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Heat and momentum transport in arbitrary mean-free path plasma with a Maxwellian lowest order distribution function

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Abstract

Expressions for the ion perpendicular viscosity as well as for the electron and ion parallel viscosities, gyroviscosities, and heat fluxes are derived for arbitrary mean-free path plasmas, in which the lowest order distribution function is a Maxwellian, by assuming the gyroradius is small compared to the shortest perpendicular scale length. The results are given in terms of a few velocity space integrals of the gyrophase independent correction to the Maxwellian, and correctly reproduce known results in the collisional limit.

I. INTRODUCTION

Expressions for the heat fluxes and viscosities are normally obtained by evaluating appropriate moments of species distribution functions to the required order in the gyroradius expansion.[1–4] To date, this has been done analytically only for short mean-free path plasmas. The better known closures were obtained by Braginskii[5] and Robinson and Bernstein[6] for plasmas with sonic or MHD flows and by Mikhailovskii and Tsypin[7] for plasmas with subsonic or diamagnetic drift flows. By assuming sonic plasma flows for all components, the early investigations[5,6] missed the important physical effect of the dependence of viscosity on heat fluxes (or temperature gradients). The later series of studies[7] took these effects into account (for ions, but not for electrons), but made unjustifiable assumptions in the derivation and thereby obtained incorrect expressions for the ion parallel and perpendicular viscosities,[8] and missed collisional contributions to gyroviscosity. Our recent work[8,9] corrects the ion treatment of Mikhailovskii and Tsypin and also derives the expressions for the electrons, to obtain the first self-consistent collisional two-fluid description for magnetized plasmas in the diamagnetic drift ordering.

In the work presented here we use the drift kinetic formalism of Hazeltine[10] as recently extended and generalized by Simakov and Catto,[11] to obtain expressions for the ion perpendicular viscosity as well as for the ion and electron parallel viscosities, gyroviscosities, and heat fluxes for arbitrary mean-free path plasmas. The ion gyroviscosity is evaluated to the same order as the (smaller) ion perpendicular viscosity. Electron perpendicular viscosity is small and usually of no interest. All the results are obtained in terms of few velocity moments of the gyrophase independent correction to the lowest order distribution function, which is assumed to be a Maxwellian. Higher order moments of the Fokker-Planck equation are employed to obtain the viscosities and heat fluxes in forms that require the minimum information about the distribution function.

Somewhat more general lowest significant order expressions for gyroviscosities have been obtained by us in Ref. 11 using a different approach. In this earlier treatment, we did the calculation for an arbitrary distribution function that is isotropic in the velocity space (i.e., independent of magnetic moment and gyrophase) to lowest order. Such lowest significant order expressions for gyroviscosities were obtained even more generally by Ramos for a collisionless plasma using a fluid approach.[12] Ramos' results do not assume velocity space isotropy of lowest order distribution functions. To the best of our knowledge, the full ion perpendicular viscosity with heat flow effects retained, as well as the gyroviscosity with higher order collisional heat flux corrections retained, have only been correctly evaluated in the short mean-free path limit,[8,9] and are unavailable for other regimes of plasma collisionality.

The ion perpendicular viscosity (as well as the higher order corrections to the ion gyroviscosity) is required to evaluate, among other things, the neoclassical and classical radial electric field in plasma confinement devices, in general, and in tokamaks, in particular.[9,13] To obtain this perpendicular viscosity in terms of species densities, flow velocities, and pressures or temperatures from the expressions derived herein, only the leading (first) order gyrophase independent correction to the Maxwellian is required. To evaluate the ion perpendicular viscosity by a direct integration of the distribution function we would need to solve for the ion distribution function to an order much higher than the first order. Indeed, we would require the ion collision frequency over ion gyrofrequency correction to the second order in gyroradius correction to the Maxwellian. Therefore, our formalism for the ion perpendicular viscosity presents a clear advantage over a purely kinetic evaluation. To obtain the lowest order gyroviscosity only the leading gyrophase independent correction to the Maxwellian is required. However, to obtain the ion gyroviscosity through the same order as the ion perpendicular viscosity, the gyrophase independent portion of the distribution function that is formally second order in the gyroradius expansion is needed.

This paper is organized as follows. In Sec. II we summarize our orderings and discuss our basic kinetic model. The various contribution to the viscosity are reviewed briefly in Sec. III. The gyroviscosity and perpendicular viscosity are evaluated in Secs. IV and V, respectively. We evaluate electron and ion heat fluxes in Sec. VI. The collisional results are recovered in Sec. VII, and a brief discussion follows in Sec. VIII.

II. ORDERINGS, ASSUMPTIONS, AND NOTATION

We consider a magnetized quasineutral electron-ion plasma and assume that the ion gyroradius ρ is small compared to both the characteristic perpendicular (to the magnetic field) equilibrium length scale $L_\perp$ and perturbation wavelength; that is,

$$\delta \equiv \frac{\rho}{L_\perp} \sim k_\perp \rho \ll 1, \tag{1}$$

where $k_\perp$ is the perpendicular wave vector. The $k_\perp \rho \ll 1$ assumption allows us to use a drift kinetic formalism instead of gyrokinetics. We allow the plasma mean-free path to be arbitrary except in Sec. VII, where we use the general formalism to recover the collisional ion results.

To obtain expressions for the ion gyroviscosity and perpendicular viscosity, we need an expression for the gyrophase dependent piece of the ion distribution function, $\tilde{f}$, which is exact through order δ^2. Such an $\tilde{f}$ was derived in Ref. 11 and is given by the sum

$$\tilde{f} = \tilde{f}^H + \tilde{f}^{NH} + \tilde{f}^C. \tag{2}$$

Here, $\tilde{f}^H$ is given by

$$\tilde{f}^H \equiv \boldsymbol{v} \cdot \left[\boldsymbol{g} - (\boldsymbol{v}_E + \boldsymbol{v}_M)\frac{1}{B}\frac{\partial \bar{f}}{\partial \mu}\right] - (\boldsymbol{v}_\perp \boldsymbol{v} \times \hat{\boldsymbol{b}} + \boldsymbol{v} \times \hat{\boldsymbol{b}}\, \boldsymbol{v}_\perp) : \boldsymbol{\nabla}\hat{\boldsymbol{b}} \frac{v_\parallel}{4\Omega B}\frac{\partial \bar{f}}{\partial \mu}, \tag{3}$$

where

$$\boldsymbol{g} \equiv \frac{1}{\Omega}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}|_{\varepsilon,\mu}\bar{f} - \boldsymbol{v}_E \frac{\partial \bar{f}}{\partial \varepsilon}. \tag{4}$$

This expression contains the first and some second order contributions and was obtained in the seminal work of Hazeltine.[10] The term

$$\tilde{f}^{NH} \equiv \frac{1}{8\Omega}\boldsymbol{v}\boldsymbol{v} : [\hat{\boldsymbol{b}} \times (\overleftrightarrow{h} + \overleftrightarrow{h}^{\mathrm{T}}) \cdot (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \cdot (\overleftrightarrow{h} + \overleftrightarrow{h}^{\mathrm{T}}) \times \hat{\boldsymbol{b}}], \tag{5}$$

with

$$\overleftrightarrow{h} \equiv \boldsymbol{\nabla}|_{\varepsilon,\mu}\boldsymbol{g} + \frac{e\boldsymbol{E}}{M}\frac{\partial \boldsymbol{g}}{\partial \varepsilon}, \tag{6}$$

is of order δ^2 and was evaluated by Simakov and Catto.[11] Finally, the order δ^2 term $\tilde{f}^C$ is given by

$$\tilde{f}^C \equiv \frac{1}{\Omega}\int \mathrm{d}\varphi[\langle C(f)\rangle_\varphi - C(f)] \approx \frac{1}{\Omega}C_{ii}^\ell(\boldsymbol{v} \cdot \boldsymbol{g} \times \hat{\boldsymbol{b}}), \tag{7}$$

with $C = C_{ii} + C_{ie}$ the ion collision operator and C_{ii}^ℓ the linearized ion-ion collision operator.

In Eqs. (2) through (7) and elsewhere, $\boldsymbol{v}$ is the velocity variable of the ion distribution function, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, $B = |\boldsymbol{B}|$, $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$, and $\Omega = eB/Mc$ is the ion gyrofrequency, with e the unit electric charge (we consider singly charged ions for simplicity), M the ion mass, and c the speed of light. The independent variables $\varepsilon = v^2/2$ and $\mu = v_\perp^2/2B$ are the kinetic energy and the magnetic moment variables of the ion distribution function, $\boldsymbol{\nabla}|_{\varepsilon,\mu}$ is the gradient with respect to the spatial variables taken at fixed ε and μ, while $\boldsymbol{v}_E = c\boldsymbol{E}\times\hat{\boldsymbol{b}}/B$ and $\boldsymbol{v}_M = \Omega^{-1}\hat{\boldsymbol{b}}\times(\mu\boldsymbol{\nabla}B + v_\parallel^2\boldsymbol{\kappa} + v_\parallel\partial\hat{\boldsymbol{b}}/\partial t)$ are the $\boldsymbol{E}\times\boldsymbol{B}$ and the magnetic drift velocities, with $\boldsymbol{\kappa} = \hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}}$ the magnetic field line curvature. The bar above the ion distribution function f indicates that it is gyrophase averaged: i.e., $\bar{f} = \langle f\rangle_\varphi$. The gyrophase average is defined as $\langle\cdots\rangle_\varphi \equiv (1/2\pi)\oint \mathrm{d}\varphi(\cdots)$, with the gyrophase φ being the third independent velocity variable, and of course $\tilde{f} = f - \bar{f}$. As usual, "parallel" and "perpendicular" refer to the direction of $\boldsymbol{B}$. We use the symbol "T" as a superscript on a dyad to denote its transpose. In addition, the double dot scalar product is defined for arbitrary vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and a dyad $\overleftrightarrow{D}$ as $\boldsymbol{ac}:\overleftrightarrow{D}\equiv \boldsymbol{c}\cdot\overleftrightarrow{D}\cdot\boldsymbol{a}$.

Expression (3) for $\tilde{f}^H$ was obtained in Ref. 10 by assuming

$$\frac{\partial}{\partial t} \sim \boldsymbol{v}\cdot\boldsymbol{\nabla} \sim \frac{e}{M}\boldsymbol{E}\cdot\boldsymbol{\nabla}_v \sim C \sim \delta\Omega. \tag{8}$$

Expression (5) for $\tilde{f}^{NH}$ was derived in Ref. 11 by using the more restrictive orderings, which we adopt herein:

$$\frac{\partial}{\partial t} \sim \delta^2\Omega, \qquad \boldsymbol{v}\cdot\boldsymbol{\nabla} \sim \frac{e}{M}\boldsymbol{E}\cdot\boldsymbol{\nabla}_v \sim C \sim \delta\Omega, \tag{9}$$

and by also assuming that the lowest order ion distribution function f_0 is isotropic in the velocity space, i.e., it does not depend on the gyrophase and magnetic moment. Expression (7) for $\tilde{f}^C$ was also obtained for an f_0 isotropic in velocity space. An expression for the gyrophase dependent portion of the electron distribution function, which is similar to that given by Eq. (2), can be derived if assumptions similar to those given by Eqs. (8) and (9) are made for electrons.

Evaluation of $\tilde{f}^{NH}$, $\tilde{f}^C$, species viscosities, and heat fluxes can be carried out without using the isotropy assumption for f_0. However, such a calculation is beyond the scope of the present treatment. Moreover, the evaluation of species viscosities and classical collisional perpendicular heat fluxes is greatly simplified for f_0 a Maxwellian,

$$f_0 = f_M = n\left(\frac{M}{2\pi T}\right)^{3/2} \exp\left(-\frac{Mv^2}{2T}\right), \tag{10}$$

where n is the plasma density and T is the ion temperature (a similar expression should be used for the lowest order electron distribution function). For example, n and T can be viewed as being advanced by the number and energy moments of the kinetic equation being solved.

In this work we use assumption (10) since, as argued in Ref. 4, it usually holds for plasmas of interest to magnetic fusion that are confined by magnetic fields with closed flux surfaces in the absence of strong external driving forces, such as neutral beams or radio-frequency waves. Use of Eq. (10) implicitly assumes that pressure anisotropy is weak and parallel flows are subsonic. Orderings (8) and (9) result in perpendicular flow velocities being subsonic as well.

