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# THEORY OF TWO POINT CORRELATION FUNCTION IN A VLASOV PLASMA

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*Abstract*

A self-consistent theory of phase space granulations, called "clumps", has been derived. These fluctuations are produced when regions of different phase space density are mixed by the fluctuating electric fields. The source term and turbulent scattering operator for these fluctuations are obtained through a renormalization of the one and two point equations for a Vlasov plasma. The proper treatement of the singular behaviour exhibited by the two-point equation, coupled with the self-consistent approach leads to a number of significant changes compared to previous formulations. We consider throughout the case of electrostatic turbulence. Our solution method is based on the concept of two disparate time scales which allow us to treat the equal time two point equation as an initial condition for its two time counterpart. The picture of a "test" clump emerges quite naturally within such a framework. The source term for the clump correlation function is identified and certain intrinsic properties determined. We give physical interpretations to the coefficients in the renormalization and demonstrate their role in energy and momentum conservation. The theory is reminiscent of Fokker-Planck analysis with which we draw numerous parallels.

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## I. INTRODUCTION

It has been theoretically predicted that non wave-like fluctuations, called clumps[1–5], are an integral element in Vlasov turbulence. These particle-like modes can be viewed as phase space granulations arising from the incompressible nature of the flow. Since the Vlasov equation conserves phase space density along particle orbits, regions of different density cannot interpenetrate. This imperfect mixing leads to a graininess of the distribution function. The resulting potential spectrum can rearrange the density gradients and in the process regenerate the turbulence. Qualitatively one can argue that if the phase space volume of a clump is sufficiently small, then the particles within the clump will be scattered turbulently as a group. This group will persist for a characteristic time period (the clump lifetime[2]) before the orbits of the individual members diverge. Thus one can view a clump as a macroparticle whose effective charge decreases with time. If the spectrum is to be self sustaining this decay has to be balanced by a source. The problem can therefore be analyzed in two steps: the first seeks the characteristic lifetime of these fluctuations while the second investigates their source. The original derivation[2] of this phenomenon relied heavily on intuition and heuristic arguments. Clearly it would be desirable to investigate this mechanism using a more systematic and hopefully rigorous procedure. As such the principal aim of this paper is to present a self-consistent renormalization of the Vlasov equation, treating the "clump" problem within such a framework. The derivation and solution of the equation for the two-point correlation function plays a central role in the theory.

A number of self-consistent renormalizations have been proposed in which the effect of clumps has been neglected or not explicitly dealt with. In particular Orzag and Kraichnan[10], Dubois and Espedal[11], and Krommes[13,14] have applied various versions of the direct interaction approximation[8,9] to the Vlasov problem. Similarly Rudakov and Tsytovich[16], developing the work of Kadomstev[15], have obtained analogous equations. None of these renormalizations have adequately described the clump problem. Our approach, which is structurally similar, develops the work of Dupree and Tetreault[6] to include self-consistency and a contribution from a "discrete" quantity such as clumps. This important contribution leads to a set of equations whose physical content and properties are quite different from other renormalized theories. If, however, we neglect the clump contribution the equations can be shown to reduce, in the appropriate limit, to weak turbulence theory[13].

Mathematically we can trace the origin of these fluctuations in the following way. Consider a renormalized "linear" equation for the fluctuation $\delta f$:

$$\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x_1} + C_{11}\right)\delta f(1) = -\frac{q}{m}\delta E(1)\frac{\partial}{\partial v_1}\langle f(1)\rangle \tag{1}$$

Here $\delta E$ is the fluctuating electric field, $\langle f\rangle$ the average distribution and $C_{11}$ (which depends only on $\langle\delta f\delta f\rangle$) is a selective summing of a certain infinite subset of non-linear terms. Physically it accounts for the perturbation of $\delta f$ away from its ballistic orbit plus other non-linear effects. In the absence of such a renormalization conventional perturbation analysis gives rise to a resonance denominator $(\omega - kv)$, where $\omega$ and $k$ describe a wave $\exp i(kx - \omega t)$ and $v$ is the particle velocity. This resonance, which is fundamental to the damping and growing of waves also leads to time secularities in the individual terms of the perturbation solution. These secularities occur becuase of the infinite interaction time, between particle and wave, implied by the vanishing of the lowest order operator

$$\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x_1} \tag{2}$$

The earliest treatement[17] of such an operator $(C_{11})$ resulted in diffusion of $\delta f$ in velocity space. This followed quite naturally from quasi-linear theory where the average distribution also obeyed a diffusion equation.

While such an approach resolves the singular inversion of (2), other *secular* contributions arise. In particular the strong mode coupling and harmonic distortion at a wave particle resonance is not properly described. If we consider the distribution function as a superposition of velocity streams then each stream will be resonant with a wave going at the same speed. In such an interaction the stream quickly develops a number of (secular) higher harmonics with complicated, seemingly random, spatial dependance. These fluctuations (clumps) then get propagated ballistically at the stream speed. An analysis of such a problem could in principle be carried out in a one point frame. However, because of the stochastic nature of the resulting $\delta f$ it is more appropriate to investigate this contribution through a statistical framework which deals with the correlation of two points at close separation. In other words we need to develop a theory for the ensemble averaged two point correlation function $\langle\delta f(1)\delta f(2)\rangle$.

One can easily obtain an equation (incorrectly as we shall see) for the correlation function $\langle\delta f(1)\delta f(2)\rangle$ by multiplying (1) by $\delta f(2)$ and *vice versa* for the equation governing $\delta f(2)$. Ensemble averaging we get

$$\left(\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}+C_{11}+C_{22}\right)\langle\delta f(1)\delta f(2)\rangle = \\ -\frac{q}{m}\langle\delta E(1)\delta f(2)\rangle\frac{\partial}{\partial v_1}\langle f(1)\rangle-\frac{q}{m}\langle\delta E(2)\delta f(1)\rangle\frac{\partial}{\partial v_2}\langle f(2)\rangle \tag{3}$$

The lowest order operator in a spatially homogeneous system is

$$\frac{\partial}{\partial t}+(v_1-v_2)\frac{\partial}{\partial x_-} \tag{4}$$

($x_-=x_1-x_2$). When this operator vanishes a secularity (singularity) occurs. In this case the divergence is due to two points experiencing the same forces so that their *relative* orbit is secular. As such one would expect the renormalization to account for the interaction of two points which are very close to each other. If we take (3) as our renormalization we find that the left-hand side operator states that two points will always diffuse independantly even when their spatial and velocity separation are extremely small. On physical grounds this cannot be correct and we would expect some terms which specifically correlate the interaction between points 1 and 2. Let us call these $C_{12}$ and $C_{21}$ so that (3) becomes

$$\left(\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}+C_{11}+C_{22}+C_{12}+C_{21}\right)\langle\delta f(1)\delta f(2)\rangle = \\ -\frac{q}{m}\langle\delta E(1)\delta f(2)\rangle\frac{\partial}{\partial v_1}\langle f(1)\rangle-\frac{q}{m}\langle\delta E(2)\delta f(1)\rangle\frac{\partial}{\partial v_2}\langle f(2)\rangle \tag{5}$$

