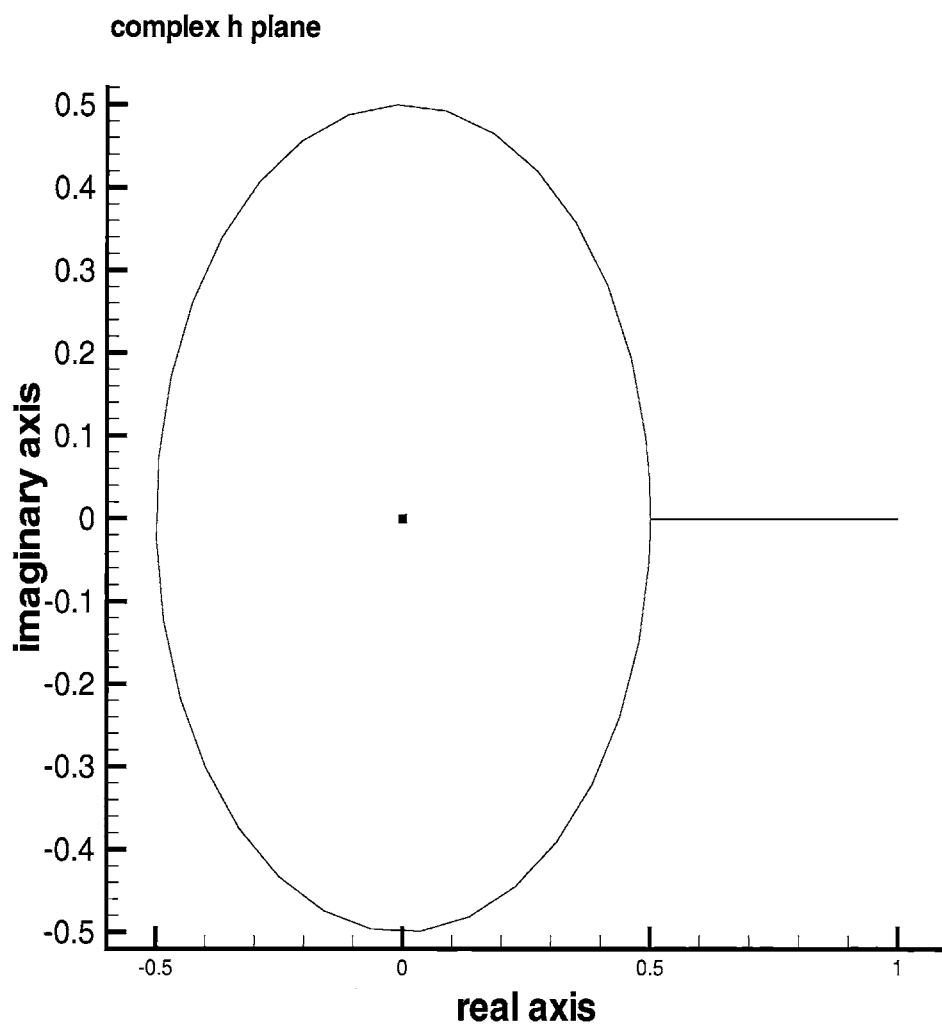


Figure 4-11: Contour in the complex h plane used to study the wavenumber coalescence with respect to the branch point located at the origin of the complex h plane.

Saddle points in the complex k plane and trajectory of the propagating wave numbers

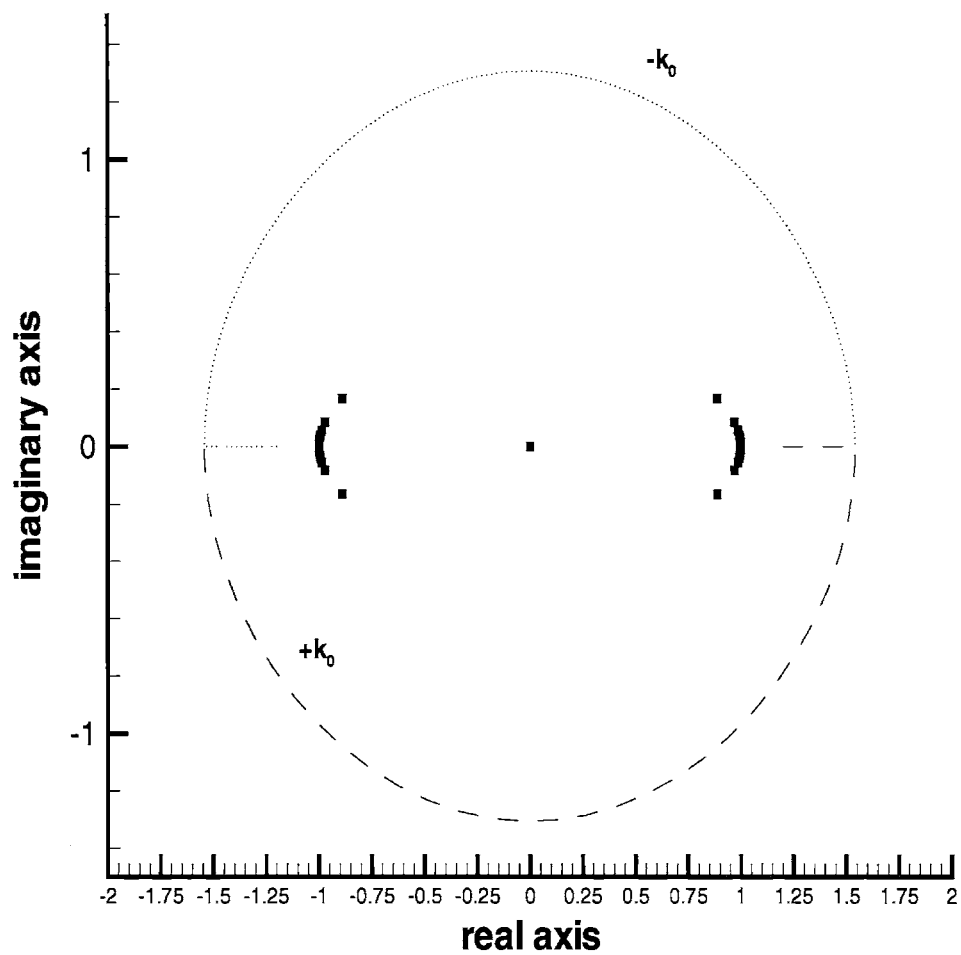


Figure 4-12: Trajectory of the wave numbers $\pm k_0$ in the complex k as we travel along the contour in the complex h plane illustrated in figure 4-11.

Trajectory of the first three evanescent wave numbers ($+k_j$ and $-k_j$)

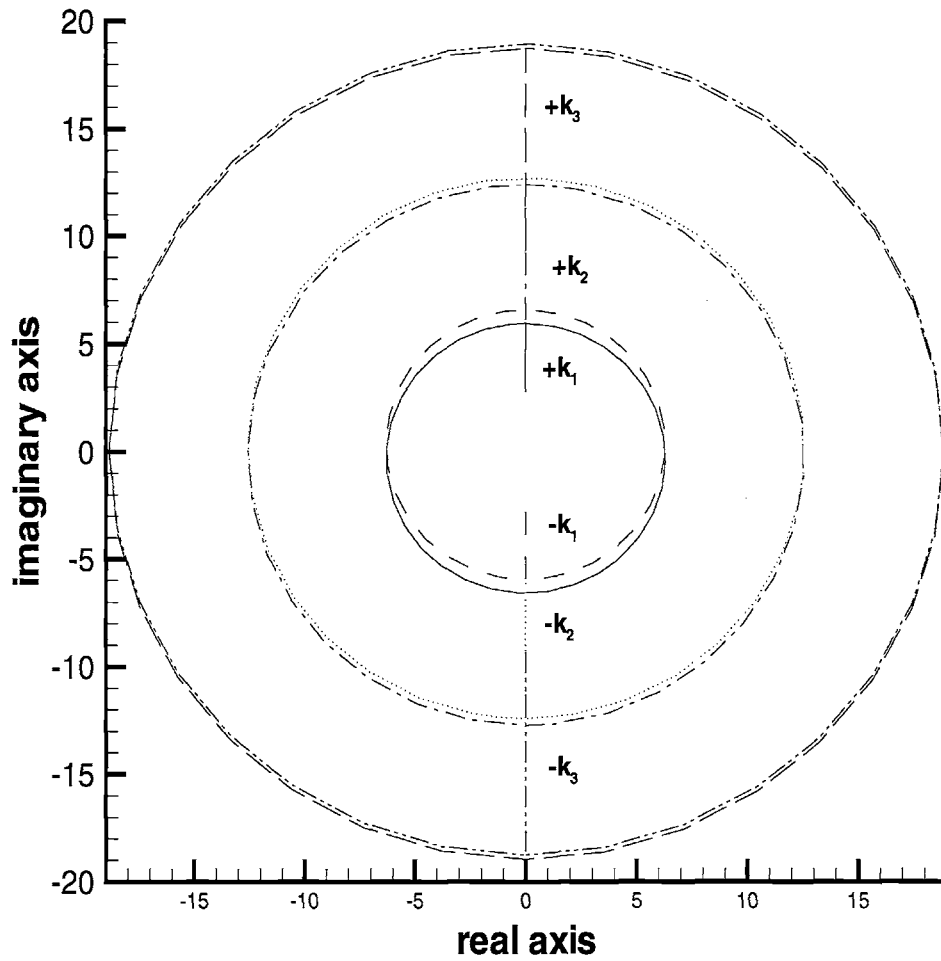


Figure 4-13: Trajectory of the wave numbers $\pm k_j$ in the complex k as we travel along the contour in the complex h plane illustrated in figure 4-11.

operator for a second order differential equation. The idea is to expand the symbol of the operator $\hat{L}$ to second order with respect to the variable p around what we will define as a wavenumber boundary.

As we mentioned before, if we consider the space coordinate x and the wavenumber k as complex variables, the dispersion relation (4.68) can be seen as an implicit mapping between the complex x plane and the complex k plane. For each value of x we have an enumerable set of wavenumbers k_j , and we can define the explicit mappings

$$\begin{aligned} x &\mapsto k_j \\ x &\rightarrow -\alpha^2 + L_0(x, k_j(x)) = 0. \end{aligned} \tag{4.87}$$

We do not have explicit expression for the mappings (4.87), but we can try to obtain approximate expressions for these mappings which are valid in a neighborhood of a branch point where two or more of the k_j coalesce to a saddle point, image of the considered branch point. For example, we would like to obtain an approximate explicit expression for the mapping (4.87) for two of the k_j . We assume that these two k_j coalesce to the saddle point k_s as we approach the branch point x_b . We expand the dispersion relation (4.68) to second order (two coalescing wave numbers) in the wavenumber around a contour k_c in the complex k plane. We do not know k_c now, we only assume that the saddle point k_s is one point of the contour k_c . We obtain the approximation

$$-\alpha^2 + L_0(x, k) \sim -\alpha^2 + L_0(x, k_c) + \frac{\partial L_0}{\partial k}(x, k_c)(k - k_c) + \frac{1}{2} \frac{\partial^2 L_0}{\partial k^2}(x, k_c)(k - k_c)^2 + O((k - k_c)^3). \tag{4.88}$$

Now we impose that the contour k_c satisfies the implicit equation

$$\frac{\partial L_0}{\partial k}(x, k_c(x)) = 0, \quad (4.89)$$

and to specify which of the implicit solutions of the above equation we are talking about, we have the condition

$$k_c(x_b) = k_s. \quad (4.90)$$

Notice that the pair (x_b, k_s) satisfies equation (4.89). So, once the pair (k_s, x_b) is specified, the wave mode boundary $k_c(x)$ is well defined. Now, the approximation (4.88) gives the approximate explicit dispersion relation

$$k = k_c(x) \pm i\sqrt{Q(x)} \quad (4.91)$$

with

$$Q(x) = \frac{-\alpha^2 + L_0(x, k_c(x))}{\frac{1}{2} \frac{\partial^2 L_0}{\partial k^2}(x, k_c)}. \quad (4.92)$$

We can obtain a closed form expression for the approximate dispersion relation (4.91) by constructing the appropriate explicit solution of equation (4.89). According to the condition (4.90) and the relation (4.72), we realize an explicit expression for the wave mode boundary

$$k_c(x) = \frac{1}{h(x)} \frac{\sigma_s}{\sigma_s^2 - 1}, \quad (4.93)$$

where σ_s is the solution of equation (4.73) which gives the saddle point k_s and the

branch point x_b . If we substitute the expressions (4.93), (4.78) and (4.71) into equation (4.87), we obtain the approximate dispersion relation

$$k = \frac{1}{h(x)} \frac{\sigma_s}{\sigma_s^2 - 1} \pm i \left\{ -2 \frac{\alpha^2 - \frac{1}{h(x)} \frac{\sigma_s^2}{\sigma_s^2 - 1}}{4\sigma_s \left[\frac{1}{\alpha^2} \frac{\sigma_s^2}{\sigma_s^2 - 1} \right]^2} \right\}^{1/2}, \quad (4.94)$$

as an explicit expression of the x variable.

Since for each branch point we have two saddle points associated with it, we have two wavenumber boundaries associated with each branch point. Examples of wavenumber boundaries are given in figures 4-14 to 4-15. In these figures, we also plot the wave numbers given by the approximate dispersion relation (equation (4.94)) against the actual wave numbers given by the full dispersion relation (4.68). The agreement between the wave numbers and their approximations given by equation (4.94) is usually good in a neighborhood of the saddle point where they coalesce. Each of these figures contains four pictures. The largest one shows the evolution of wavenumber boundaries, the evolution of the approximate wavenumbers and the evolution of the actual wavenumbers in the complex k plane as we travel in the complex h plane from $\Re\{h_b\}$ to h_b . The small figure at top right is a zoom of the portion of the large figure close to the saddle points. The right middle figure illustrates the difference between the actual and the approximate wave numbers as we travel in the complex h plane towards the branch point h_b . The bottom right figure shows the coalescence of the wave numbers towards saddle points as we approach a branch point.

The analysis above does not include the branch points in the complex x plane solutions of the equation $h(x) = 0$. These branch points are associated to the solution $\sigma = 0$ of equation (4.73), which implies that at these branch points $kh = 0$. The wave mode boundary in this case is $k_c = 0$. To obtain an approximate for of the dispersion relation, we expand the full dispersion relation, given by equation (4.68), to second order in kh with respect to $kh = 0$, which gives

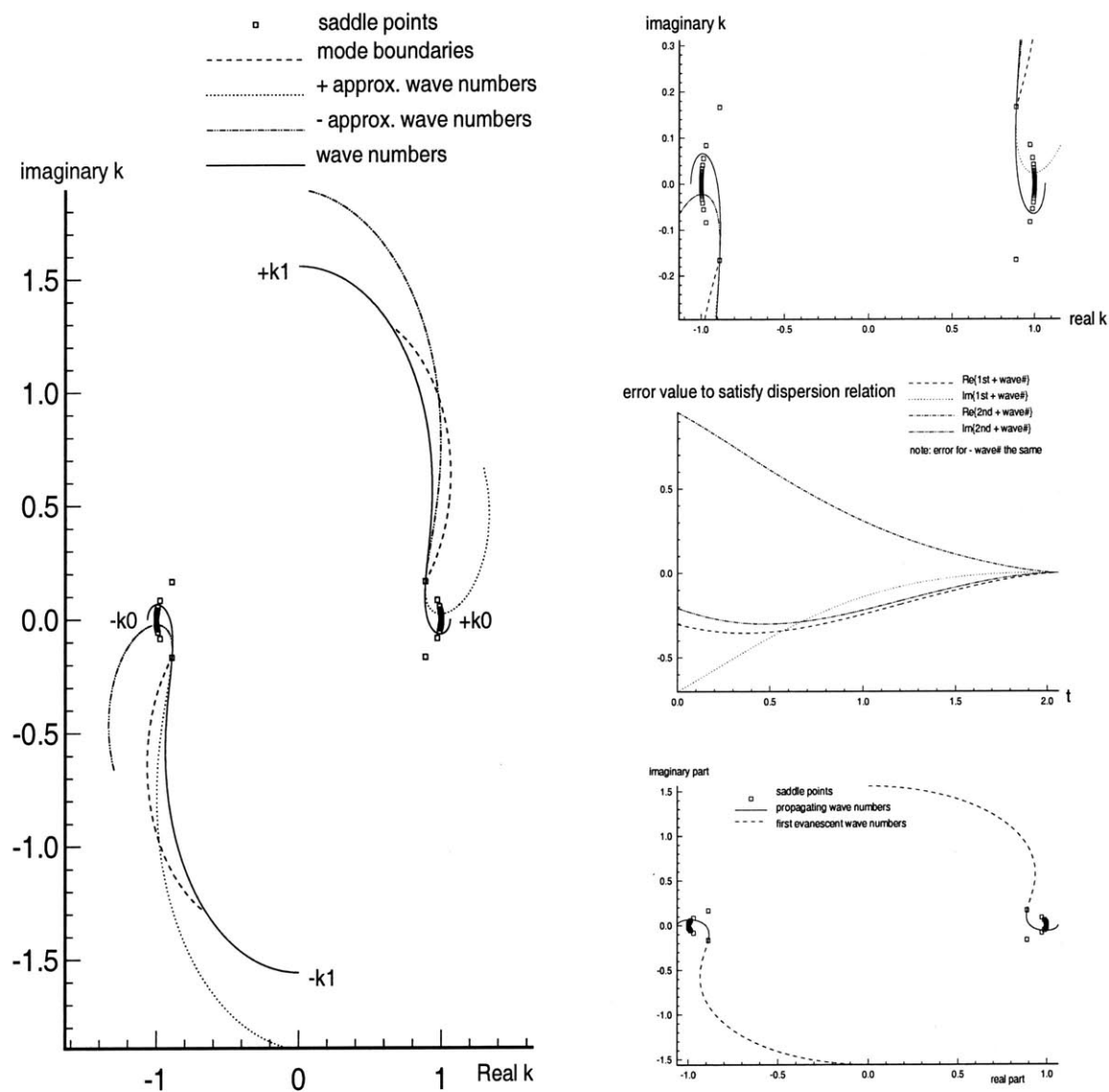


Figure 4-14: For this figure h_b is the first branch point in the complex h plane.

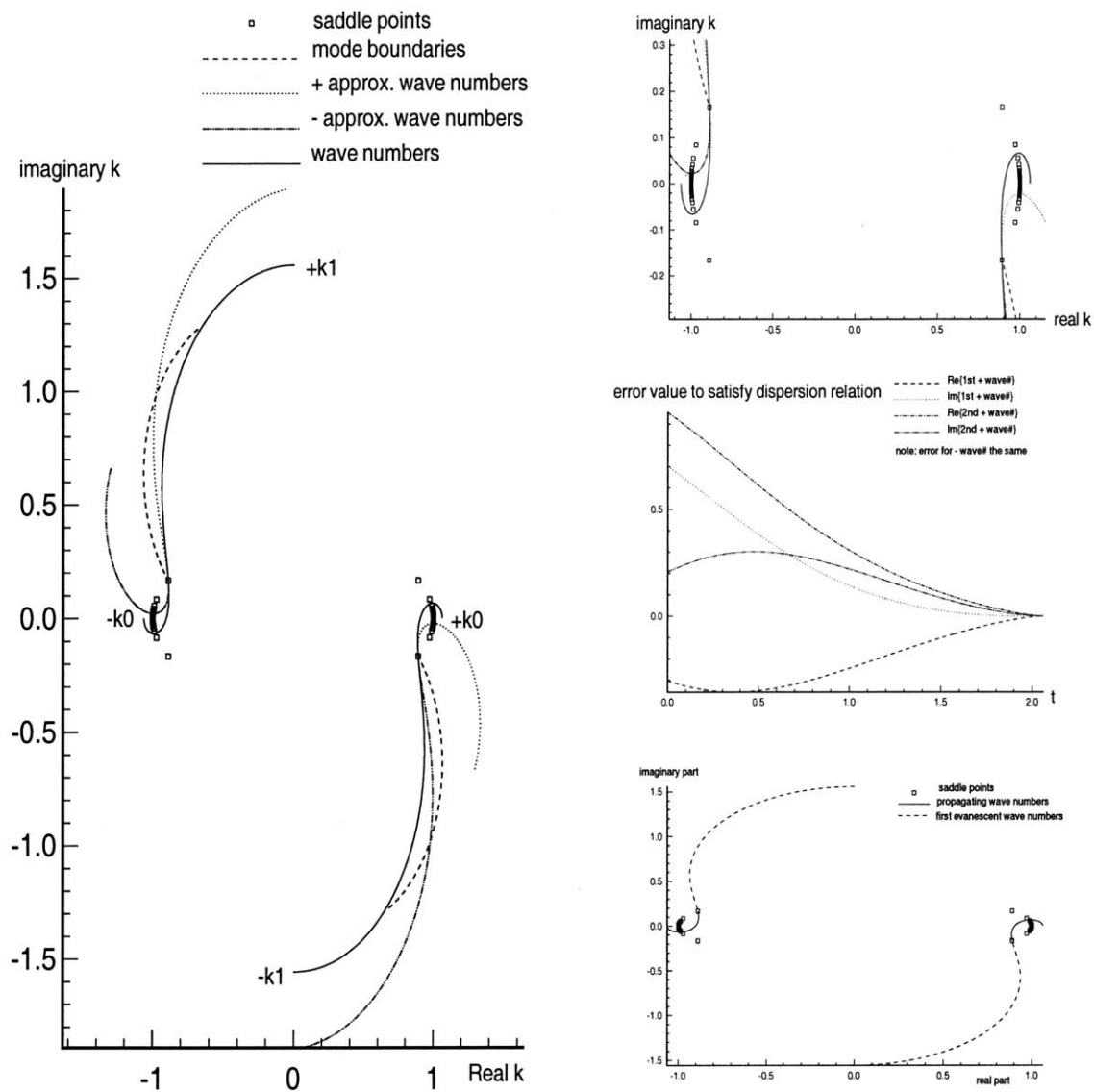


Figure 4-15: For this figure h_b is the second branch point in the complex h plane.

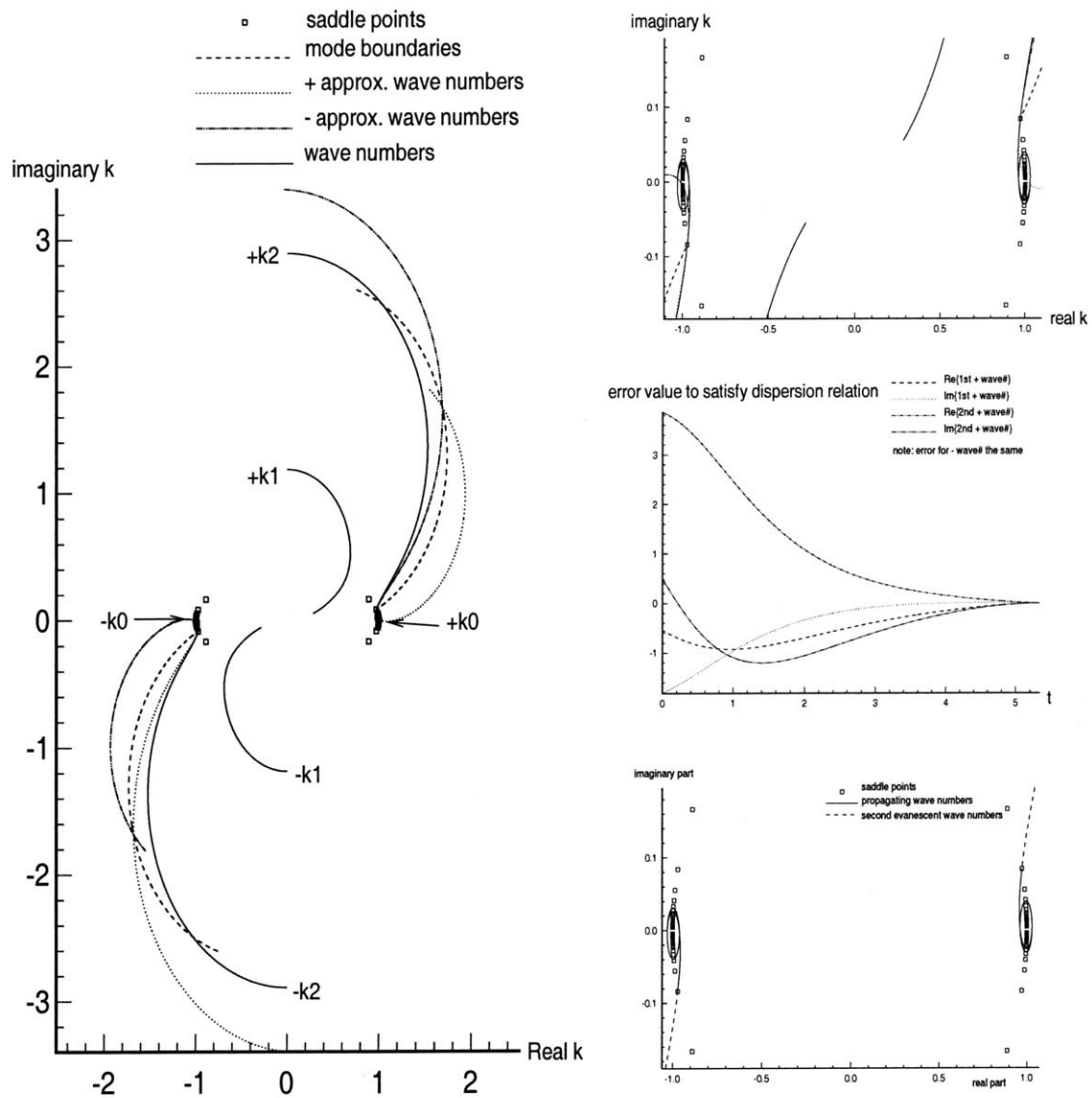


Figure 4-16: For this figure h_b is the third branch point in the complex h plane.

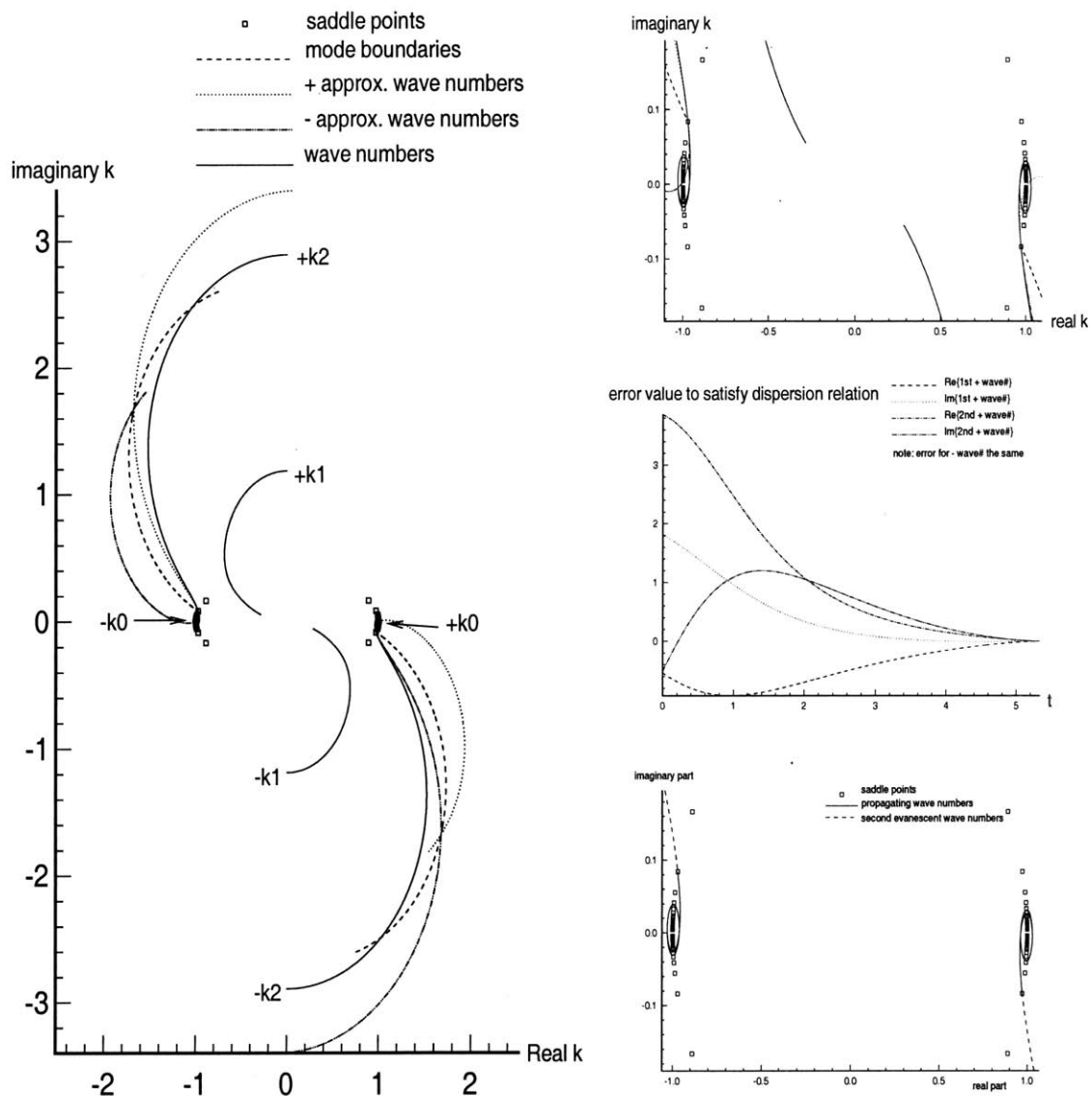


Figure 4-17: For this figure h_b is the fourth branch point in the complex h plane.

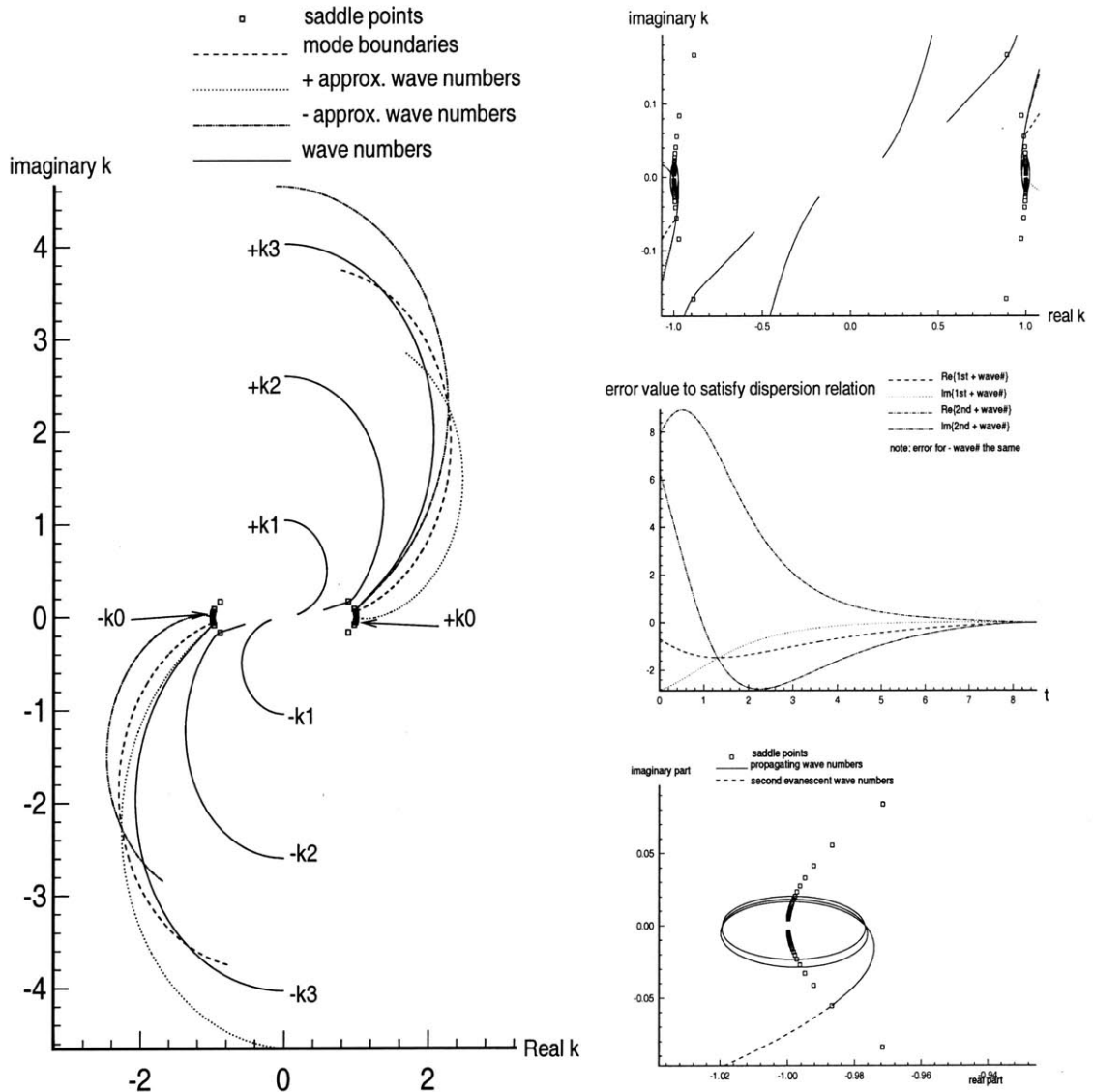


Figure 4-18: For this figure h_b is the fifth branch point in the complex h plane. For this case the right bottom figure is a zoom of the neighborhood of the set of saddle points that accumulate at $k = -1$.

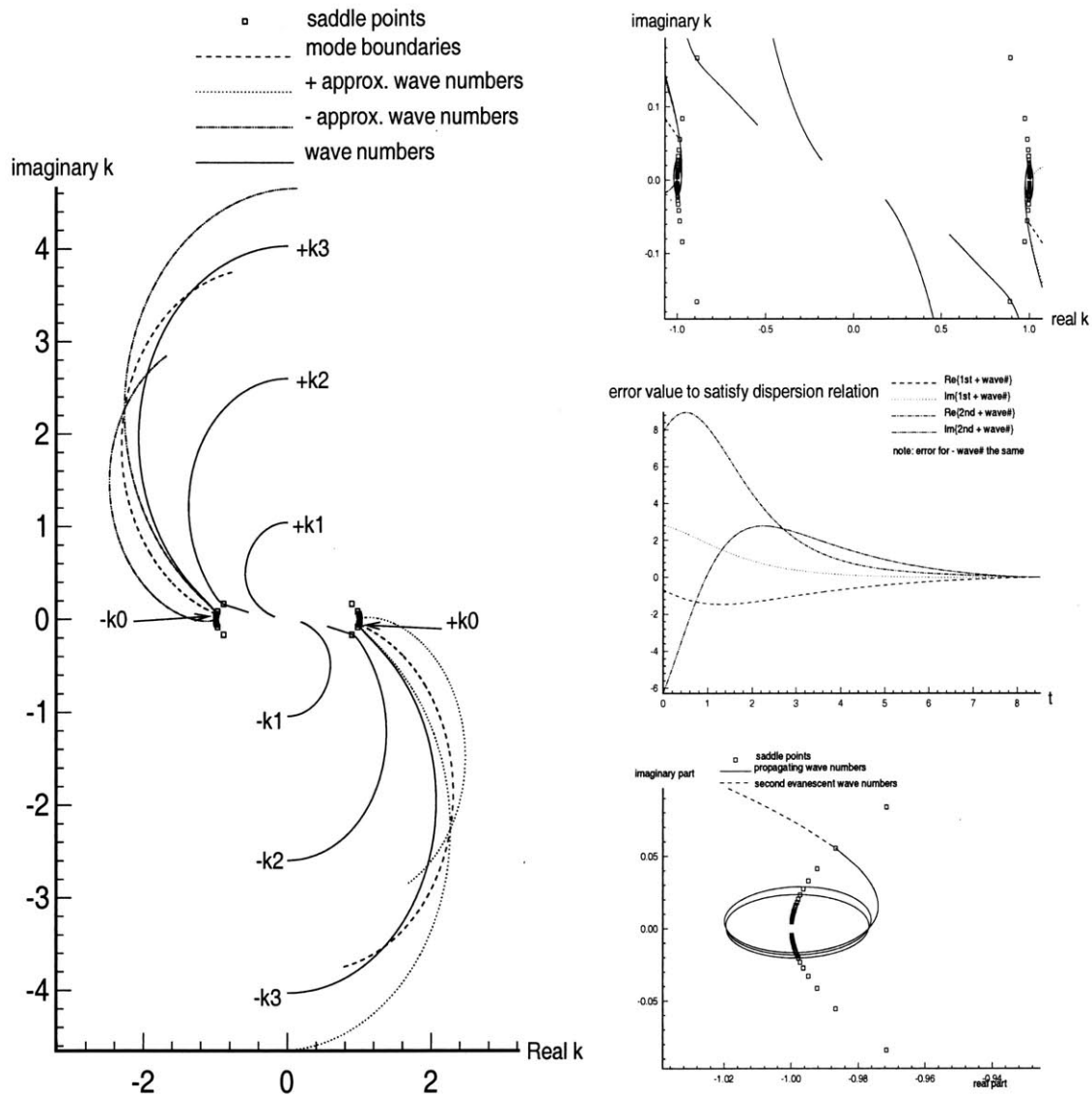


Figure 4-19: For this figure h_b is the sixth branch point in the complex h plane. For this case the right bottom figure is a zoom of the neighborhood of the set of saddle points that accumulate at $k = -1$.

$$-\alpha^2 + k(kh(x)) = 0 \implies k = \pm \frac{\alpha}{\sqrt{h(x)}}. \quad (4.95)$$

Figure 4-20 illustrates the performance of the approximate dispersion relation (4.95) with respect to the full dispersion relation (4.68) as we encircle the branch point $h = 0$. The path in the complex h plane is illustrated by figure 4-11.

4.5.3 Approximate Equations for the Free Surface Potential.

The analysis of the dispersion relation reveals which wave modes couple at the branch points of the dispersion relation, but it does not tell how the amplitude of the wave modes change. In this section we approximate the symbol $\hat{L}$ in a neighborhood of a branch point. Since we have coalescence of two wave numbers at a saddle point, image of the considered branch point, we approximate the symbol of the operator $\hat{L}$ as a symbol of a second order differential equation. The resulting approximate form of the governing equation (4.54) for the free-surface potential describes the wave mode coupling in a neighborhood of the considered branch point. The approach to approximate the operator $\hat{L}$ for branch points where a propagating wave mode couple with an evanescent wave mode differs from the approach used for the branch points where two propagating wave modes couple. We start with the approach for the branch points where a propagating and an evanescent wave mode couple. We expand the symbol of the operator $\hat{L}$ to second order in the p variable with respect to the wave mode boundary, defined by the equation (4.93). We only need the symbol $L_0(x, p)$. The approximate form of the operator $\hat{L}(\overset{2}{x}, -i\overset{1}{\beta}\frac{d}{dx}, \beta)$ in a neighborhood of (x_b, k_s) is

Trajectory of the propagating wave numbers

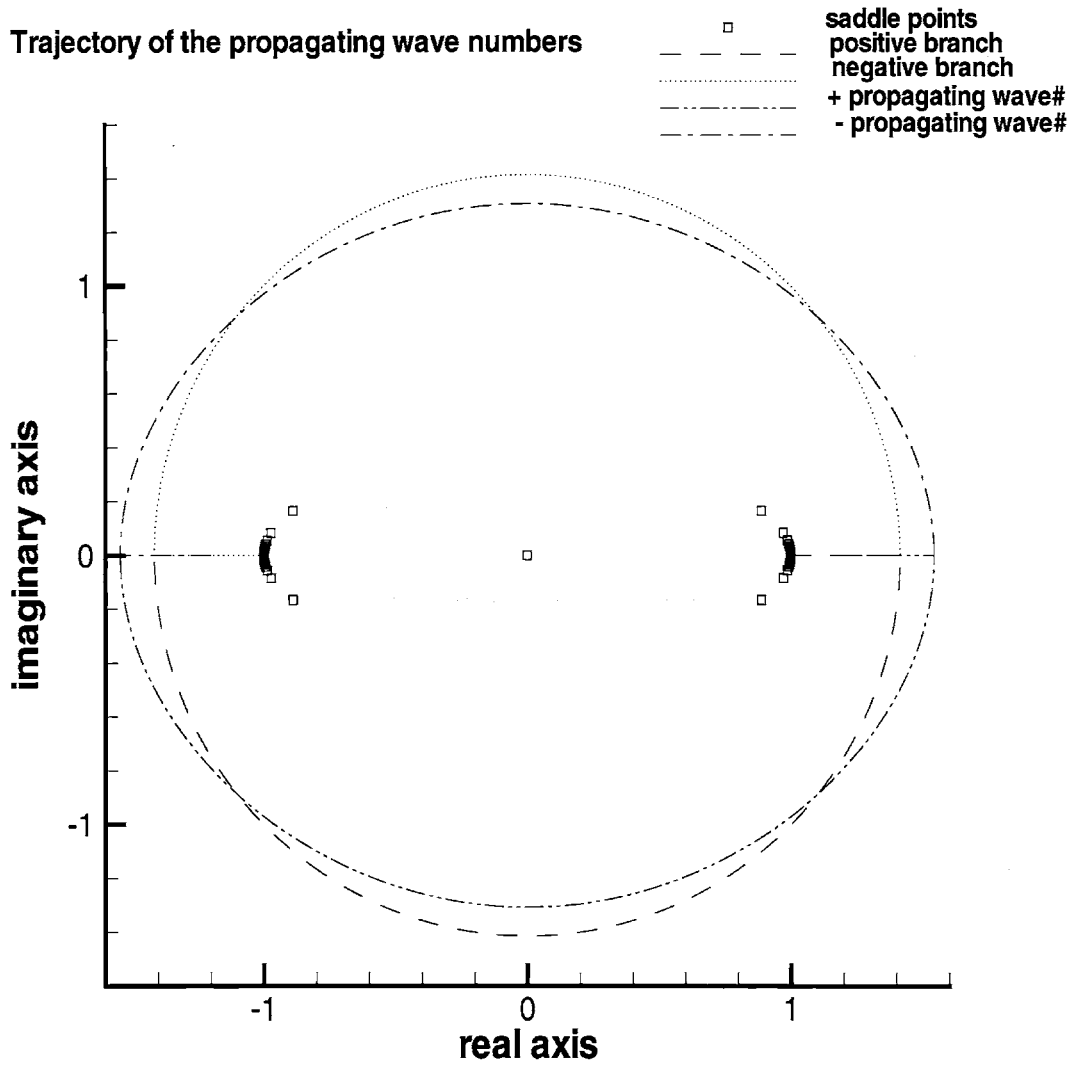


Figure 4-20: For this figure $h_b = 0$. The term positive (negative) branch is related to the positive (negative) sign in the explicit form of the approximate dispersion relation, given by equation (4.95).

$$\begin{aligned}
\hat{L}(\overset{1}{x}, -i\beta \frac{d}{dx}, \beta) \sim & L_0(x, k_c(x) + \frac{1}{2} \left(\frac{\partial^2}{\partial p^2} L_0(x, p) \Big|_{p=k_c(x)} \right) \left(-i\beta \frac{d}{dx} - k_c(x) \right)^2 \\
& L_0(x, k_c(x) + \frac{1}{2} \left(\frac{\partial^2}{\partial p^2} L_0(x, p) \Big|_{p=k_c(x)} \right) \left(-\beta^2 \frac{d^2}{dx^2} + i2\beta k_c(x) \frac{d}{dx} \right. \\
& \left. + i\beta \frac{d}{dx} k_c(x) + (k_c(x))^2 \right)
\end{aligned} \tag{4.96}$$

Now the equation (4.54) for the free surface potential $\psi(x)$ assumes the form

$$\left\{ -\alpha^2 + L_0(x, k_c(x)) + \frac{1}{2} \left(\frac{\partial^2}{\partial p^2} L_0(x, p) \Big|_{p=k_c(x)} \right) \left(-\beta^2 \frac{d^2}{dx^2} + i2\beta k_c(x) \frac{d}{dx} + i\beta \frac{d}{dx} k_c(x) + (k_c(x))^2 \right) \right\} \psi(x) = 0. \tag{4.97}$$

Next, we consider the branch points x_b that are solutions of the equation $h(x) = 0$. We need only the principal part of the symbol of the operator $\hat{L}(\overset{1}{x}, -i\beta \frac{d}{dx}, \beta)$, which is the symbol $L_0(x, p)$. In an neighborhood of a branch point x_b we expand the symbol $L_0(x, p)$ to first order in $h(x)$ with respect to $h(x) = 0$. So, the approximate form of the operator $\hat{L}(\overset{1}{x}, -i\beta \frac{d}{dx}, \beta)$ is

$$\hat{L}(\overset{1}{x}, -i\beta \frac{d}{dx}, \beta) \sim -\beta^2 \frac{d}{dx} \left(h(x) \frac{d}{dx} \right), \tag{4.98}$$

and the approximate form for the governing equation (4.54) for the free surface potential $\psi(x)$ is

$$\left\{ -\alpha^2 - \beta^2 \frac{d}{dx} \left(h(x) \frac{d}{dx} \right) \right\} \psi(x) = 0, \tag{4.99}$$

which is very similar to the shallow water equation for the free surface displacement (see chapter 4 of [44]).

4.5.3.1 Liouville-Green functions for the Approximate Equations for the Free Surface Potential.

In this section we obtain the Liouville-Green functions (WKB solution) for the approximate equations (4.97) and (4.99) for the free surface potential. We have two reasons to do so. First, we want to show that the approximate form of the transport equations (4.62), valid in a neighborhood of a branch point, coincide with the transport equations derived from the equations (4.97) and (4.99). This will be important when we match the far field solution (4.55) written in terms of the inner field variables (approximation valid in a neighborhood of a turning point) with the asymptotic expansion of the solutions of the approximate equations (4.97) and (4.99), to obtain the wave mode conversion coefficients due to the wave mode coupling at the considered turning point. Second, we use the comparison equation technique to obtain a uniformly valid asymptotic solution of the approximate equations (4.97) and (4.99), and we will need to write these asymptotic solutions in terms of the Liouville-Green functions of the approximate equations (4.97) and (4.99) to be able to do the matching with the far field asymptotic solution.

We start with the approximate equation (4.97). We proceed as follows:

- Assume a solution for the free surface potential $\psi(x)$ in the form:

$$\psi(x) = \exp\left(\frac{i}{\beta}S(x)\right)\tilde{w}(x), \quad (4.100)$$

where

$$\tilde{w}(x) = \tilde{w}_0(x) \left\{ 1 + \sum_{n=1}^{\infty} (-i\beta)^n \tilde{w}_n(x) \right\}. \quad (4.101)$$

- Approximate form of the wavenumbers DS .

$$DS = k_c(x) \pm i\sqrt{Q(x)}, \quad (4.102)$$

where $Q(x)$ was defined by the equation (4.92).

- The first order transport equation gives

$$\tilde{w}_0^\pm(x) = \exp(\mp i\frac{\pi}{4})Q(x)^{-1/4}. \quad (4.103)$$

- Transport equation of order $O(\beta^2)$ is

$$\frac{d}{dx}\tilde{w}_1^\pm(x) = -\frac{1}{2}\tilde{w}_0^\pm(x)\frac{d^2}{dx^2}\tilde{w}_0^\pm(x). \quad (4.104)$$

- Transport equation of order $O(\beta^{(n+1)})$ is

$$\begin{aligned} \tilde{w}_{n+1}^\pm(x) = & -\frac{1}{2}\int_r^x \left\{ \tilde{w}_0^\pm(t)\frac{d^2}{dt^2}\tilde{w}_0^\pm(t)\tilde{w}_n^\pm(t) + 2\tilde{w}_0^\pm(t)\frac{d}{dt}\tilde{w}_0^\pm(t)\frac{d}{dt}\tilde{w}_n^\pm(t) \right. \\ & \left. + (\tilde{w}_0^\pm(t))^2\frac{d^2}{dt^2}\tilde{w}_n^\pm(t) \right\} dt \text{ for } n \geq 1, \end{aligned} \quad (4.105)$$

where r is a reference point.

- Solutions of the transport equations (4.104) and (4.105). We first define

$$\xi(x) = \int_r^x \sqrt{Q(t)}dt, \quad (4.106)$$

and now we write the expression for the amplitudes $\tilde{w}_n^\pm(x)$ in terms of $\xi(x)$ up to $n = 4$, as follows:

$$\tilde{w}_1^\pm(x) = \pm i \int_r^x \left\{ \frac{3}{8} \left(\frac{d\xi}{dt} \right)^{-3} \left(\frac{d^2\xi}{dt^2} \right)^2 - \frac{1}{4} \left(\frac{d\xi}{dt} \right)^{-2} \frac{d^3\xi}{dt^3} \right\} dt \quad (4.107)$$

$$\tilde{w}_2^\pm(x) = \frac{1}{2}(\tilde{w}_1^\pm(x))^2 \pm \frac{i}{2} \left(\frac{d\xi}{dx} \right)^{-1} \frac{d\tilde{w}_1^\pm}{dx}, \quad (4.108)$$

$$\begin{aligned} \tilde{w}_3^\pm(x) = & \frac{1}{6}(\tilde{w}_1^\pm(x))^3 \pm \frac{i}{2} \left(\frac{d\xi}{dx} \right)^{-1} \tilde{w}_1^\pm(x) \frac{d\tilde{w}_1^\pm}{dx} + \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^2\xi}{dx^2} \frac{d\tilde{w}_1^\pm}{dx} \\ & - \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-2} \frac{d^2\tilde{w}_1^\pm}{dx^2} \pm \frac{i}{2} \int_r^x \left(\frac{d\xi}{dt} \right)^{-1} \left(\frac{d\tilde{w}_1^\pm}{dt} \right)^2 dt, \end{aligned} \quad (4.109)$$

$$\begin{aligned} \tilde{w}_4^\pm(x) = & \frac{1}{24}(\tilde{w}_1^\pm(x))^4 \pm \frac{i}{4} \left(\frac{d\xi}{dx} \right)^{-1} (\tilde{w}_1^\pm(x))^2 \frac{d\tilde{w}_1^\pm}{dx} + \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^2\xi}{dx^2} \tilde{w}_1^\pm(x) \frac{d\tilde{w}_1^\pm}{dx} \\ & - \frac{5}{8} \left(\frac{d\xi}{dx} \right)^{-2} \left(\frac{d\tilde{w}_1^\pm}{dx} \right)^2 - \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-2} \tilde{w}_1^\pm(x) \frac{d^2\tilde{w}_1^\pm}{dx^2} \mp i \frac{3}{8} \left(\frac{d\xi}{dx} \right)^{-5} \left(\frac{d^2\xi}{dx^2} \right)^2 \frac{d\tilde{w}_1^\pm}{dx} \\ & \pm \frac{i}{8} \left(\frac{d\xi}{dx} \right)^{-4} \frac{d^3\xi}{dx^3} \frac{d\tilde{w}_1^\pm}{dx} \pm i \frac{3}{8} \left(\frac{d\xi}{dx} \right)^{-4} \frac{d^2\xi}{dx^2} \frac{d^2\tilde{w}_1^\pm}{dx^2} \mp \frac{i}{8} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^3\tilde{w}_1^\pm}{dx^3} \\ & \pm \frac{i}{2} \tilde{w}_1^\pm(x) \int_r^x \left(\frac{d\xi}{dt} \right)^{-1} \left(\frac{d\tilde{w}_1^\pm}{dt} \right)^2 dt. \end{aligned} \quad (4.110)$$

Next, we consider the approximate equation (4.99). We proceed as follows:

- Assume a solution for the free surface potential $\psi(x)$ in the form given by equation (4.100) where the amplitude $\tilde{w}(x)$ has a series expansion in the parameter β given by equation (4.101).
- Approximate form of the wavenumbers DS is

$$DS = \pm \alpha \frac{1}{\sqrt{h(x)}}, \quad (4.111)$$

where $h(x)$ is the function defining the water depth with respect to the average free surface located at $y = 0$.

