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Neoclassical ion heat flux and poloidal flow in a tokamak pedestal

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In the core of a tokamak, turbulent transport normally dominates over neoclassical. The situation could be different in a high confinement (or H) mode pedestal, where the former may be suppressed by a strongly sheared equilibrium electric field. On the other hand, this very field makes conventional neoclassical results inapplicable in the pedestal by significantly modifying ion drift orbits. We present the first calculation of the banana regime neoclassical ion heat flux and poloidal flow in the pedestal accounting for the strong *ExB* drift inherent to this tokamak region. Interestingly, we find that due to the electric field the pedestal poloidal ion flow can change its direction as compared to its core counterpart. This result elucidates the discrepancy between the conventional banana regime predictions and recent experimental measurements of the impurity flow performed at Alcator C-Mod.

1 Introduction

The neoclassical theory of plasma transport considers transport processes that are due to the non-uniformity of the confining magnetic field. In the original work by Galeev and Sagdeev in 1968 {{83 Galeev, A. A. and Sagdeev R. Z. 1968}} it was pointed out that such non-uniformity results in more complicated particle trajectories as compared to simple Larmor orbits in straight magnetic field line geometry. More specifically, they observed that in toroidal magnetic fields the gyrocenters of these orbits perform cyclic motion that allow them to depart noticeably from their reference magnetic field surface. In a tokamak, particles can be classified as either trapped or passing based on the character of their gyrocenter trajectory. In particular, the poloidal projection is banana like for the former and an off-center circle with respect to the reference flux surface for the latter. These orbits of particle gyrocenters are often referred to as drift surfaces.

For ions, the characteristic size of the drift surface departure from its flux surface scales with poloidal ion gyroradius, $\rho_{pol} \equiv v_i Mc \big/ ZeB_{pol}$, where $v_i \equiv \sqrt{2T \big/ M}$ is the ion thermal speed and B_{pol} is the poloidal magnetic field. Accordingly, in the so-called "banana regime", in which collisions are rare enough for an ion to circulate several times over its neoclassical orbit before being scattered, it is ρ_{pol} that defines the elementary diffusive step in contrast to classical transport in a uniform magnetic field that is governed by a step in the Larmor radius $\rho \equiv v_i Mc \big/ ZeB$. For most tokamaks, the poloidal component of the magnetic field is much less than toroidal, making $\rho \ll \rho_{pol}$. Therefore, neoclassical transport normally dominates over classical.

Neoclassical radial transport in the core of a tokamak has been evaluated in great detail {{83 Galeev, A. A. and Sagdeev R. Z. 1968; 82 Kovrizhnikh, L. M. 1969; 9 Hinton, FL 1976; 80 Rosenbluth, MN 1972; 84 Rutherford, PH 1970; 14 Helander P and Sigmar D J 2002; 56 Hirshman, SP 1981}}. Extensions to the potato orbits near the magnetic axis have also been considered {{19 Helander, P. 2000; 20 Helander, P. 2002}}. However, all these investigations rely on the Galeev and Sagdeev equations of particle motion, making their results inapplicable to the pedestal case. The reason for this limitation is the strong electric field, present in a subsonic pedestal. It is needed to sustain ion pressure balance, making the ions nearly electrostatically confined {{69 Kagan, G. 2008}}. The strong electric field substantially modifies ion orbits, thereby requiring reconsideration of the conventional neoclassical results. The effect of strongly sheared but weak electric field on neoclassical ion transport in the banana regime, usually referred to as orbit squeezing {{21 Hazeltine, RD 1989}}, was considered in {{43 Shaing, KC 1992}}. However, this model can realistically apply only to a narrow tokamak region, because over a reasonably long distance large field shear results in a large field. Moreover, the recent calculation of the neoclassical polarization in a pedestal, carried out in {{86 Kagan, G. 2009}}, demonstrates that a shear free electric field introduces qualitatively novel features resulting in crucial kinetic implications. Therefore, we are led to investigate the full effect of strong radial electric field on the banana regime neoclassical ion heat flux and poloidal flow to gain insight into transport properties of the tokamak pedestal.

The evaluation of the neoclassical ion heat flux in the tokamak core has been done in a number of ways {{82 Kovrizhnikh, L. M. 1969; 9 Hinton, FL 1976; 84 Rutherford, PH 1970; 80 Rosenbluth, MN 1972; 14 Helander P and Sigmar D J 2002}}. Our calculation of its pedestal counterpart extends the logic outlined in {{14 Helander P and Sigmar D J 2002}} to the retention of a strong radial electric field. To do so, in the second section of this paper we review the effect of this field on particle orbits in the pedestal. The results allow us to proceed to section 3, where we utilize the linearized Rosenbluth form of the full like particle collision operator {{89 Rosenbluth, M.N. 1957}} to derive a model collision operator that is particularly convenient for describing processes near the $\vec{E} \times \vec{B}$ modified trapped-passing boundary. In the next section we employ the model to solve the kinetic equation with only the neoclassical drive term retained. This solution provides us with the first order correction to the distribution function so that we can continue to section 5 where we explicitly evaluate the neoclassical ion heat flux with the help of the moment approach. The insight developed to calculate the ion heat flux is then employed to determine the parallel ion flow in section 6. Finally, in the last section we discuss implications of these newly obtained results.

2 Particle orbits in the pedestal

Calculation of the ion orbits in the pedestal has been already presented in [{{86 Kagan, G. 2009}}, errata]. However, there were errors in the treatment of orbit squeezing because the distinction was not sharply drawn between defining the poloidal $\vec{E}\times\vec{B}$ drift at a flux surface, $u = cI\phi'(\psi)/B$, and the poloidal $\vec{E}\times\vec{B}$ drift along an ion trajectory u_*. While being insignificant for the physical implications of the results of {{86 Kagan, G. 2009}}, the distinction between u and u_* turns out to be a crucial element for understanding the physics underlying the neoclassical heat diffusion in the pedestal. Namely, u defined on a given *flux surface* is the parameter to be employed when calculating the moments of the ion distribution function in sections 5 and 6, whereas defining u_* on a given *drift surface* is necessary for estimating the neoclassical heat flux with the help of the random walk argument. Hence, in this section we briefly outline the derivation of particle orbits in the pedestal to emphasize the difference between the two.

First, we recall that in the pedestal the poloidal ion velocity is given by

$$\dot{\theta} = \left[v_{\|} + cI\phi'(\psi)/B\right]\hat{n}\cdot\nabla\theta \tag{1}$$

Next, we assume that the equilibrium electric potential $\phi_0(\psi)$ is quadratic and expand it about $\psi_* \approx \psi - Iv_{\|}/\Omega$

$$\phi_0(\psi) = \phi_0(\psi_*) + (\psi - \psi_*)\phi_0'(\psi_*) + \frac{1}{2}(\psi - \psi_*)^2\phi_0''(\psi_*). \tag{2}$$

Because of the preceding expansion, results of this section can strictly be applied only in the pedestal with a perfectly quadratic potential well. However, providing that the radial extent of the key particle orbits is much less than ρ_{pol}, we can Taylor expand the realistic equilibrium electric potential around a point on the trajectory to extend our solution to the general case.

We assume further that the radial variation of B is weak so that $B(\psi,\theta) \approx B(\psi_*,\theta)$. Then, denoting

$$\phi_* \equiv \phi_0(\psi_*), \phi_*' \equiv \phi_0'(\psi_*), \phi_*'' \equiv \phi_0''(\psi_*) \tag{3}$$

we can rewrite (1) as

$$qR_0\dot{\theta} = \left(1 + \frac{cI^2\phi_*''}{B\Omega}\right)v_{||} + \frac{cI\phi_*'}{B}, \tag{4}$$

where R_0 stands for the major radius and finite orbit effects are retained. Defining

$$u_* \equiv cI\phi_*' \big/ SB \tag{5}$$

(4) becomes

$$qR_0\dot{\theta} = S(v_{||} + u_*), \tag{6}$$

where $S \equiv 1 + cI^2\phi_*''\big/B\Omega$ is the orbit squeezing factor {{21 Hazeltine, RD 1989}}. To make analytic progress, we use an aspect ratio expansion to write

$$B = B_0(1+\varepsilon)/(1+\varepsilon\cos\theta) \approx B_0\left(1 + 2\varepsilon\sin^2\frac{\theta}{2}\right) \tag{7}$$

with $B_0 \equiv B(\theta = 0)$ and $\theta = 0$ at the outer equatorial plane. We also define $v_{||0} \equiv v_{||}(\theta = 0)$, $u_{*0} \equiv u_*(\theta = 0) = cI\phi_*'\big/S_0B_0$ and $S_0 \equiv 1 + cI^2\phi_*''\big/B_0\Omega_0$ so that $qR_0\dot{\theta}\big|_{\theta=0} = S_0(u_0 + v_{||0})$.