We derive an equation of this form in Sec. III. Let us rewrite (5) as

$$\left(\frac{\partial}{\partial t}+T_{12}\right)\langle\delta f(1)\delta f(2)\rangle = S \tag{6}$$

where $S$ represents the right-hand side of (5) and $T_{12}$ represents the renormalization plus the convective terms. We show that in the relative coordinate system $x_-, v_-$ ($v_\pm = v_1 \pm v_2, x_\pm = x_1 \pm x_2$), $T_{12}\to 0$ as $x_-, v_- \to 0$ while $S$ does not. Consequently $\langle\delta f(1)\delta f(2)\rangle$ is a very peaked function of $\{x_-, v_-\}$. The difference between (5) and (3), which represents the clump portion of the correlation function, occurs in a very locallized region of velocity space where the $C_{ij}$ terms dominate $v_-\partial/\partial x_-$. It is clear that (3), and therefore (1), does not contain this information. Thus we must conclude that there exist an important set of terms in the one point formulation which are not summed by the renormalization describing (1). Indeed we will see that the clump contribution can also be viewed as a *secular* element arising from a set of "incoherent" terms which are nominally of second order in the perturbation analysis. If $f^c$ ("coherent") is the solution to (1), the total solution must contain an added contribution $\tilde{f}$ ("incoherent") which generates the cross operators ($C_{ij}$): we therefore write $\delta f$ as

$$\delta f = f^c + \tilde{f} \tag{7}$$

While different regimes of turbulence have been characterized in the literature, we will be primarily concerned with the so called weak turbulence limit. By which we mean that the spectrum autocorrelation time ($\tau_c$) is much less than the trapping time ($\tau_{tr}$). $\tau_c$ and $\tau_{tr}$ are characterized by $\simeq (k\Delta v_{ph})^{-1}$ and $\simeq (kv_{tr})^{-1}$ where $v_{tr}^2 \simeq D\tau_{tr}$. $v_{tr}$ is the trapping width in velocity space, $\Delta v_{ph}$ is the spread in phase velocity of the fluctuations, $k$ is the average wavenumber while $D$ is the diffusion coefficient of quasi-linear theory. $q$ and $m$ are the particle charge and mass. These two time scales are closely related to another physical concept: if the "clump" is treated as a macro-particle of typical width $v_{tr}$ then $\tau_{tr}$ is the slow or "long" time scale associated with the decay of clump structure. $\tau_c$, on the other hand, represents the fast or "short" time scale which is associated with the ballistic motion of the centre of mass of the clump. These time scales have to be disparate for the concept of a clump as a *test* particle to be meaningful. If the condition $\tau_c \ll \tau_{tr}$ is satisfied then the decay of the clump will occur on a much slower time scale than the decay of the (two time) autocorerelation function. It is then approriate and expedient to handle the problem in a manner similar to the test particle model of Rosenbluth and Rostoker[18]. In Fourier space, if the clump generates a spectrum $\langle\tilde{\phi}^2\rangle_{k\omega}$ then the total shielded potential is given by

$$\langle\phi^2\rangle_{k\omega} = \frac{\langle\tilde{\phi}^2\rangle_{k\omega}}{|\epsilon_{k\omega}|^2} \tag{8}$$

where $\epsilon_{k\omega}$ is the non-linear dielectric which we derive in Sec. II. The symbol $\langle AB\rangle_{k\omega}$ is a Fourier transform on the relative coordinate $x_1$–$x_2$ and $t_1$–$t_2$ (where we have assumed temporal and spatial homogeneity).

We start in Sec. II with the derivation of a renormalized, self consistent, one point equation for an infinite spatial and temporally homogeneous electrostatic plasma. We introduce the incoherent contribution $\tilde{f}$ as an initial condition. The properties of the resulting equations are analyzed in the framework of conservation laws such as energy and momentum. In the long wavelength limit the "collision" operator reduces to a perturbed Fokker-Planck operator which conserves energy and momentum. An unperturbed version of this collision operator leads to a Lenard-Balescu like equation for the average distribution[1]

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial v}F-\frac{\partial}{\partial v}D\frac{\partial}{\partial v}\right)\langle f\rangle=0 \tag{10}$$

Here the drag ($F$) is due to the reaction of the shielding cloud on the "discrete" clump while the diffusion ($D$) results from the shielded clump spectrum. Sec. III, continues in the same vien with a derivation of the two point equation. We make use of the two time scaling ($\tau_{tr} \gg \tau_c$) to *decouple* the two time, and equal time two point equations. The result is a Markovian theory in which we use the equal time equation as an initial value for the two time equation. The analysis is carried to nominally second order in the electric field strength. The important property of phase space conservation ($T_{12} \to 0$ as $x_-, v_- \to 0$) is retained in the final equation: this result is independant of the Markovian assumption.

We can compare, schematically, the equations we derive to previous formulations in the following way. Dupree's original theory[2] and subsequent papers[19,20] considered the basic equation

$$\left(\frac{\partial}{\partial t}+T^0_{12}\right)\langle\delta f\delta f\rangle=S^0 \tag{11}$$

The zero superscripts refer to stochastic acceleration variables. For example $T^0_{12}$ was the diffusion in the relative coordinate system

$$T^0_{12}\simeq\frac{\partial}{\partial v_-}D_-\frac{\partial}{\partial v_-} \tag{12}$$

($D_- = D_{11}+D_{22}-D_{12}-D_{21}$), while $S^0$ was the $\langle\delta E\delta f\rangle\partial/\partial v\langle f\rangle$ term evaluated through the approximation $f=f^c$ only:

$$S^0\simeq D\frac{\partial^2}{\partial v_1\partial v_2}\langle f(1)\rangle\langle f(2)\rangle \tag{13}$$

The self-consistent approach which treats $\tilde{f}$ *on par* with $f^c$ changes (11) to

$$\left(\frac{\partial}{\partial t}+T^0_{12}+T^s_{12}\right)\langle\delta f\delta f\rangle=S^0+S^s \tag{14}$$

$T^s_{12}$ contains a number of complicated terms arising from the perturbation of the medium through the coupling of $\delta f$ to the background fluctuations. A systematic analysis of these contributions is carried out in the long wavelength limit where numerous cancellations between these terms and the $T^0_{12}$ operator are demonstrated on the basis of momentum and energy conservation.

The analysis of the source $S = S^0 + S^s$ is investigated in Sec. IV. A useful identification is made between the source and the relaxation of the average distribution. On the basis of this identificaton the following properties emerge. For a one species, one dimensional plasma in which $\partial/\partial t\langle f\rangle = 0$ the source term (which now resembles a Lenard-Balescu operator) is also zero. This result is directly related to the idea that in a one dimensional problem electron-electron (or ion-ion) collisions cannot relax the average distribution because of momentum constraints. If *local* momentum conservation is imposed (i. e. no waves), one can further show that for $\partial/\partial t\langle f\rangle \neq 0$ the source term is negative! Important cases exist where $S$ is positive, non zero. For example in a two species plasma where the distributions have opposite slopes (ions stationary, electrons drifting) or for a spectrum containing normal modes of the system. The latter ensures that the one dimensional collision operator is non zero, and in this case the procedure can be viewed as a correction[21] to quasi-linear theory.

To complete the analysis we require an equation for the *two time* correlation function since spectral functions such as $\langle\phi^2\rangle_{k\omega}$ require a knowledge of $\langle\phi(t_1)\phi(t_2)\rangle_k$. This last quantity appears in the evaluation of the $C_{ij}$ operators. Our basic equation is obtained quite simply by taking (1) and multiplying by $\delta f(t_2)$ to obtain

$$\left(\frac{\partial}{\partial t_1} + v_1\frac{\partial}{\partial x_1} + C_{11}\right)\langle\delta f(t_1)\delta f(t_2)\rangle = -\frac{q}{m}\langle\delta E(t_1)\delta f(t_2)\rangle\frac{\partial}{\partial v_1}\langle f\rangle \tag{15}$$

This equation is valid for $t_1 \geq t_2 \geq 0$ and is solved with the solution to (14) as an initial condition. Eqs. (15) and (5) underline our approach and solution technique. We have neglected the cross operators in (13) but not in (14). Physically this approximation is related to the idea that the clumping phenomena is intrinsically an *equal time* mechanism. It is only when two particles see the same electric field at the same point in space *and* time that a strong correlation will exist between them. Furthermore this effect is a *secular* contribution arising from the steady state (or time asymptotic) solution of (5). Thus in principle we could solve (15) with the cross terms but we would need to look at the solution as $t_1, t_2 \rightarrow \infty$ with $t_1 - t_2 \ll \tau_c$. Instead we treat the initial value problem which considers the equal time and two time equations as independant entities. In such an approach the equal time equation generates the incoherent response which then gets propagated through what, we will show, is essentially a ballistic operator to obtain its fast spectral dependance.

In Sec. V, we consider the formal solutions to the set of equations (14) and (15). We can anticipate some of the results in the following intuitive way. The distribution $f$ is conserved along a particles orbit.