- The first order transport equation gives

$$\tilde{w}_0^\pm(x) = (\pm\alpha)^{-1/2}h(x)^{-1/4}. \quad (4.112)$$

- Transport equation of order $O(\beta^2)$ is

$$\frac{d}{dx}\tilde{w}_1^\pm(x) = -\frac{1}{2}\tilde{w}_0^\pm(x)\frac{d}{dx}\left(h(x)\frac{d\tilde{w}_0^\pm}{dx}\right). \quad (4.113)$$

- Transport equation of order $O(\beta^{(n+1)})$ is

$$\begin{aligned} \tilde{w}_{n+1}^\pm(x) = & -\frac{1}{2}\int_{x^*}^x \left\{ \tilde{w}_0^\pm(t)\frac{d}{dt}\left(h(t)\frac{d\tilde{w}_0^\pm}{dt}\right)\tilde{w}_n^\pm(t) + 2h(t)\tilde{w}_0^\pm(t)\frac{d\tilde{w}_0^\pm}{dt}\frac{d}{dt}\tilde{w}_n^\pm(t) \right. \\ & \left. + (\tilde{w}_0^\pm(t))^2\frac{d}{dt}\left(h(t)\frac{d\tilde{w}_n^\pm}{dt}\right) \right\} dt \text{ for } n \geq 1. \end{aligned} \quad (4.114)$$

- Solutions of the transport equations (4.113) and (4.114). We first define

$$\xi(x) = \int_r^x \frac{1}{\sqrt{h(t)}}dt, \quad (4.115)$$

and now we write the expression for the amplitudes $\tilde{w}_n^\pm(x)$ in terms of $\xi(x)$ up to $n = 4$, as follows:

$$\tilde{w}_0^\pm(x) = (\pm\alpha)^{-1/2} \left(\frac{d\xi}{dx} \right)^{1/2}, \quad (4.116)$$

$$\tilde{w}_1^\pm(x) = (\pm\alpha)^{-1} \int_r^x \left\{ \frac{5}{8} \left(\frac{d\xi}{dt} \right)^{-3} \left(\frac{d^2\xi}{dt^2} \right)^2 - \frac{1}{4} \left(\frac{d\xi}{dt} \right)^{-2} \frac{d^3\xi}{dt^3} \right\} dt, \quad (4.117)$$

$$\tilde{w}_2^\pm(x) = \frac{1}{2}(\tilde{w}_1^\pm(x))^2 - \frac{1}{2}(\pm\alpha)^{-1} \left(\frac{d\xi}{dx} \right)^{-1} \frac{d\tilde{w}_1^\pm}{dx}, \quad (4.118)$$

$$\begin{aligned} \tilde{w}_3^\pm(x) = & \frac{1}{6}(\tilde{w}_1^\pm(x))^3 - \frac{1}{2}(\pm\alpha)^{-1} \left(\frac{d\xi}{dx} \right)^{-1} \tilde{w}_1^\pm(x) \frac{d\tilde{w}_1^\pm}{dx} - \frac{1}{4}(\pm\alpha)^{-2} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^2\xi}{dx^2} \frac{d\tilde{w}_1^\pm}{dx} \\ & + \frac{1}{4}(\pm\alpha)^{-2} \left(\frac{d\xi}{dx} \right)^{-2} \frac{d^2\tilde{w}_1^\pm}{dx^2} - \frac{1}{2}(\pm\alpha)^{-1} \int_r^x \left(\frac{d\xi}{dt} \right)^{-1} \left(\frac{d\tilde{w}_1^\pm}{dt} \right)^2 dt, \end{aligned} \quad (4.119)$$

$$\begin{aligned} \tilde{w}_4^\pm(x) = & \frac{1}{24}(\tilde{w}_1^\pm(x))^4 - \frac{1}{4}(\pm\alpha)^{-1} \left(\frac{d\xi}{dx} \right)^{-1} (\tilde{w}_1^\pm(x))^2 \frac{d\tilde{w}_1^\pm}{dx} - \frac{1}{4}(\pm\alpha)^{-2} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^2\xi}{dx^2} \tilde{w}_1^\pm(x) \frac{d\tilde{w}_1^\pm}{dx} \\ & + \frac{3}{8}(\pm\alpha)^{-2} \left(\frac{d\xi}{dx} \right)^{-2} \left(\frac{d\tilde{w}_1^\pm}{dx} \right)^2 + \frac{1}{4}(\pm\alpha)^{-2} \left(\frac{d\xi}{dx} \right)^{-2} \tilde{w}_1^\pm(x) \frac{d^2\tilde{w}_1^\pm}{dx^2} \\ & - \frac{3}{8}(\pm\alpha)^{-3} \left(\frac{d\xi}{dx} \right)^{-5} \left(\frac{d^2\xi}{dx^2} \right)^2 \frac{d\tilde{w}_1^\pm}{dx} + \frac{1}{8}(\pm\alpha)^{-3} \left(\frac{d\xi}{dx} \right)^{-4} \frac{d^3\xi}{dx^3} \frac{d\tilde{w}_1^\pm}{dx} \\ & + \frac{3}{8}(\pm\alpha)^{-3} \left(\frac{d\xi}{dx} \right)^{-4} \frac{d^2\xi}{dx^2} \frac{d^2\tilde{w}_1^\pm}{dx^2} - \frac{1}{8}(\pm\alpha)^{-3} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^3\tilde{w}_1^\pm}{dx^3} \\ & + \frac{1}{4}(\pm\alpha)^{-3} \left(\frac{d\xi}{dx} \right)^{-2} \left(\frac{d\tilde{w}_1^\pm}{dx} \right)^2. \end{aligned} \quad (4.120)$$

4.5.3.2 Uniformly Valid Asymptotic Solution of the Approximate Equations.

In the previous section we obtained the Liouville-Green functions up to fourth order for the approximate equations (4.97) and (4.99). We assumed an asymptotic solution for the free surface potential $\psi(x)$ given by equations (4.100) and (4.101). We obtained the Eikonal equation for the phase $S(x)$ and the transport equation for the amplitude corrections $\tilde{w}_n^\pm(x)$. These quantities will be important in the matching of the far field asymptotic solution (4.55) with the uniformly valid asymptotic solution of the approximate form of equation (4.54), valid in a neighborhood of a turning point. Here we derived a uniformly valid asymptotic solution for the approximate equations

(4.97) and (4.99). We are going to use the comparison equation technique. Through a change of variables

$$\psi(x) = \zeta(x)\Phi(\varphi(x)), \quad (4.121)$$

we reduce equations (4.97) and (4.99) to second order differential equations with known solution. These equations are the comparison equations. For equation (4.97), if we assume that the function $Q(x)$ is linear in a neighborhood of a turning point, the comparison equation is the Airy's equation. If $Q(x)$ is quadratic, we have Weber's equation as a comparison equation. We assume that $Q(x)$ is linear in a neighborhood of the turning point, so $\Phi(\varphi)$ is a linear combination of the solutions of the Airy's equation. For equation (4.99), we can use one of the forms of the Bessel equation as comparison equation, since we assume that $h(x)$ is a linear function of x . In this case $\Phi(\varphi)$ can be written in terms of Hankel's functions. If we substitute the change of variables (4.121) in equations (4.97) and (4.99), we obtain that $\Phi(\varphi)$ is a solution of a second order differential equation, as expected. We have some liberty to chose the functional form of $\zeta(x)$. We chose $\zeta(x)$ given in terms of $\varphi(x)$ according to the equations

$$\zeta(x) = \beta^2 \exp\left(\frac{i}{\beta} k_c(x)\right) \left(\frac{d\varphi}{dx}\right)^{-1/2} \text{ for the equation (4.97),} \quad (4.122)$$

$$\zeta(x) = \left(\frac{\varphi(x)}{h(x)\frac{d\varphi}{dx}}\right)^{1/2} \text{ for the equation (4.99).} \quad (4.123)$$

For the approximate equation (4.97) the differential equation for $\Phi(\varphi)$ is

$$\frac{d^2\Phi}{d\varphi^2} + \left[\frac{3}{4} \left(\frac{d\varphi}{dx}\right)^{-4} \left(\frac{d^2\varphi}{dx^2}\right)^2 - \frac{1}{2} \left(\frac{d\varphi}{dx}\right)^{-3} \frac{d^3\varphi}{dx^3} - \frac{1}{\beta^2} Q(x) \left(\frac{d\varphi}{dx}\right)^{-2} \right] \Phi(\varphi) = 0. \quad (4.124)$$

Since we assumed that $Q(x)$ is a linear function of x close to the turning point, the comparison equation is Airy's equation. We impose that $\Phi(\varphi)$ should satisfy the differential equation

$$\frac{d^2\Phi}{d\varphi^2} + \frac{\varphi}{\beta^2}\Phi = 0. \quad (4.125)$$

If $Q(x)$ were a quadratic function of x in a neighborhood of the turning point, we would have φ^2 instead of φ in the equation (4.125). If we compare equations (4.124) and (4.125) we obtain for $\varphi(x)$ the differential equation

$$\varphi \left(\frac{d\varphi}{dx} \right)^4 + Q(x) \left(\frac{d\varphi}{dx} \right)^2 + \beta^2 \left\{ -\frac{3}{4} \left(\frac{d^2\varphi}{dx^2} \right)^2 + \frac{1}{2} \frac{d\varphi}{dx} \frac{d^3\varphi}{dx^3} \right\} = 0. \quad (4.126)$$

For the approximate equation (4.99), the differential equation for $\Phi(\varphi)$ is

$$\begin{aligned} \beta^2 \frac{d^2\Phi}{d\varphi^2} + \beta^2 \left\{ \frac{2}{\zeta} \frac{d\zeta}{dx} \left(\frac{d\varphi}{dx} \right)^{-1} + \left(\frac{d\varphi}{dx} \right)^{-2} \frac{d^2\varphi}{dx^2} + \frac{1}{h(x)} \frac{dh}{dx} \left(\frac{d\varphi}{dx} \right)^{-1} \right\} \frac{d\Phi}{d\varphi} \\ + \left[\beta^2 \frac{1}{\zeta} \frac{d^2\zeta}{dx^2} \left(\frac{d\varphi}{dx} \right)^{-2} + \frac{\beta^2}{h(x)} \frac{dh}{dx} \frac{1}{\zeta} \frac{d\zeta}{dx} \left(\frac{d\varphi}{dx} \right)^{-2} + \frac{\alpha^2}{h(x)} \left(\frac{d\varphi}{dx} \right)^{-2} \right] \Phi(\varphi) = 0, \end{aligned} \quad (4.127)$$

and since we assumed $h(x)$ a linear function in x in a neighborhood of the turning point, the comparison equation which $\Phi(\varphi)$ should satisfy is

$$\beta^2 \varphi \frac{d^2\Phi}{d\varphi^2} + \beta^2 \frac{d\Phi}{d\varphi} + \alpha^2 \Phi(\varphi) = 0. \quad (4.128)$$

If we compare equation (4.127) with equation (4.128), we realized that $\varphi(x)$ should satisfy the differential equation

$$\begin{aligned} & \beta^2 \left\{ - \left(\frac{d\varphi}{dx} \right)^4 + 3\varphi^2 \left(\frac{d^2\varphi}{dx^2} \right)^2 - 2\varphi^2 \frac{d\varphi}{dx} \frac{d^3\varphi}{dx^3} - \varphi^2 \left(\frac{d\varphi}{dx} \right)^2 \left[\frac{d}{dx} \left(\frac{1}{h(x)} \frac{dh}{dx} \right) + \frac{1}{h(x)} \frac{d^2h}{dx^2} \right] \right\} \\ & + 4 \frac{\alpha^2}{h(x)} \varphi^2 \left(\frac{d\varphi}{dx} \right)^2 = 4\alpha^2 \varphi(x) \left(\frac{d\varphi}{dx} \right)^4. \end{aligned} \quad (4.129)$$

To find $\varphi(x)$ for the approximate equation (4.97) we have to solve the equation (4.126), and for the approximate equation (4.99) we need to solve the equation (4.129). Both equations (4.126) and (4.129) have the small parameter β , so we can obtain an asymptotic solution for $\varphi(x)$. We assume a series expansion for $\varphi(x)$ in terms of the parameter β in the form

$$\varphi(x) = \varphi_0(x) \left(1 + \sum_{n=1}^{\infty} \beta^{2n} \varphi_n(x) \right). \quad (4.130)$$

Next, we obtain the terms in the expansion for $\varphi(x)$ above for the approximate equation (4.97). We substitute the expansion for $\varphi(x)$, given by equation (4.130) into the differential equation (4.127). If we collect terms of the same order in the parameter β , we obtain a sequence of first order equations, which gives recurrence relations for the terms $\varphi_n(x)$ in the expansion (4.130). The equation for $\varphi_n(x)$ is a function of $\varphi_j(x)$, $0 \leq j < n$. We solve these equations up to fourth order in the parameter β .

- Solution of the equation of order $O(\beta^0)$ is

$$\frac{2}{3}(\varphi_0(x))^{3/2} = e^{\pm i\frac{\pi}{2}} \xi(x) \text{ or } \varphi(x) = \left(\frac{3}{2} \right)^{2/3} e^{\pm i\frac{\pi}{3}} \xi(x)^{2/3}, \quad (4.131)$$

where $\xi(x)$ is defined in terms of the function $Q(x)$ according to equation (4.106).

- Solution of the equation of order $O(\beta^2)$ is

$$\begin{aligned}
\varphi_1(x) &= \frac{1}{2} e^{\mp i \frac{\pi}{2}} \left\{ \frac{5}{36} \frac{1}{\xi(x)} + \int_r^x \left[\frac{3}{4} \left(\frac{d\xi}{dt} \right)^{-3} \left(\frac{d^2 \xi}{dt^2} \right)^2 - \frac{1}{2} \left(\frac{d\xi}{dt} \right)^{-2} \frac{d^3 \xi}{dt^3} \right] dt \right\} \\
&= \pm i \frac{5}{36} \frac{1}{\xi(x)} - \tilde{w}_1^\pm(x),
\end{aligned} \tag{4.132}$$

where $\tilde{w}_1^\pm(x)$ is defined by equation (4.107).

- Solution of the equation of order $O(\beta^4)$ is

$$\begin{aligned}
\varphi_2(x) &= \frac{i}{6} \frac{(\varphi_1(x))^2}{\xi} + \frac{5}{72} \frac{\varphi_1(x)}{\xi(x)^2} + \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-2} \frac{d^2 \varphi_1}{dx^2} + \int_r^x \left[\frac{5}{24} \frac{1}{\xi(x)^2} \frac{d\varphi}{dt} \right. \\
&\quad \left. + i \frac{5}{2} \left(\frac{d\xi}{dx} \right)^{-1} \left(\frac{d\varphi_1}{dt} \right)^2 + \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^3 \xi}{dt^3} \frac{d\varphi_1}{dt} - \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^2 \xi}{dt^2} \frac{d^2 \varphi_1}{dt^2} \right] dt \\
&= \frac{i}{6} \frac{(\tilde{w}_1^\pm(x))^2}{\xi} - \frac{5}{24} \frac{\tilde{w}_1^\pm(x)}{\xi(x)^2} - \frac{10}{243} \frac{i}{\xi(x)^3} - \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-2} \frac{d^2 \tilde{w}_1^\pm}{dx^2} \\
&\quad + \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-3} \left(\frac{d^2 \xi}{dx^2} \right) \frac{d\tilde{w}_1^\pm}{dx} + \frac{i}{2} \int_r^x \left[\left(\frac{d\xi}{dt} \right)^{-1} \left(\frac{d\tilde{w}_1^\pm}{dt} \right)^2 \right] dt.
\end{aligned} \tag{4.133}$$

Next, we do the same for the approximate equation (4.99). We substitute the expansion for $\varphi(x)$, given by equation (4.130), into the equation (4.129). As in the previous case, if we collect terms of the same order in the parameter β , we obtain a sequence of first order equations for the terms $\varphi_n(x)$ in the equation (4.130). These set of equations are also recurrence relations, in the same way we had in the previous case. We again solve the equations up to $O(\beta^4)$.

- Solution of the equation of order $O(\beta^0)$ is

$$(\varphi_0(x))^{1/2} = \pm \frac{1}{2} \xi(x) \text{ or } \varphi_0(x) = \frac{1}{4} \xi(x)^2, \tag{4.134}$$

where $\xi(x)$ is defined in terms of the function $h(x)$ according to equation (4.115).

- Solution of the equation of order $O(\beta^2)$ is

$$\begin{aligned}\varphi_1(x) &= \pm \frac{1}{8} \frac{1}{\alpha^2 \xi(x)} \pm \frac{1}{\alpha^2} \int_r^x \left[-\frac{5}{8} \left(\frac{d\xi}{dt} \right)^{-3} \left(\frac{d^2 \xi}{dt^2} \right)^2 + \frac{1}{4} \left(\frac{d\xi}{dt} \right)^{-2} \frac{d^3 \xi}{dt^3} \right] dt \\ &= \pm \frac{1}{8} \frac{1}{\alpha^2 \xi(x)} - \tilde{w}_1^\pm(x),\end{aligned}\tag{4.135}$$

where $\tilde{w}_1^\pm(x)$ is defined by equation (4.117).

- Solution of the equation of order $O(\beta^4)$ is

$$\begin{aligned}\varphi_2(x) &= \mp \frac{7}{96\alpha^4} \frac{1}{\xi(x)^3} \pm \frac{1}{2\alpha^2} \frac{(\tilde{w}_1^\pm(x))^2}{\xi(x)} + \frac{1}{\alpha^3} \left\{ \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-2} \frac{d^2 \tilde{w}_1^\pm}{dx^2} - \frac{1}{4} \left(\frac{d\xi}{dx} \right)^{-3} \frac{d^2 \xi}{dx^2} \frac{d\tilde{w}_1^\pm}{dx} \right. \\ &\quad \left. - \frac{1}{2} \int_r^x \left[\left(\frac{d\xi}{dt} \right)^{-1} \left(\frac{d\tilde{w}_1^\pm}{dt} \right)^2 \right] dt \right\}.\end{aligned}\tag{4.136}$$

Next, we discuss the solution of the comparison equation for each of the approximate equations (4.97) and (4.99). We first consider the comparison equation for the approximate (4.97). This comparison equation is given by equation (4.125). It can be solved in terms of contour integrals of the form

$$\Phi_j(\varphi) = \int_{C_j} \exp(\beta^2 \frac{t^3}{3} + \varphi t) dt, j = 1, 2, 3,\tag{4.137}$$

and the integration contours C_j in the complex t plane are illustrated in the figure 4-21 below.

Once the integration contours C_j are well defined, the functions $\Phi_j(\varphi)$ are well defined, and the solution of the approximate equation (4.97) can be written in terms

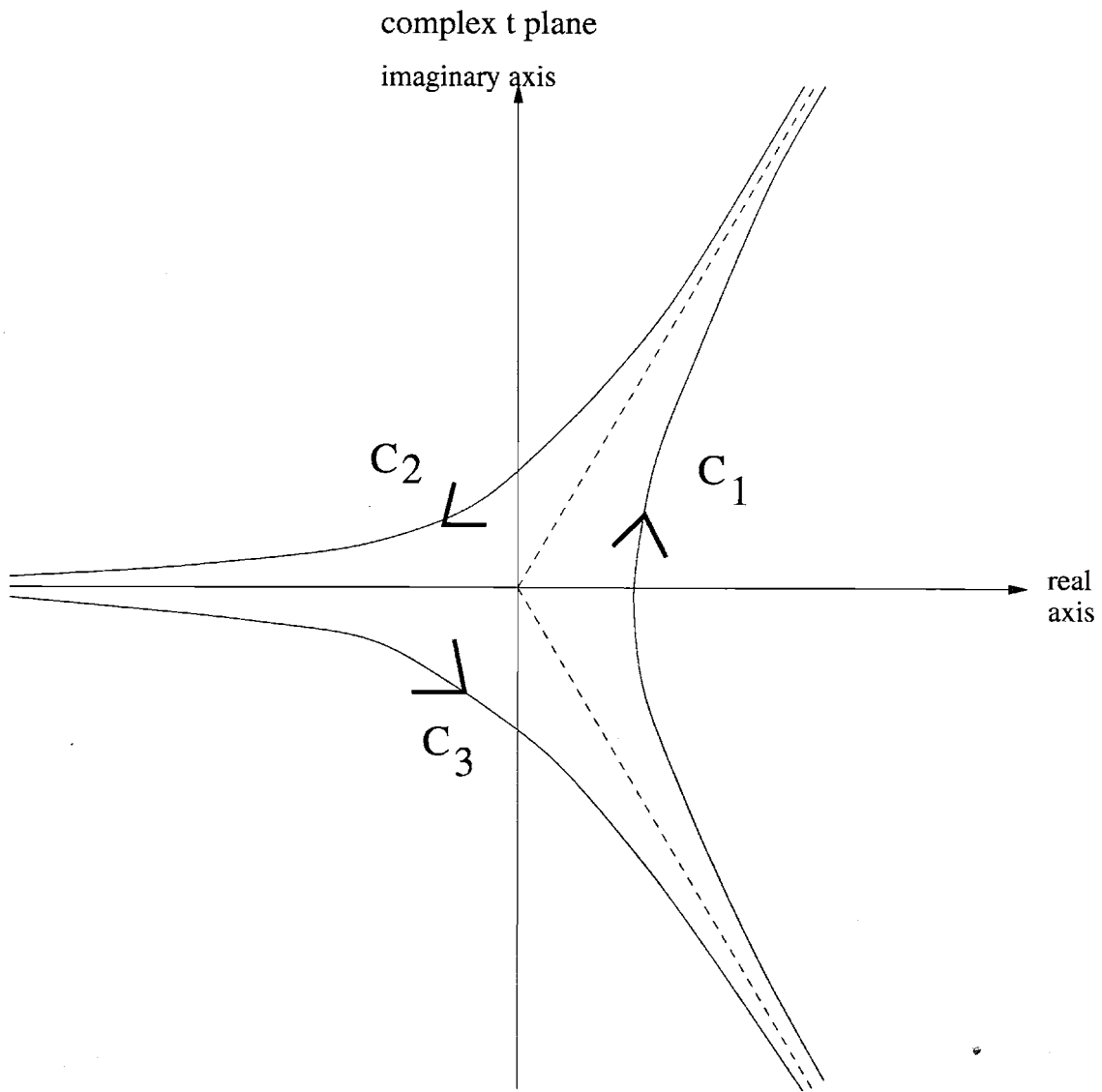


Figure 4-21: Contours C_j in the complex t plane used to define the contour integrals given by equation (4.137). The sets indicate the sense we travel along the contours as the integration is performed.

functions $\Phi_j(\varphi)$ as linear combination of two of the functions

$$\psi_j(x) = \beta^2 \exp\left(\frac{i}{\beta} k_c(x)\right) \left(\frac{d\varphi}{dx}\right)^{-1/2} \Phi_j(\varphi(x)), j = 1, 2, 3, \quad (4.138)$$

where $\varphi(x)$ is given as the series expansion (4.130) in terms of the parameter β . We evaluated this series until terms of order $O(\beta^4)$. We are actually interested in the asymptotic expansion of the solutions given by equation (4.138) as $\beta \rightarrow 0$. These asymptotic expansions are going to be matched with the far field asymptotic solution to continue them through the turning points. Before we obtain the asymptotic expansion for the solutions $\psi_j(x)$, we need to obtain the asymptotic expansion of the functions $\Phi(\varphi)$ as $\beta \rightarrow 0$. To do so, we use the method of the steepest descent, as discussed in the appendix I. Details are omitted. Here we give the uniformly valid asymptotic expansion of the functions $\psi_j(x)$ in terms of the asymptotic expansion of the functions $\Phi_j(\varphi)$. We defined the sectors S_n in the complex x plane according to the equation

$$S_n : -\frac{\pi}{3} + \frac{2n\pi}{3} < \arg\{\varphi(x)\} < \frac{\pi}{3} + \frac{2n\pi}{3}, \text{ with } n = 0, \dots, 5. \quad (4.139)$$

The asymptotic expansion of the functions $\psi_j(x)$ change as we change sectors, and they can be written as a linear combination of the functions

$$H_1(x, \beta) = -\beta^2 \left(\frac{d\varphi}{dx}\right)^{-1/2} \exp\left(\frac{2}{3} \frac{i}{\beta} \varphi(x)^{3/2} + i\frac{\pi}{4} + \frac{i}{\beta} k_c(x)\right) \sqrt{\frac{\pi}{\beta}} \varphi(x)^{-1/4} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-i)^n \Gamma(3n+1/2)}{54^n n! \Gamma(n+1/2)} \left(\frac{3}{2}\right)^n \beta^n \varphi(x)^{-3n/2} \right\}, \quad (4.140)$$

$$H_2(x, \beta) = \beta^2 \left(\frac{d\varphi}{dx}\right)^{-1/2} \exp\left(-\frac{2}{3} \frac{i}{\beta} \varphi(x)^{3/2} - i\frac{\pi}{4} + \frac{i}{\beta} k_c(x)\right) \sqrt{\frac{\pi}{\beta}} \varphi(x)^{-1/4} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(i)^n \Gamma(3n+1/2)}{54^n n! \Gamma(n+1/2)} \left(\frac{3}{2}\right)^n \beta^n \varphi(x)^{-3n/2} \right\}. \quad (4.141)$$

The asymptotic expansion of the functions $\psi_j(x)$ in terms of the functions $H_1(x, \beta)$ and $H_2(x, \beta)$ in the sectors S_n follows:

- In the sector S_0 we have:

$$\psi_1(x) \sim -H_1(x, \beta) - H_2(x, \beta), \quad (4.142)$$

$$\psi_2(x) \sim H_1(x, \beta), \quad (4.143)$$

$$\psi_3(x) \sim H_2(x, \beta). \quad (4.144)$$

- In the sector S_1 we have:

$$\psi_1(x) \sim -H_2(x, \beta), \quad (4.145)$$

$$\psi_2(x) \sim H_1(x, \beta), \quad (4.146)$$

$$\psi_3(x) \sim -H_1(x, \beta) + H_2(x, \beta). \quad (4.147)$$

- In the sector S_2 we have:

$$\psi_1(x) \sim -H_2(x, \beta), \quad (4.148)$$

$$\psi_2(x) \sim H_1(x, \beta) - H_2(x, \beta), \quad (4.149)$$

$$\psi_3(x) \sim -H_1(x, \beta). \quad (4.150)$$

- In the sector S_3 we have:

$$\psi_1(x) \sim H_1(x, \beta) - H_2(x, \beta), \quad (4.151)$$

$$\psi_2(x) \sim H_2(x, \beta), \quad (4.152)$$

$$\psi_3(x) \sim -H_1(x, \beta). \quad (4.153)$$

- In the sector S_4 we have:

$$\psi_1(x) \sim H_1(x, \beta), \quad (4.154)$$

$$\psi_2(x) \sim H_2(x, \beta), \quad (4.155)$$

$$\psi_3(x) \sim -H_1(x, \beta) - H_2(x, \beta). \quad (4.156)$$

- In the sector S_5 we have:

$$\psi_1(x) \sim H_1(x, \beta), \quad (4.157)$$

$$\psi_2(x) \sim -H_1(x, \beta) + H_2(x, \beta), \quad (4.158)$$

$$\psi_3(x) \sim -H_2(x, \beta), \quad (4.159)$$

where the function $\varphi(x)$ is given by the series expansion in β given by the equation (4.130). If we substitute this expansion for $\varphi(x)$ with $\varphi_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ given, respectively, by the equations (4.131), (4.132) and (4.133) into the equations (4.140) and (4.141), and reexpand the resulting expressions with respect to β , we obtain for the functions $H_j(x, \beta)$ the equations

$$\begin{aligned} H_1(x, \beta)|_{\pm i\xi(x)} = & -\tilde{w}_0^\mp(x) \exp(\mp \frac{1}{\beta}\xi(x) + i\frac{\pi}{4} + \frac{i}{\beta}k_c(x))\beta^{3/2}\pi^{1/2} \{1 - i\beta\tilde{w}_1^\mp(x) - \beta^2\tilde{w}_2^\mp(x) \\ & + i\beta^3\tilde{w}_3^\mp(x) + \beta^4\tilde{w}_4^\mp(x) + O(\beta^5)\}, \end{aligned} \quad (4.160)$$

$$\begin{aligned} H_2(x, \beta)|_{\pm i\xi(x)} = & \tilde{w}_0^\pm(x) \exp(\pm \frac{1}{\beta}\xi(x) + i\frac{\pi}{4} + \frac{i}{\beta}k_c(x))\beta^{3/2}\pi^{1/2} \{1 - i\beta\tilde{w}_1^\pm(x) - \beta^2\tilde{w}_2^\pm(x) \\ & + i\beta^3\tilde{w}_3^\pm(x) + \beta^4\tilde{w}_4^\pm(x) + O(\beta^5)\}. \end{aligned} \quad (4.161)$$

In the solution for $\varphi_0(x)$, we obtained two branches (see equation (4.131)). So, we actually obtained two expansion for $\varphi(x)$, and by fixing which branch of φ_0 we are using, we specify which of the two expansions for $\varphi(x)$ we are using. This reflects in the expansions for the functions $H_j(x, \beta)$ above. We appended $\pm i\xi(x)$ to these functions to specify which branch of $\varphi_0(x)$ we are using.

The expressions above for the functions $H_j(x, \beta)$ provide asymptotic expansions for the solutions $\psi_j(x)$, $j = 1, 2, 3$ of the approximate equation (4.97) in terms of the amplitude corrections $\tilde{w}_n^\pm(x)$, which will appear in the form of the far field expansion valid close to a turning point. This will result in a nice matching between the asymptotic expansion of the solution of the approximate equation and the form of the far field expansion valid in a neighborhood of a turning point up to fourth order in β , and will allow us to obtain the desired connection formulas for the far field asymptotic solution.

Next, we do the same for the approximate equation (4.99). Its comparison equation (4.128) has solutions in terms of contour integrals

$$\Phi(\varphi) = \int_C \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2} t\right) dt, \quad (4.162)$$

where the integration contour C can be one of the contours C_1 , C_{2j} and C_{3j} with $j = 1, 2, 3$. These contours are illustrated, respectively, in the figures 4-22, 4-23 and 4-24.

We define the function $\Phi_1(\varphi)$ in terms of the contour C_1 illustrated in figure 4-22. We define also two other solutions. We define $\Phi_j(\varphi)$ in terms of the contours C_{jl} , $j = 2, 3$ and $l = 1, 2, 3$. The contours C_{jl} do not define a contour integral for all values of the argument of φ , but they can be used to define contour integrals for specific ranges of the argument of φ . Once we define a contour integral solution using one of these contours (for example, use C_{21} in figure 4-23), we can continue this contour integral solution for other ranges of the argument of φ by using the other two contour (for example, contours C_{22} and C_{23} in figure 4-23). The regions where these contours can be used overlap, so in the overlapping sectors we use Cauchy's contour integral theorem to deform one contour in terms of the other (for example, deform contour C_{21} in terms of contour C_{22} or C_{23}). At the end, we obtain to functions $\Phi(\varphi)$ defined for the necessary range of the argument of φ . This way of defining the contour integral solutions was used in the second part of chapter 3.

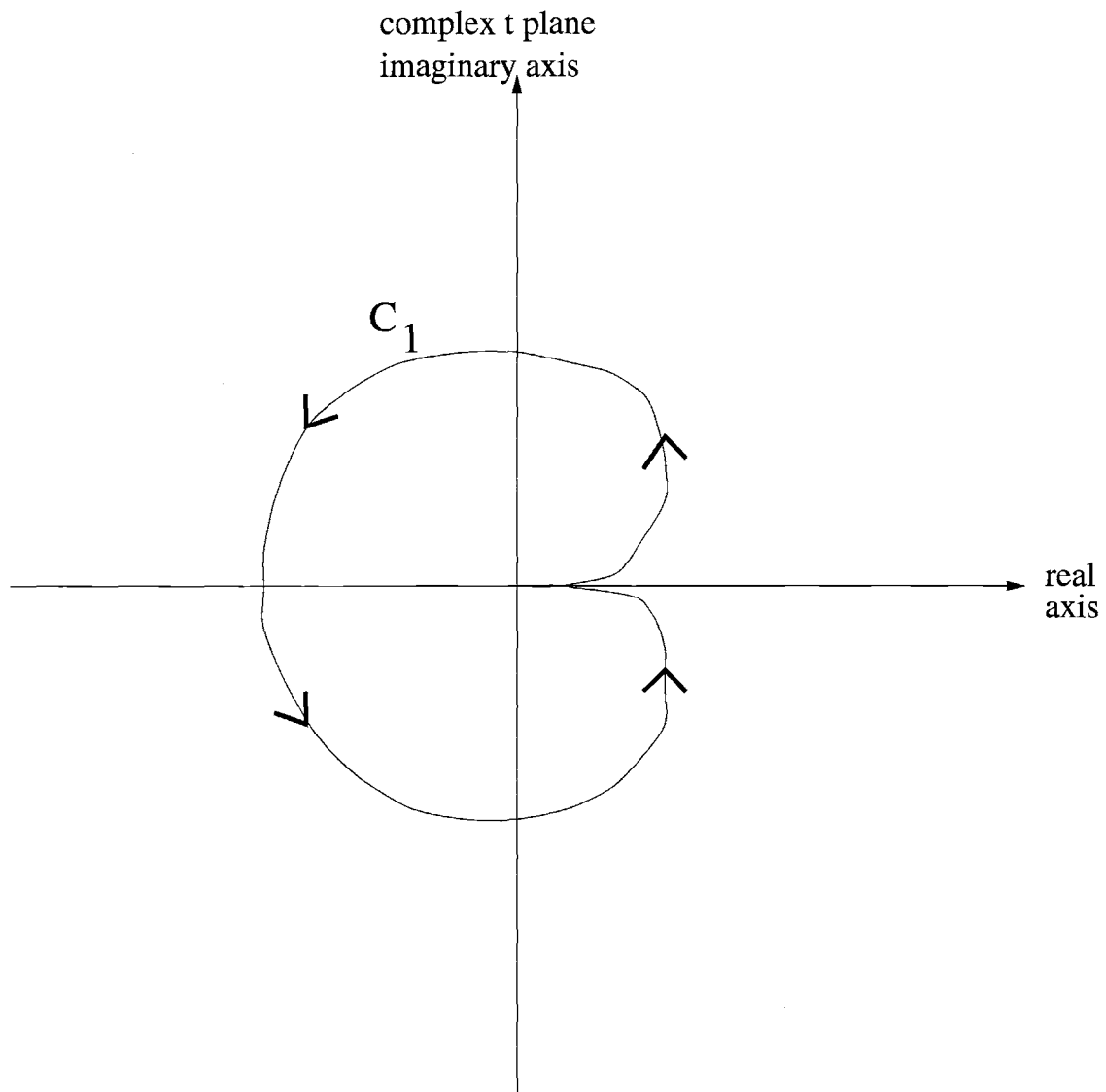


Figure 4-22: Contour C_1 in the complex t plane used to define one of the contour integrals given by equation (4.162). The sets indicate the sense we travel along the contours as the integration is performed.

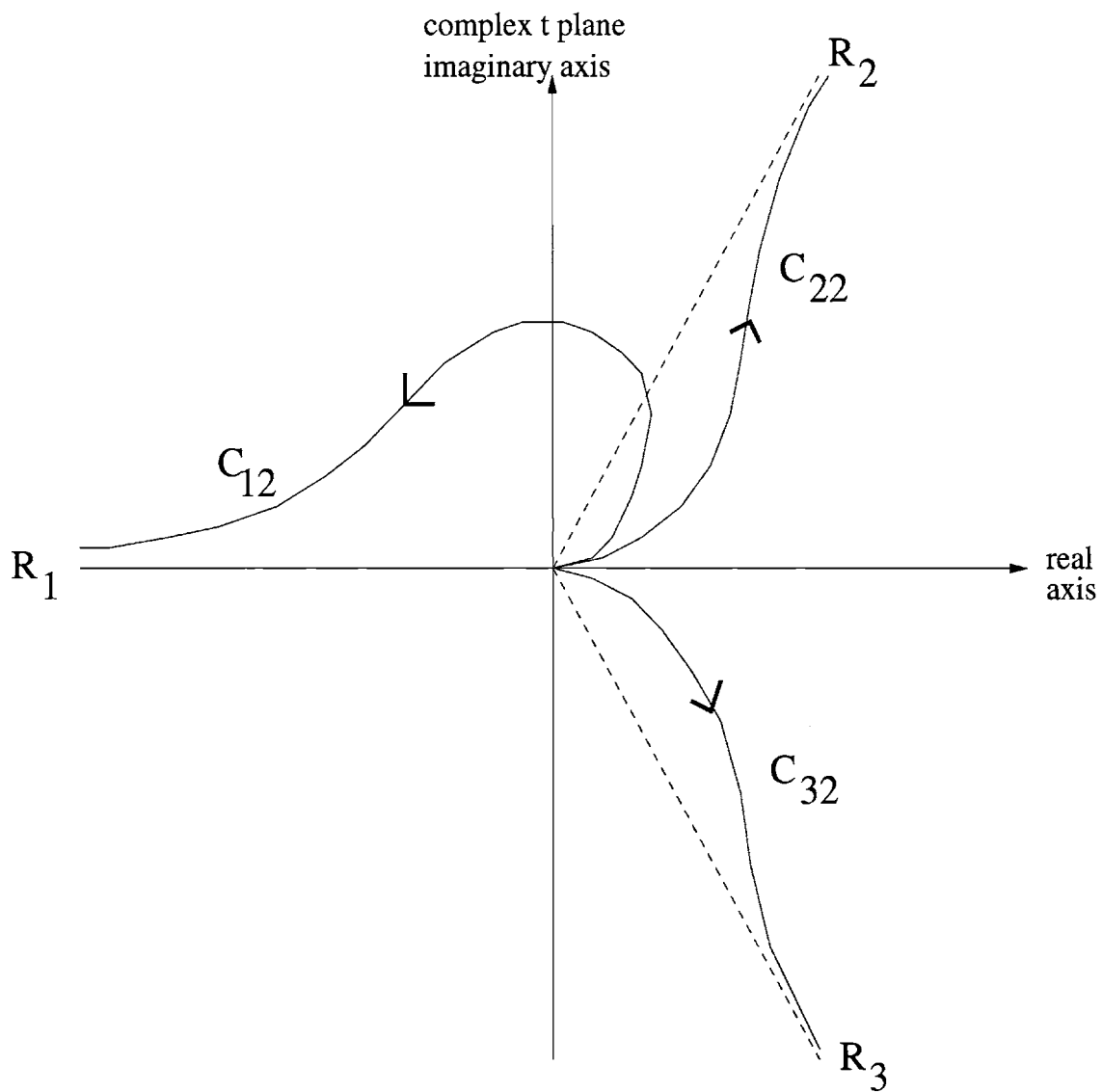


Figure 4-23: Contours $C_{2j}, j = 1, 2, 3$ in the complex t plane used to define one of the contour integrals given by equation (4.162). The sets indicate the sense we travel along the contours as the integration is performed.

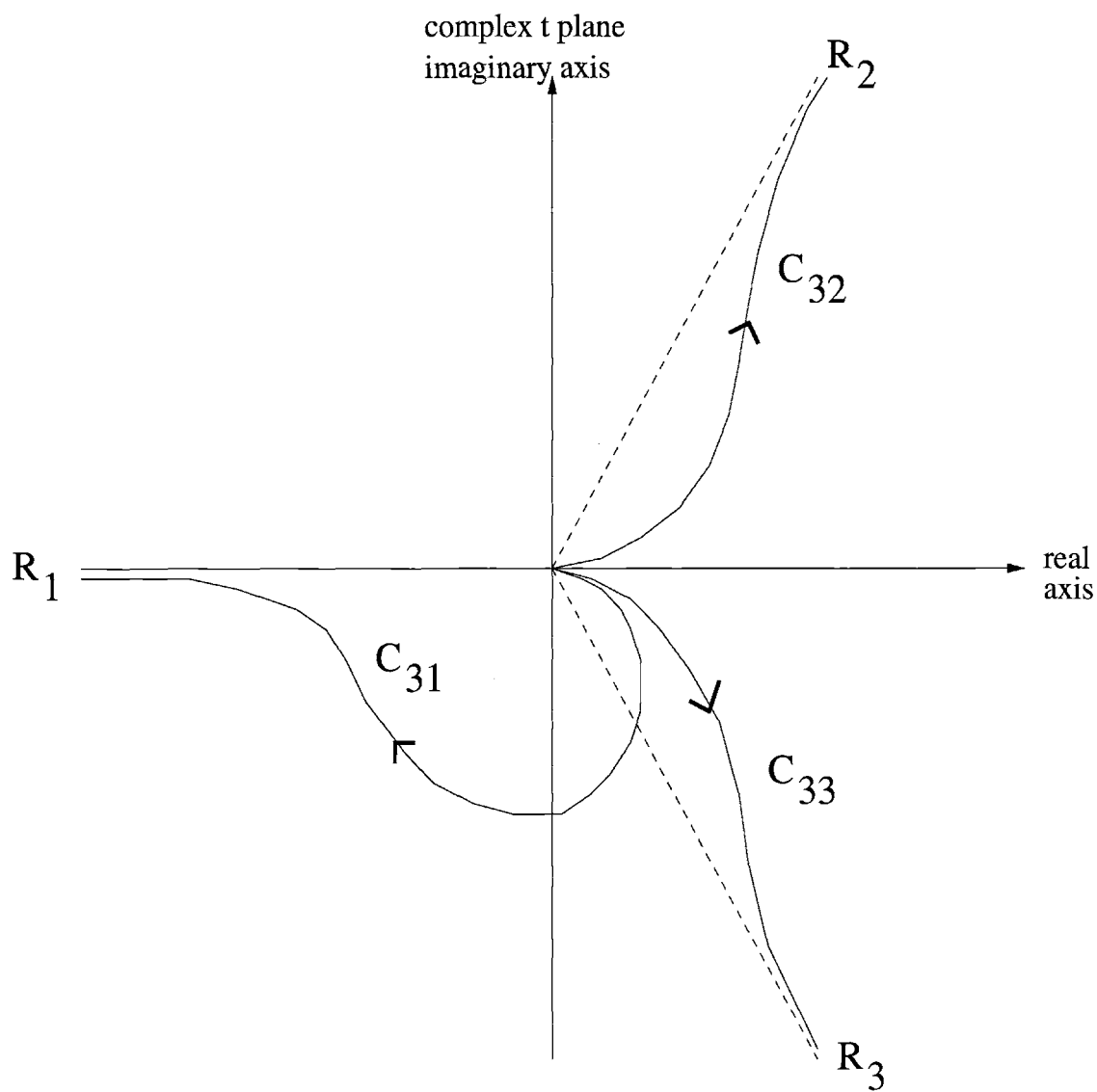


Figure 4-24: Contours $C_{3j}, j = 1, 2, 3$ in the complex t plane used to define one of the contour integrals given by equation (4.162). The sets indicate the sense we travel along the contours as the integration is performed.

In figures 4-23 and 4-24 we define the rays $R_j, j = 1, 2, 3$ in the complex t plane. The ray R_1 coincides with the negative real axis, the ray R_2 makes an angle of $\frac{\pi}{3}$ with respect to the positive real axis and the ray R_3 makes an angle of $-\frac{\pi}{3}$ with respect to the positive real axis. We use these rays to define contour integrals. The argument of the exponential in the contour integral given by equation (4.162) goes to $-\infty$ along these rays for the sectors

$$SR_1 : -\frac{\pi}{2} + 2n\pi < \arg\{\varphi\} < \frac{\pi}{2} + 2n\pi \text{ with } n \text{ as an integer number ,} \quad (4.163)$$

$$SR_2 : \frac{\pi}{6} + 2n\pi < \arg\{\varphi\} < \frac{7\pi}{6} + 2n\pi \text{ with } n \text{ as an integer number ,} \quad (4.164)$$

$$SR_3 : \frac{-\pi}{6} + 2n\pi < \arg\{\varphi\} < \frac{-7\pi}{6} + 2n\pi \text{ with } n \text{ as an integer number .} \quad (4.165)$$

So, in the region SR_j we use the ray R_j to define a contour that is used to continue or define functions $\Phi(\varphi)$. We define functions $\Phi_j(\varphi), j = 2, 3$ in the range $-\pi/2 < \arg\{\varphi\} < \pi/2$ as the contour integrals

$$\Phi_2(\varphi) = \int_{C_{21}} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2}t\right) dt, \quad (4.166)$$

$$\Phi_3(\varphi) = \int_{C_{31}} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2}t\right) dt. \quad (4.167)$$

Now the functions $\Phi_j(\varphi), j = 2, 3$ are well defined in the sector $-\pi/2 < \arg\{\varphi\} < \pi/2$. To continue them to the sector $\frac{\pi}{6} < \arg\{\varphi\} < \frac{7\pi}{6}$ (Region SR_2), we deform the contours C_{j1} in terms of the contour C_{j2} . For $\arg\{\varphi\}$ in the overlapping sector $\frac{\pi}{6} < \arg\{\varphi\} < \pi/2$, we obtain in the complex t plane a closed contour using the contours C_{j1}, C_{j2} ($j = 2, 3$) and a contour at the infinite connecting these two contours. We then we apply Cauchy's integral theorem to the contour integral. For the case $j = 3$, the resulting closed contour contains the origin of the t complex plane. The integrand of the contour integral has an essential singularity at the origin of the complex t plane. To avoid dealing with it, we add to the closed contour the contour C_1 illustrated in

figure 4-22. The resulting contour is closed, but do not encircle the origin of the complex t plane. For this range of $\arg\{\varphi\}$ we have

$$\Phi_2(\varphi) = \int_{C_{22}} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2}t\right) dt, \quad (4.168)$$

$$\Phi_3(\varphi) = \int_{C_{32} \cup -C_1} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2}t\right) dt. \quad (4.169)$$

The contour used for $\Phi_3(\varphi)$ in the range $\frac{\pi}{6} < \arg\{\varphi\} < \frac{7\pi}{6}$ is composed of the contour C_{31} and the contour C_1 , but traveled in the direction opposite to the one used in its definition in figure 4-22 as the integration is performed. Next, we continue the contour integrals defining the functions $\Phi_j(\varphi)$ to the sector $\frac{5\pi}{6} < \arg\{\varphi\} < \frac{11\pi}{6}$. We deform the contours C_{j2} in the contours C_{j3} in the same way we did before. We then obtain

$$\Phi_2(\varphi) = \int_{C_{23}} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2}t\right) dt, \quad (4.170)$$

$$\Phi_3(\varphi) = \int_{C_{33} \cup -C_1} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2}t\right) dt. \quad (4.171)$$

Next, we continue the contour integrals defining the functions $\Phi_j(\varphi)$ to the sector $\frac{3\pi}{2} < \arg\{\varphi\} < \frac{5\pi}{2}$. We deform the contours C_{j3} in the contours C_{j1} as we did before. Now, notice that when we define a closed contour using contour C_{23} and C_{21} we end up encircling the origin of the complex t plane. To avoid that we use the contour C_1 defined in figure 4-22. The resulting contour is closed and has no singularities of the integrand inside it, so we obtain

$$\Phi_2(\varphi) = \int_{C_{21} \cup -C_1} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2}t\right) dt, \quad (4.172)$$

$$\Phi_3(\varphi) = \int_{C_{31} \cup -C_1} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2}t\right) dt. \quad (4.173)$$

If we keep continuing the contour integrals in the anti-clockwise sense in the complex φ plane, we end up encircling the origin of the complex t plane and adding one more time the contour $-C_1$ to the integration contour. If we continue the contour integrals defining the functions $\Phi_j(\varphi)$ in the clockwise sense in the complex φ plane, every time we encircle the origin of the complex t plane, we add the contour C_1 to the integration contour of the contour integrals. So, we can generalize the previous results as follows:

- The functions $\Phi_j(\varphi), j = 2, 3$ in the sectors SR_1 ($-\frac{\pi}{2} + 2n\pi < \arg\{\varphi\} < \frac{\pi}{2} + 2n\pi$) are given by the contour integrals

$$\Phi_2(\varphi) = \int_{C_{21} \cup -n(C_1)} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2} t\right) dt, \quad (4.174)$$

$$\Phi_3(\varphi) = \int_{C_{31} \cup -n(C_1)} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2} t\right) dt, \quad (4.175)$$

where n assume integer values. The n in front of C_1 means how many time we encircle the origin as we continue the contour integrals defined by equations (4.166) and (4.167).

- The functions $\Phi_j(\varphi), j = 2, 3$ in the sectors SR_2 ($\frac{\pi}{6} + 2n\pi < \arg\{\varphi\} < \frac{7\pi}{6} + 2n\pi$) are given by the contour integrals

$$\Phi_2(\varphi) = \int_{C_{22} \cup -(n+1)(C_1)} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2} t\right) dt, \quad (4.176)$$

$$\Phi_3(\varphi) = \int_{C_{32} \cup -n(C_1)} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2} t\right) dt. \quad (4.177)$$

- The functions $\Phi_j(\varphi), j = 2, 3$ in the sectors SR_3 ($-\frac{\pi}{6} + 2n\pi > \arg\{\varphi\} > \frac{-7\pi}{6} + 2n\pi$) are given by the contour integrals

$$\Phi_2(\varphi) = \int_{C_{22} \cup -(n+1)(C_1)} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2} t\right) dt, \quad (4.178)$$

$$\Phi_3(\varphi) = \int_{C_{32} \cup -n(C_1)} \frac{1}{t} \exp\left(-\frac{\alpha^2}{t} + \frac{\varphi}{\beta^2} t\right) dt. \quad (4.179)$$

Now that the functions $\Phi_j(\varphi), j = 1, 2, 3$ are well defined for any range of the $\arg\{\varphi\}$, we can write solutions for the approximate equation (4.99) in terms of these functions, according to the equation

$$\psi_j(x) = \left(\frac{h(x)}{\varphi(x)} \frac{d\varphi}{dx} \right)^{-1/2} \Phi_j(\varphi(x)) \text{ for } j = 1, 2, 3, \quad (4.180)$$

where $\varphi(x)$ is given by the series expansion (4.130) in terms of the parameter β and evaluated until terms of order $O(\beta^4)$. We are actually interested in the asymptotic expansion of the functions $\psi_j(x)$, given by equation (4.180), as $\beta \rightarrow 0$. Before we obtain the asymptotic expansion of these functions, we need to obtain the asymptotic expansion of functions $\Phi_j(\varphi)$ as $\frac{|\varphi|}{\beta^2} \gg 1$. To do so, we use the method of the steepest descent, as given in the appendix I. Details are omitted. Here we give the asymptotic expansion of the functions $\psi_j(x)$ in terms of the asymptotic expansion of the functions $\Phi_j(\varphi), j = 1, 2, 3$. We defined the regions S_n in terms of the complex x variable according to the equation

$$S_n : -\pi + 2n\pi < \arg\{\varphi(x)\} < \pi + 2n\pi, \text{ with } n \text{ integer.} \quad (4.181)$$

For example, the range of x in S_0 is given by $-\pi < \arg\{\varphi(x)\} < \pi$. We need the asymptotic expansion for the functions $\psi_j(x)$ only in the regions S_{-2}, S_{-1}, S_0, S_1 and S_2 . We give the desired asymptotic expansion as a linear combination of the functions

$$H_1(x, \beta) = - \left(\frac{h(x)}{\varphi(x)} \frac{d\varphi}{dx} \right)^{-1/2} \exp(i \frac{2\alpha}{\beta} \varphi(x)^{1/2} + i \frac{\pi}{4}) \sqrt{\frac{\pi\beta}{\alpha}} \varphi(x)^{-1/4} \left\{ 1 + \sum_{n=1}^{\infty} \frac{i^n \Gamma(k+1/2)}{2^{2n} k! \Gamma(-n+1/2) \alpha^n} \beta^n \varphi(x)^{-n/2} \right\}, \quad (4.182)$$

$$H_2(x, \beta) = - \left(\frac{h(x)}{\varphi(x)} \frac{d\varphi}{dx} \right)^{-1/2} \exp(-i \frac{2\alpha}{\beta} \varphi(x)^{1/2} + i \frac{3\pi}{4}) \sqrt{\frac{\pi\beta}{\alpha}} \varphi(x)^{-1/4} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-i)^n \Gamma(k+1/2)}{2^{2n} k! \Gamma(-n+1/2) \alpha^n} \beta^n \varphi(x)^{-n/2} \right\}. \quad (4.183)$$

The asymptotic expansion of the functions $\psi_j(x)$ in the sectors S_n in terms of the functions $H_1(x, \beta)$ and $H_2(x, \beta)$ given above follows:

- In the sector S_{-2} we have

$$\psi_1(x) \sim H_1(x, \beta) - H_2(x, \beta), \quad (4.184)$$

$$\psi_2(x) \sim 3H_1(x, \beta) - 2H_2(x, \beta), \quad (4.185)$$

$$\psi_3(x) \sim 2H_1(x, \beta) - H_2(x, \beta). \quad (4.186)$$

- In the sector S_{-1} we have

$$\psi_1(x) \sim H_2(x, \beta) - H_1(x, \beta), \quad (4.187)$$

$$\psi_2(x) \sim 2H_2(x, \beta) - H_1(x, \beta), \quad (4.188)$$

$$\psi_3(x) \sim H_2(x, \beta). \quad (4.189)$$

- In the sector S_0 we have

$$\psi_1(x) \sim H_1(x, \beta) - H_2(x, \beta), \quad (4.190)$$

$$\psi_2(x) \sim H_1(x, \beta), \quad (4.191)$$

$$\psi_3(x) \sim H_2(x, \beta). \quad (4.192)$$

- In the sector S_1 we have

$$\psi_1(x) \sim H_2(x, \beta) - H_1(x, \beta), \quad (4.193)$$

$$\psi_2(x) \sim H_1(x, \beta), \quad (4.194)$$

$$\psi_3(x) \sim 2H_1(x, \beta) - H_2(x, \beta). \quad (4.195)$$

- In the sector S_2 we have

$$\psi_1(x) \sim H_1(x, \beta) - H_2(x, \beta), \quad (4.196)$$

$$\psi_2(x) \sim 2H_2(x, \beta) - H_1(x, \beta), \quad (4.197)$$

$$\psi_3(x) \sim -2H_1(x, \beta) + 3H_2(x, \beta). \quad (4.198)$$

The function $\varphi(x)$ in equations (4.182) and (4.183) is given by a the series expansion (4.130) with respect to the parameter β . If we substitute the series expansion for $\varphi(x)$ with $\varphi_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ given, respectively, by equations (4.134), (4.135) and (4.136) into the equations (4.182) and (4.183), and if we expand the resulting equations with respect to β , we realize that the functions $H_j(x, \beta)$ are given by the equations

$$\begin{aligned} H_1(x, \beta)|_{\pm\xi(x)} = & -\sqrt{\pi\beta} \exp(\pm\frac{1}{\alpha}\xi(x) + i\frac{\pi}{4})\tilde{w}_0^\pm(x) \{1 - i\beta\tilde{w}_1^\pm(x) - \beta^2\tilde{w}_2^\pm(x) \\ & + i\beta^3\tilde{w}_3^\pm(x) + \beta^4\tilde{w}_4^\pm(x) + O(\beta^5)\}, \end{aligned} \quad (4.199)$$

$$\begin{aligned} H_2(x, \beta)|_{\pm\xi(x)} = & \sqrt{\pi\beta} \exp(\mp\frac{1}{\alpha}\xi(x) + i\frac{\pi}{4})\tilde{w}_0^\mp(x) \{1 - i\beta\tilde{w}_1^\mp(x) - \beta^2\tilde{w}_2^\mp(x) \\ & + i\beta^3\tilde{w}_3^\mp(x) + \beta^4\tilde{w}_4^\mp(x) + O(\beta^5)\}. \end{aligned} \quad (4.200)$$

As in the previous case, in the solution for $\varphi_0(x)$ (given by equation (4.134)), we obtained two branches for $\varphi_0(x)$. So, we actually obtained two expansions for $\varphi(x)$. Once we chose the branch of $\varphi_0(x)$, the expansion for $\varphi(x)$ being used is specified. This reflects in the form of the expansions for the functions $H_j(x, \beta)$ above. We

append $\pm\xi(x)$ to specify which branch of $\varphi(x)$ we are using.