III. PARALLEL VISCOSITY AND FORMS FOR GYRO-VISCOSITY AND PERPENDICULAR VISCOSITY

We begin our evaluation of the ion viscosity by considering the full ion kinetic equation,

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}f) + \boldsymbol{\nabla}_v \cdot \left[\frac{e}{M}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) f\right] = C_{ii}(f) + C_{ie}(f). \tag{11}$$

We use the full Landau form for the ion-ion Fokker-Planck collision operator $C_{ii}(f)$,

$$C_{ii}(f) \equiv C_{ii}^{n\ell}(f,f) \equiv \gamma \boldsymbol{\nabla}_v \cdot \left[\int \mathrm{d}^3v'\, \boldsymbol{\nabla}_g \boldsymbol{\nabla}_g g \cdot (f'\boldsymbol{\nabla}_v f - f\boldsymbol{\nabla}_{v'} f')\right], \tag{12}$$

where $\gamma \equiv (3\sqrt{\pi}/2)(\nu/n)(T/M)^{3/2}$ with $\nu = (4\sqrt{\pi}/3)(\Lambda e^4 n/M^{1/2}T^{3/2})$ the characteristic ion collision frequency[5] (Λ is the Coulomb logarithm); f' is obtained from f

by substituting $\boldsymbol{v}'$ for $\boldsymbol{v}$, and $g \equiv |\boldsymbol{v} - \boldsymbol{v}'|$. We also employ the following approximate form for the ion-electron Fokker-Planck collision operator $C_{ie}(f)$:

$$C_{ie}(f) = \frac{1}{Mn}\boldsymbol{F} \cdot \boldsymbol{\nabla}_v f - \frac{m\nu_e}{M}\boldsymbol{V} \cdot \boldsymbol{\nabla}_v f + \frac{m\nu_e}{M}\boldsymbol{\nabla}_v \cdot \left(\frac{T_e}{M}\boldsymbol{\nabla}_v f + \boldsymbol{v} f\right). \tag{13}$$

Expression (13) is obtained by performing a mass ratio expansion of the full Landau form for C_{ie}. Here, $\boldsymbol{V}$ is the ion flow velocity, m and T_e are the electron mass and temperature, respectively, and $\nu_e = (4\sqrt{2\pi}/3)(\Lambda e^4 n/m^{1/2}T_e^{3/2})$ is the characteristic electron-ion collision frequency.[5] The electron-ion friction force is defined as $\boldsymbol{F} \equiv \int \mathrm{d}^3v\, m\boldsymbol{v}C_{ei}$, with C_{ei} the electron-ion collision operator.

Forming the $M\boldsymbol{vv}$ moment of Eq. (11) and defining the ion viscous stress tensor, $\overleftrightarrow{\pi} \equiv M \int \mathrm{d}^3v(\boldsymbol{vv} - v^2 \overleftrightarrow{I}/3)f = M \int \mathrm{d}^3v\, \boldsymbol{vv}f - p\overleftrightarrow{I}$, with p the ion pressure and $\overleftrightarrow{I}$ the unit dyad, we obtain the exact equation

$$\begin{aligned} &\frac{\partial}{\partial t}(p\overleftrightarrow{I} + \overleftrightarrow{\pi}) + \boldsymbol{\nabla} \cdot \left(M \int \mathrm{d}^3v\, \boldsymbol{vvv}f\right) - (en\boldsymbol{E} - \boldsymbol{F})\boldsymbol{V} - \boldsymbol{V}(en\boldsymbol{E} - \boldsymbol{F}) \\ &-\Omega(\overleftrightarrow{\pi} \times \hat{\boldsymbol{b}} - \hat{\boldsymbol{b}} \times \overleftrightarrow{\pi}) + \frac{2m\nu_e}{M}(p\overleftrightarrow{I} - p_e\overleftrightarrow{I} + \overleftrightarrow{\pi} - Mn\boldsymbol{VV}) = M \int \mathrm{d}^3v\, \boldsymbol{vv}C_{ii}(f), \end{aligned} \tag{14}$$

where $p_e = nT_e$ is the electron pressure. Observing that $(m/M)\nu_e \sim (m/M)^{1/2}\nu$, and anticipating that the integral on the right-hand side of Eq. (14) will contain terms of order $Mn\nu\boldsymbol{VV}$, we can safely neglect the $2mn\nu_e\boldsymbol{VV}$ term on the left-hand side.

The general solution of Eq. (14) for $\overleftrightarrow{\pi}$ is given by the sum (see, for example, Ref. 4)

$$\overleftrightarrow{\pi} = \overleftrightarrow{\pi}_{\parallel} + \overleftrightarrow{\pi}_g + \overleftrightarrow{\pi}_{\perp}, \tag{15}$$

where the *parallel viscosity* $\overleftrightarrow{\pi}_{\parallel}$ is a diagonal traceless tensor proportional to $(3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \overleftrightarrow{I})$ that cannot be determined from Eq. (14). To leading order, it follows from the definition of $\overleftrightarrow{\pi}$ and from the orderings employed requiring $\tilde{f} \sim \delta\bar{f} \ll \bar{f}$ that

$$\overleftrightarrow{\pi}_{\parallel} = M \int \mathrm{d}^3v(\boldsymbol{vv} - v^2\overleftrightarrow{I}/3)\bar{f} = (p_{\parallel} - p_{\perp})(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \overleftrightarrow{I}/3), \tag{16}$$

where $p_\parallel \equiv M\int d^3v\, v_\parallel^2 \bar{f}$ and $p_\perp \equiv M\int d^3v\, \mu B\bar{f}$ are the ion parallel and perpendicular pressures, respectively. Recall, that for an isotropic f_0, $p_\parallel - p_\perp \ll p$. We assume that $\overleftrightarrow{\pi}_\parallel$ is at most an order δ correction to the lowest order isotropic pressure tensor $p\overleftrightarrow{I}$. As will be seen shortly, the gyroviscosity and perpendicular viscosity are at most the order δ^2 corrections to $p\overleftrightarrow{I}$, so that the parallel viscosity can be the leading order piece of the viscous stress tensor. When rewritten in terms of electron quantities, Eq. (16) also describes the electron parallel viscosity.

The *gyroviscosity* $\overleftrightarrow{\pi}_g$ is given by

$$\overleftrightarrow{\pi}_g = \frac{1}{4\Omega}\left[\hat{\boldsymbol{b}}\times \overleftrightarrow{K}_g \cdot(\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\cdot \overleftrightarrow{K}_g \times\hat{\boldsymbol{b}}\right], \tag{17}$$

where

$$\overleftrightarrow{K}_g = \frac{\partial \overleftrightarrow{\pi}_\parallel}{\partial t} + \boldsymbol{\nabla}\cdot\left(M\int d^3v\, \boldsymbol{vvv}f\right) - (en\boldsymbol{E} - \boldsymbol{F})\boldsymbol{V} - \boldsymbol{V}(en\boldsymbol{E} - \boldsymbol{F}). \tag{18}$$

Finally, the *perpendicular viscosity* $\overleftrightarrow{\pi}_\perp$ is given by

$$\overleftrightarrow{\pi}_\perp = \frac{1}{4\Omega}\left[\hat{\boldsymbol{b}}\times \overleftrightarrow{K}_\perp \cdot(\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\cdot \overleftrightarrow{K}_\perp \times\hat{\boldsymbol{b}}\right], \tag{19}$$

where

$$\overleftrightarrow{K}_\perp = -M\int d^3v\, \boldsymbol{vv}C_{ii}(f). \tag{20}$$

Notice that the $[\partial p/\partial t + 2m\nu_e(p - p_e)/M]\overleftrightarrow{I}$ and $(2m\nu_e/M)\overleftrightarrow{\pi}_\parallel$ terms have been omitted in the expressions for $\overleftrightarrow{K}_g$ and $\overleftrightarrow{K}_\perp$ since they are diagonal and consequently do not contribute to $\overleftrightarrow{\pi}_g$ and $\overleftrightarrow{\pi}_\perp$. The terms $\partial(\overleftrightarrow{\pi}_g + \overleftrightarrow{\pi}_\perp)/\partial t$ and $(2m\nu_e/M)(\overleftrightarrow{\pi}_g + \overleftrightarrow{\pi}_\perp)$ are small compared to $\overleftrightarrow{K}_\perp$ and so can be dropped as well.

IV. GYROVISCOSITY

To evaluate the ion gyroviscosity we have to evaluate $\overleftrightarrow{K}_g$. When the "maximal" ordering $\lambda \sim L_\perp \sim L_\parallel$ is assumed, we must require $\nu/\Omega \sim \delta$, where λ is a particle mean-free path and $L_\parallel$ is a characteristic parallel (to the magnetic field) length scale. The leading order perpendicular viscosity is ν/Ω times smaller than the leading order gyroviscosity (i.e., of order δ^3), so to be consistent we have to evaluate the gyroviscosity to order δ^3. Doing so requires knowing $\overleftrightarrow{K}_g$ and therefore $\boldsymbol{\nabla}\cdot\left(M\int d^3v\,\boldsymbol{vvv}f\right)\approx \boldsymbol{\nabla}\cdot\left(M\int d^3v\,\langle\boldsymbol{vvv}\rangle_\varphi\bar{f}\right)+\boldsymbol{\nabla}\cdot\left[M\int d^3v\,\boldsymbol{vvv}\left(\tilde{f}^H+\tilde{f}^{NH}+\tilde{f}^C\right)\right]$ through order δ^2.

Observing that

$$\langle\boldsymbol{v}_i\boldsymbol{v}_j\boldsymbol{v}_k\rangle_\varphi=\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k\left(v_\parallel^3-\frac{3}{2}v_\parallel v_\perp^2\right)+(\delta_{ij}\hat{\boldsymbol{b}}_k+\delta_{ik}\hat{\boldsymbol{b}}_j+\delta_{jk}\hat{\boldsymbol{b}}_i)\frac{1}{2}v_\parallel v_\perp^2,\tag{21}$$

where δ_{ij} is the Kronecker delta denoting the unit dyad $\overleftrightarrow{I}$, and introducing

$$q_1\equiv\frac{1}{2}M\int d^3v\,v_\parallel v_\perp^2\bar{f},\quad q_2\equiv\frac{1}{2}M\int d^3v\,v_\parallel^3\bar{f}\tag{22}$$

we obtain

$$\left(M\int d^3v\,\langle\boldsymbol{vvv}\rangle_\varphi\bar{f}\right)_{ijk}=\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k(2q_2-3q_1)+(\delta_{ij}\hat{\boldsymbol{b}}_k+\delta_{ik}\hat{\boldsymbol{b}}_j+\delta_{jk}\hat{\boldsymbol{b}}_i)q_1.$$

Consequently,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(M\int d^3v\,\langle\boldsymbol{vvv}\rangle_\varphi\bar{f}\right)&=(2q_2-3q_1)(\hat{\boldsymbol{b}}\boldsymbol{\kappa}+\boldsymbol{\kappa}\hat{\boldsymbol{b}})+\boldsymbol{\nabla}\cdot[(2q_2-3q_1)\hat{\boldsymbol{b}}]\,\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\\&+\boldsymbol{\nabla}(q_1\hat{\boldsymbol{b}})+[\boldsymbol{\nabla}(q_1\hat{\boldsymbol{b}})]^{\mathrm{T}}+\boldsymbol{\nabla}\cdot(q_1\hat{\boldsymbol{b}})\overleftrightarrow{I}.\end{aligned}\tag{23}$$

The contribution from $\tilde{f}$ is evaluated in Appendix A:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left[M\int d^3v\,\boldsymbol{vvv}\left(\tilde{f}^H+\tilde{f}^{NH}+\tilde{f}^C\right)\right]&=\boldsymbol{\nabla}\boldsymbol{a}+(\boldsymbol{\nabla}\boldsymbol{a})^{\mathrm{T}}+\overleftrightarrow{I}\,\boldsymbol{\nabla}\cdot\boldsymbol{a}+\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}(\boldsymbol{\nabla}\cdot\boldsymbol{c})\\&+[(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}+\boldsymbol{\kappa}]\boldsymbol{c}+\boldsymbol{c}[(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}+\boldsymbol{\kappa}]+\hat{\boldsymbol{b}}[\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\boldsymbol{c}+\boldsymbol{c}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}}]\\&+[\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\boldsymbol{c}+\boldsymbol{c}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}}]\hat{\boldsymbol{b}}+\boldsymbol{\nabla}\cdot\overleftrightarrow{A},\end{aligned}\tag{24}$$

where

$$\boldsymbol{a} \equiv \frac{\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} q_4}{2\Omega} + \frac{\hat{\boldsymbol{b}} \times \boldsymbol{\kappa}}{\Omega} \left(2q_3 - \frac{5}{2}q_4\right) + p_\perp \boldsymbol{v}_E + \frac{2}{5}\boldsymbol{q}_c,$$

$$\boldsymbol{c} \equiv \frac{1}{\Omega}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} \left(2q_3 - \frac{5}{2}q_4\right) - \frac{\hat{\boldsymbol{b}} \times \boldsymbol{\kappa}}{\Omega} \left(20q_3 - \frac{35}{2}q_4 - 4q_5\right) + (p_\parallel - p_\perp)\boldsymbol{v}_E, \tag{25}$$

$$\overleftrightarrow{A}_{ijk} \equiv \frac{1}{4\Omega} \left(2q_3 - \frac{5}{2}q_4\right) \left(\hat{\boldsymbol{b}}_i \overleftrightarrow{D}_{jk} + \hat{\boldsymbol{b}}_j \overleftrightarrow{D}_{ik} + \hat{\boldsymbol{b}}_k \overleftrightarrow{D}_{ij}\right).$$

