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**GENERAL PROOF OF SYMMETRY RELATIONS
IN MODE CONVERSION WITH APPLICATION TO,
EXCITATION OF, AND EMISSION FROM,
ELECTRON BERNSTEIN WAVES**

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General Proof of Symmetry Relations in Mode Conversion With Application to, Excitation of, and Emission From, Electron Bernstein Waves

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Abstract

From general principles of wave energy flow conservation and wave energy flow in the time reversed system, we derive symmetry properties that must be satisfied by the coupling of waves in dissipation-free mode conversion regions of an inhomogeneous plasma confined by a magnetic field. The results are valid for any kinetic (Vlasov-Maxwell) full-wave description of a plasma in a magnetic field with an equilibrium distribution function symmetric in momentum parallel to the magnetic field. Thus, without solving the full-wave equations, we obtain the symmetries in transmission, reflection, and mode conversion excitation and emission. The important physical and practical consequences of the various symmetries are illustrated for O and X modes coupling to electron Bernstein waves in the vicinity of the upper-hybrid resonance, with adjoining cutoffs, relevant to plasma heating and current drive, and emission from high- β spherical tori (ST) plasmas.

1. Introduction

We consider linear mode conversions (MC) in a loss-free (LF) region of the plasma. Outside of this mode conversion region (MCR), the waves are described by their WKB eikonal form and have wave energy flows into and out of the MCR. From general properties of wave energy flow conservation and Onsager-like time reversibility on wave energy flow, we derive symmetry properties that must be satisfied by waves undergoing coupling through mode conversion [1]. For the electron cyclotron range of frequencies, we apply these results to the excitation of electron Bernstein waves (EBWs) by MC, in the vicinity of the upper-hybrid resonance (UHR), from either X or O modes at the outside edge of a plasma, as well as to the reverse mode conversion, relevant to emission, from EBWs to X and O modes. In particular, we focus on the novel circumstances relevant to EBW excitation for heating and/or current drive, and emission from EBWs for plasma diagnostics, in high- β STs (e.g., NSTX) where the MCR in the vicinity of the UHR contains the left (L), the right (R), and the ordinary (O) cutoffs [2], and the EBW experiences Doppler-shifted cyclotron resonance or Landau absorption inside the plasma, away from the MCR.

2. Symmetries in Dissipation-Free Mode Conversions

Consider a one-dimensional (in x) generic propagation and mode conversion situation in an inhomogeneous plasma, with unperturbed (equilibrium) parameters (e.g., density, temperature, and magnetic field) that vary in x , as shown schematically in Figure 1. Any Landau and/or Doppler shifted cyclotron resonance absorption is assumed to occur outside the LF-MCR.

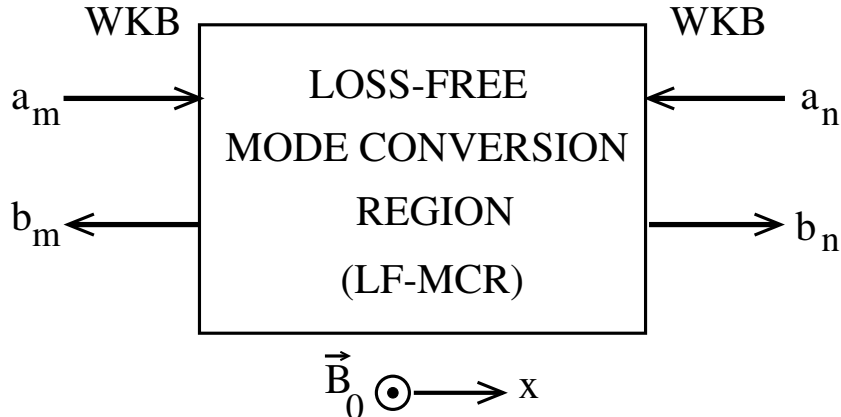


Figure 1: *Loss-free mode conversion region with incident and reflected (WKB) waves in adjoining regions; $m \equiv m_1, m_2, \dots$; $n \equiv n_1, n_2, \dots$*