Thus the value of $f$ at two neighbouring points may be quite different since these points might originally have been widely separated. Let $g_0(v, v_0, t)$ be the Green's function which solves the equation governing the average distribution function

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial v}F - \frac{\partial}{\partial v}D\frac{\partial}{\partial v}\right)g_0(v, v_0, t) = \delta(v - v_0) \tag{16}$$

and consider a small volume of phase space $x_-$, $v_-$ located at $x, v$. We define $\tau_{cl}$ as the characteristic e-folding time of the solution to (14) (i.e $\tau_{cl} \simeq T_{12}^{-1}$). Physically, if we follow the orbits of two points located about $x$, $v$ back in time, all the particles within $x_-$, $v_-$ will move together for a time $\tau_{cl}(x_-, v_-)$ (at which point they will be at coordinate $v_0$). Further back in time the orbits will have diverged and the particles will move independantly. Thus the density in the volume $x_-$, $v_-$ at time $t$ and position $v$, is approximately equal to the density of the average distribution at an earlier time $t - \tau_{cl}$ and position $v_0$. The coordinate $v_0$ is distributed according to the Green's function $g_0$ thus we can write for the fluctuations at $v, t$

$$\langle \delta f \delta f \rangle = \int dv_0 g_0(v, v_0, \tau_{cl}) f_0^2(v_0, t - \tau_{cl}) - f_0^2(v, t) \tag{17}$$

If $(D\tau_{cl})^{1/2} \ll \Delta v_{ph}$, $v_{th}$, where $v_{th}$ is the "thermal" or characteristic velocity associated with the average distribution we can expand $g_0$ to obtain the operator relation

$$\int dv_0 g_0(v, v_0, \tau_{cl}) \simeq 1 + \tau_{cl}\left[\frac{\partial}{\partial v}D\frac{\partial}{\partial v} - \frac{\partial}{\partial v}F\right] \tag{18}$$

The clump contribution is obtained by subtracting the solution to (3) from (17). If the characteristic e-folding time of (3) is $\tau_{tr}$ then the same arguments lead to

$$\langle \tilde{f}\tilde{f} \rangle \simeq [\tau_{cl} - \tau_{tr}]\left[\frac{\partial}{\partial v}D\frac{\partial}{\partial v} - \frac{\partial}{\partial v}F\right]f_0^2 \tag{19}$$

We can write (19) as $[\tau_{cl} - \tau_{tr}][S^0 + S^s]$ where $S^0$ is the diffusive part of the source and $S^s$ is the friction term. We obtain an expression similar to (19) in Sec. V. We must remember that (19) is an equal time result and to obtain spectral functions we need the two time version of $\langle \tilde{f}\tilde{f}|0\rangle$. We show that $\langle \tilde{f}\tilde{f}|t\rangle$ is obtained by propagating (19) through $g_k(v, v_0, t)$ which is a spatially *inhomogeneous* generalization of the $g_0(v, v_0, t)$ operator. In the long wave-length limit $g_k$ is a ballistic operator renormalized by terms which are equivalent to a simple iterative solution of a Fokker-Planck equation.

This system of equations is extremely complicated and at all stages we attempt to present models which explain the underlying physics. To this aim the picture of clumps being generated by the mixing of the average gradients is extremely useful. While the existence of such a mechanism can easily be justified on physical grounds some confusion has arisen on the magnitude, hence importance, of such an effect. In particular we examine the conclusion reached by Dubois *et al.*[11] who in their treatment of a version of renormalized equations for the Klimontovich system conjectured that these fluctuations were down an order of $\phi^2$ compared to the coherent response. The nominal ordering of the expansion is fully investigated and we show how to recover the correct ordering and source term in the limit of small $x_-, v_-$.

In conclusion we summarize the salient features in this work. A self-consistent renormalization of the one point and two point equations in a Vlasov plasma is performed through a procedure analogous to the direct interaction approximation. The singular element arising from phase space conservation is treated within the framework of the renormalization. Our equations are similar to those in Ref. [2] and [11]. They differ from Ref. [2] in that self-consistency is included in the formulation. Many aspects, however, of the underlying "clump" model remain the same. The equations in Ref. [11] are similar in that they contain many, but not all, of the terms (necessary for conservation laws) which are generated through our approach. If we neglect the "clump" contribution then the equations reduce to the "coherent approximation" described by Krommes and Kleva[13]. Our solution method is based on the concept of two disparate time scales which allow us to treat the equal time two point equation as an initial condition for its two time counterpart. The picture of a "test" clump emerges quite naturally within such a framework. The source term for the clump correlation function is identified and certain intrinsic properties investigated. We give physical interpretations to the coefficients in the renormalization, and demonstrate their role in energy and momentum conservation. Finally, we wish to add that the concepts and techniques proposed in this paper are of more than academic value. For example we have used a simplified version of these equations to investigate the stability boundary of ion-acoustic turbulence[7]. In a forthcoming publication we show that the theory predicts a clump spectrum that regenerates (a nonlinear instability) at electron drift velocities which are appreciably *below* those needed for the onset of linear instability.

# II. ONE POINT EQUATION

## A. One point Renormalization

Our starting point is the time honoured Vlasov equation coupled with Poisson's equation:

$$\begin{aligned} &(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{q}{m}E(x,t)\frac{\partial}{\partial v})f(x,v,t) = 0 \\ &\frac{\partial}{\partial x}E(x,t) = 4\pi n_0 q \int dv[f(x,v,t)v) - \delta(v)] \end{aligned} \tag{20}$$

$f(x, v, t)$ is the distribution function, $q$, $m$, $x$, and $v$ are the charge, mass, position, and velocity. $n_0$ is the density of the uniform background of particles with charge $q$.

If one considers the Vlasov equation as describing a fictious plasma in which the discreteness parameters $(n^{-1}, q, m)$ approach zero in such a way that $mn$, $qn$, and $nkT$ remain constant, then it is clear that this system exhibits an infinite number of degrees of freedom. We therefore seek to deal with statistical averages of the distribution function, covariance and higher order correlations. We will use $\langle \ldots \rangle$ to represent this average, which is interpreted as an ensemble average over a large number of realizations.

We write the fields as the sum of a mean plus a fluctuation:

$$\begin{aligned} f(x,v,t) =&\langle f(x,v,t)\rangle + \delta f(x,v,t) \\ E(x,t) =&\langle E(x,t)\rangle + \delta E(x,t) \end{aligned} \tag{21}$$

where $\langle \delta f\rangle = \langle \delta E\rangle = 0$. Furthermore we will assume spatial homogeneity so that the ensemble average becomes synonymous to a spatial average. In that case $\langle E\rangle = 0$ (due to charge neutrality), and $\langle f(x, v, t)\rangle = f_0(v, t)$.

To simplify the analysis we will consider a one dimensional plasma of length $L$ and proceed to the infinite case once we obtain the renormalized equations. (The multidimensional case, with weak inhomogeneities is a straightforward extension.) We expand the fluctuating part of the field and distribution function in a Fourier series

$$\begin{aligned} \delta f(v,x,t) &= \sum_k f_k(v,t)\exp ikx \qquad (k = \frac{n\pi}{L}) \\ \delta E(x,t) &= \sum_k E_k(t)\exp ikx \end{aligned} \tag{22}$$

