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Limitations of Gyrokinetics on Transport Time Scales

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Abstract. We present a new recursive procedure to find a full f electrostatic gyrokinetic equation correct to first order in an expansion of gyroradius over magnetic field characteristic length. The procedure provides new insights into the limitations of the gyrokinetic quasineutrality equation. We find that the ion distribution function must be known at least to second order in gyroradius over characteristic length to calculate the long wavelength components of the electrostatic potential self-consistently. Moreover, using the example of a steady-state θ-pinch, we prove that the quasineutrality equation fails to provide the axisymmetric piece of the potential even with a distribution function correct to second order. We also show that second order accuracy is enough if a more convenient moment equation is used instead of the quasineutrality equation. These results indicate that the gyrokinetic quasineutrality equation is not the most effective procedure to find the electrostatic potential if the long wavelength components are to be retained in the analysis.

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1. Introduction

Nonlinear gyrokinetics have proven extremely useful for studying drift turbulence in the tokamak core. In the last decade, continuum flux-tube δf models [1, 2] have been used to satisfactorily calculate the short wavelength spectrum of turbulence and the associated transport. These δf codes assume that the ion and electron distribution functions are Maxwellian at long wavelengths, and only calculate the turbulent, short wavelength δf portion of the distribution function to obtain the turbulent particle and heat transport.

In recent years there has been an increasing interest in extending these turbulence calculations to longer wavelengths and transport timescales and obtaining self-consistent radial profiles for tokamaks. The electric field is of special importance since the poloidal zonal flow [3, 4, 5, 6] induced by its radial structure can act to control the saturated amplitude of turbulence. Calculating the electric field is an incompletely solved problem even when turbulence is not considered. The axisymmetric radial electric field has only been recently found in the Pfirsch-Schlüter regime [7, 8, 9], and there has been some incomplete work on the banana regime for high aspect ratio tokamaks [10, 11]. Most results are obtained in the high flow limit [12, 13, 14, 15] that will not be considered here. Consequently, a gyrokinetic model appropriate for transport time scales has to face the unsolved challenge of providing the axisymmetric radial electric field, as well as retaining all relevant turbulence effects including its interaction with neoclassical transport (and other serious difficulties associated with large computations).

We focus on the subtleties associated with determining the long wavelength portion of the radial electric field. However, we also retain the shorter wavelength zonal flow and turbulent behavior in our electrostatic gyrokinetic model formulated to first order in a gyroradius over characteristic length expansion (current gyrokinetic codes usually work to this order only). The formalism used to find the nonlinear gyrokinetic variables is similar to the technique presented in [16, 17] for linear gyrokinetics [18, 19, 20]. Care is taken to insure that for long wavelengths, the result recovers the gyrophase dependent piece of the distribution function to second order, as already found in drift-kinetics [21, 22].

In δf gyrokinetics, a modified quasineutrality equation has been traditionally used to solve for the electrostatic potential [23, 24]. The difference between the density of gyrocenters and the real ion density, due to the effect of the short wavelength components of the electric field on the gyromotion, is adjusted to ensure quasineutrality. In the process, the short wavelength components of the electric field are determined. The calculated turbulent fluxes are reasonably close to the experimental values [25, 26]. This methodology differs strongly from the procedures used in drift-kinetics [21, 27], where some form of $\nabla \cdot \mathbf{J} = 0$ is employed to find the electrostatic potential. We carefully examine the possibility of extending the gyrokinetic approach to longer wavelengths in order to determine the axisymmetric electric field. We find that the gyrokinetic equation is not known to high enough order to give a meaningful result. This conclusion is of great importance, because GYRO [28] can be run in a global mode and several groups

[29, 30, 31] have already began to develop codes that solve for the full distribution function, and they intend to use the gyrokinetic quasineutrality equation to find the potential, including the long wavelength pieces. An issue we address is whether their results at long wavelengths will be flawed because of the limitations of the traditional gyrokinetic approach.

We are aware that gyrokinetic equations of the same or even higher order than ours (with geometrical restrictions) have been derived by different authors [32, 33, 34, 35] using Hamiltonian approaches. Our method is an alternative approach that allows us to determine the missing ingredients in the gyrokinetic distribution function and, just as importantly, the limitations of the usual gyrokinetic quasineutrality equation. These missing pieces are the main reason the gyrokinetic quasineutrality equation should not be used to find the axisymmetric electric field.

The rest of this article is organized as follows. In section 2 we present the orderings and formalism used to find the gyrokinetic variables, and we obtain the gyrokinetic equation to first order in a gyroradius over characteristic length expansion. The formalism in section 2 is required in section 3 to derive quasineutrality in gyrokinetic form and to highlight its shortcomings at long wavelengths. Details of the derivations are relegated to the appendices. Section 4 illustrates the problems of the gyrokinetic quasineutrality equation by applying the gyrokinetic approach to the simplified geometry of the θ-pinch. Finally, in section 5 we discuss our findings.

2. Gyrokinetic variables and Fokker–Planck equation

This section is devoted to the derivation of convenient non-linear gyrokinetic variables. Only electrostatic gyrokinetics is considered. We assume that the magnetic field does not change in time and that it has slow spatial variation. Since the magnetic field is constant in time, the electric field can be expressed as a function of the electrostatic potential, ϕ, by $\mathbf{E} = -\nabla\phi$. The slow spatial variation of the magnetic field implies the existence of a small parameter $\delta = \rho/L \ll 1$, with $L = |\nabla(\ln B)|^{-1}$ the characteristic length for the magnetic field and $\rho = Mcv_i/ZeB$ the ion gyroradius, where $\mathbf{B}$ and $B = |\mathbf{B}|$ are the magnetic field and the magnitude of the magnetic field, $v_i = \sqrt{2T_i/M}$ is the ion thermal velocity, Z and M are the charge number and the mass of the species of interest, and e and c are the electron charge and the speed of light.

2.1. Orderings

The characteristic frequency of the processes of interest is assumed to be the drift wave frequency $\omega \sim \omega_* \sim k_\perp \rho v_i/L$. To treat arbitrary collisionality, the ion collision frequency is assumed to be of the order of the transit time of ions, $\nu \sim v_i/L$.

We consider the drift ordering, where the $E \times B$ drift is of order δv_i. Therefore, the electrostatic potential is $O(T/e)$, where $T \sim T_i \sim T_e$, and the electric field is of order

$$\mathbf{E} = -\nabla\phi = O(T/eL). \tag{1}$$

Similarly, the spatial gradient of the distribution functions is assumed to be

$$\nabla f = O(f_M/L), \tag{2}$$

where f_M is the zeroth order distribution function. For estimates, we will assume that the zeroth order distribution function is a slowly varying Maxwellian, with the density and temperature in the Maxwellian having characteristic lengths of variation $L_{n,T} \sim L$ much larger than the ion gyroradius. Most of our results are valid for any slowly varying zeroth order distribution function, but to make estimates it is convenient to work with a Maxwellian. Moreover, it is also a reasonable assumption since we are primarily interested in the core plasma in tokamaks and other well confined plasmas.

Our gyrokinetic description must resolve both neoclassical ($k_\perp L \sim 1$) and turbulent ($k_\perp \rho \sim 1$) spatial scales. Hence, we will allow components of ϕ and f with short perpendicular wavelengths, $k_\perp L \gg 1$. Such components have a slow variation along the magnetic field: $\hat{\mathbf{n}} \cdot \nabla \sim 1/L$, with $\hat{\mathbf{n}} = \mathbf{B}/B$. The size of the short wavelength components of the electric field, ϕ_k, is determined by the ordering of the $E \times B$ drift. According to (1), the gradient of ϕ_k is $|\nabla \phi_k| = k_\perp \phi_k \sim T/eL$. This relationship sets a maximum size for ϕ_k,

$$e\phi_k/T \sim (k_\perp L)^{-1} \gtrsim \delta, \tag{3}$$

where $k_\perp L \gtrsim 1$. For $k_\perp L \sim 1$, the potential is of the order of the temperature, but as $k_\perp$ grows, the size of the corresponding potential component decreases. For $k_\perp \rho \sim 1$, the potential is given by $e\phi_k/T \sim \delta \ll 1$. We are interested in the components that have wavelengths on the order of or longer than the ion gyroradius, which means that the electrostatic potential ϕ must be determined to $O(\delta T/e)$ at least.

To treat a possible adiabatic or Maxwell-Boltzmann response, we will order the short wavelength component of the distribution function, f_k, consistent with the electrostatic potential by taking

$$f_k/f_M \sim (k_\perp L)^{-1} \gtrsim \delta. \tag{4}$$

As with the potential, the components with $k_\perp \rho \sim 1$ are $O(\delta f_M)$ so that $\nabla f_k \sim k_\perp \delta f_M \sim f_M/L$. Hence, the distribution function must be solved to $O(\delta f_M)$ or higher.

Both the potential and the distribution function may be viewed as having a slowly spatially varying piece (representing the average value in the plasma) plus some rapid oscillations of small amplitude. The zonal flows, for example, will be included in the small piece if their characteristic wavelength is comparable to the gyroradius, but their amplitude may be larger for larger wavelengths. An advantage of this view point is that the rapid spatial potential fluctuations seen by a particle in its gyromotion are small compared to the average value of the potential amplitude. Similarly, the distribution function of the gyrocenters is equal, to zeroth order, to the distribution function of the particles. The difference, coming from the rapidly oscillating pieces, is small in our ordering. Notice that the δf codes [1, 28] explicitly adopt this treatment for the components of ϕ and f that satisfy $k_\perp \rho \sim 1$, and, as in this work, they order them as $O(\delta)$.

2.2. Gyrokinetic variables

We begin by defining the Vlasov operator in the usual $\mathbf{r}$, $\mathbf{v}$ variables for an electrostatic electric field as the following total derivative

$$d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla + [-(Ze/M)\nabla\phi + \Omega\mathbf{v}\times\hat{\mathbf{n}}]\cdot\nabla_v, \tag{5}$$

where $\Omega = ZeB/Mc$ is the gyrofrequency. The Fokker–Planck equation is then simply

$$df/dt = C\{f\}, \tag{6}$$

where C is the relevant Fokker–Planck collision operator. Our goal is to change the Fokker–Planck equation to gyrokinetic variables in such a way that all the required gyrophase information is retained to higher order than standard gyrokinetic treatments [2]. Many of the algebraic details are relegated to Appendices A to C. Appendices A and B give the complete derivation of the first and second order gyrokinetic variables. To obtain the conservative form of the gyrokinetic equation the Jacobian is required. Details are presented in Appendix C. Also, our gyrokinetic variables allow us to find the gyroviscosity for long wavelengths. The calculation is shown in Appendix D and the result is the same as in [22]. We could write a higher order gyrokinetic Fokker-Planck equation based on these variables, but it would be tedious and the result is not needed. Therefore, only the variables are given to higher order.

The nonlinear gyrokinetic variables to be employed are the guiding center location $\mathbf{R}$, the kinetic energy E, the magnetic moment μ, and the gyrophase φ. These variables will be defined to higher order than is customary by employing an extension of the procedure presented in [16] for high frequency gyrokinetics. The general idea is to construct the gyrokinetic variables to higher order by adding in δ corrections such that the total derivative of a generic gyrokinetic variable Q is gyrophase independent to the desired order, and we may safely employ

$$dQ/dt \simeq \langle dQ/dt\rangle, \tag{7}$$

where the gyrophase average $\langle\ldots\rangle$ is performed holding $\mathbf{R}$, E, μ and t fixed. The gyrokinetic variable Q is expanded in powers of δ,

$$Q = Q_0 + Q_1 + Q_2 + \ldots, \tag{8}$$

where Q_0 is the lowest order gyrokinetic variable (kinetic energy, magnetic moment, etc.), and $Q_1 = O(\delta Q_0)$, $Q_2 = O(\delta^2 Q_0)$... are the order δ, δ^2 ... corrections. The first correction Q_1 is constructed so that $dQ/dt = \langle dQ/dt\rangle + O(\delta^2\Omega Q)$, while the second correction Q_2 is evaluated such that $dQ/dt = \langle dQ/dt\rangle + O(\delta^3\Omega Q)$. In principle this process can be continued indefinitely. Any Q_k can be found once the functions Q_m, for $m = 1, 2, \ldots, k-1$, are known. All the functions Q_m are constructed so that

$$\frac{dQ}{dt} \simeq \frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) = \left\langle \frac{d}{dt}(Q_0 + \ldots + Q_{k-1})\right\rangle + O(\delta^k\Omega Q_0). \tag{9}$$

Adding Q_k means adding dQ_k/dt to (9). To lowest order, $dQ_k/dt \simeq -\Omega\,\partial Q_k/\partial\varphi$, which to the requisite order leads to an equation for Q_k,

$$\frac{dQ}{dt} \simeq \frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) - \Omega\frac{\partial Q_k}{\partial\varphi} = \left\langle \frac{d}{dt}(Q_0 + \ldots + Q_{k-1})\right\rangle, \tag{10}$$

where $\langle \partial Q_k/\partial \varphi \rangle = 0$ is employed. Using (10), $Q_k = O(\delta^k Q_0)$ is found to be periodic in gyrophase and given by

$$Q_k = \frac{1}{\Omega}\int^{\varphi} d\varphi' \left[\frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) - \left\langle \frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) \right\rangle \right]. \quad (11)$$