Next, we employ energy conservation

$$E \equiv \frac{v_{\|}^2}{2} + \mu B + \frac{Ze}{M}\phi(\psi) = const. \tag{8}$$

Using (2) along with $\psi - \psi_* = I v_{\|} / \Omega$ transforms (8) into

$$\frac{S\left(v_{\|} + u_*\right)^2}{2} + \mu B - \frac{S u_*^2}{2} = const. \tag{9}$$

As a result, we can describe the particle motion solely in terms of θ and $\dot{\theta}$:

$$\frac{S\left(v_{\|} + u_*\right)^2}{2} + \mu B - \frac{S u_*^2}{2} = \frac{S_0\left(v_{\|0} + u_*\right)^2}{2} + \mu B_0 - \frac{S_0 u_{*0}^2}{2}. \tag{10}$$

In the Eq. (10), for $S < 0$ the $\left(\dot{\theta}\right)^2$ term is negative and therefore trapped particles reside on the inside of the tokamak. For what follows we assume $S > 0$ so that banana particles are localized on the outside of the tokamak as in the conventional case. Evaluating the θ dependence of u and S with the help of (7) and solving (10) for $\dot{\theta}$ we obtain

$$\left(v_{\|} + u_*\right) = \pm\left(v_{\|0} + u_{*0}\right)\sqrt{1 - \kappa^2 \sin^2\left(\theta/2\right)}, \tag{11}$$

where we assume $4\varepsilon\left(S_0 - 1\right)/S_0 \ll 1$ and define

$$\kappa^2 \equiv \frac{4\varepsilon}{S_0}\frac{u_{*0}^2 + \mu B_0}{\left(u_{*0} + v_{\|0}\right)^2} \tag{12}$$

with the trapped particles corresponding to $\kappa > 1$ and the passing to $0 < \kappa < 1$, and where $\mu B_0 \equiv v_{\perp 0}^2/2$. For $4\varepsilon/S_0 \ll 1$ the particles of interest are localized around the trapped-passing boundary.

Notice, that u_{*0} is defined through the electric field evaluated at ψ_*. So, the trapping condition (12) is for particles lying on the same drift surface. Physically, it is the collisions of these particles that matter for transit averages and, therefore, when conducting the random walk argument for the neoclassical diffusion coefficient we have to use (12) to estimate the effective collision frequency. On the other hand, when evaluating the neoclassical heat flux and poloidal flow by taking the d^3v integrals, we need to operate with quantities evaluated on a given flux surface. This distinction results in the effective collision frequency no longer being equal to the collision frequency divided by the square of the trapped particle fraction in the presence of orbit squeezing. To address this issue we introduce

$$u \equiv cI\phi'\left(\psi\right)\Big/B \tag{13}$$

and

$$u_0 \equiv cI\phi'\left(\psi_0\right)\Big/B_0\,, \tag{14}$$

where ψ_0 is the outermost point on the particle orbit. To relate u_0 and u_{*0} we use (2) to write

$$\phi_0'\left(\psi_*\right) = \phi_0'\left(\psi\right) + \phi_0''\left(\psi\right)\left(\psi_* - \psi\right). \tag{15}$$

Then, recalling (5) and (14) we find

$$v_{||} + u = S\left(v_{||} + u_*\right) \tag{16}$$

and

$$S\left(v_{||0} + u_{*0}\right) = v_{||0} + u_0\,. \tag{17}$$

Inserting (17) in (12) then gives

$$\kappa^2 \approx 4\varepsilon S_0 \frac{u_0^2 + \mu B_0}{\left(u_0 + v_{||0}\right)^2}, \tag{18}$$

where we drop terms small by $\sqrt{\varepsilon}$. Also, upon using (17), equation of motion (11) becomes

$$v_{||} + u = \left(v_{||0} + u_0\right)\sqrt{1 - \kappa^2 \sin^2\left(\theta/2\right)}, \tag{19}$$

where we set $S \approx S_0$ near the trapped-passing boundary. The preceding results are needed in the following sections to treat collisions, solve the kinetic equation, and form moments of the distribution function.

3 Model collision operator in the pedestal

We start by deriving a model of the like particle collision operator that conveniently describes the collisional transitions across the trapped-passing boundary in the pedestal. In the core of a tokamak, this boundary is a cone centered at the origin of the $\left(v_{\perp}, v_{||}\right)$ plane and therefore to retain neoclassical transport processes in the large aspect ratio limit it is sufficient to use a momentum conserving pitch-angle scattering operator. In a subsonic pedestal with a width comparable to ρ_{pol}, the dramatic density drop gives rise to a strong radial electric field to compensate the ion pressure gradient. The resulting $\vec{E} \times \vec{B}$ drift enters poloidal ion motion in leading order, thereby modifying particle orbits. Consequently, as shown on Fig 1, the trapped-passing boundary is curved and shifted so that the energy scattering component of the collision operator contributes to neoclassical transport as well.

To capture collisional processes near the trapped-passing boundary in a simple manner it is convenient to employ some function of κ^2 as an independent variable in the model collision operator. Also, it is convenient to choose variables that reduce to the conventional ones, $2\mu B_0 \big/ v^2$ and $v^2 \big/ 2$, in the limit of no electric field to make it easier to keep track of the changes associated with the pedestal case. Finally, to simplify solving the kinetic equation it is desirable to have these variables conserved along a single particle orbit.

To this end, it is useful to introduce the variables

$$W \equiv \frac{\left(v_{\|0} + u_0\right)^2}{2S_0} + \left(\mu B_0 + u_0^2\right) \text{ and } \lambda \equiv \frac{\mu B_0 + u_0^2}{W}, \tag{20}$$

as suggested by structure of (19) along with (7) and (18). As long as particle dynamics is described by equation (19), W and λ are useful constants of the motion through order $\sqrt{\varepsilon S}$. Also observe that (18) and (20) give

$$\kappa^2 = \frac{2\varepsilon\lambda}{1-\lambda} \tag{21}$$

and therefore

$$\lambda = \frac{\kappa^2}{\kappa^2 + 2\varepsilon}. \tag{22}$$

That is, λ can be defined solely in terms of κ^2 and therefore $\nabla_v \lambda$ is orthogonal to the trapped-passing boundary that is located at $\kappa = 1$. As discussed after equation (12), the range $0 < \kappa^2 < 1$ corresponds to passing particles and $\kappa^2 > 1$ to trapped so that near the trapped-passing boundary $\left(v_{\|} + u_0\right) \sim v_i \sqrt{\varepsilon}$ for $S \sim 1$. These ions are the ones of the most concern since we will need to carefully evaluate the portion of the ion distribution function localized to this region.

For many purposes it is convenient to rewrite (20) so that $v_{||} + u$ is expressed in terms of W and λ similarly to how $v_{||}$ is expressed in terms of $v^2/2$ and $2\mu B_0/v^2$ in the conventional calculation. The desired form is derived with the help of (18), (19) and (7) to obtain

$$\left(v_{||} + u\right)^2 \Big/ 2S = W\left(1 - \lambda B/B_0\right). \tag{23}$$

In the following sections this form will be found helpful for evaluating integrals over λ. Here, we employ it to check orthogonality of the variables by evaluating

$$\nabla_v W = \hat{n}\left(v_{||} + u\right)/S + \vec{v}_\perp \text{ and } \left(B/B_0\right)W\nabla_v\lambda = \left(1 - \lambda B/B_0\right)\nabla_v W - \left[\left(v_{||} + u\right)/S\right]\hat{n} \tag{24}$$

to find

$$\left(B/B_0\right)W\nabla_v\lambda \cdot \nabla_v E = \mu B\left(v_{||} + u\right)^2 \Big/ SW - \left(\lambda B/S^2 B_0\right)\left(v_{||} + u\right)^2. \tag{25}$$

In the vicinity of the trapped-passing boundary $\left(v_{||} + u\right) \sim v_i\sqrt{\varepsilon S}$ for $S \sim 1$, making W and λ nearly orthogonal. Thus, we may anticipate that once the collision operator is written in terms of these variables, the main contribution to neoclassical transport will come from the $\partial/\partial\lambda$ terms. We proceed by finding an explicit expression for such an operator.