In the WKB regions to the right and left of the LF-MCR, we let the complex field amplitudes of (e.g., forward) waves with energy flow into and out of the LF-MCR be, respectively,

$$a_p \sim \exp(ik_x x - i\omega t) \quad (1)$$

$$b_p \sim \exp(-ik_x x - i\omega t) \quad (2)$$

and normalized so that

$$|a_p|^2 = \text{energy flow density into the LF-MCR} \quad (3)$$

$$|b_p|^2 = \text{energy flow density out of the LF-MCR} \quad (4)$$

[For backward waves, retaining the normalizations in (3) and (4), the signs of the k_x 's in (1) and (2) will change.] In Figure 1, such modes on the left of the LF-MCR have $p = m$ (there can be any number of such modes: m_1, m_2, \dots), and on the right of the LF-MCR, similarly, $p = n(n_1, n_2, \dots)$. For

a weakly dissipative mode, the total wave energy flow density (electromagnetic plus kinetic) is given, in general, by [3]

$$\langle s_x \rangle_p = \left[\frac{1}{2} \text{Re} (\vec{E} \times \vec{H}^*)_x - \frac{\varepsilon_0}{4} \omega \frac{\partial \chi_{\alpha\beta}^H}{\partial k_x} E_\alpha E_\beta^* \right]_p, \quad (5)$$

where $\chi_{\alpha\beta}^H$ is the Hermitian part of the susceptibility tensor $\chi_{\alpha\beta}(\vec{k}, \omega)$ with \vec{k} and ω real [4] and the star superscript denotes the complex conjugate. Since the full-wave equations describing the LF-MCR are linear (in general, linear integro-partial-differential equation) with appropriate boundary conditions, the complex field amplitudes a_p and b_p are related by a unique scattering matrix $\overline{\overline{S}}$

$$\vec{b} = \overline{\overline{S}} \cdot \vec{a} \quad (6)$$

where \vec{b} and \vec{a} are column vectors containing complex amplitudes of all b_p and all a_p , respectively.

From *energy flow conservation* applied to the LF-MCR, we have

$$\sum_p (|a_p|^2 - |b_p|^2) = 0; \quad p = \text{all } m + \text{all } n, \quad (7)$$

where the sum is over all m 's and n 's. Using (6), we can express this as

$$\vec{a}^\dagger \cdot \left(\overline{\overline{I}} - \overline{\overline{S}}^\dagger \cdot \overline{\overline{S}} \right) \cdot \vec{a} = 0, \quad (8)$$

where the dagger superscript on $\overline{\overline{S}}$ denotes the complex-conjugate-transpose of $\overline{\overline{S}}$. Since (8) must hold true for arbitrary \vec{a} , it follows that

$$\overline{\overline{S}}^\dagger = \overline{\overline{S}}^{-1}. \quad (9)$$

Next, consider *wave energy flow under time reversibility*. For the time reversed system, the direction of time-averaged energy flow density changes sign. In other words, the reversal of time changes time-averaged energy flow into the mode conversion region to time-averaged energy flow out of the mode conversion region, and vice versa. From (5), energy flow reversal is obtained by setting [5]

$$\vec{E} \rightarrow \vec{E}^* \quad \vec{H} \rightarrow -\vec{H}^* \quad \vec{k} \rightarrow -\vec{k} \quad (10)$$

and, by (1) and (2), time reversal gives

$$a_p \rightarrow b_p^* \quad \text{and} \quad b_p \rightarrow a_p^* , \quad (11)$$

where the star superscript denotes the complex conjugate. Referring to Figure 1, the effect of time reversal is to change a to b^* and b to a^* , with arrows pointing in the same direction as indicated in the figure. Thus

$$\vec{a}^* = \overline{\overline{S}} \cdot \vec{b}^* \quad (12)$$

or, taking the complex conjugate of (12),

$$\vec{a} = \overline{\overline{S}}^* \cdot \vec{b} . \quad (13)$$

But from (6) $\vec{a} = \overline{\overline{S}}^{-1} \cdot \vec{b}$; hence

$$\overline{\overline{S}}^* = \overline{\overline{S}}^{-1} . \quad (14)$$

Combining (9) and (14), we finally obtain:

$$\overline{\overline{S}}^\dagger = \overline{\overline{S}}^* \quad \text{or equivalently} \quad \overline{\overline{S}}^T = \overline{\overline{S}} \quad (15)$$

where the T superscript on $\overline{\overline{S}}$ denotes the transpose of $\overline{\overline{S}}$. Hence, *the LF-MCR scattering matrix is symmetric*. The MC symmetries for a LF-MCR can be readily extended to 3-D propagation and mode conversion [1].

The symmetry of the LF-MC scattering matrix, $S_{ij} = S_{ji}$, entails important relationships for various power coefficients of the mode conversion process:

$$|S_{ij}|^2 = \left| \frac{b_i}{a_j} \right|^2 = \left| \frac{b_j}{a_i} \right|^2 = |S_{ji}|^2 \quad (16)$$

Thus:

- (a) For disparate modes on the two sides of the LF-MCR, the *mode conversion power coefficients* are equal.
- (b) For disparate modes on the same side of the LF-MCR, the *cross reflection* (they are in essence also *mode conversion*) *power coefficients* are equal.