More explicitly, through the first two orders, Q_1 and Q_2 are determined to be

$$Q_1 = \frac{1}{\Omega}\int^{\varphi} d\varphi' \left(\frac{dQ_0}{dt} - \left\langle \frac{dQ_0}{dt} \right\rangle \right) \quad (12)$$

and

$$Q_2 = \frac{1}{\Omega}\int^{\varphi} d\varphi' \left[\frac{d}{dt}(Q_0 + Q_1) - \left\langle \frac{d}{dt}(Q_0 + Q_1) \right\rangle \right]. \quad (13)$$

By adding Q_1 and Q_2, the total derivative of the gyrokinetic variable $Q = Q_0 + Q_1 + Q_2$ is

$$dQ/dt = \langle d(Q_0 + Q_1)/dt \rangle + O(\delta^3 \Omega Q_0). \quad (14)$$

In the reminder of this subsection, we present the gyrokinetic variables that result from this process. We begin with the kinetic energy expanded as

$$E = E_0 + E_1 + E_2 + \ldots, \quad (15)$$

where $E_0 = v^2/2$, $E_1 = O(\delta v_i^2)$ and $E_2 = O(\delta^2 v_i^2)$. We construct E_1 and E_2 such that the energy derivative is gyrophase independent to order δ,

$$dE/dt = \langle dE/dt \rangle + O(\delta^2 v_i^3/L). \quad (16)$$

The explicit details are presented in Appendices A and B. We find

$$E_1 = Ze\widetilde{\phi}/M \quad (17)$$

and

$$E_2 = (c/B)(\partial \widetilde{\Phi}/\partial t), \quad (18)$$

where $\overline{\phi}$, $\widetilde{\phi}$ and $\widetilde{\Phi}$ are functions related to the electrostatic potential. They depend on the new gyrokinetic variables. Their definitions are

$$\overline{\phi}(\mathbf{R}, E, \mu, t) = \langle \phi \rangle = \frac{1}{2\pi}\oint d\varphi \, \phi(\mathbf{r}(\mathbf{R}, E, \mu, \varphi, t), t), \quad (19)$$

$$\widetilde{\phi}(\mathbf{R}, E, \mu, \varphi, t) = \phi(\mathbf{r}(\mathbf{R}, E, \mu, \varphi, t), t) - \overline{\phi}(\mathbf{R}, E, \mu, t), \quad (20)$$

and

$$\widetilde{\Phi}(\mathbf{R}, E, \mu, \varphi, t) = \int^{\varphi} d\varphi' \, \widetilde{\phi}(\mathbf{R}, E, \mu, \varphi', t), \quad (21)$$

such that $\langle \widetilde{\Phi} \rangle = 0$. These are the same definitions used by Dubin [32].

It is important to comment on the size of these functions. Both ϕ and $\overline{\phi}$ are of the same order as the temperature for long wavelengths, but small for short wavelengths. However, $\widetilde{\phi}$ is always small as it accounts for the variation in the electrostatic potential that a particle sees as it moves in its gyromotion. Of course, since the potential is small for short wavelengths, the variation observed by the particle is also small. For long wavelengths, even though the potential is comparable to the temperature, the particle motion is small compared to the wavelength, and the variations that it sees in its motions are small. Therefore, $\widetilde{\phi} \sim \delta T/e$ for all wavelengths in our ordering, making $\widetilde{\Phi}$ small as well.

The Vlasov operator acting on E is shown in Appendix B to give

$$\frac{dE}{dt} = \left\langle \frac{dE}{dt} \right\rangle + O\left(\delta^2 \frac{v_i^3}{L}\right) = -\frac{Ze}{M}\left[\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d\right] \cdot \nabla_{\mathbf{R}}\overline{\phi} + O\left(\delta^2 \frac{v_i^3}{L}\right), \quad (22)$$

where $\mathbf{v}_d$ is the total drift velocity, composed of $E \times B$ drift and magnetic drift $\mathbf{v}_M$

$$\mathbf{v}_d = -(c/B)\nabla_{\mathbf{R}}\overline{\phi} \times \hat{\mathbf{n}} + \mathbf{v}_M, \quad (23)$$

with $\mathbf{v}_M$

$$\mathbf{v}_M = (\overline{v}_{||}^2/\Omega)\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \cdot \nabla_{\mathbf{R}}\hat{\mathbf{n}}) + (\mu/\Omega)\hat{\mathbf{n}} \times \nabla_{\mathbf{R}}B. \quad (24)$$

In the preceding equations, $\overline{v}_{||}$ is the gyrocenter parallel velocity defined by

$$\overline{v}_{||}^2/2 + \mu B(\mathbf{R}) = E. \quad (25)$$

Note that in (22), (23), (24) and (25), all the terms are given as a function of the new gyrokinetic variables, $\mathbf{R}$, E and μ.

The gyrokinetic gyrophase is obtained in a similar way as the energy by defining

$$\varphi = \varphi_0 + \varphi_1 + \ldots, \quad (26)$$

with φ_0 the original gyrophase. The details are again in Appendix A. The most important result is that $d\varphi/dt$ is gyrophase independent to order δ, that is,

$$d\varphi/dt = \langle d\varphi/dt \rangle + O(\delta^2\Omega) = -\overline{\Omega} + O(\delta^2\Omega), \quad (27)$$

where $\overline{\Omega} \simeq \Omega$ to lowest order and $\overline{\Omega}$ is constructed to be gyrophase independent through order δ. The final result for $\overline{\Omega}$ is given by (A.12).

For the guiding center position we define

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{R}_1 + \mathbf{R}_2 + \ldots, \quad (28)$$

where $\mathbf{R}_0 = \mathbf{r}$, $|\mathbf{R}_1| = O(\rho)$ and $|\mathbf{R}_2| = O(\delta\rho)$.

Proceeding with $\mathbf{R}$ in a similar manner as for E and φ we find the usual result [18]

$$\mathbf{R}_1 = \Omega^{-1}\mathbf{v} \times \hat{\mathbf{n}}. \quad (29)$$

To next order we obtain

$$\mathbf{R}_2 = \frac{1}{\Omega}\left[\left(v_{||}\hat{\mathbf{n}} + \frac{1}{4}\mathbf{v}_\perp\right)(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\left(v_{||}\hat{\mathbf{n}} + \frac{1}{4}\mathbf{v}_\perp\right)\right] \overset{\cdot}{\times} \nabla\left(\frac{\hat{\mathbf{n}}}{\Omega}\right) + \frac{v_{||}}{\Omega^2}\mathbf{v}_\perp \cdot \nabla\hat{\mathbf{n}}$$
$$+ \frac{\hat{\mathbf{n}}}{\Omega^2}\left\{v_{||}\mathbf{v}_\perp \cdot (\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}) + \frac{1}{8}[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})] : \nabla\hat{\mathbf{n}}\right\} - \frac{c}{B\Omega}\nabla_{\mathbf{R}}\widetilde{\Phi} \times \hat{\mathbf{n}}, \quad (30)$$

which is the same as [16] except for the nonlinear term given last. Our vector conventions are $\mathbf{xy}:\overleftrightarrow{\mathbf{M}} = \mathbf{y}\cdot \overleftrightarrow{\mathbf{M}} \cdot\mathbf{x}$ and $\mathbf{xy}\overset{\cdot}{\times}\overleftrightarrow{\mathbf{M}} = \mathbf{x} \times (\mathbf{y}\cdot \overleftrightarrow{\mathbf{M}})$. The Vlassov operator acting on $\mathbf{R}$ then gives

$$d\mathbf{R}/dt = \langle d\mathbf{R}/dt \rangle + O(\delta^2 v_i) = \overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d + O(\delta^2 v_i), \quad (31)$$

where $\mathbf{v}_d$ is given by (23).

The gyrokinetic magnetic moment variable is dealt with somewhat differently since we want to construct it to remain an adiabatic invariant order by order. The condition for the magnetic moment is not only that its derivative must be gyrophase independent, but that $\langle d\mu/dt\rangle$ must vanish order by order, giving

$$d\mu/dt = O(\delta^2 v_i^3/BL) \simeq 0, \tag{32}$$

for μ to remain an adiabatic invariant. We define

$$\mu = \mu_0 + \mu_1 + \mu_2 + \dots, \tag{33}$$

where $\mu_0 = v_\perp^2/2B$ is the usual lowest order result, $\mu_1 = O(\delta\mu_0)$ and $\mu_2 = O(\delta^2\mu_0)$. For μ to remain an adiabatic invariant, μ_1 and μ_2 must contain gyrophase independent contributions such that $\langle d\mu/dt\rangle = 0$ to the requisite order. Solving for μ_1 as outlined in Appendix A gives

$$\mu_1 = \frac{Ze}{MB}\widetilde{\phi} - \frac{1}{B}\mathbf{v}_\perp \cdot \mathbf{v}_M - \frac{v_{||}}{4B\Omega}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{n}} - \frac{v_{||}v_\perp^2}{2B\Omega}\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}. \tag{34}$$

To keep μ an adiabatic invariant, $\langle\mu_1\rangle = -(v_{||}v_\perp^2/2B\Omega)(\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}) \neq 0$. The calculation of μ_2 has been omitted for brevity. It turns out that the function μ_2 is not needed in what follows since we do not allow strong μ variation in f.

2.3. Fokker–Planck equation

The Fokker–Planck equation (6) becomes

$$\partial f/\partial t + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}} f + \dot{E}\,\partial f/\partial E + \dot{\mu}\,\partial f/\partial\mu + \dot{\varphi}\,\partial f/\partial\varphi = C\{f\}. \tag{35}$$

when written in gyrokinetic variables, where $\dot{Q} = dQ/dt$, and Q is any of the gyrokinetic variables. The gyroaverage of this equation is

$$\partial\overline{f}/\partial t + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\overline{f} + \dot{E}\,\partial\overline{f}/\partial E = \langle C\{f\}\rangle, \tag{36}$$

where $\overline{f}(\mathbf{R}, E, \mu, t) = \langle f\rangle$. Here, we have used that $\mathbf{R}$, E and μ are defined such that their time derivatives are gyrophase independent to the orders given by (22), (31) and (32). Therefore, in (36) we have neglected pieces that are $O(f_M\delta^2 v_i/L)$. We also have neglected the term $\langle\dot{\varphi}\,\partial f/\partial\varphi\rangle = O(\widetilde{f}\delta v_i/L)$, where $\widetilde{f} = f - \overline{f}$ is the gyrophase dependent piece of the distribution function. We will prove in the next paragraph that $\widetilde{f}$ is $O(f_M\delta\nu/\Omega)$, making all the neglected terms smaller than $f_M\delta^2 v_i/L$, and the distribution function gyrophase independent to first order, $f \simeq \overline{f}$. Notice that, due to the missing pieces, we can only obtain contributions to the distribution function that are $O(\delta f_M)$, as well as all terms with $k_\perp\rho \sim 1$.

The explicit equation for the gyrophase dependent part of the distribution function is obtained by subtracting from the full Fokker–Planck equation (35) its gyroaverage, giving to lowest order

$$-\Omega\,\partial\widetilde{f}/\partial\varphi = C\{f\} - \langle C\{f\}\rangle. \tag{37}$$

Therefore, the collisional term is the one that sets the size of $\widetilde{f}$. In many cases, the distribution function is a Maxwellian to zeroth order. This means that the collision operator vanishes to zeroth order, $C\{f\} = O(\delta\nu f_M)$, giving $C\{f\} - \langle C\{f\}\rangle = O(\delta\nu f_M)$. As a result, $\widetilde{f}$ is

$$\widetilde{f} \simeq -\frac{1}{\Omega}\int^{\varphi} d\varphi' \left(C\{f\} - \langle C\{f\}\rangle\right) = O\left(\frac{\delta\nu}{\Omega} f_M\right), \tag{38}$$

where $\nu/\Omega \ll 1$.

Using the values of dE/dt from (22) and $d\mathbf{R}/dt$ from (31), and suppressing the overbar by using $\overline{f} \simeq f$, the equation for f in gyrokinetic variables is

$$\partial f/\partial t + [\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d] \cdot [\nabla_{\mathbf{R}} f - (Ze/M)\nabla_{\mathbf{R}}\overline{\phi}\, \partial f/\partial E] = \langle C\{f\}\rangle, \tag{39}$$

where $\overline{\phi}$ is the function defined in (19), and f is gyrophase independent.