To do so we first recall the Rosenbluth form of the collision operator {{89 Rosenbluth, M.N. 1957}}

$$C_R\left\{\delta f\right\} = \nabla_v \cdot \vec{\Gamma}\left\{\delta f\right\}, \tag{26}$$

where

$$\vec{\Gamma}\left\{\delta f\right\} \equiv \gamma f_0 \nabla_v \nabla_v G_M \cdot \nabla_v\left(\delta f/f_M\right) \tag{27}$$

with f_M a stationary Maxwellian and

$$\gamma\nabla_v\nabla_v G_M = \frac{\nu_\perp}{4}\left(v^2\overleftrightarrow{I} - \vec{v}\vec{v}\right) + \frac{\nu_{||}}{2}\vec{v}\vec{v}\,. \tag{28}$$

The collision frequencies $\nu_\perp$ and $\nu_{||}$ are functions of v^2 only and defined by

$$\nu_\perp \equiv \frac{3\sqrt{2\pi}}{2x^3}\left[\mathrm{erf}\left(x\right) - \Psi\left(x\right)\right]\nu_B \text{ and } \nu_{||} \equiv \frac{3\sqrt{2\pi}}{2x^3}\Psi\left(x\right)\nu_B, \tag{29}$$

where $\nu_B = 4\pi^{1/2}Z^4e^4n_i \ln\Lambda\big/3M^{1/2}T^{3/2}$ is the Braginskii ion-ion collision frequency,

$$\Psi\left(x\right) \equiv \frac{\mathrm{erf}\left(x\right) - x\,\mathrm{erf}'\left(x\right)}{2x^2} \text{ and } \mathrm{erf}\left(x\right) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \tag{30}$$

with $x \equiv v\sqrt{M/2T} = v/v_i$.

Switching to W, λ and gyrophase φ variables and writing (26) in conservative form we obtain

$$C\left\{\delta f\right\} = \frac{1}{J}\frac{\partial}{\partial\lambda}\left(J\vec{\Gamma}\cdot\nabla_v\lambda\right) + \frac{1}{J}\frac{\partial}{\partial W}\left(J\vec{\Gamma}\cdot\nabla_v W\right) + \frac{1}{J}\frac{\partial}{\partial\varphi}\left(J\vec{\Gamma}\cdot\nabla_v\varphi\right), \tag{31}$$

where the $\partial/\partial\varphi$ term is to be set equal to zero since classical effects are ignored. Upon accounting for both signs of $\left(v_{||}+u\right)$, equation (24) gives the Jacobian of the transformation to be

$$J = \frac{d^3v}{d\varphi dW d\lambda} = \frac{2BWS}{B_0\left(v_{||}+u\right)}. \tag{32}$$

The model collision operator to be constructed will eventually be applied to $g - h_\sigma$, where h_σ denotes the neoclassical collisional drive term in the kinetic equation, whose explicit momentum conserving form convenient for our purpose we provide in the next section, while g is the neoclassical response to h_σ. In the absence of the electric field, $g - h_\sigma$ is localized around the trapped-passing boundary, so that $\partial\left[\left(g-h_\sigma\right)/f_M\right]\big/\partial\lambda \sim O\left(\varepsilon^{-1}\right)$, while $\partial\left[\left(g-h_\sigma\right)/f_M\right]\big/\partial W = O\left(1\right)$ {{14 Helander P and Sigmar D J 2002;80 Rosenbluth, MN 1972; }}. Assuming that these estimates remain appropriate in the pedestal, equations (24) and (28) give

$$\vec{\Gamma}\left\{g-h_{\sigma}\right\}\cdot\nabla_{v}\lambda \approx f_{M}\nabla_{v}\lambda\cdot\left[\frac{\nu_{\perp}}{4}\left(v^{2}\vec{\vec{I}}-\vec{v}\vec{v}\right)+\frac{\nu_{\parallel}}{2}\vec{v}\vec{v}\right]\cdot\nabla_{v}\lambda\frac{\partial}{\partial\lambda}\left(\frac{g-h_{\sigma}}{f_{M}}\right), \qquad (33)$$

where due to our orderings we may drop the $\partial/\partial W$ term. The same reasoning allows us to drop the $\partial/\partial W$ term on the right side of (31) as well.

To simplify (33) further we use that for the calculation to follow it is sufficient to account only for particles lying in the close proximity of the trapped-passing boundary and therefore we can consider $\lambda \approx B_0/B \approx 1$ to obtain

$$W^{2}\left(\nabla_{v}\lambda\right)^{2} \approx \left(v_{\parallel}+u\right)^{2}/S^{2} \qquad (34)$$

and

$$W^{2}\left(\vec{v}\cdot\nabla_{v}\lambda\right)^{2} \approx u^{2}\left(v_{\parallel}+u\right)^{2}/S^{2}\,. \qquad (35)$$

Thus, we rewrite equation (33) as

$$\vec{\Gamma}\left\{g-h_{\sigma}\right\}\cdot\nabla_{v}\lambda \approx f_{M}\frac{\left(v_{\parallel}+u\right)^{2}}{S^{2}W^{2}}\left[\frac{\nu_{\perp}}{4}v^{2}+\left(\frac{\nu_{\parallel}}{2}-\frac{\nu_{\perp}}{4}\right)u^{2}\right]\frac{\partial}{\partial\lambda}\left(\frac{g-h_{\sigma}}{f_{M}}\right) \qquad (36)$$

to obtain our pedestal collision operator to lowest order to be

$$C\left\{g-h_{\sigma}\right\}=\frac{B_{0}\left(v_{\parallel}+u\right)}{B}\frac{\partial}{\partial\lambda}\left[\frac{B}{B_{0}\left(v_{\parallel}+u\right)}\vec{\Gamma}\cdot\nabla_{v}\lambda\right]. \qquad (37)$$

The model operator defined by equations (36) - (37) must manifestly conserve momentum. To explicitly display this property, intrinsic to the full like particle collision operator, we introduce a free parameter σ to redefine $\vec{\Gamma}\cdot\nabla_{v}\lambda$ by

$$\vec{\Gamma}\left\{g-h_{\sigma}\right\}\cdot\nabla_{v}\lambda =$$

$$f_M \frac{\left(v_{\parallel} + u\right)^2}{S^2 W^2}\left[\frac{\nu_{\perp}}{4} v^2 + \left(\frac{\nu_{\parallel}}{2} - \frac{\nu_{\perp}}{4}\right) u^2\right]\frac{\partial}{\partial \lambda}\left[\frac{g-h}{f_M} - \frac{\sigma I\left(v_{\parallel} + u\right)}{\Omega T}\frac{\partial T}{\partial \psi}\right], \tag{38}$$

where

$$h_{\sigma} \equiv h + f_M \frac{\sigma I\left(v_{\parallel} + u\right)}{\Omega T}\frac{\partial T}{\partial \psi}, \tag{39}$$

with the drive term h defined at the start of the next section. Then, after solving for the first order correction to the distribution function we determine σ such that the operator given by (37) - (38) conserves momentum.

4 Passing constraint

Now that we have a convenient model of the collision operator we can solve for the first order distribution function that is responsible for the neoclassical transport. A form of kinetic equation appropriate in the pedestal case can be obtained from {{69 Kagan, G. 2008}} whose expression (11) readily provides the pedestal equation for the perturbation of the equilibrium Maxwellian f_M. Setting aside zonal flow phenomenon by omitting the $\partial/\partial t$ terms and transit averaging, we obtain the neoclassical constraint on the distribution function in the pedestal to be

$$\overline{C\left\{g-h\right\}} = 0, \tag{40}$$

where the bar on top of the full linearized like particle collision operator denotes transit averaging, g stands for the non-diamagnetic perturbation of the leading order Maxwellian and the neoclassical collisional drive h is defined by

$$h \equiv f_M \frac{I v_{\parallel}}{\Omega}\frac{M v^2}{2T^2}\frac{\partial T}{\partial \psi}, \tag{41}$$

with $\vec{B} = I\nabla\zeta + \nabla\zeta \times \nabla\psi$ our assumed magnetic field.