Eq. (20) becomes

$$\frac{\partial f_k(t)}{\partial t} + ikvf_k(t) + \frac{q}{m}E_k(t)\frac{\partial f_0}{\partial v} + \frac{q}{m}\frac{\partial}{\partial v}\sum_{k'} E_{k'}(t)f_{k-k'}(t) = 0$$

$$ikE_k(t) = 4\pi ne \int dv f_k(v,t) \tag{23}$$

$$E_k(t) = -ik\phi_k(t)$$

Conventional perturbation analysis assumes that there exists some ordering parameter $\lambda$ ($\ll 1$) which allows the solution to be written as a power series in $\lambda$:

$$f_k(t) = \lambda f_k^{(1)}(t) + \lambda^2 f_k^{(2)}(t) + \cdots$$

$$E_k(t) = \lambda E_k^{(1)}(t) + \lambda^2 E_k^{(2)}(t) + \cdots \tag{24}$$

The coefficients of the series represent succesive improvements to the previous order solution. In such an approach the non-linear term does not appear in the first order solution being nominally of second order. It is well known that expansions in terms of the resulting "free" or ballistic propagator exhibit un-acceptable time secularities. The neglect of the non-linear contribution to the linearized result is cumulative so that the level of fluctuations can be quite weak while still translating into a sizeable secular contribution. The goal of the renormalization is to extract a "collision" operator out of the non-linear term and incorporate that in the "linearized" result as a remedy. Of the infinite set of non-linear terms we will only retain those which have the same phase ("phase coherent") as the driven mode $f_k$. Let us call the coherent portion of the non-linear term, $\int dt' C_k(t-t')f_k(t')$, where $C_k$ contains the amplitude of the fluctuations but no phase information. We rewrite the Vlasov equation as

$$\frac{\partial f_k(t)}{\partial t} + ikvf_k(t) + \frac{q}{m}E_k(t)\frac{\partial f_0}{\partial v} + \int_0^t dt' C_k(t-t')f_k(t') =$$
$$-\lambda\left(\frac{q}{m}\frac{\partial}{\partial v}\sum_{k'} E_{k'}(t)f_{k-k'}(t) - \int_0^t dt' C_k(t-t')f_k(t')\right) \tag{25}$$

where the difference between the non-linear term and $C_k$ is assumed to be an order smaller than the rest of the equation. We now reinstate the perturbation expansion and associate $\lambda$ with the electric field amplitude.

Equating order by order we get

$$\frac{\partial f_k^{(1)}(t)}{\partial t}+ikvf_k^{(1)}(t)+\frac{q}{m}E_k^{(1)}(t)\frac{\partial f_0}{\partial v}+\int_0^t dt'C_k(t-t')f_k^{(1)}(t')=0 \tag{26}$$

$$\frac{\partial f_{k-k'}^{(2)}(t)}{\partial t}+ikvf_{k-k'}^{(2)}(t)+\frac{q}{m}E_{k-k'}^{(2)}(t)\frac{\partial f_0}{\partial v}+\int_0^t dt'C_{k-k'}(t-t')f_k^{(2)}(t')=$$
$$-\frac{q}{m}E_k^{(1)}(t)\frac{\partial}{\partial v}f_{k'}^{(1)*}(t)-\frac{q}{m}E_{k'}^{(1)*}(t)\frac{\partial}{\partial v}f_k^{(1)}(t) \tag{27}$$

and

$$\int_0^t dt'C_k(t-t')f_k^{(1)}(t')=\left(\frac{q}{m}\frac{\partial}{\partial v}\sum_{k'}(E_{k'}^{(1)}f_{k-k'}^{(2)}+E_{k-k'}^{(2)}f_{k'}^{(1)}+\cdots)+(E_{k'}^{(1)}f_{k-k'}^{(1)})\right)_{Phase\ Coherent} \tag{28}$$

where we have included factors up to second order. It is important to note that only the $E^{(1)}f^{(2)}$ and $E^{(2)}f^{(1)}$ terms will give the correct phase dependance for $C_k$. The last term cannot contribute phase coherently since the fluctuation $f_{k-k'}^{(1)}$ cannot be decomposed into ones driven by $k$ and $k'$. This is illustrated by the right-hand side of (27) where, of the infinite set of non-linear terms, we have only retained the subset which when iterated in (28) will give terms proportional to $f_k(t)$ or $E_k(t)$ We anticipate this last observation by writing

$$\int_0^t dt'C_k(t-t')f_k(t')=\int_0^t dt'C_k^f(t-t')f_k(t')+\frac{q}{m}\int_0^t dt'E_k(t')\frac{\partial}{\partial v}C_k^\phi(t-t') \tag{29}$$

Eq. (26) is solved by *defining* the "coherent" ($f_k^c$) and "incoherent" ($\tilde{f}_k$) responses through the following separation:

$$\frac{\partial f_k^{c(1)}(t)}{\partial t}+ikvf_k^{c(1)}(t)+\int_0^t dt'C_k^f(t-t')f_k^{c(1)}(t')=$$
$$-\frac{q}{m}E_k^{(1)}(t)\frac{\partial f_0}{\partial v}-\frac{q}{m}\int_0^t dt'E_k^{(1)}(t')\frac{\partial}{\partial v}C_k^\phi(t-t') \tag{30}$$

and

$$\frac{\partial \tilde{f}_k^{(1)}(t)}{\partial t}+ikv\tilde{f}_k^{(1)}(t)+\int_0^t dt'C_k^f(t-t')\tilde{f}_k^{(1)}(t')=0 \tag{31}$$

with initial conditions $f_k^{c(1)}(t=0)=f_k^{c(1)}(0)$ and $\tilde{f}_k^{(1)}(t=0)=\tilde{f}_k^{(1)}(0)$. Note that (31) and (32) add up to the original equation (26). This division tracks linear response theory. $f^c$ is associated with the induced fields which shield perturbations in the plasma. In this case, however, the ballistic operator is renormalized through what we will show is a Fokker-Planck operator and the average distribution through $C^{\phi}$. We can neglect the initial condition $f_k^c(0)$ by setting the lower limit of the $dt'$ integral in (30) to $-\infty$. This presumes that the coherent initial condition decays very quickly $O(\tau_c)$. We will show, however, that the ballistic contribution arising from the popagation of the initial condition $\tilde{f}_k(0)$ decays on a much longer time scale so that such a stratagem is not particulary useful. Instead we define a backward equation for $f_k(-t)$ where we use $C_k(t-t') = C_k(|t-t'|)\mathrm{sgn}(t-t')$. $\tilde{f}_k(0)$ *cannot* be obtained in an iterative way. In fact the exact structure of $\tilde{f}_k(0)$ is far too complicated and in practice we will only need the correlation function $\langle \tilde{f}(0)\tilde{f}(0)\rangle_k$. This quantity, which is extremely localized in velocity, can be obtained from a solution of the *equal time* two point equation for small separation.

If we define the Green's function $g_k(t)$ through

$$\begin{aligned} &\frac{\partial g_k(t)}{\partial t} + ikvg_k(t) + \int_0^t dt' C_k^f(t-t')g_{k'}(t') = 0 \quad t>0 \\ &g_k(t{=}0^+) = 1 \\ &g_k(t) = 0 \qquad t<0 \end{aligned} \tag{32}$$

and the relevant transforms (for the fast time scale) as

$$\begin{aligned} g_{k\omega} &= \int_0^{\infty} dt g_k(t) \exp i\omega t \\ f_k(t) &= \sum_{\omega} f_{k\omega} \exp -i\omega t \qquad (\omega = \frac{m\pi}{T}) \end{aligned} \tag{33}$$

we can synthesise the one point results in the following form

$$f_{k\omega}^{(1)} = \tilde{f}_{k\omega}^{(1)} + f_{k\omega}^{c(1)} = \tilde{f}_{k\omega}^{(1)} + \frac{q}{m} g_{k\omega} ik\phi_{k\omega}^{(1)} \frac{\partial \overline{F}_{k\omega}}{\partial v} \tag{34a}$$

where

$$-i(\omega - kv + iC_{k\omega}^f)g_{k\omega} = 1, \qquad C_{k\omega}^f = \int_0^{\infty} dt C_k^f(t) \exp i\omega t \tag{34b}$$

and

$$\overline{F}_{k\omega} = f_0 + C_{k\omega}^{\phi}, \qquad C_{k\omega}^{\phi} = \int_0^{\infty} dt C_k^{\phi}(t) \exp i\omega t \tag{34c}$$

are respectively, the renormalized Green function and "equivalent " background distribution[13]. If we use Poisson's equation we get

$$\phi^{(1)}_{k\omega} = \frac{4\pi ne}{|k|^2} \int dv f^{(1)}_{k\omega} \tag{34d}$$

$$\tilde{\phi}^{(1)}_{k\omega} = \frac{4\pi ne}{|k|^2} \int dv \tilde{f}^{(1)}_{k\omega} \tag{34e}$$

which yields

$$\phi^{(1)}_{k\omega} = \frac{\tilde{\phi}^{(1)}_{k\omega}}{\epsilon_{k\omega}} \tag{34f}$$