The gyrokinetic equation can be also written in conservative form. To do so, the Jacobian of the gyrokinetic transformation is needed. Conservation of particles in phase space requires the Jacobian of the transformation, $J = \partial(\mathbf{r}, \mathbf{v})/\partial(\mathbf{R}, E, \mu, \varphi)$, to satisfy

$$\partial J/\partial t + \nabla_{\mathbf{R}} \cdot (\dot{\mathbf{R}} J) + \partial(\dot{E}J)/\partial E + \partial(\dot{\mu}J)/\partial\mu + \partial(\dot{\varphi}J)/\partial\varphi = 0. \tag{40}$$

(This is the equality $\nabla \cdot \dot{\mathbf{r}} + \nabla_v \cdot \dot{\mathbf{v}} = 0$ written in gyrokinetic variables). Employing this property, equation (35) can be written in conservative form by multiplying it by J to obtain

$$\frac{\partial}{\partial t}(Jf) + \nabla_{\mathbf{R}} \cdot \left(\dot{\mathbf{R}} Jf\right) + \frac{\partial}{\partial E}\left(\dot{E}Jf\right) + \frac{\partial}{\partial \mu}\left(\dot{\mu}Jf\right) + \frac{\partial}{\partial \varphi}\left(\dot{\varphi}Jf\right) = JC\{f\}. \tag{41}$$

The gyroaverage of this equation is

$$\partial(J\overline{f})/\partial t + \nabla_{\mathbf{R}} \cdot (\dot{\mathbf{R}}J\overline{f}) + \partial(\dot{E}J\overline{f})/\partial E = J\langle C\{f\}\rangle. \tag{42}$$

We have taken into account that the Jacobian J is independent of φ to the order of interest, as can be seen by using (40). The equation for the gyrophase dependent part of the Jacobian is obtained by subtracting from (40) its gyroaverage. Notice that $J - \langle J\rangle$ depends on the differences $\dot{\mathbf{R}} - \langle\dot{\mathbf{R}}\rangle$, $\dot{E} - \langle\dot{E}\rangle$..., and those differences are small by definition of the gyrokinetic variables. The gyrophase-dependent part of the Jacobian is estimated to be $J - \langle J\rangle = O(\delta^2 B/v_i)$. Finally, we substitute $d\mathbf{R}/dt$ and dE/dt in (42) to get

$$\frac{\partial}{\partial t}(Jf) + \nabla_{\mathbf{R}} \cdot \{Jf[\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d]\} - \frac{\partial}{\partial E}\left\{Jf\frac{Ze}{M}[\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d] \cdot \nabla_{\mathbf{R}}\overline{\phi}\right\} = J\langle C\{f\}\rangle. \tag{43}$$

The calculation of the Jacobian is described in Appendix C. The final result is

$$J = \partial(\mathbf{r}, \mathbf{v})/\partial(\mathbf{R}, E, \mu, \varphi) = B(\mathbf{R})/\overline{v}_{||} + (Mc/Ze)(\hat{\mathbf{n}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{n}}). \tag{44}$$

In Appendix C we also prove that J satisfies the gyroaverage of (40).

Similar gyrokinetic equations to (39) and (43) can be found for the gyrokinetic variables $\mathbf{R}$, $\overline{v}_{||}$ and μ, where $\overline{v}_{||}$ is defined by (25). From (22), (25), (31) and $d\mu/dt \simeq 0$ we find

$$\dot{\overline{v}}_{||} = -[\hat{\mathbf{n}}(\mathbf{R}) + (\overline{v}_{||}/\Omega)\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \cdot \nabla_{\mathbf{R}}\hat{\mathbf{n}})] \cdot \nabla_{\mathbf{R}}[\mu B(\mathbf{R}) + Ze\overline{\phi}/M], \tag{45}$$

which gives the gyrokinetic equation

$$\partial f/\partial t + [\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d] \cdot \nabla_{\mathbf{R}} f + \dot{\overline{v}}_{||}\, \partial f/\partial \overline{v}_{||} = \langle C\{f\}\rangle. \quad (46)$$

This gyrokinetic equation can be written in conservative form by noticing that the new Jacobian is given by

$$J_{\overline{v}_{||}} = \partial(\mathbf{r}, \mathbf{v})/\partial(\mathbf{R}, \overline{v}_{||}, \mu, \varphi) = B(\mathbf{R}) + (Mc\overline{v}_{||}/Ze)(\hat{\mathbf{n}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{n}}). \quad (47)$$

Using the new Jacobian, we can write

$$\partial(J_{\overline{v}_{||}} f)/\partial t + \nabla_{\mathbf{R}} \cdot \{J_{\overline{v}_{||}} f [\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d]\} + \partial(J_{\overline{v}_{||}} f \dot{\overline{v}}_{||})/\partial \overline{v}_{||} = J_{\overline{v}_{||}} \langle C\{f\}\rangle. \quad (48)$$

3. Quasineutrality equation for a gyrokinetic distribution function

The distribution function f_i in Poisson's equation,

$$-\nabla^2 \phi = 4\pi e \left[Z \int d^3v\, f_i(\mathbf{R}, E, \mu, t) - n_e(\mathbf{r}, t) \right], \quad (49)$$

is obtained from (39) or (43). Therefore, it is known as a function of the gyrokinetic variables. The distribution function can be rewritten more conveniently as a function of $\mathbf{r} + \Omega^{-1}\mathbf{v} \times \hat{\mathbf{n}}$ and $\mathbf{v}$ by Taylor expanding. However, it is important to remember that there are missing pieces of order δ^2 in the distribution function since terms of this order must be neglected to derive (39) and (43). Thus, the expansion can only be carried out to the order where the distribution function is totally known, resulting in

$$f_i(\mathbf{R}, E, \mu, t) = f_i(\mathbf{R}_g, E_0, \mu_0, t) + E_1\, \partial f_i/\partial E_0 + \mu_1\, \partial f_i/\partial \mu_0 + O(\delta^2 f_M). \quad (50)$$

Notice that $\mathbf{R}_g \equiv \mathbf{r} + \Omega^{-1}\mathbf{v} \times \hat{\mathbf{n}}$ cannot be Taylor-expanded if we allow $k_\perp \rho \sim 1$. In addition to this Taylor expansion, we also take into account that the turbulent wavelengths we are interested in are usually much larger than the Debye length. Thus, the term in the left side of Poisson's equation may be neglected. The resulting quasineutrality equation reduces to

$$\frac{Z^2 e}{M} \int d^3v\, \widetilde{\phi} \left(\frac{\partial f_i}{\partial E_0} + \frac{1}{B}\frac{\partial f_i}{\partial \mu_0} \right) \simeq -Z\hat{N}_i(\mathbf{r}, t) + n_e(\mathbf{r}, t). \quad (51)$$

where n_e is the electron density, the function $\hat{N}_i$ is the ion guiding center number density when the effect of the electrostatic potential is extracted, and terms of order $O(\delta^2 n_e)$ and $O(n_e \lambda_D^2/L\rho)$ have been neglected, where $\lambda_D = \sqrt{T/4\pi e^2 n_e}$ is the Debye length and $k_\perp \rho \sim 1$. It is convenient to write $\hat{N}_i$ as

$$\hat{N}_i(\mathbf{r}, t) = \int d^3v\, f_i(\mathbf{R}_g, E_0, \mu_0, t) + \frac{1}{\Omega} \hat{\mathbf{n}} \cdot \nabla \times \hat{\mathbf{n}} \int d^3v\, v_{||} f_i. \quad (52)$$

The last integral comes from the term $\mu_1 \partial f_i/\partial \mu_0$ upon integrating by parts and is negligible when f_i is a stationary Maxwellian to lowest order. In the higher order integrals involving $\widetilde{\phi}$ in (51), the function f_i must be taken only to lowest order. Moreover, according to our ordering, the corrections arising from using $\mathbf{R}_g$ instead of $\mathbf{r}$ are small in these two terms because, even though we allow small wavelengths, the amplitude of the fluctuations with small wavelengths is assumed to be of the next order.

Therefore, in the higher order integrals, only the long wavelength distribution function depending on $\mathbf{R}_g \simeq \mathbf{r}$ need be retained.

Equation (51) may be used to calculate ϕ for wavelengths of the order of the gyroradius, including zonal flow, as is normally done in δf turbulence codes such as GS2 [1] or GYRO [28]. However, the equation is not useful for long wavelengths. In the limit $k_\perp L \sim 1$, the average of $\widetilde{\phi}$ holding $\mathbf{r}$, $v_{||}$ and $v_\perp$ fixed becomes the same order as terms already neglected since

$$\widetilde{\phi}(\mathbf{r} + \Omega^{-1}(\mathbf{v} \times \hat{\mathbf{n}}), E_0, \mu_0, \varphi_0, t) = \widetilde{\phi}(\mathbf{r}, E_0, \mu_0, \varphi_0, t) + O(\delta^2 T/e). \tag{53}$$

As a result, the terms on the left side of (51) vanish to the order the equation has been derived, leaving

$$Z\hat{N}_i(\mathbf{r}, t) = n_e(\mathbf{r}, t) \tag{54}$$

as the quasineutrality equation. This equation does not depend at all on ϕ. Therefore, we cannot solve for the correct ϕ at long wavelengths. Either a moment description or a more accurate gyrokinetic treatment are required to solve for ϕ.

It is possible to obtain a higher order long wavelength quasineutrality equation if the ion distribution function is assumed to be known to high enough order. The resulting equation in the long wavelength limit, assuming a Maxwellian distribution to lowest order, is

$$\nabla \cdot \left(\frac{Zcn_i}{B\Omega} \nabla_\perp \phi \right) - \frac{ZMc^2 n_i}{2T_i B^2} |\nabla_\perp \phi|^2 = n_e - Z\hat{N}_i, \tag{55}$$

where $\hat{N}_i$ is defined to higher order,

$$\hat{N}_i(\mathbf{r}, t) = \int d^3v\, f_i(\mathbf{r}, E_0, \mu_0, t) \left(1 + \frac{v_{||}}{\Omega} \hat{\mathbf{n}} \cdot \nabla \times \hat{\mathbf{n}} \right) + (\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{n}}\hat{\mathbf{n}}) : \frac{\nabla\nabla p_i}{2M\Omega^2}. \tag{56}$$

Even though this equation is correct, it is only useful if we are able to evaluate the missing $O(\delta^2)$ pieces in f_i that are of the same order as the left side in (55). Equations (39) or (43) miss these pieces. The derivation of (55) is shown in Appendix E.

4. Gyrokinetic solution of the θ-pinch at long wavelengths

In the θ-pinch, the magnetic field is given by $\mathbf{B} = B(r)\hat{\mathbf{n}}$, where here $\hat{\mathbf{n}}$ is a constant unit vector in the axial direction, and r is the radial coordinate in cylindrical geometry. For long wavelengths, we can find the gyrokinetic equation to order δ^2. The simplified geometry of the magnetic field yields more manageable expressions for the gyrokinetic variables, i.e., μ_1 and $\mathbf{R}_2$ become

$$\mu_1 = Ze\widetilde{\phi}/MB - (v_\perp^2/2B^2\Omega)(\mathbf{v} \times \hat{\mathbf{n}}) \cdot \nabla B \tag{57}$$

and

$$\mathbf{R}_2 = (4B\Omega^2)^{-1} [\mathbf{v}_\perp \mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{n}})(\mathbf{v} \times \hat{\mathbf{n}})] \cdot \nabla B, \tag{58}$$

where the term $(c/B\Omega)\nabla_{\mathbf{R}}\widetilde{\Phi}\times\hat{\mathbf{n}}$ has been neglected because we assume that $k_\perp L \sim 1$. Using $\mathbf{R}_2$, we calculate the gyroaverage of $\dot{\mathbf{R}}$,

$$\langle\dot{\mathbf{R}}\rangle \simeq \langle v_{||}\rangle\hat{\mathbf{n}} + \langle(\mu_0/\Omega)\hat{\mathbf{n}}\times\nabla B - (c/B)\nabla\phi\times\hat{\mathbf{n}} + \mathbf{v}_\perp\cdot\nabla\mathbf{R}_2 - (Ze/M)\nabla\phi\cdot\nabla_v\mathbf{R}_2\rangle, \quad (59)$$

where we have used the fact that in a θ-pinch $\hat{\mathbf{n}}\cdot\nabla B = 0$ to write $\mathbf{v}\cdot\nabla\mathbf{R}_2 = \mathbf{v}_\perp\cdot\nabla\mathbf{R}_2$. The gyroaverages are performed by employing the long wavelength approximation $\nabla\phi \simeq \nabla_{\mathbf{R}}\phi - \Omega^{-1}(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla_{\mathbf{R}}\phi$ and the relation $\langle\mathbf{v}_\perp\mathbf{v}_\perp\mathbf{v}_\perp\rangle = 0$, to get

$$\langle\dot{\mathbf{R}}\rangle \simeq \langle v_{||}\rangle\hat{\mathbf{n}} + [\mu/\Omega(\mathbf{R})]\hat{\mathbf{n}}\times\nabla_{\mathbf{R}}B - [c/B(\mathbf{R})]\nabla_{\mathbf{R}}\phi\times\hat{\mathbf{n}}. \quad (60)$$

The gyroaverage of $\dot{E}$ is found by using (B.3) to write

$$\dot{E} = -(Ze/M)(\mathbf{v}\cdot\nabla\phi - d\widetilde{\phi}/dt) \simeq -(Ze/M)(\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\overline{\phi} + \dot{\mu}\,\partial\overline{\phi}/\partial\mu - \partial\widetilde{\phi}/\partial t), \quad (61)$$

where we employ that $\partial/\partial t \sim \delta v_i/L$ for long wavelengths and $\partial\phi/\partial E = O(\delta^3 M/e)$ in the θ-pinch. Considering that $\langle\dot{\mu}\rangle = O(\delta^2 v_i^3/BL)$ and $\partial\overline{\phi}/\partial\mu = O(\delta MB/e)$, the gyroaverage of (61) is calculated to be

$$\langle\dot{E}\rangle \simeq -(Ze/M)\langle\dot{\mathbf{R}}\rangle\cdot\nabla_{\mathbf{R}}\overline{\phi}. \quad (62)$$