Written in this form our passing constraint looks identical to the conventional one {{14 Helander P and Sigmar D J 2002}}. Notice, however, that in contrast to the core case transit averaging in the pedestal must keep the distinction between the flux function ψ and canonical angular momentum $\psi_* \approx \psi - Iv_{\|}\big/\Omega$ and account for the effect of the $\vec{E} \times \vec{B}$ drift in leading order of the poloidal ion velocity. That is, transit averaging of a quantity Q is now defined by

$$\overline{Q} \equiv \oint_* \frac{d\theta Q}{v_{\|} + u} \Bigg/ \oint_* \frac{d\theta}{v_{\|} + u}, \tag{42}$$

where the integration on the right side must be performed holding ψ_*, μ and total energy fixed as indicated by the $*$ subscript on the integral. In the core of a tokamak u is negligible compared to $v_{\|}$ and ψ is approximately constant along the ion trajectory. In such a case, equations (40) - (42) reduce to the conventional ones by dropping the u terms and $*$ subscript on the trajectory integrals.

To proceed with solving for g we use the number, momentum, and energy conservation properties of the linearized like particle collision operator to replace the drive term h by

$$h \equiv f_M \frac{I\left(v_{\|} + u\right)}{\Omega} \frac{M\left(v^2 + u^2\right)}{2T^2} \frac{\partial T}{\partial \psi}. \tag{43}$$

For $\lambda \approx 1$ equation (20) gives

$$\frac{v^2 + u^2}{2} \approx W \tag{44}$$

so that for the trapped and barely passing particles (43) becomes

$$h \approx f_M \frac{I\left(v_{\|} + u\right)}{\Omega} \frac{MW}{T^2} \frac{\partial T}{\partial \psi}. \tag{45}$$

Analogously to the conventional case, for the trapped particles $g = 0$ since the drive term vanishes upon transit averaging over a complete bounce. The goal of the remainder of this section is therefore to solve (40) for g in the passing region of the $\left(W,\lambda\right)$ space.

Employing our model collision operator (37) - (38) along with the fact that λ and W are constants of the motion to the requisite order we obtain

$$\oint_* d\theta\left(v_{\|}+u\right)\frac{\partial}{\partial\lambda}\left[\frac{g}{f_M}-\frac{I\left(v_{\|}+u\right)}{\Omega}\frac{M\left(W-T\sigma/M\right)}{T^2}\frac{\partial T}{\partial\psi}\right]=0\,. \tag{46}$$

In the banana regime g/f_M is independent of θ to leading order giving

$$\frac{\partial}{\partial\lambda}\left(\frac{g}{f_M}\right)\oint_* d\theta\left(v_{\|}+u\right)=\oint_* d\theta\left(v_{\|}+u\right)\frac{\partial}{\partial\lambda}\left[\frac{IM\left(v_{\|}+u\right)\left(W-T\sigma/M\right)}{\Omega T^2}\frac{\partial T}{\partial\psi}\right]. \tag{47}$$

We observe that due to (23)

$$\left(v_{\|}+u\right)\frac{\partial\left(v_{\|}+u\right)}{\partial\lambda}=-SW\frac{B}{B_0}\,. \tag{48}$$

Thus, setting $B/B_0\approx 1$ we obtain

$$\left.\frac{\partial}{\partial\lambda}\left(\frac{g}{f_M}\right)\right|_p\approx-\frac{SIMW\left(W-T\sigma/M\right)}{\left\langle v_{\|}+u\right\rangle\Omega_0 T^2}\frac{\partial T}{\partial\psi}, \tag{49}$$

where angle brackets denote the flux surface average such that

$$\left\langle v_{\|}+u\right\rangle\equiv\frac{1}{2\pi}\oint_* d\theta\left(v_{\|}+u\right). \tag{50}$$

Now we can verify the localization assumption made to derive our model collision operator. To do so we form

$$\frac{\partial}{\partial\lambda}\left(\frac{g-h_\sigma}{f_M}\right)=\frac{SIMW\left(W-\sigma T/M\right)}{\Omega_0 T^2}\frac{\partial T}{\partial\psi}\left(\frac{1}{v_\parallel+u}-\frac{1}{\left\langle v_\parallel+u\right\rangle}\right). \tag{51}$$

To estimate the expression on the right side of (51) we flux surface average it and notice that

$$\left\langle\frac{1}{v_\parallel+u}\right\rangle-\frac{1}{\left\langle v_\parallel+u\right\rangle}\propto\left\langle\frac{1}{\sqrt{1-\kappa^2\sin^2\theta/2}}\right\rangle-\frac{1}{\left\langle\sqrt{1-\kappa^2\sin^2\theta/2}\right\rangle}. \tag{52}$$

We also observe from (21) that at the trapped-passing boundary $\lambda=1/\left(1+2\varepsilon\right)$ when $\kappa^2=1$. However, once λ leaves the ε vicinity of the trapped-passing boundary, κ^2 becomes small and we can Taylor expand the expression on the right side of (52) to find

$$\left\langle\frac{1}{\sqrt{1-\kappa^2\sin^2\theta/2}}\right\rangle-\frac{1}{\left\langle\sqrt{1-\kappa^2\sin^2\theta/2}\right\rangle}\sim O\left(\kappa^4\right)\ll 1. \tag{53}$$

Thus, for the particles of interest, the λ derivative of the function inside the collision operator in (40) indeed goes like $O\left(\varepsilon^{-1}\right)$, justifying our dropping of the $\partial/\partial W$ terms in the equations (31) and (33).

It is necessary to emphasize that because of approximations made, solution (49) is only valid for $\lambda\approx 1$ and should not be applied in the freely passing particle region. Fortunately, the integral for the ion heat flux calculated in section 4 involves the expression (51) rather than $\partial g/\partial\lambda$ alone, thereby making the freely passing particles unimportant for the final result. Evaluation of the neoclassical parallel ion flow does not have the same advantage and therefore requires the alternate treatment presented in section 5.

Next, we have to ensure momentum conservation by choosing an appropriate value of σ. That is, we have to find σ such that

$$\left\langle \int \frac{d^3v}{B} v_{\|} C\left\{ g-h \right\} \right\rangle = \left\langle \int \frac{d^3v}{B} \left(v_{\|} + u \right) C\left\{ g-h \right\} \right\rangle = 0\,, \tag{54}$$

where the number conservation property of the full like particle collision operator is employed. To evaluate the right side of (54) we approximate $C\left\{ g-h \right\}$ by (37) and use (38) along with (51) to write

$$\vec{\Gamma}\left\{ \frac{g-h_\sigma}{f_M} \right\} \cdot \nabla_v \lambda = f_M \frac{\left(u+v_{\|} \right)^2}{S^2 W^2} \times$$

$$\left[\frac{\nu_\perp}{4} v^2 + \left(\frac{\nu_{\|}}{2} - \frac{\nu_\perp}{4} \right) u^2 \right] \frac{SIMW\left(W - T\sigma/M \right)}{\Omega_0 T^2} \frac{\partial T}{\partial \psi} \left(\frac{1}{v_{\|}+u} - \frac{1}{\left\langle v_{\|}+u \right\rangle} \right). \tag{55}$$

Then, we recall (32) and (37) and integrate (54) by parts over λ to obtain

$$\int dW d\lambda f_M W \left(W - T\sigma/M \right) \left(\frac{1}{\left\langle v_{\|}+u \right\rangle} - \left\langle \frac{1}{v_{\|}+u} \right\rangle \right) \left[\frac{\nu_\perp}{4} v^2 + \left(\frac{\nu_{\|}}{2} - \frac{\nu_\perp}{4} \right) u^2 \right] = 0\,, \tag{56}$$

where (48) is used to find $\partial\left(v_{\|}+u \right)/\partial\lambda$. Using (23) we now complete the integration over λ in (63) by employing the technique of {{80 Rosenbluth, MN 1972; }}:

$$\int d\lambda \left(\frac{1}{\sqrt{1-\lambda B/B_0}} - \frac{1}{\left\langle \sqrt{1-\lambda B/B_0} \right\rangle} \right) =$$

$$\int_{u_0^2/W}^{B_0/B} \frac{d\lambda}{\sqrt{1-\lambda B/B_0}} - \int_{u_0^2/W}^{B_0/B} \frac{d\lambda}{\left\langle \sqrt{1-\lambda B/B_0} \right\rangle} \approx 1.38\sqrt{2\varepsilon}\,, \tag{57}$$

where the lower limit of integration obtained from (20) is unimportant because freely passing particles do not contribute to the leading order result. Then, (56) reduces to

$$\int dW W^{1/2}\left(W-\sigma T/M\right)\left[\nu_{\perp}v^2+\left(2\nu_{\parallel}-\nu_{\perp}\right)u^2\right]f_M=0. \tag{58}$$

Finally, we introduce a new variable of integration $y\equiv M\left(v^2-u^2\right)/2T=M\left(W-u^2\right)/T$ in (58) and solve for σ to obtain

$$\sigma=\frac{\int_0^\infty dy e^{-y}\left(y+Mu^2/T\right)^{3/2}\left[\nu_{\perp}y+\nu_{\parallel}\left(Mu^2/T\right)\right]}{\int_0^\infty dy e^{-y}\left(y+Mu^2/T\right)^{1/2}\left[\nu_{\perp}y+\nu_{\parallel}\left(Mu^2/T\right)\right]}, \tag{59}$$

where frequencies $\nu_{\perp}$ and $\nu_{\parallel}$ are defined in terms of $x=\sqrt{y+Mu^2/2T}$ by (29).