Here,

$$q_3 \equiv \frac{1}{4}M \int \mathrm{d}^3v\, v^2 v_\perp^2 \bar{f}, \quad q_4 \equiv \frac{1}{4}M \int \mathrm{d}^3v\, v_\perp^4 \bar{f}, \quad q_5 \equiv \frac{1}{4}M \int \mathrm{d}^3v\, v^4 \bar{f}, \tag{26}$$

$$\boldsymbol{q}_c \equiv -\frac{2\,p\,\nu}{M\Omega^2}\boldsymbol{\nabla}_\perp T, \tag{27}$$

and

$$\overleftrightarrow{D} = \hat{\boldsymbol{b}} \times [\boldsymbol{\nabla}\hat{\boldsymbol{b}} + (\boldsymbol{\nabla}\hat{\boldsymbol{b}})^{\mathrm{T}}] \cdot (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \cdot [\boldsymbol{\nabla}\hat{\boldsymbol{b}} + (\boldsymbol{\nabla}\hat{\boldsymbol{b}})^{\mathrm{T}}] \times \hat{\boldsymbol{b}}. \tag{28}$$

Using results (16), (23), and (24) to evaluate $\overleftrightarrow{K}_g$ from Eq. (18), neglecting $\boldsymbol{F}_\perp$ as small by (ν_e/Ω_e) as compared with $en\boldsymbol{E}_\perp$, where $\Omega_e \equiv eB/mc$ is the electron gyrofrequency, and employing Eq. (17), we finally arrive at the following expression for the ion gyroviscous stress tensor, which is exact through δ^3:

$$\begin{aligned} \overleftrightarrow{\pi}_g = {} & Mn \left[\boldsymbol{V}_\parallel \boldsymbol{v}_E + \frac{1}{4}(\boldsymbol{V}_\perp \boldsymbol{v}_E - \boldsymbol{V} \times \hat{\boldsymbol{b}}\ \boldsymbol{v}_E \times \hat{\boldsymbol{b}})\right] + \frac{1}{\Omega}(en\boldsymbol{E}_\parallel - \boldsymbol{F}_\parallel)\boldsymbol{V} \times \hat{\boldsymbol{b}} \\ & + \frac{2q_2 - 3q_1}{\Omega}\,\hat{\boldsymbol{b}}\,\hat{\boldsymbol{b}} \times \boldsymbol{\kappa} + \frac{1}{\Omega}\left[(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}} \times \boldsymbol{c}\,\hat{\boldsymbol{b}} - (\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\times\hat{\boldsymbol{b}})\boldsymbol{c}\,\hat{\boldsymbol{b}} + (\overleftrightarrow{I} - \hat{\boldsymbol{b}}\,\hat{\boldsymbol{b}})\cdot(\boldsymbol{\nabla}\times\boldsymbol{c})\hat{\boldsymbol{b}}\right] \\ & + \frac{1}{4\Omega}\left\{\hat{\boldsymbol{b}} \times \left[\boldsymbol{\kappa}\,\boldsymbol{c} + \boldsymbol{c}\,\boldsymbol{\kappa} + \boldsymbol{\nabla}(\boldsymbol{a} + q_1\hat{\boldsymbol{b}}) + \boldsymbol{\nabla}(\boldsymbol{a} + q_1\hat{\boldsymbol{b}})^{\mathrm{T}} + \boldsymbol{\nabla}\cdot \overleftrightarrow{A}\right] \cdot (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\,\hat{\boldsymbol{b}})\right\} \\ & + \frac{p_\parallel - p_\perp}{\Omega}\hat{\boldsymbol{b}}\,\hat{\boldsymbol{b}} \times \frac{\partial\hat{\boldsymbol{b}}}{\partial t} + \text{Transpose}. \end{aligned} \tag{29}$$

All quantities in this expression should be evaluated to an accuracy which ensures an overall accuracy of δ^3.

Expression (29) can be simplified considerably if only δ^2 accuracy is required. In this case quantities q_1, q_2, $\boldsymbol{a}$, $\boldsymbol{c}$, $\overleftrightarrow{A}$ can be evaluated to accuracy δ, i.e. quantities q_3,

q_4, q_5, $p_\|$, and $p_\perp$ can be evaluated for $\bar{f}_0 = f_M$, and the last term in the expression for $\boldsymbol{a}$ can be neglected. Then, $\boldsymbol{a} = \Omega^{-1}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}(pT/M) + p\boldsymbol{v}_E$, $\boldsymbol{c} = 0$, and $\overleftrightarrow{A} = 0$, giving

$$\overleftrightarrow{\pi}_g \rightarrow Mn\left[\boldsymbol{V}_\|\boldsymbol{v}_E + \frac{1}{4}(\boldsymbol{V}_\perp\boldsymbol{V}_\perp - \boldsymbol{V}_\perp \times \hat{\boldsymbol{b}}\,\boldsymbol{V}_\perp \times \hat{\boldsymbol{b}})\right] + \frac{2q_2 - 3q_1}{\Omega}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \times \boldsymbol{\kappa} \tag{30}$$
$$+\frac{1}{4\Omega}\hat{\boldsymbol{b}} \times (\overleftrightarrow{N} + \overleftrightarrow{N}^{\mathrm{T}}) \cdot (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) + \text{Transpose},$$

with

$$\overleftrightarrow{N} = p\boldsymbol{\nabla}\boldsymbol{V}_\perp + \frac{2}{5}\boldsymbol{\nabla}\boldsymbol{q}_\perp + \boldsymbol{\nabla}(q_1\hat{\boldsymbol{b}}). \tag{31}$$

In these expressions, to the requisite order $\boldsymbol{V}_\|$,

$$\boldsymbol{V}_\perp = c\frac{\boldsymbol{E} \times \hat{\boldsymbol{b}}}{B} + \frac{\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}p}{Mn\Omega}, \qquad \boldsymbol{q}_\perp = \frac{5p}{2M\Omega}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}T \tag{32}$$

are the lowest order parallel and perpendicular ion flow velocities, and heat flow, respectively. To obtain expression (30) we also used the fact that to the order required $\hat{\boldsymbol{b}} \cdot (en\boldsymbol{E} - \boldsymbol{\nabla}p - \boldsymbol{F}) \approx 0$.

The ion gyroviscous stress tensor given by Eqs. (30) and (31) is exactly the same (for the case $f_0 = f_M$) as the expression for $\overleftrightarrow{\pi}_g$ obtained in Ref. 11 by directly evaluating $M\int \mathrm{d}^3v(\boldsymbol{v}\boldsymbol{v} - v^2\overleftrightarrow{I}/3)\tilde{f} \approx M\int \mathrm{d}^3v\,\boldsymbol{v}\boldsymbol{v}(\tilde{f}^H + \tilde{f}^{NH})$ since $\tilde{f}^C$ does not contribute. In addition, it does not contain any terms proportional to $(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \overleftrightarrow{I}/3)$. As a result, to second order in the δ expansion $\tilde{f}$ does not contribute to the parallel viscosity. Therefore, we conclude that expression (16) for $\overleftrightarrow{\pi}_\|$ is exact through at least order δ^2.

Reference 11 gives a somewhat more general expression for $\overleftrightarrow{\pi}_g$ (exact through order δ^2), which is obtained for an arbitrary velocity space isotropic f_0 (as opposed to a Maxwellian). Moreover, it shows that Eqs. (30) and (31) also describe the *electron* gyroviscous stress tensor if the ion quantities are replaced with the corresponding electron quantities. Indeed, our gyroviscosities are also consistent with the general collisionless results obtained in Ref. 12 for an arbitrary f_0.

V. ION PERPENDICULAR VISCOSITY

The ion perpendicular viscosity is given by the combination of Eqs. (19) and (20). To evaluate it we need to know the leading order non-vanishing result for $M \int \mathrm{d}^3 v\, \boldsymbol{v}\boldsymbol{v} C_{ii}(f)$. Noticing that

$$\langle \boldsymbol{v}_i \boldsymbol{v}_j \rangle_\varphi = v_\parallel^2 \hat{\boldsymbol{b}}_i \hat{\boldsymbol{b}}_j + (v_\perp^2/2)(\delta_{ij} - \hat{\boldsymbol{b}}_i \hat{\boldsymbol{b}}_j), \tag{33}$$

and also that any terms in $\overleftrightarrow{K}_\perp$, which are proportional to $\overleftrightarrow{I}$ or $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$, do not contribute to $\overleftrightarrow{\pi}_\perp$, we conclude that to obtain the leading order non-trivial result we only need

$$M \int \mathrm{d}^3 v\, \boldsymbol{v}\boldsymbol{v} [C_{ii}^{\ell}(\tilde{f}_1 + \tilde{f}_2) + C_{ii}^{n\ell}(f_1, f_1) - C_{ii}^{n\ell}(\bar{f}_1, \bar{f}_1)]. \tag{34}$$

Here, $C_{ii}^{\ell}(f)$ is the linearized (about a Maxwellian) form of the ion-ion collision operator,

$$C_{ii}^{\ell}(f) \equiv \gamma \boldsymbol{\nabla}_v \cdot \left\{ f_M \int \mathrm{d}^3 v'\, f_M' \boldsymbol{\nabla}_g \boldsymbol{\nabla}_g g \cdot [\boldsymbol{\nabla}_v (f/f_M) - \boldsymbol{\nabla}_{v'} (f'/f_M')] \right\}, \tag{35}$$

where f' and f_M' are obtained from f and f_M, respectively, by substituting $\boldsymbol{v}'$ for $\boldsymbol{v}$, and $C_{ii}^{n\ell}(f, f) \equiv C_{ii}(f)$ is the full bilinear form of the ion-ion collision operator given by Eq. (12). The symbols f_1 and f_2 are used to denote the first and the second order in δ portions of the ion distribution function.

A. LINEARIZED CONTRIBUTION C_{ii}^{ℓ}

First, we evaluate the contribution from the linearized collision operator

$$\overleftrightarrow{K}_\perp^{\ell} \equiv -M \int \mathrm{d}^3 v\, \boldsymbol{v}\boldsymbol{v} C_{ii}^{\ell}(\tilde{f}_1 + \tilde{f}_2). \tag{36}$$

Using the self-adjointness of $C_{ii}^{\ell}(f)$ we rewrite Eq. (36) as

$$\overset{\leftrightarrow}{K}{}_{\perp}^{\ell} = -M \int \mathrm{d}^3 v \, (\tilde{f}_1 + \tilde{f}_2) f_M^{-1} C_{ii}^{\ell}(\boldsymbol{v}\boldsymbol{v} f_M), \tag{37}$$

where[8]

$$C_{ii}^{\ell}(\boldsymbol{v}\boldsymbol{v} f_M) = \nu F(x) f_M \left(\boldsymbol{v}\boldsymbol{v} - \frac{1}{3} v^2 \overset{\leftrightarrow}{I} \right), \tag{38}$$

with

$$F(x) \equiv -\frac{9\sqrt{\pi}}{\sqrt{2} x^3} \left[\left(1 - \frac{3}{2x^2} \right) E(x) + \frac{3}{2x} E'(x) \right], \tag{39}$$

$x \equiv \sqrt{Mv^2/2T}$, $E(x) \equiv (2/\sqrt{\pi}) \int_0^x \mathrm{d}t \exp(-t^2)$ the error function, and $E'(x) \equiv \mathrm{d}E(x)/\mathrm{d}x$. Substituting $\tilde{f}$ from Eq. (2) for $\tilde{f}_1 + \tilde{f}_2$, taking into account

$$(\boldsymbol{v}_{\perp}\boldsymbol{v} \times \hat{\boldsymbol{b}} + \boldsymbol{v} \times \hat{\boldsymbol{b}}\, \boldsymbol{v}_{\perp}) : \boldsymbol{\nabla}\hat{\boldsymbol{b}} = -4 v_{\parallel} \boldsymbol{v} \cdot \hat{\boldsymbol{b}} \times \boldsymbol{\kappa} + \frac{1}{2} \boldsymbol{v} \cdot \overset{\leftrightarrow}{D} \cdot \boldsymbol{v}, \tag{40}$$

where $\overset{\leftrightarrow}{D}$ is defined in Eq. (28), and using Eqs. (21), (A2), and (A8) we obtain

$$\begin{aligned} \overset{\leftrightarrow}{K}{}_{\perp}^{\ell} = & -\nu M \int \mathrm{d}^3 v \, \frac{v_{\parallel} v_{\perp}^2}{2} F(x) \, \hat{\boldsymbol{b}} \left[\boldsymbol{g} - \left(\boldsymbol{v}_E + \frac{\mu}{\Omega} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B \right) \frac{1}{B} \frac{\partial \bar{f}}{\partial \mu} + \frac{\nu}{\Omega^2} (\boldsymbol{\nabla}_{\perp} \ln T) Q(x) f_M \right] \\ & - \frac{\nu}{8\Omega} M \int \mathrm{d}^3 v \, F(x) \left[\frac{v_{\perp}^4}{8} \overset{\leftrightarrow}{M} + v_{\perp}^2 \left(v_{\parallel}^2 - \frac{v_{\perp}^2}{4} \right) \hat{\boldsymbol{b}} (\hat{\boldsymbol{b}} \cdot \overset{\leftrightarrow}{M}) \right] + \text{Transpose}. \end{aligned} \tag{41}$$