Thus, the gyrokinetic equation to order $O(\delta^2)$ is

$$\partial f_i/\partial t + \langle\dot{\mathbf{R}}\rangle\cdot[\nabla_{\mathbf{R}}f_i - (Ze/M)\nabla_{\mathbf{R}}\overline{\phi}\,\partial f_i/\partial E] = \langle C\{f_i\}\rangle. \quad (63)$$

We have neglected the derivative $\partial f_i/\partial\mu$ because the distribution function is Maxwellian to zeroth order and $\langle\dot{\mu}\rangle$ is already small by definition of μ. For an axisymmetric steady state solution, the terms on the left side of (63) vanish, the second term because the gyrocenter parallel and perpendicular drifts, $\langle\dot{\mathbf{R}}\rangle$, remain in surfaces of constant f_i and ϕ (therefore $\langle v_{||}\rangle$ need not be evaluated to second order). Thus, equation (63) becomes $\langle C\{f_i\}\rangle = 0$. Such an equation can be solved for a simplified collision operator. We use a Krook operator, $C\{f_i\} = -\nu(f_i - f_M)$, with constant collision frequency ν and a shifted Maxwellian,

$$f_M = n_i(M/2\pi T_i)^{3/2}\exp[-M(\mathbf{v}-\mathbf{V}_i)^2/2T_i], \quad (64)$$

where n_i, T_i and $\mathbf{V}_i$ are functions of the position $\mathbf{r}$. We assume that the parallel average velocity, $V_{i||} = \hat{\mathbf{n}}\cdot\mathbf{V}_i$, is zero and we order $\mathbf{V}_i$ as $O(\delta v_i)$ to obtain

$$f_M \simeq f_{M0}[1 + M\mathbf{v}_\perp\cdot\mathbf{V}_i/T_i + M^2(\mathbf{v}_\perp\cdot\mathbf{V}_i)^2/2T_i^2 - MV_i^2/2T_i], \quad (65)$$

with

$$f_{M0} = n_i(M/2\pi T_i)^{3/2}\exp(-Mv^2/2T_i). \quad (66)$$

With the Krook operator, the gyrokinetic solution is

$$\begin{aligned} f_i = \langle f_M\rangle = \overline{f}_M\Bigg[1 &- \frac{x_\perp^2}{n_i}\nabla\cdot\left(\frac{cn_i}{B\Omega}\nabla_\perp\phi\right) + \frac{Mc^2}{2T_iB^2}(2-x_\perp^2)|\nabla_\perp\phi|^2 - \frac{x_\perp^2}{2n_iM\Omega^2}\nabla_\perp^2 p_i \\ &+ \frac{x_\perp^2}{2T_iM\Omega^2}\left(\frac{35}{4} - 7x^2 + x^4\right)|\nabla_\perp T_i|^2 + \frac{2x_\perp^2}{MB\Omega^2}\left(\frac{5}{2} - x^2\right)\nabla_\perp B\cdot\nabla_\perp T_i \\ &+ \frac{c}{T_iB\Omega}\nabla_\perp\phi\cdot\nabla_\perp T_i\left(\frac{5}{2} - x_\perp^2 - x^2\right) + \frac{x_\perp^2}{2M\Omega^2}\left(x^2 - \frac{5}{2}\right)\nabla_\perp^2 T_i + \frac{MV_i^2}{2T}(x_\perp^2 - 1)\Bigg], \end{aligned} \quad (67)$$

where $x^2 = Mv^2/2T_i \simeq ME/T_i$, $x_\perp^2 = Mv_\perp^2/2T_i \simeq M\mu B/T_i$ and

$$\overline{f}_M = n_i(\mathbf{R})[M/2\pi T_i(\mathbf{R})]^{3/2} \exp[-ME/T_i(\mathbf{R})]. \tag{68}$$

To obtain this equation we have employed

$$\mathbf{V}_i = (n_i M\Omega)^{-1}\hat{\mathbf{n}} \times \nabla p_i - (c/B)\nabla\phi \times \hat{\mathbf{n}}. \tag{69}$$

The distribution function in (67) has been calculated by using a gyrokinetic equation that is correct to order δ^2 for both the left side and the gyroaveraged collision operator. Using the definitions $\mathbf{R} = \mathbf{r} + \mathbf{R}_1 + \mathbf{R}_2$, $E = E_0 + E_1 + E_2$ and $\mu = \mu_0 + \mu_1$, and the gyroaverage collisional piece $\widetilde{f}_i$ given in (38), we can find the distribution function f_i in $\mathbf{r}$, $\mathbf{v}$ variables to order $O(\delta^2 f_{M0})$. As a check, the same solution has been also obtained without resorting to gyrokinetics to order $O(\delta^2 f_{M0})$.

If we had gyroaveraged $C\{f_i\}$ only to order δ, as most gyrokinetic models do, the solution would have been simply

$$f_i \simeq \overline{f}_M. \tag{70}$$

Substituting this solution into (55), we find the inconsistent result

$$\nabla \cdot \left(\frac{Zcn_i}{B\Omega} \nabla_\perp \phi \right) - \frac{ZMc^2 n_i}{2T_i B^2} |\nabla_\perp \phi|^2 = n_e - Zn_i - \frac{Z}{2M\Omega^2} \nabla_\perp^2 p_i. \tag{71}$$

However, this quasineutrality equation is very different from the one we obtain by using the full $O(\delta^2 f_{M0})$ solution in (67), which simply gives

$$Zn_i = n_e. \tag{72}$$

Therefore, the gyrokinetic quasineutrality equation reduces to the quasineutrality condition when the exact $O(\delta^2)$ distribution function of (67) is employed. Equation (71) is wrong because the $O(\delta)$ result of (70) is either inducing an $O(\delta^2)$ charge difference or imposing the non-physical condition

$$\nabla \cdot \left(\frac{Zcn_i}{B\Omega} \nabla_\perp \phi \right) - \frac{ZMc^2 n_i}{2T_i B^2} |\nabla_\perp \phi|^2 = - \frac{Z}{2M\Omega^2} \nabla_\perp^2 p_i. \tag{73}$$

The difference between (71) and (72), given by (73), originates in $O(\delta^2)$ terms that should have been cancelled by pieces of the distribution function of the same order.

The θ-pinch example illustrates the problem of using a lower order gyrokinetic equation than needed, but it also highlights another issue. The potential does not appear in the quasineutrality equation (72), and, therefore, it cannot be found using it. This result is not surprising since in the simplified problem presented here the dependence of f_i with the axisymmetric potential appears only in the orders $O(\delta^3)$ and $O(\delta^2\nu/\Omega)$, as we will prove shortly. It is possible, of course, that if turbulence is included in the problem, an axisymmetric zonal flow component of the potential appears in the quasineutrality equation at $O(\delta^2)$, but in the absence of turbulence, the axisymmetric component cannot be solved from quasineutrality even if the distribution function is known to $O(\delta^2)$.

In the θ-pinch in absence of turbulence, the electrostatic potential is obtained from a moment equation. The moment approach has the advantage of showing the effect of the potential in the ion density without having to calculate the distribution function to higher order than $O(\delta^2 f_{M0})$. The methodology we use here is presented for screw pinches and dipolar configurations in [36]. Equation (72) is equivalent to $\nabla\cdot\mathbf{J} = 0$, where $\mathbf{J}$ is the current density. In the case of the axisymmetric θ-pinch, this is equivalent to $J_r = 0$, where J_r is the radial component of the current density. In order to find J_r, we use conservation of azimuthal angular momentum to get

$$\nabla \cdot (cr \overleftrightarrow{\boldsymbol{\pi}} \cdot \hat{\boldsymbol{\theta}}) = r(\mathbf{J} \times \mathbf{B}) \cdot \hat{\boldsymbol{\theta}}, \tag{74}$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction and $\overleftrightarrow{\boldsymbol{\pi}}$ is the ion viscosity, given by

$$\overleftrightarrow{\boldsymbol{\pi}} = M \int d^3v \; f_i \left(\mathbf{vv} - \frac{v^2}{3} \overleftrightarrow{\mathbf{I}} \right). \tag{75}$$

Since $(\mathbf{J} \times \mathbf{B}) \cdot \hat{\boldsymbol{\theta}} = -BJ_r = 0$, equation (74) reduces to $r^{-1}\partial(r^2\pi_{r\theta})/\partial r = 0$. In a case without sources or sinks of momentum, the final equation for the potential is $\pi_{r\theta} = 0$. Finding $\pi_{r\theta}$ directly from the distribution function requires a higher order solution than the one provided by the $O(\delta^2)$ gyrokinetic equation used so far. However, that problem can be circumvented by using a moment approach, similar to the one in [37]. The moment equation for the gyroviscosity is

$$\Omega(\overleftrightarrow{\boldsymbol{\pi}} \times \hat{\mathbf{n}} - \hat{\mathbf{n}} \times \overleftrightarrow{\boldsymbol{\pi}}) = \overleftrightarrow{\mathbf{K}}_g + \overleftrightarrow{\mathbf{K}}_\perp, \tag{76}$$

with

$$\overleftrightarrow{\mathbf{K}}_g = \nabla \cdot \left(M \int d^3v \; \mathbf{vvv} f_i \right) + Zen_i(\nabla\phi\mathbf{V}_i + \mathbf{V}_i\nabla\phi) \tag{77}$$

and

$$\overleftrightarrow{\mathbf{K}}_\perp = -M \int d^3v \; \mathbf{vv}C\{f_i\}. \tag{78}$$

According to this relation,

$$\pi_{r\theta} = (K_{g,rr} - K_{g,\theta\theta} + K_{\perp,rr} - K_{\perp,\theta\theta})/4\Omega. \tag{79}$$

The gyrokinetic solution in (67) is high enough order to calculate $\pi_{r\theta}$ by this moment approach. The gyrokinetic variables $\mathbf{R} = \mathbf{r} + \mathbf{R}_1 + \mathbf{R}_2$ and $E = E_0 + E_1 + E_2$ must be Taylor expanded to get a second order distribution function dependent on the variables $\mathbf{r}$, $\mathbf{v}$. Actually, it turns out that only the gyrophase dependent part of the distribution function, $f_i - \langle f_i \rangle$, is needed, where here the gyroaverage is done holding $\mathbf{r}$, $v_{||}$ and $v_\perp$ fixed. This gyrophase dependent part is calculated in Appendix D, and the result is

$$\begin{aligned}(f_i - \langle f_i \rangle)_g = \mathbf{v} \cdot \mathbf{g}_\perp - [\mathbf{v}_d \cdot \mathbf{v} + (v_{||}/4\Omega)(\mathbf{v}_\perp\mathbf{v} \times \hat{\mathbf{n}} + \mathbf{v} \times \hat{\mathbf{n}}\mathbf{v}_\perp) : \nabla\hat{\mathbf{n}}]B^{-1}\partial f_i/\partial\mu_0 \\ + \Omega^{-1}[(v_{||}\hat{\mathbf{n}} + \mathbf{v}_\perp/4)\mathbf{v} \times \hat{\mathbf{n}} + \mathbf{v} \times \hat{\mathbf{n}}(v_{||}\hat{\mathbf{n}} + \mathbf{v}_\perp/4)] : \overleftrightarrow{\mathbf{h}},\end{aligned} \tag{80}$$

where the subindex g stands for non-collisional, and where

$$\mathbf{g}_\perp = \Omega^{-1}\hat{\mathbf{n}} \times [\nabla f_i - (Ze/M)\nabla\phi \, \partial f_i/\partial E_0] \tag{81}$$

and

$$\overleftrightarrow{\mathbf{h}} = \nabla\mathbf{g}_\perp - (Ze/M)\nabla\phi \, \partial\mathbf{g}_\perp/\partial E_0. \tag{82}$$

We also need to add the gyrophase dependent piece given by (38). For the Krook operator it becomes

$$(f_i - \langle f_i \rangle)_c = -(\nu/2\Omega^2) f_{M0} (Mv^2/T_i - 5)\mathbf{v}_\perp \cdot \nabla \ln T_i. \tag{83}$$

When all these factors are taken into account, we find

$$K_{g,rr} - K_{g,\theta\theta} = -r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{2\nu p_i}{M\Omega^2}\frac{\partial T_i}{\partial r}\right) \tag{84}$$

and

$$K_{\perp,rr} - K_{\perp,\theta\theta} = -\frac{\nu r p_i}{\Omega}\frac{\partial}{\partial r}\left[\frac{c}{rB}\left(\frac{\partial \phi}{\partial r} + \frac{1}{Zen_i}\frac{\partial p_i}{\partial r}\right)\right] - \frac{\nu r}{\Omega}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{p_i}{M\Omega}\frac{\partial T_i}{\partial r}\right). \tag{85}$$

Using these results, $\pi_{r\theta} = 0$ gives

$$c\left(\frac{\partial \phi}{\partial r} + \frac{1}{Zen_i}\frac{\partial p_i}{\partial r}\right) = rB(r)\int_0^r dr'\,\frac{U(r')}{r'}\left[\frac{\partial}{\partial r'}\ln B(r') - \frac{3}{2}\frac{\partial}{\partial r'}\ln\left(\frac{p_i(r')U(r')}{r'}\right)\right], \tag{86}$$

where $U = (2/M\Omega)(\partial T_i/\partial r)$. Note the difference between this equation and (73). In particular, notice that for an isothermal f_{M0}, $\partial T_i/\partial r = 0$, a radial Maxwell-Boltzmann response is recovered from (86) as expected, but this is not a feature of the non-physical forms (71) and (73).