In the absence of the background electric field $u=0$ and $x^2=y$ so that

$$\sigma=\frac{\int_0^\infty dx e^{-x^2}x^3\left[\operatorname{erf}\left(x\right)-\Psi\left(x\right)\right]}{\int_0^\infty dx e^{-x^2}x\left[\operatorname{erf}\left(x\right)-\Psi\left(x\right)\right]}=\frac{\sqrt{2}}{2\left[\sqrt{2}-\ln\left(1+\sqrt{2}\right)\right]}\approx 1.33 \tag{60}$$

which agrees with the conventional result {{80 Rosenbluth, MN 1972; 9 Hinton, FL 1976; 14 Helander P and Sigmar D J 2002}}.

5 Neoclassical ion heat flux in the pedestal

Here we proceed by calculating the neoclassical radial ion heat flux in the pedestal using the moment approach {{14 Helander P and Sigmar D J 2002}} so we need only evaluate

$$\left\langle \vec{q}\cdot\nabla\psi\right\rangle=-\frac{McIT}{Ze}\left\langle\int\frac{d^3v}{B}\left(\frac{Mv^2}{2T}-\frac{5}{2}\right)v_{\parallel}C\left\{g-h\right\}\right\rangle. \tag{61}$$

To perform the integration on the right side of (61) we first employ the number, momentum and energy conservation properties of the linearized collision operator to rewrite it as

$$\left\langle \vec{q}\cdot\nabla\psi \right\rangle = -\frac{McIT}{Ze}\left\langle \int \frac{d^3v}{B}\frac{M\left(v^2+u^2\right)\left(v_{\|}+u\right)}{2T} f_M C\left\{g-h\right\}\right\rangle. \tag{62}$$

Now we can continue in a manner similar to the one used in the previous section to find σ. That is, we again replace $C\left\{g-h\right\}$ with approximation (37), then use (55) inside the collision operator and integrate the result by parts using (48) and (44). Then, (62) transforms into

$$\left\langle \vec{q}\cdot\nabla\psi \right\rangle = -\frac{4\pi M^2 I^2 S}{\Omega_0^2 T^2}\frac{\partial T}{\partial\psi}\int dW d\lambda f_M W^2\left(W - T\sigma/M\right)\left(\frac{1}{\left\langle v_{\|}+u\right\rangle} - \left\langle \frac{1}{v_{\|}+u}\right\rangle\right)\times \\ \left[\frac{\nu_{\perp}}{4}v^2 + \left(\frac{\nu_{\|}}{2} - \frac{\nu_{\perp}}{4}\right)u^2\right] \tag{63}$$

and we can carry out the λ integration with the help of (57) to obtain

$$\left\langle \vec{q}\cdot\nabla\psi \right\rangle = -1.38\frac{\pi M^2 I^2}{\Omega_0^2 T^2}\frac{\partial T}{\partial\psi}\sqrt{\varepsilon S}\int dW W^{3/2}\left(W - T\sigma/M\right)\times \\ \left[\nu_{\perp}v^2 + \left(2\nu_{\|} - \nu_{\perp}\right)u^2\right] f_M\,. \tag{64}$$

Finally, we again substitute y for W to find

$$\left\langle \vec{q}\cdot\nabla\psi \right\rangle = -1.38\frac{n_i T I^2 e^{-Mu^2/2T}}{\sqrt{2\pi}\,\Omega_0^2 M}\frac{\partial T}{\partial\psi}\sqrt{\varepsilon S}\int_0^{\infty} dy e^{-y}\left(y + Mu^2/T\right)^{3/2}\times \\ \left(y + Mu^2/T - \sigma\right)\left[\nu_{\perp}y + \nu_{\|}\left(Mu^2/T\right)\right], \tag{65}$$

where n_i stands for the ion density and the parameter σ is provided by equation (59).

To proceed with the analysis we insert expression (29) for the collision frequencies into (65) to obtain

$$\left\langle \vec{q}\cdot\nabla\psi\right\rangle = -4.14e^{-\left(u/v_i\right)^2}\frac{n_i\nu_B TI^2}{\Omega_0^2 M}\frac{\partial T}{\partial\psi}\sqrt{\varepsilon S}\int\limits_0^\infty dye^{-y}\left[y+2\left(u/v_i\right)^2\right]^{3/2}\times$$

$$\left[y+2\left(u/v_i\right)^2-\sigma\right]\left[y+\left(u/v_i\right)^2\right]^{-3/2}\left\{y\left[\mathrm{erf}\left(x\right)-\Psi\left(x\right)\right]+2\left(u/v_i\right)^2\Psi\left(x\right)\right\}. \tag{66}$$

First, we consider the conventional limit in which $u=0$ and $S=1$. In this case, σ is given by (60) and $y=x^2$ so that (66) becomes

$$\left\langle \vec{q}\cdot\nabla\psi\right\rangle \approx -4.14n_i\frac{\nu_B TI^2\sqrt{\varepsilon}}{\Omega_0^2 M}\frac{\partial T}{\partial\psi}\int\limits_0^\infty dxe^{-x^2}x^3\left(x^2-1.33\right)\left[\mathrm{erf}\left(x\right)-\Psi\left(x\right)\right]\approx$$

$$-1.35n_i\nu_B\frac{TI^2\sqrt{\varepsilon}}{\Omega_0^2 M}\frac{\partial T}{\partial\psi} \tag{67}$$

in agreement with the usual neoclassical result {{14 Helander P and Sigmar D J 2002; 9 Hinton, FL 1976; 80 Rosenbluth, MN 1972}}. Now we can write the full result (66) in a normalized form as

$$\left\langle \vec{q}\cdot\nabla\psi\right\rangle = -1.35n_i\nu_B\frac{TI^2\sqrt{\varepsilon S}}{\Omega_0^2 M}\frac{\partial T}{\partial\psi}G\left(u\right), \tag{68}$$

where $G\left(u\right)$ is given by

$$G\left(u\right)=1.53e^{-\left(u/v_i\right)^2}\int\limits_0^\infty dye^{-y}\left[y+2\left(u/v_i\right)^2\right]^{3/2}\left[y+2\left(u/v_i\right)^2-\sigma\right]\times$$

$$\left[y+\left(u/v_i\right)^2\right]^{-3/2}\left\{y\left[\mathrm{erf}\left(x\right)-\Psi\left(x\right)\right]+2\left(u/v_i\right)^2\Psi\left(x\right)\right\} \tag{69}$$

so that $G\left(0\right)=1$. The dependence of the normalized neoclassical heat flux on u is plotted in Fig 2. Notice, that as u goes beyond unity $G\left(u\right)$ decays exponentially with the electric field. As in the problem of the zonal flow in the pedestal {{86 Kagan, G. 2009}}, this decay is due to the trapped-passing boundary shifting towards the tail of the ion distribution function, thereby making the number of particles contributing to neoclassical heat flux negligible once the electric field is large enough as sketched in Fig 1.