$\epsilon_{k\omega}$ is the non-linear dielectric given by

$$\epsilon_{k\omega} = 1 - i\frac{\omega_p^2}{|k|^2} \int dv g_{k\omega} k \frac{\partial}{\partial v} \overline{F}_{k\omega} \tag{34g}$$

Given the set (34) the next step in the calculation is to obtain explicit expressions for the coefficients in the collision operators $C^f_{k\omega}$ and $C^\phi_{k\omega}$. This requires the quantities $f^{(2)}_{k-k'}(t)$ and $E^{(2)}_{k-k'}(t)$, which are functions of $\phi^{(1)}_{k'}$ and $f^{(1)}_{k'}$. Because $C_k$ is constructed to contain no phases the modes at $k'$ will only appear as products of the form $\sum_{k'} \langle \phi^{(1)*}_{k'}(t') f^{(1)}_{k'}(t)\rangle$ and $\sum_{k'} \langle \phi^{(1)*}_{k'}(t') \phi^{(1)}_{k'}(t)\rangle$. Assuming time stationarity we can write, for example, (in the limit $L, T \to \infty$)

$$\sum_{k'} \langle \phi^{(1)}_{k'*}(t) f^{(1)}_{k'}(t')\rangle = \sum_{k',\omega'} \langle \phi^{(1)*}_{k'\omega'} f^{(1)}_{k'\omega'}\rangle \exp -i\omega'(t-t') \to \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} \langle \phi f\rangle_{k'\omega'} \exp -i\omega'(t-t') \tag{35}$$

where

$$\langle \phi f\rangle_{k'\omega'} = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx e^{i(\omega' t - k' x)} \langle \phi(x+x', t+t') f(v_2, x', t')\rangle \tag{36}$$

We can use the set (34) to obtain

$$\langle \phi^*_{k'\omega'} f_{k'\omega'}\rangle = \frac{q}{m} \frac{\langle \tilde{\phi}^*_{k'\omega'} \tilde{\phi}_{k'\omega'}\rangle}{|\epsilon_{k'\omega'}|^2} i g_{k'\omega'} k' \frac{\partial}{\partial v} \overline{F}_{k'\omega'} + \frac{\langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k\omega}\rangle}{|\epsilon_{k'\omega'}|^2} \epsilon_{k'\omega'} \tag{37a}$$

Making use of (35), (37a) is equivalent to

$$\langle \phi f \rangle_{k'\omega'} = \frac{\langle \tilde{\phi}^2 \rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} i g_{k'\omega'} k' \frac{\partial}{\partial v} \overline{F}_{k'\omega'} + \frac{\langle \tilde{\phi} \tilde{f} \rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} \epsilon_{k'\omega'} \tag{37b}$$

where we have expressed all quantities in terms of velocity moments of the incoherent correlation function $\langle \tilde{f}\tilde{f} \rangle_{k'\omega'}$. This quantity can be obtained quite simply by noting that

$$\langle \tilde{f}(1)\tilde{f}(2) \rangle_{k'\omega'} = \left( \langle \tilde{f}(1,\omega')\tilde{f}(2,0) \rangle_{k'} + \langle \tilde{f}(1,0)\tilde{f}(2,-\omega') \rangle_{k'} \right) \tag{38}$$

where

$$\tilde{f}(\omega') = \int_0^\infty dt f(t) \exp i\omega' t \tag{39}$$

Using Eq. (31) we immediately get

$$\langle \tilde{f}(1)\tilde{f}(2) \rangle_{k'\omega'} = [g_{k'\omega'}(1) + g^*_{k'\omega'}(2)] \langle \tilde{f}(0)\tilde{f}(0) \rangle_{k'} \tag{40}$$

Note that if the turbulence is weak this reduces to the familiar result of ballistic propagation. If the fluctuations are localized in velocity such that $\langle \tilde{f}\tilde{f} \rangle \simeq \delta(v_1 - v_2)$, (40) reduces to $2Re\, g_{k'\omega'} \langle \tilde{f}\tilde{f} \rangle_{k'}$. (37) and (40) are obtained in a slightly different way in Sec. V., where we treat the *total* response $\langle f(t) f(t) \rangle$ as an initial value for $\langle f(t_1) f(t_2) \rangle$.

Proceeding with the renormalization we partition the expression for $f^{(2)}_{k-k'}$ into $f^{c(2)}_{k-k'}(t)$ and $\tilde{f}^{(2)}_{k-k'}(t)$ through

$$f^{c(2)}_{k-k'}(t) = \int_0^t dt' g_{k-k'}(t-t') \times i\frac{q}{m}\left( i(k-k')\frac{\partial}{\partial v}(f_0 + C^{\phi}_{k-k'}(t'))\phi^{(2)}_{k-k'}(t') - k'\frac{\partial f^{(1)}_k(t')}{\partial v}\phi^{(1)*}_{k'}(t') + k\frac{\partial f^{c(1)*}_{k'}(t')}{\partial v}\phi^{(1)}_k(t') \right) \tag{41}$$

and

$$\tilde{f}^{(2)}_{k-k'}(t) = \int_0^t dt' g_{k-k'}(t-t') i\frac{q}{m} k \frac{\partial \tilde{f}^{(1)*}_{k'}(t')}{\partial v}\phi^{(1)}_k(t') \tag{42}$$

we have assumed that the only initial condition is $\tilde{f}^{(1)}_k(0)$. Note that $\tilde{f}^{(2)}_{k-k'}$ is a *perturbative* quantity. It represents the modification, on the ballistic time scale, of the *non perturbative* quantity $\tilde{f}^{(1)}_{k'}$ through the action of the electric field $\phi^{(1)}_k$.

Using (34),(36) and (37) coupled with Poisson's equation in (28) we get, after transforming, the following set of coefficients:

$$C_{k\omega}f_{k\omega} = C^{f}_{k\omega}f_{k\omega} - \frac{q}{m}ik\frac{\partial}{\partial v}C^{\phi}_{k\omega}\phi_{k\omega} \tag{43}$$

where $C^{f}_{k\omega}f_{k\omega}$ is defined through:

$$C^{f}_{k\omega}f_{k\omega} \equiv -\frac{\partial}{\partial v}\Big(D_{k\omega}\frac{\partial}{\partial v} - F_{k\omega}\Big)f_{k\omega} - \frac{\partial}{\partial v}\Big((d^{f} + d^{t})\frac{\partial}{\partial v} - \frac{\partial}{\partial v}(\mathcal{F}^{f} + \mathcal{F}^{t})\Big)\bar{f} \tag{44}$$

The various symbols in equation (44) are given by

$$D_{k\omega} = \frac{q^2}{m^2}\sum_{k',\omega'} g_{k-k',\omega-\omega'}k'k'\frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \tag{45}$$

$$F_{k\omega} = -\frac{q}{m}\sum_{k',\omega'} i\Big(k'\tilde{f}^{(2)}_{k-k',\omega-\omega'}\frac{\tilde{\phi}^{(1)}_{k'\omega'}}{\epsilon_{k'\omega'}} + (k-k')\tilde{f}^{(1)}_{k'\omega'}\frac{\tilde{\phi}^{(2)}_{k-k',\omega-\omega'}}{\epsilon_{k-k',\omega-\omega'}}\Big)\frac{1}{f_{k\omega}} \tag{46}$$

$$\begin{aligned} d^{t}\frac{\partial\bar{f}}{\partial v} = &-\frac{q^2}{m^2}\sum_{k',\omega'} ik'(k-k')\frac{\tilde{\phi}^{(1)}_{k'\omega'}}{\epsilon_{k'\omega'}}\frac{\tilde{\phi}^{(2)}_{k-k',\omega-\omega'}}{\epsilon_{k-k',\omega-\omega'}} \\ &\times\Big(\frac{\partial\overline{F}_{k-k',\omega-\omega'}}{\partial v}g_{k-k',\omega-\omega'} + \frac{\partial\overline{F}_{k'\omega'}}{\partial v}g_{k'\omega'}\Big) \end{aligned} \tag{47}$$