As in the previous case, the expressions (4.199) and (4.200) provide asymptotic expansions for the solutions $\psi_j(x)$ of the approximate equation (4.99) in terms of the amplitude corrections $\bar{w}_n^\pm(x)$, which will appear in the form of the asymptotic far field solution, valid close to a turning point. As a result, we will obtain a nice matching between the appropriate form of the far field asymptotic solution, valid in a neighborhood of a turning point, and the asymptotic expansion of the approximate equation (4.99) up to fourth order in the parameter β .

Next, we discuss how to obtain the scattering coefficients due to the presence of turning points. These quantities are obtained through the matching between the asymptotic solutions of the approximate equations (4.97) and (4.99) and the appropriate form of the far field asymptotic solution, valid in a neighborhood of a turning point.

4.5.4 Matching Process at a Turning Point.

Here we discuss the matching between the Liouville-Green functions (far field asymptotic solutions) representing the wave modes that couple at a particular branch point with the asymptotic expansion of the solutions of the approximate equations (4.97) and (4.99). If at the considered turning point we have the coupling between two propagating wave modes, we use the asymptotic expansion of the solutions of the approximate equation (4.99). If at the considered branch points the coupling is between a propagating wave mode and an evanescent wave mode, we use the asymptotic expansions of the solutions of the approximate equation (4.97). We present here the general results.

We consider the Liouville-Green functions written in the form given by equation (4.66) instead of the form given by equation (4.55) with $w(x)$ represented by the equation (4.60). The approximate form of the Liouville-Green functions given by equation (4.55), valid in a neighborhood of a turning point, is going to be used when we perform the matching with the asymptotic expansion of the solution of the approximate equations (4.97) and (4.99). The form given by equation (4.66) is going

to be used to write the boundary conditions for the turning points and to write the asymptotic solution of equation (4.54) far from the turning points.

In the complex h plane, the branch points of the dispersion relation for uniform depth, assumed locally valid, appear as complex conjugate numbers (see figures 4-2). The depth function $h(x)$ gives real positive values for all real values of the variable x . We assume that it is almost every where analytic, except for countable number of singularities in the complex x plane (no singularities in the real axis). These restrictions implies that the function $h(x)$ is symmetric with respect to the real axis of the complex x plane. So, the image of the complex conjugate branch points in the complex h plane through the depth function are complex conjugate numbers in the complex x plane. The branch point located at $h = 0$ has also complex conjugate numbers as images in the complex x plane. So, the turning points appear in pairs, and are complex numbers. We define each pair of complex conjugate turning points as a turning point problem. For each turning point problem we have to solve a connection problem associated with it. The solution of the connection problem gives local wave reflection and wave mode conversion coefficients. These are used to build a local transfer matrix, which connects the wave mode amplitudes before and after the considered turning point problem, and the local transfer matrices are used to build a global transfer matrix which relates the wave mode amplitudes at the beginning with the amplitude of the wave modes at the end of the non-uniform part of the bottom. To illustrate the turning point problems, we consider the depth function

$$h(x) = \gamma - \frac{\epsilon}{\beta} H(x) \text{ with } H(x) = \tanh(x). \quad (4.201)$$

The branch points of the uniform system dispersion (assumed locally valid) in the complex x plane for the depth function given by equation (4.201) is illustrated in figure 4-25 below

According to the figure 4-25, we have an enumerable set of branch points. The poles located at $x = \pm i\frac{\pi}{2}$ are the two acumulation points of the set of branch points.

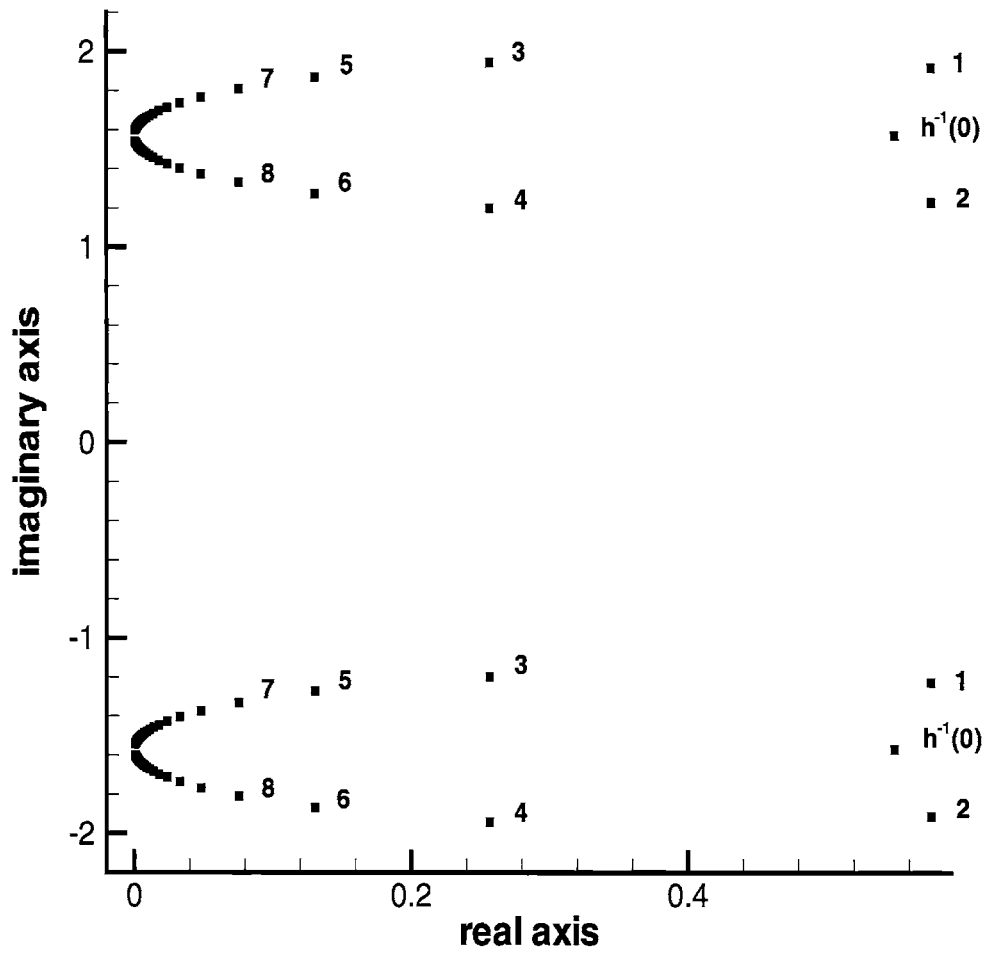


Figure 4-25: Image of the branch points in the complex h plane given in figure 4-2 through the depth function given by equation (4.201) in the complex x plane. The numbering for the branch points above corresponds to the numbering in figure 4-2. The branch points image of the branch point $h = 0$ is labeled $h^{-1}(0)$. Parameters are $\beta = 0.1$, $\gamma = 1.0$ and $\epsilon = 0.05$.

In our asymptotic analysis we consider only the branch points closer to the real axis in the complex x plane and which are not too close one to the other. We consider too close branch points these which are at a distance of less than one wave length of the incident wave from the neighboring branch points. In the figure above, the wave length of the incident wave corresponds in the complex x plane to a distance equal to the value of the parameter β . So the branch points we are going to consider in our analysis are the branch points labeled 1, 2, 3, 4, 5, 6, 7 and 8, which are closer to the real axis of the complex x plane, and the two branch points labeled $h^{-1}(0)$. Their distance from the neighboring branch points is of the order of the the wave length of the incident wave or more. To take into account the branch points close to the poles $\pm i\frac{\pi}{2}$ we have to deal with a cluster of an enumerable set of branch points, which is a very challenging problem, and we are going to avoid it. The finite set of branch points mentioned above form pairs of complex conjugate numbers. These pairs are the turning point problems we are going to deal with in our asymptotic analysis.

We define branch cuts in the complex x plane. For each pair of branch points being considered in our analysis, we define the branch cut as the part of the Stokes lines emanating from each branch point and reaching the real axis of the complex x plane. Due to the symmetry of the topology of the branch points with respect to the real axis, these two Stokes lines cross the real axis at the same point. So, this two pieces of Stokes lines perform a contour connecting both branch points. For a branch point, the Stokes-lines are defined as

$$\Im\{\Delta S_{kl}\} = 0, \quad (4.202)$$

where

$$\Delta S_{kl}(x) = \int_{x_b}^x \left\{ \frac{dS_k}{dt} - \frac{dS_l}{dt} \right\} dt. \quad (4.203)$$

The lower limit (x_b) in the integral above is the considered branch point. The

wavenumbers $\frac{dS_k}{dx}$ and $\frac{dS_l}{dx}$ are given by the uniform depth dispersion relation given by equation (4.57), and they are one of the two pairs of wavenumbers which coalesce at the considered branch point. For example, for the even branch points, we have that

$$\Delta S_{0l}(x) = \int_{x_b}^x \left\{ \frac{dS_0}{dt} + \frac{dS_l}{dt} \right\} dt, \quad (4.204)$$

since the wavenumber $\frac{dS_0}{dx}(-\frac{dS_0}{dx})$ coalesce with the wavenumber $-\frac{dS_l}{dx}(\frac{dS_l}{dx})$. For the odd branch points we have that

$$\Delta S_{0l}(x) = \int_{x_b}^x \left\{ \frac{dS_0}{dt} - \frac{dS_l}{dt} \right\} dt, \quad (4.205)$$

since the wavenumber $\frac{dS_0}{dx}(-\frac{dS_0}{dx})$ coalesce with the wavenumber $\frac{dS_l}{dx}(-\frac{dS_l}{dx})$. For the pairs of branch points labeled $h^{-1}(0)$ the Stokes lines are given by

$$\Im\{\Delta S_{00}\} = \Im\left\{2 \int_{h^{-1}(0)}^x \frac{dS_0}{dt} dt\right\} = 0. \quad (4.206)$$

From branch points where we have coalescence of pairs of branch points, we have three Stokes lines emanating from it. At this type of branch point, the dispersion relation has a square root branch point. For the branch points $h^{-1}(x)$ we have only one Stokes line emanating from it. At this type of branch point, the dispersion relation coefficients have a simple pole.

For the branch points not considered in the asymptotic analysis, we consider branch cuts joining these points to the infinite in the complex x plane without crossing the real axis. Example of the branch cuts mentioned above are given in figure 4-26. We approximate the branch-cuts based on Stokes lines by straight lines connecting a considered branch point and its complex conjugate. This is usually a good approximation. The exception are the branch points $h^{-1}(0)$. For the example considered, the branch cut associated with these branch points cannot be approximated

by a straight line and it is illustrated in figure 4-26. We split the real axis in intervals (A_n, A_{n+1}) , as illustrated in figure 4-26. Each of these intervals contains the real part of a pair of complex conjugate turning points, denoted B_n , as illustrated in the figure 4-26.

Next, we consider each turning point problem separately. At each turning point we have the coupling of a propagating wave mode with an evanescent wave mode. The exception are the turning points where the two propagating wave modes couple, which are, for example, the turning points 8 and 9 in the figure 4-26. If at the turning points enumerated $2n$ and $2n+1$ we have coupling between evanescent and propagating wave modes, we consider boundary conditions of left and right incidence of propagating and evanescent wave modes at the points A_n and A_{n+1} . If at the turning points enumerated $2n$ and $2n+1$ we have coupling between the two propagating wave modes, we consider only boundary conditions of left and right incidence of propagating wave modes at the points A_n and A_{n+1} . These boundary conditions are given in the section 4.5.4.1.

4.5.4.1 Boundary Condition.

Here we consider the boundary conditions for a pair of complex conjugate branch points. We write the boundary conditions in terms of the Liouville-Green functions in the form given by the equation (4.66). We write the phase $S(x)$, the amplitude $v_0(x)$ and the phase corrections $v_n(x)$ also as functions of a reference point r and the wavenumber they are associated with. The reference point r is important to specify because these quantities are given by definite integrals where the reference point is the lower limit of integration. For example, the Liouville-Green function up to fourth order in β associated with the wavenumber $+k_0$ (right going wave mode) is now written as

$$\psi(x) \sim v_0(+k_0, r, x) \exp\left(\frac{i}{\beta}S(+k_0, r, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, r, x)\right), \quad (4.207)$$

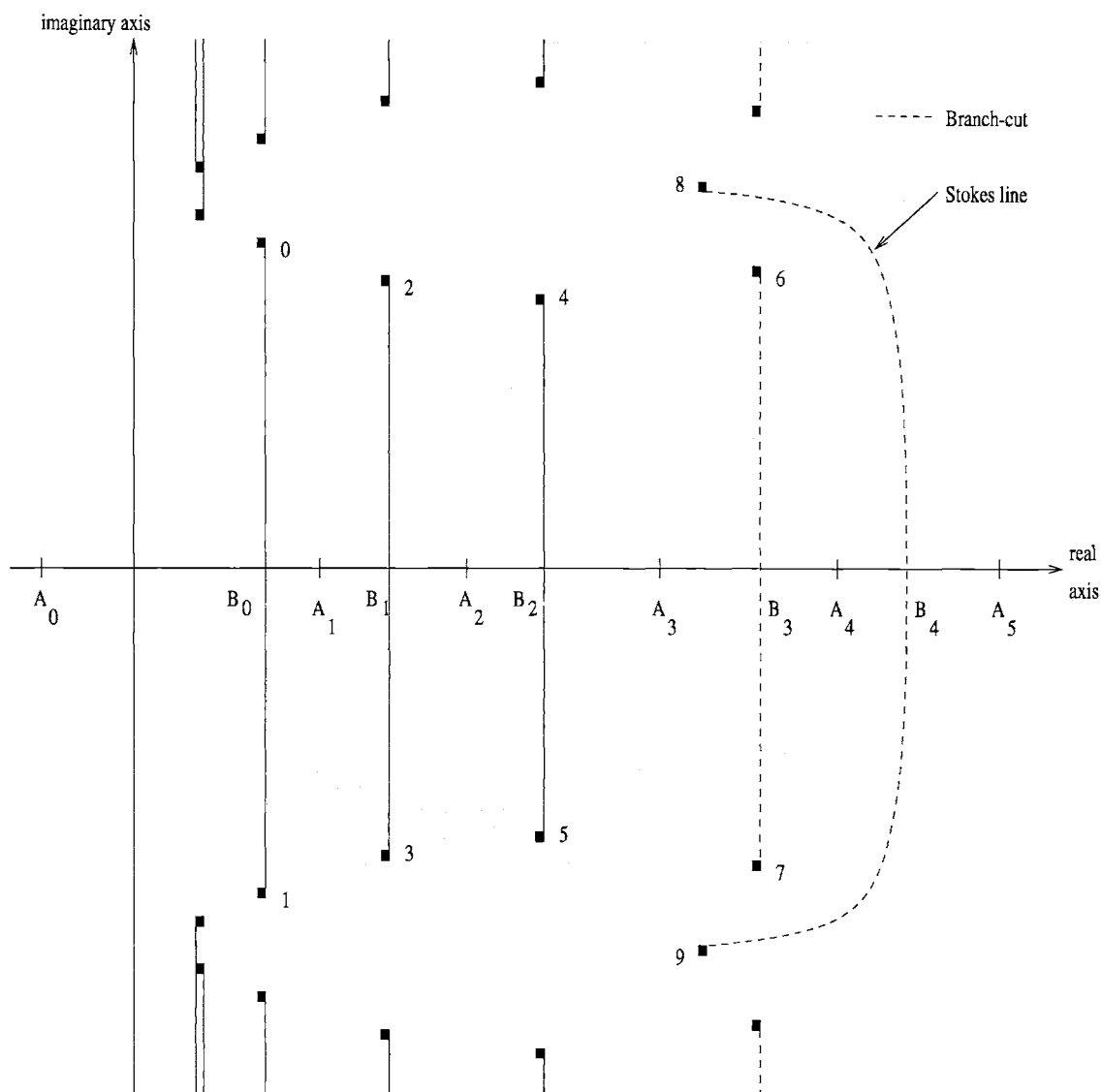


Figure 4-26: Sketch in the complex x plane of the branch points considered in the asymptotic analysis for the depth function given by equation (4.201) and given previously in figure 4-25. The branch points considered in the asymptotic analysis are numbered. Their enumeration is given according to the interval (A_n, A_{n+1}) that contains the crossing of the branch-cut (Stokes line) emanating from them and crossing the real axis of the complex x plane. The - - - - - lines represents the branch cuts.

and the Liouville-Green function up to fourth order in β associated with the evanescent wavenumber $-k_l$ (left l -th evanescent wave mode) is now written as

$$\psi(x) \sim v_0(-k_l, r, x) \exp\left(\frac{i}{\beta} S(-k_l, r, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, r, x)\right). \quad (4.208)$$

We first consider a pair of turning points enumerated as $2n$ and $2n+1$ where we have coupling between a propagating and an evanescent wave modes. We assume that at the turning point $2n$ ($2n+1$) we have coalescence between the wavenumbers $\pm k_0$ and the wavenumbers $\mp k_l$ ($\pm k_l$) (see section 4.5.2.1). The possible boundary conditions for this case follows.

- Left incidence of a propagating mode.
 - For $A_n < x < B_n$ we have an incident left propagating wave mode (wavenumber $+k_0$), a right propagating wave mode (wavenumber $-k_0$) and a left evanescent wave mode (wavenumber $-k_l$). We have

$$\begin{aligned} \psi(x) \sim & A v_0(+k_0, A_n, x) \exp\left(\frac{i}{\beta} S(+k_0, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, A_n, x)\right) \\ & + R_n^- A v_0(-k_0, A_n, x) \exp\left(\frac{i}{\beta} S(-k_0, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_n, x)\right) \\ & + C_n^- A v_0(-k_l, B_n, x) \exp\left(\frac{i}{\beta} S(-k_l, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, B_n, x)\right), \end{aligned} \quad (4.209)$$

where A is the amplitude of the incident wave and R_n^- is the reflection coefficient for left wave incidence at the point A_n . C_n^- is the left evanescent wave mode conversion coefficient at the point B_n due to left wave incidence.

- For $B_n < x < A_{n+1}$ we have a transmitted right going wave and a right evanescent wave mode. We have

$$\begin{aligned}
\psi(x) \sim & T_n^- Av_0(+k_0, B_n, x) \exp\left(\frac{i}{\beta} S(+k_0, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, B_n, x)\right) \\
& + D_n^- Av_0(+k_l, B_n, x) \exp\left(\frac{i}{\beta} S(+k_l, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_l, B_n, x)\right),
\end{aligned} \tag{4.210}$$

where T_n^- and D_n^- are, respectively, the transmission and right evanescent wave mode conversion coefficients at the point B_n due to left wave incidence.

- Right incidence of a propagating mode.
 - For $A_n < x < B_n$ we have a transmitted left going wave and a left evanescent wave mode. We have

$$\begin{aligned}
\psi(x) \sim & T_n^+ Av_0(-k_0, B_n, x) \exp\left(\frac{i}{\beta} S(-k_0, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, B_n, x)\right) \\
& + C_n^+ Av_0(-k_l, B_n, x) \exp\left(\frac{i}{\beta} S(-k_l, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, B_n, x)\right),
\end{aligned} \tag{4.211}$$

where T_n^+ and C_n^+ are, respectively, the transmission and left evanescent wave mode conversion coefficients at the point B_n due to right wave incidence.

- For $B_n < x < A_{n+1}$ we have an incident right going wave mode (wavenumber $-k_0$), a left going wave mode (wavenumber $+k_0$) and a right evanescent wave mode (wavenumber k_l). Then we can write

$$\begin{aligned}
\psi(x) \sim & Av_0(-k_0, x) \exp\left(\frac{i}{\beta}S(-k_0, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_{n+1}, x)\right) \\
& + R_n^+ Av_0(+k_0, x) \exp\left(\frac{i}{\beta}S(+k_0, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, A_{n+1}, x)\right) \\
& + D_n^+ Av_0(k_l, B_n, x) \exp\left(\frac{i}{\beta}S(k_l, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(k_l, B_n, x)\right),
\end{aligned} \tag{4.212}$$

where A is the amplitude of the incident wave and R_n^+ is the reflection coefficient for right wave incidence at the point A_{n+1} . D_n^+ is the right evanescent wave mode conversion coefficient at the point B_n due to right wave incidence.

- Left incidence of an evanescent mode.

- For $A_n < x < B_n$ we have an incident right evanescent wave mode (wavenumber $+k_l$), a left propagating wave mode (wavenumber $-k_0$) and a left evanescent wave mode (wavenumber $-k_l$). Then we can write

$$\begin{aligned}
\psi(x) \sim & Av_0(+k_l, A_n, x) \exp\left(\frac{i}{\beta}S(+k_l, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_l, A_n, x)\right) \\
& + \tilde{R}_n^- Av_0(-k_0, A_n, x) \exp\left(\frac{i}{\beta}S(-k_0, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_n, x)\right) \\
& + \tilde{C}_n^- Av_0(-k_l, B_n, x) \exp\left(\frac{i}{\beta}S(-k_l, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, B_n, x)\right),
\end{aligned} \tag{4.213}$$

where A is the amplitude of the left incident evanescent wave mode and $\tilde{R}_n^-$ is the wave mode conversion coefficient due to the left incidence of an evanescent wave mode at the point A_n . $\tilde{C}_n^-$ is the reflection coefficient at the point B_n due to the left incidence of an evanescent wave mode.

- For $B_n < x < A_{n+1}$ we have a transmitted right going wave and a right evanescent wave mode. Then we can write

$$\begin{aligned}\psi(x) \sim & \tilde{T}_n^- Av_0(+k_0, B_n, x) \exp\left(\frac{i}{\beta} S(+k_0, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, B_n, x)\right) \\ & + \tilde{D}_n^- Av_0(+k_l, B_n, x) \exp\left(\frac{i}{\beta} S(+k_l, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_l, B_n, x)\right),\end{aligned}\tag{4.214}$$

where $\tilde{T}_n^-$ and $\tilde{D}_n^-$ are, respectively, the right wave mode conversion coefficient and the transmission coefficient at the point B_n due to the left incidence of an evanescent wave mode.

- Right incidence of an evanescent wave mode.

- For $A_n < x < B_n$ we have a wave mode conversion into a left going wave and the transmission of a left evanescent wave mode. We then write

$$\begin{aligned}\psi(x) \sim & \tilde{T}_n^+ Av_0(-k_0, B_n, x) \exp\left(\frac{i}{\beta} S(-k_0, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, B_n, x)\right) \\ & + \tilde{C}_n^+ Av_0(-k_l, B_n, x) \exp\left(\frac{i}{\beta} S(-k_l, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, B_n, x)\right),\end{aligned}\tag{4.215}$$

where $\tilde{T}_n^+$ and $\tilde{C}_n^+$ are, respectively, the left wave mode conversion coefficient and the transmission coefficients at the point B_n due to the right incidence of an evanescent wave mode.

- For $B_n < x < A_{n+1}$ we have the right incidence of an evanescent wave mode (wavenumber $-k_l$), a right going wave mode (wavenumber $+k_0$) and a reflected right evanescent wave mode (wavenumber k_l). We then write

$$\begin{aligned}
\psi(x) \sim & Av_0(-k_l, x) \exp\left(\frac{i}{\beta} S(-k_l, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, A_{n+1}, x)\right) \\
& + \tilde{R}_n^+ Av_0(+k_0, x) \exp\left(\frac{i}{\beta} S(+k_0, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, A_{n+1}, x)\right) \\
& + \tilde{D}_n^+ Av_0(k_l, B_n, x) \exp\left(\frac{i}{\beta} S(k_l, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(k_l, B_n, x)\right),
\end{aligned} \tag{4.216}$$

where A is the amplitude of the right incident evanescent wave mode and $\tilde{R}_n^+$ is the wave mode conversion coefficient at the point A_{n+1} due to the right incidence of an evanescent wave mode. $\tilde{D}_n^+$ is the right reflection coefficient at the point B_n due to the right incidence of an evanescent wave mode.

Next, we consider a pair of turning points enumerated as $2n$ and $2n + 1$ where we have coupling between the two propagating wave modes. The possible boundary conditions for this case follows.

- Left incidence.

- For $A_n < x < B_n$ we have an incident right propagating wave mode (wavenumber $+k_0$) and a right propagating wave mode (wavenumber $-k_0$). We have

$$\begin{aligned}
\psi(x) \sim & Av_0(+k_0, A_n, x) \exp\left(\frac{i}{\beta} S(+k_0, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, A_n, x)\right) \\
& + R_n^- Av_0(-k_0, A_n, x) \exp\left(\frac{i}{\beta} S(-k_0, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_n, x)\right),
\end{aligned} \tag{4.217}$$

where A is the amplitude of the incident wave and R_n^- is the reflection coefficient for left wave incidence at the point A_n .

- For $B_n < x < A_{n+1}$ we have a transmitted right going wave. We have

$$\psi(x) \sim T_n^- A v_0(+k_0, B_n, x) \exp\left(\frac{i}{\beta} S(+k_0, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, B_n, x)\right), \quad (4.218)$$

where T_n^- is the transmission coefficient at the point B_n due to left wave incidence.

- Right incidence.

- For $A_n < x < B_n$ we have a transmitted left going wave.

$$\psi(x) \sim T_n^+ A v_0(-k_0, B_n, x) \exp\left(\frac{i}{\beta} S(-k_0, B_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, B_n, x)\right), \quad (4.219)$$

where T_n^+ is the transmission coefficient at the point B_n due to right wave incidence.

- For $B_n < x < A_{n+1}$ we have an incident left going wave mode (wavenumber $-k_0$) and a reflected right going wave mode (wavenumber $+k_0$).

$$\begin{aligned} \psi(x) \sim & A v_0(-k_0, x) \exp\left(\frac{i}{\beta} S(-k_0, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_{n+1}, x)\right) \\ & + R_n^+ A v_0(+k_0, x) \exp\left(\frac{i}{\beta} S(+k_0, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, A_{n+1}, x)\right), \end{aligned} \quad (4.220)$$

where A is the amplitude of the incident wave and R_n^+ is the reflection coefficient at the point A_{n+1} for right wave incidence.

4.5.4.2 Local Form of the Liouville-Green Functions.

Here we continue the Liouville-Green functions used to specify the boundary conditions given in the section 4.5.4.1 along the contour that goes from the real axis (point B_n) to points neighbor to the turning point labeled $2n$. This is necessary to perform the matching between the Liouville-Green functions and the asymptotic solution of the appropriate approximate form of the equation (4.54) for the free surface potential. This matching furnishes the reflection, transmission and wave mode conversion coefficients due to the pair of turning points labeled $2n$ and $2n + 1$.

For the matching purpose we could use the turning point $2n + 1$ instead, but the results are basically the same. So, to solve the connection problem for a turning point problem, we consider the turning points in the upper part of the complex x plane. In the figure 4-27 we illustrate the contour used to continue the Liouville-Green function from the real axis to the neighbor points C_n and D_n of the turning point $2n$ with $n = 2$ and $n = 3$.

In the way illustrated above to continue the Liouville-Green functions we do not cross the branch cuts, so we stay in the same Riemann sheet all the time. We do not continue the Liouville-Green functions around the branch cuts, so, for example, the wavenumber $+k_0$ for x along the real axis is always associated with the right propagating wave mode at both sides of the branch cuts. We just continue the Liouville-Green functions to the points C_n and D_n , so we can match these functions with the asymptotic solution of the appropriate approximate form of the equation (4.54) for the free surface potential.

In a neighborhood of the points C_n and D_n we need to obtain the approximate form of the Liouville-Green functions, so we can match them with the asymptotic solution of the approximate equations (4.97) and (4.99). This is done in the Appendix M. The approximate form of the Liouville-Green function obtained in the Appendix M were written in terms of the wave numbers given by the appropriate approximate form of the uniform system dispersion relation, valid in a neighborhood of the turning point $2n$. This approximate dispersion relation gives two wave numbers only. If the turning

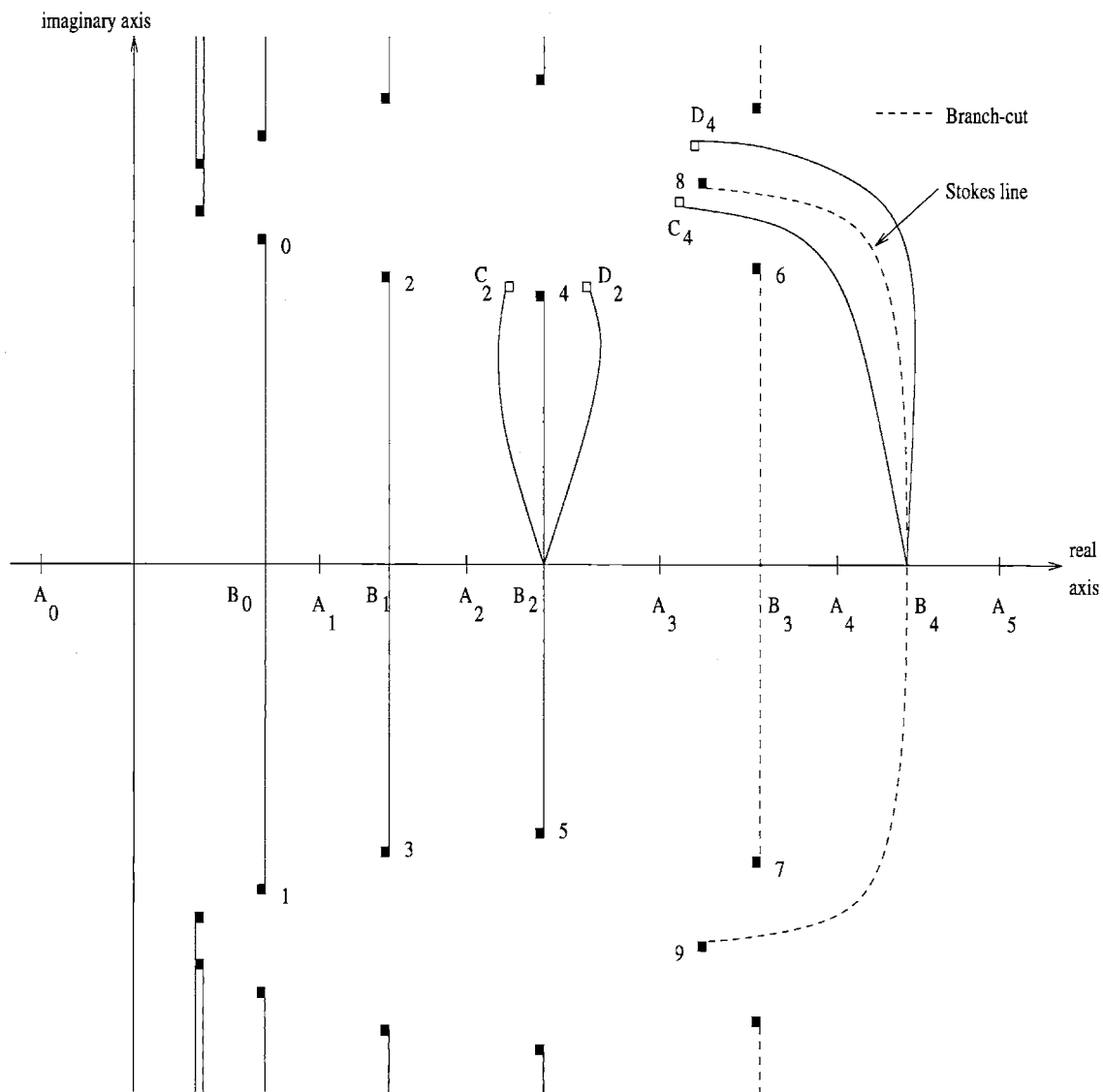


Figure 4-27: Sketch in the complex x plane of the contour used to continue the Liouville-Green functions from the real axis (point B_n) to point C_n and D_n neighbors to the turning point $2n$ with $n = 2$ and $n = 4$. The - - - - - lines represent the branch cuts. The ——— line represents a Stoke-line.

point $2n$ is of the type where we have coalescence of propagating and evanescent wave modes, we have to consider four wave modes according to the boundary conditions given by equations (4.209) to (4.216). Each turning point of this type is associated with two saddle points in the complex wave number plane, and with two wave mode boundaries, each of them emanating from one of the two saddle points. The turning point $2n$ has image in the lower part of the complex h plane according to figure 4-2, and the branch points lying in this part of the complex plane h have saddle points in the second and fourth quadrants of the complex k plane (see figure 4-3). Therefore, the wave mode boundaries associated with the turning point $2n$ are in the second and fourth quadrant (see figures 4-14 to 4-19). The turning point $2n + 1$, complex conjugate of turning point $2n$, has saddle points associated to it in the complex k plane which are complex conjugate numbers of the saddle points associated with the turning point $2n$. So, the wave mode boundaries associated with the turning point $2n + 1$ are complex conjugates of the wave mode boundaries associated with the turning point $2n$ (see figures 4-14 to 4-19). We define as $k_c(x)(-k_c(x))$ the wave mode boundary lying in the first (third) quadrant of the complex k plane, and as $\bar{k}_c(x)(-\bar{k}_c(x))$ as the mode boundaries lying in the fourth (second) quadrant.

The wave mode boundaries associated with turning point $2n$ are the wave mode boundaries $\bar{k}_c(x)$ and $-\bar{k}_c(x)$ defined above. Now we can write the two approximate forms of the dispersion relation, valid in a neighborhood of the turning point $2n$. To each of the wave mode boundaries $\bar{k}_c(x)$ and $-\bar{k}_c(x)$ we have an approximate equation of the type given by equation (4.97). Now is clear that at the turning point $2n$ we have the coalescent of two pairs of wave numbers. The wave number $+k_0(-k_0)$ coalesce to the wavenumber $-k_l(+k_l)$ at the saddle point located in the fourth (second) quadrant (see for example, figures 4-15 ($l = 1$), 4-17 ($l = 2$) and 4-19 ($l = 3$)). Below follows the correspondence between the wavenumbers given by the approximate dispersion relation and the wavenumbers given by the full dispersion relation. For the cases of left wave incidence we continue the local form of the Liouville-Green functions from left to right (point C_n to point D_n)

- At left of the turning point $2n$ (point C_n) we have:

$$-\bar{k}_c(x) - i\sqrt{Q(x)} \rightarrow -k_0 \quad (4.221)$$

$$-\bar{k}_c(x) + i\sqrt{Q(x)} \rightarrow +k_l \quad (4.222)$$

$$\bar{k}_c(x) - i\sqrt{Q(x)} \rightarrow -k_l \quad (4.223)$$

$$\bar{k}_c(x) + i\sqrt{Q(x)} \rightarrow k_0 \quad (4.224)$$

- At right of the turning point $2n$ (point D_n) we have:

$$-\bar{k}_c(x) + i\sqrt{Q(x)} \rightarrow -k_0 \quad (4.225)$$

$$-\bar{k}_c(x) - i\sqrt{Q(x)} \rightarrow +k_l \quad (4.226)$$

$$\bar{k}_c(x) + i\sqrt{Q(x)} \rightarrow -k_l \quad (4.227)$$

$$\bar{k}_c(x) - i\sqrt{Q(x)} \rightarrow k_0 \quad (4.228)$$

and the square root of the function $Q(x)$ in the equation above is always the positive branch. For the case of right incidence, we continue the local form of the Liouville-Green functions from right to left (point D_n to point C_n).

- At left of the turning point $2n$ (point C_n) we have:

$$-\bar{k}_c(x) + i\sqrt{Q(x)} \rightarrow -k_0 \quad (4.229)$$

$$-\bar{k}_c(x) - i\sqrt{Q(x)} \rightarrow +k_l \quad (4.230)$$

$$\bar{k}_c(x) + i\sqrt{Q(x)} \rightarrow -k_l \quad (4.231)$$

$$\bar{k}_c(x) - i\sqrt{Q(x)} \rightarrow k_0 \quad (4.232)$$

- At right of the turning point $2n$ (point D_n) we have:

$$-\bar{k}_c(x) - i\sqrt{Q(x)} \rightarrow -k_0 \quad (4.233)$$

$$-\bar{k}_c(x) + i\sqrt{Q(x)} \rightarrow +k_l \quad (4.234)$$

$$\bar{k}_c(x) - i\sqrt{Q(x)} \rightarrow -k_l \quad (4.235)$$

$$\bar{k}_c(x) + i\sqrt{Q(x)} \rightarrow k_0 \quad (4.236)$$

The objective is to continue the Liouville-Green functions describing the boundary conditions specified in the previous section towards the points C_n and D_n . Boundary conditions specified at the left side of the turning point $2n$ are continued until the point C_n , and the boundary conditions specified at the right side of $2n + 1$ are continued until the point D_n . For a neighborhood of the point C_n we can approximate the Liouville-Green functions in the forms

- At turning point $2n$ we have coupling of a propagating and an evanescent wave mode.

$$\begin{aligned} & \tilde{w}_0^\pm(\pm k_0, x) \exp \left(\frac{i}{\beta} S(\pm k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_0, A_n, B_n) \right. \\ & + \frac{i}{\beta} S(\pm k_0, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_0, B_n, C_n) \pm \frac{i}{\beta} \int_{C_n}^x \bar{k}_c(t) dt \\ & \left. \pm \frac{1}{\beta} \xi(C_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(\pm k_0, x, C_n) + O(\beta^5) \right\}, \end{aligned}$$

and

$$\begin{aligned} & \tilde{w}_0^\pm(-k_m, x) \exp \left(\frac{i}{\beta} S(-k_m, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_m, B_n, C_n) \right. \\ & \left. \pm \frac{i}{\beta} \int_{C_n}^x \bar{k}_c(t) dt \pm \frac{1}{\beta} \xi(x, C_n) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(\pm k_m, x, C_n) + O(\beta^5) \right\} \end{aligned}$$

with $m = 0, l$,

where $\xi(x, r)$ is given by equation (4.106) with $r = C_n$.

- At turning point $2n$ we have coupling of the two propagating wave modes.

$$\begin{aligned} & \tilde{w}_0^\pm(\pm k_0, x) \exp \left(\frac{i}{\beta} S(\pm k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_0, A_n, B_n) \right. \\ & \quad \left. + \frac{i}{\beta} S(\pm k_0, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_0, B_n, C_n) \right. \\ & \quad \left. \pm \frac{1}{\beta} \xi(C_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(\pm k_0, x, C_n) + O(\beta^5) \right\}, \end{aligned}$$

and

$$\begin{aligned} & \tilde{w}_0^\pm(-k_0, x) \exp \left(\frac{i}{\beta} S(-k_0, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, B_n, C_n) \right. \\ & \quad \left. \pm \frac{1}{\beta} \xi(C_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(-k_0, x, C_n) + O(\beta^5) \right\}, \end{aligned}$$

where $\xi(x, r)$ in the expression above is given by equation (4.115) with $r = C_n$.

For a neighborhood of the point D_n we can approximate the Liouville-Green functions in the forms

- At turning point $2n$ we have coupling of a propagating and an evanescent wave mode.

$$\begin{aligned} & \tilde{w}_0^\pm(\pm k_0, x) \exp \left(\frac{i}{\beta} S(\pm k_0, A_{n+1}, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_0, A_{n+1}, B_n) \right. \\ & \quad \left. + \frac{i}{\beta} S(\pm k_0, B_n, D_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_0, B_n, D_n) \pm \frac{i}{\beta} \int_{D_n}^x \bar{k}_c(t) dt \right. \\ & \quad \left. \pm \frac{1}{\beta} \xi(D_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(\pm k_0, x, D_n) + O(\beta^5) \right\}, \end{aligned}$$

and

$$\begin{aligned} & \tilde{w}_0^\pm(k_m, x) \exp \left(\frac{i}{\beta} S(k_m, B_n, D_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_m, B_n, D_n) \right. \\ & \quad \left. \pm \frac{i}{\beta} \int_{D_n}^x \bar{k}_c(t) dt \pm \frac{1}{\beta} \xi(x, D_n) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(\pm k_m, x, D_n) + O(\beta^5) \right\} \end{aligned}$$

with $m = 0, l$,

where $\xi(x, r)$ is given by equation (4.106) with $r = D_n$.

- At turning point $2n$ we have coupling of the two propagating wave modes.

$$\begin{aligned} & \tilde{w}_0^\pm(\pm k_0, x) \exp \left(\frac{i}{\beta} S(\pm k_0, A_{n+1}, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_0, A_{n+1}, B_n) \right. \\ & \quad \left. + \frac{i}{\beta} S(\pm k_0, B_n, D_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_0, B_n, D_n) \right. \\ & \quad \left. \pm \frac{1}{\beta} \xi(D_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(\pm k_0, x, D_n) + O(\beta^5) \right\} \end{aligned}$$

and

$$\begin{aligned} & \tilde{w}_0^\pm(k_0, x) \exp \left(\frac{i}{\beta} S(k_0, B_n, D_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, B_n, D_n) \right. \\ & \quad \left. \pm \frac{1}{\beta} \xi(D_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(k_0, x, D_n) + O(\beta^5) \right\}, \end{aligned}$$

where $\xi(x, r)$ in the expression above is given by equation (4.115) with $r = D_n$.

Notice that the form given by the equation (4.100) appear in the expressions

above. The final result is the exponential of the appropriate phase integral with end point at C_n or at D_n times the appropriate form of equation (4.100). The local form of the Liouville-Green function is then matched with the asymptotic expansion of the solution of the approximate equations (4.97) and (4.99), which results in the reflection, transmission and wave mode coupling coefficients for a turning point problem.

4.5.4.3 Matching Process at a Turning Point Problem.

In this section we are going to illustrate the matching process at the turning point $2n$. We assume that at this turning point we have coupling between the propagating wave mode and an evanescent wave mode. We consider just the case of left incidence of a propagating wave mode. For the other boundary conditions we only give the final result (the scattering coefficients) in the next section. Now we write the boundary conditions given by equations (4.209) and (4.210) in the neighborhood of the points C_n and D_n . We have

- For $A_n < \Re\{x\} < B_n$ we can write

$$\begin{aligned}
\psi(x) \sim & A\tilde{w}_0^-(+k_0, x) \exp \left(\frac{i}{\beta} S(+k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, A_n, B_n) \right. \\
& + \frac{i}{\beta} S(+k_0, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, B_n, C_n) + \frac{i}{\beta} \int_{C_n}^x \bar{k}_c(t) dt \\
& \left. - \frac{1}{\beta} \xi(C_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(+k_0, x, C_n) + O(\beta^5) \right\} \\
& + R_n^- A\tilde{w}_0^+(-k_0, x) \exp \left(\frac{i}{\beta} S(-k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_n, B_n) \right. \\
& + \frac{i}{\beta} S(-k_0, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, B_n, C_n) - \frac{i}{\beta} \int_{C_n}^x \bar{k}_c(t) dt \\
& \left. + \frac{1}{\beta} \xi(C_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^+(-k_0, x, C_n) + O(\beta^5) \right\} \\
& + C_n^- A\tilde{w}_0^+(-k_l, x) \exp \left(\frac{i}{\beta} S(-k_l, B_n, D_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, B_n, D_n) \right. \\
& \left. + \frac{i}{\beta} \int_{C_n}^x \bar{k}_c(t) dt + \frac{1}{\beta} \xi(C_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(+k_0, x, C_n) + O(\beta^5) \right\},
\end{aligned}$$

where A is the amplitude of the incident wave and R_n^- is the reflection coefficient for left wave incidence at the point A_n . C_n^- is the left evanescent wave mode conversion coefficient at the point B_n due to left wave incidence.

- For $B_n < \Re\{x\} < A_{n+1}$ we can write

$$\begin{aligned}
\psi(x) \sim & T_n^- A \tilde{w}_0^+(+k_0, x) \exp \left(\frac{i}{\beta} S(+k_0, B_n, D_n) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, B_n, D_n) \right. \\
& \left. + \frac{i}{\beta} \int_{D_n}^x \bar{k}_c(t) dt + \frac{1}{\beta} \xi(D_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(+k_0, x, D_n) + O(\beta^5) \right\} \\
& + D_n^- A \tilde{w}_0(+k_l, x) \exp \left(\frac{i}{\beta} S(+k_l, B_n, D_n) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_l, B_n, D_n) \right. \\
& \left. - \frac{i}{\beta} \int_{D_n}^x \bar{k}_c(t) dt + \frac{1}{\beta} \xi(D_n, x) \right) \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^\pm(+k_l, x, D_n) + O(\beta^5) \right\},
\end{aligned} \tag{4.237}$$

where T_n^- and D_n^- are, respectively, the transmission and right evanescent wave mode conversion coefficients at the point B_n due to left wave incidence.

Next, we are going to chose the appropriate solution of the approximate equation (4.97). At the considered branch point we know that $\arg\{\gamma\} \sim \pi/2$ ($\gamma = dQ/dx$ at $x = x_{2n}$), so the matching between the asymptotic expansion of the local solution and the WKB waves happens in the regions S_0 and S_2 defined by the equation (4.139). The appropriate local solution is

$$\psi(x) \sim B H_2^-(x), \tag{4.238}$$

where the expression for the function $H_2^-(x)$ is given by equation (4.141). The asymptotic expansion of this function in the regions S_0 and S_2 has the form

- In the region S_0 we have

$$H_2^-(x) \sim -\tilde{w}_0^+(x) \exp \left(\frac{1}{\beta} \xi(x, D_n) + i \frac{\pi}{4} \right) \sqrt{\pi} \beta^{1/6} \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^+(x, D_n) + O(\beta^5) \right\} \tag{4.239}$$

- In the region S_2 we have

$$\begin{aligned}
H_2^-(x) &\sim -\tilde{w}_0^+(x) \exp\left(\frac{1}{\beta}\xi(x, D_n) + i\frac{\pi}{4}\right) \sqrt{\pi}\beta^{1/6} \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^+(x, D_n) + O(\beta^5) \right\} \\
&\sim -\tilde{w}_0^-(x) \exp\left(-\frac{1}{\beta}\xi(x, D_n) + i\frac{\pi}{4}\right) \sqrt{\pi}\beta^{1/6} \left\{ 1 + \sum_{n=1}^4 (-i\beta)^n \tilde{w}_n^-(x, D_n) + O(\beta^5) \right\}
\end{aligned} \tag{4.240}$$

Next, we are going to perform the matching between the asymptotic form of the local solution and the WKB waves in the regions S_0 and S_2 . The matching in the region S_0 gives the equation

$$-B\sqrt{\pi}\beta^{1/6} \exp(i\frac{\pi}{4}) = T_n^- A \exp\left(\frac{i}{\beta}S(+k_0, B_n, D_n) + \sum_{n=1}^4 (-i\beta)^n v_n(+k_0, B_n, D_n) + \frac{i}{\beta} \int_{D_n}^x \bar{k}_c(t) dt\right), \tag{4.241}$$

with $x \in S_0$, and the matching at the region S_2 gives the equations

$$-B\sqrt{\pi}\beta^{1/6} \exp(i\frac{\pi}{4}) = C_n^- A \exp\left(\frac{i}{\beta}S(-k_l, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, B_n, C_n) + \frac{i}{\beta} \int_{C_n}^x \bar{k}_c(t) dt\right), \tag{4.242}$$

$$-B\sqrt{\pi}\beta^{1/6} \exp(i\frac{\pi}{4}) = A \exp\left(\frac{i}{\beta}S(k_0, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, B_n, C_n) + \frac{i}{\beta} \int_{C_n}^x \bar{k}_c(t) dt\right), \tag{4.243}$$

with $x \in S_2$.

Now we solve the matching equations above to obtain the value of the constant B and the expression for the coefficients T_n^- and C_n^- . We obtain

$$B = -\pi^{-1/2} \beta^{-1/6} \exp(-i\frac{\pi}{4}) A \exp \left(\frac{i}{\beta} S(k_0, B_n, C_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, B_n, C_n) + \frac{i}{\beta} \int_{C_n}^x \bar{k}_c(t) dt \right), \quad (4.244)$$

and for the scattering coefficients T_n^- and C_n^- we obtain the expressions

$$\begin{aligned} T_n^- = & \exp \left(\frac{i}{\beta} S(k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_n, B_n) + \frac{i}{\beta} S(k_0, B_n, C_n) \right. \\ & + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, B_n, C_n) - \frac{i}{\beta} S(k_0, B_n, D_n) \\ & \left. - \sum_{n=1}^4 (-i\beta)^n v_n(k_0, B_n, D_n) + \frac{i}{\beta} \int_{C_n}^{x^-} \bar{k}_c(t) dt - \frac{i}{\beta} \int_{D_n}^{x^+} \bar{k}_c(t) dt \right), \quad (4.245) \end{aligned}$$

$$\begin{aligned} C_n^- = & \exp \left(\frac{i}{\beta} S(k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_n, B_n) + \frac{i}{\beta} S(k_0, B_n, C_n) \right. \\ & + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, B_n, C_n) - \frac{i}{\beta} S(-k_l, B_n, D_n) - \sum_{n=1}^4 (-i\beta)^n v_n(-k_l, B_n, D_n) \\ & \left. + \frac{i}{\beta} \int_{C_n}^{x^-} \bar{k}_c(t) dt - \frac{i}{\beta} \int_{D_n}^{x^+} \bar{k}_c(t) dt \right), \quad (4.246) \end{aligned}$$

where $x^- \in S_2$ and $x^+ \in S_0$. We need to regularize the integrals appearing in the expressions for the scattering coefficients T_n^- and C_n^- . A first step is to set $C_n = D_n$. We can chose this point at the branch cut, before reaching the turning point (C_n at left of the cut and D_n at right of the cut), or in some appropriate neighborhood of the turning point. As a result, we can write

$$\frac{i}{\beta} \int_{C_n}^{x^-} \bar{k}_c(t) dt - \frac{i}{\beta} \int_{D_n}^{x^+} \bar{k}_c(t) dt = -\frac{i}{\beta} \int_{x^-}^{x^+} \bar{k}_c(t) dt. \quad (4.247)$$

We can move the points x^+ and x^- close to the branch-cut, such that the contour connecting these points is a circle enclosing the turning point. We call this contour

Γ_{n1} , and now we write

$$\frac{i}{\beta} \int_{C_n}^{x^-} \bar{k}_c(t) dt - \frac{i}{\beta} \int_{D_n}^{x^+} \bar{k}_c(t) dt = -\frac{i}{\beta} \int_{\Gamma_{n1}} \bar{k}_c(t) dt. \quad (4.248)$$

We assumed $C_n = D_n$ with these points laying at the branch cut. The contour connecting the point B_n to these points can be collapsed over the branch-cut. The wavenumbers $\pm k_0$ is the same at both sides of the branch-cut. Therefore, the contour integrals $S(\pm k_0, B_n, C_n)$ should be equal to the phase integral $S(\pm k_0, B_n, D_n)$, and the phase integral $v_n(\pm k_0, B_n, C_n)$ is also equal to the phase integral $v_n(\pm k_0, B_n, D_n)$. As a result, the expression for the scattering coefficient T_n^- simplifies to

$$T_n^- = \exp \left(\frac{i}{\beta} S(k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_n, B_n) - \frac{i}{\beta} \int_{\Gamma_{n1}} \bar{k}_c(t) dt \right). \quad (4.249)$$

As we continue the wavenumber k_0 around the turning point, it assumes values equal to the continuation of the wavenumber $-k_l$ (k_0 coalesce to $-k_l$ at the considered turning point) from the real axis (point B_n). The continuation of the wavenumber k_0 at the other side of the branch-cut coincides with the wavenumber k_l , so we can write the phase integrals in the expression for the scattering coefficient C_n^- as phase integrals over a single contour Γ_{n3} , which emanates from the left of point B_n at the real axis, encircles the turning point x_{2n} and ends at the right side of point B_n at the real axis. The expression for the scattering coefficient C_n^- assume the form

$$C_n^- = \exp \left(\frac{i}{\beta} S(k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_n, B_n) + \frac{i}{\beta} S(k_0, \Gamma_{n3}) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, \Gamma_{n3}) \right) \quad (4.250)$$

The contour Γ_{n1} and Γ_{n3} are illustrated in figure 4-28 bellow.

The approximate form of the governing equation for the free-surface potential valid in a neighborhood of the considered turning point and given as a second order

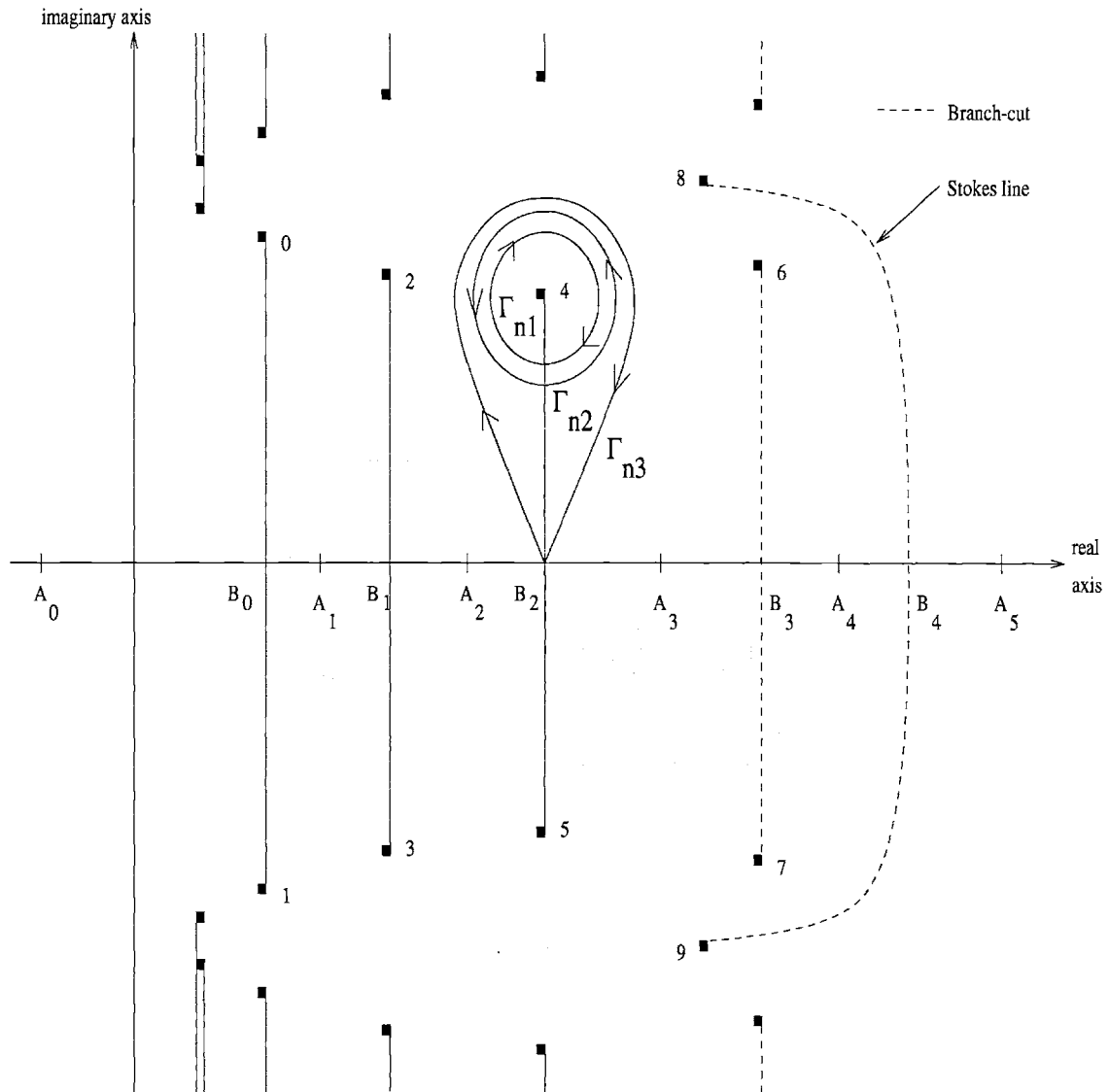


Figure 4-28: Sketch in the complex x plane of the contour Γ_n ($n = 2$) used to evaluate the contour integral in the expression for the scattering coefficients at the second turning point problem. The direction of integration is indicated by the arrows. The line - - - - - represent branch cuts.

differential equation was not able to connect the reflected wave and the right evanescent wave with the incident wave. Therefore, according to this approximation we do not have wave reflection and we do not have mode conversion to the right evanescent wave mode. To capture wave reflection and wave conversion to the right evanescent wave mode we need a higher order equation to approximate the free-surface potential. We need a fourth order differential equation, since at the considered turning point we have coalescence of two pairs of wavenumbers, which implies the coupling of four wave modes.