6 Poloidal ion flow in the pedestal

Based on the technique of the previous sections it is also possible to evaluate the poloidal ion flow in the pedestal. The ion velocity $\vec{V}_i$ is defined by

$$n_i \vec{V}_i \equiv \int d^3 v \vec{v} f \tag{70}$$

giving

$$n_i \vec{V}_i = -\frac{cR}{Ze}\left(\frac{\partial p}{\partial \psi} + Zen_i \frac{\partial \phi}{\partial \psi}\right)\hat{\zeta} + \hat{n}\int d^3 v v_{\|} g \tag{71}$$

where the two toroidally directed terms on the right side are diamagnetic and $\vec{E} \times \vec{B}$ while the parallel term is neoclassical, with $\hat{\zeta}$ the toroidal unit vector. To proceed it is convenient to rewrite (71) further as

$$\vec{V}_i = \omega R \hat{\zeta} - \hat{n}\frac{u}{n_i}\int d^3 v g + \frac{\hat{n}}{n_i}\int d^3 v \left(v_{\|} + u\right) g\,, \tag{72}$$

where $\omega \equiv -c\left(\partial\phi/\partial\psi\right) - \left(c/Zen_i\right)\left(\partial p/\partial\psi\right)$ with $p = n_i T$.

It is shown in Appendix A that to leading order in $\sqrt{\varepsilon S}$

$$\int d^3 v g = 0\,. \tag{73}$$

Therefore, we only have to evaluate the last integral on the right side of (72). At this point, the previously mentioned difference between the treatment of the neoclassical ion heat flux and poloidal flow enters. The former is carried by the trapped and barely passing particles, which is mathematically manifested by the integrand in (61) being localized at $\lambda \approx 1$. This feature is what justifies our procedure of evaluating the ion heat flux because our model collision operator, as well as the resulting solution for the distribution function (49), is derived under the same localization assumption.

However, the integral for the neoclassical ion flow does not have this property. As in the conventional case, freely passing particles are expected to contribute to give the final answer to zeroth order in the $\sqrt{\varepsilon S}$ expansion. To ensure validity of the calculation we rewrite the last integral on the right side of (72) as follows

$$\int d^3v\left(v_{\|} + u\right)g = \int d^3v\left(v_{\|} + u\right)\left(g - h_\sigma\right) + \int d^3v\left(v_{\|} + u\right)h_\sigma \approx$$

$$-4\pi\int dW d\lambda W\lambda \frac{\partial\left(g - h_\sigma\right)}{\partial\lambda} + \int d^3v\left(v_{\|} + u\right)h_\sigma\,, \tag{74}$$

where the integration by parts over λ is completed upon employing the Jacobian (32) with B/B_0 set equal unity. The first integral on the right side of (74) involves the function $\partial\left(g - h_\sigma\right)/\partial\lambda$ that satisfies our localization assumption as discussed after equation (51), thereby making appropriate our solution (49) for $\partial g/\partial\lambda$. The second integral on the right side of (74) only involves h_σ defined by equations (39), (43), and (59) that are valid for the freely passing particles as well as for the trapped and barely passing. Thus, equation (74) is appropriate for evaluating the neoclassical ion flow in a self-consistent manner.

To proceed with finding an explicit expression for the ion flow we notice that the first integral on the right side of (74) can be evaluated in the same way as the neoclassical ion heat flux in section 5 to give a non-zero answer at first order in $\sqrt{\varepsilon S}$. Based on the conventional calculation we anticipate that the overall result for the neoclassical flow is of zeroth order in this expansion parameter and therefore it is the second integral on the right side of (74) that contributes to the leading order flow, whereas the first one is negligible and of the same order as the $\int d^3vg$ already ignored.

Having made the preceding comments, the calculation becomes straightforward. We use (39) and (43) to write

$$\int d^3v\left(v_{\|}+u\right)h_{\sigma}=\frac{I}{\Omega T}\frac{\partial T}{\partial\psi}\int d^3v\left(v_{\|}+u\right)^2\left[\frac{M\left(v^2+u^2\right)}{2T}-\sigma\right]f_M\,, \tag{75}$$

that is easily evaluated to obtain

$$\int d^3v\left(v_{\|}+u\right)h_{\sigma}=\frac{n_iI}{M\Omega_0}\frac{\partial T}{\partial\psi}\left[\left(5/2-\sigma\right)+2\left(1-\sigma\right)\left(u/v_i\right)^2+2\left(u/v_i\right)^4\right]. \tag{76}$$

To recover the conventional result, we insert (60) for σ to obtain

$$\frac{1}{n_i}\int d^3v\left(v_{\|}+u\right)h=\frac{7I}{6\Omega_0M}\frac{\partial T}{\partial\psi}\approx 1.17\frac{I}{\Omega_0M}\frac{\partial T}{\partial\psi}, \tag{77}$$

matching the answer given in {{14 Helander P and Sigmar D J 2002}}. To write (77) in a normalized form we introduce

$$J\left(u\right)\equiv\frac{6}{7}\left[\left(5/2-\sigma\right)+2\left(1-\sigma\right)\left(u/v_i\right)^2+2\left(u/v_i\right)^4\right] \tag{78}$$

such that (72) becomes

$$\vec{V}_i-\omega R\hat{\zeta}\approx\frac{\hat{n}}{n_i}\int d^3v\left(v_{\|}+u\right)g=\frac{7I}{6\Omega_0M}\frac{\partial T}{\partial\psi}J\left(u\right)\hat{n} \tag{79}$$

with $J\left(0\right)=1$. Thus, recalling equation (72) we find the poloidal ion flow in the pedestal to be

$$V_i^{pol}=\frac{7IB_{pol}}{6\Omega_0MB}\frac{\partial T}{\partial\psi}J\left(u\right)\approx\frac{7J\left(u\right)}{6\Omega_0M}\frac{\partial T}{\partial r}, \tag{80}$$

where r stands for the minor radius of a given flux surface.

As shown on Fig 3, this normalized neoclassical flow does not decay exponentially with u/v_i in contrast to the behavior of the neoclassical heat flux found in the previous section and neoclassical polarization discussed in {{86 Kagan, G. 2009}}. This aspect can be understood by observing that, unlike neoclassical ion heat flux and polarization, the leading order neoclassical ion flow is provided by the freely passing particles, making this flow persist even if the trapped and barely passing particle populations are diminished by a strong electric field. The replacement of (41) by (43) in (39) is due to the electric field modified trapped-passing boundary along with the need to conserve momentum during ion-ion collisions which changes σ from its usual value and gives rise to the integral J. We discuss this result further in the next section.

7 Discussion

In the preceding sections, we introduce the technique for evaluating the neoclassical ion behavior in the presence of a strong background electric field and use it to explicitly calculate the neoclassical ion heat flux and poloidal flow in the pedestal. A key step is the construction of the model collision operator (37) - (38) to replace the pitch angle scattering operator employed in the conventional calculation. The need for choosing a different model to describe collisions in the pedestal is due to the electric field modifying the trapped-passing boundary in velocity space, thereby making the conventional operator inadequate for the particles that contribute the most to neoclassical ion heat flux.

Importantly, the effects of the electric field modified trapped-passing boundary impact the freely passing particles along with the need to conserve momentum in the like particle collisions. As a result, the neoclassical poloidal ion flow, carried by these particles, is rather sensitive to the electric field (though independent of orbit squeezing). Due to this sensitivity the ion flow can change direction within the pedestal as indicated by the sign change of $J\left(u\right)$ for $u > 0.6v_i$ as plotted in Fig 3. This new feature may explain the results of the experiments performed in the Alcator C-Mod by Kenneth Marr and coauthors {{96 Marr K, Lipschultz B , McDermott R, Catto P , Simakov A, Hutchinson I, Hughes J, Reinke M}} in which the absolute values of the observed banana regime ion flows are much bigger than those predicted by conventional formulas. Of course, their measurements focus on the impurity ion velocities, whereas the theory presented here applies to the background ions. However, for the purposes of an estimate we can neglect the effect of the electric field on the more collisional impurity orbits. In such a case, the net velocity of impurity ions can be evaluated given that of the background ions with the help of the following formula {{56 Hirshman, SP 1981; 93 Kim, YB 1991; 92 Helander, P. 2001; 91 Catto, P.J. 2006}}

$$V_z^{pol} = V_i^{pol} - \frac{cIB_{pol}}{eB^2}\left(\frac{1}{n_i}\frac{\partial p_i}{\partial \psi} - \frac{1}{Zn_z}\frac{\partial p_z}{\partial \psi}\right), \tag{81}$$

where we have dropped all terms small in $\sqrt{\varepsilon}$. Employing the conventional banana regime formula for the poloidal ion flow in (81) gives that V_i^{pol} and the sum of the diamagnetic terms have opposite signs, thereby resulting in a relatively low prediction for the impurity flow. However, for a more realistic pedestal case of $u/v_i > 0.6$ the newly obtained expression (80) gives the terms on the right side of (81) adding to make V_z^{pol} larger. Thus, accounting for the effect of the electric field on the background ion orbits is expected to lead to better agreement between the theoretical and experimental results for the impurity ion flow. Also, because the neoclassical *electron* current depends on the net ion velocity {{14 Helander P and Sigmar D J 2002}}, the bootstrap current will be increased in the pedestal.