- (c) For the same modes on the two sides of the LF-MCR, the *transmission power coefficients* are equal.

3. Mode Conversion of O and/or X Modes to Electron Bernstein Waves (EBW) at the Upper-Hybrid Resonance (UHR)

We assume that the Doppler-shifted electron cyclotron resonance is outside the MCR containing the UHR, and thus the MCR is taken as loss-free. We also consider an example of situations relevant to high- β plasmas where the slow X-mode cutoff occurs inside the MCR [2], as illustrated by the local dispersion relation in Figure 2. The local dispersion relation in Figure 2 is sketched for two sets of values for (ω, k_z) to indicate important differences in the two optimal couplings through MC: X-EBW and O-EBW.

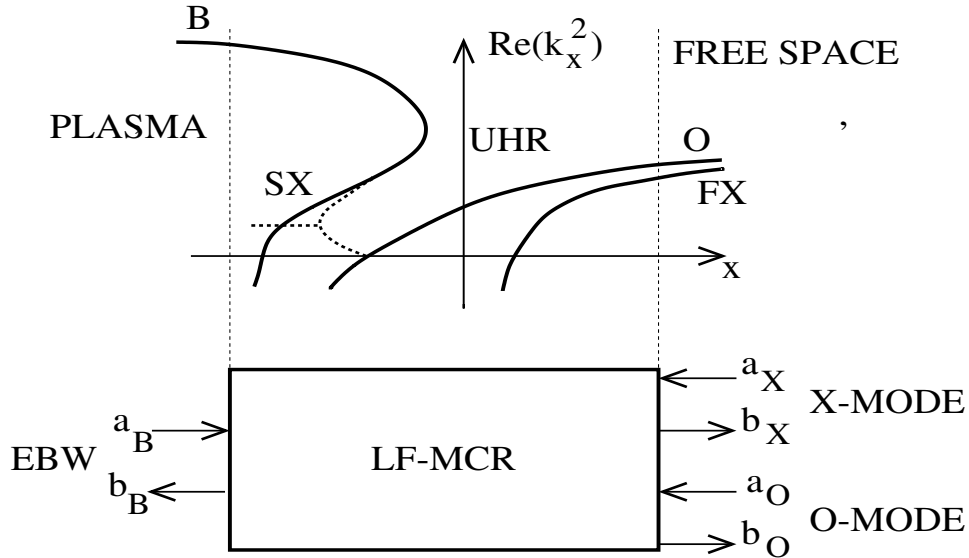


Figure 2: Sketch of the local dispersion relation for O- and/or X-EBW mode conversion near UHR. Two (ω, k_z) situations are depicted; solid lines for a case when the O-mode cutoff is well-separated from the SX cutoff, and solid lines with SX branch modified as shown by dashed line when the SX cutoff coincides with the O-mode cutoff.

The scattering matrix is given by:

$$\begin{pmatrix} b_B \\ b_X \\ b_O \end{pmatrix} = \begin{pmatrix} S_B & S_{BX} & S_{BO} \\ S_{XB} & S_X & S_{XO} \\ S_{OB} & S_{OX} & S_O \end{pmatrix} \begin{pmatrix} a_B \\ a_X \\ a_O \end{pmatrix} \quad (17)$$

The symmetries required by (15) give three relations: $S_{BX} = S_{XB}$, $S_{BO} = S_{OB}$, and $S_{XO} = S_{OX}$. The first entails $|S_{BX}|^2 = |S_{XB}|^2$, i.e.,

that mode conversion power excitation of EBWs by X-modes is the same as MC power excitation of X-modes by EBWs (emission). The second, $|S_{BO}|^2 = |S_{OB}|^2$, is of a similar nature for the O-EBW mode conversion power and EBW emission MC to O-mode. The last relation states a reflection symmetry: the power reflected out of the X-mode by an excitation of the O-mode, $|S_{XO}|^2$, is the same as the power reflected out of the O-mode by an excitation of the X-mode, $|S_{OX}|^2$.

An approximate kinetic formulation of the full-wave problem of coupling among O, X, and EBWs near the UHR has been solved computationally [2]. Details are given in an adjoining paper at this conference [6]. The results exhibit all of the above symmetries exactly.

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[5] Note: for real \vec{k} and ω , $\chi_{\alpha\beta}^H(-\vec{k}, \omega) = \chi_{\alpha\beta}^H(\vec{k}, \omega)$ for any kinetic (Vlasov) description of a plasma in a magnetic field and with an equilibrium momentum distribution function symmetric in parallel (to the magnetic field) momentum; see references in [4].

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