4.5.4.4 Reflection, Transmission and Mode Conversion Coefficients for a Turning Point Problem.

The product of the matching process at each turning point problem are the wave reflection, transmission and wave mode conversion coefficients, which are given in their final form in this section. We distinguish between turning points where propagating modes coalesce with evanescent modes and turning points where the two propagating modes coalesce. We first give the reflection, transmission and wave mode conversion coefficients due to the boundary conditions given in section 4.5.4.1 for the first type of turning point problem mentioned above.

- Case of left incidence of a propagating wave mode. No wave reflection is obtained, just wave transmission and coupling with the left evanescent wave mode only.

$$T_n^- = \exp \left(\frac{i}{\beta} S(k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_n, B_n) - \frac{i}{\beta} \int_{\Gamma_{n1}} \bar{k}_c(t) dt \right) \quad (4.251)$$

$$C_n^- = \exp \left(\frac{i}{\beta} S(k_0, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_n, B_n) + \frac{i}{\beta} S(k_0, \Gamma_{n3}) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, \Gamma_{n3}) \right) \quad (4.252)$$

$$R_n^- = 0 \quad (4.253)$$

$$D_n^- = 0 \quad (4.254)$$

where Γ_n is the closed contour in the complex x plane which encircles the turning point $2n$ and starts and ends up at point B_n , as illustrated in figure 4-28 below for the case $n = 2$.

- Case of right incidence of a propagating wave mode. No wave reflection is obtained, just wave transmission and coupling with the right evanescent wave mode only.

$$T_n^+ = \exp \left(\frac{i}{\beta} S(-k_0, A_{n+1}, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_{n+1}, B_n) - \frac{i}{\beta} \int_{\Gamma_{n1}} \bar{k}_c(t) dt \right) \quad (4.255)$$

$$C_n^+ = 0 \quad (4.256)$$

$$R_n^+ = 0 \quad (4.257)$$

$$D_n^+ = \exp \left(\frac{i}{\beta} S(-k_0, A_{n+1}, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_{n+1}, B_n) - \frac{i}{\beta} S(-k_0, \Gamma_{n3}) - \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, \Gamma_{n3}) \right) \quad (4.258)$$

- Case of left incidence of an evanescent wave mode. Wave reflection is obtained,

but no wave transmission. The incident evanescent wave mode is transmitted, but no left evanescent wave mode obtained.

$$\tilde{T}_n^- = 0 \quad (4.259)$$

$$\tilde{C}_n^- = 0 \quad (4.260)$$

$$\begin{aligned} \tilde{R}_n^- = \exp & \left(\frac{i}{\beta} S(k_l, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_l, A_n, B_n) \right. \\ & - \frac{i}{\beta} S(-k_0, A_n, B_n) - \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_n, B_n) \\ & \left. - \frac{i}{\beta} S(-k_0, \Gamma_{n3}) - \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, \Gamma_{n3}) \right) \end{aligned} \quad (4.261)$$

$$\tilde{D}_n^- = \exp \left(\frac{i}{\beta} S(k_l, A_n, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(k_l, A_n, B_n) + \frac{i}{\beta} \int_{\Gamma_{n1}} \bar{k}_c(t) dt \right) \quad (4.262)$$

- Case of right incidence of an evanescent wave mode. Wave reflection is obtained, but no wave transmission. The incident evanescent wave mode is transmitted, but no right evanescent wave mode obtained.

$$\tilde{T}_n^+ = 0 \quad (4.263)$$

$$\tilde{C}_n^+ = -\exp\left(\frac{i}{\beta}S(-k_l, A_{n+1}, B_n) + \sum_{n=1}^4(-i\beta)^n v_n(-k_l, A_{n+1}, B_n) - \frac{i}{\beta} \int_{\Gamma_{n2}} \bar{k}_c(t) dt\right) \quad (4.264)$$

$$\begin{aligned} \tilde{R}_n^+ = & -\exp\left(\frac{i}{\beta}S(-k_l, A_{n+1}, B_n) + \sum_{n=1}^4(-i\beta)^n v_n(-k_l, A_{n+1}, B_n) \right. \\ & \left. - \frac{i}{\beta}S(k_0, A_{n+1}, B_n) - \sum_{n=1}^4(-i\beta)^n v_n(-k_0, A_{n+1}, B_n) \right) \\ & + \frac{i}{\beta}S(k_0, \Gamma_n) + \sum_{n=1}^4(-i\beta)^n v_n(-k_l, \Gamma_{n3}) \quad (4.265) \end{aligned}$$

$$\tilde{D}_n^+ = 0 \quad (4.266)$$

Next, we give the reflection and transmission coefficients for the boundary conditions specified in section 4.5.4.1 due to a turning point problem where the two propagating wave modes couple.

- Case of left wave incidence at the n -th turning point problem.

$$T_n^- = -\exp\left(\frac{i}{\beta}S(k_0, A_n, B_n) + \sum_{n=1}^4(-i\beta)^n v_n(k_0, A_n, B_n)\right), \quad (4.267)$$

$$\begin{aligned} R_n^- = & -2\exp\left(\frac{i}{\beta}S(k_0, A_n, B_n) + \sum_{n=1}^4(-i\beta)^n v_n(k_0, A_n, B_n) \right. \\ & \left. - \frac{i}{\beta}S(-k_0, A_n, B_n) - \sum_{n=1}^4(-i\beta)^n v_n(-k_0, A_n, B_n) \right. \\ & \left. + \frac{i}{\beta}S(k_0, \Gamma_{n3}) + \sum_{n=1}^4(-i\beta)^n v_n(k_0, \Gamma_{n3})\right), \quad (4.268) \end{aligned}$$

The integration Γ_{n3} for this case is illustrated in figure 4-29 for the turning point problem $n = 4$.

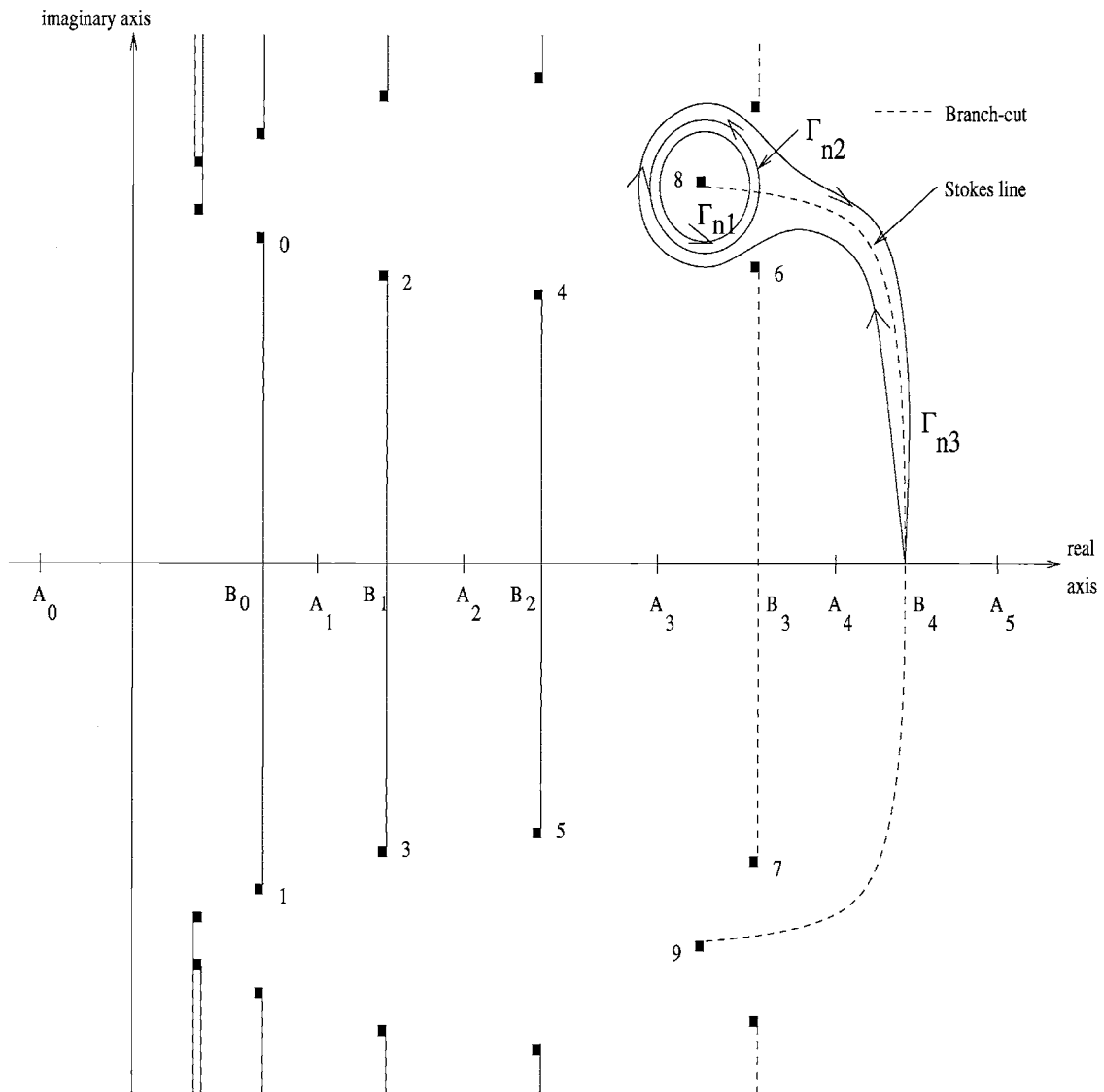


Figure 4-29: Sketch in the complex x plane of the contour Γ_{n3} ($n = 4$) used to evaluate the contour integral in the expression for the scattering coefficients at the fourth turning point problem. The direction of integration is indicated by the arrows. The lines - - - - - represent branch cuts.

- Case of right wave incidence at the n -th turning point problem.

$$T_n^+ = -\exp\left(\frac{i}{\beta}S(-k_0, A_{n+1}, B_n) - \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_{n+1}, B_n)\right), \quad (4.269)$$

$$\begin{aligned} R_n^+ = -2 \exp & \left(\frac{i}{\beta}S(-k_0, A_{n+1}, B_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_{n+1}, B_n) \right. \\ & - \frac{i}{\beta}S(k_0, A_{n+1}, B_n) - \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_{n+1}, B_n) \\ & \left. - \frac{i}{\beta}S(-k_0, \Gamma_{n3}) - \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, \Gamma_{n3}) \right) \end{aligned} \quad (4.270)$$

Next, we represent the scattering at a turning point problem through a transfer matrix that relates the wave modes amplitude at A_n to the wave modes amplitude at A_{n+1} . We use the scattering coefficients obtained above to derive the expression for the elements of the transfer matrix representing the scattering at a desired turning point problem. This gives a local transfer matrix. The product of the local transfer matrices furnishes the global transfer matrix, and the global scattering coefficients can be obtained from the global transfer matrix. In the next section we are going to obtain the local transfer matrix for the n th turning point problem.

4.5.5 Local Transfer Matrix.

We consider the n th turning point problem located at $A_n < \Re\{x\} < A_{n+1}$. The asymptotic representation of the solution is given as a superposition of the WKB waves, as given below.

- For $A_n < \Re\{x\} < B_n$ we have

$$\begin{aligned}
\psi(x) \sim & F_{0,n}^+ v_0(k_0, x) \exp \left(\frac{i}{\beta} S(k_0, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_n, x) \right) \\
& + F_{0,n}^- v_0(-k_0, x) \exp \left(\frac{i}{\beta} S(-k_0, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_n, x) \right) \\
& + \sum_{m=1}^M \left\{ F_{m,n}^+ v_0(k_m, x) \exp \left(\frac{i}{\beta} S(k_m, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(k_m, A_n, x) \right) \right. \\
& \left. + F_{m,n}^- v_0(-k_m, x) \exp \left(\frac{i}{\beta} S(-k_m, A_n, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_m, A_n, x) \right) \right\}
\end{aligned} \tag{4.271}$$

- For $B_n < \Re\{x\} < A_{n+1}$ we have

$$\begin{aligned}
\psi(x) \sim & F_{0,n+1}^+ v_0(k_0, x) \exp \left(\frac{i}{\beta} S(k_0, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, A_{n+1}, x) \right) \\
& + F_{0,n+1}^- v_0(-k_0, x) \exp \left(\frac{i}{\beta} S(-k_0, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, A_{n+1}, x) \right) \\
& + \sum_{m=1}^M \left\{ F_{m,n+1}^+ v_0(k_m, x) \exp \left(\frac{i}{\beta} S(k_m, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(k_m, A_{n+1}, x) \right) \right. \\
& \left. + F_{m,n+1}^- v_0(-k_m, x) \exp \left(\frac{i}{\beta} S(-k_m, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_m, A_{n+1}, x) \right) \right\},
\end{aligned} \tag{4.272}$$

where M is the number of wave modes considered in the asymptotic representation for the free-surface potential.

The local transfer matrix relates the amplitude of the wave modes at A_n with the amplitude of the wave modes at A_{n+1} . So, we can write

$$\begin{Bmatrix} F_{0,n+1}^+ \\ F_{0,n+1}^- \\ F_{1,n+1}^+ \\ F_{1,n+1}^- \\ \dots \\ F_{M,n+1}^+ \\ F_{M,n+1}^- \end{Bmatrix} = [M_n] \begin{Bmatrix} F_{0,n}^+ \\ F_{0,n}^- \\ F_{1,n}^+ \\ F_{1,n}^- \\ \dots \\ F_{M,n}^+ \\ F_{M,n}^- \end{Bmatrix}. \quad (4.273)$$

Next, we would like to obtain the elements of the local transfer matrix $[M_n]$. At each turning point problem there is always the coupling between a propagating mode and an evanescent mode or the coupling between the two propagating modes. As a result, the majority of the elements of the local transfer matrix are zero. The exception are the diagonal elements and the non-diagonal elements which represents the coupling among wave modes. The diagonal elements are derived below. For the wave modes that do not couple we can write

$$\begin{aligned} F_{m,n}^\pm v_0(\pm k_m, x) \exp \left(\frac{i}{\beta} S(\pm k_m, A_n, A_{n+1}) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_m, A_n, A_{n+1}) \right. \\ \left. + \frac{i}{\beta} S(\pm k_m, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_m, A_{n+1}, x) \right) = \\ F_{m,n+1}^\pm v_0(\pm k_m, x) \exp \left(\frac{i}{\beta} S(\pm k_m, A_{n+1}, x) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_m, A_{n+1}, x) \right), \end{aligned} \quad (4.274)$$

which implies a relation between the wave mode amplitudes before (A_n) and after (A_{n+1}) the turning point problem, given by the equation

$$F_{m,n+1}^\pm = \exp \left(\frac{i}{\beta} S(\pm k_m, A_n, A_{n+1}) + \sum_{n=1}^4 (-i\beta)^n v_n(\pm k_m, A_n, A_{n+1}) \right) F_{m,n}^\pm. \quad (4.275)$$

From this relation we obtain the diagonal terms of the local transfer matrices that are not associated with wave modes that couple at the considered turning point problem. The expression for these elements follows:

$$M_{2m+1,2m+1} = \exp \left(\frac{i}{\beta} S(k_m, A_n, A_{n+1}) + \sum_{n=1}^4 (-i\beta)^n v_n(k_m, A_n, A_{n+1}) \right), \quad (4.276)$$

$$M_{2m+2,2m+2} = \exp \left(\frac{i}{\beta} S(-k_m, A_n, A_{n+1}) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_m, A_n, A_{n+1}) \right). \quad (4.277)$$

For a turning point problem where only the two propagating wave modes couple, we have only the boundary conditions of left and right wave incidence. This allow us to obtain more four elements of the local transfer matrix. If we apply the boundary conditions of left and right wave incidence to the matrix equation (4.273), we obtain the system of equations

$$\left\{ \begin{array}{cc} (T_n^-)^* & R_n^- \\ 0 & 1 \\ F_{1,n+1}^+ & F_{1,n+1}^+ \\ F_{1,n+1}^- & F_{1,n+1}^- \\ \dots & \\ F_{M,n+1}^+ & F_{M,n+1}^+ \\ F_{M,n+1}^- & F_{M,n+1}^- \end{array} \right\} = [M_n] \left\{ \begin{array}{cc} 1 & 0 \\ R_n^- & (T_n^+)^* \\ F_{1,n+1}^+ & F_{1,n+1}^+ \\ F_{1,n+1}^- & F_{1,n+1}^- \\ \dots & \\ F_{M,n+1}^+ & F_{M,n+1}^+ \\ F_{M,n+1}^- & F_{M,n+1}^- \end{array} \right\}, \quad (4.278)$$

which allow us to derive the expressions for the first two diagonal terms of matrix $[M_n]$ and other two non-diagonal elements which describe the coupling between the two propagating wave modes. These elements are

$$(M_n)_{11} = (T_n^-)^* - \frac{R_n^+ R_n^-}{(T_n^+)^*}, \quad (4.279)$$

$$(M_n)_{21} = -\frac{R_n^-}{(T_n^+)^*}, \quad (4.280)$$

$$(M_n)_{12} = \frac{R_n^+}{(T_n^+)^*}, \quad (4.281)$$

$$(M_n)_{22} = \frac{1}{(T_n^+)^*}, \quad (4.282)$$

where the expression for the scattering coefficients R_n^- and R_n^+ are given, respectively, by equations (4.268) and (4.270). The scattering coefficients $(T_n^\pm)^*$ are given in terms of the scattering coefficients $T_n^\pm$ according to the equations

$$(T_n^-)^* = T_n^- \exp\left(\frac{i}{\beta} S(-k_0, B_n, A_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, B_n, A_n)\right), \quad (4.283)$$

$$(T_n^+)^* = T_n^+ \exp\left(\frac{i}{\beta} S(k_0, B_n, A_{n+1}) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, B_n, A_{n+1})\right), \quad (4.284)$$

where the coefficients $T_n^\pm$ are given by equations (4.267) and (4.269). Next, we consider a turning point problem where the propagating modes couple with an evanescent mode (actually two modes, one that is evanescent to the right and the other evanescent to the left, but the same wavenumber). For this case, we have to consider four boundary conditions. They are the incidence of a left and right propagating wave and the left and right incidence of the evanescent wave mode. For this case, we assume that the l th evanescent wave mode couple with the propagating wave modes. After we apply the boundary conditions, the matrix equation (4.273) assumes the form

$$\left\{ \begin{array}{cccc}
(T_n^-)^* & R_n^- & (\tilde{T}_n^-)^* & (\tilde{T}_n^+)^* \\
0 & 1 & 0 & 0 \\
F_{1,n+1}^+ & F_{1,n+1}^+ & F_{1,n+1}^+ & F_{1,n+1}^+ \\
F_{1,n+1}^- & F_{1,n+1}^- & F_{1,n+1}^- & F_{1,n+1}^- \\
\cdots & & & \\
(D_n^-)^* & (D_n^+)^* & (\tilde{D}_n^-)^* & (\tilde{D}_n^+)^* \\
0 & 0 & 0 & 1 \\
\cdots & & & \\
F_{M,n+1}^+ & F_{M,n+1}^+ & F_{M,n+1}^+ & F_{M,n+1}^+ \\
F_{M,n+1}^- & F_{M,n+1}^- & F_{M,n+1}^- & F_{M,n+1}^-
\end{array} \right\} = [M_n] \left\{ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
R_n^- & (T_n^+)^* & \tilde{R}_n^- & \tilde{R}_n^+ \\
F_{1,n+1}^+ & F_{1,n+1}^+ & F_{1,n+1}^+ & F_{1,n+1}^+ \\
F_{1,n+1}^- & F_{1,n+1}^- & F_{1,n+1}^- & F_{1,n+1}^- \\
\cdots & & & \\
0 & 0 & 1 & 0 \\
(C_n^-)^* & (C_n^+)^* & (\tilde{C}_n^-)^* & (\tilde{C}_n^+)^* \\
\cdots & & & \\
F_{M,n+1}^+ & F_{M,n+1}^+ & F_{M,n+1}^+ & F_{M,n+1}^+ \\
F_{M,n+1}^- & F_{M,n+1}^- & F_{M,n+1}^- & F_{M,n+1}^-
\end{array} \right\}, \quad (4.285)$$

This system of equations allow us to obtain the two first diagonal terms, the $2l + 1$ th and the $2l + 2$ th diagonal terms, and the non-diagonal terms that describe the coupling process among wave modes. The expressions for these matrix elements are given below.

$$(M_n)_{2l+2,2l+2} = \frac{(T_n^+)^*}{(\tilde{C}_n^-)^*(T_n^+)^* - \tilde{R}_n^+(C_n^+)^*} \quad (4.286)$$

$$(M_n)_{2l+2,2} = - \frac{(C_n^+)^*}{(\tilde{C}_n^-)^*(T_n^+)^* - \tilde{R}_n^+(C_n^+)^*} \quad (4.287)$$

$$(M_n)_{2l+2,2l+1} = \frac{\tilde{R}_n^- - (\tilde{C}_n^-)^*(T_n^+)^*}{(\tilde{C}_n^-)^*(T_n^+)^* - \tilde{R}_n^+(C_n^+)^*} \quad (4.288)$$

$$(M_n)_{2l+2,1} = \frac{R_n^- - (C_n^-)^*(T_n^+)^*}{(\tilde{C}_n^-)^*(T_n^+)^* - \tilde{R}_n^+(C_n^+)^*} \quad (4.289)$$

$$(M_n)_{2l+1,2l+2} = - \frac{(D_n^+)^*\tilde{R}_n^+ - (T_n^+)^*(\tilde{D}_n^+)^*}{(\tilde{C}_n^-)^*(T_n^+)^* - \tilde{R}_n^+(C_n^+)^*} \quad (4.290)$$

$$(M_n)_{2l+1,2} = \frac{(\tilde{D}_n^+)^*}{\tilde{R}_n^+} + \frac{(\tilde{C}_n^-)^*}{\tilde{R}_n^+} \left[\frac{(D_n^+)^*\tilde{R}_n^+ - (T_n^+)^*(\tilde{D}_n^+)^*}{(\tilde{C}_n^-)^*(T_n^+)^* - \tilde{R}_n^+(C_n^+)^*} \right] \quad (4.291)$$

$$(M_n)_{2l+1,2l+1} = (\tilde{D}_n^-)^* - \frac{\tilde{R}_n^- (\tilde{D}_n^+)^*}{\tilde{R}_n^+} - \left(\frac{\tilde{R}_n^- (\tilde{C}_n^+)^*}{\tilde{R}_n^+} - (\tilde{C}_n^-)^* \right) \left[\frac{(D_n^+)^* \tilde{R}_n^+ - (T_n^+)^* (\tilde{D}_n^+)^*}{(\tilde{C}_n^-)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \right] \quad (4.292)$$

$$(M_n)_{2l+1,1} = (\tilde{D}_n^-)^* - \frac{\tilde{R}_n^- (\tilde{D}_n^+)^*}{\tilde{R}_n^+} - \left(\frac{\tilde{R}_n^- (\tilde{C}_n^+)^*}{\tilde{R}_n^+} - (\tilde{C}_n^-)^* \right) \left[\frac{(D_n^+)^* \tilde{R}_n^+ - (T_n^+)^* (\tilde{D}_n^+)^*}{(\tilde{C}_n^-)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \right] \quad (4.293)$$

$$(M_n)_{2,2l+2} = - \frac{\tilde{R}_n^+}{(\tilde{C}_n^+)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \quad (4.294)$$

$$(M_n)_{2,2} = \frac{(\tilde{C}_n^+)^*}{(\tilde{C}_n^+)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \quad (4.295)$$

$$(M_n)_{2,2l+1} = \frac{(\tilde{C}_n^-)^* \tilde{R}_n^+ - \tilde{R}_n^- (\tilde{C}_n^+)^*}{(\tilde{C}_n^+)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \quad (4.296)$$

$$(M_n)_{2,1} = \frac{(\tilde{C}_n^-)^* \tilde{R}_n^+ - R_n^- (\tilde{C}_n^+)^*}{(\tilde{C}_n^+)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \quad (4.297)$$

$$(M_n)_{12} = \frac{(\tilde{C}_n^+)^* R_n^+ - (\tilde{T}_n^+)^* (C_n^+)^*}{(\tilde{C}_n^+)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \quad (4.298)$$

$$(M_n)_{1,2l+2} = \frac{(\tilde{T}_n^+)^*}{(\tilde{C}_n^+)^*} - \frac{\tilde{R}_n^+}{(\tilde{C}_n^+)^*} \left[\frac{(\tilde{C}_n^+)^* R_n^+ - (\tilde{T}_n^+)^* (C_n^+)^*}{(\tilde{C}_n^+)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \right] \quad (4.299)$$

$$(M_n)_{1,2l+1} = (\tilde{T}_n^-)^* - \left(R_n^- + \frac{(C_n^-)^* (\tilde{T}_n^+)^*}{(\tilde{C}_n^+)^*} - \frac{(C_n^-)^* (\tilde{R}_n^+)^*}{(\tilde{C}_n^+)^*} \right) \left[\frac{(\tilde{C}_n^+)^* R_n^+ - (\tilde{T}_n^+)^* (C_n^+)^*}{(\tilde{C}_n^+)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \right] \quad (4.300)$$

$$(M_n)_{11} = (T_n^-)^* - \left(R_n^- + \frac{(C_n^-)^* (\tilde{T}_n^+)^*}{(\tilde{C}_n^+)^*} - \frac{(C_n^-)^* (\tilde{R}_n^+)^*}{(\tilde{C}_n^+)^*} \right) \left[\frac{(\tilde{C}_n^+)^* R_n^+ - (\tilde{T}_n^+)^* (C_n^+)^*}{(\tilde{C}_n^+)^* (T_n^+)^* - \tilde{R}_n^+ (C_n^+)^*} \right], \quad (4.301)$$

where the scattering coefficients without * in the equations above have their expressions given in section 4.5.4.4. The scattering coefficients with the symbol * are given below according to the equations

$$(T_n^-)^*[(\tilde{T}_n^-)^*] = T_n^-[\tilde{T}_n^-] \exp\left(\frac{i}{\beta} S(-k_0, B_n, A_n) + \sum_{n=1}^4 (-i\beta)^n v_n(-k_0, B_n, A_n)\right), \quad (4.302)$$

$$(T_n^+)^*[(\tilde{T}_n^+)^*] = T_n^+[\tilde{T}_n^+] \exp\left(\frac{i}{\beta} S(k_0, B_n, A_{n+1}) + \sum_{n=1}^4 (-i\beta)^n v_n(k_0, B_n, A_{n+1})\right), \quad (4.303)$$

$$\begin{Bmatrix} (C_n^-)^*[(\tilde{C}_n^-)^*] \\ (D_n^-)^*[(\tilde{D}_n^-)^*] \end{Bmatrix} = \begin{Bmatrix} C_n^-[\tilde{C}_n^-] \\ (D_n^-[\tilde{D}_n^-] \end{Bmatrix} \exp\left(\frac{i}{\beta} S(\mp k_l, B_n, A_n) + \sum_{n=1}^4 (-i\beta)^n v_n(\mp k_l, B_n, A_n)\right), \quad (4.304)$$

$$\begin{Bmatrix} (C_n^+)^*[(\tilde{C}_n^+)^*] \\ (D_n^+)^*[(\tilde{D}_n^+)^*] \end{Bmatrix} = \begin{Bmatrix} C_n^+[\tilde{C}_n^+] \\ (D_n^+[\tilde{D}_n^+] \end{Bmatrix} \exp\left(\frac{i}{\beta} S(\mp k_l, B_n, A_{n+1}) + \sum_{n=1}^4 (-i\beta)^n v_n(\mp k_l, B_n, A_{n+1})\right), \quad (4.305)$$

and the expression for the scattering coefficients appearing on the right of the equations above are given in section 4.5.4.4.

4.6 Global Transfer Matrix.

The global transfer matrix is just the product of the local transfer matrices describing the wave mode coupling along the bottom due to its variation. If we assume that we have N turning point problems for a variable bottom, the global transfer matrix $[M]$ is given by the equation

$$[M] = \prod_{n=1}^N [M_n]. \quad (4.306)$$

The global transfer matrix relates the wave mode amplitudes at the beginning of the variable bottom with the wave mode amplitudes at the end of the variable bottom. If A_1 and A_{N+1} are, respectively, the initial and final positions of the variable bottom,

we can relate the wave mode amplitudes according to the equation

$$\begin{Bmatrix} F_{0,N+1}^+ \\ F_{0,N+1}^- \\ F_{1,N+1}^+ \\ F_{1,N+1}^- \\ \dots \\ F_{M,N+1}^+ \\ F_{M,N+1}^- \end{Bmatrix} = [M] \begin{Bmatrix} F_{0,1}^+ \\ F_{0,1}^- \\ F_{1,1}^+ \\ F_{1,1}^- \\ \dots \\ F_{M,1}^+ \\ F_{M,1}^- \end{Bmatrix}. \quad (4.307)$$

This equation can be used to obtain the global scattering coefficients at the positions A_1 and A_{n+1} . For the condition of left wave incidence (wave incidence at A_1), we have the equation

$$\begin{Bmatrix} T_G^- \\ 0 \\ D_{1,G}^- \\ 0 \\ \dots \\ D_{M,G}^- \\ 0 \end{Bmatrix} = [M] \begin{Bmatrix} 1 \\ R_G^- \\ 0 \\ C_{1,G}^- \\ \dots \\ 0 \\ C_{M,G}^- \end{Bmatrix}, \quad (4.308)$$

and for the condition of right wave incidence (wave incidence at A_{N+1}), we have the equation

$$\begin{Bmatrix} R_G^+ \\ 1 \\ D_{1,G}^+ \\ 0 \\ \dots \\ D_{M,G}^+ \\ 0 \end{Bmatrix} = [M] \begin{Bmatrix} 0 \\ T_G^+ \\ 0 \\ C_{1,G}^+ \\ \dots \\ 0 \\ C_{M,G}^+ \end{Bmatrix}. \quad (4.309)$$

The coefficients $R_G^\pm, T_G^\pm, C_{m,G}^\pm$ and $D_{m,G}^\pm$ with $m = 1, \dots, M$ are the global scattering coefficients.

4.7 Discussion and Conclusions.

In this chapter we extended the wave refraction model for linear surface gravity waves propagating over an one-dimensional non-uniform bottom topography to incorporate wave reflection and wave mode conversion due to the bottom non-uniformity. The wave reflection is exponentially small, as suggests the asymptotic approximations for the wave scattering coefficients given in section 4.5.4.4. We were able to obtain asymptotic expressions for the reflection coefficient and to obtain asymptotic estimates for the coefficients related to the coupling of the propagating wave field with the evanescent wave field.

The asymptotic expansion given by equation (4.55) breaks down at the branch points of the uniform bottom dispersion relation, which gives the phase $S(x)$. If we take the effects of the branch points into account, we obtain wave reflection and wave mode conversion. To do so, we consider the branch points closer to the real axis, and we approximate the governing equation (4.54) in a neighborhood of the considered branch points. We approximate it as a second order differential equation since at the branch points the wave numbers associated with the coupling wave modes coalesce in pairs. So, it seems reasonable to use a second order differential equation to

describe the wave mode coupling. For the branch points where the two propagating wave modes coalesce, the approximation of the governing equation (4.54) by a second order equation is enough to describe the wave mode coupling (wave reflection) which happens at this branch points. The other possibility of wave mode coupling at the branch points is the coupling of each propagating wave mode with an evanescent wave mode (evanescent wave modes with wavenumber of the same modulus, but out of phase). For this case, the approximation of the governing equation (4.55) by a second order differential equation was not enough to describe the wave mode coupling which happens at this type of branch point, according to the results for the scattering coefficients at this type of branch points given in section 4.5.4.4. For this type of branch points we have coalescence of actually four wavenumbers, coalescence of two pairs of wavenumbers. Therefore, a fourth order differential equation seems more adequate to describe the wave mode coupling which happens at this type of turning point. This is not an easy task and was left as future work.

In the second part of chapter 3 we considered the wave interaction with a non-uniform Euler-Bernoulli beam. We applied the same asymptotic technique developed in this chapter to obtain asymptotic expressions for the scattering coefficients. At the branch points of the uniform beam dispersion relation we observed the coupling of four wave modes, similar to what happened at the second type of branch points appearing in the linear surface gravity waves problem. We approximated the governing equation for this problem in a neighborhood of its branch points by a fourth order differential equation. We were able to solve the approximate equation for the case of non-uniformity in the beam flexural rigidity which is linear in a neighborhood of the branch points. We chose this problem to get a feeling of the difficulties involved in this kind of analysis. Numerical evaluation of the asymptotic expressions for the reflection and wave mode conversion coefficients was left as future work. The asymptotic expressions for the scattering coefficients are exponential of phase integrals, which are contour integrals of the wavenumber in the complex x plane. So, to evaluate numerically the scattering coefficients we need a numerical quadrature method to compute contour integrals. These asymptotic expressions allow us to obtain the scattering

coefficients for slowly varying topography of the order of many wavelengths of the incident wave, which is not easy to obtain with standard numerical methods due to the size of the discretization of the physical domain.

A proposed numerical method to generate results to compare with the numerical evaluation of the asymptotic approximations is based on the bottom discretization by steps. So, the bottom is represented by a sequence of small (size much smaller than the wavelength of the incident wave) shelves. The boundary value problem for linear surface gravity waves potential function over each shelf has a closed form solution. The solution over the whole discretized bottom is obtained by matching the velocity field and the potential function at the edges of each shelf. This continuity condition at each step result in a set of equations for the scattering coefficients at this step. If we consider all equations obtained through the continuity conditions at each step, we obtain a linear system of equations which represents the wave scattering along the variable bottom, and the solution of this system of equations gives the scattering coefficients along the discretized bottom. We are interested in the reflection and transmission coefficients at the first and last steps of the bottom discretization. These are the global reflection and transmission coefficients we can use to compare with the numerical evaluation of the asymptotic expressions for the scattering coefficient.

Chapter 5

Summary and Future Work.

As discussed in the thesis introduction, waves trying to propagate through a non-uniform medium undergo many partial reflections and transmission from the scattering centers in the non-uniform medium and the various transmitted and reflected waves, which have a random de-phasing, interfere with each other. If the non-uniformity is strong enough, the incident wave will be completely reflected as it tries to propagate through the medium. The amplitude of the wave disturbance along the medium shows an exponential decay, and the rate of decay is defined as the length scale of this wave interference effect, which is denominated localization phenomenon in the literature.

In the context of water waves, localization phenomenon has been observed experimentally by Belzons et al [3]. They considered the interaction of linear gravity waves with a static one-dimensional random bottom. Devillard et al [19] performed theoretical calculations for the same kind of bottom used in the experiments to provide comparison between measured and predicted localization lengths. There were two limitations for the observation of the localization phenomenon. The length scale of the dissipation could be smaller than the localization length and the bottom length can be larger than the localization length, so in both situations localization is not observable. Besides these limitations, theoretical calculations in Devillard et al [19] for random bottoms used in the experiment predicted a range of wave frequencies where the localization length should be observable for the given experiment set up.

For this range of wave frequencies, good agreement between the experiments and the theoretical prediction was observed. Since the localization length for this range of wave frequencies was of the order of the bottom length, fluctuations in the transmission (reflection) coefficient were observed. For very large systems these fluctuations are not present. The eigenmodes for very large systems are localized, but for a finite system they are turned into resonances. Since these resonant modes are trapped along the non-uniform medium, the incident wave could pass by tunneling for the corresponding resonant frequency. This implies a sample-to-sample dependence of the transmission (reflection). In other words, the system vibratory response shows a large sensitivity with respect to the non-uniformity variation, which opens the possibility of designing the non-uniformity to achieve a desired vibratory response, like minimizing wave transmission.

The main objective of this thesis was to predict how to tune the medium non-uniformity (bottom non-uniformity in the case of surface gravity waves) such that the interaction of the incident wave field with the medium non-uniformity generates a desired scattered wave field. In other words, we would like to be able to design the system non-uniformity such that a desired vibratory response is achieved. A particular design problem of interest would be to predict the minimum amount possible of changes to the non-uniformity of the medium needed to minimize wave transmission to a desired level for a given range of wave frequencies.

Triantafyllou and Triantafyllou [65] provided ideas on how to succeed in the design problem mentioned above. They showed that localization in disordered repetitive systems is associated with large sensitivity of both natural frequency and natural modes with respect to small disorder change. By allowing the disorder parameters to become complex, the system eigenvalue equation can be seen as a complex mapping from the complex parameter space to the complex frequency plane. They showed that the high modal sensitivity is caused by the presence of branch points of this complex mapping. At the image of a branch point in the complex frequency plane, a saddle point, two or more natural frequencies coalesce. A real system, with parameters close to a branch point of the eigenvalue equation, should have to a large or small

degree a large modal sensitivity. The closer the branch points lay to the real axis, the larger is the sensitivity of the vibratory response of the real system. The higher the number of frequencies coalescing as we approach a branch point, the higher the system sensitivity for disorder configurations close to this branch point. So, under the design perspective, we should look for the disorder configuration which gives the higher number of branch points with higher number of frequency coalescence and which are closer to the real part of the parameter space. Then, we project this disorder configuration in the real part of the parameter space, and we search in its vicinity for the real disorder which minimizes wave transmission, for example.

The ideas mentioned in the previous paragraph should work for finite systems described by a set of differential equations or a set of partial differential equations under various boundary conditions. For the interaction of linear surface gravity waves with a non-uniform bottom we have a semi-infinite or infinite domain, and instead of only boundary conditions we have also radiation conditions. For this class of problem the eigenvalue spectrum is a continuum with the possibility of having also a discrete part, associated with trapped modes due to the bottom non-uniformity in the case of linear surface gravity waves. So, the methodology previously described to handle the design problem mentioned above is very complex to use in the problem of the interaction of surface gravity waves with a non-uniform bottom.

We consider a different strategy to face the design problem mentioned above. We use an asymptotic approach. We obtained asymptotic expressions for the scattering coefficients. These asymptotic expressions revealed the functional dependence of the scattering coefficients with respect to the variation of the medium non-uniformity. This knowledge can be used to tune the medium non-uniformity, for example, to minimize wave transmission to a desired level for a given range of wave frequencies. Before applying this asymptotic approach to the interaction problem of linear surface gravity waves with a non-uniform bottom, we consider two different physical problems, namely, a chain of non-uniform coupled pendulums and the vibration problem of a beam under tensile force. These problems are simpler than the water waves problem and they can be seen as steps towards solving the difficulties which we have

to face in the water waves problem.

In chapter 2 we consider one-dimensional chains of repetitive systems of the size of the order of many wavelengths. An one-dimensional chain of coupled pendulums is an example of such type of systems. This type of systems support only two wave modes, a right and a left propagating wave. So, the coupling between the wave modes due to the medium non-uniformity give rise only to wave reflection, which is a much simpler problem than the water waves problem where we have an enumerable set of wave modes and coupling among them. Besides the simplicity of this type of systems, they already have key features appearing in the water wave problem like wave mode conversion.

We applied an asymptotic theory for wave propagation along non-uniform chains of repetitive systems with respect to the slowness of the variation of the system parameters. This type of systems are governed by second order difference equations, and the asymptotic method used was the WKB method. The contributions made in chapter 2 are:

- Improvement of the WKB method for second order difference equations. Before the work presented in chapter 2, a procedure to continue (connection formula) the asymptotic solutions through complex turning points of difference equations was not available. We derived connection formulas for pairs of complex conjugate first order turning points, for a cluster of a pair of complex or real first order turning points and for a complex or real second order turning point.
- A systematic procedure to construct asymptotic solution for second order difference equations with a large number of turning points. This procedure is a blend of the WKB method and the method of transfer matrices.
- Qualitative understanding of the localization phenomenon. For a one-dimensional chain of uniform repetitive systems, the dispersion relation provides the system passband. The larger the coupling, the larger the passband. Wave propagation happens only for frequencies inside the system passband. In the context of the WKB method, the uniform system dispersion relation is locally valid. Due

to the non-uniformity along the chain, the size of the passband given by the uniform system dispersion relation varies along the chain. This leads to an effective passband for the whole chain. Waves with frequency outside the effective passband but inside the the uniform system passband have exponentially small transmission and are strongly localized. Therefore, for whatever small disorder in the system, there will always exist a boundary layer close to both edges of the uniform system passband, and waves with frequency in these layers are strongly localized. For weak coupling, the effective passband may be very small or even cease to exist for a given disorder, and the strong wave localization observed can be explained by the WKB theory, which is a new result.

The work presented in chapter 2 is published in reference Burr et al [12]. This work can be extended in different ways:

- Extension of the WKB method for second order difference equations to high order. Difference equations can be represented as pseudo-differential equations. The asymptotic analysis of a pseudo-differential equation is very similar to the asymptotic analysis of differential equations, and all the machinery for differential equations can be applied to pseudo-differential equations almost directly.
- Extension of the WKB method to higher order difference equations or system of difference equations. These equations represents chains of repetitive systems where the interaction of the subsystems is not restricted to the two nearest neighbors.
- Design problem. With the development of higher order asymptotic approximations for the local reflection and transmission coefficients along the system, we may be able to predict more accurately the global scattering coefficients for more faster variations of the parameters along the chain of repetitive systems. This allow us to predict more accurately the wave interaction with more fast varying non-uniformities in the system, where strong localization may occur even in the present of relatively strong coupling. This high order asymptotic

approximation could be an important tool to succeed in the design problem, since it reveals the dependence of the scattering coefficients with respect to the non-uniformity variation.

In chapter 3 we consider the effects of the non-uniformity in the vibration response of beams under tensile force. We consider the Euler-Bernoulli beam model to study the effects of non-uniformity on beam vibration. This chapter has two parts. In the first part, we derived a simpler equation governing the beam vibratory response. This equation is asymptotic to the non-uniformity of the beam. As the steepness of the non-uniformity decreases, the simple equation recovers the vibratory behavior predicted by the full governing equation. Under certain restrictions, viz. of having the constant tension and a constant product of the mass per unit length and flexural rigidity, the simple equation has a Helmholtz-like form and is an exact governing equation for the beam vibration response. In summary, the contributions of the first part of chapter 3 are:

- A simpler equation (second order differential equation) governing the vibration response of non-uniform beams. For general non-uniformity, this simple equation is asymptotic with respect to the steepness of the non-uniformity, but under special conditions mentioned above, it is an exact governing equation.
- Design problem. Under the restrictions of having constant tension and a constant product of the mass per unit length and flexural rigidity, the simpler equation is exact and has a Helmholtz-like form, and inverse scattering methods for the Helmholtz-like equations can be applied to design the beam non-uniformity such that desired scattering properties, like minimizing the wave transmission to a desired degree for a given range of wave frequencies, are achieved. The restrictions mentioned above are not commonly met in practice, but such beams can be designed and built.

The work presented in the first part of chapter 3 is published in reference Burr et al [13].

In the second part of chapter 3, we applied an asymptotic theory to study wave propagation along a non-uniform Euler-Bernoulli beam under tension. The governing equation for this wave propagation problem in the frequency domain is a fourth order differential equation. Therefore, this medium support four wave modes. There is a left and a right propagating wave mode and a right and a left evanescent wave mode. The evanescent wave modes have only a local character and they couple with the propagating modes due to the non-uniformity in the beam. We have energy transfer from the propagating to the evanescent wave modes due to the presence of non-uniformity along the beam. The asymptotic theory used was the WKB method and we consider non-uniformity with only first order turning points and simple poles. We studied for the given type of non-uniformity how the wave modes couple at the turning points and singularities. The contributions of the second part of chapter 3 are:

- High order asymptotic approximations for the scattering coefficients. In the context of the WKB method, wave reflection and wave mode conversion happens due to the presence of turning points and singularities like poles. We obtained an asymptotic approximation for the local reflection, transmission and wave mode conversion coefficients due to the presence of first order turning points and poles, which is a new result. We model each turning point problem as a transfer matrix, such that the wave interaction problem with the non-uniformity in the beam is represented by a transfer matrix, product of all local transfer matrices. From the global transfer matrix we obtained the global scattering coefficient.
- Asymptotic estimates of the energy transfer between the propagating and evanescent wave modes. At the turning point and singularities, coupling between propagating and evanescent wave modes was observed, and we obtained asymptotic estimates for the amplitudes of the excited evanescent modes (wave mode conversion coefficients). This is a new result in the context of wave propagation along non-uniform beams.

Regarding future work, the second part of chapter 3 can be extended in different ways.

- Comparison between the asymptotic results and full numerical simulation. We built in the second part of chapter 3 the asymptotic theory for wave propagation along non-uniform beams with first order turning points and poles, but we did not provide comparison between the asymptotic results and full numerical simulations, what was left as future work.
- More general turning points and singularities. We consider only first order turning point and poles. To consider the effect of high order turning points and other types of singularities, we need a different approach than the one used in the second part of chapter 3, since the local form of the governing equation (3.95) has no closed form solution, at least in terms of contour integrals, and matching of the local solution with the local form of the WKB waves is not possible.

The good results obtained through the asymptotic approach in chapter 2 stimulated us to extend the asymptotic approach to the interaction of surface gravity waves with a non-uniform bottom topography. This is done in chapter 4. The asymptotic method used (the WKB method) is a refraction theory and do not predict wave reflection or wave mode conversion, unless the effects of the turning points are taken into account, as we did, and we obtained asymptotic estimates for the wave mode conversion coefficients (wave reflection is a particular case of wave mode conversion). Wave reflection for surface gravity waves in the context of the WKB method is not a new result in the literature (Meyer [46]), but wave mode conversion between the propagating modes and the evanescent modes, as we obtained in chapter 4 is a new result. The contributions of chapter 4 are:

- The interaction pattern between wave modes. In the context of the asymptotic theory, the solution is represented as a superposition of plane waves with variable phase and amplitude. We have two propagating wave modes and an

enumerable set of evanescent wave modes, which couple with the two propagating wave modes due to the presence of non-uniformity. The coupling between wave modes happens at the turning points, which are the branch points of the uniform depth dispersion relation. We obtained the turning points and we studied how the wave modes couple at each turning point. Results of the wave mode interaction pattern are given in section 4.5.2.1.

- Wave mode conversion coefficients. At the turning points we have coupling between a propagating wave mode and an evanescent wave mode. We obtained asymptotic estimates for the transfer of energy between the propagating wave mode and the evanescent wave mode, which are the wave mode conversion coefficients. Wave reflection is a particular case, and it happens at the turning points where the two propagating modes couple. The asymptotic estimates for the wave mode conversion coefficients are a new result and they are given in section 4.5.4.4.
- A more complete asymptotic representation of the wave field. We represented the free surface potential as a superposition of plane waves with variable phase and amplitude. We were able to describe the coupling among the wave modes due to the presence of non-uniformity. At each relevant turning point (closer to the real axis), we have coupling between the wave modes. The coupling between wave modes at a relevant turning point is modeled by a transfer matrix, so the wave interaction with the non-uniform bottom is given as a sequence of transfer matrices, and the global effect of this interaction is given by the product of all local transfer matrices.

Regarding future work, what was done in chapter 4 need and can be extended as outlined below:

- Comparison of the asymptotic results with results from the full numerical simulation. The asymptotic expressions for the wave mode conversion coefficients are given in terms of exponentials of phase integrals. These integrals, in general,

do not have closed form solution, and they need to be evaluated numerically. To access the effectiveness of the asymptotic approximations for the wave mode conversion coefficients, we need to evaluate numerically the phase integrals and compare the values of the asymptotic approximation of the wave mode conversion coefficients with the values of the wave mode conversion coefficients obtained through direct numerical simulation of the wave interaction with a variable bottom.

- More complex turning point configurations. The turning points appear as complex conjugate numbers in the complex space parameterization plane. In chapter 4 we considered the case where the turning points are far from the real axis, so they can be considered individually. In the case where the pairs of complex turning points approach the real axis, we have to consider both turning points together. This leads to more complicated connection formulas and more complicated expressions for the wave mode conversion coefficients.
- Fourth order differential equation describing wave mode conversion. To describe the wave interaction in a neighborhood of a turning point, we used second order differential equations. This approach was okay for the case where the two propagating modes couple, but it was not satisfactory for the case where we have coupling between a propagating mode and an evanescent mode. Actually, for the turning point problems where coupling between propagating and evanescent modes happens, we have the interaction of four wave modes. Each propagating wave mode couple with an evanescent wave mode (evanescent wave modes with wavenumber with the same absolute value). So, a fourth order differential equation is necessary to describe the wave mode interaction in a neighborhood of this type of branch point.
- Extension of the asymptotic approach to two dimensional bottoms. In chapter 4 we just considered one dimensional bottoms. A natural extension of this work would be to consider two dimensional bottoms. The asymptotic solution for the free surface potential given in section 4.5 can be easily extended to

two-dimensional bottoms. The uniform bottom dispersion relation in the case of two-dimensional bottoms gives the wavenumbers as vectors, defining a vector field. The lines tangent to the vector field are called rays, over which the wave disturbance propagates. Along a ray, the wave propagation problem is again one-dimensional. So, we can apply the machinery developed in chapter 4 to consider the coupling between the propagating part of the wave field and the evanescent part of the wave field and also exponentially small wave reflection. The turning points in the one-dimensional case form complex caustics in the two-dimensional case. At these caustics we would have generation of local evanescent wave fields or exponentially small wave reflection, leading to a more complete picture of the wave interaction with the bottom topography.

Appendix A

HIGH ORDER WKB METHOD FOR SYSTEMS OF FIRST ORDER DIFFERENCE EQUATIONS.

In this appendix we discuss how to derive the Liouville-Green functions for a system of first order difference equations. The first step is to reduce the given system of difference equations to a high order difference equation. The order of this difference equation is equal to the dimension of the system of first order difference equations. Then, we describe a procedure to obtain the Liouville-Green functions for a high order difference equation. For a difference equation of order n we have n linearly independent Liouville-Green functions, which form an asymptotic base for the solution of the high order difference equation. The asymptotic solution is not valid at the turning points. At these points in the index variable complex plane the Liouville-Green functions are not linearly independent and do not form an asymptotic base for the solution.

In terms of wave propagation, the WKB asymptotic solution gives only wave refraction. To incorporate wave reflection and wave mode conversion in the asymptotic

theory, we have to take the effects of the turning points. We do that by continuing the asymptotic solution through the turning points. This continuation process yields what is called connection formulas in the literature. Example of connection formulas for second order difference equations were given in chapter 2. Regarding turning points for high order difference equations, we do not discuss how to continue an asymptotic solution through turning points here, we just describe how to obtain the equation whose solutions are the turning points of a given high order difference equations. We also point out another way to obtain asymptotic solutions for difference equations. We show that difference equations are pseudo-differential operators. Asymptotics for pseudo-differential operators are discussed in chapter 4 of this thesis and also in Maslov & Nazaikinskii (1988) and in Maslov & Fedoriuk (1981).

A.1 Reduction of first order difference equations to a recurrence relation.

We consider a system of first order difference equations in the form

$$\begin{Bmatrix} x_1(j+1) \\ x_2(j+1) \\ \dots \\ x_n(j+1) \end{Bmatrix} = \begin{bmatrix} a_{1,1}(j) & a_{1,2}(j) & \dots & a_{1,n}(j) \\ a_{2,1}(j) & a_{2,2}(j) & \dots & a_{2,n}(j) \\ \dots & \dots & \dots & \dots \\ a_{n,1}(j) & a_{n,2}(j) & \dots & a_{n,n}(j) \end{bmatrix} \begin{Bmatrix} x_1(j) \\ x_2(j) \\ \dots \\ x_n(j) \end{Bmatrix} \quad (\text{A.1})$$

We describe the procedure to reduce the system of n difference equations (A.1) to a difference equation of n -th order. This is a recurrence algorithm with $n - 1$ steps. We describe the two first steps and the k -th step where $2 < k < n$. At the $n - 1$ -th step we obtain a difference equation of n -th order.