In this expression,

$$Q(x) \equiv -\frac{3\sqrt{2\pi}}{x} \left[\left(1 - \frac{5}{2x^2} \right) E(x) + \frac{5}{2x} E'(x) \right], \tag{42}$$

$$\overset{\leftrightarrow}{M} \equiv \hat{\boldsymbol{b}} \times (\overset{\leftrightarrow}{h}_1 + \overset{\leftrightarrow}{h}{}_1^{\mathrm{T}}) \cdot (\overset{\leftrightarrow}{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\overset{\leftrightarrow}{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \cdot (\overset{\leftrightarrow}{h}_1 + \overset{\leftrightarrow}{h}{}_1^{\mathrm{T}}) \times \hat{\boldsymbol{b}}, \tag{43}$$

with

$$\overset{\leftrightarrow}{h}_1 \equiv \overset{\leftrightarrow}{h} - \frac{v_{\parallel}}{B} \frac{\partial \bar{f}}{\partial \mu} \boldsymbol{\nabla} \hat{\boldsymbol{b}}, \tag{44}$$

and we neglected the small term $(v_{\parallel}/\Omega) \hat{\boldsymbol{b}} \times \partial \hat{\boldsymbol{b}} / \partial t$ in $\boldsymbol{v}_M$.

To simplify Eq. (41) we notice that

$$M\int \mathrm{d}^3v\,\frac{v_\parallel v_\perp^2}{2}F(x)\boldsymbol{\nabla}|_{\varepsilon,\mu}\bar{f} = \frac{6}{5}\left(\boldsymbol{\nabla}q_6 - q_6\boldsymbol{\nabla}\ln B^2 + q_7\boldsymbol{\nabla}\ln T\right),$$
$$M\int \mathrm{d}^3v\,\frac{v_\parallel v_\perp^2}{2}F(x)\left(\frac{\partial\bar{f}}{\partial\varepsilon}+\frac{1}{B}\frac{\partial\bar{f}}{\partial\mu}\right) = -\frac{6M}{5T}\,q_8,$$
$$M\int \mathrm{d}^3v\,\frac{v_\parallel v_\perp^2}{2}F(x)\left(\frac{\mu}{\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\ln B\right)\frac{\partial\bar{f}}{\partial\mu} = -\frac{6\,q_6}{5\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\ln B^2,$$
$$M\int \mathrm{d}^3v\,\frac{v_\parallel v_\perp^4}{4}F(x)\frac{1}{B}\frac{\partial\bar{f}}{\partial\mu} = -\frac{12}{5}\,q_6,$$
$$M\int \mathrm{d}^3v\,v_\parallel v_\perp^2\left(v_\parallel^2-\frac{v_\perp^2}{4}\right)F(x)\frac{1}{B}\frac{\partial\bar{f}}{\partial\mu} = -\frac{12}{5}\,q_9,$$

where we define the following integrals:

$$q_6 \equiv \frac{5}{12}\,M\int \mathrm{d}^3v\,v_\parallel v_\perp^2 F(x)\bar{f},$$
$$q_7 \equiv \frac{5}{24}\,M\int \mathrm{d}^3v\,v_\parallel v_\perp^2 xF'(x)\bar{f},$$
$$q_8 = \frac{5}{6}\,M\int \mathrm{d}^3v\,v_\parallel\left[\frac{v_\perp^2}{4}F'(x)+\frac{T}{M}F(x)\right]\bar{f}, \tag{45}$$
$$q_9 = \frac{5}{6}\,M\int \mathrm{d}^3v\,v_\parallel\left(v_\parallel^2-\frac{3}{2}\,v_\perp^2\right)F(x)\bar{f}.$$

Also, we evaluate terms involving $\overleftrightarrow{h}$ for $\bar{f}=f_0=f_M$ to find

$$M\int \mathrm{d}^3v\,v_\perp^2\left(v_\parallel^2-\frac{v_\perp^2}{4}\right)F(x)\,\overleftrightarrow{h}=0,$$
$$M\int \mathrm{d}^3v\,\frac{v_\perp^2}{4}F(x)\,\overleftrightarrow{h}=\frac{12}{5}\Bigg[-\boldsymbol{\nabla}\left(p\,\boldsymbol{V}_\perp+\frac{1}{10}\,\boldsymbol{q}_\perp\right)$$
$$+\boldsymbol{\nabla}\ln T\left(\frac{3}{4}\,p\,\boldsymbol{V}_\perp+\frac{9}{40}\,\boldsymbol{q}_\perp\right)+\frac{e\boldsymbol{E}}{T}\left(p\,\boldsymbol{V}_\perp-\frac{3}{10}\,\boldsymbol{q}_\perp\right)\Bigg],$$

with $\boldsymbol{V}_\perp$ and $\boldsymbol{q}_\perp$ given by Eq. (32). As a result, we obtain

$$\overleftrightarrow{K}_\perp^{\ell}=\frac{3\nu}{10\Omega}\left[\hat{\boldsymbol{b}}\times\overleftrightarrow{W}\cdot(\overleftrightarrow{I}+3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})-(\overleftrightarrow{I}+3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\cdot\overleftrightarrow{W}\times\hat{\boldsymbol{b}}\right], \tag{46}$$

where

$$\overleftrightarrow{W}\equiv\boldsymbol{\nabla}\left(p\,\boldsymbol{V}_\perp+\frac{1}{10}\,\boldsymbol{q}_\perp-q_6\,\hat{\boldsymbol{b}}\right)-\boldsymbol{\nabla}\ln T\left(\frac{3}{4}\,p\,\boldsymbol{V}_\perp+\frac{9}{40}\,\boldsymbol{q}_\perp+q_7\,\hat{\boldsymbol{b}}\right) \tag{47}$$
$$-\frac{e\boldsymbol{E}}{T}\left(p\,\boldsymbol{V}_\perp-\frac{3}{10}\,\boldsymbol{q}_\perp-q_8\,\hat{\boldsymbol{b}}\right)-q_9\,\boldsymbol{\kappa}\,\hat{\boldsymbol{b}}+\text{Transpose}.$$

Substituting $\overset{\leftrightarrow}{K}^{\ell}_{\perp}$ from Eq. (46) into Eq. (19) we finally obtain (see Ref. 8 for the detailed procedure)

$$\overset{\leftrightarrow}{\pi}^{\ell}_{\perp} = -\frac{3\nu}{10\Omega^2}\Big[\overset{\leftrightarrow}{W} + 3\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\overset{\leftrightarrow}{W}) + 3(\hat{\boldsymbol{b}}\cdot\overset{\leftrightarrow}{W})\hat{\boldsymbol{b}} \tag{48}$$
$$+\frac{1}{2}(\overset{\leftrightarrow}{I} - 15\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})(\hat{\boldsymbol{b}}\cdot\overset{\leftrightarrow}{W}\cdot\hat{\boldsymbol{b}}) - \frac{1}{2}(\overset{\leftrightarrow}{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\overset{\leftrightarrow}{W}:\overset{\leftrightarrow}{I}\Big].$$

Notice, that the right-hand side of Eq. (48) does not contain any terms proportional to $(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \overset{\leftrightarrow}{I}/3)$ and has zero trace. The diagonal contributions represented by the last two terms on the right-hand side of Eq. (48) of this work were retained in Ref. 9, but ignored as small in Eqs. (62) and (63) of the collisional treatment of Ref. 8 [although they were retained in the equation between Eqs. (61) and (62) of Ref. 8]. This was done there because the two terms represent small corrections to the scalar pressure and pressure anisotropy. Moreover, in a tokamak the perpendicular viscosity is likely to only be important in evaluating conservation of toroidal angular momentum, where it can be needed to determine an axisymmetric portion of the radial electric field. In this case these diagonal terms play no role as found in Ref. 13.

B. BILINEAR CONTRIBUTION $C^{n\ell}_{ii}$

To evaluate the contribution from the nonlinear piece of the ion-ion collision operator,

$$\overset{\leftrightarrow}{K}^{n\ell}_{\perp} \equiv -M\int \mathrm{d}^3v\,\boldsymbol{v}\boldsymbol{v}[C^{n\ell}_{ii}(f_1, f_1) - C^{n\ell}_{ii}(\bar{f}_1, \bar{f}_1)], \tag{49}$$

we use expression (12) for $C^{n\ell}_{ii}(f, f)$ to rewrite $\overset{\leftrightarrow}{K}^{n\ell}_{\perp}$ as

$$\overset{\leftrightarrow}{K}^{n\ell}_{\perp} = -6\gamma M\int \mathrm{d}^3v\,(2\bar{f}_1 + \tilde{f}_1)\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v G + 4\gamma M\int \mathrm{d}^3v\left[(2\bar{f}_1 + \tilde{f}_1)\left(\int \mathrm{d}^3v'\tilde{f}'_1/g\right)\right]\overset{\leftrightarrow}{I}, \tag{50}$$

where $G \equiv \int \mathrm{d}^3v' \tilde{f}_1' g$, and $\tilde{f}_1'$ is obtained from $\tilde{f}_1$ by substituting $\boldsymbol{v}'$ for $\boldsymbol{v}$. Since the $\overleftrightarrow{I}$ and $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$ terms in the expression for $\overleftrightarrow{K}_{\perp}^{n\ell}$ do not contribute to $\overleftrightarrow{\pi}_{\perp}$ we will neglect the last term in Eq. (50) as well as all other $\overleftrightarrow{I}$ and $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$ terms arising in the process of evaluating the first term in Eq. (50).

Evaluating $\tilde{f}_1$ from Eq. (3) by substituting $f_0 = f_M$ for $\bar{f}$,

$$\tilde{f}_1 = \boldsymbol{v} \cdot \frac{M}{T} \left[\boldsymbol{V}_{\perp} - \frac{2\boldsymbol{q}_{\perp}}{5p} L_1^{3/2}(x^2) \right] f_M, \tag{51}$$

where $L_1^{3/2}(x^2)$ is a generalized Laguerre polynomial and $\boldsymbol{V}_{\perp}$ and $\boldsymbol{q}_{\perp}$ are given by Eq. (32), and employing the result in the definition of G, we obtain

$$G = -\boldsymbol{\nabla}_v \cdot \left(\boldsymbol{V}_{\perp} G_M + \frac{2T}{5pM} \boldsymbol{q}_{\perp} H_M \right), \tag{52}$$

where

$$G_M \equiv \int \mathrm{d}^3v' f_M'\, g = nv \left[\left(1 + \frac{1}{2x^2} \right) E(x) + \frac{1}{2x} E'(x) \right] \tag{53}$$

and

$$H_M \equiv \int \mathrm{d}^3v' \frac{f_M'}{g} = \frac{n}{v} E(x) \tag{54}$$

are the Rosenbluth potentials for a Maxwellian.[14] Then, the relevant piece of $\overleftrightarrow{K}_{\perp}^{n\ell}$ can be rewritten as

$$\overleftrightarrow{K}_{\perp}^{n\ell} = 6\gamma M \int \mathrm{d}^3v (2\bar{f}_1 + \tilde{f}_1) \left(\boldsymbol{V}_{\perp} \cdot \boldsymbol{\nabla}_v \boldsymbol{\nabla}_v \boldsymbol{\nabla}_v G_M + \frac{2T}{5pM} \boldsymbol{q}_{\perp} \cdot \boldsymbol{\nabla}_v \boldsymbol{\nabla}_v \boldsymbol{\nabla}_v H_M \right), \tag{55}$$

where for an arbitrary function $L(v)$ and vector $\boldsymbol{d}$

$$\boldsymbol{d} \cdot \boldsymbol{\nabla}_v \boldsymbol{\nabla}_v \boldsymbol{\nabla}_v L = (\overleftrightarrow{I}\, \boldsymbol{d} \cdot \boldsymbol{v} + \boldsymbol{d}\boldsymbol{v} + \boldsymbol{v}\boldsymbol{d}) \frac{1}{v} \frac{\mathrm{d}}{\mathrm{d}v} \left(\frac{1}{v} \frac{\mathrm{d}L}{\mathrm{d}v} \right) + \boldsymbol{v}\boldsymbol{v} \frac{\boldsymbol{d} \cdot \boldsymbol{v}}{v} \frac{\mathrm{d}}{\mathrm{d}v} \left[\frac{1}{v} \frac{\mathrm{d}}{\mathrm{d}v} \left(\frac{1}{v} \frac{\mathrm{d}L}{\mathrm{d}v} \right) \right]. \tag{56}$$