In the above equations $\tilde{f}^{(2)}_{k-k',\omega-\omega'}$ and $\tilde{\phi}^{(2)}_{k-k',\omega-\omega'}$ are *defined* through

$$\begin{aligned} \tilde{f}^{(2)}_{k-k',\omega-\omega'} &= \frac{q}{m}g_{k-k',\omega-\omega'}ik\phi^{(1)}_{k\omega}\frac{\partial\tilde{f}^{(1)*}_{k'\omega'}}{\partial v} \\ \tilde{\phi}^{(2)}_{k-k',\omega-\omega'} &= \frac{4\pi ne}{|k-k'|^2}\int dv\tilde{f}^{(2)}_{k-k',\omega-\omega'} \end{aligned} \tag{48}$$

The remaining terms satisfy

$$d^f \frac{\partial \overline{f}}{\partial v} = \frac{q^2}{m^2}\omega_p^2 \sum_{k',\omega'} ik' \frac{k-k'}{|k-k'|^2} \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \int dv' \frac{g_{k-k',\omega-\omega'}}{\epsilon_{k-k',\omega-\omega'}} k' \frac{\partial f_{k\omega}}{\partial v'}$$

$$\times \left( \frac{\partial \overline{F}_{k-k',\omega-\omega'}}{\partial v} g_{k-k',\omega-\omega'} + \frac{\partial \overline{F}_{k'\omega'}}{\partial v} g_{k'\omega'} \right) \tag{49}$$

$$(\mathfrak{F}^l + \mathfrak{F}^f)\overline{f} = -\frac{q}{m}\omega_p^2 \sum_{k',\omega'} k' \frac{k-k'}{|k-k'|^2} \frac{\langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'} \rangle}{\epsilon^*_{k'\omega'}\epsilon_{k-k',\omega-\omega'}} \int dv' g_{k-k',\omega-\omega'} \frac{\partial f_{k\omega}}{\partial v'} \tag{50}$$

The $C^{\phi}_{k\omega}$ operator is defined by:

$$C^{\phi}_{k\omega}\phi_{k\omega} \equiv (\beta_{k\omega} + \gamma_{k\omega} + \delta_{k\omega})\phi_{k\omega} \tag{51}$$

where

$$\beta_{k\omega} = \frac{q^2}{m^2} \sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} g_{k-k',\omega-\omega'} \frac{\partial}{\partial v} g^*_{k'\omega'} \frac{\partial}{\partial v} \overline{F}^*_{k'\omega'} \tag{52}$$

$$\gamma_{k\omega} = \frac{q^2}{m^2} \sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \omega_p^2 \frac{i(k-k')}{|k-k'|^2} \int dv' \frac{g_{k-k',\omega-\omega'}}{\epsilon_{k-k',\omega-\omega'}} \frac{\partial}{\partial v'} g^*_{k'\omega'} \frac{\partial}{\partial v'} \overline{F}^*_{k'\omega'}$$

$$\times \left( \frac{\partial \overline{F}_{k-k',\omega-\omega'}}{\partial v} g_{k-k',\omega-\omega'} + \frac{\partial \overline{F}_{k'\omega'}}{\partial v} g_{k'\omega'} \right) \tag{53}$$

$$\delta_{k\omega} = \frac{q}{m}\omega_p^2 \sum_{k',\omega'} k' \frac{k-k'}{|k-k'|^2} \frac{\langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'} \rangle}{\epsilon^*_{k'\omega'}\epsilon_{k-k',\omega-\omega'}} \int dv' g_{k-k',\omega-\omega'} \frac{\partial}{\partial v'} g^*_{k'\omega'} \frac{\partial}{\partial v'} \overline{F}^*_{k'\omega'} \tag{54}$$

It is helpful to use the first two columns of Fig. (1.0), which indicate diagramatically the steps in the iterative process, to see the origin of the various terms in the renormalization. Time stationarity and an implicit assumption of steady state are used throughout the formulation.

One can obtain the spatially (and temporally) homogeneous counter part of the $C_{k\omega}$ "collision" operator by considering the equation for the average distribution function $f_0$:

$$\frac{\partial}{\partial t} f_0 = -\frac{q}{m}\frac{\partial}{\partial v} \sum_{k',\omega'} ik' \langle \phi^*_{k'\omega'} f_{k'\omega'} \rangle \tag{55}$$

We can use (37) to recast (55) into

$$\frac{\partial f_0}{\partial t} = -C_0 f_0 = -\frac{\partial}{\partial v} J(v)$$

where

$$J(v) = \omega_p^2 \frac{q}{m} \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} \frac{k'k'}{|k'|^2} \frac{1}{|\epsilon_{k'\omega'}|^2} \int dv' (\langle \tilde{\phi} \tilde{f}(v) \rangle_{k'\omega'} Re[g_{k'\omega'}(v') \frac{\partial \overline{F}_{k'\omega'}}{\partial v'}] - \langle \tilde{\phi} \tilde{f}(v') \rangle_{k'\omega'} Re[g_{k'\omega'}(v) \frac{\partial \overline{F}_{k'\omega'}}{\partial v}]) \quad (56)$$

In direct analogy to (44) this can also be written as

$$C_0 f_0 = -\frac{\partial}{\partial v} D \frac{\partial}{\partial v} \bar{f} + \frac{\partial}{\partial v} F \bar{f} \quad (57)$$

The physical interpretations and properties of these collision operators are investigated in the next section.

## B. Properties and Interpretation of the "Collision" Integrals

The spatially homogeneous collision operator, $C_0$, is readily interpreted in terms of conventional Fokker-Planck analysis. For example it is fairly easy to see that for a discrete particle spectrum

$$\langle \tilde{f}(1) \tilde{f}(2) \rangle = n^{-1} \delta(x_1 - x_2 - v_1(t_1 - t_2)) \delta(v_1 - v_2) \langle f \rangle \quad (58)$$

and taking $\overline{F}_{k\omega} \simeq f_0$, $g_{k\omega} \simeq 1/(\omega - kv + i\epsilon)$ (56) would reduce to the Lenard-Balescu collision intergral.

In this case, however, the source of fluctuations is the "discrete" clump. The first term in (56) is the dynamical friction due to the shielding cloud acting on the discrete fluctuation. The second is the diffusion of Quasi-Linear theory. We note that one of the effects of the renormalization is to introduce additional friction and diffusion coefficients in the equation for the the average distribution function. For example the friction term instead of being driven by the gradient of $f_0$ only, contains contributions from the gradient of $C^{\phi}_{k\omega}$. By the same token, the diffusive process rearranges $\overline{F}_{k\omega}$ rather than $f_0$.

If we define

$$N \equiv n \int dv, \qquad M \equiv nm \int dv\, v, \qquad E \equiv \frac{1}{2} nm \int dv\, v^2$$

as respectively the number, momentum and energy operators it is straightforward to show that

$$\frac{\partial}{\partial t}Nf_0 = \frac{\partial}{\partial t}Mf_0 = \frac{\partial}{\partial t}Ef_0 = 0$$

The first two properties are self evident from the structure of equation (56). The third follows from the equation of continuity: $\int dv(\omega' - k'v)Re g_{k'\omega'}\langle\tilde{f}\tilde{f}\rangle_{k'} = 0$. Total energy conservation (kinetic plus potential) is achieved[7] by relaxing the adiabatic assumption, $\partial/\partial t f_0 \simeq 0$, in the derivation of the one point results (34).

The operator $C^f_{k\omega}$ has been written in the suggestive form of (44) to emphasize the physical origins of the individual terms. One can gain considerable insight by looking at the long wavelength limit of this expression. For simplicity, and to make contact with previous theories[22], we will consider the discrete particle case where the self correlation is given by (58). We will once again assume $\overline{F}_{k\omega} \simeq f_0$ and take $g_{k\omega} \simeq 1/(\omega - kv + i\epsilon)$. These assumptions do not make any of the underlying physics less general: in particular the conservation properties described by (64) can be proved *independantly* of these assumptions.