Our new result for the banana regime ion heat flux possesses the same qualitative feature as the neoclassical polarization discussed in {{86 Kagan, G. 2009}}. Namely, the neoclassical ion heat flux given by (68) decays exponentially in u. Obviously, this behavior is again explained by the fact that the trapped particle region is shifted to the tail of the Maxwellian distribution once the electric field is large enough. We observe that the qualitative modifications of the pedestal case as compared to the conventional one are due to the parameter u, while the orbit squeezing parameter S only enters algebraically. In other words, it is the magnitude of the radial electric field rather than its shear that is the central quantity governing neoclassical phenomena in the pedestal.

The case of substantial electric field shear in the absence of a significant electric field itself was considered by Shaing and Hazeltine in {{43 Shaing, KC 1992}}. It is important to emphasize that their problem formulation is not appropriate for most of flux surfaces in the pedestal, with the only possible exceptions being the very top or very bottom of this region, where the electric field can be considered small. Our calculation for the ion neoclassical heat flux therefore captures the more important physics of the electric field, while still retaining orbit squeezing. Notice, that the heat conductivity given in {{43 Shaing, KC 1992}} has the factor of $S^{3/2}$ in the denominator contradicting our equation (68) which is proportional $S^{1/2}$ instead. This contradiction is due to subtle differences in the treatment of the like particle collision operator as explained in detail in Appendix B. In particular, these differences result in our finding the effective ion-ion collision frequency to be $\sim \nu_B S / \varepsilon$, rather than the Shaing and Hazeltine estimate of $\sim \nu_B / \varepsilon S$. Here, the two alternative expressions for κ given by equations (12) and (18) takes on special significance. The usual estimate of the effective collision frequency for the trapped and barely passing ions is based on the observation that they need only scatter by a small angle $\Delta\theta$ proportional to the trapped particle fraction. The effective frequency can then be estimated by $\nu_B / (\Delta\theta)^2$. In the conventional kinetic calculation we can consider $\psi \approx \psi_*$ and $S_0 = 1$ when estimating the size of the boundary layer for the distribution function to obtain $\Delta\theta \sim \sqrt{\varepsilon}$. However, in the pedestal $S_0 \neq 1$ and the difference between ψ and ψ_* must be accounted for since the transit averages are over ion trajectories holding ψ_* fixed. Therefore, it is the boundary layer in the ψ_* variables that now matters. Consequently, the corresponding $\Delta\theta$ should therefore be estimated from (12) to scale as $\sqrt{\varepsilon / S_0}$, thereby recovering the above mentioned $\nu_B S / \varepsilon$ estimate for the effective collision frequency. We provide the further comparison of our technique against that of {{43 Shaing, KC

1992}} in Appendix B to show that their treatment of momentum conservation is responsible for their obtaining a different result that corresponds to the $\sqrt{\varepsilon S_0}$ estimate for $\Delta\theta$.

In brief, the banana regime ion heat conductivity we derive accounts for the presence of a strong radial electric field. As this electric field is inevitably present in tokamak regions such as a pedestal, it is this newly derived expression that has to be used there instead of the conventional formula. Moreover, as the parallel ion flow is altered by the electric field, we expect the bootstrap current to be enhanced in the pedestal.

Appendix A The integral $\int d^3vg$

When evaluating the parallel ion flow we used (73) to neglect one of the integrals on the right side of (72). Integrals of this type do not appear in the conventional case and require special treatment as presented here.

We start by switching to W and λ variables,

$$\int d^3vg = 4\pi\int \frac{dWd\lambda BW}{SB_0\left(v_{\|}+u\right)}g = -4\pi\int dWd\lambda g\frac{\partial\left(v_{\|}+u\right)}{\partial\lambda}, \tag{A.1}$$

where (48) is used to obtain the integral in the expression on the right side of (A.1). Before integrating by parts it is convenient to rewrite (A.1) as

$$\int d^3vg = -4\pi\int dWd\lambda g\frac{\partial\left[\left(v_{\|}+u\right)-\sqrt{2W/S_0}\right]}{\partial\lambda}$$

$$= -4\pi\int dWd\lambda\sqrt{2W/S}g\frac{\partial\left(\sqrt{1-\lambda B/B_0}-1\right)}{\partial\lambda}. \tag{A.2}$$

Then, observing that

$$\int_p d\lambda g\frac{\partial\left(\sqrt{1-\lambda B/B_0}-1\right)}{\partial\lambda} = \left(\sqrt{1-\lambda B/B_0}-1\right)g\Big|_{\text{freely passing}}^{\text{trapped-passing}}$$

$$-\int_p d\lambda\left(\sqrt{1-\lambda B/B_0}-1\right)\frac{\partial g}{\partial\lambda}, \tag{A.3}$$

and, using that $g = 0$ at the trapped-passing boundary as well as $\lambda \to 0$ for the freely passing particles, we transform (A.2) into

$$\int d^3 vg = 4\pi \int dWd\lambda \sqrt{2W/S_0}\left(\sqrt{1-\lambda B/B_0}-1\right)\frac{\partial g}{\partial \lambda}. \tag{A.4}$$

Next, we insert (49) into (A.4) to find

$$\int d^3 vg = \frac{4\pi IM}{\Omega_0 T^2 \sqrt{S}}\frac{\partial T}{\partial \psi}\int dWd\lambda W\left(W - T\sigma/M\right)\left(\frac{\sqrt{1-\lambda B/B_0}-1}{\left\langle\sqrt{1-\lambda B/B_0}\right\rangle}\right). \tag{A.5}$$

Replacing the λ variable with κ^2 using (22), along with the observation that

$$d\lambda = \frac{2\varepsilon d\kappa^2}{\kappa^2 + 2\varepsilon}, \tag{A.6}$$

equation (A.5) becomes

$$\int d^3 vg \approx \frac{8\pi IM\varepsilon}{\Omega_0 T^2 S}\frac{\partial T}{\partial \psi}\int dWd\kappa^2 \frac{W\left(W - T\sigma/M\right)}{\kappa^2 + 2\varepsilon}\times$$

$$\frac{\left[\sqrt{1-\kappa^2 \sin^2\left(\theta/2\right)} - \sqrt{1+\kappa^2 S_0/2\varepsilon}\right]}{\left\langle\sqrt{1-\kappa^2\sin^2\left(\theta/2\right)}\right\rangle}, \tag{A.7}$$

where (19) is used for $\left(v_{||} + u\right)$ and the κ^2 integral is only over the passing $\left(0 < \kappa^2 < 1\right)$ region.

To leading order

$$\frac{\left[\sqrt{1-\kappa^2 \sin^2\left(\theta/2\right)} - \sqrt{1+\kappa^2/2\varepsilon}\right]}{\left\langle\sqrt{1-\kappa^2\sin^2\left(\theta/2\right)}\right\rangle} \approx \frac{\pi\left[1-\sqrt{1+\kappa^2/2\varepsilon}\right]}{2E\left(\kappa\right)}, \tag{A.8}$$

where the elliptic function in the denominator changes from $\pi/2$ at $\kappa = 0$ to 1 at $\kappa = 1$. Our goal is to demonstrate that integral (A.1) is small in $\sqrt{\varepsilon/S}$. For this purpose we can replace $E(\kappa)$ with $E(0) = \pi/2$ since this does not change the order of the estimate for (A.1). Thus, the integral over κ^2 in (A.7) is approximately evaluated to give

$$\int_0^1 \frac{d\kappa^2}{\left(\kappa^2 + 2\varepsilon/S\right)} \frac{\pi\left[1 - \sqrt{1 + \kappa^2 S/2\varepsilon}\right]}{2E(\kappa)} \approx$$

$$-\sqrt{\frac{S}{2\varepsilon}} \int_0^1 \frac{d\kappa^2\left(\sqrt{2\varepsilon/S} - \sqrt{\kappa^2 + 2\varepsilon/S}\right)}{\left(\kappa^2 + 2\varepsilon/S\right)} \approx 2\sqrt{\frac{S}{2\varepsilon}}\,. \tag{A.9}$$

Hence, noticing that in (A.7) the integral (A.9) is preceded by a factor of ε/S we obtain the desired result (73) to leading order in the expansion parameter, $\int d^3vg \propto \sqrt{\varepsilon/S} \ll 1$.