Substituting expression (51) for $\tilde{f}_1$ into Eq. (55), averaging over angles in velocity space by using Eq. (A2), noting that for an arbitrary function $L(v)$

$$\int \mathrm{d}^3v\, L(v) \boldsymbol{v}\boldsymbol{v} = \frac{1}{3} \overleftrightarrow{I} \int \mathrm{d}^3v\, v^2 L(v), \tag{57}$$

and continuing to drop terms proportional to $\overleftrightarrow{I}$ and $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$, we obtain

$$
\begin{aligned}
6\gamma M\int \mathrm{d}^3v\tilde{f}_1\left(\boldsymbol{V}_\perp\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v\boldsymbol{\nabla}_vG_M+\frac{2T}{5pM}\boldsymbol{q}_\perp\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v\boldsymbol{\nabla}_vH_M\right) &= \\
\frac{2\gamma M^2}{T}\int \mathrm{d}^3v\,v f_M\left\{\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}G_M}{\mathrm{d}v}\right)+\frac{v^2}{5}\frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}G_M}{\mathrm{d}v}\right)\right]\right\}& \\
\times\left\{\boldsymbol{V}_\perp\left[\boldsymbol{V}_\perp-\frac{2\boldsymbol{q}_\perp}{5p}L_1^{3/2}(x^2)\right]+\left[\boldsymbol{V}_\perp-\frac{2\boldsymbol{q}_\perp}{5p}L_1^{3/2}(x^2)\right]\boldsymbol{V}_\perp\right\}& \\
+\frac{4\gamma M}{5p}\int \mathrm{d}^3v\,v f_M\left\{\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}H_M}{\mathrm{d}v}\right)+\frac{v^2}{5}\frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}H_M}{\mathrm{d}v}\right)\right]\right\}& \\
\times\left\{\boldsymbol{q}_\perp\left[\boldsymbol{V}_\perp-\frac{2\boldsymbol{q}_\perp}{5p}L_1^{3/2}(x^2)\right]+\left[\boldsymbol{V}_\perp-\frac{2\boldsymbol{q}_\perp}{5p}L_1^{3/2}(x^2)\right]\boldsymbol{q}_\perp\right\}&.
\end{aligned}
\tag{58}
$$

Evaluating the integrals (by using, for example, MATHEMATICA) we arrive at

$$
\begin{aligned}
6\gamma M\int \mathrm{d}^3v\tilde{f}_1\left(\boldsymbol{V}_\perp\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v\boldsymbol{\nabla}_vG_M+\frac{2T}{5pM}\boldsymbol{q}_\perp\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v\boldsymbol{\nabla}_vH_M\right) &= \\
-\frac{6}{5}Mn\nu\left[\boldsymbol{V}_\perp\boldsymbol{V}_\perp-\frac{3}{10p}\left(\boldsymbol{V}_\perp\boldsymbol{q}_\perp+\boldsymbol{q}_\perp\boldsymbol{V}_\perp\right)+\frac{3}{20p^2}\boldsymbol{q}_\perp\boldsymbol{q}_\perp\right]&.
\end{aligned}
\tag{59}
$$

Observing that $\langle\boldsymbol{v}\rangle_\varphi=v_\parallel\hat{\boldsymbol{b}}$ and using Eq. (21), we obtain in a similar way (again dropping $\overleftrightarrow{I}$ and $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$ terms)

$$
\begin{aligned}
12\gamma M\int \mathrm{d}^3v\bar{f}_1\left(\boldsymbol{V}_\perp\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v\boldsymbol{\nabla}_vG_M+\frac{2T}{5pM}\boldsymbol{q}_\perp\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v\boldsymbol{\nabla}_vH_M\right) &= \\
q_{10}(\boldsymbol{V}_\perp\hat{\boldsymbol{b}}+\hat{\boldsymbol{b}}\boldsymbol{V}_\perp)+\frac{2q_{11}}{5p}(\boldsymbol{q}_\perp\hat{\boldsymbol{b}}+\hat{\boldsymbol{b}}\boldsymbol{q}_\perp)&,
\end{aligned}
\tag{60}
$$

where

$$
\begin{aligned}
q_{10}&\equiv 12\gamma M\int \mathrm{d}^3v\,\frac{v_\parallel}{v}\left\{\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}G_M}{\mathrm{d}v}\right)+\frac{v_\perp^2}{2}\frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}G_M}{\mathrm{d}v}\right)\right]\right\}\bar{f}_1, \\
q_{11}&\equiv 12\gamma M\int \mathrm{d}^3v\,\frac{v_\parallel}{v}\left\{\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}H_M}{\mathrm{d}v}\right)+\frac{v_\perp^2}{2}\frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}H_M}{\mathrm{d}v}\right)\right]\right\}\bar{f}_1.
\end{aligned}
\tag{61}
$$

Consequently, continuing to use the definitions for $\boldsymbol{V}_\perp$ and $\boldsymbol{q}_\perp$ given by Eq. (32) results in

$$
\begin{aligned}
\overleftrightarrow{K}_\perp^{n\ell}=-\frac{6}{5}Mn\nu\left[\boldsymbol{V}_\perp\boldsymbol{V}_\perp-\frac{3}{10p}\left(\boldsymbol{V}_\perp\boldsymbol{q}_\perp+\boldsymbol{q}_\perp\boldsymbol{V}_\perp\right)+\frac{3}{20p^2}\boldsymbol{q}_\perp\boldsymbol{q}_\perp\right]& \\
+q_{10}(\boldsymbol{V}_\perp\hat{\boldsymbol{b}}+\hat{\boldsymbol{b}}\boldsymbol{V}_\perp)+\frac{2q_{11}}{5p}(\boldsymbol{q}_\perp\hat{\boldsymbol{b}}+\hat{\boldsymbol{b}}\boldsymbol{q}_\perp)&
\end{aligned}
\tag{62}
$$

and we obtain

$$
\begin{aligned}
\overleftrightarrow{\pi}^{n\ell}_{\perp} = & -\frac{3Mn\nu}{10\Omega}(\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp\boldsymbol{V}_\perp+\boldsymbol{V}_\perp\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp)-\frac{9M\nu}{200pT\Omega}(\hat{\boldsymbol{b}}\times\boldsymbol{q}_\perp\boldsymbol{q}_\perp+\boldsymbol{q}_\perp\hat{\boldsymbol{b}}\times\boldsymbol{q}_\perp) \\
& +\frac{9M\nu}{100\,T\Omega}(\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp\boldsymbol{q}_\perp+\boldsymbol{q}_\perp\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp+\hat{\boldsymbol{b}}\times\boldsymbol{q}_\perp\boldsymbol{V}_\perp+\boldsymbol{V}_\perp\hat{\boldsymbol{b}}\times\boldsymbol{q}_\perp) \\
& +\frac{q_{10}}{\Omega}(\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp\hat{\boldsymbol{b}}+\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp)+\frac{2q_{11}}{5p\Omega}(\hat{\boldsymbol{b}}\times\boldsymbol{q}_\perp\hat{\boldsymbol{b}}+\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\times\boldsymbol{q}_\perp).
\end{aligned} \tag{63}
$$

The general expression for the ion perpendicular viscosity $\overleftrightarrow{\pi}_{\perp}=\overleftrightarrow{\pi}^{\ell}_{\perp}+\overleftrightarrow{\pi}^{n\ell}_{\perp}$ is therefore given by the sum of Eqs. (48) and (63) with $\overleftrightarrow{W}$ given by Eq. (47) and quantities q_6 through q_{11} defined by Eqs. (45) and (61).

VI. HEAT FLUXES

In this section, we give expressions for plasma heat fluxes. We define the ion heat flux as

$$
\boldsymbol{q}\equiv\int \mathrm{d}^3v\,\left(\frac{1}{2}Mv^2-\frac{5}{2}T\right)\boldsymbol{v}f, \tag{64}
$$

and the electron heat flux in a similar way. We consider ions first and treat electrons next.

Recalling the definitions of q_1 and q_2 given by Eq. (22) we see that the parallel heat flux is

$$
q_\parallel\equiv\boldsymbol{q}\cdot\hat{\boldsymbol{b}}=q_1+q_2-\frac{5}{2}pV_\parallel. \tag{65}
$$

To evaluate the perpendicular component of the ion heat flux it is convenient to employ the $Mv^2\boldsymbol{v}/2$ moment of the full kinetic equation. When the "maximal" ordering $\lambda\sim L_\perp\sim L_\parallel$ is assumed, $\nu/\Omega\sim\delta$ must again be used. Consequently, we have to evaluate all the terms in this moment equation to the same order as that of the (small) terms leading to the perpendicular classical collisional heat flux. Keeping this in mind we arrive at the following equation for the ion perpendicular heat flux,

$$\boldsymbol{\nabla}\cdot\left(\int \mathrm{d}^3v\,\frac{1}{2}Mv^2\boldsymbol{v}\boldsymbol{v}f\right)-\frac{e}{M}\boldsymbol{E}\cdot\left(\frac{5}{2}p\overleftrightarrow{I}+\overleftrightarrow{\pi}\right)$$
$$-\Omega\left(\frac{5}{2}p\boldsymbol{V}+\boldsymbol{q}\right)\times\hat{\boldsymbol{b}}=\int \mathrm{d}^3v\,\frac{1}{2}Mv^2\boldsymbol{v}C,$$

where we ignored time derivative terms as being of higher order and used the approximation $\overleftrightarrow{\pi}\approx\overleftrightarrow{\pi}_{\|}$, which is exact to the order required. Employing the momentum conservation equation to the same order to remove the $\boldsymbol{V}\times\hat{\boldsymbol{b}}$ term yields

$$\Omega\hat{\boldsymbol{b}}\times\boldsymbol{q}+\boldsymbol{\nabla}\cdot\left[\int \mathrm{d}^3v\,\left(\frac{1}{2}Mv^2-\frac{5}{2}T\right)\boldsymbol{v}\boldsymbol{v}f\right]+\frac{5\boldsymbol{\nabla}T}{2M}\cdot\left(p\overleftrightarrow{I}+\overleftrightarrow{\pi}_{\|}\right) \tag{66}$$
$$-\frac{e}{M}\boldsymbol{E}\cdot\overleftrightarrow{\pi}_{\|}=\int \mathrm{d}^3v\,\left(\frac{1}{2}Mv^2-\frac{5}{2}T\right)\boldsymbol{v}C,$$

where upon using expression (51) for $\tilde{f}_1$ we can write through order δ

$$\int \mathrm{d}^3v\,\left(\frac{1}{2}Mv^2-\frac{5}{2}T\right)\boldsymbol{v}\boldsymbol{v}f=q_3\overleftrightarrow{I}+(2q_5-3q_3)\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}-\frac{5T}{2M}\left(p\overleftrightarrow{I}+\overleftrightarrow{\pi}_{\|}\right). \tag{67}$$

To evaluate the term on the right-hand side of Eq. (66) we first notice that only the linearized contribution from the ion-ion collision operator is required to the order we require, giving

$$\int \mathrm{d}^3v\,\left(\frac{1}{2}Mv^2-\frac{5}{2}T\right)\boldsymbol{v}C\approx\int \mathrm{d}^3v\,\frac{1}{2}Mv^2\boldsymbol{v}C_{ii}^{\ell}(f_1).$$

Using self-adjointness of the linearized ion-ion collision operator and the result[15]

$$C_{ii}^{\ell}\left(\frac{1}{2}Mv^2\boldsymbol{v}f_M\right)=\nu TQ(x)f_M\boldsymbol{v}, \tag{68}$$

with $Q(x)$ defined by Eq. (42), we next write

$$\int \mathrm{d}^3v\,\frac{1}{2}Mv^2\boldsymbol{v}C_{ii}^{\ell}(f_1)=\nu T\int \mathrm{d}^3v\,\boldsymbol{v}f_1Q(x).$$

Employing $\tilde{f}_1$ from Eq. (51) and evaluating the integrals we finally obtain

$$\int \mathrm{d}^3v\,\frac{1}{2}Mv^2\boldsymbol{v}C_{ii}^{\ell}(f_1)=\hat{\boldsymbol{b}}\,\nu T\int \mathrm{d}^3v\,Q(x)v_{\|}\bar{f}_1-\frac{2p\nu}{M\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T. \tag{69}$$