- First step. We assume that at least one $a_{1,l}(j) \neq 0$ for all j of interest. For $l = l_1$ our assumption is satisfied. We write the first line of the system of difference equations (A.1) in the form

$$x_{l_1}(j) = \frac{1}{a_{1,l_1}(j)} \left\{ x_1(j+1) - \sum_{m=1, m \neq l_1}^n a_{1,m}(j) x_m(j) \right\}. \quad (\text{A.2})$$

Next, we rewrite lines 2 to n , except the l_1 -th line, of the system of difference equations (A.1) (rewrite $x_n(j+1)$) using equation (A.2) for $x_{l_1}(j)$. We obtain

$$x_m(j+1) = f_n^1(j) x_1(j+1) + \sum_{q=1, q \neq l_1}^n g_{m,q}^1(j) x_q(j) \quad (\text{A.3})$$

where we define

$$f_n^1(j) = \frac{a_{n,l_1}(j)}{a_{1,l_1}(j)} \quad (\text{A.4})$$

$$g_{m,q}^1(j) = a_{m,q}(j) - \frac{a_{m,l_1}(j)}{a_{1,l_1}(j)} a_{1,q}(j) \quad (\text{A.5})$$

Now, we substitute equation (A.2) into the l_1 -th line of the system of difference equations (A.1) to obtain

$$\begin{aligned} x_1(j+2) - \sum_{m=1, m \neq l_1}^n a_{1,m}(j+1) x_m(j+1) &= \sum_{m=1, m \neq l_1}^n a_{l_1,m}(j) a_{1,l_1}(j+1) x_m(j) \\ &+ \frac{a_{1,l_1}(j+1) a_{l_1,l_1}(j)}{a_{1,l_1}(j)} \left\{ x_1(j+1) - \sum_{m=1, m \neq l_1}^n a_{1,m}(j) x_m(j) \right\} \end{aligned} \quad (\text{A.6})$$

We substitute equation (A.3) in the equation (A.6) to obtain the identity

$$x_1(j+2) + c_1^1(j) x_1(j+1) - \sum_{m=1, m \neq l_1}^n b_m^1(j) x_m(j) = 0 \quad (\text{A.7})$$

and we define

$$c_1^1(j) = - \sum_{m=1, m \neq l_1}^n a_{1,m}(j+1) f_m^1(j) - \frac{a_{1,l_1}(j+1) a_{l_1,l_1}(j)}{a_{1,l_1}(j)} \quad (\text{A.8})$$

$$b_m^1(j) = a_{1,l_1}(j+1) a_{l_1,n}(j) - \frac{a_{1,n}(j) a_{1,l_1}(j+1) a_{l_1,l_1}(j)}{a_{1,l_1}(j)} + \sum_{q=1, q \neq l_1}^n a_{1,q}(j) g_{q,m}^1(j) \quad (\text{A.9})$$

The first step is finalized.

- Second step. From equation (A.9), we chose l_2 such that $b_{l_2}^1(j) \neq 0$ for all j of interest. Now we write equation (A.9) in the form

$$x_{l_2}(j) = \frac{1}{b_{l_2}^1(j)} \left\{ x_1(j+1) + c_1^1(j+1) x_1(j+1) - \sum_{m=1, m \neq l_1, l_2} b_m^1(j) x_m(j) \right\} \quad (\text{A.10})$$

We update the equation for $x_{l_1}(j)$. We substitute equation (A.10) into equation (A.2) to obtain

$$x_{l_1}(j) = \frac{1}{a_{1,l_1}(j)} \left\{ \sum_{q=1}^2 e_q^{l_1, l_2}(j) x_1(j+q) - \sum_{m=1, m \neq l_1, l_2}^n d_m^{l_1, l_2}(j) x_m(j) \right\} \quad (\text{A.11})$$

and we define

$$e_2^{l_1, l_2} = - \frac{a_{1,l_2}(j)}{b_{l_2}^1(j)} \quad (\text{A.12})$$

$$e_1^{l_1, l_2} = 1 - \frac{a_{1,l_2}(j)}{b_{l_2}^1(j)} c_1^1(j) \quad (\text{A.13})$$

$$d_m^{l_1, l_2} = a_{1,m}(j) - a_{1,l_2}(j) \frac{b_m^1(j)}{b_{l_2}^1(j)} \quad (\text{A.14})$$

Next, we update the expressions for $x_m(j)$, $m \neq 1, l_1, l_2$ using equations (A.11) and (A.10). We obtain

$$x_m(j+1) = \sum_{q=1}^2 f_{m,q}^2 x_1(j+q) + \sum_{q=1, q \neq l_1, l_2}^n g_{m,q}^2 x_q(j), m \neq 1, l_1, l_2 \quad (\text{A.15})$$

and we define

$$f_{m,2}^2(j) = \frac{a_{m,l_1}(j)}{a_{1,l_1}(j)} e_2^{l_1, l_2} + \frac{a_{m,l_2}(j)}{b_{l_2}^1(j)} \quad (\text{A.16})$$

$$f_{m,1}^2(j) = \frac{a_{m,l_1}(j)}{a_{1,l_1}(j)} e_1^{l_1, l_2} + \frac{a_{m,l_2}(j)}{b_{l_2}^1(j)} c_1^1(j) \quad (\text{A.17})$$

$$g_{m,q}^2(j) = a_{m,q}(j) - \frac{a_{n,l_1}(j)}{a_{1,l_1}(j)} d_q^{l_1, l_2}(j) - \frac{a_{m,l_2}(j)}{b_{l_2}^1(j)} b_q^1(j) \quad (\text{A.18})$$

Next, we substitute equations (A.11) and (A.10) into the l_2 -th line of the system of difference equations (A.1) to obtain

$$\begin{aligned} x_1(j+3) + c_1^1(j+1)x_1(j+2) - \sum_{m=1, m \neq l_1, l_2}^n b_m^1(j+1)x_m(j+1) = \\ \sum_{m=1, m \neq l_1, l_2}^n a_{l_2,m}(j)b_{l_2}^1(j+1)x_m(j) + \frac{a_{l_2,l_2}(j)}{a_{1,l_1}(j)} \left\{ \sum_{q=1}^2 e_q^{l_1, l_2}(j)x_1(j+q) - \right. \\ \left. \sum_{m=1, m \neq l_1, l_2}^n d_m^{l_1, l_2}(j)x_m(j) \right\} + \frac{a_{l_2,l_2}(j)}{b_{l_2}^1(j)} \left\{ x_1(j+2) + c_1^1(j)x_1(j+1) - \sum_{m=1, m \neq l_1, l_2}^n b_m^1(j)x_m(j) \right\} \end{aligned} \quad (\text{A.19})$$

Next, we substitute equation (A.15) into the equation (A.19) to obtain the equality

$$x_1(j+3) + \sum_{q=1}^2 c_q^2(j)x_1(j+q) - \sum_{m=1, m \neq l_1, l_2}^n b_m^2(j)x_m(j) = 0 \quad (\text{A.20})$$

and we define

$$c_2^2(j) = c_1^1(j+1) - \sum_{m=1, m \neq l_1, l_2}^n b_m^1(j+1) f_{m,2}^2(j) - \frac{a_{l_2, l_1}(j)}{a_{1, l_1}(j)} e_2^{l_1, l_2}(j) - \frac{a_{l_2, l_2}(j)}{b_{l_2}^1(j)} \quad (\text{A.21})$$

$$c_1^2(j) = - \sum_{m=1, m \neq l_1, l_2}^n b_m^1(j+1) f_{m,1}^2(j) - \frac{a_{l_2, l_2}(j)}{a_{1, l_1}(j)} e_1^{l_1, l_2}(j) - \frac{a_{l_2, l_2}(j)}{b_{l_2}^1(j)} c_1^1(j) \quad (\text{A.22})$$

$$b_m^2(j) = b_m^1(j+1) - a_{l_2, m}(j) b_{l_2}^1(j+1) + \frac{a_{l_2, l_1}(j)}{a_{1, l_1}(j)} d_m^{l_1, l_2}(j) + \frac{a_{l_2, l_2}(j)}{b_{l_2}^1(j)} b_n^1(j) \quad (\text{A.23})$$

and the second step is finished.

- Third step. Now we chose l_3 such that the element $b_{l_3}^2(j)$ in equation (A.20) is non-zero for the interval of j of interest. After this choice is made, we have

$$x_{l_3}(j) = \frac{1}{b_{l_3}^2(j)} \left\{ x_1(j+3) + \sum_{q=1}^2 c_q^2(j) x_1(j+q) - \sum_{m=1, m \neq l_1, l_2, l_3}^n b_m^2(j) x_m(j) \right\} \quad (\text{A.24})$$

Next, we update the expressions for $x_{l_1}(j)$ (equation (A.11)) and $x_{l_2}(j)$ (equation (A.10)) using equation (A.24). We obtain

$$x_{l_1}(j) = \frac{1}{a_{1, l_1}(j)} \left\{ \sum_{q=1}^3 e_q^{l_1, l_3}(j) x_1(j+q) - \sum_{m=1, m \neq l_1, l_2, l_3}^n d_m^{l_1, l_3}(j) x_m(j) \right\} \quad (\text{A.25})$$

$$x_{l_2}(j) = \frac{1}{b_{1, l_2}(j)} \left\{ \sum_{q=1}^3 e_q^{l_2, l_3}(j) x_1(j+q) - \sum_{m=1, m \neq l_1, l_2, l_3}^n d_m^{l_2, l_3}(j) x_m(j) \right\} \quad (\text{A.26})$$

and we define

$$e_3^{l_1, l_3}(j) = -\frac{d_{l_3}^{l_1, l_2}(j)}{b_{l_3}^2(j)} \quad (\text{A.27})$$

$$e_2^{l_1, l_3}(j) = e_2^{l_1, l_2}(j) - \frac{d_{l_3}^{l_1, l_2}(j)}{b_{l_3}^2(j)} c_2^2(j) \quad (\text{A.28})$$

$$e_1^{l_1, l_3}(j) = e_1^{l_1, l_2}(j) - \frac{d_{l_3}^{l_1, l_2}(j)}{b_{l_3}^2(j)} c_1^2(j) \quad (\text{A.29})$$

$$e_3^{l_2, l_3}(j) = -\frac{b_{l_3}^1(j)}{b_{l_3}^2(j)} \quad (\text{A.30})$$

$$e_2^{l_2, l_3}(j) = 1 - \frac{b_{l_3}^1(j)}{b_{l_3}^2(j)} c_2^2(j) \quad (\text{A.31})$$

$$e_1^{l_2, l_3}(j) = c_1^1(j) - \frac{b_{l_3}^1(j)}{b_{l_3}^2(j)} c_1^1(j) \quad (\text{A.32})$$

$$d_m^{l_1, l_3}(j) = d_m^{l_2, l_3} - \frac{d_{l_3}^{l_1, l_2}(j)}{b_{l_3}^2(j)} b_m^2(j) \quad (\text{A.33})$$

$$d_m^{l_2, l_3}(j) = b_m^1 - \frac{b_{l_3}^1(j)}{b_{l_3}^2(j)} b_m^2(j) \quad (\text{A.34})$$

Next, we update the expression for $x_m(j+1)$, $m \neq l_1, l_2, l_3$. We substitute the expressions for $x_{l_1}(j)$, $x_{l_2}(j)$ and $x_{l_3}(j)$ (respectively, equations (A.25), (A.26) and (A.24)) into equation (A.15). We obtain

$$x_m(j+1) = \sum_{q=1}^3 f_{m,q}^3(j) x_1(j+q) + \sum_{q=1, q \neq l_1, l_2, l_3}^n g_{m,q}^3(j) x_q(j) \quad (\text{A.35})$$

and we define

$$f_{m,3}^3(j) = \frac{a_{m,l_1}(j)}{a_{1,l_1}(j)} e_3^{l_1, l_3}(j) + \frac{a_{m,l_2}(j)}{b_{l_2}^1} e_3^{l_2, l_3}(j) + \frac{a_{m,l_3}(j)}{b_{l_3}^2(j)} \quad (\text{A.36})$$

$$f_{m,q}^3(j) = \frac{a_{m,l_1}(j)}{a_{1,l_1}(j)} e_q^{l_1, l_3}(j) + \frac{a_{m,l_2}(j)}{b_{l_2}^1(j)} e_q^{l_2, l_3}(j) + \frac{a_{m,l_3}(j)}{b_{l_3}^2(j)} c_q^2(j) \quad (\text{A.37})$$

$$g_{m,q}^3(j) = a_{m,q}(j) - \frac{a_{m,l_1}(j)}{a_{1,l_1}(j)} d_q^{l_1, l_3}(j) - \frac{a_{m,l_2}(j)}{b_{l_2}^1(j)} d_q^{l_2, l_3}(j) - \frac{a_{m,l_3}(j)}{b_{l_3}^2(j)} b_q^2(j) \quad (\text{A.38})$$

We substitute equation (A.24) into the l_3 -th equation of the system of difference

equations (A.1). We also use equations (A.25) and (A.26). The final result is

$$\begin{aligned}
& x_1(j+4) + \sum_{q=1}^2 c_q^2(j+1)x_1(j+q+1) - \sum_{m=1, m \neq l_1, l_2, l_3}^n b_m^2(j+1)x_m(j+1) = \\
& b_{l_3}^2(j+1) \left\{ \sum_{m=1, m \neq l_1, l_2, l_3}^n a_{l_3, m}(j)x_m(j) + \sum_{q=1}^3 \frac{a_{l_3, l_q}(j)}{b_{l_q}^{q-1}(j)} \left[\sum_{m=1}^3 e_m^{l_q, l_3}(j)x_1(j+m) - \right. \right. \\
& \left. \left. \sum_{m=1, m \neq l_1, l_2, l_3}^n d_m^{l_q, l_3}(j)x_m(j) \right] \right\}
\end{aligned} \tag{A.39}$$

and we define

$$e_3^{l_3, l_3}(j) = 1 \tag{A.40}$$

$$e_m^{l_3, l_3}(j) = c_m^2(j) \tag{A.41}$$

$$d_m^{l_3, l_3}(j) = b_m^2(j) \tag{A.42}$$

$$b_{l_1}^0(j) = a_{1, l_1}(j) \tag{A.43}$$

We substitute equation (A.35) into the equation (A.39) to obtain the equality

$$x_1(j+4) + \sum_{q=1}^3 c_q^3(j)x_1(j+q) - \sum_{m=1, m \neq l_1, l_2, l_3}^n b_m^3(j)x_m(j) = 0 \tag{A.44}$$

and we define

$$c_m^3(j) = c_{m-1}^2(j+1) - \sum_{q=1, q \neq l_1, l_2, l_3}^n b_q^2(j+1) f_{q,m}^3(j) + \sum_{q=1}^3 \frac{a_{l_3, l_q}(j)}{b_{l_q}^{q-1}(j)} e_m^{l_q, l_3}(j) \quad (\text{A.45})$$

$$b_m^3(j) = b_{l_3}^2(j+1) a_{l_3, n}(j) - \sum_{q=1}^3 \frac{a_{l_3, l_q}(j)}{b_{l_q}^{q-1}(j)} d_q^{l_q, l_3}(j) + \sum_{q=1, q \neq l_1, l_2, l_3}^n b_q^2(j+1) g_{q,m}^3(j) \quad (\text{A.46})$$

- The k -th step. Now we generalize the third step. We consider $3 < k < n-1$. As in the previous step, at the equation that is the end result of the $k-1$ -th step, we chose a l_k such that $b_{l_k}^{k-1}(j) \neq 0$ for the interval of j we are interested in. Now we write

$$x_{l_k}(j) = \frac{1}{b_{l_k}^{k-1}} \left\{ x_1(j+k) + \sum_{q=1}^{k-1} c_q^{k-1}(j) x_1(j+q) - \sum_{m=1, m \neq l_1, l_2, \dots, l_k}^n b_m^{k-1}(j) x_m(j) \right\} \quad (\text{A.47})$$

We rewrite $x_{l_1}(j)$ to $x_{l_{k-1}}(j)$ using equation (A.47). The result is

$$x_{l_q}(j) = \frac{1}{b_{l_q}^{q-1}(j)} \left\{ \sum_{m=1}^k e_m^{l_q, l_k}(j) x_1(j+m) - \sum_{m=1, m \neq l_1, l_2, \dots, l_k}^n d_m^{l_q, l_k}(j) x_m(j) \right\} \text{ with } q = 1, \dots, k-1 \quad (\text{A.48})$$

and we define

$$e_k^{l_q, l_k}(j) = -\frac{d_{l_k}^{l_1, l_{k-1}}(j)}{b_{l_k}^{k-1}(j)} \text{ with } q = 1, \dots, k-2 \quad (\text{A.49})$$

$$e_k^{l_{k-1}, l_k}(j) = -\frac{b_{l_k}^{k-2}(j)}{b_{l_k}^{k-1}(j)} \quad (\text{A.50})$$

$$e_m^{l_q, l_k}(j) = e_m^{l_q, l_{k-1}}(j) - \frac{d_{l_k}^{l_q, l_{k-1}}(j)b_m^{k-1}(j)}{b_{l_k}^{k-1}(j)} \text{ with } m = 1, \dots, k-2 \quad (\text{A.51})$$

$$d_m^{l_q, l_k}(j) = d_m^{l_q, l_{k-1}}(j) - \frac{d_{l_k}^{l_q, l_{k-1}}(j)b_m^{k-1}(j)}{b_{l_k}^{k-1}(j)} \text{ with } m = 1, \dots, k-2 \quad (\text{A.52})$$

$$e_{k-1}^{l_{k-1}, l_k}(j) = 1 - \frac{b_{l_k}^{k-2}(j)}{b_{l_k}^{k-1}(j)} c_{k-1}^{k-1}(j) \quad (\text{A.53})$$

$$e_m^{l_{k-1}, l_k}(j) = c_m^{k-2}(j) - \frac{b_{l_k}^{k-2}(j)}{b_{l_k}^{k-1}(j)} c_m^{k-1}(j) \quad (\text{A.54})$$

$$d_m^{l_{k-1}, l_k}(j) = b_m^{k-2}(j) - \frac{b_{l_k}^{k-2}(j)}{b_{l_k}^{k-1}(j)} b_m^{k-1}(j) \quad (\text{A.55})$$

$$b_{l_1}^0(j) = a_{1, l_1}(j) \quad (\text{A.56})$$

Next, we update the expression for $x_m(j+1)$ given by the system of difference equations (A.1) using equation (A.48). We obtain

$$x_m(j+1) = \sum_{q=1}^k f_{m,q}^k(j) x_1(j+q) + \sum_{q=1, q \neq l_1, l_2, \dots, l_k}^n g_{m,q}^k(j) x_q(j) \text{ with } q \neq l_1, \dots, l_k \quad (\text{A.57})$$

and we define

$$f_{m,k}^k(j) = \frac{a_{m, l_k}(j)}{b_{l_k}^{k-1}(j)} + \sum_{q=1}^{k-1} \frac{a_{m, l_q}(j)}{b_{l_q}^{q-1}(j)} e_k^{l_q, l_k}(j) \quad (\text{A.58})$$

$$f_{m,q}^k(j) = \frac{a_{m, l_q}(j)}{b_{l_k}^{k-1}(j)} c_q^{k-1}(j) + \sum_{s=1}^{k-1} \frac{a_{m, l_s}(j)}{b_{l_s}^{s-1}(j)} e_q^{l_s, l_k}(j) \quad (\text{A.59})$$

$$g_{m,q}^k(j) = a_{m,q}(j) - \frac{a_{m, l_k}(j)}{b_{l_k}^{k-1}(j)} b_q^{k-1}(j) - \sum_{s=1}^{k-1} \frac{a_{m, l_s}(j)}{b_{l_s}^{s-1}(j)} d_q^{l_s, l_k}(j) \quad (\text{A.60})$$

Next, we rewrite the l_k -th line of the system of difference equations (A.1) using equations (A.57) and (A.48). The result of these substitutions is

$$x_1(j+k+1) + \sum_{q=1}^k c_q^k(j) x_1(j+q) - \sum_{m=1, m \neq l_1, \dots, l_k}^n b_m^k(j) x_m(j) = 0 \quad (\text{A.61})$$

and we define

$$c_q^k(j) = c_{q-1}^{k-1}(j+1) - \sum_{s=1, s \neq l_1, \dots, l_k}^n b_s^{k-1}(j) f_{s,q}^k(j) + \sum_{s=1}^k \frac{b_{l_k}^{k-1}(j+1) a_{l_k, l_s}(j)}{b_{l_s}^{s-1}(j)} e_m^{l_s, l_k}(j) \quad (\text{A.62})$$

$$b_m^k(j) = b_{l_k}^{k-1}(j+1) a_{l_k, m}(j) + \sum_{q=1, q \neq l_1, \dots, l_k}^n b_q^{k-1}(j+1) g_{q,m}^k(j) - \sum_{q=1}^k \frac{b_{l_k}^{k-1}(j+1) a_{l_k, l_q}(j)}{b_{l_q}^{q-1}(j)} d_m^{l_q, l_k}(j) \quad (\text{A.63})$$

$$e_k^{l_k, l_k}(j) = 1 \quad (\text{A.64})$$

$$e_m^{l_k, l_k}(j) = c_m^{k-1}(j) \text{ with } m = 1, \dots, k-1 \quad (\text{A.65})$$

$$b_{l_1}^0(j) = a_{1, l_1}(j) \quad (\text{A.66})$$

$$d_m^{l_k, l_k}(j) = b_m^{k-1}(j) \quad (\text{A.67})$$

This finalizes the k -th step. If $k = n-1$, equation (A.61) can be written as

$$x_1(j+n) + \sum_{q=1}^{n-1} c_q^{n-1}(j) x_1(j+q) - b_1^{n-1}(j) x_1(j) = 0. \quad (\text{A.68})$$

This is the desired n -th order difference equation. Suppose we have the solution for the difference equation (A.68). We would like to obtain the solution for the system of difference equations (A.1) from the solution of the difference equation (A.68). This can be accomplished by taking a look at the final $n-1$ -th step of the procedure described above. At the beginning of the $n-1$ -th step we write the last equation in the $n-2$ -th step in the form

$$x_{l_{n-1}}(j) = \frac{1}{b_{l_{n-1}}^{n-2}(j)} \left\{ x_1(j+n-1) + \sum_{m=1}^{n-2} c_m^{n-2}(j) x_1(j+m) - b_1^{n-2}(j) x_1(j) \right\} \quad (\text{A.69})$$

Notice that this equation is a function of $x_1(j+q)$, $q = 0, 1, \dots, n-1$. So, if the solution of the difference equation (A.68) is known, we can obtain, through equation (A.69), the solution for the l_{n-1} -th dependent variable in the system of difference equations (A.1). To obtain the solution of the other dependent variables, we need to write $x_{l_1}(j), \dots, x_{l_{n-3}}(j)$ and $x_{l_{n-2}}(j)$ in terms of $x_1(j)$. To accomplish that, we use equation (A.69) to update the equations for $x_{l_1}(j), \dots, x_{l_{n-2}}(j)$ in the $n-1$ -th step. This gives

$$x_{l_q}(j) = \frac{1}{b_{l_q}^{q-1}(j)} \left\{ \sum_{m=1}^{n-1} e_m^{l_q, l_{n-1}}(j) x_1(j+m) - d_1^{l_q, l_{n-1}}(j) x_1(j) \right\} \quad \text{with } q = 1, \dots, n-2 \quad (\text{A.70})$$

where the expression for the coefficients that appear in this equation is given in the k -step of the procedure to reduce a n -dimensional system of first order difference equations to a n -th order difference equation. Equation (A.70) gives the dependent variable $x_{l_q}(j)$, $q = 1, \dots, n-1$ in terms of the dependent variable $x_1(j)$, solution of the high order difference equation (A.69).

A.2 High order Liouville-Green functions for a high order difference equation.

The objective of this section is to obtain asymptotic solutions (Liouville-Green functions) of a high order difference equation. First, we write our n -th order difference equation (A.69) in the form

$$x(j+n) + \sum_{m=0}^{n-1} c_m(j+m)x(j+m) = 0 \quad (\text{A.71})$$

Second, we consider a change of the dependent variable. From the new form of the equation (A.71) we obtain an iterative scheme to obtain the high order approximations for the new dependent variable. The first approximation for the new dependent variable are the roots of the polynomial obtained from the difference equation (A.71) by making the substitution $x(j+n) = z^n$. We consider the change of the dependent variable

$$x(j) = \prod_{l=j_0}^j u(l) \quad (\text{A.72})$$

Now we substitute the change of variable (A.72) into the equation (A.71). The result is

$$u(j+n) \dots u(j+1) + \sum_{m=1}^{n-1} c_m(j) \prod_{l=1}^m u(j+l) + c_0(j) = 0 \quad (\text{A.73})$$

We write this equation in a more appropriate form, which is

$$\begin{aligned} & \frac{u(j+n)}{u(j+n-1)} \left(\frac{u(j+n-1)}{u(j+n-2)} \right)^2 \left(\frac{u(j+n-2)}{u(j+n-3)} \right)^3 \dots \left(\frac{u(j+2)}{u(j+1)} \right)^{n-1} u(j+1)^n + \\ & \sum_{m=1}^{n-1} c_m(j) \frac{u(j+m)}{u(j+m-1)} \left(\frac{u(j+m-1)}{u(j+m-2)} \right)^2 \dots \left(\frac{u(j+2)}{u(j+1)} \right)^{m-1} u(j+1)^m + c_0(j) = 0 \end{aligned} \quad (\text{A.74})$$

Each product in the equation (A.74) can be written as the exponential of a sum logarithms of the functions $u(j+m)$. So, we have

$$\exp \left(\Delta \ln u(j+n-1) + 2\Delta \ln u(j+n-2) + \sum_{m=1}^{n-3} m\Delta \ln u(j+m) \right) u(j+1)^n + \sum_{m=1}^{n-1} c_m(j) \exp \left(\sum_{k=1}^{m-1} k\Delta \ln u(j+m-k) \right) u(j+1)^m + c_0(j) = 0 \quad (\text{A.75})$$

We expand the exponentials in the equation (A.75). This results in the equation

$$\left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{m=1}^{n-1} m\Delta \ln u(j+n-m) \right)^k \right\} u(j+1)^n + \sum_{l=1}^{n-1} c_l(j) \left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{m=1}^{l-1} m\Delta \ln u(j+l-m) \right)^k \right\} u(j+1)^l + c_0(j) = 0 \quad (\text{A.76})$$

We assume that the coefficients of the difference equation (A.71) varies slowly with respect to the index variable j such that we have

$$\left| \frac{u(j+m)}{u(j+m-1)} \right| \sim 1 \text{ for } m = 1, \dots, n \Rightarrow |\Delta \ln u(j+m-1)| \ll 1 \text{ for } m = 1. \quad (\text{A.77})$$

So, we can solve equation (A.76) in an iterative way. The 0-th order approximation for $u(j+1)$, denoted $u^{(0)}(j+1)$, is just the solution of the polynomial

$$u(j+1)^n + \sum_{m=1}^{n-1} c_m(j) u(j+1)^m + c_0(j) = 0 \quad (\text{A.78})$$

For each j , the polynomial equation (A.78) in $u(j+1)$ has n roots, denoted $u_l^{(0)}(j+1)$, $l = 1, \dots, n$. High order approximations can be obtained from equation (A.76) by iteration. The first order approximation $u_l^{(1)}(j+1)$ is the solution of the first iteration

of equation (A.76), given by

$$\begin{aligned} & \left\{ 1 + \left(\sum_{m=1}^{n-1} m \Delta \ln u_l^{(0)}(j+n-m) \right) \right\} u(j+1)^n + \\ & \sum_{q=1}^{n-1} c_q(j) \left\{ 1 + \left(\sum_{m=1}^{q-1} m \Delta \ln u_l^{(0)}(j+q-m) \right) \right\} u(j+1)^q + c_0(j) = 0 \end{aligned} \quad (\text{A.79})$$

This is a polynomial for $u(j+1)$ since $u_l^{(0)}(j+m)$, $m = 1, \dots, n$ is known. This polynomial has n roots. As the first order approximation for $u_l(j+1)$, we chose the root of the polynomial equation (A.79) which is closer to $u_l^{(0)}(j+1)$. Here we assume that the roots of the zero-th order approximation are not to close, so we can trace them as the independent variable j varies or as we iterate equation (A.76) to obtain higher order approximations for solutions $u_l(j+1)$ of the difference equation (A.73). If (A.71) is a difference equation of fourth order or lower order, analytic solutions for the roots of the polynomial equations (A.78) and (A.79) are known. In this case, we have analytic approximations for the solutions of the difference equation (A.71). For higher orders, the roots of the polynomial equations (A.78) and (A.79) have to be obtained numerically. We can generalize equation (A.79). The polynomial equation given the k -th order approximation for $u_l(j+1)$ is

$$\begin{aligned} & \left\{ 1 + \sum_{q=1}^k \frac{1}{q!} \left(\sum_{m=1}^{n-1} m \Delta \ln u_l^{(k-q)}(j+n-m) \right)^q \right\} u(j+1)^n + \\ & \sum_{s=1}^{n-1} c_s(j) \left\{ 1 + \sum_{q=1}^k \frac{1}{q!} \left(\sum_{m=1}^{l-1} m \Delta \ln u_l^{(k-q)}(j+l-m) \right)^q \right\} u(j+1)^s + c_0(j) = 0 \end{aligned} \quad (\text{A.80})$$

From the k -th order approximation for $u_l(j+1)$ we can obtain the k -th approximation for the solutions of the difference equation (A.71) using the change of variable (A.72). The approximate solution for $x(j)$ is a linear combination of

$$\prod_{q=j_0}^j u_l^{(k)}(q); \quad l = 1, \dots, n. \quad (\text{A.81})$$

In other words, the k -th order approximation $x(j)$ has the form

$$x(j) = \sum_{l=1}^n A_l \exp \left(\sum_{q=j_0}^j \ln u_l^{(k)}(q) \right) \quad (\text{A.82})$$

where the constants A_l are determined by imposing initial or boundary conditions.

A.3 Turning Points.

Here we discuss how to obtain the turning points of a high order difference equation. From the previous section, the zero-th order approximations $u_l^{(0)}$ were roots of the polynomial equation (A.78). If two or more of the roots of the polynomial equation (A.78) coalesce, we lose one or more of the zero-th order approximations $u_l^{(0)}(j+1)$. Therefore, at the values of index variable j where we have coalescence of the roots of the polynomial equation (A.8), the asymptotic solution built in the previous section is not valid. Let us substitute $u(j+1)$ in equation (A.78) by the variable z . We obtain the polynomial

$$p(z, j) = \sum_{m=0}^n c_{n-m}(j) z^{n-m} \quad (\text{A.83})$$

where $c_n(j) = 1$. Coalescence of two roots of the polynomial equation $p(z, j) = 0$ happens at the solutions of the polynomial equations

$$p(z, j) = 0 \quad \text{and} \quad \frac{\partial p}{\partial z}(z, j) = 0, \quad (\text{A.84})$$

which have as solutions the value of the variable z for which the roots coalesce and points in the complex j plane where the two roots coalesce. The polynomial equation $p(z, j) = 0$ can be seen as a mapping from the complex j plane into the complex z plane. This mapping may have branch points in the complex j plane and their images, called saddle points, in the complex z plane. The solutions of equations (A.84) are the branch points of $p(z, j) = 0$ in the j complex plane and the respective saddle points in the complex z plane. As the index variable j approaches a branch point (which are the desired turning point), two or more roots of the polynomial equation (A.83) approach a saddle point. The coalescence of a larger number of roots, let say, coalescence of k roots happens at the solutions of the system of equations

$$p(z, j) = 0 \quad \text{and} \quad \frac{\partial^m p}{\partial z^m}(z, j) = 0 \quad \text{for } m = 1, \dots, k \quad (\text{A.85})$$

The solutions of the set of equations (A.85) are also solutions of the set of equations (A.84), which is the condition for coalescence of two roots. So, to obtain the turning points of the difference equation we basically need to solve only the system of equations (A.84).

Two polynomials $p(z, j)$ and $q(z, j)$ are relatively prime polynomials when they have no common factors except constants. This is the case between $p(z, j)$ and $\frac{\partial p}{\partial z}(z, j)$. From the theory of algebraic functions (see Alfors (1978)) , we know that if polynomials $p(z, j)$ and $q(z, j)$ are relatively prime polynomials, there is only a finite number of values j_0 for which the equations $p(z, j) = 0$ and $q(z, j) = 0$ have a common root. This fact implies that we have a finite number of turning points for the difference equation (A.71). From equations (A.84) we can obtain a single equation whose solutions are the turning points. This equation can be obtained by the following division algorithm.

$$\begin{aligned}
p(z, j) &= q_0(z, j) \frac{\partial p}{\partial z}(z, j) + R_1(z, j) \\
\frac{\partial p}{\partial z}(z, j) &= q_1(z, j) R_1(z, j) + R_2(z, j) \\
R_1(z, j) &= q_2(z, j) R_2(z, j) + R_3(z, j) \\
&\dots\dots\dots \\
R_{n-3}(z, j) &= q_{n-2}(j) R_{n-2}(z, j) + R_{n-1}(j)
\end{aligned}$$

where n is the degree of the polynomial $p(z, j)$ and the desired equation is the term $R_{n-1}(j)$ obtained at the end of the division algorithm. This term is called the resultant from the polynomials $p(z, j)$ and $\frac{\partial p}{\partial z}(z, j)$. Next, we apply this algorithm to the n -th order polynomial

$$p(z, j) = \sum_{m=0}^n c_{n-m}(j) z^{n-m} \quad (\text{A.86})$$

and its derivative $\frac{\partial p}{\partial z}(z, j)$. The first step of the algorithm gives

$$R_1(z, j) = \sum_{m=2}^n \left\{ \frac{m}{n} c_{n-m}(j) - \frac{(n-m+1)}{n^2} \frac{a_{n-1}(j) a_{n-m+1}(j)}{a_n(j)} \right\} z^{n-m} \quad (\text{A.87})$$

Now we define

$$b_{n-m}^1(j) = \left\{ \frac{m}{n} c_{n-m}(j) - \frac{(n-m+1)}{n^2} \frac{a_{n-1}(j) a_{n-m+1}(j)}{a_n(j)} \right\}. \quad (\text{A.88})$$

The second step of the division algorithm gives

$$R_2(z, j) = \sum_{m=3}^n b_{n-m}^2 z^{n-m}, \quad (\text{A.89})$$

and we define

$$b_{n-m}^2(j) = (n-m+1)c_{n-m+1}(j) - \frac{na_n(j)b_{n-m}^1(j)}{b_{n-2}^1(j)} - \frac{b_{n-m}^1(j)}{b_{n-2}^1(j)} \left[(n-1)a_{n-1}(j) - \frac{na_n(j)b_{n-3}^1(j)}{b_{n-2}^1(j)} \right]. \quad (\text{A.90})$$

The third step of the division algorithm gives

$$R_3(z, j) = \sum_{m=4}^n b_{n-m}^3(j) z^{n-m}, \quad (\text{A.91})$$

and we define

$$b_{n-m}^3(j) = b_{n-m}^1(j) - \frac{b_{n-2}^1(j)}{b_{n-3}^2(j)} b_{n-m}^2(j) - \frac{b_{n-m}^2(j)}{b_{n-3}^2(j)} \left(b_{n-3}^1(j) - \frac{b_{n-2}^1(j)b_{n-4}^1(j)}{b_{n-3}^2(j)} \right). \quad (\text{A.92})$$

Now we can generalize. The k -th ($3 < k \leq n-2$) step of the division algorithm gives

$$R_k(z, j) = \sum_{m=k+1}^n b_{n-m}^k(j) z^{n-m}, \quad (\text{A.93})$$

and we define

$$b_{n-m}^k(j) = b_{n-m}^{k-2}(j) - \frac{b_{n-k+1}^{k-2}(j)}{b_{n-k}^{k-1}(j)} b_{n-m-1}^{k-1}(j) - \frac{b_{n-m}^{k-1}(j)}{b_{n-k}^{k-1}(j)} \left(b_{n-k}^{k-2}(j) - \frac{b_{n-k+1}^{k-2}(j)}{b_{n-k}^{k-2}(j)} b_{n-k-1}^{k-1}(j) \right) \quad (\text{A.94})$$

The last step (the $n-1$ -th step) gives the resultant between the polynomials $p(z, j)$ and $\frac{\partial p}{\partial z}(z, j)$, which has the expression

$$R_{n-1}(j) = b_0^{n-2}(j) - \frac{b_0^{n-1}(j)}{b_1^{n-1}(j)} \left(b_0^{n-2}(j) - \frac{b_2^{n-2}(j)}{b_1^{n-1}(j)} b_0^{n-1}(j) \right) \quad (\text{A.95})$$

Equation (A.95) is only a function of the index variable j , and from it we can obtain the turning points of the difference equation (A.71).

A.4 Difference Equations as Pseudo Differential Equations.

Differential equations can be written as pseudo-differential equations, according to Maslov & Nazaikinskii (1988). For example, we can write the second order difference equation (2.1) as a pseudo-differential operator. To start, we expand the second order difference operator in equation (2.1) in Taylor series.

$$\begin{aligned}
z_{j+1} - 2z_j + z_{j-1} &= \sum_{m=0}^{\infty} \left(\frac{1}{m!} \frac{d^m z}{dj^m}(j) \right) - 2z_j + \sum_{m=0}^{\infty} \left((-1)^m \frac{1}{m!} \frac{d^m z}{dj^m}(j) \right) \\
&= \sum_{m=1}^{\infty} \frac{2}{(2m)!} \frac{d^{2m} z}{dj^{2m}}(j) \\
&= 2 \cosh\left(\frac{d}{dj}\right) z(j) - 2z(j)
\end{aligned} \tag{A.96}$$

This implies that the second order difference equation (A.96) is given as the pseudo-differential operator

$$\cosh\left(\frac{d}{dj}\right) z(j) - \left(\frac{1}{2}Q(j) - 1\right) z(j) = 0 \tag{A.97}$$

High order difference equations can also be written as pseudo-differential equations. This simplifies the asymptotic analysis for difference equations, since the asymptotic methods for differential equations can be applied to pseudo-differential equations. In chapter 4 we apply the concept of pseudo-differential operators to develop an asymptotic theory for water waves interacting with a non-uniform bottom topography.

Appendix B

PROCEDURE TO OBTAIN THE TURNING POINTS OF Q_j .

For a given sequence Q_j , we describe a procedure on how to obtain the turning points in the complex j plane. We consider real and complex first order turning points and second order turning points, which result from the coalescence of a pair of real or complex first order turning points. We impose the restrictions on the sequence Q_j given in section 3.2. The procedure to obtain the turning points follows.

- We assume Q_j to be a bounded sequence that oscillates around a fixed value Q_0 , so we can write

$$Q_j = Q_0 + \chi_j, \text{ for } j = j_0 \text{ to } j_0 + M + 1 \quad (\text{B.1})$$

with χ_j given by some functional form for complex values of j . If χ_j is known only at the integer values of j from j_0 to $j_0 + M + 1$, we can represent it by a discrete Fourier series

$$\chi_j = \sum_{l=1}^M \hat{\chi}_l \sin \left[\frac{(j - j_0) l \pi}{M + 1} \right], \text{ for } j = j_0 \text{ to } j_0 + M + 1 \quad (\text{B.2})$$

where $\hat{\chi}_l$ is the discrete Fourier transform of the sequence χ_j . This is the natural way to extend χ_j to the whole complex j plane.

- Maxima and minima of the extension of sequence Q_j to the complex j plane for real j . These points are solutions of the equation

$$\frac{dQ_j}{dj} = 0, \text{ for } j \text{ Real} \quad (\text{B.3})$$

Let us label the solutions of (B.3) as j^* . We have to check if the points j^* are maxima or minima. With this point in mind we need to evaluate $\frac{d^2Q(j^*)}{dj^2}$. If we obtain that

$$\begin{aligned} \frac{d^2Q(j^*)}{dj^2} &> 0 \rightarrow j^* \text{ is a minimum} \\ \frac{d^2Q(j^*)}{dj^2} &< 0 \rightarrow j^* \text{ is a maximum} \end{aligned}$$

Let us call the points j that are maxima as j^{*+} and those that are minima as j^{*-} .

- Search for the real turning points first. To see if there is any real turning point, we just need to check if any of the maxima or minima j^* satisfies

$$Q(j^{*-}) \leq 0 \quad (\text{B.4})$$

or

$$Q(j^{*+}) \geq 4 \quad (\text{B.5})$$

If the inequality (B.4) is satisfied for some j^{*-} , we have real turning points that satisfy the first turning point condition. If the inequality (B.5) is satisfied for some j^{*+} , we have real turning points that satisfy the second turning point condition. These turning points are solutions of

$$Q_0 - 2(1 + (-1)^k) + \chi_j = 0, \text{ for } j_0 \leq j \leq j_0 + M + 1 \quad (\text{B.6})$$

We have that $k=1$ refers to the first turning point condition, and $k=2$ refers to the second turning point condition. If at each of the maxima j^{*+} or minima j^{*-}

one of the inequalities (B.4) or (B.5) is satisfied, then all the turning points are real. If for some $j^{*\pm}$ an equality of the type (B.4) or (B.5) is verified, instead of inequality, we have a real second order turning point which coincides with this particular $j^{*\pm}$. If for some $j^{*\pm}$, none of the inequalities (B.4) or (B.5) is verified, then we have a pair of complex conjugate turning points.

- Search for complex turning points. All the points of the form $j^{*\pm}$ that do not satisfy (B.4) or (B.5) have a pair of complex conjugate turning points associated with it, and we use this fact to obtain the pairs of complex conjugate turning points. Since we are dealing with complex turning points, let us write $j = x + i y$. A complex turning point satisfies the system of equations

$$Q_0 - 2(1 + (-1)^k) + \Re\{\chi_l\} = 0, \text{ with } k=1 \text{ and } k=2 \quad (\text{B.7})$$

$$\Im\{\chi_l\} = 0 \quad (\text{B.8})$$

To obtain the turning points solutions of the systems of equations (B.7)-(B.8), we consider the maximum (minimum) $j^{*+}(j^{*-})$ associated with the turning point we want to obtain. We use the maximum (minimum) $j^{*+}(j^{*-})$ as a starting value and follow the path described implicitly by (B.8) for $y > 0$, until (B.7) is satisfied. This gives one of the desired complex turning point related to the maximum (minimum) $j^{*+}(j^{*-})$. The other desired turning point is just the complex conjugate of the previous turning point. Complex turning points come as pairs of complex conjugate numbers in the j plane.

The procedure described above can be implemented in a code to find all turning points.

Appendix C

CONNECTION FORMULAS.

In this appendix we discuss in more detail the connection formulas for the considered turning point problems, which are:

- A pair of complex first order turning points.
- A cluster of two almost coalescing real and complex first order turning points.

C.1 CONNECTION FORMULAE FOR A PAIR OF COMPLEX FIRST ORDER TURNING POINTS.

In this section we derive the connection formulae for TPP1. The solution of the approximate form of equation (2.1), valid in a neighborhood of a first order real or complex turning point, is given just below. The leading term of its asymptotic expansion for arbitrary ν and $|\gamma| \gg 1$ with $-\pi < \arg\{\gamma\} < \pi$ is given in the next section. The results given in the next section can be applied directly for the case $k = 1$. The solutions $H_j(\gamma, \nu)$ for the case $k=1$ are defined in terms of contour integrals, as follows.

$$H_j(\gamma, \nu) = \int_{C_j} \exp(-2\gamma[-z + \sin z] + i\nu z) dz, \quad \text{for } j = 1, 2, 3 \quad (\text{C.1})$$

where $\nu = (j - j_{*k})$. The contours C_j are given in terms of the $\arg\{\gamma\}$, as follows.

- For $\arg(\gamma) \in \text{I quadrant}$ we have C_1 and C_3 asymptotic to $-\pi^- + i\infty$, C_1 and C_2 asymptotic to $0^- - i\infty$ and C_2 and C_3 asymptotic to $\pi^- + i\infty$.
- For $\arg(\gamma) \in \text{II quadrant}$ we have C_1 and C_3 asymptotic to $-\pi^+ + i\infty$, C_1 and C_2 asymptotic to $0^+ - i\infty$ and C_2 and C_3 asymptotic to $\pi^+ + i\infty$.
- For $\arg(\gamma) \in \text{III quadrant}$ we have C_1 and C_3 asymptotic to $-\pi^+ - i\infty$, C_1 and C_2 asymptotic to $0^+ + i\infty$ and C_2 and C_3 asymptotic to $\pi^+ - i\infty$.
- For $\arg(\gamma) \in \text{IV quadrant}$ we have C_1 and C_3 asymptotic to $-\pi^- - i\infty$, C_1 and C_2 asymptotic to $0^- + i\infty$ and C_2 and C_3 asymptotic to $\pi^- - i\infty$.

To deal with the case $k=2$, we consider the change of variable

$$w_j = z_j \exp(\pm i\pi j) \quad (\text{C.2})$$

which factors out the highest wavenumber wave component and reduces equation (2.32) for $k=2$ to the form

$$w_{j+1} - 2w_j + w_{j-1} - \alpha_2(j - j_{*2}) = 0 \quad (\text{C.3})$$

The solution of equation (C.3) in terms of contour integrals is the same as in the case $k = 1$, but with $\gamma = \frac{i}{\alpha_2}$. Therefore, we only need to study the solution of (2.32) with $k = 1$ and then use the change of variable (C.2) to deal with the case $k = 2$.

The solutions $H_j(\gamma, \nu)$ of the approximate equation (2.1) will be used in section C.1.2 to connect a given linear combination of Liouville-Green functions through turning points and Stokes lines. The connection formulae are obtained by matching the asymptotic form of the appropriate function $H_j(\gamma, \nu)$ for large $|\gamma|$ with the expansion of the Liouville-Green function in terms of the local variable ν .

C.1.1 Leading Term of the Asymptotic Expansion of the Functions $H_j(\gamma, \nu)$.

Here we present the leading term of the asymptotic expansion of the functions $H_j(\gamma, \nu)$, $j = 1, 2, 3$ for large $|\gamma|$, $|\nu| < |\gamma|$ and with $-\pi < \arg\{\gamma\} < \pi$. The functions $H_j(\gamma, \nu)$ were defined in the previous section in terms of contour integrals. To obtain the asymptotic expansion of these functions, we deformed the integration contour C_j in terms of the steepest descent paths of the function

$$p(z; \gamma, \nu) = 2e^{i\arg\{\gamma\}}[-z + \sin(z)] - i\frac{\nu}{2|\gamma|}z. \quad (\text{C.4})$$

Then, we applied Watson's Lemma (Olver 1997) to the integrals along the steepest descent paths, which resulted in the desired asymptotic expansion. Here we only give the leading term of the asymptotic expansions of the functions $H_j(\gamma, \nu)$.

C.1.1.1 Leading term of the asymptotic expansion of $H_j(\gamma, \nu)$ for $-\pi < \arg\{\gamma\} < 0$.

Now we consider $-\pi < \arg\{\gamma\} < 0$. For this range of $\arg\{\gamma\}$ we give the leading term of the asymptotic expansion of the functions $H_j(\gamma, \nu)$. First, we define some quantities of interest. Instead of giving the leading term of the asymptotic expansions as we vary $\arg\{\nu\}$, we use the variable β , defined by:

$$\beta = \arg\{\nu\} - \frac{1}{3}\arg\{\gamma\} \quad (\text{C.5})$$

The branch cut in the ν complex plane is a line ranging from 0 to ∞ , with argument given as a function of $\arg\{\gamma\}$, as follows

$$\arg\{\nu\} = \arg\{\gamma\} + \frac{\pi}{2} \quad (\text{C.6})$$

For $-\pi < \arg\{\gamma\} < 0$ we have that the branch cut lies in the first and fourth quadrants, and for $0 < \arg\{\gamma\} < \pi$ we have that the branch cut lies in the second and

third quadrants. Next, we consider the leading terms of the asymptotic expansions of the function $H_1(\gamma, \nu)$.

C.1.1.1.1 Leading term of the asymptotic expansions of $H_1(\gamma, \nu)$ when $-\pi < \arg\{\gamma\} < 0$.

Here we give the leading term of the asymptotic expansion of the function $H_1(\gamma, \nu)$ as β varies in the range $\frac{\pi}{2} + \frac{2}{3} \arg\{\gamma\} < \beta < \frac{5\pi}{2} + \frac{2}{3} \arg\{\gamma\}$.

- Case $-\pi < \arg\{\gamma\} < -\frac{\pi}{2}$ and $\frac{11\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3} \arg\{\gamma\}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.7})$$

This leading term dominant for this range of β .

- Case $-\pi < \arg\{\gamma\} < -\frac{\pi}{2}$ and $\frac{\pi}{2} + \frac{2}{3} \arg\{\gamma\} < \beta < \frac{\pi}{2}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.8})$$

This leading term is recessive for the range $\frac{\pi}{2} + \frac{2}{3} \arg\{\gamma\} < \beta < \frac{\pi}{6}$ and it is dominant for the range $\frac{\pi}{6} < \beta < \frac{\pi}{2}$.

- Case $-\frac{\pi}{2} < \arg\{\gamma\} < 0$ and $\frac{11\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3} \arg\{\gamma\}$ follows:

$$H_1(\gamma, \nu) \sim -\frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.9})$$

This leading term is dominant for the range $\frac{11\pi}{6} < \beta < \frac{13\pi}{6}$, and it is recessive for the range $\frac{13\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3} \arg\{\gamma\}$.

- Case $-\frac{\pi}{2} < \arg\{\gamma\} < 0$ and $\frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{2}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.10})$$

This leading term is recessive for this range of β .

- Case $\frac{\pi}{2} < \beta < \frac{7\pi}{6}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.11})$$

This leading term recessive for $\frac{\pi}{2} < \beta < \frac{5\pi}{6}$ and it is dominant for $\frac{5\pi}{6} < \beta < \frac{7\pi}{6}$.

- Case $\frac{7\pi}{6} < \beta < \frac{11\pi}{6}$ follows:

$$\begin{aligned} H_1(\gamma, \nu) \sim & \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ & - \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \end{aligned} \quad (\text{C.12})$$

The first part of this leading term is dominant for the range $\frac{7\pi}{6} < \beta < \frac{3\pi}{2}$, and it is recessive for the range $\frac{3\pi}{2} < \beta < \frac{11\pi}{6}$. The opposite is true for the second term of this leading term.

C.1.1.1.2 Leading term of the asymptotic expansions of $H_2(\gamma, \nu)$ when $-\pi < \arg\{\gamma\} < 0$.

Here we give the leading term of the asymptotic expansion of the function $H_2(\gamma, \nu)$ as β varies in the range $\frac{5\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\}$.

- Case $-\pi < \arg\{\gamma\} < -\frac{\pi}{2}$ and $\frac{11\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3}\arg\{\gamma\}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.13})$$

This leading term is recessive for this range of β .

- Case $-\pi < \arg\{\gamma\} < -\frac{\pi}{2}$ and $\frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{2}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.14})$$

This leading term is dominant for the range $\frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{6}$ and it is recessive for the range $\frac{\pi}{6} < \beta < \frac{\pi}{2}$.

- Case $-\frac{\pi}{2} < \arg\{\gamma\} < 0$ and $\frac{11\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3}\arg\{\gamma\}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.15})$$

This leading term is recessive for the range $\frac{11\pi}{6} < \beta < \frac{13\pi}{6}$, and it is dominant for the range $\frac{13\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3}\arg\{\gamma\}$.

- Case $-\frac{\pi}{2} < \arg\{\gamma\} < 0$ and $\frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{2}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.16})$$

This leading term is recessive for this range of β .

- Case $\frac{\pi}{2} < \beta < \frac{7\pi}{6}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ - \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.17)$$

The first (second) part of this leading term dominant (recessive) for $\frac{\pi}{2} < \beta < \frac{5\pi}{6}$, and it is recessive (dominant) for $\frac{5\pi}{6} < \beta < \frac{7\pi}{6}$.

- Case $\frac{7\pi}{6} < \beta < \frac{11\pi}{6}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.18)$$

This leading term is dominant for the range $\frac{7\pi}{6} < \beta < \frac{3\pi}{2}$, and it is recessive for the range $\frac{3\pi}{2} < \beta < \frac{11\pi}{6}$.

C.1.1.1.3 Leading term of the asymptotic expansions of $H_3(\gamma, \nu)$ when $-\pi < \arg\{\gamma\} < 0$.

Here we give the leading term of the asymptotic expansion of the function $H_3(\gamma, \nu)$ as β varies in the range $\frac{5\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\}$.

- Case $-\pi < \arg\{\gamma\} < -\frac{\pi}{2}$ and $\frac{11\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3}\arg\{\gamma\}$ follows:

$$H_3(\gamma, \nu) \sim \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ + \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.19)$$

The first part of this leading term is dominant for this range of β , and the second term is recessive for this range of β .

- Case $-\pi < \arg\{\gamma\} < -\frac{\pi}{2}$ and $\frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{2}$ follows:

$$H_3(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ + \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.20})$$

The first part of this leading term is dominant for the range $\frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{6}$ and it is recessive for the range $\frac{\pi}{6} < \beta < \frac{\pi}{2}$. For the second part of this leading term the opposite is true.

- Case $-\frac{\pi}{2} < \arg\{\gamma\} < 0$ and $\frac{11\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3}\arg\{\gamma\}$ follows:

$$H_3(\gamma, \nu) \sim \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ + \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.21})$$

The first part of this leading term is recessive for the range $\frac{11\pi}{6} < \beta < \frac{13\pi}{6}$, and it is dominant for the range $\frac{13\pi}{6} < \beta < \frac{5\pi}{2} + \frac{2}{3}\arg\{\gamma\}$. The opposite is true for the second part of this leading term.

- Case $-\frac{\pi}{2} < \arg\{\gamma\} < 0$ and $\frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < \frac{\pi}{2}$ follows:

$$H_3(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ + \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.22})$$

The first part of this leading term is recessive for this range of β , and the second part of this leading term is dominant for this range of β .

- Case $\frac{\pi}{2} < \beta < \frac{7\pi}{6}$ follows:

$$H_3(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.23})$$

This leading term is dominant for $\frac{\pi}{2} < \beta < \frac{5\pi}{6}$, and it is recessive for $\frac{5\pi}{6} < \beta < \frac{7\pi}{6}$.

- Case $\frac{7\pi}{6} < \beta < \frac{11\pi}{6}$ follows:

$$H_3(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.24})$$

This leading term is recessive for the range $\frac{7\pi}{6} < \beta < \frac{3\pi}{2}$, and it is dominant for the range $\frac{3\pi}{2} < \beta < \frac{11\pi}{6}$.

C.1.1.2 Leading term of the asymptotic expansion of $H_j(\gamma, \nu)$ for $\pi > \arg\{\gamma\} > 0$.