5. Discussion

We have found an electrostatic gyrokinetic equation that provides the solution to $O(\delta f_M)$ for both long and short wavelengths. Furthermore, the gyrokinetic variables are found to high enough order to provide, at long wavelengths, the gyrophase dependent part of the distribution function to $O(\delta^2 f_M)$ (allowing us to recover the gyroviscosity).

The gyrokinetic equation is complemented by a quasineutrality equation that might be expected to provide the electrostatic potential in a self-consistent calculation. However, we demonstrate that it is unable to retain the long wavelength components of the potential if the distribution function is only exact to $O(\delta f_M)$. The traditional gyrokinetic approach is based on adjusting the potential each timestep according to its effect on the gyromotion of the particles, while the gyrocenter motion is given by the potential in the previous timestep. This procedure gives the right potential for short wavelengths, on the order $k_\perp \rho \sim 1$, since $O(\delta)$ accuracy is adequate, but fails as longer and longer wavelengths are included in the analysis because their effect on the gyromotion is averaged out and $O(\delta^2)$ and higher modifications to the Maxwellian are required. It might seem that keeping more terms in the gyrokinetic equation to obtain a higher order solution for the distribution function would be enough to find the potential, but finding such a gyrokinetic equation for general geometry is difficult and its solution by numerical means requires high numerical precision since terms smaller than $O(\delta^2 f_M)$ must be recovered without appreciable error to calculate the full axisymmetric potential to lowest order.

The θ-pinch solution shows that the quasineutrality condition fails to provide a solution in the steady state without turbulence even when a $O(\delta^2 f_M)$ solution is used.

In the absence of a moment description, the axisymmetric potential can be only obtained if the distribution function is known to $O(\delta^2\nu/\Omega)$. This behavior is not exclusive to the θ-pinch, as the same problem is found for up-down symmetric tokamaks [22], where contributions to the distribution function of $O(\delta^3)$ and $O(\delta^2\nu/\Omega)$ must be retained to determine the axisymmetric radial electric field.

In our minds, the best solution to these problems is a combined kinetic and moment approach that solves a gyrokinetic equation and a group of moment equations at the same time. The potential will be given in this case not by a gyrokinetic quasineutrality equation, but by $\nabla\cdot\mathbf{J} = 0$. The calculation would be able to retain neoclassical viscosity effects and the turbulent Reynolds stress that must be allowed to compete to determine the potential. A moment approach usually has the advantage that it requires a lower order distribution function [7]. A simplified example of this approach is the solution of the θ-pinch presented in this paper, where the electrostatic potential is finally given by the conservation of azimuthal angular momentum, which in turn is an integrated form of $\nabla \cdot \mathbf{J} = 0$.

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Appendix A. First order gyrokinetic variables

In this appendix the detailed calculation of the gyrokinetic variables is carried out to first order. It is convenient to express any term that contains the electrostatic potential ϕ in gyrokinetic variables, mainly because we are not able to Taylor expand the electrostatic potential components with $k_\perp\rho \sim 1$. In order to do so, we will develop some useful relations involving the potential ϕ in the first section of this appendix. With these relations, the first order corrections, $\mathbf{R}_1$, E_1, μ_1 and φ_1, are derived.

Useful relations for ϕ. We first derive all possible gyrokinetic partial derivatives of ϕ and their relation to one another. To do so, only $\mathbf{R} = \mathbf{r} + \Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}} + O(\delta^2 L)$ is needed.

The derivative respect to the gyrocenter position is

$$\nabla_{\mathbf{R}}\phi(\mathbf{r}) = \nabla\phi + \nabla_{\mathrm{R}}(\mathbf{r}-\mathbf{R})\cdot\nabla\phi = \nabla\phi + O(\delta T/eL) \simeq \nabla\phi. \tag{A.1}$$

The derivative respect to the energy is

$$\partial\phi/\partial E = \partial(\mathbf{r}-\mathbf{R})/\partial E \cdot \nabla\phi = O(\delta^2 M/e) \simeq 0, \tag{A.2}$$

since $\mathbf{r} - \mathbf{R}$ only depends on E at $O(\delta^2 L)$.

Using $\mathbf{r} - \mathbf{R} \propto \sqrt{\mu}(\hat{\mathbf{e}}_1 \sin\varphi - \hat{\mathbf{e}}_2 \cos\varphi)$, the derivatives with respect to μ and φ can be calculated to be

$$\partial\phi/\partial\mu = \partial(\mathbf{r} - \mathbf{R})/\partial\mu \cdot \nabla\phi \simeq -(Mc/Zev_\perp^2)(\mathbf{v} \times \hat{\mathbf{n}}) \cdot \nabla\phi \tag{A.3}$$

and

$$\partial\phi/\partial\varphi = \partial(\mathbf{r} - \mathbf{R})/\partial\varphi \cdot \nabla\phi \simeq -\Omega^{-1}\mathbf{v}_\perp \cdot \nabla\phi. \tag{A.4}$$

We will need a more accurate relationship than (A.4) for the second order corrections. It will be developed in Appendix B.

Calculation of $\mathbf{R}_1$. The first order correction $\mathbf{R}_1$ is given by (12), where in this case, $Q_0 = \mathbf{R}_0 = \mathbf{r}$. The total derivative of $\mathbf{R}_0$ is $d\mathbf{R}_0/dt = \mathbf{v} = v_{||}\hat{\mathbf{n}} + \mathbf{v}_\perp$, and its gyroaverage gives $\langle d\mathbf{R}_0/dt\rangle = v_{||}\hat{\mathbf{n}} + O(\delta v_i)$. By employing $\mathbf{v}_\perp = \partial(\mathbf{v} \times \hat{\mathbf{n}})/\partial\varphi_0$, equation (12) gives (29).

Calculation of E_1. The first order correction E_1 is given by (12), where $Q_0 = E_0 = v^2/2$ and $dQ_0/dt = dE_0/dt = -(Ze/M)\mathbf{v} \cdot \nabla\phi$. It is convenient to write E_1 as a function of $\mathbf{R}$, E, μ and φ. To do so, we use (A.1) and (A.4) to find

$$-\mathbf{v} \cdot \nabla\phi = -v_{||}\hat{\mathbf{n}} \cdot \nabla\phi - \mathbf{v}_\perp \cdot \nabla\phi \simeq -v_{||}\hat{\mathbf{n}} \cdot \nabla_{\mathbf{R}}\phi + \Omega\, \partial\phi/\partial\varphi. \tag{A.5}$$

Notice that $\hat{\mathbf{n}} \cdot \nabla_{\mathbf{R}}\widetilde{\phi} \ll \hat{\mathbf{n}} \cdot \nabla_{\mathbf{R}}\overline{\phi}$ because $\widetilde{\phi}$ is smaller than $\overline{\phi}$. As a result, $dE_0/dt = -(Ze/M)v_{||}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{R}}\overline{\phi} + (Ze\Omega/M)\partial\phi/\partial\varphi$ and $\langle dE_0/dt\rangle = -(Ze/M)v_{||}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{R}}\overline{\phi}$, and equation (12) gives (17).

Calculation of μ_1. Calculating μ_1 requires more work than calculating any of the other first order corrections since we want μ to be an adiabatic invariant to all orders of interest. This requirement imposes two conditions to μ_1. One of them is similar to the requirements already imposed to $\mathbf{R}_1$ and E_1, $d\mu_0/dt - \Omega\, \partial\mu_1/\partial\varphi = \langle d\mu_0/dt\rangle \equiv 0$, but there is an additional condition making $\mu_0 + \mu_1$ an adiabatic invariant to first order,

$$\langle d(\mu_0 + \mu_1)/dt\rangle = O(\delta^2 v_i^3/BL). \tag{A.6}$$

The solution to both conditions is given by

$$\mu_1 = \frac{1}{\Omega}\int^{\varphi} d\varphi' \left(\frac{d\mu_0}{dt} - \left\langle\frac{d\mu_0}{dt}\right\rangle\right) + \langle\mu_1\rangle. \tag{A.7}$$

Notice that the only difference with the result in (12) is that the gyrophase independent term, $\langle\mu_1\rangle$, must be retained, making it possible to satisfy condition (A.6).

The total derivative for $\mu_0 = v_\perp^2/(2B)$ is

$$\begin{aligned} d\mu_0/dt = &-(Ze/MB)\mathbf{v}_\perp \cdot \nabla\phi - (v_\perp^2/2B)\mathbf{v}_\perp \cdot \nabla\ln B - (v_{||}^2/B)\hat{\mathbf{n}} \cdot \nabla\hat{\mathbf{n}} \cdot \mathbf{v}_\perp \\ &- (v_{||}/2B)[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{n}})(\mathbf{v} \times \hat{\mathbf{n}})] : \nabla\hat{\mathbf{n}}, \end{aligned} \tag{A.8}$$

where we have used the relations $\langle\mathbf{v}_\perp\mathbf{v}_\perp\rangle = (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{n}}\hat{\mathbf{n}})$ and

$$\mathbf{v}_\perp\mathbf{v}_\perp - \langle\mathbf{v}_\perp\mathbf{v}_\perp\rangle = [\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{n}})(\mathbf{v} \times \hat{\mathbf{n}})]/2. \tag{A.9}$$

Notice that the gyrophase independent terms in (A.8) cancel exactly due to $\hat{\mathbf{n}}\cdot\nabla\ln B + \nabla\cdot\hat{\mathbf{n}} = 0$, making μ_0 an adiabatic invariant to zeroth order. The term that contains ϕ in (A.8) is rewritten as a function of the gyrokinetic variables by using (A.4), to give $-(Ze/MB)\mathbf{v}_\perp\cdot\nabla\phi = (Ze\Omega/MB)\partial\phi/\partial\varphi$.

Applying (A.7), μ_1 is found to be given by (34). To get this result, we have employed $\mathbf{v}_\perp = \partial(\mathbf{v}\times\hat{\mathbf{n}})/\partial\varphi_0$ and

$$\partial[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\mathbf{v}_\perp]/\partial\varphi_0 = 2[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})]. \quad \text{(A.10)}$$

The average value $\langle\mu_1\rangle = -(v_{||}v_\perp^2/2B\Omega)(\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}})$ was chosen to insure that condition (A.6) is satisfied. In previous works [38, 20], it has been noticed that solving (A.6) may be avoided and replaced by imposing the relation $E = (d\mathbf{R}/dt\cdot\hat{\mathbf{n}}(\mathbf{R}))^2/2 + \mu B(\mathbf{R})$ on the gyrokinetic variables. This procedure works in this case, and allows us to find $\langle\mu_1\rangle$. We have checked that this value satisfies condition (A.6), but this derivation is omitted here because of its length.

Calculation of φ_1. The first order correction φ_1 is given by (12), where $Q_0 = \varphi_0$. The zeroth order gyrophase φ_0 is defined by $\mathbf{v}_\perp = v_\perp(\hat{\mathbf{e}}_1\cos\varphi_0 + \hat{\mathbf{e}}_2\sin\varphi_0)$, where $v_\perp$ is the magnitude of the perpendicular velocity and $\hat{\mathbf{e}}_1(\mathbf{r},t)$ and $\hat{\mathbf{e}}_2(\mathbf{r},t)$ are two orthonormal vector fields constructed such that $\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{n}}$. According to this definition, upon using $v_\perp^2\nabla_v\varphi_0 = -\mathbf{v}\times\hat{\mathbf{n}}$ and $v_\perp^2\nabla\varphi_0 = v_{||}\nabla\hat{\mathbf{n}}\cdot(\mathbf{v}\times\hat{\mathbf{n}}) + v_\perp^2\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1$, the total derivative of φ_0 is

$$d\varphi_0/dt = -\overline{\Omega} - (Z^2e^2/M^2c)\partial\widetilde{\phi}/\partial\mu - (\mathbf{v}\times\hat{\mathbf{n}})\cdot[\nabla\ln\Omega + (v_{||}^2/v_\perp^2)\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}} - \hat{\mathbf{n}}\times\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1] + (v_{||}/2v_\perp^2)[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{n}}, \quad \text{(A.11)}$$

where we have used (A.9), and the potential $\phi(\mathbf{r},t)$ and the gyrofrequency $\Omega(\mathbf{r})$ have been written as functions of the gyrokinetic variables by using (A.3) and $\Omega(\mathbf{r}) \simeq \Omega(\mathbf{R}) + (\mathbf{r}-\mathbf{R})\cdot\nabla\Omega$, respectively. The function $\overline{\Omega}$ is given by

$$\overline{\Omega} = \Omega(\mathbf{R}) + (v_{||}/2)\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}} - v_{||}\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 + (Z^2e^2/M^2c)\partial\overline{\phi}/\partial\mu. \quad \text{(A.12)}$$

Upon gyroaveraging (A.11) we obtain $\langle d\varphi_0/dt\rangle = -\overline{\Omega} + O(\delta^2\Omega)$. Finally, φ_1 is obtained from (12) by employing $\mathbf{v}\times\hat{\mathbf{n}} = -\partial\mathbf{v}_\perp/\partial\varphi_0$ and (A.10), giving

$$\varphi_1 = -(Ze/MB)\partial\widetilde{\Phi}/\partial\mu - \Omega^{-1}\mathbf{v}_\perp\cdot[\nabla\ln B + (v_{||}^2/v_\perp^2)\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}} - \hat{\mathbf{n}}\times\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1] - (v_{||}/4\Omega v_\perp^2)[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})] : \nabla\hat{\mathbf{n}}, \quad \text{(A.13)}$$

where $\widetilde{\Phi}$ is the function defined in (21). As a result, the total derivative of φ is $d\varphi/dt = \langle d\varphi_0/dt\rangle + O(\delta^2\Omega) = -\overline{\Omega} + O(\delta^2\Omega)$.