Taking results (67) and (69) into account, we arrive at the following equation for $\boldsymbol{q}_\perp$:

$$\Omega\,\hat{\boldsymbol{b}}\times\boldsymbol{q}+\boldsymbol{\nabla}q_3-\frac{5T}{2M}\left[\boldsymbol{\nabla}p-\frac{1}{3}\boldsymbol{\nabla}(p_\parallel-p_\perp)\right]+\boldsymbol{\kappa}\left[(2q_5-3q_3)-\frac{5T}{2M}(p_\parallel-p_\perp)\right]$$
$$+\frac{e\boldsymbol{E}}{3M}(p_\parallel-p_\perp)+\hat{\boldsymbol{b}}\,\boldsymbol{\nabla}\cdot[\hat{\boldsymbol{b}}(2q_5-3q_3)]-\frac{5T}{2M}\hat{\boldsymbol{b}}\,\boldsymbol{\nabla}\cdot[\hat{\boldsymbol{b}}(p_\parallel-p_\perp)] \qquad (70)$$
$$-\frac{e\boldsymbol{E}_\parallel}{M}(p_\parallel-p_\perp)-\hat{\boldsymbol{b}}\,\nu T\int \mathrm{d}^3v\,Q(x)v_\parallel\bar{f}_1+\frac{2p\nu}{M\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T\approx 0.$$

Crossing by $\hat{\boldsymbol{b}}$, we finally obtain

$$\boldsymbol{q}_\perp=\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}q_3}{\Omega}-\frac{5T}{2M\Omega}\hat{\boldsymbol{b}}\times\left[\boldsymbol{\nabla}p-\frac{1}{3}\boldsymbol{\nabla}(p_\parallel-p_\perp)\right]-\frac{1}{3}\boldsymbol{v}_E(p_\parallel-p_\perp) \qquad (71)$$
$$+\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\kappa}}{\Omega}\left[(2q_5-3q_3)-\frac{5T}{2M}(p_\parallel-p_\perp)\right]-\frac{2p\nu}{M\Omega^2}\boldsymbol{\nabla}_\perp T.$$

Following the same procedure for electrons we find that expression (65) for the parallel ion heat flux also holds for the parallel electron heat flux if ion quantities are replaced with the corresponding electron quantities. The perpendicular electron heat flux can be found from the equation that is equivalent to the ion Eq. (66), but with the right-hand side given by

$$\int \mathrm{d}^3v\left(\frac{1}{2}mv^2-\frac{5}{2}T_e\right)\boldsymbol{v}\left[C^\ell_{ee}(f_{1e})+C^\ell_{ei}(f_{1e})\right]. \qquad (72)$$

Here, C^ℓ_{ee} and C^ℓ_{ei} are linearized electron-electron and electron-ion collision operators, respectively. The former is given by the electron analog of Eq. (35) with $\gamma_e=(3\sqrt{2\pi}/4)(\nu_e/n)(T_e/m)^{3/2}$. The latter is given to the required order by the expression

$$C^\ell_{ei}(f_{1e})=L_e(f_{1e})+\frac{2\gamma_e mn}{T_e v^3}\boldsymbol{v}\cdot\boldsymbol{V}f_{Me}, \qquad (73)$$

where

$$L_e(f_{1e})=\gamma_e n\boldsymbol{\nabla}_v\cdot(\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v v\cdot\boldsymbol{\nabla}_v f_{1e}) \qquad (74)$$

is the Lorentz operator and f_{Me} is the electron analog of the ion Maxwellian distribution function (10).

By analogy with the ion Eq. (71), electron-electron collisions result in a $-(\sqrt{2}p_e\nu_e/m\Omega_e^2)\boldsymbol{\nabla}_\perp T_e$ contribution to the classical perpendicular electron heat flux; the $\sqrt{2}$ difference arising purely from the difference in definitions of ν and ν_e. The contribution from the Lorentz collision operator can be easily evaluated by taking into account its self-adjointness, noticing that

$$L_e\left[\left(\frac{1}{2}mv^2-\frac{5}{2}T_e\right)\boldsymbol{v}f_{Me}\right]=2\gamma_e p_e L_1^{3/2}\left(\frac{mv^2}{2T_e}\right)\frac{\boldsymbol{v}}{v^3}f_{Me}, \tag{75}$$

employing $\tilde{f}_{1e}$ (since $\bar{f}_{1e}$ does not contribute to $\boldsymbol{q}_{e\perp}$), which is given by Eq. (51) with ion quantities replaced by electron ones, and performing the velocity-space integration. As a result, we obtain

$$\frac{1}{\Omega_e}\hat{\boldsymbol{b}}\times\int \mathrm{d}^3v\left(\frac{1}{2}mv^2-\frac{5}{2}T_e\right)\boldsymbol{v}L_e(f_{1e})=-\frac{13p_e\nu_e}{4m\Omega_e^2}\boldsymbol{\nabla}_\perp T_e+\frac{3p_e\nu_e}{2\Omega_e}\hat{\boldsymbol{b}}\times\boldsymbol{V}_e, \tag{76}$$

where $\boldsymbol{V}_e$ is the electron flow velocity. Finally, the $(2\gamma_e mn/T_e v^3)\boldsymbol{v}\cdot\boldsymbol{V}f_{Me}$ piece of the electron-ion collision operator results in the $-(3p_e\nu_e/2\Omega_e)\hat{\boldsymbol{b}}\times\boldsymbol{V}$ contribution to $\boldsymbol{q}_{e\perp}$. Putting everything together, we obtain

$$\begin{aligned}\boldsymbol{q}_{e\perp}\approx&-\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}q_{3e}}{\Omega_e}+\frac{5T_e}{2m\Omega_e}\hat{\boldsymbol{b}}\times\left[\boldsymbol{\nabla}p_e-\frac{1}{3}\boldsymbol{\nabla}(p_{\|e}-p_{\perp e})\right]\\&-\frac{1}{3}\boldsymbol{v}_E(p_{\|e}-p_{\perp e})-\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\kappa}}{\Omega_e}\left[(2q_{5e}-3q_{3e})-\frac{5T_e}{2m}(p_{\|e}-p_{\perp e})\right]\\&-\left(\frac{13}{4}+\sqrt{2}\right)\frac{p_e\nu_e}{m\Omega_e^2}\boldsymbol{\nabla}_\perp T_e-\frac{3p_e\nu_e}{2\Omega_e}\hat{\boldsymbol{b}}\times(\boldsymbol{V}-\boldsymbol{V}_e).\end{aligned} \tag{77}$$

VII. RECOVERING COLLISIONAL LIMIT

The general ion expressions for $\overleftrightarrow{\pi}_g$, $\overleftrightarrow{\pi}_\perp$, and $\boldsymbol{q}$ obtained in the previous sections describe plasma of arbitrary collisionality, provided that the leading order distribution function is a Maxwellian. Consequently, they should recover the known collisional expressions[8] for the ion gyroviscosity, perpendicular viscosity, and heat flux. Of course, the same should be true for the electron quantities as well.

To see that this is indeed the case we evaluate $\overleftrightarrow{\pi}_g$, $\overleftrightarrow{\pi}_\perp$, and $\boldsymbol{q}$ by employing the standard[16,17] short mean-free path result for $\bar{f}^{coll}$ as is given in the Appendix B. Then, we obtain through second order in δ

$$q_1 = \frac{2}{3}q_2 = pV_\parallel + \frac{2}{5}q_\parallel, \tag{78}$$

with $q_\parallel = -(125p/32M\nu)\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}T$ and $V_\parallel$ the second order accurate parallel ion flow velocity. Moreover, through first order in δ

$$q_3 = \frac{5pT}{2M}, \quad q_4 = \frac{2pT}{M}, \quad q_5 = \frac{15pT}{4M}, \quad p_\parallel = p_\perp = p. \tag{79}$$

Then, through second order in δ we obtain

$$\boldsymbol{a} = p\boldsymbol{V}_\perp + \frac{2}{5}(\boldsymbol{q}_\perp + \boldsymbol{q}_c), \quad \boldsymbol{c} = 0, \quad \overleftrightarrow{A} = 0, \tag{80}$$

where $\boldsymbol{V}_\perp$ and $\boldsymbol{q}_\perp$ are given by Eq. (32). Using Eq. (29) we then find that

$$\overleftrightarrow{\pi}_g \rightarrow \overleftrightarrow{\pi}_g\,|_{\rm coll} + \Delta\overleftrightarrow{\pi}_g, \tag{81}$$

where

$$\overleftrightarrow{\pi}_g\,|_{\rm coll} \equiv \frac{1}{4\Omega}\hat{\boldsymbol{b}}\times\left[p\boldsymbol{\nabla}(\boldsymbol{V}_\parallel + \boldsymbol{V}_\perp) + \frac{2}{5}\boldsymbol{\nabla}\left(\boldsymbol{q}_\parallel + \boldsymbol{q}_\perp + \boldsymbol{q}_c\right) + \text{Transpose}\right]\cdot(\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})$$
$$+\text{Transpose} \tag{82}$$

is the standard Mikhailovskii-Tsypin[7] and Catto-Simakov[8] short mean-free path ion gyroviscosity (as generalized in Ref. 9 by adding $\boldsymbol{q}_c$ to the heat flux) and the terms left over are simply

$$\Delta\overleftrightarrow{\pi}_g \equiv Mn(\boldsymbol{V}_\parallel + \boldsymbol{V}_\perp)(\boldsymbol{V}_\parallel + \boldsymbol{V}_\perp) - Mn\left[V_\parallel^2\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \frac{1}{2}V_\perp^2(\overleftrightarrow{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\right]. \tag{83}$$

It can be shown using $\bar{f}^{coll}$ from Appendix B that n represents the plasma density through second order in δ and p (or T) represents ion pressure (or temperature) through first order in δ. It follows from the fact that $p_{\|} = p_{\perp}$ through first order in δ and from ion momentum conservation that $\boldsymbol{V}_{\perp}$ represents ion perpendicular flow velocity accurately through the second order in δ. Eqs. (65) and (78) give $q_{\|}$ accurately through second order in δ. Finally, Eqs. (71) and (79) predict that the sum $\boldsymbol{q}_{\perp} + \boldsymbol{q}_c$ gives the ion perpendicular heat flux through second order in δ as well, so that our expressions (65) and (71) recover the standard Braginskii's result for the ion heat flux,[5] which is accurate through second order in δ. Consequently, Eqs. (82) and (83) can be rewritten in more compact forms, accurate through third order in δ, as

$$\overleftrightarrow{\pi}_g |_{\text{coll}} \equiv \frac{1}{4\Omega}\hat{\boldsymbol{b}} \times \left[p\boldsymbol{\nabla V} + p(\boldsymbol{\nabla V})^{\mathrm{T}} + \frac{2}{5}\boldsymbol{\nabla q} + \frac{2}{5}(\boldsymbol{\nabla q})^{\mathrm{T}} \right] \cdot (\overleftrightarrow{\boldsymbol{I}} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) + \text{Transpose}$$

and

$$\Delta\overleftrightarrow{\pi}_g \equiv Mn\boldsymbol{VV} - Mn\left[V_{\|}^2\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \frac{1}{2}V_{\perp}^2(\overleftrightarrow{\boldsymbol{I}} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \right],$$

where n, p, $\boldsymbol{V}$, and $\boldsymbol{q}$ are the total plasma density, ion pressure, flow velocity, and heat flux.