Now we consider $\pi > \arg\{\gamma\} > 0$. For this range of $\arg\{\gamma\}$ we give the leading term of the asymptotic expansion of the functions $H_j(\gamma, \nu)$. For this range of $\arg\{\gamma\}$ we have that the branch cut in the ν complex plane lies in the second and third quadrants. The argument of the line that represents the branch cut is given by (C.6). For this range of $\arg\{\gamma\}$ we have that $\frac{\pi}{2} + \arg\{\gamma\} > \arg\{\nu\} > -\frac{3\pi}{2} + \arg\{\gamma\}$. This range of $\arg\{\nu\}$ translates in terms of the β variable as $\frac{\pi}{2} + \frac{2}{3}\arg\{\gamma\} > \beta > -\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\}$.

We start with the function $H_1(\gamma, \nu)$, and then we consider functions $H_2(\gamma, \nu)$ and $H_3(\gamma, \nu)$.

C.1.1.2.1 Leading term of the asymptotic expansions of $H_1(\gamma, \nu)$ when $\pi > \arg\{\gamma\} > 0$.

- Case $-\frac{\pi}{6} < \beta < \frac{\pi}{2}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} + \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.25)$$

The first (second) part of this leading term is dominant (recessive) for $-\frac{\pi}{6} < \beta < \frac{\pi}{6}$, and it is recessive (dominant) for $\frac{\pi}{6} < \beta < \frac{\pi}{2}$.

- Case $0 < \arg\{\gamma\} < \frac{\pi}{2}$ with $-\frac{3\pi}{2} < \beta < -\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.26)$$

For this range of β we have that this leading term is recessive.

- $0 < \arg\{\gamma\} < \frac{\pi}{2}$ with $-\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < -\frac{5\pi}{6}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.27)$$

This leading term is recessive for $-\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < -\frac{7\pi}{6}$ and it is dominant for $-\frac{7\pi}{6} < \beta < -\frac{5\pi}{6}$.

- Case $\frac{\pi}{2} < \arg\{\gamma\} < \pi$ with $-\frac{3\pi}{2} < \beta < -\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.28)$$

This leading term is recessive for $-\frac{3\pi}{2} < \beta < -\frac{7\pi}{6}$, and it is dominant for $-\frac{7\pi}{6} < \beta < -\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\}$.

- Case $\frac{\pi}{2} < \arg\{\gamma\} < \pi$ with $-\frac{3\pi}{2} + \frac{2}{3} \arg\{\gamma\} < \beta < -\frac{\pi}{2}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.29})$$

This leading term is dominant for this range of β .

- Case $-\frac{5\pi}{6} < \beta < -\frac{\pi}{6}$ follows:

$$H_1(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.30})$$

This leading term is dominant for $-\frac{5\pi}{6} < \beta < -\frac{\pi}{2}$ and it is recessive for $-\frac{\pi}{2} < \beta < -\frac{\pi}{6}$.

C.1.1.2.2 Leading term of the asymptotic expansions of $H_2(\gamma, \nu)$ when $\pi > \arg\{\gamma\} > 0$.

- Case $-\frac{\pi}{6} < \beta < \frac{\pi}{2}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.31})$$

This leading term is recessive for $-\frac{\pi}{6} < \beta < \frac{\pi}{6}$, and it is dominant for $\frac{\pi}{6} < \beta < \frac{\pi}{2}$.

- Case $0 < \arg\{\gamma\} < \frac{\pi}{2}$ with $-\frac{3\pi}{2} < \beta < -\frac{3\pi}{2} + \frac{2}{3} \arg\{\gamma\}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.32})$$

For this range of β we have that this leading term is dominant.

- $0 < \arg\{\gamma\} < \frac{\pi}{2}$ with $-\frac{3\pi}{2} + \frac{2}{3} \arg\{\gamma\} < \beta < -\frac{5\pi}{6}$ follows:

$$H_2(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.33})$$

This leading term is dominant for $-\frac{3\pi}{2} + \frac{2}{3} \arg\{\gamma\} < \beta < -\frac{7\pi}{6}$ and it is recessive for $-\frac{7\pi}{6} < \beta < -\frac{5\pi}{6}$.

- Case $\frac{\pi}{2} < \arg\{\gamma\} < \pi$ with $-\frac{3\pi}{2} < \beta < -\frac{3\pi}{2} + \frac{2}{3} \arg\{\gamma\}$ follows:

$$H_2(\gamma, \nu) \sim -\frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.34})$$

This leading term is dominant for $-\frac{3\pi}{2} < \beta < -\frac{7\pi}{6}$, and it is recessive for $-\frac{7\pi}{6} < \beta < -\frac{3\pi}{2} + \frac{2}{3} \arg\{\gamma\}$.

- Case $\frac{\pi}{2} < \arg\{\gamma\} < \pi$ with $-\frac{3\pi}{2} + \frac{2}{3} \arg\{\gamma\} < \beta < -\frac{\pi}{2}$ follows:

$$H_2(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.35})$$

This leading term is recessive for this range of β .

- Case $-\frac{5\pi}{6} < \beta < -\frac{\pi}{6}$ follows:

$$H_2(\gamma, \nu) \sim \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} - \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.36})$$

The first (second) part of this leading term is recessive (dominant) for $-\frac{5\pi}{6} < \beta < -\frac{\pi}{2}$ and it is dominant (recessive) for $-\frac{\pi}{2} < \beta < -\frac{\pi}{6}$.

C.1.1.2.3 Leading term of the asymptotic expansions of $H_3(\gamma, \nu)$ when $\pi > \arg\{\gamma\} > 0$.

- Case $-\frac{\pi}{6} < \beta < \frac{\pi}{2}$ follows:

$$H_3(\gamma, \nu) \sim -\frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.37})$$

This leading term is dominant for $-\frac{\pi}{6} < \beta < \frac{\pi}{6}$, and it is recessive for $\frac{\pi}{6} < \beta < \frac{\pi}{2}$.

- Case $0 < \arg\{\gamma\} < \frac{\pi}{2}$ with $-\frac{3\pi}{2} < \beta < -\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\}$ follows:

$$\begin{aligned} H_3(\gamma, \nu) \sim & -\frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ & + \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \end{aligned} \quad (\text{C.38})$$

For this range of β we have that the first (second) part of this leading term is recessive (dominant).

- $0 < \arg\{\gamma\} < \frac{\pi}{2}$ with $-\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < -\frac{5\pi}{6}$ follows:

$$\begin{aligned} H_3(\gamma, \nu) \sim & -\frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ & - \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \end{aligned} \quad (\text{C.39})$$

The first (second) part of this leading term is recessive (dominant) for $-\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < -\frac{7\pi}{6}$ and it is dominant (recessive) for $-\frac{7\pi}{6} < \beta < -\frac{5\pi}{6}$.

- Case $\frac{\pi}{2} < \arg\{\gamma\} < \pi$ with $-\frac{3\pi}{2} < \beta < -\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\}$ follows:

$$H_3(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ + \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.40})$$

The first part of this leading term is recessive (dominant) for $-\frac{3\pi}{2} < \beta < -\frac{7\pi}{6}$, and it is dominant (recessive) for $-\frac{7\pi}{6} < \beta < -\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\}$.

- Case $\frac{\pi}{2} < \arg\{\gamma\} < \pi$ with $-\frac{3\pi}{2} + \frac{2}{3}\arg\{\gamma\} < \beta < -\frac{\pi}{2}$ follows:

$$H_3(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \\ - \frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.41})$$

The first part of this leading term is dominant, and the second part of it is recessive for this range of β .

- Case $-\frac{5\pi}{6} < \beta < -\frac{\pi}{6}$ follows:

$$H_3(\gamma, \nu) \sim -\frac{1}{2}\sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2|\gamma|e^{i\arg\{\gamma\}}[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2}\arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (\text{C.42})$$

This leading term is recessive for $-\frac{5\pi}{6} < \beta < -\frac{\pi}{2}$ and it is dominant for $-\frac{\pi}{2} < \beta < -\frac{\pi}{6}$.

C.1.2 LOCAL APPROXIMATION OF THE LIOUVILLE-GREEN FORMULA.

The second step in obtaining the connection formula is to write the Liouville-Green functions in terms of the local variable, namely α , defined as

$$\alpha = 1 + i \frac{\nu}{2\gamma} \quad (\text{C.43})$$

We consider the boundary conditions of left and right incidence discussed in section 4.4. We write them in terms of the local variable α . The form of the Liouville-Green function in terms of the local variable α changes with respect to which condition (2.30) the turning points of the turning point problem are considered to satisfy.

C.1.2.1 Pair of Complex Conjugate First Order Turning Points.

In this section we consider two boundary conditions for TPP1, namely left and right wave incidence. We give these boundary conditions as a linear combination of the Liouville-Green functions written in terms of the local variable α . First, we consider TPP1 with turning points (see box (A) in figure 2-2) that satisfy the first turning point condition. We use the same notation as in section 4.4.1, namely, we have $a_l < a_{l+1}$ as boundary points, $j_{*1,l}$ and $\bar{j}_{*1,l}$ as the pair of complex conjugate turning points and $r_l = \Re\{j_{*1,l}\}$ ($a_l < r_l < a_{l+1}$) as a reference point in the real axis. For left incidence we have

- For $a_l < j < r_l$ we have an incident plus a reflected wave

$$\begin{aligned} z_j \sim \frac{1}{(\alpha^2 - 1)^{1/4}} \Big\{ & A \exp(-i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} - iI(a_l, r_l) - iI(r_l, \bar{j}_{*1,l})) \\ & + R^- A \exp(i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} + iI(a_l, r_l) + iI(r_l, \bar{j}_{*1,l})) \Big\} \end{aligned} \quad (\text{C.44})$$

- For $r_l < j < a_{l+1}$ we have a transmitted wave

$$z_j \sim \frac{T^- A}{(\alpha^2 - 1)^{1/4}} \exp(-i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} - iI(a_l, r_l) - iI(r_l, \bar{j}_{*1,l})) \quad (\text{C.45})$$

For the case of left incidence we approximate the Liouville-Green functions with respect to the complex turning point $\bar{j}_{*1,l}$, since the transmitted wave is exponentially small in the region bounded by the real axis, by the Stokes line that connects both turning points and crosses the real axis and by the Stokes line that emanates from $\bar{j}_{*1,l}$ and goes to infinity for $j > \Re\{\bar{j}_{*1,l}\}$. Since the transmitted wave is exponentially small in the region described above and specified by the boundary conditions, it is uniquely defined in this region. This fact allow us to continue it to other regions with a common turning point ($\bar{j}_{*1,l}$ in this case) by using the asymptotic form of the solution of the approximate form of (2.1), valid in a neighborhood of $\bar{j}_{*1,l}$ (see figure 2-2). If the transmitted wave was exponentially large, it would not be defined uniquely by the boundary conditions, since we always can add the other exponentially small Liouville-Green function and the boundary condition is still satisfied. This is actually true for most of the complex j plane. The exceptions are the real axis and the Anti-Stokes lines (Lines that emanate from the turning points defined by $\Re\{\int_r^j \theta_n dn\} = 0$, with r as a turning point), where the Liouville-Green functions are of the same order of magnitude. Along these lines, boundary conditions specify uniquely the appropriate linear combination of Liouville-Green functions. Next, we consider the boundary condition of right incidence.

- For $a_l < j < r_l$ we have a transmitted wave

$$z_j \sim \frac{T^+ A}{(\alpha^2 - 1)^{1/4}} \exp(i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} + iI(r_l, j_{*1,l})) \quad (\text{C.46})$$

- For $r_l < j < a_{l+1}$ we have an incident plus a reflected wave

$$z_j \sim \frac{1}{(\alpha^2 - 1)^{1/4}} \left\{ A \exp(i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} + iI(a_{l+1}, r_l) + iI(r_l, j_{*1,l})) \right. \\ \left. + R^+ A \exp(-i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} - iI(a_{l+1}, r_l) - iI(r_l, j_{*1,l})) \right\} \quad (\text{C.47})$$

and here we chose to expand the Liouville-Green functions with respect to $j_{*1,l}$ for the same reasons mentioned in the case of left incidence. The transmitted wave is exponentially small in the region bounded by the real axis, by the Stokes line that connects both turning points and cross the real axis and by the Stokes line that emanates from $j_{*1,l}$ and goes to infinity for $j < \Re\{j_{*1,l}\}$.

For TPP1 with turning points that satisfy the second turning point condition, we have that the phase term in the Liouville-Green functions is written in terms of the variable α as

$$\exp(\pm i \int_r^j \theta_n dn) \sim \begin{cases} \pm i\pi\nu + i2\gamma[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] & \text{for } \Im\{\alpha\} \geq 0 \\ \mp i\pi\nu - i2\gamma[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] & \text{for } \Im\{\alpha\} < 0 \end{cases} \quad (\text{C.48})$$

The term $\pm i\pi\nu(\pm i\pi(j - j_{*2,l}))$ represents the wave with the highest possible wave number. This is due to the fact that in a neighborhood of a turning point that satisfies the second turning point condition, the wave number θ_n is in a neighborhood of the upper edge of the uniform system passband. If we take into account the change of dependent variable (C.2), we can factor out the term $\pm i\pi j$. For waves that propagate to the left we chose the plus sign in (C.2), and for waves propagating to right we chose the minus sign in (C.2). In terms of the dependent variable w_j , the boundary conditions for left incidence are similar to (C.44) and (C.45). The difference lies in that we consider the turning point $j_{*2,l}$, instead of $\bar{j}_{*2,l}$. The reason is that in the appropriate neighborhood of $j_{*2,l}$, the transmitted wave is exponentially small. In the following four expressions, $r_l = \Re\{j_{*2,l}\}$. The boundary condition for

left incidence, in terms of w_j , follows bellow.

- For $a_l < j < r_l$ we have an incident plus a reflected wave

$$w_j \sim \frac{1}{(\alpha^2 - 1)^{1/4}} \left\{ A \exp(-i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} - iI(a_l, r_l) - iI(r_l, j_{*2,l}) + i\pi j_{*2,l}) + R^- A \exp(i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} + iI(a_l, r_l) + iI(r_l, j_{*2,l}) - i\pi j_{*2,l}) \right\} \quad (C.49)$$

- For $r_l < j < a_{l+1}$ we have a transmitted wave

$$w_j \sim \frac{T^- A}{(\alpha^2 - 1)^{1/4}} \exp(-i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} - iI(a_l, r_l) - iI(r_l, j_{*2,l}) - i\pi j_{*2,l}) \quad (C.50)$$

The right incidence boundary condition is similar to (C.46) and (C.47). The difference lies that in this case we consider the turning point $\bar{j}_{*2,l}$ instead of $j_{*2,l}$. The reason is the same as the one mentioned above, i.e. the transmitted wave is exponentially small in the neighborhood of $\bar{j}_{*2,l}$. The boundary condition follows.

- For $a_l < j < r_l$ we have a transmitted wave

$$w_j \sim \frac{T^+ A}{(\alpha^2 - 1)^{1/4}} \exp(i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} + iI(r_l, \bar{j}_{*2,l}) - i\pi \bar{j}_{*2,l}) \quad (C.51)$$

- For $r_l < j < a_{l+1}$ we have an incident plus a reflected wave

$$w_j \sim \frac{1}{(\alpha^2 - 1)^{1/4}} \left\{ A \exp(i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} + iI(a_{l+1}, r_l) + iI(r_l, \bar{j}_{*2,l}) - i\pi \bar{j}_{*2,l}) + R^+ A \exp(-i2\gamma[\sqrt{\alpha^2 - 1} - \alpha \ln(\alpha + \sqrt{\alpha^2 - 1})] + i\frac{\pi}{4} - iI(a_{l+1}, r_l) - iI(r_l, \bar{j}_{*2,l}) + i\pi \bar{j}_{*2,l}) \right\} \quad (C.52)$$

The next step is to define the appropriate solution to the locally valid equations (2.32) (k equal to 1) and (C.48). We consider (2.32) as the locally valid equation when turning points of TPP1 satisfy the first turning point condition (2.30). When turning points of TPP1 satisfy the second turning point condition, we consider the locally valid equation (C.48). After we obtain the appropriate solutions of the locally valid equation in terms of the function $Hj(\gamma, \nu)$, we match these solutions with the expansion of the Liouville-Green formulae given in the sections above. The result of the matching process are the reflection/transmission coefficients given by (2.49) to (2.64). The steps mentioned in this paragraph are given in the next section.

C.1.3 MATCHING PROCESS.

First we give the leading term of the asymptotic expansion of the solution of the locally valid equations (2.32) and (C.48), appropriate for the matching with the Liouville-Green formulae for complex first order turning point. For second order turning points, we already gave the solution of the appropriate locally valid equation in section 4.3.1, so here we only give the asymptotic expansion of this solution.

C.1.3.1 Leading Term of The Asymptotic Expansion of The Solution of The Approximate Equation in a Neighborhood of a Complex First Order Turning Point.

The complex first order turning point satisfies the first or the second turning point condition (2.30). If it satisfies the first turning point condition, the leading term of the asymptotic expansion of the appropriate solution of (2.32) with k=1 is given below in terms of the functions

$$A(\gamma, \nu) = \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(-i2\gamma[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] - i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.53)$$

$$B(\gamma, \nu) = \frac{1}{2} \sqrt{\frac{\pi}{|\gamma|}} \frac{\exp(i2\gamma[\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \sqrt{\alpha^2 - 1}] + i\frac{\pi}{4} - \frac{i}{2} \arg\{\gamma\})}{(\alpha^2 - 1)^{1/4}} \quad (C.54)$$

The leading term of the asymptotic expansion for the appropriate solution of (2.32) with $k=1$ for $|\gamma| \gg 1$ and $|\nu| \leq |\gamma|$ follows.

- For left incidence boundary condition (Turning point $\bar{j}_{*1,l}$).

$$z_j \sim -C B(\gamma, \nu), \text{ for } 0 < \arg\{\gamma\} < \frac{\pi}{2} \text{ and } \frac{\pi}{6} < \beta < \frac{\pi}{2} \quad (\text{C.55})$$

$$C (A(\gamma, \nu) - B(\gamma, \nu)), \text{ for } 0 < \arg\{\gamma\} < \frac{\pi}{2} \text{ and } \frac{\pi}{2} < \beta < \frac{5\pi}{6}$$

$$z_j \sim -C B(\gamma, \nu), \text{ for } 0 \leq \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } \frac{\pi}{6} < \beta < \frac{\pi}{2} \quad (\text{C.56})$$

$$C (A(\gamma, \nu) - B(\gamma, \nu)), \text{ for } 0 \leq \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } \frac{\pi}{2} < \beta < \frac{5\pi}{6}$$

where β is defined by

$$\beta = \arg\{\nu\} - \frac{1}{3} \arg\{\gamma\} \quad (\text{C.57})$$

in terms of $\arg\{\nu\}$ and $\arg\{\gamma\}$.

- For right incidence boundary condition (Turning point $j_{*1,l}$).

$$z_j \sim -C A(\gamma, \nu), \text{ for } \frac{\pi}{2} \leq \arg\{\gamma\} \leq \pi \text{ and } \frac{5\pi}{6} < \beta < \frac{7\pi}{6} \quad (\text{C.58})$$

$$-C (A(\gamma, \nu) + B(\gamma, \nu)), \text{ for } \frac{\pi}{2} \leq \arg\{\gamma\} \leq \pi \text{ and } \frac{7\pi}{6} < \beta < \frac{3\pi}{2}$$

$$z_j \sim -C A(\gamma, \nu), \text{ for } -\pi < \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } -\frac{\pi}{2} < \beta < \frac{-\pi}{6} \quad (\text{C.59})$$

$$C (A(\gamma, \nu) + B(\gamma, \nu)), \text{ for } -\pi < \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } -\frac{\pi}{6} < \beta < \frac{\pi}{6}$$

If the first order complex turning point satisfies the second turning point condition, then the appropriate solution of (C.48) follows.

- For left incidence boundary condition (Turning point $j_{*2,l}$).

$$w_j \sim C (-A(\gamma, \nu) + B(\gamma, \nu)), \text{ for } -\pi < \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } -\frac{\pi}{2} < \beta < \frac{-\pi}{6} \quad (\text{C.60})$$

$$\begin{aligned} & -C A(\gamma, \nu), \text{ for } -\pi < \arg\{\gamma\} < -\frac{\pi}{2} \text{ and } -\frac{\pi}{6} < \beta < \frac{\pi}{6} \\ w_j \sim C (A(\gamma, \nu) - B(\gamma, \nu)), & \text{ for } \frac{\pi}{2} \arg\{\gamma\} \leq \pi \text{ and } \frac{5\pi}{6} < \beta < \frac{7\pi}{6} \quad (\text{C.61}) \\ & C A(\gamma, \nu), \text{ for } \frac{\pi}{2} \arg\{\gamma\} \leq \pi \text{ and } \frac{7\pi}{6} < \beta < \frac{3\pi}{2} \end{aligned}$$

- For right incidence boundary condition (Turning point $\bar{j}_{*2,l}$).

$$w_j = C (A(\gamma, \nu) + B(\gamma, \nu)), \text{ for } 0 \geq \arg\{\gamma\} > \frac{\pi}{2} \text{ and } \frac{\pi}{6} < \beta < \frac{\pi}{2} \quad (\text{C.62})$$

$$C B(\gamma, \nu), \text{ for } 0 \geq \arg\{\gamma\} > \frac{\pi}{2} \text{ and } \frac{\pi}{6} < \beta < \frac{5\pi}{6}$$

$$w_j = C (A(\gamma, \nu) + B(\gamma, \nu)), \text{ for } 0 > \arg\{\gamma\} > -\frac{\pi}{2} \text{ and } \frac{\pi}{6} < \beta < \frac{\pi}{2} \quad (\text{C.63})$$

$$C B(\gamma, \nu), \text{ for } 0 > \arg\{\gamma\} > -\frac{\pi}{2} \text{ and } \frac{\pi}{2} < \beta < \frac{5\pi}{6}$$

The next step is to match the asymptotic expansions, for $|\gamma| \rightarrow \infty$, of the appropriate solutions of the locally valid equations (2.32) and (C.48) with the linear combination of expansions of the Liouville-Green functions in terms of the local variable α . The matching process is described in the next section.

C.1.3.2 Matching Process for Complex First Order Turning Points.

Here we describe how to match the expansion of the Liouville-Green functions in terms of the local variable α with the asymptotic expansion of the appropriate solution of the locally valid equations (2.32) and (C.48), which were given in the previous section. First, we consider TPP1 (see box (A) of figure 2-2 with turning points that satisfy the first turning point condition (2.30). The boundary conditions to consider follow below.

- Matching for the left incidence boundary condition. We have to match (C.44),

section B.1.1, with (C.55) if $0 < \arg\{\gamma\} < \frac{\pi}{2}$ with $\frac{\pi}{2} < \beta < \frac{5\pi}{6}$ or with (C.56) if $0 \geq \arg\{\gamma\} > -\frac{\pi}{2}$ with $\frac{\pi}{2} < \beta < \frac{5\pi}{6}$, given in section B.2. This matching gives

$$\frac{1}{2}\sqrt{\frac{\pi}{\gamma}}C = A \exp(i\frac{\pi}{4} - iI(a_l, r_l) - iI(r_l, \bar{j}_{*1,l})) \quad (\text{C.64})$$

$$\frac{1}{2}\sqrt{\frac{\pi}{\gamma}}C = R^- A \exp(i\frac{\pi}{4} + iI(a_l, r_l) + iI(r_l, \bar{j}_{*1,l})) \quad (\text{C.65})$$

We have two equations and two unknowns, so we have the constants C and R^- . To obtain the transmission coefficient we need to match (C.45) with (C.55) or (C.56), according to the value of $\arg\{\gamma\}$ and $\frac{\pi}{6} < \beta < \frac{\pi}{2}$. This gives the transmission coefficient T^- in terms of the constant C given by (C.64). The output of the matching process are the coefficients T^- and R^- given by (2.49) and (2.50).

- Matching for right incidence boundary condition. We have to match (C.47), section B.1.1, with (C.58) with $\frac{3\pi}{2} < \beta < \frac{7\pi}{6}$ or (C.59) with $-\frac{\pi}{6} < \beta < \frac{\pi}{6}$, according with the value of $\arg\{\gamma\}$. This matching gives the coefficients C and R^+ . To obtain the transmission coefficient we have to match (C.46), section B.1.1, with (C.58) with $\frac{5\pi}{6} < \beta < \frac{3\pi}{2}$ or (C.59) with $-\frac{\pi}{2} < \beta < -\frac{\pi}{6}$, according with the value of $\arg\{\gamma\}$. This gives the transmission coefficient T^+ . The final result of this matching are the reflection R^+ and transmission T^+ coefficients given by (2.51) and (2.52).

Next we consider the right and left incidence boundary conditions for TPP1 with turning points that satisfies the second turning point condition (2.30).

- Matching for left incidence boundary condition. We match (C.49), section B.1.1, with (C.60) with $-\frac{\pi}{2} < \beta < -\frac{\pi}{6}$ or (C.61) with $\frac{5\pi}{6} < \beta < \frac{7\pi}{6}$, according to the value of $\arg\{\gamma\}$. This matching gives the reflection coefficient R^- and the coefficient C . To obtain the transmission coefficient T^- , we match (C.50) with (C.60) with $-\frac{\pi}{6} < \beta < \frac{\pi}{6}$ or (C.61) with $\frac{7\pi}{6} < \beta < \frac{3\pi}{2}$, according to the value of $\arg\{\gamma\}$. The final result are the coefficients R^- and T^- , given by (2.53) and (2.54).

- Matching for right incidence boundary condition. We match (C.52), section B.1.1, with (C.62) with $\frac{\pi}{6} < \beta < \frac{\pi}{2}$ or (C.63) with $\frac{\pi}{6} < \beta < \frac{\pi}{2}$, according to the value of $\arg\{\gamma\}$. This matching gives the reflection coefficient R^+ and the coefficient C . To obtain the transmission coefficient T^+ , we match (C.51) with (C.62) with $\frac{\pi}{2} < \beta < \frac{5\pi}{6}$ or (C.63) with $\frac{\pi}{2} < \beta < \frac{5\pi}{6}$, according to the value of $\arg\{\gamma\}$. The final result are the coefficients R^+ and T^+ , given by (2.55) and (2.56).

C.2 CONNECTION FORMULAE FOR A PAIR OF ALMOST COALESCING REAL AND COMPLEX FIRST ORDER TURNING POINTS.

In this appendix we describe how to obtain the connection formulae for TPP3 and TPP4. We also consider TPP5, which is the limiting case of TPP3 or TPP4. Results for TPP5 follows from the results for TPP3 or TPP4 by putting $b_{k,l} = 0$. First, we give the Liouville-Green functions in terms of the local variable α , defined for this case as

$$\alpha = -(-1)^k + \psi_{k,l}[(j - a_{k,l})^2 - b_{k,l}^2] \quad (\text{C.66})$$

where $a_{k,l}$ and $b_{k,l}$ for TPP3 are, respectively, the real and imaginary part of a complex turning point that satisfies the k -th turning point condition. In the case of TPP4, $a_{k,l}$ is the average of the two turning points and $b_{k,l}$ is half of the modulus of the difference between turning points.

C.2.1 APPROXIMATE FORM OF THE LIOUVILLE-GREEN FUNCTIONS.

For TPP3 and TPP4 (see boxes (A) and (B) in figure 2-2) we consider the boundary condition of left and right incidence. Here we give the appropriate linear combination

of Liouville-Green functions in terms of the local variable α , given by (C.66), that satisfy the boundary conditions mentioned above. We use the same notation as in section 4.4 regarding the reference points $r_{1,l}$ and $r_{2,l}$. For TPP3, we have $r_{1,l} = r_{2,l} = r_l$ (r_l is the real part of the pair of complex conjugate turning points). For TPP4, we have $r_{1,l} = j_{*k,l} < r_{2,l} = \hat{j}_{*k,l}$ ($j_{*k,l}$ and $\hat{j}_{*k,l}$ are the pair of real turning points). We define $f_l = r_l$ ($f_l = \frac{j_{*k,l} + \hat{j}_{*k,l}}{2}$) when we consider TPP3 (TPP4).

- Left incidence boundary condition. We use the same notation as in section 4.4.

The boundary points are $a_l < a_{l+1}$.

- For $a_l < j < r_{1,l}$ we have an incident plus a reflected wave.

$$\begin{aligned}
z_j \sim & A 2^{i\frac{3}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2}} (-1)^{-i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2} \mu_{k,l}^{-i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{4}} |j - f_l|^{-i\frac{1}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \\
& \exp(-i\frac{1}{4}\sqrt{\mu_{k,l}}|j - f_l|^2 - iI(a_l, f_l) + i\pi f_l \delta_{k,2}) \\
& + R^- A 2^{-i\frac{3}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2}} (-1)^{i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2} \mu_{k,l}^{i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{4}} |j - f_l|^{i\frac{1}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \\
& \exp(i\frac{1}{4}\sqrt{\mu_{k,l}}|j - f_l|^2 + iI(a_l, f_l) - i\pi f_l \delta_{k,2}), \text{ as } j \rightarrow r_{1,l}^- \text{ and } b_{k,l} \rightarrow 0
\end{aligned} \tag{C.67}$$

- For $r_{2,l} < j < a_{l+1}$ we have a transmitted wave.

$$\begin{aligned}
z_j \sim & T^- A 2^{i\frac{3}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2}} (-1)^{-i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2} \mu_{k,l}^{-i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{4}} |j - f_l|^{-i\frac{1}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \\
& \exp(-i\frac{1}{4}\sqrt{\mu_{k,l}}|j - f_l|^2 + i\pi f_l \delta_{k,2}), \text{ as } j \rightarrow r_{2,l}^+ \text{ and } b_{k,l} \rightarrow 0
\end{aligned} \tag{C.68}$$

- Right incidence boundary condition. We use the same notation as for the left incidence boundary condition.

- For $a_l < j < r_{1,l}$ we have a transmitted wave.

$$\begin{aligned}
z_j \sim & T^+ A 2^{-i\frac{3}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2}} (-1)^{i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2} \mu_{k,l}^{i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{4}} |j - f_l|^{i\frac{1}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \\
& \exp(i\frac{1}{4}\sqrt{\mu_{k,l}}|j - f_l|^2 - i\pi f_l \delta_{k,2}), \text{ as } j \rightarrow r_{1,l}^- \text{ and } b_{k,l} \rightarrow 0
\end{aligned} \tag{C.69}$$

– For $r_{2,l} < j < a_{l+1}$ we have an incident and a reflected wave.

$$\begin{aligned}
z_j \sim & A 2^{-i\frac{3}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2}} (-1)^{i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2} \mu_{k,l}^{i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{4}} |j - f_l|^{i\frac{1}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \\
& \exp(i\frac{1}{4}\sqrt{\mu_{k,l}}|j - f_l|^2 + iI(a_{l+1}, f_l) - i\pi f_l \delta_{k,2}) \\
& + R^+ A 2^{i\frac{3}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2}} (-1)^{-i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2} \mu_{k,l}^{-i\frac{1}{8}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{4}} |j - f_l|^{-i\frac{1}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \\
& \exp(-i\frac{1}{4}\sqrt{\mu_{k,l}}|j - f_l|^2 - iI(a_{l+1}, f_l) + i\pi f_l \delta_{k,2}), \text{ as } j \rightarrow r_{2,l}^+ \text{ and } b_{k,l} \rightarrow 0
\end{aligned} \tag{C.70}$$

For TPP5 (see box (C) of figure 2-2), we just need to make $b_{k,l} = 0$ in the expressions above to obtain the linear combination of the Liouville-Green formulae in terms of the local variable α for the boundary conditions of left and right incidence.

C.2.2 ASYMPTOTIC EXPANSION OF THE SOLUTION OF THE APPROXIMATE EQUATION.

The approximate equation for TPP3 or TPP4 is given by (2.35). This equation is related to the solutions of the Mathieu equation, so closed form solution is not available. We proceed to approximate (2.35) by a second order differential equation, as discussed in section 4.3.2. We obtain a differential equation which has solution in terms of the parabolic cylinder functions. As a result, an approximate solution of the difference equation (2.35) is given by (2.42) and (2.43). In this section, we give the asymptotic expansion of the approximate solutions (2.42) and (2.43) for $|j - r_l| \gg 1$ (case TPP3, $r_l = \Re\{j_{*k,l}\}$) or $|j - \frac{j_{*k,l} + \hat{j}_{*k,l}}{2}| \gg 1$ (case TPP4). The term $|j - r_l|$ appears in the asymptotic expansion of (2.42) and (2.43) when we have TPP3, and the term $|j - \frac{j_{*k,l} + \hat{j}_{*k,l}}{2}|$ appears in the asymptotic expansion of (2.42) and (2.43) when we have TPP4. We define $f_l = r_l$ ($f_l = \frac{j_{*k,l} + \hat{j}_{*k,l}}{2}$) if we consider TPP3 (TPP4). For TPP3 (TPP4), we have the asymptotic expansion for (2.42) and (2.43) that follows. In the expressions above, the upper (lower) sign is for the case TPP3 (TPP4).

$$\begin{aligned}
z(j) \sim & \left\{ A_1(-1)^{\pm \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} + B_1 \right\} \exp\left(\pm \frac{\pi}{16}\sqrt{\mu_{k,l}}(b_{k,l})^2 + i\frac{\pi}{8} + \frac{\sqrt{\mu_{k,l}}}{4}e^{i\frac{\pi}{2}}|j-f_l|^2\right) \\
& \mu_{k,l}^{\pm \frac{i}{16}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{8}}|j-f_l|^{\pm \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} - A_1 \frac{\sqrt{2\pi}}{\Gamma(\mp \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2})} \exp\left(\mp \frac{5\pi}{16}\sqrt{\mu_{k,l}}(b_{k,l})^2 - i\frac{3\pi}{8}\right. \\
& \left. + \frac{\sqrt{\mu_{k,l}}}{4}e^{-i\frac{\pi}{2}}|j-f_l|^2\right)(-1)^{\mp \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \mu_{k,l}^{\mp \frac{i}{16}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{8}}|j-f_l|^{\mp \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}}, \text{ as} \\
& |j-f_l| \gg 1 \text{ and } j-f_l < 0
\end{aligned} \tag{C.71}$$

$$\begin{aligned}
z(j) \sim & \left\{ B_1(-1)^{\pm \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} + A_1 \right\} \exp\left(\pm \frac{\pi}{16}\sqrt{\mu_{k,l}}(b_{k,l})^2 + i\frac{\pi}{8}\right) \mu_{k,l}^{\pm \frac{i}{16}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{8}} \\
& |j-f_l|^{\pm \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \exp\left(\frac{\mu_{k,l}^{1/2}}{4}e^{i\frac{\pi}{2}}|j-f_l|^2\right) \\
& - B_1 \frac{\sqrt{2\pi}}{\Gamma(\mp \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 + \frac{1}{2})} \exp\left(\mp \frac{5\pi}{16}\sqrt{\mu_{k,l}}(b_{k,l})^2 - i\frac{3\pi}{8}\right)(-1)^{\mp \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \mu_{k,l}^{\mp \frac{i}{16}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{8}} \\
& |j-f_l|^{\mp \frac{i}{4}\sqrt{\mu_{k,l}}(b_{k,l})^2 - \frac{1}{2}} \exp\left(\frac{\mu_{k,l}^{1/2}}{4}e^{-i\frac{\pi}{2}}|j-f_l|^2\right), \text{ as } |j-f_l| \gg 1 \text{ and } j-f_l > 0
\end{aligned} \tag{C.72}$$

C.2.3 MATCHING PROCESS FOR A PAIR OF ALMOST COALESCING COMPLEX CONJUGATE FIRST ORDER TURNING POINTS.

In this section we describe the matching process which leads to the connection formulae for TPP3 (see figure 2-2). We consider right and left incidence boundary conditions. We do not distinguish the connection formulae with respect to which conditions (2.30) the turning points satisfy. The connection formulae we discuss here are valid for turning points that satisfy the first or second turning point condition.

- Left incidence boundary condition. At first, we match the appropriate linear combination of the expansion of the Liouville-Green functions in terms of the local variable α , given by (C.67) with the asymptotic expansion of the approximate solution of the locally valid difference equation, given by (C.71). This

matching results in two equations with three unknowns, namely, the reflection coefficient R^- and the coefficients A_1 and B_1 . To obtain equations in terms of the coefficients A_1 , B_1 and T^- (transmission coefficient), we match (C.68) (expansion of the “transmitted wave” in terms of the local variable α) with (C.72). Therefore, we have four equations and four unknowns. This system of equations leads to the reflection and transmission coefficients R^- and T^- , given by (2.61) and (2.62).

- Right incidence boundary condition. At first, we step match (C.69) with the asymptotic expansion of the approximate solution of the locally valid difference equation, given by (C.71). This matching results in two equations with three unknowns, namely, the transmission coefficient T^+ and the coefficients A_1 and B_1 . To obtain equations in terms of the coefficients A_1 , B_1 and R^+ (reflection coefficient), we match (C.70) with (C.72). Therefore, we have four equations and four unknowns. This system of equations leads to the reflection and transmission coefficients R^+ and T^+ , given by (2.63) and (2.64).

C.2.4 MATCHING PROCESS FOR A PAIR OF ALMOST COALESCING REAL FIRST ORDER TURNING POINTS.

We consider TPP4 (see figure 2-2), with the turning points satisfying the first or second turning point condition (2.30). Besides this, we proceed the same way as in the previous section.

- Left incidence boundary condition. At first, we match (C.67) with (C.71). The matching results in two equations with three unknowns, namely, the reflection coefficient R^- and the coefficients A_1 and B_1 . To obtain equations in terms of the coefficients A_1 , B_1 and T^- , we match (C.68) with (C.72). Therefore, we have four equations and four unknowns. This system of equations leads to the reflection and transmission coefficients R^- and T^- , given by (2.61) and (2.62).
- Right incidence boundary condition. At first, we match (C.69) with (C.71).

The matching results in two equations with three unknowns, namely, the transmission coefficient T^+ and the coefficients A_1 and B_1 . Second, we match (C.70) with (C.72) and obtain two equations in terms of the coefficients A_1 , B_1 and T^+ . Therefore, we have four equations and four unknowns. This system of equations leads to the reflection and transmission coefficients R^+ and T^+ , given by (2.63) and (2.64).

Appendix D

Governing Equation for Quantities $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$ and Elements of the matrix $M(s)$.

Here, we present the governing equations for the dependent variables $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$ derived in section 3.1.3.1. This set of equations follows.

$$\frac{d\tilde{A}}{ds} + \frac{d\tilde{B}}{ds} + \frac{d\tilde{C}}{ds} + \frac{d\tilde{D}}{ds} - i k_1(s)\tilde{A}(s) + i k_1(s)\tilde{B}(s) \quad (\text{D.1})$$

$$-k_2(s)\tilde{C}(s) + k_2(s)\tilde{D}(s) = 0, \quad (\text{D.2})$$

$$\begin{aligned} & i k_1(s) \frac{d\tilde{A}}{ds} - i k_1(s) \frac{d\tilde{B}}{ds} + k_2(s) \frac{d\tilde{C}}{ds} - k_2(s) \frac{d\tilde{D}}{ds} \\ & + \left(i \frac{dk_1}{ds} + k_1^2(s) \right) \tilde{A}(s) + \left(-i \frac{dk_1}{ds} + k_1^2(s) \right) \tilde{B}(s) \\ & + \left(\frac{dk_2}{ds} - k_2^2(s) \right) \tilde{C}(s) - \left(\frac{dk_2}{ds} - k_2^2(s) \right) \tilde{D}(s) = 0, \end{aligned} \quad (\text{D.3})$$

$$\begin{aligned}
& \left([-k_1^2(s) - ik_1(s)(k_2^2(s) - k_1^2(s))] \frac{d\tilde{A}}{ds} + [-k_1^2(s) + ik_1(s)(k_2^2(s) - k_1^2(s))] \frac{d\tilde{B}}{ds} \right. \\
& + [k_2^2(s) - k_2(s)(k_2^2(s) - k_1^2(s))] \frac{d\tilde{C}}{ds} + [k_2^2(s) + k_2(s)(k_2^2(s) - k_1^2(s))] \frac{d\tilde{D}}{ds} \Big) ei(s) \\
& - \left(ei(s) \left[2 k_1(s) \frac{dk_1}{ds} + i(k_2^2(s) - k_1^2(s)) \frac{dk_1}{ds} - i k_1^3(s) \right] + k_1^2(s) \frac{dei}{ds} \right) \tilde{A}(s) \\
& - \left(ei(s) \left[2 k_1(s) \frac{dk_1}{ds} - i(k_2^2(s) - k_1^2(s)) \frac{dk_1}{ds} + i k_1^3(s) \right] + k_1^2(s) \frac{dei}{ds} \right) \tilde{B}(s) \\
& + \left(ei(s) \left[2 k_2(s) \frac{dk_2}{ds} - (k_2^2(s) - k_1^2(s)) \frac{dk_2}{ds} - k_2^3(s) \right] + k_2^2(s) \frac{dei}{ds} \right) \tilde{C}(s) \\
& + \left(ei(s) \left[2 k_2(s) \frac{dk_2}{ds} + (k_2^2(s) - k_1^2(s)) \frac{dk_2}{ds} + k_2^3(s) \right] + k_2^2(s) \frac{dei}{ds} \right) \tilde{D}(s) = 0.
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
& \frac{d}{ds} (-i k_1(s)(k_1^2(s) + ei(s)[k_2^2(s) - k_1^2(s)])) (\tilde{A}(s) - \tilde{B}(s)) \\
& + \frac{d}{ds} (k_2(s)(k_2^2(s) - ei(s)[k_2^2(s) - k_1^2(s)])) (\tilde{C}(s) - \tilde{D}(s)) \\
& + (-i k_1(s)(k_1^2(s) + ei(s)[k_2^2(s) - k_1^2(s)])) \left(\frac{d\tilde{A}}{ds} - \frac{d\tilde{B}}{ds} \right) \\
& + (k_2(s)(k_2^2(s) - ei(s)[k_2^2(s) - k_1^2(s)])) \left(\frac{d\tilde{C}}{ds} - \frac{d\tilde{D}}{ds} \right) = 0
\end{aligned} \tag{D.5}$$

These set of equations can be written in matrix form, according to equation (3.15). The elements of the matrix, denoted as $\mathbf{M}(s)$, of this system of equations are given in terms of the non-dimensional flexural rigidity $ei(s)$, non-dimensional tensile force $\bar{P}(s)$ and in terms of the wavenumbers $k_1(s)$ and $k_2(s)$, which are given, respectively, by equations (3.10) and (3.11). The equations for the elements of matrix $\mathbf{M}(s)$ follows:

$$M_{11}(s) = -\frac{1}{2} \frac{dk_1}{k_1(s)} \frac{ds}{ds} + i k_1(s) - \frac{1}{ei(s)[k_1^2(s) + k_2^2(s)]} \left\{ \frac{d}{ds}(ei(s)k_1^2(s)) + \frac{1}{2} \frac{d\bar{P}}{ds} \right\}, \quad (D.6)$$

$$M_{21}(s) = \frac{1}{2} \frac{dk_1}{k_1(s)} \frac{ds}{ds} + \frac{1}{2} \frac{d\bar{P}}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{ds}{ds}, \quad (D.7)$$

$$M_{31}(s) = \frac{k_1(s)}{2ei k_2(s)[k_1^2(s) + k_2^2(s)]} \left\{ 2 ei(s)(k_2(s) + i k_1(s)) \frac{dk_1}{ds} + k_1(s)(k_2(s) + i k_1(s)) \frac{dei}{ds} + i \frac{d\bar{P}}{ds} \right\}, \quad (D.8)$$

$$M_{41}(s) = M_{31}^*(s), \quad (D.9)$$

$$M_{12}(s) = M_{21}(s), \quad (D.10)$$

$$M_{22}(s) = M_{11}^*(s), \quad (D.11)$$

$$M_{32}(s) = M_{41}(s), \quad (D.12)$$

$$M_{42}(s) = M_{31}(s), \quad (D.13)$$

$$M_{13}(s) = \frac{k_2(s)}{2 ei(s)k_1(s)[k_1^2(s) + k_2^2(s)]} \left\{ 2 ei(s)(k_1(s) - i k_2(s)) \frac{dk_2}{ds} + k_2(s)(k_1(s) - i k_2(s)) \frac{dei}{ds} + i \frac{d\bar{P}}{ds} \right\}, \quad (D.14)$$

$$M_{23}(s) = M_{13}^*(s), \quad (D.15)$$

$$M_{33}(s) = \frac{-1}{2} \frac{dk_2}{k_2(s)} \frac{ds}{ds} + k_2(s) - \frac{1}{ei(s)[k_1^2(s) + k_2^2(s)]} \left\{ \frac{d}{ds}(ei(s)k_2^2(s)) - \frac{1}{2} \frac{d\bar{P}}{ds} \right\}, \quad (D.16)$$

$$M_{43}(s) = \frac{1}{2} \frac{dk_2}{k_2(s)} \frac{ds}{ds} - \frac{1}{2} \frac{d\bar{P}}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{ds}{ds}, \quad (D.17)$$

$$M_{14}(s) = M_{13}^*(s), \quad (D.18)$$

$$M_{24}(s) = M_{13}(s), \quad (D.19)$$

$$M_{34}(s) = M_{43}(s), \quad (\text{D.20})$$

$$M_{44}(s) = \frac{-1}{2} \frac{dk_2}{k_2(s)} \frac{dk_2}{ds} - k_2(s) - \frac{1}{ei(s)[k_1^2(s) + k_2^2(s)]} \left\{ \frac{d}{ds}(ei(s)k_2^2(s)) - \frac{1}{2} \frac{d\bar{P}}{ds} \right\}. \quad (\text{D.21})$$

The symbol $M_{jk}^*(s)$ stands for the complex conjugate of the matrix element $M_{jk}(s)$.

Appendix E

Order of Magnitude of the Elements of Matrix $\mathbf{M}(s)$.

Here, we discuss in detail the order of magnitude of the elements of matrix $\mathbf{M}(s)$ in the system of equations (3.15), which governs the evolution of the wave modes amplitudes along the non-uniform beam. As mentioned in section 3.1.3.2, we consider two regimes. In the first regime, we consider the limit $\theta \rightarrow 0$, where θ is defined in section 3.1.3.2 as the ratio between the restoring force due to the tensile load and the bending moment. In the second regime, we consider the order of magnitude of the restoring force due to the tensile force to be of the same or larger order of magnitude than the order of magnitude of the bending moment ($\theta \geq 1$). The order of magnitude of the terms that compose the elements of matrix $\mathbf{M}(s)$ are given in terms of the non-dimensional quantities $\nu, \theta, \varepsilon, \varepsilon', \delta'$ and λ , which were defined in the first part of section 3.1.3.2.

E.1 Regime $\theta \rightarrow 0$.

First, we need to estimate the order of magnitude of the wavenumbers $k_1(s)$ and $k_2(s)$ and their derivatives, and in terms of the quantities $\nu(s)$ and $\theta(s)$, they are given by the equations

$$k_1(s) = \sqrt{-\frac{\theta(s)}{2} + \frac{1}{2}\sqrt{\theta^2(s) + 4\nu(s)}}, \quad (\text{E.1})$$

$$k_2(s) = \sqrt{\frac{\theta(s)}{2} + \frac{1}{2}\sqrt{\theta^2(s) + 4\nu(s)}}, \quad (\text{E.2})$$

$$\frac{dk_1}{ds} = \frac{1}{k_1(s)} \left\{ -\frac{1}{2} \frac{d\theta}{ds} + \frac{1}{4\sqrt{\theta^2(s) + 4\nu(s)}} \left[2\theta(s) \frac{d\theta}{ds} + 4 \frac{d\nu}{ds} \right] \right\}, \quad (\text{E.3})$$

$$\frac{dk_2}{ds} = \frac{1}{k_2(s)} \left\{ \frac{1}{2} \frac{d\theta}{ds} + \frac{1}{4\sqrt{\theta^2(s) + 4\nu(s)}} \left[2\theta(s) \frac{d\theta}{ds} + 4 \frac{d\nu}{ds} \right] \right\}, \quad (\text{E.4})$$

Then, we discuss the order of magnitude of the elements of the matrix $\mathbf{M}(s)$. As noted in section 3.1.3.2, we need only to discuss the order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$. The equations for these seven elements is given in Appendix D.

In this regime, according to equations (E.1) and (E.2), the order of magnitude of the wavenumbers are given as:

$$k_1(s) \sim k_2(s) \sim O(\nu^{1/4}) \quad (\text{E.5})$$

and

$$k_1^2(s) + k_2^2(s) \sim O(\nu^{1/2}). \quad (\text{E.6})$$

To estimate the order of magnitude of the derivatives of the wavenumbers, we need the order of magnitude of the derivative of the quantities $\nu(s)$ and $\theta(s)$. According to equations (3.18) and (3.19), we have:

$$\frac{d\nu}{ds} \sim \underbrace{\frac{\nu(s)}{m(s)} \frac{dm}{ds}}_{O(\nu\epsilon\Lambda)} - \underbrace{\frac{\nu(s)}{ei(s)} \frac{dei}{ds}}_{O(\nu\epsilon'\Lambda)} \sim O(\max\{\nu\epsilon\Lambda, \nu\epsilon'\Lambda\}), \quad (\text{E.7})$$

$$\frac{d\theta}{ds} \sim \underbrace{\frac{1}{ei(s)} \frac{d\bar{P}}{ds}}_{O(\delta\Lambda)} - \underbrace{\frac{\theta(s)}{ei(s)} \frac{dei}{ds}}_{O(\theta\epsilon'\Lambda)} \sim O(\max\{\delta\Lambda, \theta\epsilon'\Lambda\}). \quad (\text{E.8})$$

According to equations (E.3), (E.4) and the estimates above, the estimates for the order of magnitude of the derivatives of the wavenumbers are given by the equation

$$\frac{dk_1}{ds} \sim \frac{dk_2}{ds} \sim O(\max\{\nu^{1/4}\epsilon\Lambda, \nu^{1/4}\epsilon'\Lambda, \nu^{-1/4}\delta\Lambda\}). \quad (\text{E.9})$$

The analysis of the order of magnitude of the basic elements of matrix $\mathbf{M}(s)$ follows:

$$M_{11}(s) = - \underbrace{\frac{1}{2 k_1(s)} \frac{dk_1}{ds}}_{O(\max\{\epsilon\Lambda, \epsilon'\Lambda, \nu^{-1/2}\delta\Lambda\})} + i \underbrace{k_1(s)}_{O(\nu^{1/4})} - \underbrace{\frac{2 k_1(s)}{[k_1^2(s) + k_2^2(s)]} \frac{dk_1}{ds}}_{O(\max\{\epsilon\Lambda, \epsilon'\Lambda, \nu^{-1/2}\delta\Lambda\})} \\ - \underbrace{\frac{k_1^2(s)}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\epsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\nu^{-1/2}\delta\Lambda)}, \quad (\text{E.10})$$

$$M_{21}(s) = \underbrace{\frac{1}{2 k_1(s)} \frac{dk_1}{ds}}_{O(\max\{\epsilon\Lambda, \epsilon'\Lambda, \nu^{-1/2}\delta\Lambda\})} + \underbrace{\frac{1}{2 ei(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\nu^{-1/2}\delta\Lambda)}, \quad (\text{E.11})$$

$$M_{31}(s) = \underbrace{\frac{k_1(s)(k_2(s) + i k_1(s))}{k_2[k_1^2(s) + k_2^2(s)]} \frac{dk_1}{ds}}_{O(\max\{\epsilon'\Lambda, \epsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} + \underbrace{\frac{k_1(s)(k_1(s)k_2(s) + i k_1^2(s))}{2ei(s)k_2[k_1^2 + k_2^2]} \frac{dei}{ds}}_{O(\nu^{-1/4}\epsilon'\Lambda)} \\ + \underbrace{\frac{i}{2} \frac{k_1(s)}{ei(s)k_2(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\nu^{-1/2}\delta\Lambda)}, \quad (\text{E.12})$$

$$\begin{aligned}
M_{13}(s) = & \underbrace{\frac{k_2(s)(k_1(s) - i k_2(s))}{k_1(s)[k_1^2(s) + k_2^2(s)]} \frac{dk_2}{ds}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/4}\delta\Lambda\})} + \underbrace{\frac{k_2(s)(k_1(s)k_2(s) - i k_2^2(s))}{ei(s)k_1(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\varepsilon'\Lambda)} \\
& + \underbrace{\frac{i}{2} \frac{k_2(s)}{ei(s)k_1(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\nu^{-1/2}\delta\Lambda)}(s), \tag{E.13}
\end{aligned}$$

$$\begin{aligned}
M_{33}(s) = & \underbrace{\frac{-1}{2} \frac{dk_2}{k_2(s) ds}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} + \underbrace{k_2(s)}_{O(\nu^{1/4})} - \underbrace{\frac{k_2(s)}{[k_1^2(s) + k_2^2(s)]} \frac{dk_2}{ds}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} \\
& - \underbrace{\frac{k_2^2(s)}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\varepsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\nu^{-1/2}\delta\Lambda)} \tag{E.14}
\end{aligned}$$

$$\begin{aligned}
M_{43}(s) = & \underbrace{\frac{1}{2} \frac{dk_2}{k_2(s) ds}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} + \underbrace{\frac{1}{2} \frac{d\bar{P}}{ei(s)[k_1^2(s) + k_2^2(s)] ds}}_{O(\nu^{-1/2}\delta\Lambda)}, \tag{E.15}
\end{aligned}$$

$$\begin{aligned}
M_{44}(s) = & \underbrace{\frac{-1}{2} \frac{dk_2}{k_2(s) ds}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} - \underbrace{k_2(s)}_{O(\nu^{1/4})} - \underbrace{\frac{k_2(s)}{[k_1^2(s) + k_2^2(s)]} \frac{dk_2}{ds}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} \\
& - \underbrace{\frac{k_2^2(s)}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\varepsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\nu^{-1/2}\delta\Lambda)}. \tag{E.16}
\end{aligned}$$

The order of magnitude of one element $M_{jk}(s)$ above is given by the order of magnitude of the term with largest order of magnitude among the components of the element in question, which result in equations (3.29) to (3.35).

E.2 Regime $\theta \geq 1$.