We show in the Appendix that in the limit of $k \to 0$, with $\lim_{k,\omega\to 0} f_{k\omega} = f_0^1$, equation (44) for $C^f_{k\omega}$ reduces to

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial v}F - \frac{\partial}{\partial v}D\frac{\partial}{\partial v}\right)f_0^1 = \left(-\frac{\partial}{\partial v}(\mathcal{F}^t + \mathcal{F}^f) + \frac{\partial}{\partial v}(d^t + d^f)\frac{\partial}{\partial v}\right)f_0^0 \tag{59}$$

We can reproduce this equation from a simple linearization of the Lenard-Balescu equation. Let us write the average distribution function as a series expansion

$$f_0 = f_0^0 + f_0^1 + \dots \tag{60}$$

$f_0^0$ represents the background distribution, while $f_0^1$ represents the $0'th$ fourier component of the fluctuations. Typically, $f_0^1$ could be a disturbance due to a very long wave-length fluctuation (such as an eigen-mode of the dispersion relation). The presence of this small amplitude disturbance implies that the dielectric $\epsilon$ will have a perturbative component due to $f_0^1$. That is we can write

$$\epsilon_{k\omega} = \epsilon^0_{k\omega} + \chi^1_{k\omega} + \dots = 1 + \chi^0_{k\omega} + \chi^1_{k\omega} + \dots \tag{61}$$

where $\chi$ is the standard succeptibility, defined by

$$\chi^1_{k\omega} = \frac{\omega_p^2}{|k|^2} \int dv \, \frac{k\partial/\partial v_1 f_0^1}{\omega - kv + i\delta} \tag{62}$$

The fluctuating fields can be expanded in a similar fashion. In this case the "1" superscript might represent the result of ballistic motion,while the "2" superscript the distortion to these orbits due to the presence of $f_0^1$. We note that in the spirit of a test particle picture we would expect second order perturbed quantities to be made up of two distinct physical processes: the first would affect the "test" particle while the second would affect the "field" particle. Schematically if $\phi \simeq \phi^{test} e^{-r/\lambda}$ then the perturbation will affect *both* $\phi^{test}$ and the coherent (shielding) response.

If we linearize

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial v} F f_0 + \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0 \tag{63a}$$

$$F f_0 = \frac{q}{m} \sum_{k',\omega'} k' \frac{\epsilon^i_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} \langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'} \rangle \qquad D = \sum_{k',\omega'} \frac{q^2}{m^2} \pi \delta(\omega' - k'v) \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \tag{63b}$$

according to the above prescription we will immediately recover (59) with the same *coefficients* as obtained through the renormalization. The details of the calculation are presented in the Appendix.

The physical interpretation of the terms is now simple: $D$ and $F$ are the standard diffusion and friction coefficients in the absence of the perturbation. $d$ is the modification to the diffusion coefficient due to the perturbation. As previously indicated it consists of two terms, one decribing the rearrangement of the test particles (perturbation of the orbits: $d^t$), the other describing the distortion of the shielding cloud or field particles ($d^f$). $\mathfrak{F}$ is the modification to the drag coeifficient, and likewise has two components.

This distinction is important in terms of energy and momentun conservation. It is straight-forward to show by taking the $v^2$, and $v$ moments of equation (59) that the conservation properties are achieved through the following cancellation of individual terms:

$$\begin{aligned} (N;M;E)\Big(\frac{\partial}{\partial v} F f_0^1 - \frac{\partial}{\partial v} d^t \frac{\partial}{\partial v} f_0^0\Big) &= 0 \\ (N;M;E)\Big(\frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0^1 - \frac{\partial}{\partial v} \mathfrak{F}^t f_0^0\Big) &= 0 \\ (N;M;E)\Big(\frac{\partial}{\partial v} \mathfrak{F}^f f_0^0 - \frac{\partial}{\partial v} d^f \frac{\partial}{\partial v} f_0^0\Big) &= 0 \end{aligned} \tag{64}$$

Field and test perturbations balance independantly.

We are left with a physical picture of the operator $C^f_{k\omega}$: it describes the divergence of "test" particles away from their ballistic orbit due to their interaction with the electric fields of "field" particles. Because it is a self-consistent calculation these same particles, to conserve momentum and energy, act back on the plasma. In the absence of the "clump" contribution $\tilde{f}$, the only terms to survive are $D_{k\omega}$ and $d^f$. These coefficients have appeared in several theories[11-13] and are the "diffusion" and "polarization" terms of Krommes and Kleva[13]. In that limit energy and momentum conservation are obtained through the additional requirement $\epsilon^0_{k\omega} = \chi^1_{k\omega} = 0$. Clearly when $k \neq 0$ there are more complicated effects taking place. In particular the probing nature of the $k$ wave vector through the $k - k'$ convolution is not self evident. However we still believe these interpretations are helpful in understanding the fundamental actions of this operator.

The $C^\phi_{k\omega}$ operator unfortunately eludes such a straighforward interpretation. The beta term is the velocity equivalent of the drift wave "$\beta$" term which appears in Ref. [6]. Krommes and Kleva[13] have obtained the $\beta_{k\omega}$ and $\gamma_{k\omega}$ terms in what they refer to as a "coherent" approximation to e direct interaction approximation. They interpret these as a ponderomotive renormalization of the background distribution, while Dubois[12] refers to the same elements as sources of "quasi-particles". We have not been able to find a simple physical interpretation to these terms.

In one dimension, when we take $\overline{F}_{k\omega} \simeq f_0$ and $g_{k\omega} \simeq 1/(\omega - kv)$, the collision operator exhibits one further property; namely

$$\lim_{k,\omega \to 0} \qquad C_{k\omega} f_{k\omega} \to 0 \tag{65}$$

The various terms in the $C^f_{k\omega}$ operator cancel in the same pairs as in Eq. (64), while the $\beta_{k\omega}$, $\gamma_{k\omega}$ and $\delta_{k\omega}$ in the $C^\phi_{k\omega}$ operator can also be shown to pair, and cancel. This cancellation is easily reconcilled on physical grounds. Collision like processes cannot change the average distribution in one dimension since momentum constraints insure that an encounter between two particles moving at $v$ and $v'$ will result in the same division in velocity after the collision. In higher dimensions or for different mass encounters this is not the case. Equally, keeping the broadened resonance functions etc. leads to a non-zero operator since this is equivalent to taking *three body* encounters into account. Finally we point out that a plasma has the added capability, in the presence of a wave, of transmitting momentum through non-resonant interactions. This would also invalidate the previous considerations; the effect, however, is not included in our collision operator since we do not consider the zeroes of the dielectric.

## III. TWO POINT EQUATION

In this section we obtain equations for the equal time and two time two point equations. The main feature of the two point renormalization is to capture any contribution from the $E^{(1)}f^{(1)}$ terms which were excluded in Sec. II. We go to a two point formulation because it is only by squaring such terms that their phases can be made to cancel.

### A. Phase Space Conservation

Let us start by considering the exact two point equation (with spatial homogeneity)

$$\left(\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}\right)\langle\delta f(1)\delta f(2)\rangle=-\frac{q}{m}\langle\delta E(1)\delta f(2)\rangle\frac{\partial}{\partial v_1}\langle f\rangle \\ -\frac{q}{m}\frac{\partial}{\partial v_1}\langle\delta E(1)\delta f(2)\delta f(1)\rangle+(1\leftrightarrow 2) \tag{66}$$

A standard weak turbulence expansion would assume that $|\delta E|^2 \ll mv_{th}^2$, and use $\delta E$ as an expansion parameter. The linearized solution, which neglects the "third" order terms has a non-integrable singularity as $v_1$ approaches $v_2$. (For a stationary state the left hand side goes to zero while the right hand side does not.)