Similarly, to the requisite order we find

$$q_6 = -\left(pV_{\|} + \frac{7}{100}q_{\|} \right), \quad q_7 = \frac{3}{4}\left(pV_{\|} + \frac{77}{300}q_{\|} \right),$$
$$q_8 = -\left(pV_{\|} - \frac{1}{4}q_{\|} \right), \quad q_9 = 0,$$
$$q_{10} = -\frac{3}{4}Mn\nu\left(\frac{8}{5}V_{\|} - \frac{2}{5p}q_{\|} \right), \quad q_{11} = \frac{69}{80}Mn\nu\left(\frac{24}{23}V_{\|} - \frac{2}{5p}q_{\|} \right),$$

so that

$$\overleftrightarrow{\pi}_{\perp} = \overleftrightarrow{\pi}_{\perp}^{\ell} |_{\text{coll}} + \overleftrightarrow{\pi}_{\perp}^{n\ell} |_{\text{coll}} + \Delta\overleftrightarrow{\pi}_{\perp}^{\ell} + \Delta\overleftrightarrow{\pi}_{\perp}^{n\ell}, \tag{84}$$

where

$$\overleftrightarrow{\pi}^{\ell}_{\perp}|_{\rm coll} \equiv -\frac{3\nu}{10\Omega^2}\Bigg[\overleftrightarrow{W}|_{\rm coll} + 3\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot \overleftrightarrow{W}|_{\rm coll}) + 3(\hat{\boldsymbol{b}}\cdot \overleftrightarrow{W}|_{\rm coll})\hat{\boldsymbol{b}} \tag{85}$$

$$+\frac{1}{2}(\overleftrightarrow{I} - 15\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})(\hat{\boldsymbol{b}}\cdot \overleftrightarrow{W}|_{\rm coll}\cdot\hat{\boldsymbol{b}}) - \frac{1}{2}(\overleftrightarrow{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\overleftrightarrow{W}|_{\rm coll} : \overleftrightarrow{I}\Bigg]$$

with

$$\overleftrightarrow{W}|_{\rm coll} \equiv p\boldsymbol{\nabla}\boldsymbol{V} + \frac{2}{5}\boldsymbol{\nabla}\boldsymbol{q} - \frac{3[p\boldsymbol{\nabla}\boldsymbol{q} - (\boldsymbol{\nabla}p)\boldsymbol{q}]}{10p} \tag{86}$$

$$-\frac{3p\boldsymbol{\nabla}\boldsymbol{q}_{\parallel} + 5(\boldsymbol{\nabla}p)\boldsymbol{q}_{\parallel}}{100p} - \frac{\boldsymbol{\nabla}T(90\boldsymbol{q} - 13\boldsymbol{q}_{\parallel})}{400T} + \text{Transpose}$$

and

$$\overleftrightarrow{\pi}^{n\ell}_{\perp}|_{\rm coll} \equiv -\frac{9M\nu}{200pT\Omega}\left[\hat{\boldsymbol{b}}\times\boldsymbol{q}\left(\boldsymbol{q} + \frac{31}{15}\boldsymbol{q}_{\parallel}\right) + \left(\boldsymbol{q} + \frac{31}{15}\boldsymbol{q}_{\parallel}\right)\hat{\boldsymbol{b}}\times\boldsymbol{q}\right]. \tag{87}$$

When the diagonal terms in Eq. (85) are ignored as small corrections to the scalar pressure and parallel viscosity, the first two terms in Eq. (84) are the same as the Catto-Simakov[8] short mean-free path result for the ion perpendicular viscosity. Retaining the diagonal terms recovers Eq. (16) of Ref. 9.

The remaining terms in Eq. (84) are

$$\Delta\overleftrightarrow{\pi}^{\ell}_{\perp} \equiv -\frac{3\nu}{10\Omega^2}\Bigg[\Delta\overleftrightarrow{W} + 3\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\Delta\overleftrightarrow{W}) + 3(\hat{\boldsymbol{b}}\cdot\Delta\overleftrightarrow{W})\hat{\boldsymbol{b}} \tag{88}$$

$$+\frac{1}{2}(\overleftrightarrow{I} - 15\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})(\hat{\boldsymbol{b}}\cdot\Delta\overleftrightarrow{W}\cdot\hat{\boldsymbol{b}}) - \frac{1}{2}(\overleftrightarrow{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\Delta\overleftrightarrow{W} : \overleftrightarrow{I}\Bigg]$$

with

$$\Delta\overleftrightarrow{W} \equiv (\boldsymbol{\nabla}p - en\boldsymbol{E})\left(\boldsymbol{V} - \frac{3\boldsymbol{q}}{10p} + \frac{\boldsymbol{q}_{\parallel}}{20p}\right) - \frac{3}{4}n\boldsymbol{\nabla}T\boldsymbol{V} + \text{Transpose} \tag{89}$$

and

$$\Delta\overleftrightarrow{\pi}^{n\ell}_{\perp} \equiv -\frac{3Mn\nu}{10\Omega}\left[\hat{\boldsymbol{b}}\times\boldsymbol{V}_{\perp}\left(\boldsymbol{V} + 3\boldsymbol{V}_{\parallel} - \frac{3\boldsymbol{q} + 7\boldsymbol{q}_{\parallel}}{10p}\right) - \frac{3\hat{\boldsymbol{b}}\times\boldsymbol{q}_{\perp}}{10p}\left(\boldsymbol{V} + 3\boldsymbol{V}_{\parallel}\right)\right] \tag{90}$$

$$+\text{Transpose}.$$

These extra terms in $\overleftrightarrow{\pi}_g$ and $\overleftrightarrow{\pi}_\perp$ are attributable to the differences in the definitions of the viscosity used in Ref. 8 and herein. In Ref. 8 $\overleftrightarrow{\pi}_g$ and $\overleftrightarrow{\pi}_\perp$ were defined in terms of the random velocity variable $\boldsymbol{w} \equiv \boldsymbol{v} - \boldsymbol{V}$, while here we employ the full velocity variable $\boldsymbol{v}$. To demonstrate that $\Delta \overleftrightarrow{\pi}_g$, $\Delta \overleftrightarrow{\pi}_\perp^{\ell}$, and $\Delta \overleftrightarrow{\pi}_\perp^{n\ell}$ are indeed due to the use of the different variables we notice that for any operator $L(\boldsymbol{v})$ and functions $f_v(\boldsymbol{v})$, $f_w(\boldsymbol{w})$

$$\int \mathrm{d}^3v\, L(\boldsymbol{v}) f_v(\boldsymbol{v}) - \int \mathrm{d}^3w\, L(\boldsymbol{w}) f_w(\boldsymbol{w}) = \int \mathrm{d}^3v\, L(\boldsymbol{v})[f_v(\boldsymbol{v}) - f_w(\boldsymbol{v})]$$
$$\equiv \int \mathrm{d}^3v\, L(\boldsymbol{v}) \Delta f(\boldsymbol{v}).$$

Noticing that $\Delta \overleftrightarrow{\pi}_g = M \int \mathrm{d}^3v\, \boldsymbol{vv} \Delta \tilde{f}(\boldsymbol{v})$, using expression (B5) for $\Delta f^{coll}(\boldsymbol{v})$, employing Eqs. (21), (33), and (A2), and evaluating velocity space integrals we easily obtain result (83) through second order in δ.

Next, we consider $\Delta \overleftrightarrow{\pi}_\perp^{\ell}$. Substituting result (B5) into the expression for the difference between $\overleftrightarrow{K}_\perp^{\ell}$ written in $\boldsymbol{v}$ and $\boldsymbol{w}$ variables,

$$\Delta \overleftrightarrow{K}_\perp^{\ell} \equiv -M \int \mathrm{d}^3v\, f_M^{-1} \Delta \tilde{f} C_{ii}^{\ell}(\boldsymbol{vv} f_M), \tag{91}$$

using Eqs. (21), (33), (38), and (A2), and evaluating velocity space integrals we find

$$\Delta \overleftrightarrow{K}_\perp^{\ell} = \frac{6}{5} Mn\nu \left[\boldsymbol{VV} - \frac{3}{10p} (\boldsymbol{Vq} + \boldsymbol{qV}) + \frac{1}{20p} \left(\boldsymbol{Vq}_\parallel + \boldsymbol{q}_\parallel \boldsymbol{V} \right) \right]. \tag{92}$$

This result should coincide to the order required with the result for $\Delta \overleftrightarrow{K}_\perp^{\ell}$ obtained by substituting $\Delta \overleftrightarrow{W}$ from Eq. (89) for $\overleftrightarrow{W}$ in Eq. (46). To see that this is indeed the case we first recall that $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$ terms in $\Delta \overleftrightarrow{W}$ do not contribute to $\Delta \overleftrightarrow{K}_\perp^{\ell}$ and can therefore be omitted. We then rewrite Eq. (89) by recalling that the parallel, as well as the perpendicular, friction is small in the collisional case so that the leading order ion momentum equation is simply $\boldsymbol{\nabla} p - en\boldsymbol{E} \approx Mn\Omega \boldsymbol{V}_\perp \times \hat{\boldsymbol{b}}$. We use this in the first term of Eq. (89) and simplify the second term by noticing that

$$\nabla T = -\frac{2M\Omega}{5p}\left(\hat{\boldsymbol{b}} \times \boldsymbol{q}_\perp + \frac{16\nu}{25\Omega}\boldsymbol{q}_\parallel\right) \approx \frac{2M\Omega}{5p}\hat{\boldsymbol{b}} \times \boldsymbol{q}_\perp,$$

where the $\boldsymbol{q}_\parallel$ contribution may be neglected since it results in the $\boldsymbol{q}_\parallel$ pieces of the second term that are small by $(\nu/\Omega) \sim \delta \ll 1$ as compared to the $\boldsymbol{q}_\parallel$ piece of the first term. Substituting the expression for $\Delta \overleftrightarrow{W}$ thus simplified into Eq. (46) we recover result (92) as expected.

Finally, we will demonstrate that $\Delta \overleftrightarrow{\pi}_\perp^{n\ell}$ as given by Eq. (90) is also due to the difference between $\boldsymbol{v}$ and $\boldsymbol{w}$ variables. Indeed, employing the formalism developed in Sec. VB we can write

$$\Delta \overleftrightarrow{K}_\perp^{n\ell} = -6\gamma M \int \mathrm{d}^3 v\, [(2\bar{f}_{1w} + \tilde{f}_{1w})\nabla_v \nabla_v (\Delta G) + (2\Delta\bar{f}_1 + \Delta\tilde{f}_1)\nabla_v \nabla_v G], \quad (93)$$

where f_{1w} is given by Eq. (B3), Δf_1 is given by the first term on the right-hand side of Eq. (B5), and $\Delta G \equiv \int \mathrm{d}^3 v' \Delta \tilde{f}_1' g = -\boldsymbol{V}_\perp \cdot \nabla_v G_M$. Evaluating the integrals and substituting the result obtained for $\Delta \overleftrightarrow{K}_\perp^{n\ell}$ into Eq. (19) recovers Eq. (90), completing the demonstration.

VIII. DISCUSSION

In the preceding sections, expressions for the ion and electron parallel viscosities and gyroviscosities, the ion perpendicular viscosity, and electron and ion parallel and perpendicular heat fluxes are derived for arbitrary mean-free path plasmas. The results for the ion perpendicular viscosity and species heat fluxes are obtained in forms requiring $\bar{f}_1$ only. If the ion gyroviscosity is required through the same order as the ion perpendicular viscosity then both $\bar{f}_1$ and $\bar{f}_2$ are needed. For the lowest order expression for the gyroviscosity it is sufficient to know $\bar{f}_1$.

The gyroviscosities are given by Eqs. (29) and (25) with quantities q_1 through q_5 defined by Eqs. (22) and (26), whereas the ion perpendicular viscosity is the sum of Eqs. (48) and (63) with $\overleftrightarrow{W}$ given by Eq. (47) and quantities q_6 through q_{11} defined by Eqs. (45) and (61). Parallel viscosities as given by Eq. (16) require knowledge of $\bar{f}$ to first or second order, depending on the accuracy desired. It is shown in Sec. VII that the general expressions for the ion gyroviscosity and perpendicular viscosity recover the correct short mean-free path limits as expected.

Ion and electron parallel heat fluxes are given in terms of q_1 and q_2 by Eq. (65), whereas ion and electron perpendicular heat fluxes are given by Eqs. (71) and (77), respectively. The lowest order perpendicular heat fluxes are given by the standard diamagnetic expressions, as expected.

These viscosity and heat flux expressions make it possible to obtain a practical hybrid fluid-kinetic closure consisting of the usual Hazeltine's drift kinetic equation[4,10,11] and the conservation of number, momentum, and energy equations for each species.

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Appendix A. EVALUATION OF $\boldsymbol{\nabla} \cdot (M \int \mathrm{d}^3v\, \boldsymbol{vvv} \tilde{f})$

To evaluate $M \int \mathrm{d}^3v\, \boldsymbol{vvv}\tilde{f}$ with $\tilde{f} = \tilde{f}^H + \tilde{f}^{NH} + \tilde{f}^C$ it is convenient to rewrite

$$\tilde{f}^H + \tilde{f}^{NH} = \tilde{f}^v + \tilde{f}^{vv}, \tag{A1}$$

where

$$\tilde{f}^v \equiv \boldsymbol{v} \cdot \boldsymbol{d}, \quad \tilde{f}^{vv} \equiv \frac{1}{8\Omega}\boldsymbol{vv} : \overleftrightarrow{\mathsf{M}},$$

with

$$\boldsymbol{d} \equiv \boldsymbol{g} - \left(\boldsymbol{v}_E + \frac{\mu}{\Omega}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B + \frac{v_\parallel}{\Omega}\hat{\boldsymbol{b}} \times \frac{\partial \hat{\boldsymbol{b}}}{\partial t} \right) \frac{1}{B}\frac{\partial \bar{f}}{\partial \mu}$$

and $\overleftrightarrow{\mathsf{M}}$ given by Eq. (43).