We proceed in the same way as in the section above. First, we estimate the order of magnitude of the wavenumbers $k_1(s)$ and $k_2(s)$ and their derivatives. Then, we analyze the order of magnitude of the basic elements of matrix $\mathbf{M}(s)$.

In this regime, according to equations (E.1) and (E.2), the order of magnitude of the wavenumbers is given by:

$$k_1(s) \sim O(\nu^{1/4}), \quad (\text{E.17})$$

$$k_2(s) \sim O(\theta^{1/2}), \quad (\text{E.18})$$

and

$$k_1^2(s) + k_2^2(s) \sim O(\theta). \quad (\text{E.19})$$

To estimate the order of magnitude of the derivatives of the wavenumbers, we need the order of magnitude of the derivative of the quantities $\nu(s)$ and $\theta(s)$. The order of magnitude of these quantities are exactly the same obtained in the previous section, and given by equations (E.7) and (E.8).

According to equations (E.3), (E.4) and the estimates for the derivative of the quantities $\nu(s)$ and $\theta(s)$, the estimates for the order of magnitude of the derivatives of the wavenumbers are given by the equations:

$$\frac{dk_1}{ds} \sim O(\max\{\nu^{-1/4}\delta\Lambda, \nu^{-1/4}\theta\epsilon'\Lambda, \theta^{-1}\nu^{3/4}\epsilon\Lambda\}), \quad (\text{E.20})$$

$$\frac{dk_2}{ds} \sim O(\max\{\theta^{-1/2}\delta\Lambda, \theta^{1/2}\epsilon'\Lambda, \theta^{-3/2}\nu\epsilon\Lambda\}). \quad (\text{E.21})$$

The analysis of the order of magnitude of the seven elements $M_{11}(s), M_{21}(s), M_{31}(s), M_{13}(s), M_{33}(s), M_{43}(s)$ and $M_{44}(s)$ follows:

$$\begin{aligned} M_{11}(s) = & - \underbrace{\frac{1}{2} \frac{dk_1}{k_1(s)} \frac{dk_1}{ds}}_{O(\max\{\nu^{-1/2}\delta\Lambda, \nu^{-1/2}\theta\epsilon'\Lambda, \theta^{-1}\nu^{1/2}\epsilon\Lambda\})} + i \underbrace{\frac{k_1(s)}{O(\nu^{1/4})}}_{O(\nu^{1/4})} - \underbrace{\frac{2 k_1(s)}{[k_1^2(s) + k_2^2(s)]} \frac{dk_1}{ds}}_{O(\max\{\theta^{-1}\delta\Lambda, \epsilon'\Lambda, \theta^{-2}\nu\epsilon\Lambda\})} \\ & - \underbrace{\frac{k_1^2(s)}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\theta\nu^{1/2}\epsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\theta^{-1}\delta\Lambda)}, \end{aligned} \quad (\text{E.22})$$

$$M_{21}(s) = \underbrace{\frac{1}{2} \frac{dk_1(s)}{k_1(s)} \frac{ds}{ds}}_{O(\max\{\nu^{-1/2}\delta\Lambda, \nu^{-1/2}\theta\epsilon'\Lambda, \theta^{-1}\nu^{1/2}\epsilon\Lambda\})} + \underbrace{\frac{1}{2} \frac{d\bar{P}}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{ds}{ds}}_{O(\theta^{-1}\delta\Lambda)}, \quad (\text{E.23})$$

$$M_{31}(s) = \underbrace{\frac{k_1(s)(k_2(s) + i k_1(s))}{k_2(s)[k_1^2(s) + k_2^2(s)]} \frac{dk_1}{ds}}_{O(\max\{\theta^{-1}\delta\Lambda, \epsilon'\Lambda, \theta^{-2}\nu\epsilon\Lambda\})} + \underbrace{\frac{k_1(s)(k_1(s)k_2(s) + i k_1^2(s))}{2 ei(s)k_2(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\theta^{-1}\nu^{1/4}\epsilon'\Lambda)} + \underbrace{\frac{i}{2} \frac{k_1(s)}{2 ei(s)k_2(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\nu^{1/4}\delta\Lambda\theta^{-3/2})}(s), \quad (\text{E.24})$$

$$M_{13}(s) = \underbrace{\frac{k_2(s)(k_1(s) - i k_2(s))}{k_1(s)[k_1^2(s) + k_2^2(s)]} \frac{dk_2}{ds}}_{O(\max\{\nu^{-1/4}\theta^{-1/2}\delta\Lambda, \nu^{-1/4}\theta^{1/2}\epsilon'\Lambda, \nu^{3/4}\theta^{-3/2}\epsilon\Lambda\})} + \underbrace{\frac{k_2(s)(k_1(s)k_2(s) - i k_2^2(s))}{ei(s)k_1(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\theta^{1/2}\nu^{-1/4}\epsilon'\Lambda)} + \underbrace{i \frac{k_2(s)}{2 ei(s)k_1(s)[k_1^2(s) + k_2^2(s)]} \frac{d\bar{P}}{ds}}_{O(\theta^{-1/2}\nu^{-1/4}\delta\Lambda)}, \quad (\text{E.25})$$

$$M_{33}(s) = \underbrace{\frac{-1}{2} \frac{dk_2}{k_2(s)} \frac{ds}{ds}}_{O(\max\{\theta^{-1}\delta\Lambda, \epsilon'\Lambda, \theta^{-2}\nu\epsilon\Lambda\})} + \underbrace{\frac{k_2(s)}{O(\theta^{1/2})}}_{O(\theta^{1/2})} - \underbrace{\frac{k_2(s)}{[k_1^2(s) + k_2^2(s)]} \frac{dk_2}{ds}}_{O(\max\{\theta^{-1}\delta\Lambda, \epsilon'\Lambda, \theta^{-2}\nu\epsilon\Lambda\})} - \underbrace{\frac{k_2^2(s)}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\epsilon'\Lambda)} + \underbrace{\frac{1}{2} \frac{d\bar{P}}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{ds}{ds}}_{O(\theta^{-1}\delta\Lambda)}, \quad (\text{E.26})$$

$$M_{43}(s) = \underbrace{\frac{1}{2} \frac{dk_2}{k_2(s)} \frac{ds}{ds}}_{O(\max\{\theta^{-1}\delta\Lambda, \epsilon'\Lambda, \theta^{-2}\nu\epsilon\Lambda\})} + \underbrace{\frac{1}{2} \frac{d\bar{P}}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{ds}{ds}}_{O(\theta^{-1}\delta\Lambda)}, \quad (\text{E.27})$$

$$M_{44}(s) = \underbrace{\frac{-1}{2} \frac{dk_2}{k_2(s)} \frac{ds}{ds}}_{O(\max\{\theta^{-1}\delta\Lambda, \epsilon'\Lambda, \theta^{-2}\nu\epsilon\Lambda\})} - \underbrace{\frac{k_2(s)}{O(\theta^{1/2})}}_{O(\theta^{1/2})} - \underbrace{\frac{k_2(s)}{[k_1^2(s) + k_2^2(s)]} \frac{dk_2}{ds}}_{O(\max\{\theta^{-1}\delta\Lambda, \epsilon'\Lambda, \theta^{-2}\nu\epsilon\Lambda\})} - \underbrace{\frac{k_2^2(s)}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{dei}{ds}}_{O(\epsilon'\Lambda)} + \underbrace{\frac{1}{2} \frac{d\bar{P}}{ei(s)[k_1^2(s) + k_2^2(s)]} \frac{ds}{ds}}_{O(\theta^{-1}\delta\Lambda)}. \quad (\text{E.28})$$

The order of magnitude of the elements $M_{jk}(s)$ above is given by the order of magnitude of the component with largest order of magnitude among the components of the element in question, which result in equations (3.37) to (3.43).

Appendix F

Decoupling of the Propagating and Evanescent Part of the Wavefield.

Here, we show that under the restriction of the tensile force and the product of the mass per unit length by the flexural rigidity being constants, the matrix elements $M_{jk}(s)$ and $M_{kj}(s)$ ($j = 1, 2$ and $k = 3, 4$) are zero.

Under the restrictions mentioned above, the wavenumbers $k_1(s)$ and $k_2(s)$ are given by the equations

$$k_1(s) = \frac{1}{ei(s)^{1/2}} \left\{ -\frac{\bar{P}}{2} + \frac{1}{2} \sqrt{\bar{P}^2 + 4\omega^2 C} \right\}^{1/2}, \quad (\text{F.1})$$

$$k_2(s) = \frac{1}{ei(s)^{1/2}} \left\{ \frac{\bar{P}}{2} + \frac{1}{2} \sqrt{\bar{P}^2 + 4\omega^2 C} \right\}^{1/2}. \quad (\text{F.2})$$

The derivatives of the wavenumbers can be obtained from equations (F.1) and (F.2).

C is the value of $m(s)ei(s)$ when this product is restricted to a constant value. In other words,

$$m(s) ei(s) = C. \quad (\text{F.3})$$

Based on the expressions above for the wavenumbers $k_1(s)$ and $k_2(s)$, we realize that

$$2 \operatorname{ei}(s)(k_2(s) + i k_1(s)) \frac{dk_1}{ds} + (k_1(s)k_2(s) + i k_1^2(s)) \frac{dei}{ds} = 0, \quad (\text{F.4})$$

$$2 \operatorname{ei}(s)(k_2(s) - i k_1(s)) \frac{dk_1}{ds} + (k_1(s)k_2(s) - i k_1^2(s)) \frac{dei}{ds} = 0. \quad (\text{F.5})$$

Equations (F.4) and (F.5) plus the restriction of constant non-dimensional tensile force imply that the elements $M_{jk}(s)$ and $M_{kj}(s)$ ($j = 1, 2$ and $k = 3, 4$) are zero, as we can see through their expressions in appendix E.

In this case, the second order governing equation (3.49) assumes the Helmholtz-like form (3.65).

Appendix G

Second Order Equation with Analytic Solution.

The second order governing equation assumes the form given by equation (3.65). The index of refraction $\tilde{n}(s)$ is given by

$$\tilde{n}(s) = \left\{ S - N \frac{\exp(k_g \lambda(s + s_0))}{1 + \exp(k_g \lambda(s + s_0))} - 4M \frac{\exp(k_g \lambda(s + s_0))}{(1 + \exp(k_g \lambda(s + s_0)))^2} \right\}^{1/2}. \quad (\text{G.1})$$

The solution of the second order governing equation (3.65) for this index of refraction is given in terms of hyper-geometric functions in the form

$$\chi(s) = \frac{1}{\sqrt{k_g \lambda \exp(k_g \lambda(s - s_0))}} (1 + \exp(k_g \lambda(s - s_0)))^{\delta + \beta - \gamma + 1} 2(D F_5 + B F_6), \quad (\text{G.2})$$

with F_5 and F_6 as the hyper-geometric functions:

$$F_5 = z^{-\delta} F(\delta, \delta - \gamma + 1, \delta - \beta + 1; 1/z), \quad (\text{G.3})$$

$$F_6 = z^{-\beta} F(\beta, \beta - \gamma + 1, \beta - \delta + 1; 1/z). \quad (\text{G.4})$$

The constants δ, β and γ are given in terms of the constants $\tilde{\Omega}(\omega, \bar{P}, 1)$, $k_g \lambda$, S , N and M according to the equations (3.81), (3.82) and (3.83).

The hyper-geometric functions $F_5(s)$ and $F_6(s)$ (see Brekhovskikh [10], pages 54 and 55) were used in the solution (G.2) due to the chosen radiation conditions of left wave incidence and due to the sign of the parameter k_g , which is positive according to the caption of figure 3-5. The reflection and transmission coefficients are given in terms of Gamma functions, according to equations (3.79) and (3.80).

Appendix H

Reflection and Transmission Coefficients for The Bernoulli-Euler Beam.

We consider an Euler-Bernoulli beam of three pieces. At each piece, the material and geometrical properties are constants. There is a finite piece bounded by two semi-infinite pieces with the same material and geometrical properties. The material and geometrical properties for the finite piece differs from the two semi-infinite pieces. The governing equation for wave propagation along the beam is given by equation (3.6). The non-dimensional functions representing the mass per unit length and the flexural rigidity are, respectively, the functions $m(s)$ and $ei(s)$, given in this case by

$$m(s) = \begin{cases} 1 & \text{for } s < 0 \text{ and } s > \bar{L}, \\ m_1 & \text{for } 0 < s < \bar{L}, \end{cases} \quad (\text{H.1})$$

$$ei(s) = \begin{cases} 1 & \text{for } s < 0 \text{ and } s > \bar{L}, \\ ei_1 & \text{for } 0 < s < \bar{L}. \end{cases} \quad (\text{H.2})$$

The general solution for the governing equation (3.6) is given by

$$y(s) = \begin{cases} A_1 e^{ik_1 s} + B_1 e^{-ik_1 s} + C_1 e^{k_2 s} + D_1 e^{-k_2 s} & \text{for } s < 0, \\ A_2 e^{ik_3 s} + B_2 e^{-ik_3 s} + C_2 e^{k_4 s} + D_2 e^{-k_4 s} & \text{for } 0 \leq s \leq \bar{L}, \\ A_3 e^{ik_1(s-\bar{L})} + B_3 e^{-ik_1(s-\bar{L})} + C_3 e^{k_2(s-\bar{L})} + D_3 e^{-k_2(s-\bar{L})} & \text{for } s > \bar{L}. \end{cases} \quad (\text{H.3})$$

The wave numbers k_1 and k_3 are given by equation (3.10), and the wave numbers k_2 and k_4 are given by the equation (3.11). $ei(s)$ and $m(s)$ in the equations (3.10) and (3.11) are substituted by their values given, respectively, by equations (H.1) and (H.2). At the discontinuity of the coefficients of the governing equation (3.6), we have the matching conditions

$$y(\eta^+) = y(\eta^-), \quad (\text{H.4})$$

$$y_s(\eta^+) = y_s(\eta^-), \quad (\text{H.5})$$

$$ei(\eta^+) y_{ss}(\eta^+) = ei(\eta^-) y_{ss}(\eta^-), \quad (\text{H.6})$$

$$-\bar{P} y_s(\eta^+) + ei(\eta^+) y_{sss}(\eta^+) = -\bar{P} y_s(\eta^-) + ei(\eta^-) y_{sss}(\eta^-), \quad (\text{H.7})$$

where $\eta = 0$ and $\eta = \bar{L}$. If we substitute the general solution given by equation (H.3) in the matching conditions (H.4)-(H.7) for $\eta = 0$ and $\eta = \bar{L}$, we obtain the following set of systems of equations for the wave modes amplitudes along the beam:

$$[\mathbf{M}(k_1, k_2, ei(0^-))] \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = [\mathbf{M}(k_3, k_4, ei(0^+))][\mathbf{R}(k_3, k_4, 0)] \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} \text{ at } \eta = 0, \quad (\text{H.8})$$

$$[\mathbf{M}(k_3, k_4, ei(\bar{L}^-))][\mathbf{R}(k_3, k_4, \bar{L})] \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} = [\mathbf{M}(k_1, k_2, ei(\bar{L}^+))] \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{Bmatrix} \text{ at } \eta = \bar{L}. \quad (\text{H.9})$$

The matrices $\mathbf{M}(k_a, k_b, ei)$ and $\mathbf{R}(k_a, k_b, s)$ are given by the equations

$$[\mathbf{M}(k_a, k_b, ei(s))] = \begin{bmatrix} 1 & 1 & 1 \\ ik_a & -ik_a & k_b \\ ei(s)(ik_a)^2 & ei(s)(-ik_a)^2 & ei(s)(k_b)^2 \\ -i\bar{P}k_a + ei(s)(ik_a)^3 & i\bar{P}k_a + ei(s)(-ik_a)^3 & -\bar{P}k_b + ei(s)(k_b)^3 \\ 1 \\ -k_b \\ ei(s)(-k_b)^2 \\ \bar{P}k_b + ei(s)(-k_b)^3 \end{bmatrix} \quad (\text{H.10})$$

$$[\mathbf{R}(k_a, k_b, s)] = \begin{bmatrix} \exp(ik_a s) & 0 & 0 & 0 \\ 0 & \exp(-ik_a s) & 0 & 0 \\ 0 & 0 & \exp(k_b s) & 0 \\ 0 & 0 & 0 & \exp(k_b s) \end{bmatrix} \quad (\text{H.11})$$

Next, let us consider the radiation condition of left incidence, which is given by

$$\begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ R^- \\ C^- \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} T^- \\ 0 \\ 0 \\ D^- \end{Bmatrix}. \quad (\text{H.12})$$

If we impose the radiation conditions (H.12) to the system of equations (H.8) and (H.9), we obtain the reflection coefficient R^- and the transmission coefficient T^- by

solving numerically the resulting system of equations, which follows:

$$\begin{aligned} \begin{Bmatrix} 1 \\ R^- \\ C^- \\ 0 \end{Bmatrix} &= [\mathbf{M}(k_1, k_2, ei(0^-))]^{-1} [\mathbf{M}(k_3, k_4, ei(0^+)) [\mathbf{R}(k_3, k_4, 0)] \\ &\quad [\mathbf{R}(k_3, k_4, \bar{L})]^{-1} [\mathbf{M}(k_3, k_4, ei(\bar{L}^-))]^{-1} [\mathbf{M}(k_1, k_2, ei(\bar{L}^+))] \begin{Bmatrix} T^- \\ 0 \\ 0 \\ D^- \end{Bmatrix}. \end{aligned} \quad (\text{H.13})$$

Appendix I

Outline of the Steepest Descent Method.

To apply the steepest descent method we deform the integration contours in terms of the steepest descent paths. We close a contour which is the union of the original integration contour with the appropriate steepest descent paths and apply Cauchy's integral theorem, and if there are no singularities inside the closed contour, we obtain the original contour integral as a linear combination of integrals over the steepest descent paths. If there is any singularity inside the closed contour, the original contour integral is given as a linear combination of integrals over the steepest descent paths plus $2\pi i$ times the sum of the residue of the singularities inside the closed contour. We then apply Watson's Lemma to the integrals over the steepest descent paths to obtain the asymptotic expansion of these integrals. As a result, the asymptotic expansion of the original contour integral is given as a linear combination of the asymptotic expansion of the integrals over the steepest descent paths plus $2\pi i$ times the sum of the residue of the singularities inside the closed contour mentioned above.

Details on how to perform the asymptotic expansion of the integrals over the steepest descent paths using Watson's Lemma are discussed in the references Olver [51], Bleistein & Handelsman [6] and Copson [16].

I.1 Asymptotic Expansion of the Functions $H_l(\delta, \lambda, \varepsilon, \alpha)$.

In this section we give the asymptotic expansion of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$. They are defined as integrals over the steepest descent paths of the function $f(t, \delta, \varepsilon, \alpha)$ on the complex t plane. The function $f(t, \delta, \varepsilon, \alpha)$ is given by equation (3.194).

We applied Watson's Lemma to the integrals over the steepest descent paths to obtain the asymptotic expansion of the functions $H_l(\delta, \lambda, \varepsilon, \alpha)$, which are given below.

$$H_1(\delta, \lambda, \varepsilon, \alpha) \sim -\exp\left(\frac{1}{|\lambda|}f(t_1, \delta, \varepsilon, \alpha) - i\frac{\theta''}{4} + i\frac{\theta}{4} - i\frac{\theta'}{4} + i\frac{\beta}{2}\right) \sqrt{\frac{\pi\lambda}{2\varepsilon}} \left(\frac{\rho}{\rho'\rho''}\right)^{1/4} \left\{1 + \sum_{n=1}^{\infty} D_{n1}(\delta)\lambda^n\right\} \quad (\text{I.1})$$

$$H_2(\delta, \lambda, \varepsilon, \alpha) \sim -\exp\left(\frac{1}{|\lambda|}f(t_2, \delta, \varepsilon, \alpha) - i\frac{\theta''}{4} + i\frac{\theta}{4} - i\frac{\theta'}{4} + i\frac{\beta}{2} - i\frac{\pi}{2}\right) \sqrt{\frac{\pi\lambda}{2\varepsilon}} \left(\frac{\rho}{\rho'\rho''}\right)^{1/4} \left\{1 + \sum_{n=1}^{\infty} D_{n2}(\delta)\lambda^n\right\} \quad (\text{I.2})$$

$$H_3(\delta, \lambda, \varepsilon, \alpha) \sim -\exp\left(\frac{1}{|\lambda|}f(t_3, \delta, \varepsilon, \alpha) - i\frac{\theta'''}{4} + i\frac{\theta}{4} - i\frac{\theta'}{4} + i\frac{\beta}{2} - i\frac{\pi}{2}\right) \sqrt{\frac{\pi\lambda}{2\varepsilon}} \left(\frac{\rho}{\rho'\rho'''}\right)^{1/4} \left\{1 + \sum_{n=1}^{\infty} D_{n3}(\delta)\lambda^n\right\} \quad (\text{I.3})$$

$$H_4(\delta, \lambda, \varepsilon, \alpha) \sim -\exp\left(\frac{1}{|\lambda|}f(t_4, \delta, \varepsilon, \alpha) - i\frac{\theta''}{4} + i\frac{\theta}{4} - i\frac{\theta'}{4} + i\frac{\beta}{2}\right) \sqrt{\frac{\pi\lambda}{2\varepsilon}} \left(\frac{\rho}{\rho'\rho'''}\right)^{1/4} \left\{1 + \sum_{n=1}^{\infty} D_{n4}(\delta)\lambda^n\right\} \quad (\text{I.4})$$

where t_j is a saddle point of the function $f(t, \delta, \varepsilon, \alpha)$ and the polar coordinate systems (ρ, θ) , (ρ', θ') , (ρ'', θ'') and (ρ''', θ''') were defined in the second part of chapter 3. The coefficients D_{nj} up to fourth order are given below.

- Coefficients $D_{1j}(\delta)$ follows:

$$D_{1j}(\delta) = \frac{(-1)^j \sqrt{2} e^{i\beta}}{96 \varepsilon^3} \left\{ \frac{\sqrt{\alpha - \sqrt{\alpha^2 + 4\delta}}}{\sqrt{\delta}(\alpha^2 + 4\delta)^2} (52\delta\sqrt{\alpha^2 + 4\delta} + 2\alpha^3 + 8\alpha\delta - 7\alpha^2\sqrt{\alpha^2 + 4\delta}) \right\}$$

for $j = 1, 2$ (I.5)

$$D_{1j}(\delta) = \frac{(-1)^j \sqrt{2} e^{i\beta}}{96 \varepsilon^3} \left\{ \frac{\sqrt{\alpha + \sqrt{\alpha^2 + 4\delta}}}{\sqrt{\delta}(\alpha^2 + 4\delta)^2} (-52\delta\sqrt{\alpha^2 + 4\delta} + 2\alpha^3 + 8\alpha\delta + 7\alpha^2\sqrt{\alpha^2 + 4\delta}) \right\}$$

for $j = 3, 4$ (I.6)

- Coefficients $D_{2j}(\delta)$ follows:

$$D_{2j}(\delta) = -\frac{5}{9216} \frac{e^{i2\beta}}{\varepsilon^6} \left\{ \frac{-11664\alpha\delta^2 + 3536\delta^2\sqrt{\alpha^2 + 4\delta} + 2120\alpha^3\delta + 27\alpha^5}{\delta(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right. \\ \left. + \frac{-1912\delta\alpha^2\sqrt{\alpha^2 + 4\delta} - 27\alpha^4\sqrt{\alpha^2 + 4\delta}}{\delta(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right\} \text{ for } j = 1, 2 \quad (\text{I.7})$$

$$D_{2j}(\delta) = -\frac{5}{9216} \frac{e^{i2\beta}}{\varepsilon^6} \left\{ \frac{-11664\alpha\delta^2 - 3536\delta^2\sqrt{\alpha^2 + 4\delta} + 2120\alpha^3\delta + 27\alpha^5}{\delta(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right. \\ \left. + \frac{1912\delta\alpha^2\sqrt{\alpha^2 + 4\delta} + 27\alpha^4\sqrt{\alpha^2 + 4\delta}}{\delta(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right\} \text{ for } j = 3, 4 \quad (\text{I.8})$$

- Coefficients $D_{3j}(\delta)$ follows:

$$D_{3j}(\delta) = \frac{(-1)^j 35 \sqrt{2} e^{i3\beta}}{2654208 \varepsilon^9} \left\{ \frac{\sqrt{\alpha - \sqrt{\alpha^2 + 4\delta}}}{\delta^{3/2}(\alpha^2 + 4\delta)^2} \frac{(-243\alpha^7\sqrt{\alpha^2 + 4\delta} + 3935552\delta^3\alpha\sqrt{\alpha^2 + 4\delta})}{(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right. \\ + \frac{(-1477264\alpha^3\delta^2\sqrt{\alpha^2 + 4\delta} + 6076\alpha^5\delta\sqrt{\alpha^2 + 4\delta} + 45376\alpha^4\delta^2 - 728\alpha^6\delta + 243\alpha^8)}{(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \\ \left. + \frac{(-165632\delta^4 + 167296\alpha^2\delta^3)}{(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right\} \text{ for } j = 1, 2 \quad (\text{I.9})$$

$$D_{3j}(\delta) = \frac{(-1)^j 35 \sqrt{2} e^{i3\beta}}{2654208 \varepsilon^9} \left\{ \frac{\sqrt{\alpha + \sqrt{\alpha^2 + 4\delta}}}{\delta^{3/2}(\alpha^2 + 4\delta)^2} \frac{(243\alpha^7 \sqrt{\alpha^2 + 4\delta} - 3935552\delta^3 \alpha \sqrt{\alpha^2 + 4\delta})}{(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right. \\ \left. \frac{(+1477264\alpha^3\delta^2 \sqrt{\alpha^2 + 4\delta} - 6076\alpha^5\delta \sqrt{\alpha^2 + 4\delta} + 45376\alpha^4\delta^2 - 728\alpha^6\delta + 243\alpha^8)}{(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right. \\ \left. \frac{(-165632\delta^4 + 167296\alpha^2\delta^3)}{(\alpha^6 + 12\alpha^4\delta + 48\alpha^2\delta^2 + 64\delta^3)} \right\} \text{ for } j = 3, 4 \quad (\text{I.10})$$

• Coefficients $D_{4j}(\delta)$ follows:

$$D_{4j}(\delta) = -\frac{35}{254803968} \frac{e^{i4\beta}}{\varepsilon^{12}} \frac{1}{F_{14}(\delta)} \left\{ F_{24}(\delta) + F_{34}(\delta) \sqrt{\alpha^2 + 4\delta} \right\} \text{ for } j = 1, 2 \quad (\text{I.11})$$

$$D_{4j}(\delta) = -\frac{35}{254803968} \frac{e^{i4\beta}}{\varepsilon^{12}} \frac{1}{F_{14}(\delta)} \left\{ F_{24}(\delta) - F_{34}(\delta) \sqrt{\alpha^2 + 4\delta} \right\} \text{ for } j = 3, 4 \quad (\text{I.12})$$

where

$$F_{14}(\delta) = \delta^2(\alpha^{12} + 24\alpha^{10}\delta + 240\alpha^8\delta^2 + 1280\alpha^6\delta^3 + 3840\alpha^4\delta^4 + 6144\alpha^2\delta^5 + 4096\delta^6) \quad (\text{I.13})$$

$$F_{24}(\delta) = 485514\alpha^8\delta - 133189952\alpha^6\delta^2 + 4587092928\alpha^4\delta^3 - 13584880896\delta^4\alpha^2 \\ + 32805\alpha^{10} + 2896079360\delta^5 \quad (\text{I.14})$$

$$F_{34}(\delta) = -419904\alpha^7\delta - 32805\alpha^9 + 4368436992\alpha\delta^4 - 2916477440\alpha^3\delta^3 + 131839456\alpha^5\delta^2 \quad (\text{I.15})$$

Appendix J

Recurrence Relations for the Symbols $A_m^n(x, p)$ and $B_m^n(x, p)$.

Here we give the recurrence relation for the symbols $A_m^n(x, p)$ and $B_m^n(x, p)$ with $m = 0, \dots, n$ and for $n = 1, 2, \dots$. The boundary value problems for the symbols $R_n(x, y, p)$ with $n \geq 1$ are non-homogeneous. So, we assume that the symbols $R_n(x, y, p)$ can be represented in the form

$$R_n(x, y, p) = A(x, y, p) \cosh(|p|(y + h(x))) + B(x, y, p) \sinh(|p|(y + h(x))) + \tilde{R}_n(x, p) \quad (\text{J.1})$$

where the symbol $\tilde{R}(x, y, p)$ satisfies the non-homogeneous differential equation (4.46). This symbol has the form

$$\tilde{R}_n(x, y, p) = \sum_{m=0}^n \left\{ \tilde{A}_m^n y^m \cosh(|p|(y + h(x))) + \tilde{B}_m^n y^m \sinh(|p|(y + h(x))) \right\} \quad (\text{J.2})$$

After we solve the non-homogeneous differential equation (4.46) in the y variable, and obtain the symbol $\tilde{R}(x, y, p)$, we substitute the equation (J.1) into the boundary conditions (4.47) and (4.48). This gives the expression for the symbols $A(x, y, p)$ and

$B(x, y, p)$. Next, we compare the resultant expression for the symbol $R_n(x, y, p)$, given by equation (J.1), with the equation (4.51). This comparison gives the expression for the symbols $A_m^n(x, p)$ and $B_m^n(x, p)$ with $m = 0, \dots, n$. For $n = 0$ we have the symbol $A_0^0(x, p)$ given by equation (4.50) and the symbol $B_0^0(x, p) = 0$. For $n = 1$, the particular solution of the non-homogeneous differential equation (4.43) gives

$$\tilde{A}_1^1(x, p) = -p \frac{dH}{dx} A_0^0(x, p) \quad (\text{J.3})$$

$$\tilde{B}_1^1(x, p) = \frac{\partial A_0^0}{\partial x}(x, p) \quad (\text{J.4})$$

$$\tilde{A}_0^1(x, p) = -\frac{1}{2p} \frac{\partial A_0^0}{\partial x}(x, p) \quad (\text{J.5})$$

$$\tilde{B}_0^1(x, p) = \frac{1}{2} \frac{dH}{dx} A_0^0(x, p) \quad (\text{J.6})$$

The symbol ∇ stands here for $\frac{\partial}{\partial x}$. After we obtain the symbols $A(x, y, p)$ and $B(x, y, p)$ for the case $n = 1$, the comparison of the expressions for the symbol $R_1(x, y, p)$ gives

$$A_1^1(x, p) = \tilde{A}_1^1(x, p) \quad (\text{J.7})$$

$$B_1^1(x, p) = \tilde{B}_1^1(x, p) \quad (\text{J.8})$$

$$A_0^1(x, p) = \frac{\tanh(ph(x))}{p} \tilde{A}_1^1(x, p) - h(x) \tanh(ph(x)) \tilde{B}_1^1(x, p) \quad (\text{J.9})$$

$$B_0^1(x, p) = -\frac{dH}{dx} A_0^0(x, p) + h(x) \tilde{B}_0^1(x, p) - \frac{\tilde{A}_1^1(x, p)}{|p|} \quad (\text{J.10})$$

For the case $n > 1$ we obtain the recurrence relation for symbols $A_m^n(x, p)$ and $B_m^n(x, p)$ with $m = 0, \dots, n$.

$$A_m^n(x, p) = \tilde{A}_m^n(x, p), \quad (\text{J.11})$$

$$B_m^n(x, p) = \tilde{B}_m^n(x, p), \quad (\text{J.12})$$

where:

1. Symbols $\tilde{A}_n^n(x, p)$ and $\tilde{B}_n^n(x, p)$.

$$\tilde{A}_n^n(x, p) = \frac{1}{n} \left\{ \frac{\partial B_{n-1}^{n-1}}{\partial x}(x, p) - p \frac{dH}{dx} A_{n-1}^{n-1}(x, p) \right\} \quad (\text{J.13})$$

$$\tilde{B}_n^n(x, p) = \frac{1}{n} \left\{ \frac{\partial A_{n-1}^{n-1}}{\partial x}(x, p) - p \frac{dH}{dx} B_{n-1}^{n-1}(x, p) \right\} \quad (\text{J.14})$$

2. For $0 < k < n$ and both k and $n - 1$ even or both k and $n - 1$ odd.

$$\tilde{A}_n^k(x, p) = -\frac{(n-1)!}{k!} \frac{1}{2^{n-k}|p|^{n-k}} \left\{ \frac{\partial A_{n-1}^{n-1}}{\partial x}(x, p) - p \frac{dH}{dx} B_{n-1}^{n-1}(x, p) \right\} + \tilde{C}_k^n(x, p) \quad (\text{J.15})$$

$$\tilde{B}_n^k(x, p) = -\frac{(n-1)!}{k!} \frac{1}{2^{n-k}p^{n-k}} \left\{ \frac{\partial B_{n-1}^{n-1}}{\partial x}(x, p) - p \frac{dH}{dx} A_{n-1}^{n-1}(x, p) \right\} + \tilde{D}_k^n(x, p) \quad (\text{J.16})$$

3. For $0 < k < n$ and k odd and $n - 1$ even or both k even and $n - 1$ odd.

$$\tilde{A}_n^k(x, p) = -\frac{(n-1)!}{k!} \frac{1}{2^{n-k}|p|^{n-k}} \left\{ \frac{\partial B_{n-1}^{n-1}}{\partial x}(x, p) - p \frac{dH}{dx} A_{n-1}^{n-1}(x, p) \right\} + \tilde{C}_k^n(x, p) \quad (\text{J.17})$$

$$\tilde{B}_n^k(x, p) = -\frac{(n-1)!}{k!} \frac{1}{2^{n-k}|p|^{n-k}} \left\{ \frac{\partial A_{n-1}^{n-1}}{\partial x}(x, p) - p \frac{dH}{dx} B_{n-1}^{n-1}(x, p) \right\} + \tilde{D}_k^n(x, p) \quad (\text{J.18})$$

4. Symbols $\tilde{C}_n^n(x, p)$ and $\tilde{D}_n^n(x, p)$.

$$\begin{aligned}
\tilde{C}_k^n(x, p) = & - \sum_{l=k, k+2, \dots, M_A} \frac{l!}{k!} \frac{1}{2^{l-k+1} |p|^{l-k+1}} \left\{ \frac{\partial A_l^{n-1}}{\partial x}(x, p) - p \frac{dH}{dx} B_l^{n-1}(x, p) \right. \\
& + \frac{1}{2|p|} \frac{\partial^2 A_l^{n-2}}{\partial x^2}(x, p) - \frac{dH}{dx} \frac{\partial B_l^{n-2}}{\partial x}(x, p) + \frac{p}{2} \left(\frac{dH}{dx} \frac{dH}{dx} \right) A_l^{n-2}(x, p) \\
& \left. - \frac{1}{2} \frac{d^2 H}{dx^2} B_l^{n-2}(x, p) \right\} + \sum_{l=k+1, k+3, \dots, M_B} \frac{l!}{k!} \frac{1}{2^{l-k+1} p^{l-k+1}} \left\{ \frac{\partial B_l^{n-1}}{\partial x}(x, p) \right. \\
& - p \frac{dH}{dx} A_l^{n-1}(x, p) + \frac{1}{2|p|} \frac{\partial^2 B_l^{n-2}}{\partial x^2}(x, p) - \frac{dH}{dx} \frac{\partial A_l^{n-2}}{\partial x}(x, p) \\
& + \frac{p}{2} \left(\frac{dH}{dx} \frac{dH}{dx} \right) B_l^{n-2}(x, p) - \frac{1}{2} \frac{d^2 H}{dx^2} A_l^{n-2}(x, p) \left. \right\} + \frac{1}{k} \left\{ \frac{\partial B_{k-1}^{n-1}}{\partial x}(x, p) \right. \\
& - p \frac{dH}{dx} A_{k-1}^{n-1}(x, p) + \frac{1}{2p} \frac{\partial^2 B_{k-1}^{n-2}}{\partial x^2}(x, p) - \frac{dH}{dx} \frac{\partial A_{k-1}^{n-2}}{\partial x}(x, p) \\
& \left. + \frac{p}{2} \left(\frac{dH}{dx} \frac{dH}{dx} \right) B_{k-1}^{n-2}(x, p) - \frac{1}{2} \frac{d^2 H}{dx^2} A_{k-1}^{n-2}(x, p) \right\} \quad (J.19)
\end{aligned}$$

$$\begin{aligned}
\tilde{D}_k^n(x, p) = & \sum_{l=k+1, k+3, \dots, M_B} \frac{l!}{k!} \frac{1}{2^{l-k+1} |p|^{l-k+1}} \left\{ \frac{\partial A_l^{n-1}}{\partial x}(x, p) - p \frac{dH}{dx} B_l^{n-1}(x, p) \right. \\
& + \frac{1}{2|p|} \frac{\partial^2 A_l^{n-2}}{\partial x^2}(x, p) - \frac{dH}{dx} \frac{\partial B_l^{n-2}}{\partial x}(x, p) + \frac{|p|}{2} \left(\frac{dH}{dx} \frac{dH}{dx} \right) A_l^{n-2}(x, p) \\
& \left. - \frac{1}{2} \frac{d^2 H}{dx^2} B_l^{n-2}(x, p) \right\} - \sum_{l=k, k+2, \dots, M_A} \frac{l!}{k!} \frac{1}{2^{l-k+1} p^{l-k+1}} \left\{ \frac{\partial B_l^{n-1}}{\partial x}(x, p) \right. \\
& - p \frac{dH}{dx} A_l^{n-1}(x, p) + \frac{1}{2|p|} \frac{\partial^2 B_l^{n-2}}{\partial x^2}(x, p) - \frac{dH}{dx} \frac{\partial A_l^{n-2}}{\partial x}(x, p) \\
& + \frac{p}{2} \left(\frac{dH}{dx} \frac{dH}{dx} \right) B_l^{n-2}(x, p) - \frac{1}{2} \frac{d^2 H}{dx^2} A_l^{n-2}(x, p) \left. \right\} + \frac{1}{k} \left\{ \frac{\partial A_{k-1}^{n-1}}{\partial x}(x, p) \right. \\
& - p \frac{dH}{dx} B_{k-1}^{n-1}(x, p) + \frac{1}{2p} \frac{\partial^2 A_{k-1}^{n-2}}{\partial x^2}(x, p) - \frac{dH}{dx} \frac{\partial B_{k-1}^{n-2}}{\partial x}(x, p) \\
& \left. + \frac{p}{2} \left(\frac{dH}{dx} \frac{dH}{dx} \right) A_{k-1}^{n-2}(x, p) - \frac{1}{2} \frac{d^2 H}{dx^2} B_{k-1}^{n-2}(x, p) \right\} \quad (J.20)
\end{aligned}$$

with

$$M_A = \begin{cases} O(n-2) & \text{if } k \text{ is odd} \\ E(n-2) & \text{if } k \text{ is even} \end{cases} \quad (\text{J.21})$$

$$M_B = \begin{cases} E(n-2) & \text{if } k \text{ is odd} \\ O(n-2) & \text{if } k \text{ is even} \end{cases} \quad (\text{J.22})$$

where $O(n)$ means the largest odd number less than or equal to n and $E(n)$ means the largest even number less than or equal to n .

5. Symbols $\tilde{A}_0^n(x, p)$ and $\tilde{B}_0^n(x, p)$.

$$\begin{aligned} A_0^n(x, p) = & \frac{\tanh(ph(x))}{p} \left\{ \sum_{m=1}^n m(-1)^{m-1} (h(x))^{m-1} \tilde{A}_m^n(x, p) \right\} + \sum_{m=1}^n (-1)^m (h(x))^m \tilde{B}_m^n(x, p) \\ & + i \frac{\tanh(ph(x))}{p} \frac{dH}{dx} p \left\{ \sum_{m=0}^{n-1} (-1)^m (h(x))^m A_m^{n-1}(x, p) \right\} \\ & + i \frac{\tanh(ph(x))}{p} \sum_{m=0}^{n-2} \left\{ (-1)^m (h(x))^m \left[\frac{dH}{dx} \frac{\partial A_m^{n-2}}{\partial x}(x, p) - p \left(\frac{dH}{dx} \frac{dH}{dx} \right) B_m^{n-2}(x, p) \right] \right\} \end{aligned} \quad (\text{J.23})$$

$$\begin{aligned} B_0^n(x, p) = & -\frac{1}{p} \left\{ \sum_{m=1}^n m(-1)^{m-1} (h(x))^{m-1} \tilde{A}_m^n(x, p) \right\} - \sum_{m=1}^n (-1)^m (h(x))^m \tilde{B}_m^n(x, p) \\ & - \frac{i}{p} \frac{dH}{dx} p \left\{ \sum_{m=0}^{n-1} (-1)^m (h(x))^m A_m^{n-1}(x, p) \right\} \\ & - \frac{i}{p} \sum_{m=0}^{n-2} \left\{ (-1)^m (h(x))^m \left[\frac{dH}{dx} \frac{\partial A_m^{n-2}}{\partial x}(x, p) - p \left(\frac{dH}{dx} \frac{dH}{dx} \right) B_m^{n-2}(x, p) \right] \right\} \end{aligned} \quad (\text{J.24})$$

Based on the recurrence relation for the symbols $A_m^n(x, p)$ and $B_m^n(x, p)$ we give the Maple V routine GNSRN(n) that generates the expression for the symbol $R_n(x, y, p)$. The input for this routine in the parameter n , which is the order of the approximation this symbol is associated with. The routine code follows:

```

GNSRN:=proc(n) local u,v,u1,u2,v1,v2,w,t,q,i1,i2,i3;
#This procedure generates the explicit equation for the symbol  $R_n(x, y, p)$ 
if n = 0 then
    q:=cosh(p*(y+h(x)))/cosh(p*h(x));
else
    u1[1]:=1/cosh(p*h(x));
    v1[1]:=0;
    for i1 from 1 to n do
        w[i1]:=GNESRN(i1);
        for i2 from 0 to i1 do
            u[1+i2]:=coeff(coeff(w[i1],cosh(p*(y+h(x))),1),y,i2);
            v[1+i2]:=coeff(coeff(w[i1],sinh(p*(y+h(x))),1),y,i2);
            for i3 from 0 to i1-1 do
                if i1=1 then
                    u[1+i2]:=eval(subs(A_[i3,i1-1](x,p)=u1[i3+1],u[1+i2]));
                    u[1+i2]:=eval(subs(B_[i3,i1-1](x,p)=v1[i3+1],u[1+i2]));
                    v[1+i2]:=eval(subs(A_[i3,i1-1](x,p)=u1[i3+1],v[1+i2]));
                    v[1+i2]:=eval(subs(B_[i3,i1-1](x,p)=v1[i3+1],v[1+i2]));
                else
#from the previous symbol
                    u[1+i2]:=eval(subs(A_[i3,i1-1](x,p)=u1[i3+1],u[1+i2]));
                    u[1+i2]:=eval(subs(B_[i3,i1-1](x,p)=v1[i3+1],u[1+i2]));
                    v[1+i2]:=eval(subs(A_[i3,i1-1](x,p)=u1[i3+1],v[1+i2]));
                    v[1+i2]:=eval(subs(B_[i3,i1-1](x,p)=v1[i3+1],v[1+i2]));
#from the second previous symbol
                    if i3 < i1-1 then
                        u[1+i2]:=eval(subs(A_[i3,i1-2](x,p)=u2[i3+1],u[1+i2]));
                        u[1+i2]:=eval(subs(B_[i3,i1-2](x,p)=v2[i3+1],u[1+i2]));
                        v[1+i2]:=eval(subs(A_[i3,i1-2](x,p)=u2[i3+1],v[1+i2]));
                        v[1+i2]:=eval(subs(B_[i3,i1-2](x,p)=v2[i3+1],v[1+i2]));
                    end if
                end if
            end for
        end for
    end for
end if
end proc

```



```

fi;

fi;

od;
u[1+i2]:=simplify(subs(diff(h(x),x$1)=-diff(H(x),x$1),
    subs(diff(h(x),x$2)=-diff(H(x),x$2),u[1+i2])));
v[1+i2]:=simplify(subs(diff(h(x),x$1)=-diff(H(x),x$1),
    subs(diff(h(x),x$2)=-diff(H(x),x$2),v[1+i2])));
u[1+i2]:=collect(collect(u[1+i2],cosh(p*h(x))),sinh(p*h(x)));
v[1+i2]:=collect(collect(v[1+i2],cosh(p*h(x))),sinh(p*h(x)));
od;
for i2 from 0 to i1-1 do
    u2[i2+1]:=u1[i2+1];
    v2[i2+1]:=v1[i2+1];
od;
for i2 from 0 to i1 do
    u1[i2+1]:=u[i2+1];
    v1[i2+1]:=v[i2+1];
od;
od;
q:=0;
for i1 from 0 to n do
    q:=q+u[i1+1]*cosh(p*(y+h(x)))*(yi1)+v[i1+1]*sinh(p*(y+h(x)))*(yi1);
od;
fi;
end:

```

This procedure uses the Maple V procedure GNESRN(n) which solves the boundary value problem for the symbols $R_n(x, y, p)$. The input parameter is the variable n . the procedure code follows:

```

GNESRN:=proc(n) local i1,i2,i3,u,v,w,t,q;
  if n = 0 then
    q:=A_[0,0](x,p)*cosh(p*(y+h(x)));
  elif n = 1 then
    v[1]:=A_[0,0](x,p)*cosh(p*(y+h(x)));
    v[2]:=2* p*subs(diff(h(x),x$1)=-diff(H(x),x$1),diff(v[1],x$1)/p);
    u[1]:=int(v[2]*cosh(p*(y+h(x))),y);
    u[2]:=int(-v[2]*sinh(p*(y+h(x))),y);
    w[1]:=simplify(u[2]*cosh(p*(y+h(x)))+u[1]*sinh(p*(y+h(x)))
      ,{sinh(p*y)*cosh(p*h(x))+cosh(p*y)*sinh(p*h(x))=sinh(p*(y+h(x))),
        cosh(p*y)*cosh(p*h(x))+sinh(p*y)*sinh(p*h(x))=cosh(p*(y+h(x)))});
    w[1]:=subs(sinh(p*(y+h(x)))^ 2=-1+cosh(p*(y+h(x)))^ 2,w[1]);
    for i1 from 1 to 3 do
      w[1]:=collect(w[1],cosh(p*(y+h(x)))^ i1);
    od;
    w[1]:=subs(cosh(p*(y+h(x)))^ 3=cosh(3*p*(y+h(x)))/4
      +(3/4)*cosh(p*(y+h(x))),subs(cosh(p*(y+h(x)))^
        sinh(p*(y+h(x)))^ 2=cosh(3*p*(y+h(x)))/4
        -(1/4)*cosh(p*(y+h(x))),w[1]));
    w[2]:=eval(subs(y=0,w[1]));
    t[1]:=-diff(H(x),x$1)*A_[0,0](x,p)-eval(subs(y=-h(x),diff(w[1],y$1)/p));
    t[2]:=-t[1]*tanh(p*h(x))-w[2]/cosh(p*h(x));
    t[3]:=t[2]*cosh(p*(y+h(x)))+t[1]*sinh(p*(y+h(x)))+w[1];
    t[4]:=collect(collect(t[3],cosh(p*(y+h(x)))),sinh(p*(y+h(x))));
    q:=0;
    for i1 from 0 to 1 do
      q:=q+simplify(coeff(coeff(t[4],cosh(p*(y+h(x))),1),y,i1))*
        (y^ i1)*cosh(p*(y+h(x)))+simplify(coeff(coeff(t[4],
          sinh(p*(y+h(x))),1),y,i1))*(y^ i1)*sinh(p*(y+h(x)));
    od;

```

elif n > 1 then

for i1 from 1 to 2 do

$$v[i1] := A_{[0,n-i1]}(x,p) * \cosh(p*(y+h(x))) + B_{[0,n-i1]}(x,p) * \sinh(p*(y+h(x)));$$

for i2 from 1 to n-i1 do

$$v[i1] := v[i1] + A_{[i2,n-i1]}(x,p) * (y^{i2}) * \cosh(p*(y+h(x))) + B_{[i2,n-i1]}(x,p) * (y^{i2}) * \sinh(p*(y+h(x)));$$

od;

od;

$$u[1] := 2*p*subs(diff(h(x),x$1)=-diff(H(x),x$1),diff(v[1],x$1))+ \\ subs(diff(h(x),x$1)=-diff(H(x),x$1),subs(diff(h(x),x$2)=-diff(H(x),x$2),diff(v[2],x$2)));$$

$$u[2] := -diff(H(x),x$1)*p*v[1]-diff(H(x),x$1)*subs(diff(h(x),x$1)=-diff(H(x),x$1),diff(v[2],x$1));$$

$$w[2] := subs(\cosh(p*y+p*h(x))=\cosh(p*(y+h(x))),\sinh(p*y+p*h(x))=\sinh(p*(y+h(x))),int(simplify(\cosh(p*(y+h(x)))*(u[1]/p)),y));$$

$$w[1] := subs(\cosh(p*y+p*h(x))=\cosh(p*(y+h(x))),\sinh(p*y+p*h(x))=\sinh(p*(y+h(x))),int(simplify(-\sinh(p*(y+h(x)))*(u[1]/p)),y));$$

#

particular solution

#

$$t[1] := simplify(w[1]*\cosh(p*(y+h(x))) + w[2]*\sinh(p*(y+h(x))), \\ \{\sinh(p*y)*\cosh(p*h(x)) + \cosh(p*y)*\sinh(p*h(x)) = \sinh(p*(y+h(x))), \\ \cosh(p*y)*\cosh(p*h(x)) + \sinh(p*y)*\sinh(p*h(x)) = \cosh(p*(y+h(x)))\});$$

$$t[1] := subs(\sinh(p*(y+h(x)))^2 = -1 + \cosh(p*(y+h(x)))^2, t[1]);$$

for i1 from 1 to 3 do

$$t[1] := collect(t[1], \cosh(p*(y+h(x))))^{i1};$$

od;

$$t[1] := subs(\cosh(p*(y+h(x)))^3 = \cosh(3*p*(y+h(x)))/4 \\ + (3/4)*\cosh(p*(y+h(x))), subs(\cosh(p*(y+h(x)))^*$$

```

sinh(p*(y+h(x)))^2=cosh(3*p*(y+h(x)))/4
-(1/4)*cosh(p*(y+h(x))),t[1]));

#
# particular solution at the y=0
#
t[2]:=eval(subs(y=0,t[1]));

#
# d/dy of the particular solution at y = -h(x)
#
t[3]:=eval(subs(y=-h(x),diff(t[1],y$1)));

#
# coefficient of sinh(p*(y+h(x)))
#
t[4]:=eval(subs(y=-h(x),u[2]));
t[5]:=simplify(-(t[3]-t[4])/p);
t[7]:=t[5]+coeff(coeff(t[1],sinh(p*(y+h(x))),1),y,0);

#
# coefficient of cosh(p*(y+h(x)))
#
t[6]:=simplify(-t[5]*sinh(p*h(x))/cosh(p*h(x))-t[2]/cosh(p*h(x)));
t[6]:=collect(collect(t[6],cosh(p*h(x))),sinh(p*h(x)));
t[6]:=t[6]+coeff(coeff(t[1],cosh(p*(y+h(x))),1),y,0);

#
# built of symbol R_n(x,y,p,beta)
#
q:=(t[6]*cosh(p*(y+h(x)))+(t[7]*sinh(p*(y+h(x))));
for il from 1 to n do
q:=q+simplify(coeff(coeff(t[1],cosh(p*(y+h(x))),1),y,il))*(y^(il))*
cosh(p*(y+h(x)))+simplify(coeff(coeff(t[1],
sinh(p*(y+h(x))),1),y,il))*(y^(il))*

```

```
sinh(p*(y+h(x)));  
    od;  
fi;  
end:
```

Appendix K

Expression for the Symbols $L_n(x, p)$.

Here we give the expression for the symbols $L_n(x, p)$ for $n > 1$. We give explicit expressions until $n = 3$, and for larger values we give just the general form of these symbols. We have

$$L_1(x, p) = -\frac{dH}{dx} \frac{p}{\cosh^2 \phi} + h(x) p^2 \frac{dH}{dx} \frac{\tanh \phi}{\cosh^2 \phi} \quad (\text{K.1})$$

$$\begin{aligned} L_2(x, p) = & \left[3p^2 h(x) \left(\frac{dH}{dx} \right)^2 - \frac{1}{2} h(x)^2 p^2 \frac{d^2 H}{dx^2} \right] \frac{1}{\cosh^2 \phi} + \left[-4p^2 h(x) \left(\frac{dH}{dx} \right)^2 + h(x)^2 p^2 \frac{d^2 H}{dx^2} \right] \\ & \frac{1}{\cosh^4 \phi} + \left[-(p^3 h(x)^2 + p) \left(\frac{dH}{dx} \right)^2 + h(x) p \frac{d^2 H}{dx^2} \right] \frac{\tanh \phi}{\cosh^2 \phi} \\ & + \left[3p^3 h(x)^2 \left(\frac{dH}{dx} \right)^2 \right] \frac{\tanh \phi}{\cosh^4 \phi} \end{aligned} \quad (\text{K.2})$$

$$\begin{aligned}
L_3(x, p) = & \left[(4p^2h(x) + \frac{2}{3}h(x)^3p^4 - ph(x)) \left(\frac{dH}{dx} \right)^3 + (p^2h(x) - \frac{7}{2}p^2h(x)^2) \frac{dH}{dx} \frac{d^2H}{dx^2} \right. \\
& + \left. \frac{p^3}{6}h(x)^3 \frac{d^3H}{dx^3} \right] \frac{\tanh \phi}{\cosh^2 \phi} + \left[(-8p^2h(x) - 10h(x)^3p^3) \left(\frac{dH}{dx} \right)^3 + \frac{19}{2}p^2h(x)^2 \frac{dH}{dx} \frac{d^2H}{dx^2} \right. \\
& - \left. p^2h(x)^3 \frac{d^3H}{dx^3} \right] \frac{\tanh \phi}{\cosh^4 \phi} + 15p^4h(x)^3 \left(\frac{dH}{dx} \right)^3 \frac{\tanh \phi}{\cosh^6 \phi} \\
& + \left[(2p^2h(x)^2 - \frac{14}{3}h(x)^2p^3 - p) \left(\frac{dH}{dx} \right)^3 + (p^3h(x)^3 + \frac{3}{2}ph(x)) \frac{dH}{dx} \frac{d^2H}{dx^2} \right. \\
& - \left. \frac{p}{6}h(x)^2 \frac{d^3H}{dx^3} \right] \frac{1}{\cosh^2 \phi} + \left[(-3p^2h(x)^2 + 22h(x)^2p^3 + p) \left(\frac{dH}{dx} \right)^3 \right. \\
& - \left. (\frac{5}{2}ph(x) + \frac{19}{2}p^3h(x)^3) \frac{dH}{dx} \frac{d^2H}{dx^2} + \frac{p}{2}h(x)^2 \frac{d^3H}{dx^3} \right] \frac{1}{\cosh^4 \phi} \\
& + \left[-19p^3h(x)^2 \left(\frac{dH}{dx} \right)^3 + 10p^3h(x)^3 \frac{dH}{dx} \frac{d^2H}{dx^2} \right] \frac{1}{\cosh^6 \phi} \\
& \dots
\end{aligned} \tag{K.3}$$

$$L_n(x, p) = \sum_{l=1}^n \left\{ \phi_l^n(p, h(x), DH, \dots, D^n H) \frac{\tanh \phi}{\cosh^{2l} \phi} + \psi_l^n(p, h(x), DH, \dots, D^n H) \frac{1}{\cosh^{2l} \phi} \right\} \tag{K.4}$$

where the $\phi = ph(x)$ and $D^n = \frac{d^n}{dx^n}$. We also give a Maple V procedure to compute the expression for the symbols $L_n(x, p)$. It is called GNSLN(n) and it uses the procedure GSRN(n) given in the appendix J. The parameter n is related to the index n in $L_n(x, p)$. The code of the procedure follows:

```

GNSLN:=proc(n) local q;
# This procedure evaluates the explicit form of the symbol  $L_n(x, p)$ 
#
q:=diff(GNSRN(n), y$1);
q:=eval(subs(y=0, q));
q:=simplify(q);
q:=collect(collect(q, cosh(p*h(x))), sinh(p*h(x)));

```


Appendix L

Derivation of the Transport Equations.