Appendix B. Second order gyrokinetic variables

To construct the gyrokinetic variables to second order, higher order relations than the ones developed in Appendix A are needed to express ϕ as a function of the gyrokinetic variables. These extended relations are deduced in the first part of this appendix. Using them, the second order corrections $\mathbf{R}_2$ and E_2 are calculated. The magnetic moment and the gyrophase are not required to higher order.

More useful relations for ϕ. To calculate the second order corrections, $\mathbf{v}\cdot\nabla\phi$ must be expressed in gyrokinetic variables to order $O(\delta T v_i/eL)$. The total time derivative for ϕ in $\mathbf{r}$, $\mathbf{v}$ variables is

$$d\phi/dt = \partial\phi/\partial t|_{\mathbf{r}} + \mathbf{v}\cdot\nabla\phi, \tag{B.1}$$

while as a function of the new gyrokinetic variables it becomes

$$d\phi/dt = \partial\phi/\partial t|_{\mathbf{R},\,E,\,\mu,\,\varphi} + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\phi + \dot{E}\,\partial\phi/\partial E + \dot{\varphi}\,\partial\phi/\partial\varphi. \tag{B.2}$$

Combining these equations gives an equation for $\mathbf{v}\cdot\nabla\phi$,

$$-\mathbf{v}\cdot\nabla\phi = (\partial\phi/\partial t|_{\mathbf{r}} - \partial\phi/\partial t|_{\mathbf{R},\,E,\,\mu,\,\varphi}) - \dot{E}\,\partial\phi/\partial E - \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\phi - \dot{\varphi}\,\partial\phi/\partial\varphi, \tag{B.3}$$

where the left side of the equation is of order $O(Tv_i/eL)$. We analyze the right side term by term. Noticing that $\phi(\mathbf{r},t) = \phi(\mathbf{R}+(\mathbf{r}-\mathbf{R}),t)$, the partial derivatives with respect to time give the negligible contribution $(\partial\phi/\partial t|_{\mathbf{r}} - \partial\phi/\partial t|_{\mathbf{R},\,E,\,\mu,\,\varphi}) = -\partial(\mathbf{r}-\mathbf{R})/\partial t\cdot\nabla\phi = O(\delta^2 Tv_i/eL)$, since the time derivative of $\mathbf{r}-\mathbf{R}$ can only be of order $O(\omega\delta^2 L)$ for a static magnetic field. The partial derivative with respect to E is estimated in (A.2). Applying it to our equation, we find that it is also negligible, $\dot{E}\,\partial\phi/\partial E = O(\delta^2 Tv_i/eL)$. The total derivative $\dot{\mathbf{R}}$ has two different components, which we will calculate in detail in the next paragraph. These components are the parallel velocity of the gyrocenter, $\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R})$, of order v_i, and the drift velocity, $\mathbf{v}_d$, of order δv_i. Using this information, we find $\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R})\cdot\nabla_{\mathbf{R}}\phi = O(Tv_i/eL)$ and $\mathbf{v}_d\cdot\nabla_{\mathbf{R}}\phi = O(\delta Tv_i/eL)$. Finally, the last term in the right side of (B.3) is $\dot{\varphi}\,\partial\phi/\partial\varphi = O(Tv_i/eL)$, since $\dot{\varphi}\sim\Omega$ and $\partial\phi/\partial\varphi = \partial\widetilde{\phi}/\partial\varphi \sim \delta T/e$ according to (20). Neglecting all the terms smaller than $O(\delta)$ compared to $\mathbf{v}\cdot\nabla\phi$, equation (B.3) becomes

$$-\mathbf{v}\cdot\nabla\phi = -\overline{v}_{||}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{R}}\phi - \mathbf{v}_d\cdot\nabla_{\mathbf{R}}\phi + \overline{\Omega}\,\partial\widetilde{\phi}/\partial\varphi. \tag{B.4}$$

Calculation of $\mathbf{R}_2$. The second order correction $\mathbf{R}_2$ is given by (13), where $Q_0 = \mathbf{R}_0 = \mathbf{r}$ and $Q_1 = \mathbf{R}_1 = \Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}$. The total time derivative of $\mathbf{R}_0 + \mathbf{R}_1$ is

$$d(\mathbf{R}_0 + \mathbf{R}_1)/dt = v_{||}\hat{\mathbf{n}} - \mathbf{v}\cdot\nabla(\hat{\mathbf{n}}/\Omega)\times\mathbf{v} - (c/B)\nabla\phi\times\hat{\mathbf{n}}, \tag{B.5}$$

and its gyroaverage may be written as $\langle d(\mathbf{R}_0 + \mathbf{R}_1)/dt\rangle = \overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d$, where $\overline{v}_{||} = \langle v_{||}\rangle + (v_\perp^2/2\Omega)(\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}})$, and $\mathbf{v}_d$ has been already defined in (23). The function $\overline{v}_{||}$ can be written as a function of the gyrokinetic variables. We express $v_{||}$ as a function of the lowest order gyrokinetic variables, expand about these lowest order gyrokinetic variables, and insert $\mathbf{R}_1$, μ_1 and E_1 to obtain

$$\begin{aligned} v_{||} = \sqrt{2(E_0 - \mu_0 B(\mathbf{r}))} &\simeq \sqrt{2(E - \mu B(\mathbf{R}))} - (v_\perp^2/2\Omega)\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}} \\ &- (v_{||}/\Omega)(\mathbf{v}\times\hat{\mathbf{n}})\cdot(\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}) - (4\Omega)^{-1}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{n}}. \end{aligned} \tag{B.6}$$

Finally, gyroaveraging and using $\langle\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\mathbf{v}_\perp\rangle = 0$ [a result that is deduced from (A.9)] give $\overline{v}_{||} = \sqrt{2(E - \mu B(\mathbf{R}))}$, which can be rewritten as (25).

Using (B.5) and (B.6), Taylor expanding $\hat{\mathbf{n}}(\mathbf{r})$ about $\mathbf{R}$ and inserting the result into (13) gives (30) and (31). To integrate, $\mathbf{v}\times\hat{\mathbf{n}} = -\partial\mathbf{v}_\perp/\partial\varphi_0$ and (A.10) have been used.

Calculation of E_2. Equation (13) gives E_2, where $Q_0 = E_0 = v^2/2$ and $Q_1 = E_1 = Ze\widetilde{\phi}/M$. The total derivative of $E_0 = v^2/2$ can be expressed as a function of the new gyrokinetic variables to the requisite order by using (B.4) to obtain

$$dE_0/dt = -(Ze/M)\mathbf{v}\cdot\nabla\phi \simeq (Ze/M)[\overline{\Omega}\,\partial\widetilde{\phi}/\partial\varphi - (\overline{v}_{||}\hat{\mathbf{n}} + \mathbf{v}_d)\cdot\nabla_{\mathbf{R}}\phi]. \quad \text{(B.7)}$$

From the definition of $E_1 = Ze\widetilde{\phi}/M$, use of gyrokinetic variables yields

$$dE_1/dt = (Ze/M)[\partial\widetilde{\phi}/\partial t + (\overline{v}_{||}\hat{\mathbf{n}} + \mathbf{v}_d)\cdot\nabla_{\mathbf{R}}\widetilde{\phi} - \overline{\Omega}\,\partial\widetilde{\phi}/\partial\varphi]. \quad \text{(B.8)}$$

Adding both contributions together leaves

$$d(E_0 + E_1)/dt = -(Ze/M)(\overline{v}_{||}\hat{\mathbf{n}} + \mathbf{v}_d)\cdot\nabla_{\mathbf{R}}\overline{\phi} + (Ze/M)\partial\widetilde{\phi}/\partial t. \quad \text{(B.9)}$$

As a result, E_2 is as shown in (18), and to this order, dE/dt is given by (22).

Appendix C. Jacobian of the gyrokinetic transformation

The inverse of the Jacobian is

$$\frac{1}{J} = \left|\begin{array}{ccc:c:c:c} \ddots & & & \vdots & \vdots & \vdots \\ & \nabla\mathbf{R} & & \nabla E & \nabla\mu & \nabla\varphi \\ & & \ddots & \vdots & \vdots & \vdots \\ \hdashline \ddots & & & \vdots & \vdots & \vdots \\ & \nabla_v\mathbf{R} & & \nabla_v E & \nabla_v\mu & \nabla_v\varphi \\ & & \ddots & \vdots & \vdots & \vdots \end{array}\right| = \left|\begin{array}{ccc:c:c:c} \ddots & & & \vdots & \vdots & \vdots \\ & \nabla\mathbf{R} & & \nabla E & \nabla\mu & \nabla\varphi \\ & & \ddots & \vdots & \vdots & \vdots \\ \hdashline \ddots & & & \vdots & \vdots & \vdots \\ & \mathbf{0} & & \partial E & \partial\mu & \partial\varphi \\ & & \ddots & \vdots & \vdots & \vdots \end{array}\right|. \quad \text{(C.1)}$$

Employing that the terms in the left columns of the first form are to first approximation $\nabla\mathbf{R} \simeq \overleftrightarrow{\mathbf{I}}$ and $\nabla_v\mathbf{R} \simeq \Omega^{-1}\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{n}}$, the determinant is simplified by combining linearly the rows in the matrix to determine the second form, where

$$\partial(\ldots) = \nabla_v(\ldots) - \Omega^{-1}\hat{\mathbf{n}}\times\nabla(\ldots). \quad \text{(C.2)}$$

The second form of (C.1) can be simplified by noticing that the lower left piece of the matrix is zero. Thus, the determinant may be written as

$$J^{-1} = \det(\nabla\mathbf{R})[\partial E\cdot(\partial\mu\times\partial\varphi)]. \quad \text{(C.3)}$$

We analyze the two determinants on the right side independently. The matrix $\nabla\mathbf{R}$ is $\overleftrightarrow{\mathbf{I}} + \nabla(\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}} + \mathbf{R}_2)$. Hence, $\det(\nabla\mathbf{R}) = 1 + \nabla\cdot(\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}} + \mathbf{R}_2)$. The Jacobian must be obtained to first order only. The only important term to that order in $\mathbf{R}_2$ is the term that contains the potential ϕ, since its gradient may be large, but $\nabla\cdot\mathbf{R}_2 \simeq \nabla\cdot[(c/B\Omega)\nabla_{\mathbf{R}}\widetilde{\Phi}\times\hat{\mathbf{n}}] \simeq 0$. Therefore, the determinant of $\nabla\mathbf{R}$ becomes

$$\det(\nabla\mathbf{R}) = 1 - \mathbf{v}\cdot\nabla\times(\hat{\mathbf{n}}/\Omega). \quad \text{(C.4)}$$

For the second determinant in (C.3), we evaluate the columns of the matrix ∂E, $\partial\mu$ and $\partial\varphi$ to the order of interest, using

$$\partial E = \mathbf{v} + \nabla_v E_1 - \Omega^{-1}\hat{\mathbf{n}}\times\nabla E, \quad \text{(C.5)}$$

$$\partial\mu = \mathbf{v}_\perp/B + \nabla_v\mu_1 - \Omega^{-1}\hat{\mathbf{n}}\times\nabla\mu \quad \text{(C.6)}$$

and

$$\partial\varphi = -v_\perp^{-2}\mathbf{v}\times\hat{\mathbf{n}} + \nabla_v\varphi_1 - \Omega^{-1}\hat{\mathbf{n}}\times\nabla\varphi. \tag{C.7}$$

The determinant becomes

$$\begin{aligned}\partial E\cdot(\partial\mu\times\partial\varphi) \simeq\ & v_{||}/B(\mathbf{r}) + \hat{\mathbf{n}}\cdot\nabla_v E_1/B + (\nabla_v\mu_1 - \Omega^{-1}\hat{\mathbf{n}}\times\nabla\mu)\cdot[(v_{||}/v_\perp^2)\mathbf{v}_\perp - \hat{\mathbf{n}}]\\ & -(v_{||}/B)(\nabla_v\varphi_1 - \Omega^{-1}\hat{\mathbf{n}}\times\nabla\varphi)\cdot(\mathbf{v}\times\hat{\mathbf{n}}).\end{aligned} \tag{C.8}$$