Appendix B Comparison to Shaing and Hazeltine

Here, using a streamlined notation that ignores all irrelevant functions (such as B, Ω, W, I, $\partial f_M/\partial\psi$, …) we illustrate the subtle difference between our solution and that of Shaing and Hazeltine {{43 Shaing, KC 1992}} for the localized piece of the distribution function. To do so we define $\omega \equiv v_{\|} + u = S\left(v_{\|} + u_*\right) \equiv S\omega_*$, and employ $\omega^2/2S = \left(1 - \lambda\right)$ to write $\omega\,\partial\omega/\partial\lambda = -S$ and $\partial/\partial\lambda = -\left(S/\omega\right)\partial/\partial\omega$. Then, the constraint equation to be solved in our variables is

$$\left\langle \omega \right\rangle \partial g/\partial\lambda = \left\langle \omega\,\partial\omega/\partial\lambda \right\rangle, \tag{B.1}$$

while in the variables of Shaing and Hazeltine our equation becomes

$$\left(\left\langle \omega \right\rangle/\omega\right)\partial g/\partial\omega = 1\,, \tag{B.2}$$

where the derivatives and averages are to be taken at fixed ψ_*, and we write $\psi_* = \psi - v_{\|}$ and $f = \psi_* - \psi - u + g = g - \omega$. In terms of f we find the localized piece of the distribution function to be

$$\partial f \big/ \partial \omega = \left(\omega \big/ \left\langle \omega \right\rangle \right) - 1 \,. \tag{B.3}$$

Shaing and Hazeltine write $f_{SH} = -\psi + g_{SH}$ and solve

$$\left(\left\langle \omega \right\rangle \big/ \omega \right) \partial g_{SH} \big/ \partial \omega = C \tag{B.4}$$

and rewrite this equation in terms of f_{SH} to obtain

$$\partial f_{SH} \big/ \partial \omega = C \left(\omega \big/ \left\langle \omega \right\rangle \right) - \partial \psi \big/ \partial \omega \,, \tag{B.5}$$

where C is a constant and the ω derivative of ψ must be performed holding $\psi_* = \psi - v_{\|}$ fixed. To perform this derivative Shaing and Hazeltine implicitly assume that they can replace ψ by $\psi + u_*$ since $\partial u_* \big/ \partial \omega = 0$ at fixed ψ_*. Consequently, they find

$$\partial \psi \big/ \partial \omega = \partial \left(\psi + u_* \right) \big/ \partial \omega = \partial \left(v_{\|} + u_* \right) \big/ \partial \omega = \partial \omega_* \big/ \partial \omega = 1 \big/ S \,. \tag{B.6}$$

As a result, they obtain

$$\partial f_{SH} \big/ \partial \omega = C \left(\omega \big/ \left\langle \omega \right\rangle \right) - 1 \big/ S \,. \tag{B.7}$$

The boundary condition gives them $C = 1 \big/ S$ and the localized piece of the distribution function becomes

$$\partial f_{SH} \big/ \partial \omega = \left[\left(\omega \big/ \left\langle \omega \right\rangle \right) - 1 \right] \big/ S \,, \tag{B.8}$$

which has the extra $1/S$ factor when compared to (B.3). This extra $1/S$ results from u_* being implicitly inserted, instead of including the u factor by employing $f_{SH} = -\psi - u + g_{SH}$ at the outset. By inserting the u at the start we are making use of the energy conservation property of the ion-ion collision operator, namely $C\left\{uv^2 f_M\right\} = 0$. Inserting u_* later gives an error because $C\left\{u_* v^2 f_M\right\} \neq 0$ since u_* has hidden velocity space dependence through $\phi'\left(\psi_*\right)$. The erroneous $1/S$ factor gets squared in the Shaing and Hazeltine evaluation of the ion heat flux, thereby accounting for the difference between our result and theirs.

Bibliography

Figures

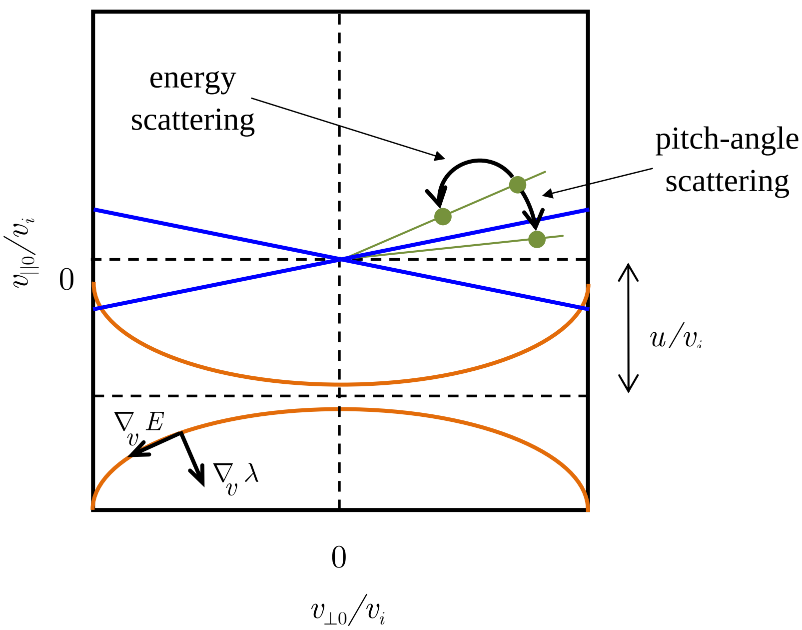

FIG 1. The trapped-passing boundary in the core (upper curves) and pedestal (lower curves) cases. In the pedestal, energy scattering can take an ion across the trapped passing boundary, whereas in the core this transition is solely due to pitch-angle scattering. Therefore, we employ a new set of variables, W and λ, such that the gradient of the former is parallel to the trapped-passing boundary in the pedestal and gradient of the latter is perpendicular to it. As a result, our model collision operator (37)-(38) involves the $\partial/\partial\lambda$ terms only. Notice that in the pedestal, the trapped region axis of symmetry is shifted by u/v_i from its core counterpart. As the equilibrium Maxwellian is still centered at $v_{||}$=0, we therefore expect neoclassical ion heat flux to decay exponentially for u/v_i>1.

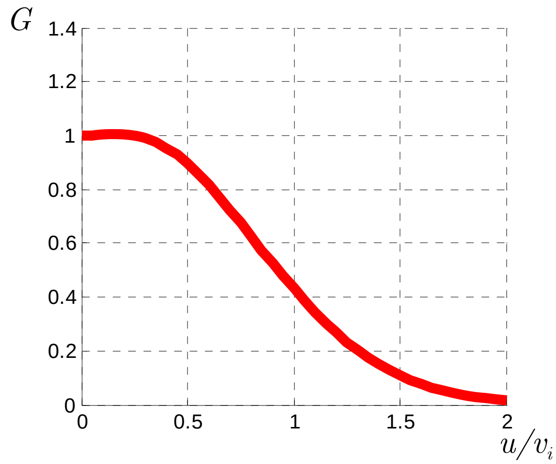

FIG 2. Normalized banana regime ion heat flux as a function of the equilibrium electric field.

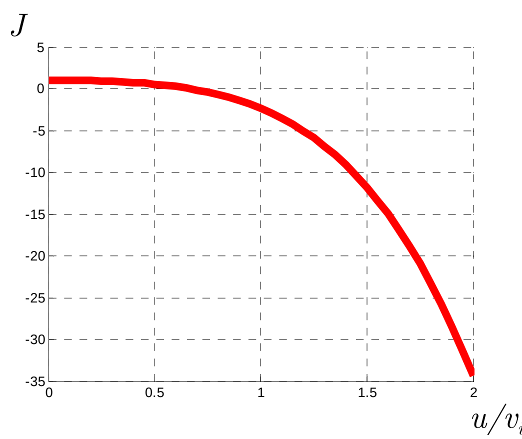

FIG 3. Normalized poloidal ion flow as a function of the equilibrium electric field.