Noticing that

$$\langle \boldsymbol{v}_i \boldsymbol{v}_j \boldsymbol{v}_k \boldsymbol{v}_l \rangle_\varphi = \hat{\boldsymbol{b}}_i \hat{\boldsymbol{b}}_j \hat{\boldsymbol{b}}_k \hat{\boldsymbol{b}}_l \left(v_\parallel^4 - 3v_\parallel^2 v_\perp^2 + \frac{3}{8}v_\perp^4 \right) + (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\frac{1}{8}v_\perp^4 \tag{A2}$$
$$+(\delta_{ij}\hat{\boldsymbol{b}}_k\hat{\boldsymbol{b}}_l + \delta_{ik}\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_l + \delta_{il}\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k + \delta_{jk}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_l + \delta_{jl}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_k + \delta_{kl}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j) \left(\frac{1}{2}v_\parallel^2 v_\perp^2 - \frac{1}{8}v_\perp^4 \right)$$

we obtain

$$M \int \mathrm{d}^3v\, \boldsymbol{v}_i \boldsymbol{v}_j \boldsymbol{v}_k \tilde{f}^v = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \left(M \int \mathrm{d}^3v \frac{1}{8}v_\perp^4 d_l \right) + \tag{A3}$$
$$(\delta_{il}\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k + \delta_{jl}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_k + \delta_{kl}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j) \left[M \int \mathrm{d}^3v \left(\frac{v_\parallel^2 v_\perp^2}{2} - \frac{v_\perp^4}{8} \right) d_l \right],$$

where it is easy to show using integration by parts that

$$M \int \mathrm{d}^3v \frac{1}{8}v_\perp^4 \boldsymbol{d} = \frac{1}{2\Omega}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} q_4 + p_\perp \boldsymbol{v}_E + \frac{q_1}{\Omega}\hat{\boldsymbol{b}} \times \frac{\partial \hat{\boldsymbol{b}}}{\partial t}, \tag{A4}$$
$$M \int \mathrm{d}^3v \left(\frac{v_\parallel^2 v_\perp^2}{2} - \frac{v_\perp^4}{8} \right) \boldsymbol{d} = \frac{1}{\Omega}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} \left(2q_3 - \frac{5}{2}q_4 \right) + (p_\parallel - p_\perp)\boldsymbol{v}_E + \frac{2q_2 - 3q_1}{\Omega}\hat{\boldsymbol{b}} \times \frac{\partial \hat{\boldsymbol{b}}}{\partial t}.$$

Since q_1 and q_2 are at most of order δ and $\partial/\partial t \sim \delta^2 \Omega$ we can safely neglect the last terms in both of Eqs. (A4).

Observing that

$$\langle \boldsymbol{v}_i\boldsymbol{v}_j\boldsymbol{v}_k\boldsymbol{v}_l\boldsymbol{v}_m\rangle_\varphi = \hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k\hat{\boldsymbol{b}}_l\hat{\boldsymbol{b}}_m\left(v_\parallel^5 - 5v_\parallel^3 v_\perp^2 + \frac{15}{8}v_\parallel v_\perp^4\right) +$$
$$(\delta_{ij}\hat{\boldsymbol{b}}_k\hat{\boldsymbol{b}}_l\hat{\boldsymbol{b}}_m + \delta_{ik}\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_l\hat{\boldsymbol{b}}_m + \delta_{il}\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k\hat{\boldsymbol{b}}_m + \delta_{im}\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k\hat{\boldsymbol{b}}_l + \delta_{jk}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_l\hat{\boldsymbol{b}}_m +$$
$$\delta_{jl}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_k\hat{\boldsymbol{b}}_m + \delta_{jm}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_k\hat{\boldsymbol{b}}_l + \delta_{kl}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_m + \delta_{km}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_l + \delta_{lm}\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k)\left(\frac{1}{2}v_\parallel^3 v_\perp^2 - \frac{3}{8}v_\parallel v_\perp^4\right)$$
$$[\hat{\boldsymbol{b}}_i(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}) + \hat{\boldsymbol{b}}_j(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) \tag{A5}$$
$$+\hat{\boldsymbol{b}}_k(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) + \hat{\boldsymbol{b}}_l(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})$$
$$\hat{\boldsymbol{b}}_m(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\frac{1}{8}v_\parallel v_\perp^4,$$

we find for an arbitrary velocity space isotropic f_0 (so that the portion of $\overleftrightarrow{M}$ due to $\overleftrightarrow{h}$ does not contribute):

$$M\int \mathrm{d}^3v\, \boldsymbol{v}_i\boldsymbol{v}_j\boldsymbol{v}_k\tilde{f}^{vv} = \tag{A6}$$
$$-\frac{1}{\Omega}\left(20q_3 - \frac{35}{2}q_4 - 4q_5\right)[\hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_j(\hat{\boldsymbol{b}}\times\boldsymbol{\kappa})_k + \hat{\boldsymbol{b}}_i\hat{\boldsymbol{b}}_k(\hat{\boldsymbol{b}}\times\boldsymbol{\kappa})_j + \hat{\boldsymbol{b}}_j\hat{\boldsymbol{b}}_k(\hat{\boldsymbol{b}}\times\boldsymbol{\kappa})_i]$$
$$+\frac{1}{\Omega}\left(2q_3 - \frac{5}{2}q_4\right)\left[\delta_{ij}(\hat{\boldsymbol{b}}\times\boldsymbol{\kappa})_k + \delta_{ik}(\hat{\boldsymbol{b}}\times\boldsymbol{\kappa})_j + \delta_{jk}(\hat{\boldsymbol{b}}\times\boldsymbol{\kappa})_i + \frac{1}{4}\left(\hat{\boldsymbol{b}}_i\overleftrightarrow{D}_{jk} + \hat{\boldsymbol{b}}_j\overleftrightarrow{D}_{ik} + \hat{\boldsymbol{b}}_k\overleftrightarrow{D}_{ij}\right)\right],$$

where $\overleftrightarrow{D}$ is defined by Eq. (28). To obtain Eq. (A6) for an arbitrary $\bar{f}$ we employ the definitions of Eqs. (26) and

$$\int \mathrm{d}^3v\, M\left(\frac{1}{2}v_\parallel^4 v_\perp^2 - \frac{3}{8}v_\parallel^2 v_\perp^4\right)\frac{1}{B}\frac{\partial\bar{f}}{\partial\mu} = 20q_3 - \frac{35}{2}q_4 - 4q_5, \tag{A7}$$
$$\int \mathrm{d}^3v\, M\frac{1}{8}v_\parallel^2 v_\perp^4\frac{1}{B}\frac{\partial\bar{f}}{\partial\mu} = -2q_3 + \frac{5}{2}q_4.$$

To evaluate $M\int \mathrm{d}^3v\, \boldsymbol{vvv}\tilde{f}^C$ we first notice that for $f_0 = f_M$, Eqs. (7) and (68) give

$$\tilde{f}^C \approx \frac{\nu}{\Omega^2}Q(x)f_M\boldsymbol{v}\cdot\boldsymbol{\nabla}_\perp \ln T. \tag{A8}$$

We then employ Eq. (A2) and $\int_0^{+\infty}\mathrm{d}x x^6 Q(x)\exp(-x^2) = -3\sqrt{\pi}/4$ to obtain

$$M\int \mathrm{d}^3v\, \boldsymbol{vvv}\tilde{f}^C \approx \frac{2}{5}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})(\boldsymbol{q}_c)_l, \tag{A9}$$

where $\boldsymbol{q}_c$ is the classical collisional ion heat flux for short mean-free path plasma defined in Eq. (27).

Employing Eqs. (A1), (A3), (A4), (A6), and (A9) we arrive at the final expression given by Eqs. (24) and (25).

Appendix B. COLLISIONAL ION DISTRIBUTION FUNCTION

The ion distribution function for short mean-free path plasmas was evaluated in Ref. 8 through order δ^2 using the *random velocity variable* $\boldsymbol{w} \equiv \boldsymbol{v} - \boldsymbol{V}$. The gyrophase averaged portion of this distribution function is given by

$$\bar{f}_w^{coll} = f_{Mw} + \bar{f}_{1w}^{coll} + \bar{f}_{2w}^{coll} + \cdots \tag{B1}$$

with

$$\begin{aligned} f_{Mw} &= n\left(\frac{M}{2\pi T}\right)^{3/2} \exp\left(-\frac{Mw^2}{2T}\right), \\ \bar{f}_{1w}^{coll} &= -\frac{2M}{5pT}\left[L_1^{3/2}(x_w^2) - \frac{4}{15}L_2^{3/2}(x_w^2)\right] q_\parallel w_\parallel f_{Mw}, \qquad \text{(B2)} \\ \bar{f}_{2w}^{coll} &= \left\{a_2 L_2^{1/2}(x_w^2) + a_3 L_3^{1/2}(x_w^2) + P_2(w_\parallel/w)x_w^2\left[b_0 L_0^{5/2}(x_w^2) + b_1 L_1^{5/2}(x_w^2)\right]\right\} f_{Mw}, \end{aligned}$$

where $q_\parallel = -(125p/32M\nu)\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} T$, and a_2, a_3, b_0, and b_1 are known coefficients, $P_2(w_\parallel/w)$ is a Legendre polynomial, $x_w^2 \equiv (Mw^2/2T)$, and $L_i^{j+1/2}(x_w^2)$, $i = 0, 1, 2, 3$, $j = 0, 1, 2$, are generalized Laguerre polynomials. In addition, the lowest order gyrophase dependent portion of the ion distribution function is given by

$$\tilde{f}_{1w}^{coll} = -\frac{2M}{5pT}\boldsymbol{q}_\perp \cdot \boldsymbol{w}_\perp L_1^{3/2}(x_w^2) f_{Mw},$$

so that

$$f_{1w}^{coll} = \bar{f}_{1w}^{coll} + \tilde{f}_{1w}^{coll} = -\frac{2M}{5pT}\boldsymbol{w} \cdot \left[\boldsymbol{q} L_1^{3/2}(x_w^2) - \frac{4}{15}\boldsymbol{q}_\parallel L_2^{3/2}(x_w^2)\right] f_{Mw}, \tag{B3}$$

where $\boldsymbol{q} = \boldsymbol{q}_\parallel + \boldsymbol{q}_\perp$, with $\boldsymbol{q}_\perp = (5p/2M\Omega)\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T$ to the required order.

In this work, the gyrophase averaged portion of the ion distribution function written in terms of the *full velocity variable* $\boldsymbol{v}$ is required through order δ^2. To obtain an expression for this distribution function we first notice that if $f_v(\boldsymbol{v})$ and $f_w(\boldsymbol{w})$ are used to denote exactly the same distribution function but written in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$ variables, respectively, then $f_w(\boldsymbol{v} - \boldsymbol{V}) = f_v(\boldsymbol{v})$. If $|\boldsymbol{V}|/v \ll 1$ can be assumed, then Taylor expanding this equality gives

$$\Delta f(\boldsymbol{v}) \equiv f_v(\boldsymbol{v}) - f_w(\boldsymbol{v}) = -\boldsymbol{V} \cdot \boldsymbol{\nabla}_w f_w(\boldsymbol{w})|_{\boldsymbol{w}\to\boldsymbol{v}} + \frac{1}{2}\boldsymbol{V}\boldsymbol{V} : \boldsymbol{\nabla}_w \boldsymbol{\nabla}_w f_w(\boldsymbol{w})|_{\boldsymbol{w}\to\boldsymbol{v}} + \cdots . \tag{B4}$$

Since for a short mean-free path plasma $f_w(\boldsymbol{w})$ is given through first order in δ by the sum of f_{Mw} and f_{1w}^{coll} from Eq. (B3) we obtain from Eq. (B4) through second order in δ

$$\begin{aligned} \Delta f^{coll}(\boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{V} \frac{M}{T} f_M - \frac{MV^2}{2T} f_M \\ + \boldsymbol{V} \cdot \frac{2M}{5pT} \left[\boldsymbol{q} L_1^{3/2}(x^2) - \boldsymbol{q}_\parallel \frac{4}{15} L_2^{3/2}(x^2) \right] f_M \\ + \boldsymbol{v}\boldsymbol{v} : \frac{M^2}{2T^2} \left\{ \boldsymbol{V}\boldsymbol{V} - \frac{4}{5p} \boldsymbol{V} \left[\boldsymbol{q} L_1^{5/2}(x^2) - \boldsymbol{q}_\parallel \frac{4}{15} L_2^{5/2}(x^2) \right] \right\} f_M, \end{aligned} \tag{B5}$$

where $\boldsymbol{V}$ in the first term on the right-hand side must be evaluated through second order in δ, while elsewhere the lowest order expression for $\boldsymbol{V}$ with $\boldsymbol{V}_\perp$ given by Eq. (32) can be employed, and $x^2 \equiv (Mv^2/2T)$.

It follows from $f_v(\boldsymbol{v}) = f_w(\boldsymbol{v}) + \Delta f(\boldsymbol{v})$ that the gyrophase averaged portion of the ion distribution function in $\boldsymbol{v}$ variables through second order in δ is given by Eqs. (B1) and (B2) with $\boldsymbol{w} \to \boldsymbol{v}$ and the gyrophase averaged Eq. (B5): $\bar{f}_v(\boldsymbol{v}) = \bar{f}_w(\boldsymbol{v}) + \Delta \bar{f}^{coll}(\boldsymbol{v})$.

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