From the definitions of E_1, μ_1 and φ_1, we find their gradients in velocity space. We need the gradients in velocity space of $\widetilde{\phi}$ and $\partial\widetilde{\Phi}/\partial\mu$. The gradient $\nabla_v\widetilde{\phi}$ is given by

$$\begin{aligned}\nabla_v\widetilde{\phi} &= \nabla_v E\,\partial\widetilde{\phi}/\partial E + \nabla_v\mu\,\partial\widetilde{\phi}/\partial\mu + \nabla_v\varphi\,\partial\widetilde{\phi}/\partial\varphi + \nabla_v\mathbf{R}\cdot\nabla_{\mathbf{R}}\widetilde{\phi}\\ &= B^{-1}\mathbf{v}_\perp\,\partial\widetilde{\phi}/\partial\mu - v_\perp^{-2}\mathbf{v}\times\hat{\mathbf{n}}\,\partial\widetilde{\phi}/\partial\varphi + \Omega^{-1}\hat{\mathbf{n}}\times\nabla_{\mathbf{R}}\widetilde{\phi}.\end{aligned} \tag{C.9}$$

The gradient $\nabla_v(\partial\widetilde{\Phi}/\partial\mu)$ is found in a similar way. The gradients in real space are only to be obtained to zeroth order. However, some terms of the first order quantities that contain ϕ are important because they have steep gradients. Considering this, we find

$$\nabla E = (Ze/M)\nabla\widetilde{\phi}, \tag{C.10}$$

$$\nabla\mu = -(v_\perp^2/2B)\nabla\ln B - (v_{||}/B)\nabla\hat{\mathbf{n}}\cdot\mathbf{v}_\perp + (Ze/MB)\nabla\widetilde{\phi} \tag{C.11}$$

and

$$\nabla\varphi = (v_{||}/v_\perp^2)\nabla\hat{\mathbf{n}}\cdot(\mathbf{v}\times\hat{\mathbf{n}}) + \nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 - (Ze/MB)\nabla(\partial\widetilde{\Phi}/\partial\mu). \tag{C.12}$$

Due to the preceding considerations, equation (C.8) becomes

$$\partial E\cdot(\partial\mu\times\partial\varphi) = [\overline{v}_{||}/B(\mathbf{r})](1 + \Omega^{-1}\mathbf{v}_\perp\cdot\nabla\times\hat{\mathbf{n}}), \tag{C.13}$$

where we have employed (B.6) to express $v_{||}$ as a function of the gyrokinetic variables and $(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{v}_\perp - \mathbf{v}_\perp\cdot\nabla\hat{\mathbf{n}}\cdot(\mathbf{v}\times\hat{\mathbf{n}}) = v_\perp^2\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}$, a result that is deduced from

$$v_\perp^2(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{n}}\hat{\mathbf{n}}) = \mathbf{v}_\perp\mathbf{v}_\perp + (\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}}). \tag{C.14}$$

Combining (C.4) and (C.13), the Jacobian of the transformation is found to be as given by (44). Notice that to this order $J = \langle J\rangle$ as required by (40).

Finally, we prove that J satisfies the gyroaverage of (40) to the required order, namely,

$$\partial J/\partial t + \nabla_{\mathbf{R}}\cdot\{J[\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d]\} - (Ze/M)\partial\{J[\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d]\cdot\nabla_{\mathbf{R}}\overline{\phi}\}/\partial E = 0, \tag{C.15}$$

where $\mathbf{v}_d$ is given by (23). To first order, we obtain

$$J[\overline{v}_{||}\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d] \simeq \mathbf{B}(\mathbf{R}) + (B\mu/\overline{v}_{||}\Omega)\hat{\mathbf{n}}\times\nabla_{\mathbf{R}}B + (B\overline{v}_{||}/\Omega)\nabla_{\mathbf{R}}\times\hat{\mathbf{n}} - (c/\overline{v}_{||})\nabla_{\mathbf{R}}\overline{\phi}\times\hat{\mathbf{n}}, \tag{C.16}$$

where we have employed $\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{n}}) = (\nabla_{\mathbf{R}}\times\hat{\mathbf{n}})_\perp$. Inserting (C.16) into (C.15) and remembering that $\overline{v}_{||} = \sqrt{2(E - \mu B(\mathbf{R}))}$ is enough to prove that (C.15), and thus (40) gyroaveraged, are satisfied by the Jacobian to first order.

Appendix D. Calculation of gyroviscosity at long wavelengths

Here we show how to obtain the gyroviscosity at long wavelengths ($k_\perp \rho \ll 1$ and $k_\perp L \sim 1$). To simplify the calculation, the distribution function of ions will be assumed to be Maxwellian to lowest order, i.e., $f_i(\mathbf{R}, E, \mu, t) \simeq \overline{f}_M(\mathbf{R}, E, t)$, where $\overline{f}_M$ is given by (68). To recover the gyroviscosity, the ion distribution function must be written in $\mathbf{r}$, $\mathbf{v}$ variables. Taylor expanding f_i to $O(\delta^2 \overline{f}_M)$ gives

$$f_i(\mathbf{R}, E, \mu, t) \simeq f_i(\mathbf{r}, E_0, \mu_0, t) + (\mathbf{R}_1 + \mathbf{R}_2)\cdot\nabla f_i + (E_1 + E_2)\partial f_i/\partial E_0 + \mu_1 \partial f_i/\partial \mu_0 \\ + (1/2)\mathbf{R}_1\mathbf{R}_1 : \nabla\nabla\overline{f}_M + (1/2)E_1^2 \partial^2 \overline{f}_M/\partial E_0^2 + E_1 \mathbf{R}_1 \cdot \nabla(\partial \overline{f}_M/\partial E_0). \quad \text{(D.1)}$$

Here, $f_i \simeq \overline{f}_M$ is used in the higher order terms and the contribution of the collisional piece $\widetilde{f}$, given by (38), is intentionally ignored because its only effect is a small classical transport contribution [8].

The gyroviscosity depends only on the gyrophase dependent part of f_i and according to equation (27) of [22] may be written as

$$\overleftrightarrow{\boldsymbol{\pi}}_g = \int d^3v\, M\mathbf{v}\mathbf{v}\,(f_i - \langle f_i \rangle)\,, \quad \text{(D.2)}$$

where here the gyroaverage $\langle f_i \rangle$ is now performed holding $\mathbf{r}$, $v_{||}$ and $v_\perp$ fixed. Employing (D.1), the gyrophase dependent part of f_i is found to be

$$f_i - \langle f_i \rangle = (\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}} + \mathbf{R}_2)\cdot\nabla f_i + [(Ze/M)(\widetilde{\phi} - \langle\widetilde{\phi}\rangle) + E_2](\partial f_i/\partial E_0) \\ + (4\Omega^2)^{-1}[(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}}) - \mathbf{v}_\perp\mathbf{v}_\perp] : \nabla\nabla\overline{f}_M + (Z^2e^2/2M^2)(\widetilde{\phi}^2 - \langle\widetilde{\phi}^2\rangle)\partial^2\overline{f}_M/\partial E_0^2 \\ + (c/B)[\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{n}}) - \langle\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{n}})\rangle]\cdot\nabla(\partial\overline{f}_M/\partial E_0) + (\mu_1 - \langle\mu_1\rangle)(\partial f_i/\partial\mu_0), \quad \text{(D.3)}$$

where we have used (A.9) to rewrite $(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}}) - \langle(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})\rangle$.

The function $\widetilde{\phi}$ must be written as a function of the $\mathbf{r}$, $\mathbf{v}$ variables. To do so, first we will write $\phi(\mathbf{r}, t)$ as a function of the gyrokinetic variables by Taylor expansion to $O(\delta^2 T/e)$ to find

$$\phi(\mathbf{r}, t) \simeq \phi(\mathbf{R}, t) - \mathbf{R}_1\cdot\nabla_\mathbf{R}\phi + (1/2)\mathbf{R}_1\mathbf{R}_1 : \nabla_\mathbf{R}\nabla_\mathbf{R}\phi - \mathbf{R}_2\cdot\nabla_\mathbf{R}\phi. \quad \text{(D.4)}$$

The second term in the right side of the equation needs to be re-expanded in order to express ϕ as a self-consistent function of the gyrokinetic variables to the right order. The function $\mathbf{R}_1$ is, to $O(\delta\rho)$,

$$\mathbf{R}_1 = \Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}} \simeq [\sqrt{2\mu B(\mathbf{R})}/\Omega(\mathbf{R})][\hat{\mathbf{e}}_1(\mathbf{R})\sin\varphi - \hat{\mathbf{e}}_2(\mathbf{R})\cos\varphi] - \Delta\boldsymbol{\rho}, \quad \text{(D.5)}$$

where the function $\Delta\boldsymbol{\rho}(\mathbf{R}, E, \mu, \varphi, t)$ is $O(\delta\rho)$, but its exact form is not needed here. Combining (D.4) and (D.5), $\widetilde{\phi}$ is found to be

$$\widetilde{\phi} = \phi - \langle\phi\rangle \simeq -[\sqrt{2\mu B(\mathbf{R})}/\Omega(\mathbf{R})][\hat{\mathbf{e}}_1(\mathbf{R})\sin\varphi - \hat{\mathbf{e}}_2(\mathbf{R})\cos\varphi]\cdot\nabla_\mathbf{R}\phi - \mathbf{R}_2\cdot\nabla_\mathbf{R}\phi \\ + (\Delta\boldsymbol{\rho} - \langle\Delta\boldsymbol{\rho}\rangle)\cdot\nabla_\mathbf{R}\phi + (4\Omega^2)^{-1}[(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}}) - \mathbf{v}_\perp\mathbf{v}_\perp] : \nabla_\mathbf{R}\nabla_\mathbf{R}\phi. \quad \text{(D.6)}$$

We require $\widetilde{\phi}$ as a function of $\mathbf{r}$, $\mathbf{v}$ and t. Taylor expanding the first term in (D.6) gives

$$\widetilde{\phi} \simeq -\Omega^{-1}(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\phi - \langle\Delta\boldsymbol{\rho}\rangle\cdot\nabla\phi - \mathbf{R}_2\cdot\nabla\phi - (v_\perp^2/4\Omega^2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{n}}\hat{\mathbf{n}}) : \nabla\nabla\phi \\ -(2\Omega^2)^{-1}(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}}) : \nabla\nabla\phi. \quad \text{(D.7)}$$

This result allows us to rewrite some of the terms in (D.3) in a more convenient way. For example, we obtain

$$\widetilde{\phi} - \langle\widetilde{\phi}\rangle = -\Omega^{-1}(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\phi - \mathbf{R}_2\cdot\nabla\phi + [\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})] : (\nabla\nabla\phi)/4\Omega^2. \quad (D.8)$$

For the higher order terms, we can simply use the lowest order result $\widetilde{\phi} \simeq -\Omega^{-1}(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\phi$, which leads to

$$\widetilde{\phi}^2 - \langle\widetilde{\phi}^2\rangle = -(2\Omega^2)^{-1}[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})] : (\nabla\phi\nabla\phi) \quad (D.9)$$

and

$$\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{n}}) - \langle\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{n}})\rangle = (2\Omega)^{-1}\nabla\phi\cdot[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})]. \quad (D.10)$$

Using these expressions, the gyrophase dependent part of the distribution function becomes

$$f_i - \langle f_i\rangle = \mathbf{v}\cdot\mathbf{g}_\perp + (\mu_1 - \langle\mu_1\rangle)\partial f_i/\partial\mu_0 + \mathbf{R}_2\cdot\mathbf{G} + E_2\,\partial f_i/\partial E_0$$
$$+ (4\Omega^2)^{-1}[(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}}) - \mathbf{v}_\perp\mathbf{v}_\perp] : [\nabla\mathbf{G} - (Ze/M)\nabla\phi\,\partial\mathbf{G}/\partial E_0], \quad (D.11)$$

where $\mathbf{g}_\perp$ is given in (81) and

$$\mathbf{G} = \nabla f_i - (Ze/M)\nabla\phi\,\partial f_i/\partial E_0. \quad (D.12)$$

Thus, $\mathbf{g}_\perp = \Omega^{-1}\hat{\mathbf{n}}\times\mathbf{G}$. In the long wavelength limit, $\partial/\partial t \ll v_i/L$, so E_2 as given in (18) is negligible since it contains a time derivative. Also, the zeroth order Fokker-Planck equation for the ion distribution function is

$$v_{||}\hat{\mathbf{n}}\cdot\mathbf{G} \equiv v_{||}\hat{\mathbf{n}}\cdot[\nabla f_i - (Ze/M)\nabla\phi\,\partial f_i/\partial E_0] = C\{f_i\} = 0, \quad (D.13)$$

since the ion distribution function is assumed to be Maxwellian to zeroth order. This condition is important in (D.11) because it implies that the components of $\mathbf{R}_2$ that are parallel to the magnetic field do not enter $f_i - \langle f_i\rangle$. Therefore, employing the definition of $\mathbf{R}_2$ in (30) and using the fact that for long wavelenghts $[c/(B\Omega)]\nabla_{\mathbf{R}}\widetilde{\Phi}\times\hat{\mathbf{n}}$ is negligible, we obtain

$$\mathbf{R}_2\cdot\mathbf{G} = \Omega^{-1}[(v_{||}\hat{\mathbf{n}} + \mathbf{v}_\perp/4)\mathbf{v}\times\hat{\mathbf{n}} + \mathbf{v}\times\hat{\mathbf{n}}(v_{||}\hat{\mathbf{n}} + \mathbf{v}_\perp/4)] : [\nabla(\hat{\mathbf{n}}/\Omega)\times\mathbf{G}]$$
$$+ (v_{||}/\Omega^2)\mathbf{v}_\perp\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{G}, \quad (D.14)$$