In this appendix we discuss in detail the derivation of the transport equations for the amplitude corrections $w_n(x)$ in chapter 4 and for the amplitude corrections $w_n(z)$ in chapter 3. The transport equations required in chapter 4 are more general than the transport equations that appear in chapter 3. We derive the transport equations for the more general case, and then we particularize them for the needs of chapter 3. We also give Maple V scripts to generate the transport equations for the amplitude correction $w_n(x)$ (chapter 4) and $w_n(z)$ (chapter 3) up to any order needed (any desired value of the index n of the amplitude correction).

The symbol of an ordinary differential operator of order N is a polynomial, as illustrated by the equation below

$$\begin{array}{ccc} \text{Operator} & \text{Operator symbol} & \\ \sum_{n=0}^N a_n(x) \frac{d^n}{dx^n} & \implies & \sum_{n=0}^N (i)^n a_n(x) p^n \end{array} \quad (\text{L.1})$$

where $p = -i \frac{d}{dx}$. We can define operators based on more general symbols than polynomials. These are denoted pseudo-differential operators and are defined in terms

of a Fourier integral. The action of the operator $\hat{L}$ over a function $\phi(x)$ is defined through its symbol $L(x, p)$ according to

$$\hat{L}\phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(ixp) L(x, p) \hat{\phi}(p) dp$$

where $\hat{\phi}(p)$ is the Fourier transform of the function $\phi(x)$.

We are interested in differential operators and pseudo-differential operators which have a small parameter associated with the D (differential) operator, like the operators in chapters 3 and 4. The operator $\hat{L}$ of chapter 3 with symbol given by equation (3.100) is an example of a differential operator with a small parameter ε , and the operators $\hat{R}$ and $\hat{L}$ of chapter 4 are examples of pseudo-differential operators with a small parameter β . The symbol of operator $\hat{R}$ is given by equations (4.37) and (4.51), and the symbol of operator $\hat{L}$ of chapter 4 is given by equation (4.53). These operators are associated with ordinary differential (chapter 3) or pseudo-differential equations (chapter 4) of the form

$$\hat{L}(\overset{1}{x}, -i\beta \overset{2}{\frac{d}{dx}}, \beta)y(x) = 0 \tag{L.2}$$

For the function $y(x)$ solution of the pseudo-differential equation (L.2) we assume the form

$$y(x) = \exp(\frac{i}{\beta} S(x)) w(x) \tag{L.3}$$

where $S(x)$ is the phase function and $w(x)$ is the amplitude function. If we substitute the equation (L.3) for $y(x)$ into the equation (L.2), we obtain

$$\exp\left(\frac{i}{\beta}S(x)\right)\hat{L}\left(x,\frac{dS}{dx}-i\beta\frac{d}{dx},\beta\right)w(x)=0 \quad (\text{L.4})$$

If we expand this pseudo-differential equation in terms of the parameter β , the term of order $O(\beta^0)$ gives the Eikonal equation for the function $S(x)$, as discussed in chapters 3 and 4. What is left gives transport equation for the amplitude function $w(x)$. Here we derive the transport equation for the amplitude function $w(x)$. Further, we assume that we can write

$$w(x) = w_0(x) \left\{ 1 + \sum_{n=1}^{\infty} w_n(x) \right\}, \quad (\text{L.5})$$

and here we also derive the transport equation for the amplitude corrections $w_n(x)$, $n = 0, 1, \dots$. We assume that the symbol of the operator $\hat{L}$ in the pseudo-differential equation (L.3) has a series expansion in the small parameter β , as follows

$$L(x, p, \beta) = \sum_{n=0}^M (-i\beta)^n L_n(x, p) \text{ where } M \text{ may be finite (chapter 3) or infinite (chapter 4)} \quad (\text{L.6})$$

The symbols $L_n(x, p)$ are polynomials or more general analytic functions with respect to the p variable. So, without loss of generality we can assume that the symbols $L_n(x, p)$ are represented by a Taylor series in the p variable, as follows:

$$L_n(x, p) = \sum_{m=0}^{\infty} (i)^m a_{n,m}(x) p^m. \quad (\text{L.7})$$

Next, we describe the steps necessary to obtain the transport equation for the amplitude $w(x)$.

- First, we obtain the general for the action of the operator $\frac{d^n}{dx^n}$ over the function $y(x)$ given by equation (L.3). We write

$$y(x) = u(x)w(x), \text{ where } u(x) = \exp\left(\frac{i}{\beta}S(x)\right). \quad (\text{L.8})$$

Then we can write

$$\frac{d^n y}{dx^n} = \sum_{q=0}^n \frac{n!}{(n-q)!q!} \frac{d^{n-q}u}{dx^{n-q}} \frac{d^q w}{dx^q}, \quad (\text{L.9})$$

where $\frac{d^{n-q}u}{dx^{n-q}}$ is given in terms of the function $S(x)$ by the equation

$$\frac{d^{n-q}u}{dx^{n-q}} = \left\{ \sum_{l=1}^{(n-q)} \frac{1}{l!} \left(\frac{i}{\beta}\right)^l \left[\sum_{j_l+\dots+j_1=(n-q)} \frac{(n-q)!}{j_l! \dots j_1!} D^{j_l} S \times \dots \times D^{j_1} S \right] \right\} u(x), \quad (\text{L.10})$$

where the symbol D stands for $\frac{d}{dx}$. Equations (L.9) and (L.10) allow us to write

$$\exp\left(-\frac{i}{\beta}S(x)\right)(-i\beta)^n \frac{d^n y}{dx^n} = \left\{ (-i\beta)^n \frac{d^n}{dx^n} + (-i)^n \sum_{q=0}^{n-1} \frac{1}{q!} \left[\sum_{l=1}^q \frac{n!}{l!} (i)^l \beta^{n-l} b_{l,q} \frac{d^q}{dx^q} \right] \right\} w(x), \quad (\text{L.11})$$

where $b_{l,q}$ is defined as

$$b_{l,q} = \left\{ \sum_{j_l+\dots+j_1=(n-q)} \frac{(n-q)!}{j_l! \dots j_1!} D^{j_l} S \times \dots \times D^{j_1} S \right\} \quad (\text{L.12})$$

We rearrange equation (L.11) such that we can write it as a power series in the

parameter $-i\beta$.

$$\exp\left(-\frac{i}{\beta}S(x)\right)(-i\beta)^n \frac{d^n y}{dx^n} = \left\{ (-i\beta)^n \frac{d^n}{dx^n} + \sum_{l=0}^{n-1} (-i\beta)^l \left[\sum_{q=0}^l \frac{n!}{(n-l)!q!} b_{n-l,q} \frac{d^q}{dx^q} \right] \right\} w(x) \quad (\text{L.13})$$

- Second, we assumed that the operators we are working with have symbols that are analytic functions with respect to the p variable, so the symbol of these operators admit a Taylor series expansion

$$L(x, p) = \sum_{n=0}^{\infty} (i)^n a_n(x) p^n \quad (\text{L.14})$$

With the help of equation (L.13), the action of an operator with symbol given by equation (L.14) over the function $y(x)$ given by equation (L.3) can now be written as

$$\exp\left(-\frac{i}{\beta}S(x)\right) \hat{L}(x, -i\beta \frac{d}{dx}) y(x) = \sum_{l=0}^{\infty} (-i\beta)^l \left[\sum_{n=l+1}^{\infty} (i)^n a_n(x) \left(\sum_{q=0}^l \frac{n!}{(n-l)!q!} b_{n-l,q} \frac{d^q}{dx^q} w \right) + (i)^l a_l(x) \frac{d^l w}{dx^l} \right] \quad (\text{L.15})$$

The term corresponding to $l = 0$ in the summation over the index l in the equation (L.15) can be written as

$$\sum_{n=1}^{\infty} (i)^n a_n(x) \prod_{q=1}^n DS(x) + a_0(x) = L(x, DS) \quad (\text{L.16})$$

The term corresponding to $l = 1$ in the summation over the index l in the

equation (L.15) can be written as

$$\begin{aligned}
& -i\beta \left\{ \sum_{n=2}^{\infty} (i)^n a_n(x) \left[\frac{n(n-1)}{2} (DS)^{n-2} D^2 S + n \left(\prod_{q=1}^{n-1} DS(x) \right) \frac{dw}{dx} \right] + i a_1(x) \frac{dw}{dx} \right\} = \\
& -i\beta \left\{ \frac{1}{2} \frac{\partial^2 L}{\partial p^2}(x, p)|_{p=DS} + \frac{\partial L}{\partial p}(x, p)|_{p=DS} \frac{dw}{dx} \right\}
\end{aligned} \tag{L.17}$$

The term corresponding to $l = 2$ in the summation over the index l in the equation (L.15) can be written as

$$\begin{aligned}
& -i\beta \left\{ \sum_{n=3}^{\infty} (i)^n a_n(x) \left[\frac{n(n-1)(n-2)}{3!0!} (DS)^{n-3} D^3 S + \frac{n(n-1)(n-2)}{2!1!} (DS)^{n-3} D^2 S \frac{dw}{dx} \right. \right. \\
& \left. \left. + \frac{n(n-1)}{2} (DS)^{n-2} \frac{d^2 w}{dx^2} \right] + \sum_{n=4}^{\infty} \frac{n!}{0!2!2!(n-4)!} (DS)^{n-4} (D^2 S)^2 - a_2(x) \frac{d^2 w}{dx^2} \right\} = \\
& -i\beta \left\{ \frac{1}{3!} \frac{\partial^3 L}{\partial p^3}(x, p)|_{p=DS} D^3 S + \frac{1}{2!} \frac{\partial^3 L}{\partial p^3}(x, p)|_{p=DS} D^2 S \frac{dw}{dx} + \frac{1}{2!} \frac{\partial^2 L}{\partial p^2}(x, p)|_{p=DS} \frac{d^2 w}{dx^2} \right. \\
& \left. + \frac{1}{2!2!2!} \frac{\partial^4 L}{\partial p^4}(x, p)|_{p=DS} (D^2 S)^2 \right\}
\end{aligned} \tag{L.18}$$

Below we give a Maple V routine that generates the expression for the l th term in the first summation present in the right hand side of equation (L.15) in terms of the derivatives of the symbol $L(x, p)$ with respect to the p variable.

```

FEXPSA:= proc (F, G, y, n)
    local k, m, u, v, p, q, t, check, i1, i2, i3, i4;
    if    n = 0 then
        u := subs    (p = D(G)(x), F(x, p))*y(x);
    else
        u :=  D(@@n)[2](F) (x, D(G)(x))*diff(y(x), x$ n)/n!;
    for k from 0 to n-1 do

```

```

for m to n-k do
  if m=1 then
    v := (D@@(n+1)) [2](F)(x,D(G)(x))*
          (D@@(n-k+1))(G)(x)/(k!*(n-k+1)!)
  else
    i1 := (n-k+1)m-1;
    for i2 from 0 to i1 do
      i3 := i2;
      for i4 to m do
        p[i4] := modp(i3,n-k+1)+1;
        i3 := round(i3/(n-k+1));
      od;
      check := 0;
      for i4 to m do
        if p[i4] = 1 then check := 1 fi;
      od;
      if check = 0 then
        q := 0;
        for i4 to m do
          q := q+p[i4];
        od;
        if q = n-k+m then
          t := (D@@(n+m))[2](F)
                (x,D(G)(x))/(k!*m!);
          for i4 to m do
            t := t*(D@@(p[i4]))
                  (G)(x)/p[i4]!
          od;
          v := v+t;
        fi;
      fi;
    od;
  fi;

```

```

                                fi;
                                od;
                                fi;
                                od;
                                if k = 0 then u := u+v*y(x) else u := u+v*diff(y(x),x$ k) fi;
                                od;
                                fi;
end:

```

In the procedure given above, F is the letter representing the symbol, L in our case, G is the letter representing the phase function S and y is the amplitude function w , and l is the index number. If we want the 3rd term in the first summation in the right hand side of the equation (L.15), we just type in the Maple V prompt FEXPSA(L,S,w,3);

We realize that we can write equation (L.15) in terms of the partial derivatives of the symbol $L(x, p)$ with respect to the variable p . So, equation (L.15) is written in the form

$$\begin{aligned}
 \exp\left(-\frac{i}{\beta}S(x)\right)\hat{L}\left(x, -i\beta\frac{d}{dx}\right)y(x) = \\
 L(x, DS) + \sum_{l=1}^{\infty} (-i\beta)^l \left\{ \sum_{k=0}^{l-1} \left[\sum_{m=1}^{l-k} \left(\sum_{\substack{j_m+\dots+j_1=(l-k)+m \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k!j_m! \dots j_1!m!} \right. \right. \right. \\
 \left. \left. \left. \frac{\partial^{l+m} L}{\partial p^{l+m}}(x, p)|_{p=DS} \right) \right] \frac{d^k w}{dx^k} + \frac{1}{l!} \frac{\partial^l L}{\partial p^l}(x, p)|_{p=DS} \frac{d^l w}{dx^l} \right\}
 \end{aligned}
 \tag{L.19}$$

The action of the operator $\hat{L}$ over the function $y(x)$ given by the equation above is for the case where the symbol $L(x, p)$ has an infinite Taylor series expansion with respect to the p variable. This is the case if $L(x, p)$ is an analytic

transcendental function, but when $L(x, p)$ is just a polynomial in the p variable, its Taylor series expansion with respect to the p variable has a finite number of terms equal to the order of the operator $\hat{L}$. If the Taylor expansion of the symbol $L(x, p)$ with respect to the p variable has N terms, equation (L.19) assumes the form

$$\exp(-\frac{i}{\beta}S(x))\hat{L}(\frac{d}{dx})y(x) =$$

$$L(x, DS) + \sum_{l=1}^{N-1} (-i\beta)^l \left\{ \sum_{k=0}^{l-1} \left[\sum_{m=1}^{\min l-k, N-l} \left(\sum_{\substack{j_m+\dots+j_1=(l-k)+m \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k! j_m! \dots j_1! m!} \right. \right. \right.$$

$$\left. \left. \frac{\partial^{l+m} L}{\partial p^{l+m}}(x, p)|_{p=DS} \right) \right] \frac{d^k w}{dx^k} + \frac{1}{l!} \frac{\partial^l L}{\partial p^l}(x, p)|_{p=DS} \frac{d^l w}{dx^l} \right\}$$

$$+ \frac{(-i\beta)^N}{N!} \frac{\partial^N L}{\partial p^N}(x, p)|_{p=DS} \frac{d^N w}{dx^N}$$

(L.20)

We give also a Maple V procedure to generate the terms of the first summation in the right hand side of equation (L.20). It follows below.

```
FEXPSB:=proc(F,G,y,n,l) local k,m,u,v,p,q,t,check,i1,i2,i3,i4;
# n - order of the term of the transport equation
# l - order of the symbol with respect to the variable p
#
if n = 0 then u:=subs(p=D(G)(x),F(x,p))*y(x);
elif 0 < n and n < l+1 then
u:=(1/n!)*(D@@n)[2](F(x,D(G)(x))*diff(y(x),x$n);
for k from 0 to n-1 do
for m from 1 to n-k do
if m = 1 then
if n < l then
```

```

v:=(1/(k!*(n-k+1)!))*(D@@(n+1))[2](F)(x,D(G)(x))
*(D@@(n-k+1))(G)(x);
else
v:=0;
fi;
else
i1:=(n-k+1)**m-1;
for i2 from 0 to i1 do
i3:=i2;
for i4 from 1 to m do
p[i4]:=modp(i3,(n-k+1))+1;
i3:=round(i3/(n-k+1));
od;
check:=0;
for i4 from 1 to m do
q:=q+p[i4];
od;
if q = n-k+m then
if n+m ; l+1 then
t:=(D@@(n+m))[2](F)
(x,D(G)(x))/(k!*m!);
for i4 from 1 to m do
t:=t*(D@@p[i4])(G)(x)/
(p[i4]!) od;
else
t:=0;
fi;
v:=v+t;
fi;
fi;

```

```

                                od;
                                fi;
                                od;
                                if k = 0 then
                                    u:=u+(v)*y(x);
                                else
                                    u:=u+(v)*diff(y(x),x$k);
                                fi;
                                od;
                                else
                                    u:=0;
                                fi;
                                end:

```

- Now, we consider that the symbol $L(x, p)$ is of the form given by equation (L.6). We substitute the symbol $L(x, p, \beta)$ into the equation (L.19) and we rewrite it such that we obtain a series in the parameter β . The end result is given by the equation

$$\begin{aligned}
 \exp\left(-\frac{i}{\beta}S(x)\right)\hat{L}(\hat{x}, -i\beta\frac{d}{dx})y(x) &= L_0(x, DS) + \\
 &\sum_{l=1}^{\infty}(-i\beta)^n \left(\sum_{n=0}^{l-1} \left\{ \sum_{k=0}^{(l-n)-1} \left(\sum_{m=1}^{(l-k)-n} \left[\sum_{\substack{j_m+\dots+j_1=(l-k)-n+m \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k!j_m! \dots j_1!m!} \right. \right. \right. \\
 &\quad \left. \left. \frac{\partial^{l+m-n}}{\partial p^{l+m-n}} L_n(x, p) \right|_{p=DS} D^{j_l} S \times \dots \times D^{j_1} S \times D^p \right] \right) \frac{d^k}{dx} w(x) \\
 &\quad \left. + \frac{1}{(l-n)!} \frac{\partial^{(l-n)}}{\partial p^{(l-n)}} L_n(x, p) \right|_{p=DS} \frac{d^{(l-n)}}{dx} w(x) \Big\} + L_l(x, DS)w(x) \Big),
 \end{aligned} \tag{L.21}$$

and for the case that M is finite in equation (L.6), we obtain a similar equation. Below we give a Maple V procedure that evaluate the terms of the summation with respect to the index l in the right hand side of equation (L.21).

```
GNTPEQ:=proc(G,y,n) local u,v,p,il;
    for il from 0 to n do
        u := L_[il];
        if il = 0 then
            if n=0 then
                v:=FEXPSA(u,G,y,n);
            else v:=(FEXPSA(u,G,y,n));
            fi;
        else
            if il > n then
                p:=( FEXPSA(u,G,y,(n-il)) );
            else p:=FEXPSA(u,G,y,n-il);
            fi;
            v:=v+p;
        fi;
    od;
end;
```

Notice that the procedure given above uses the Maple V procedure FEXPSA previously defined. In the procedure above, G is the letter representing the phase function S , y represents the amplitude function w , and n stands for the index l . If we want, for example, the 3rd term in the first summation in the right hand side of the equation (L.21), we just type in the Maple V prompt `GNTPEQ(S,w,3);`

For the case that M is finite in equation (L.6), we give a Maple V procedure

that evaluate the terms of order $(-i\beta)^n$ of the respective transport equation, analogous to equation (L.21). This Maple V procedure follows below.

```
GNTPEQB:=proc(G,y,n,l) local u,v,p,i1,i2;
# l          - order of the symbol with respect to the variable p
# n          - order of the term of the transport equation
              if n > 2 then i2:= 2 else i2:= n fi;
              for i1 from 0 to i2 do
                  u:=R_[i1];
                  if i1 = 0 then
                      if n =0 then v:=FEXPSB(u,G,y,n,l)
                      else v:=(FEXPSB(u,G,y,n,l)) fi;
                  else
                      if i1 < n then p:=(FEXPSB(u,G,y,(n-i1),l-i1))
                      else p:=FEXPSB(u,G,y,n-i1,l-i1) fi;
                      v:=v+p;
                  fi;
              od;
end:
```

In this Maple V procedure the index n specifies the order of the term in the parameter β (term of order $O(\beta^n)$) and l specify the order of the symbol $L_0(x, p)$ in equation (L.6) with respect to the variable p . In the procedure GNTPEQB it is assumed that the order of the symbols $L_n(x, p)$ with respect to the p variable is $(l-n)$ (n is the index appearing in equation (L.6). G and y stands for, respectively, the phase function $S(x)$ and the amplitude $w(x)$. This procedure was used to provide the transport equation (3.113).

The resulting transport equation is a differential equation of infinite order when the symbols $L_n(x, p)$ are transcendental analytic functions. To make progress in solving this transport equation, we consider the expansion

$$w(x) = \sum_{q=0}^{\infty} (i\beta)^q F_q(x) \quad (\text{L.22})$$

We substitute the expansion for the wave amplitude $w(x)$ into the transport equation (L.21). This leads to a sequence of first order transport equations for the wave amplitude corrections $F_q(x)$, $q = 0, 1, 2, \dots$. The sequence of transport equations has the form

$$\begin{aligned} \sum_{l=1}^{\infty} (-i\beta)^l \left\{ \sum_{q=1}^l \left[\sum_{n=0}^{q-1} \left(\sum_{k=0}^{q-1-n} \left\{ \sum_{m=1}^{q-k-n} \left[\sum_{\substack{j_m + \dots + j_1 = q-k+m-n \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k! j_m! \dots j_1! m!} \right. \right. \right. \right. \right. \\ \left. \left. \left. \frac{\partial^{q+m-n}}{\partial p^{q+m-n}} L_n(x, p) \right|_{p=DS} D^{j_m} S \times \dots \times D^{j_1} S \right] \right\} D^k F_{l-q} \right) \\ + \frac{1}{(q-n)!} \frac{\partial^{q-n}}{\partial p^{q-n}} L_n(x, p) \Big|_{p=DS} D^{q-n} F_{l-q}(x) \Big] + L_l(x, DS) F_0(x) \Big\} = 0 \end{aligned} \quad (\text{L.23})$$

If we collect terms of the same order in β , we obtain a sequence of first order differential equations for the amplitude corrections $F_q(x)$, but before doing that, we assume that the wave amplitude corrections are given in the form

$$F_q(x) = w_0(x) w_n(x) \quad \text{for } q \geq 1, \quad (\text{L.24})$$

$$F_0(x) = w_0(x), \quad (\text{L.25})$$

which implies now that the wave amplitude $w(x)$ can be written as

$$w(x) = w_0(x) \left\{ 1 + \sum_{n=1}^{\infty} (-i\beta)^n w_n(x) \right\}. \quad (\text{L.26})$$

This form for the expansion of the wave amplitude $w(x)$ in terms of the parameter β

gives a transport equation of the form

$$\begin{aligned}
& \sum_{l=1}^{\infty} (-i\beta)^l \left\{ \sum_{q=1}^{l-1} \left[\sum_{n=0}^{q-1} \left(\sum_{k=0}^{q-1-n} \left\{ \sum_{m=1}^{q-k-n} \left[\sum_{\substack{j_m+\dots+j_1=q-k+m-n \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k!j_m! \dots j_1!m!} \right. \right. \right. \right. \right. \\
& \left. \left. \left. \frac{\partial^{q+m-n}}{\partial p^{q+m-n}} L_n(x, p) \Big|_{p=DS} D^{j_m} S \times \dots \times D^{j_1} S \right] \right\} \left(\sum_{j=0}^k \frac{k!}{(k-j)!j!} \frac{d^{k-j}}{dx^{k-j}} w_0(x) \frac{d^j}{dx^j} w_{l-q}(x) \right) \right) \\
& + \frac{1}{(q-n)!} \frac{\partial^{q-n}}{\partial p^{q-n}} L_n(x, p) \Big|_{p=DS} \left(\sum_{j=0}^{q-n} \frac{(q-n)!}{(q-n-j)!j!} \frac{d^{q-n-j}}{dx^{q-n-j}} w_0(x) \frac{d^j}{dx^j} w_{l-q}(x) \right) \\
& + \sum_{n=0}^{l-1} \left(\sum_{k=0}^{l-1-n} \left\{ \sum_{m=1}^{l-k-n} \left[\sum_{\substack{j_m+\dots+j_1=q-k+m-n \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k!j_m! \dots j_1!m!} \frac{\partial^{l+m-n}}{\partial p^{l+m-n}} L_n(x, p) \Big|_{p=DS} \right. \right. \right. \\
& \left. \left. \left. D^{j_m} S \times \dots \times D^{j_1} S \right] \right\} \frac{d^k}{dx^k} w_0(x) + \frac{1}{(l-n)!} \frac{\partial^{l-n}}{\partial p^{l-n}} L_n(x, p) \Big|_{p=DS} \frac{d^{l-n}}{dx^{l-n}} w_0(x) \right) \\
& \left. + L_l(x, DS) w_0(x) \right\} = 0
\end{aligned}
\tag{L.27}$$

We collect terms of the same order in the parameter β in the transport equation (L.23).

This leads to a sequence of transport equations for the wave amplitude corrections $w_n(x)$, $n = 0, 1, \dots$. The transport equation of order $O(\beta^N)$ is given by the equation

$$\begin{aligned}
& \sum_{l=1}^{N-1} \left[\sum_{n=0}^{l-1} \left(\sum_{k=0}^{l-1-n} \left\{ \sum_{m=1}^{l-k-n} \left[\sum_{\substack{j_m+\dots+j_1=l-k+m-n \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k!j_m! \dots j_1!m!} \frac{\partial^{l+m-n}}{\partial p^{l+m-n}} L_n(x, p) \Big|_{p=DS} \right. \right. \right. \\
& \quad \left. \left. \left. D^{j_m} S \times \dots \times D^{j_1} S \right\} \left(\sum_{j=0}^k \frac{k!}{(k-j)!j!} \frac{d^{k-j}}{dx^{k-j}} w_0(x) \frac{d^j}{dx^j} w_{N-l}(x) \right) \right) \right. \\
& \quad \left. + \frac{1}{(l-n)!} \frac{\partial^{l-n}}{\partial p^{l-n}} L_n(x, p) \Big|_{p=DS} \left(\sum_{j=0}^{l-n} \frac{(l-n)!}{(l-n-j)!j!} \frac{d^{l-n-j}}{dx^{l-n-j}} w_0(x) \frac{d^j}{dx^j} w_{N-l}(x) \right) \right] \Bigg\} \\
& \quad + \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1-n} \left\{ \sum_{m=1}^{N-k-n} \left[\sum_{\substack{j_m+\dots+j_1=N-k+m-n \\ j_m, \dots, j_1 \geq 2}} \frac{1}{k!j_m! \dots j_1!m!} \frac{\partial^{N+m-n}}{\partial p^{N+m-n}} L_n(x, p) \Big|_{p=DS} \right. \right. \right. \\
& \quad \left. \left. \left. D^{j_m} S \times \dots \times D^{j_1} S \right\} D^k w_0(x) + \frac{1}{(N-n)!} \frac{\partial^{N-n}}{\partial p^{N-n}} L_n(x, p) \Big|_{p=DS} D^{N-n} w_0(x) \right) \\
& \quad + L_N(x, DS) w_0(x) + \frac{1}{2} \frac{\partial^2}{\partial p^2} L_0(x, p) \Big|_{p=DS} D^2 S w_0(x) \\
& \quad + \frac{\partial}{\partial p} L_0(x, p) \Big|_{p=DS} ((D w_0(x)) w_{N-1}(x) + w_0(x) D w_{N-1}(x)) + L_1(x, DS) w_0(x) w_{N-1}(x) = 0
\end{aligned} \tag{L.28}$$

For $N = 0$, we have the transport equation for the main term of the wave amplitude. The transport equation for $w_0(x)$ is

$$\frac{\partial}{\partial p} L_0(x, p) \Big|_{p=DS} \frac{d}{dx} w_0(x) + \left(\frac{1}{2!} \frac{\partial^2}{\partial p^2} L_0(x, p) \Big|_{p=DS} D^2 S + L_1(x, DS) \right) w_0(x) = 0 \tag{L.29}$$

The transport equations of order $O(\beta^N)$ can be written as non-homogeneous first order differential equations of the form

$$\begin{aligned}
\frac{d}{dx}w_{N-1}(x) = & -\frac{1}{\frac{\partial}{\partial p}L_0(x,p)|_{p=DS}w_0(x)} \left[\sum_{l=2}^{N-1} \left\{ \sum_{n=0}^{l-1} \left(\sum_{k=0}^{l-1-n} \left[\sum_{j=0}^k \left\{ \sum_{m=1}^{l-k-n} \left(\sum_{\substack{j_m+\dots+j_1=l-k+m-n \\ j_m,\dots,j_1 \geq 2}} \right. \right. \right. \right. \right. \right. \\
& \left. \left. \left. \frac{1}{(k-j)!j!j_m!\dots j_1!m!} \frac{\partial^{l+m-n}}{\partial p^{l+m-n}} L_n(x,p)|_{p=DS} D^{j_m} S \times \dots \times D^{j_1} S \right) \right\} \right. \\
& \left. \frac{d^{k-j}}{dx^{k-j}} w_0(x) \frac{d^j}{dx^j} w_{N-l}(x) \right] + \sum_{j=0}^{l-n} \left[\frac{1}{(l-n-j)!j!} \frac{\partial^{l-n}}{\partial p^{l-n}} L_n(x,p)|_{p=DS} \right. \\
& \left. \frac{d^{l-n-j}}{dx^{l-n-j}} w_0(x) \frac{d^j}{dx^j} w_{N-l}(x) \right] \left. \right\} + \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1-n} \left[\sum_{m=1}^{N-k-n} \left(\sum_{\substack{j_m+\dots+j_1=N-k+m-n \\ j_m,\dots,j_1 \geq 2}} \right. \right. \right. \right. \\
& \left. \left. \left. \frac{1}{k!j_m!\dots j_1!m!} \frac{\partial^{N+m-n}}{\partial p^{N+m-n}} L_n(x,p)|_{p=DS} D^{j_m} S \times \dots \times D^{j_1} S \right) \right] \frac{d^k}{dx^k} w_0(x) \right. \\
& \left. + \frac{1}{(N-n)!} \frac{\partial^{N-n}}{\partial p^{N-n}} L_n(x,p)|_{p=DS} \frac{d^{N-n}}{dx^{N-n}} w_0(x) \right) + L_N(x, DS) \Big]
\end{aligned} \tag{L.30}$$

This transport equation is valid for the case where the symbols $L_n(x, p)$ are transcendental analytic functions of the variable p . When the symbol $L(x, p, \beta)$ is given by a finite sum of symbols $L_n(x, p)$ which are polynomials in the variable p (see chapter 3), the transport equation for the amplitude corrections $w_n(x)$ is of the form given by equation (3.122) of chapter 3.

We also provide a Maple V procedure which gives the transport equation (L.30) for the amplitude corrections $w_n(x)$, $n = 0, 1, \dots$. This procedure is named GNTPEQ2, and it follows below.

```

GNTPEQ2:=proc(G,y,n,key) local v,p,t,w,il;
# key =      0 - transport equation
# key =      1 - right side of the transport equation
if n= 0 then
    il:= 1;
    t:=y-[n0] ;

```

```

v:=GNTPEQ(G,t,i1);
if key = 0 then
    diff(y_[n](x),x$1)=solve(v,diff(y_[0](x),x$1));
else
    solve(v,diff(y_[0](x),x$1));
fi;
else
    for i1 from 1 to n+1 do
        if n+1-i1 = 0 then
            t:=y_[n+1-i1];
        elif i1 = 1 then
            t:=y_[n+1-i1];
        else
            t:=y_[0]*y_[n+1-i1];
        fi;
        if i1 = 1 then
            v:=diff(t(x),x$1)/y_[0](x);
        else
            if i1 < n+1 then p:=( GNTPEQ(G,t,i1) )
            else p:=GNTPEQ(G,t,i1) fi;
            v:=v+p;
        fi;
    od;
    if key = 0 then
        diff(y_[n](x),x$1)=solve(v,diff(y_[n](x),x$1));
    else
        solve(v,diff(y_[n](x),x$1));
    fi;
fi;
end:

```

Notice that this procedure uses the previously defined Maple V procedure GNTPEQ. If we want the transport equation, for example, for the amplitude correction $w_3(x)$, we just need to type in the Maple V prompt GNTPEQ2(S,w,3,0);, and if we just want the right hand side of equation (L.30) for, let us say, $N - 1 =$, we type at the Maple V prompt GNTPEQ2(S,w,3,1);. When the symbol $L(x, p, \beta)$ is given by a finite sum of symbols $L_n(x, p)$ which are polynomials in the variable p , the transport equation for the amplitude correction $w_n(x)$ can be computed symbolically by using the Maple V procedure GNTPEQ2B, which follows.

```
GNTPEQ2B:=proc(G,y,n,l,key) local v,p,t,w,i1,i2;
# key = 0      - transport equation
# key = 1      - right side of the transport equation
    if n = 0 then
        i1:=1;
        t:=y-[n];
        v:=GNPEQB(G,t,i1,l);
        if key = 0 then
            D(y-[n])(x)=solve(v,diff(y-[0](x),x$1));
        else
            solve(v,diff(y-[0](x),x$1));
        fi;
    else
        for i1 from 1 to n+1 do
            if n+1-i1 = 0 then
                t:=y-[n+1-i1];
            elif i1 = 1 then
                t:=y-[n+1-i1];
            else
                t:=y-[0]*y-[n+1-i1];
            fi;
        end do;
    end if;
end proc;
```

```

fi;
if i1 = 1 then
    v:=diff(t(x),x$1)/y-[0](x);
else
    if i1 ≤ n+1 then p:=( GNTPEQB(G,t,i1,l) )
    else p:=GNTPEQB(G,t,i1,l) fi;
    v:=v+p;
fi;
od;
if key = 0 then
    v:=convert(solve(v,diff(y-[n](x),x$1)),D);
    if n > 1 then
        for i1 from 1 to n-1 do
            for i2 from 0 to min(n+1,l) do
                v:=collect(v,(D@@i2)(y-[i1])(x));
            od;
        od;
    fi;
    for i1 from 0 to n+1 do
        v:=collect(v,(D@@i1)(y-[0])(x));
    od;
    D(y-[n])(x)=v;
else
    solve(v,diff(y-[n](x),x$1));
fi;
fi;
end:

```

This Maple V procedure generated the transport equations for the amplitude correc-

tions $w_{1j}(z)$ and $w_{2j}(z)$ in chapter 3. The letters G and y in the procedure entry stands for, respectively, the phase function $S(x)$ and the wave amplitude $w(x)$. The index n specifies for which amplitude correction we want to obtain the transport equation. the index l specifies the order of the symbol $L_0(x, p)$ with respect to the variable p . The same assumptions made in GNTPEQB regarding the symbols $L_n(x, p)$ are also valid for this procedure.

Appendix M

Local Form of the Liouville-Green functions.

In this appendix we derived the approximate form of the Liouville-Green functions of the form given by equation (4.55) with $w(x)$ given as a series expansion in the parameter β according to equation (4.60). This approximate form of the Liouville-Green functions is valid in a neighborhood of the turning points. We differentiate between the turning points where one propagating wave mode couple with one evanescent wave mode and turning points where the two propagating modes coalesce, since the approximate form of the Liouville-Green function is not the same for both cases.

The approach here is not to approximate the Liouville-Green functions themselves. We first find the approximate form of the Eikonal equation (4.57) and the approximate form of the transport equations (4.61) and (4.62) which are valid in a neighborhood of a turning point. The Liouville-Green functions have a singularity at a turning point (become unbounded at these points). therefore, the approximate form of the transport equations (4.61) and (4.62) in a neighborhood of a turning point is just the part of the transport equations which give the leading term of the Liouville-Green functions at the turning point. Second, we obtain the approximate form of the phase function. The wave numbers coalescing at the considered turning point are solutions of the approximate dispersion relation given by equation (4.91) or given by the equation (4.95), so the phase function follows directly from these equations.

Third, we obtain the approximate form of the amplitude corrections $w_n(x)$ from the approximate transport equations. This procedure results in the approximate form of the Liouville-Green functions, valid in a neighborhood of a turning point.

M.1 Approximate Liouville-Green Functions for Turning Points where a Propagating and an Evanescent Wave Modes Couple.

We start the procedure described above by approximating the Eikonal equation for the wave numbers $\frac{dS}{dx}$.

- Approximate form of the phase function $S(x)$. In section 4.5.2.2 the approximate dispersion relation is given by equation (4.91), so the approximate Eikonal equation is

$$L_0(x, k_c(x)) + \frac{1}{2} \frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=k_c(x)} (DS - k_c(x))^2 = 0 \quad (\text{M.1})$$

The approximate form of the phase function $S(x)$ follows as

$$S(x) = \int_{x^*}^x \left(k_c(t) \pm \left\{ -2 \frac{-\alpha^2 + L_0(t, k_c(t))}{\frac{\partial^2 L_0}{\partial p^2}(t, p)|_{p=k_c(t)}} \right\}^{1/2} \right) dt \quad (\text{M.2})$$

- Approximate form of the high order symbols in a neighborhood of a branch point. The idea is to reduce the symbol of the operator $\hat{L}(x, -i\beta \frac{d}{dx}, \beta)$ to a symbol of a second order equation, since in a neighborhood of a branch point only two wave modes coalesce. The symbol of this operator is given as a series expansion in the parameter β according to the equation (4.38). For $(p, x) = (k_s, x_b)$ (saddle point and its associated branch point), we have that

$$ph(x) = \frac{\sigma}{(\sigma)^2 - 1} \quad (\text{M.3})$$

$$\tanh(ph(x)) = \sigma \quad (\text{M.4})$$

$$\frac{1}{\cosh^2(ph(x))} = 1 - (\sigma)^2 \quad (\text{M.5})$$

and from the general expression for the symbols $L_n(x, p)$ given by equation (4.52), we can see that

$$L_0(x_b, k_s) = 0 \quad \text{and} \quad \frac{\partial}{\partial p} L_0(x_b, p)|_{k_s} = 0 \quad (\text{M.6})$$

$$L_n(x_b, k_s) \neq 0 \text{ for } n \geq 1 \quad \text{and} \quad \frac{\partial}{\partial p} L_n(x_b, p)|_{p=k_s} \neq 0 \text{ for } n \leq 1 \quad (\text{M.7})$$

We approximate the symbol $L_0(x, p)$ in the same way we approximated it to obtain the approximate equation (4.97) for the free surface potential $\psi(x)$. So we have

$$L_0(x, p) \sim L_0(x, k_c(x)) + \frac{\partial^2 L_0}{\partial p^2}(x, k_c(x))(p - k_c(x))^2 \text{ as } x \rightarrow x_b(p \rightarrow k_s) \quad (\text{M.8})$$

where $k_c(x)$ is the wave mode boundary defined in section 4.5.2.2, and given by the equation (4.93). The higher order symbols are approximated in the form given by the equation

$$L_n(x, p) \sim L_n(x, k_c(x)) + \frac{\partial L_n}{\partial p}(x, k_c(x))(p - k_c(x)) \quad (\text{M.9})$$

- For each of the transport equations, we are going to obtain its approximate form as $(x, p) \rightarrow (x_b, k_s)$. To do so, we need the order of magnitude of the derivatives of the wavenumber DS , and the order of magnitude of $w_0(x)$ and its derivatives as $(x, p) \rightarrow (x_b, k_s)$. For the derivatives of the wavenumbers, the leading term as $(x, p) \rightarrow (x_b, k_s)$ is:

$$D^n S \sim - \frac{(2n-5)!}{2^{(n-3)}(n-3)!} \frac{\left(\frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=DS}\right)^{(n-2)} \left(\frac{\partial}{\partial p} L_0(x, p)|_{p=DS}\right)^{(n-1)}}{\left(\frac{\partial}{\partial p} L_0(x, p)|_{p=DS}\right)^{(2n-3)}} \quad \text{as } x \rightarrow x_b(p \rightarrow k_s) \quad (\text{M.10})$$

For the derivatives of function $w_0(x)$, the leading term is:

$$\frac{d^n}{dx^n} w_0(x) \sim \left\{ \sum_{i=1}^n \frac{1}{i!} \left[\sum_{j_1+\dots+j_n=n} \frac{n!}{j_1! \dots j_n!} \frac{(2i-1)!}{2^{(2i-1)}(i-1)!} \right] \right\} \frac{\left(\frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=DS}\right)^n \left(\frac{\partial}{\partial x} L_0(x, p)|_{p=DS}\right)^n}{\left(\frac{\partial}{\partial p} L_0(x, p)|_{p=DS}\right)^{(4n+1)/2}} \quad (\text{M.11})$$

Now we can use the approximations given by equations (M.10) and (M.11) to obtain the leading term of the transport equations (4.61) and (4.62) as $(x, p) \rightarrow (x_b, k_s)$. The leading term of these transport equations follows below.

- Leading term of the transport equation of order $O(\beta^2)$ as $(x, p) \rightarrow (x_b, k_s)$.

$$\frac{d}{dx} w_1(x) \sim -\frac{1}{2} \left(\frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=DS} \right) w_0(x) \frac{d^2}{dx^2} w_0(x) \quad (\text{M.12})$$

From this result we can say that:

$$w_1(x) \sim O \left(\left[\frac{\partial}{\partial p} L_0(x, p)|_{p=DS} \right]^{-3} \right) \quad (\text{M.13})$$

$$\frac{d}{dx} w_1(x) \sim O \left(\left[\frac{\partial}{\partial p} L_0(x, p)|_{p=DS} \right]^{-5} \right) \quad (\text{M.14})$$

$$\frac{d^2}{dx^2} w_1(x) \sim O \left(\left[\frac{\partial}{\partial p} L_0(x, p)|_{p=DS} \right]^{-7} \right) \quad (\text{M.15})$$

as $(x, p) \rightarrow (x_b, k_s)$. If we proceed in the same way with the transport equation of order $O(\beta^3)$ as we did with the transport equation of order $O(\beta^2)$, and if we take into account equations (M.13) to (M.15), we realize that

$$w_2(x) \sim O \left(\left[\frac{\partial L_0}{\partial p}(x, p)|_{p=DS} \right]^{-6} \right) \quad (\text{M.16})$$

We can repeat this procedure again with the transport equation of $O(\beta^4)$ and high order equations. This leads to the estimate

$$w_n(x) \sim O \left(\left[\frac{\partial L_0}{\partial p}(x, p)|_{p=DS} \right]^{-3n} \right). \quad (\text{M.17})$$

- Leading term of the transport equation of order $O(\beta^{(n+1)})$ as $(x, p) \rightarrow (x_b, k_s)$.

$$\begin{aligned} \frac{d}{dx} w_n(x) \sim \frac{1}{2} \left(\frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=DS} \right) & \left\{ -w_0^2(x) \frac{d^2}{dx^2} w_{n-1}(x) - 2w_0(x) \frac{d}{dx} w_0(x) \frac{d}{dx} w_{n-1}(x) \right. \\ & \left. - w_0 \frac{d^2}{dx^2} w_0(x) w_{n-1}(x) \right\} \text{ for } n \leq 2. \end{aligned} \quad (\text{M.18})$$

We are going to show below that the recurrence relation above is the same recurrence relation we obtain when we apply the WKB scheme to a second

order differential equation without first order derivatives. The first step is to approximate $w_0(x)$ as $(x, p) \rightarrow (x_b, k_s)$. According to equation (4.63), we know that

$$w_0(x) = \left(\frac{\partial}{\partial p} L_0(x, p)|_{p=DS} \right)^{-1/2}.$$

According to equation (M.8), $w_0(x)$ can be approximated as

$$w_0(x) \sim \left(\frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=k_c(x)} (DS - k_c(x)) \right)^{-1/2}. \quad (\text{M.19})$$

As $(x, p) \rightarrow (x_b, k_s)$, the wavenumber DS , according to equation (4.91), is approximated as

$$DS = k_c(x) \pm iQ(x)^{1/2}.$$

As a result, the expression for $w_0(x)$ reduces to

$$w_0(x) \sim \left(\frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=k_c(x)} (\pm iQ(x)^{1/2}) \right)^{-1/2} \quad (\text{M.20})$$

and

$$\begin{aligned} \frac{d}{dx} w_0(x) \sim & -\frac{1}{2} \left(\frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=k_c(x)} (\pm iQ(x)^{1/2}) \right)^{-3/2} \frac{\partial^2}{\partial p^2} L_0(x, p)|_{p=k_c(x)} \\ & \left(\pm \frac{i}{2} Q(x)^{-1/2} \frac{dQ}{dx} \right) \end{aligned} \quad (\text{M.21})$$

$$\begin{aligned} \frac{d^2}{dx^2} w_0(x) \sim & \left(\frac{\partial^2}{\partial p^2} L_0(x, p) \Big|_{p=k_c(x)} \right)^{-1/2} \left\{ \frac{3}{4} (\pm i Q(x)^{1/2})^{-5/2} \left[\pm \frac{i}{2} Q(x)^{-1/2} \frac{dQ}{dx} \right]^2 \right. \\ & \left. - \frac{1}{2} (\pm i Q(x)^{1/2})^{-3/2} \left[\mp \frac{i}{4} Q(x)^{-3/2} \left(\frac{dQ}{dx} \right)^2 \right] \right\} \end{aligned} \quad (\text{M.22})$$

If we go back to the transports equations (M.12) and (M.18), and if we substitute these expressions, we will see that we recover the leading term of the transport equations for a second order differential equation. The form of the transport equations (M.12) and (M.18) is

$$\begin{aligned} \frac{d}{dx} w_1(x) \sim & -\frac{1}{2} (\pm i Q(x)^{1/2})^{-1/2} \left\{ \frac{3}{4} (i Q(x)^{1/2})^{-5/2} \left[\pm \frac{i}{2} Q(x)^{-1/2} \frac{dQ}{dx} \right]^2 \right. \\ & \left. - \frac{1}{2} (\pm i Q(x)^{1/2})^{-3/2} \left[\mp \frac{i}{4} Q(x)^{-3/2} \left(\frac{dQ}{dx} \right)^2 \right] \right\} \end{aligned} \quad (\text{M.23})$$

$$\begin{aligned} \frac{d}{dx} w_n(x) \sim & -\frac{1}{2} \frac{1}{\pm i Q(x)^{1/2}} \frac{d^2}{dx^2} w_{n-1}(x) + \frac{1}{2} (\pm i Q(x)^{1/2})^{-2} \left(\pm \frac{i}{2} Q(x)^{-1/2} \frac{dQ}{dx} \right) \frac{d}{dx} w_{n-1}(x) \\ & - \frac{1}{2} (\pm i Q(x)^{1/2})^{-1/2} \left\{ \frac{3}{4} (\pm i Q(x)^{1/2})^{-5/2} \left[\pm \frac{i}{2} Q(x)^{-1/2} \frac{dQ}{dx} \right]^2 \right. \\ & \left. - \frac{1}{2} (\pm i Q(x)^{1/2})^{-3/2} \left[\mp \frac{i}{4} Q(x)^{-3/2} \left(\frac{dQ}{dx} \right)^2 \right] \right\} w_{n-1}(x) \end{aligned} \quad (\text{M.24})$$

- Asymptotic solution for the approximate equation (4.97). In section 4.5.3.1, we obtained the Liouville-Green functions (WKB approximation) for the approximate equation (4.97). The phase function obtained in section 4.5.3.1 is the same phase function we obtained here. The leading term of the transport equations

(4.104) and (4.105) for the amplitudes $\tilde{w}^\pm(x)$ are match with the equations (M.23) and (M.24). This can be shown by calculating the leading term of the first and second derivatives of $\tilde{w}_0^\pm(x)$, which follows

$$\frac{d}{dx}\tilde{w}_0(x) \sim (\pm iQ(x)^{1/2})^{-3/2} \left(\pm \frac{i}{2}Q(x)^{-1/2}\frac{dQ}{dx} \right), \quad (\text{M.25})$$

$$\begin{aligned} \frac{d^2}{dx^2}\tilde{w}_0(x) \sim & \pm \frac{3i}{8}(\pm iQ(x)^{1/2})^{-5/2}Q(x)^{-1}\frac{dQ}{dx} \left(\frac{dQ}{dx} \right)^2 \\ & \mp \frac{i}{4}(\pm iQ(x)^{1/2})^{-3/2} \left[-\frac{1}{2}Q(x)^{-3/2} \left(\frac{dQ}{dx} \right)^2 \right], \end{aligned} \quad (\text{M.26})$$

and by substituting equations (M.25) and (M.26) into the transport equation (4.105).

Therefore, close to a turning point where one propagating mode couple with a evanescent wave mode, Liouville-Green functions for the free surface potential governing equation (4.54) assume the form of the Liouville-Green functions for the approximate equation (4.97). These Liouville-Green functions were obtained in section 4.5.3.1.

M.2 Approximate Liouville-Green Functions for Turning Points where the two a Propagating Wave Modes Couple.

The procedure to approximate the Liouville-Green Function in a neighborhood of a turning point where the two propagating wave modes couple follows the same steps as in the case of turning points where propagating modes couple with evanescent wave modes. In this case the symbol $L_0(x, p)$ is approximated according to the equation

$$L_0(x, p) \sim p(h(x)p), \quad (\text{M.27})$$

and the uniform system dispersion relation is approximated according to equation (4.95). For this case, we also need the approximate form of the symbol $L_1(x, p)$, which is given by the equation

$$L_1(x, p) \sim -\frac{dH}{dx}p. \quad (\text{M.28})$$

Besides the necessity to include the symbol $L_1(x, p)$ in the derivation of the local form of the Liouville-Green function, the steps to perform this task are basically the same steps followed in the previous section.

- Approximate form of the phase function $S(x)$. The wavenumber $DS(x)$ is the solution of the local form of the dispersion relation, given by the equation

$$-\alpha^2 + (DS(x))^2 h(x) = 0. \quad (\text{M.29})$$

This equation has two solutions, and they give two possible phase functions

$$S(x) = \pm \int_{x^*}^x \frac{\alpha}{\sqrt{h(t)}} dt \quad (\text{M.30})$$

where x^* is the considered branch point of the uniform depth dispersion relation.

- We need the local form of the symbols $L_0(x, p)$ and $L_1(x, p)$, which were given above. The derivatives of the phase $S(x)$ close to the considered branch point have the order of magnitude

$$D^n S(x) \sim O(h(x)^{-(2n-1)/2}) \text{ as } x \rightarrow x^* \text{ which implies } h \rightarrow 0. \quad (\text{M.31})$$

The function $w_0(x)$ is approximated as according to the equation

$$w_0(x) \sim \{\pm 2\alpha h(x)^{1/2}\}^{1/2} \sim O(h^{-1/4}) \text{ as } h \rightarrow 0. \quad (\text{M.32})$$

The order of magnitude of its high order derivative is given according to the equation

$$\frac{d^n w_0}{dx^n}(x) \sim O(h^{-\frac{4n+1}{4}}) \text{ as } h \rightarrow 0. \quad (\text{M.33})$$

- Based on the approximations above for the phase $S(x)$, for its derivatives, for the function $w_0(x)$ and for its derivatives, we give below the approximate form of the transport equations for the amplitudes corrections $w_n(x)$. The transport equation of $O(\beta^2)$ is

$$\frac{dw_1}{dx} \sim -\frac{1}{2} \left(\frac{\partial^2 L_0}{\partial p^2}(x, p)|_{p=DS} \right) w_0(x) \frac{d^2 w_0}{dx^2} - w_0(x) \frac{dw_0}{dx} \left(\frac{\partial L_1}{\partial p}(x, p)|_{p=DS} \right), \quad (\text{M.34})$$

and the transport equation of order $O(\beta^{(n+1)})$ in a neighborhood of the considered branch point assume the form

$$\begin{aligned}
\frac{dw_{n+1}}{dx} \sim & -w_0^2(x) \left[\frac{1}{2} \left(\frac{\partial^2 L_0}{\partial p^2}(x, p)|_{p=DS} \right) \frac{d^2 w_n}{dx^2} + \left(\frac{\partial L_1}{\partial p}(x, p)|_{p=DS} \right) \frac{dw_n}{dx} \right] \\
& - w_0 \frac{dw_0}{dx} \left[\left(\frac{\partial^2 L_0}{\partial p^2}(x, p)|_{p=DS} \right) \frac{dw_n}{dx} + \left(\frac{\partial L_1}{\partial p}(x, p)|_{p=DS} \right) w_n \right] \\
& - w_0(x) \frac{d^2 w_0}{dx^2} \left(\frac{1}{2} \frac{\partial^2 L_0}{\partial p^2}(x, p)|_{p=DS} \right) w_n(x); n \geq 2
\end{aligned}
\tag{M.35}$$

If we solve the transport equations above, we obtain the correction amplitudes given by equations (4.116) to (4.120). So, close to a branch point where the two propagating modes couple, the amplitude corrections $w_n(x)$ are approximated by the amplitude corrections $w_n^\pm(x)$.

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