Equation (D.14) can be written in a more recognizable manner by using

$$[\hat{\mathbf{n}}(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\hat{\mathbf{n}}] : \overleftrightarrow{\mathbf{h}} = -\Omega^{-1}\hat{\mathbf{n}}\cdot[\nabla\mathbf{G} - (Ze/M)\nabla\phi\,\partial\mathbf{G}/\partial E_0]\cdot\mathbf{v}_\perp$$
$$+ [\hat{\mathbf{n}}(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\hat{\mathbf{n}}] : [\nabla(\hat{\mathbf{n}}/\Omega)\times\mathbf{G}], \quad (D.15)$$

where $\overleftrightarrow{\mathbf{h}}$ is given in (82). The first term in the right side of (D.15) can be further simplified by using (D.13) to obtain

$$\hat{\mathbf{n}}\cdot[\nabla\mathbf{G} - (Ze/M)\nabla\phi\,\partial\mathbf{G}/\partial E_0]\cdot\mathbf{v}_\perp = \mathbf{v}_\perp\cdot\nabla\mathbf{G}\cdot\hat{\mathbf{n}} = -\mathbf{v}_\perp\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{G}. \quad (D.16)$$

As a result, (D.14) becomes

$$\mathbf{R}_2\cdot\mathbf{G} = (v_{||}/\Omega)[\hat{\mathbf{n}}(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\hat{\mathbf{n}}] : \overleftrightarrow{\mathbf{h}}$$
$$+ (4\Omega)^{-1}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{n}}) + (\mathbf{v}\times\hat{\mathbf{n}})\mathbf{v}_\perp] : [\nabla(\hat{\mathbf{n}}/\Omega)\times\mathbf{G}]. \quad (D.17)$$

The gyrophase dependent part of the ion distribution function can now be explicitly written as in (80). This is exactly the same gyrophase-dependent distribution function found in [22]. Therefore, the same gyroviscosity as found there will be obtained.

Appendix E. Quasineutrality equation at long wavelengths

In this Appendix we obtain the gyrokinetic quasineutrality equation at long wavelengths ($k_\perp \rho \ll 1$ and $k_\perp L \sim 1$). To simplify the calculation, the ion distribution function is assumed to be Maxwellian to lowest order. In quasineutrality, the ion distribution function is again written in $\mathbf{r}$, $\mathbf{v}$ variables, as has already been done in Appendix D, in (D.1). The ion density is

$$n_i(\mathbf{r}, t) = \int d^3v\, f_i(\mathbf{R}, E, \mu, t). \tag{E.1}$$

This density can be calculated to $O(\delta^2 n_i)$ by using (D.1). Some of the terms are zero because the integral over gyrophase, φ_0, is zero; for example, $\int d^3v\, (\mathbf{R}_1 + \mathbf{R}_2) \cdot \nabla f_i = 0 = \int d^3v\, E_2\, \partial f_i/\partial E_0$. The ion density becomes

$$n_i \simeq \hat{N}_i + \int d^3v\, \frac{Ze\widetilde{\phi}}{M} \left[\frac{\partial f_i}{\partial E_0} + \frac{1}{B}\frac{\partial f_i}{\partial \mu_0} + \frac{Ze\widetilde{\phi}}{2M}\frac{\partial^2 \overline{f}_M}{\partial E_0^2} + \mathbf{R}_1 \cdot \nabla \left(\frac{\partial \overline{f}_M}{\partial E_0} \right) \right], \tag{E.2}$$

where $\hat{N}_i(\mathbf{r}, t)$ is the ion gyrocenter density, defined as the portion of the ion density independent of $\widetilde{\phi}$ and given by

$$\hat{N}_i = \int d^3v\, f_i(\mathbf{r}, E_0, \mu_0, t) - \int d^3v\, \frac{v_{||} v_\perp^2}{2B\Omega} \hat{\mathbf{n}} \cdot \nabla \times \hat{\mathbf{n}} \frac{\partial f_i}{\partial \mu_0} + \int d^3v\, \frac{1}{2} \mathbf{R}_1 \mathbf{R}_1 : \nabla\nabla \overline{f}_M. \tag{E.3}$$

The formula for $\hat{N}_i$ can be simplified. The second term in the right side of the equation is proportional to $\int d^3v\, (v_{||} v_\perp^2/2B) \partial f_i/\partial \mu_0$. This integral is simplified by changing to the variables $E_0 = v^2/2$, $\mu_0 = v_\perp^2/2B$ and φ_0 and integrating by parts,

$$-\int d^3v\, \frac{v_{||} v_\perp^2}{2B} \frac{\partial f_i}{\partial \mu_0} = -B \sum_\sigma \int dE_0\, d\mu_0\, d\varphi_0\, \sigma \mu_0 \frac{\partial f_i}{\partial \mu_0} = \int d^3v\, v_{||} f_i, \tag{E.4}$$

where $\sigma = v_{||}/|v_{||}|$ is the sign of the parallel velocity, the summation in front of the integral indicates that the integral must be done for both signs of $v_{||}$, and we have used the equality $d^3v = dE_0\, d\mu_0\, d\varphi_0\, B/|v_{||}|$. The third term in the right side of (E.3) is proportional to

$$M \int d^3v\, (\mathbf{v} \times \hat{\mathbf{n}})(\mathbf{v} \times \hat{\mathbf{n}}) : \nabla\nabla \overline{f}_M = (\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{n}}\hat{\mathbf{n}}) : \nabla\nabla p_i. \tag{E.5}$$

The final function $\hat{N}_i$ reduces to the result shown in (56).

In (E.2), $\widetilde{\phi}$ appears in several integrals. We perform these integrals by integrating first in the gyroangle to simplify the expressions. For the integral $\int d^3v\, \widetilde{\phi} (\partial f_i/\partial E_0 + B^{-1} \partial f_i/\partial \mu_0)$, only the gyrophase integral $\int_0^{2\pi} \widetilde{\phi}\, d\varphi_0$ is needed, but $\widetilde{\phi}$ must be known as a function of the variables $\mathbf{r}$, $\mathbf{v}$ to $O(\delta^2 T/e)$ to be consistent with the order of the Taylor expansion. This has already been done in Appendix D and the result is given in (D.7), leading to

$$\frac{1}{2\pi} \int_0^{2\pi} \widetilde{\phi}\, d\varphi_0 \simeq -\langle \Delta \boldsymbol{\rho} \rangle \cdot \nabla \phi - \frac{v_\perp^2}{2\Omega^2} (\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{n}}\hat{\mathbf{n}}) : \nabla\nabla\phi. \tag{E.6}$$

with $\Delta\boldsymbol{\rho}$ the difference shown in (D.5). The function $\Delta\boldsymbol{\rho}$ is found by Taylor expanding to $O(\delta^2 L)$, giving

$$\Delta\boldsymbol{\rho} = -(2\Omega^2)^{-1}(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\ln B + (Mc/Zev_\perp^2)\mu_1(\mathbf{v}\times\hat{\mathbf{n}}) + \Omega^{-1}\varphi_1\mathbf{v}_\perp + (v_\perp/\Omega^2)(\mathbf{v}\times\hat{\mathbf{n}})\cdot(\sin\varphi_0\nabla\hat{\mathbf{e}}_1 - \cos\varphi_0\nabla\hat{\mathbf{e}}_2). \tag{E.7}$$

To simplify this equation, the gradients of the unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are expressed in the alternate forms $\nabla\hat{\mathbf{e}}_1 = -(\nabla\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{n}} - (\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_2$ and $\nabla\hat{\mathbf{e}}_2 = -(\nabla\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\hat{\mathbf{n}} + (\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$, giving

$$\Delta\boldsymbol{\rho} = -(2\Omega^2)^{-1}(\mathbf{v}\times\hat{\mathbf{n}})(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\ln B + (Mc/Zev_\perp^2)\mu_1(\mathbf{v}\times\hat{\mathbf{n}}) + \Omega^{-1}\varphi_1\mathbf{v}_\perp - \Omega^{-2}(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\hat{\mathbf{n}}\cdot(\mathbf{v}\times\hat{\mathbf{n}})\hat{\mathbf{n}} - \Omega^{-2}\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1, \tag{E.8}$$

and its gyroaverage

$$\langle\Delta\boldsymbol{\rho}\rangle = -(v_\perp^2/\Omega^2)\nabla_\perp\ln B - (v_{||}^2/\Omega^2)\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}} - (c/B\Omega)\nabla_\perp\phi - (v_\perp^2/2\Omega^2)(\nabla\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}, \tag{E.9}$$

where we have used

$$\langle\mu_1(\mathbf{v}\times\hat{\mathbf{n}})\rangle = -(cv_\perp^2/2B^2)\nabla_\perp\phi - (v_\perp^4/4B\Omega)\nabla_\perp\ln B - (v_\perp^2 v_{||}^2/2B\Omega)\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}} \tag{E.10}$$

and

$$\langle\varphi_1\mathbf{v}_\perp\rangle = -(c/2B)\nabla_\perp\phi - (v_\perp^2/2\Omega)\nabla_\perp\ln B - (v_{||}^2/2\Omega)\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}} + (v_\perp^2/2\Omega)\hat{\mathbf{n}}\times\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1. \tag{E.11}$$

These expressions are found by using the definitions of μ_1 and φ_1, given by (34) and (A.13), and employing the lowest order expressions $\widetilde{\phi} \simeq -\Omega^{-1}(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\phi$, $\widetilde{\Phi} = \int^\varphi \widetilde{\phi}\, d\varphi' \simeq \Omega^{-1}\mathbf{v}_\perp\cdot\nabla\phi$ and $\partial\widetilde{\Phi}/\partial\mu \simeq (Mc/Zev_\perp^2)\mathbf{v}_\perp\cdot\nabla\phi$.

Substituting (E.9) in (E.6) gives

$$\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\phi}\, d\varphi_0 = -\frac{v_\perp^2}{2}\nabla\cdot\left(\frac{1}{\Omega^2}\nabla_\perp\phi\right) + \left(v_{||}^2 - \frac{v_\perp^2}{2}\right)\frac{1}{\Omega^2}\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\nabla\phi + \frac{c}{B\Omega}|\nabla_\perp\phi|^2, \tag{E.12}$$

where we have used

$$\nabla\cdot(\Omega^{-2}\nabla_\perp\phi) = \Omega^{-2}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{n}}\hat{\mathbf{n}}) : \nabla\nabla\phi - 2\Omega^{-2}\nabla\ln B\cdot\nabla_\perp\phi - \Omega^{-2}\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\nabla\phi - \Omega^{-2}(\hat{\mathbf{n}}\cdot\nabla\phi)\nabla\cdot\hat{\mathbf{n}}. \tag{E.13}$$

Note that the gyroaverage of $\widetilde{\phi}$ is $O(\delta^2 T/e)$, which means that the integral is $O(\delta^2 n_i)$, and the lowest order distribution function, $\overline{f}_M$, can be used to write $\partial f_i/\partial E_0 \simeq -(M/T_i)\overline{f}_M$ and $\partial f_i/\partial\mu_0 \simeq 0$. All these simplifications lead to the final result

$$\frac{Ze}{M}\int d^3v\,\widetilde{\phi}\left(\frac{\partial f_i}{\partial E_0} + \frac{1}{B}\frac{\partial f_i}{\partial\mu_0}\right) = n_i\nabla\cdot\left(\frac{c}{B\Omega}\nabla_\perp\phi\right) - \frac{Mc^2 n_i}{T_i B^2}|\nabla_\perp\phi|^2, \tag{E.14}$$

The other two integrals in (E.2) can be done by using the lowest order results $\widetilde{\phi} \simeq -\Omega^{-1}(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\phi$ and $f_i \simeq \overline{f}_M$. The integrals are

$$\int d^3v\,\frac{Z^2e^2}{2M^2}\widetilde{\phi}^2\frac{\partial^2\overline{f}_M}{\partial E_0^2} = \frac{Mc^2 n_i}{2T_i B^2}|\nabla_\perp\phi|^2 \tag{E.15}$$

and

$$\int d^3v\,\frac{c}{B}\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{n}})\cdot\nabla\left(\frac{\partial\overline{f}_M}{\partial E_0}\right) = \frac{c}{B\Omega}\nabla n_i\cdot\nabla_\perp\phi. \tag{E.16}$$

Using (E.14), (E.15) and (E.16), (E.2) becomes

$$n_i = \hat{N}_i + \nabla \cdot \left(\frac{cn_i}{B\Omega} \nabla_\perp \phi \right) - \frac{Mc^2 n_i}{2T_i B^2} |\nabla_\perp \phi|^2, \tag{E.17}$$

where $\hat{N}_i$ is given by (56). Therefore, the quasineutrality condition is as shown in (55).

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