Consider the *exact* equation (containing the triplets): since the singular behaviour arises for small separation we can change to "+, −" coordinates, and neglect the "+" contribution. Using $x_\pm = x_1 \pm x_2$, $v_\pm = v_1 \pm v_2$, and $t_\pm = t_1 \pm t_2$, we have in the limit of $\{x_-,\ v_- \to 0\}$ the following property

$$\lim_{x_-,v_-\to 0}\quad \frac{\partial}{\partial t}\langle\delta f\delta f|x_-,v_-,t\rangle = S(x_-,t) \tag{67}$$

with

$$S=-\frac{q}{m}\langle\delta E(1)\delta f(2)\rangle\frac{\partial}{\partial v_1}\langle f\rangle-\langle\delta E(2)\delta f(1)\rangle\frac{\partial}{\partial v_2}\langle f\rangle \tag{68}$$

The non-linear terms cancel in the relative coordinate. Thus if the renormalization contains $C_{11}(\equiv C_k(1))$ and $C_{22}$ $(\equiv C_k(2))$ only, one of the properties of the *exact* equation indicates that there exist other contributions from the non-linear term which are, not only of the same order as $C_{11}$, but actually *cancel* these elements for small separation.

It is not hard to trace the origin of this behaviour. The Vlasov equation preserves phase space density along particle orbits and the singular behaviour is just an alternative way of formulating that same statement. Consider the exact distribution $f(x, v, t)$; the conservation property can be stated as

$$\frac{d}{dt} f(x(t), v(t), t) = 0 \tag{69}$$

where the differential is now taken along the particles orbit. Multiplying the above equation by $f$, ensemble averaging and integrating over the velocity coordinate we get

$$\int dv \left(\frac{\partial}{\partial t}\langle \delta f^2 \rangle + \frac{\partial}{\partial t}\langle f \rangle^2\right) = 0 \tag{70}$$

Using

$$\frac{\partial}{\partial t}\langle f \rangle = -\frac{q}{m}\frac{\partial}{\partial v}\langle \delta E \delta f \rangle \tag{71}$$

and integrating the second term by parts we get

$$\int dv \left(\frac{\partial}{\partial t}\langle \delta f^2 \rangle + 2\frac{q}{m}\langle \delta E \delta f \rangle \frac{\partial}{\partial v}\langle f \rangle\right) = 0 \tag{72}$$

This is consistent with Eq. (66) in the limit of $v_-, x_- \to 0$. The important point to note is that no perturbative scheme to *any* order will get rid of the singular effect. It is entrenched as a basic property of the equation. We can even go further and state that any approximate set of equations which does not conserve this property is incapable of describing small scale fluctuations in a plasma.

Thus one of the properties which the final renormalized result must exhibit is that the left-hand side of (67), which for finite $x_-, v_-$ we write as $(\partial/\partial t + T_{12})$, reduces to $\partial/\partial t$ as $\{x_-, v_- \to 0\}$. We see from (70) and (72) that $(\partial/\partial t + T_{12})$ preserves the square of the fluctuating part of the distribution while the right-hand side of (67) is related to the conservation of $\langle f \rangle^2$. Physically this is a natural division and allows us to identify $S$, defined through (68), as the source of fluctuations. For finite $x_-, v_-$ the left-hand side of (67) is some non-linear operator which acts on the fluctuations through the self consistent interactions of the turbulent electric fields set up by the fluctuations. This operator might destroy ( through turbulent diffusion, ballistic motion, etc.) the spectrum or enhance it through some kind of non-linear instability. The right-hand side of (67), on the other hand, does not act on the fluctuations directly, but through the indirect mechanism of changing the average distribution. When the gradients of the average distribution are modified a mixing process occurs as elements of phase space rearrange to generate the new average distribution. The rearrangement creates new fluctuations and a steady state can be envisioned as a result of the competition between creation (right-hand side) and destruction (left-hand side) of the fluctuations.

## B. Two Point Renormalization: Two Time

The two point renormalization is performed by taking the one point equation of Sec. II for $f_k(t_1)$, multiplying by $f_k^*(t_2)$ and ensemble averaging the result. In this case we will retain the non-linear terms proportional to $\lambda$ in (25) as our ultimate goal is to obtain an equal time equation for $\langle f_k(1) f_k(2)\rangle$. We will show that these terms (for $t_1 = t_2$) are part of the mechanism which generates the clump spectrum.

We write the equation for $\langle f_k(t_1) f_k^*(t_2)\rangle$ as:

$$\frac{\partial}{\partial t_1}\langle f_k(t_1)f_k^*(t_2)\rangle + ikv_1\langle f_k(t_1)f_k^*(t_2)\rangle + \frac{q}{m}\langle E_k(t_1)f_k^*(t_2)\rangle\frac{\partial f_0}{\partial v_1} = \\ -\int_0^{t_1} dt' C_{11}(k, t_1-t')\langle f_k(t')f_k^*(t_2)\rangle - \int_0^{t_2} dt' C_{12}(k, t_1-t', t'-t_2)\langle f_k(t')f_k^*(t_2)\rangle \tag{73}$$

where

$$\int_0^{t_2} dt' C_{12}(k, t_1-t', t'-t_2)\langle f_k^{(1)}(t')f_k^{(1)*}(t_2)\rangle \equiv \frac{q}{m}\frac{\partial}{\partial v_1}\sum_{k'}\langle E_{k'}^{(1)}(t_1)f_{k-k'}^{(1)}(t_1)f_k^{(2)*}(t_2)\rangle \tag{74}$$

Let us note the following points: (1) $C_{11}(k, t-t')$ is the $C_k$ operator of Sec. II. (2) In (74) the lowest order contribution to $f_k^*(t_2)$ is $f_k^{(2)*}(t_2)$. Here there is an implicit assumption that the phases of the terms with "(1)" superscripts are randomly phased so that the $\langle E^{(1)} f^{(1)} f^{(1)}\rangle$ term does not contribute.

We select, as before, only those terms out of $f_k^{(2)*}(t_2)$ whose phases will cancel the phases of $E_{k'}^{(1)}$ and $f_{k-k'}^{(1)}$. We can use the one point results of Sec. II, Eqs. (41) and (42), transcribed for the mode $k$ rather than $k-k'$. When this result is substituted in (74) coupled with Poisson's equation we obtain the *two time*, two point, equation. The domain of validity of the equation is $t_1 > t_2 \geq 0$.

Unless $t_1 \simeq t_2$ (with $t_1 + t_2 \gg \tau_{tr}$) we will consider the $C_{12}$ operator as a second order contribution: the two time equation reduces to a sum of one point equations and we recover the results of Sec. II In the Markovian limit one can show that the cross operators are a function of $\exp ik(x_- - v_+ t_-)$. Thus when $t_1 = t_2$ and $x_- \simeq 0$ these terms approach their one point counterparts and become central to the clumping mechanism. We return to this important difference between the equal and two time equations in Sec. V.

## C. Two Point Renormalization: Equal Time

To obtain the equal time equation we take the one point equation for $f(t_2)$, perform the same excercise as in the previous section, ensemble average, and add the result to (73). We use

$$\frac{\partial}{\partial t}\langle f(1,t)f(2,t)\rangle \equiv \langle f(1,t)\frac{\partial}{\partial t}f(2,t)\rangle + \langle f(2,t)\frac{\partial}{\partial t}f(1,t)\rangle$$

The next step is to take the limit $t_1, t_2 \to \infty$ for the arguments in the time integrals of the collision operators. This is consistent with our two time scaling procedure in which we assume that $\partial/\partial t_+ \ll \partial/\partial t_-$. We will assume that the resulting equations are still valid for weak departures from steady state and stationarity.

We expand field and distribution in a Fourier series in space and time as given by (22) and (33). Using (73) and taking the limit $t_1 = t_2 = \infty$ we get the following equal time equation:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + ik(v_1 - v_2)\right)\langle f_k(1)f_k^*(2)|t\rangle + \left(\sum_\omega (C_{11}^f + C_{12}^f)\langle f_{k\omega}(1)f_{k\omega}^*(2)|t\rangle + (1 \leftrightarrow 2)\right) = S_k \\ + \left(\frac{q}{m}ik\frac{\partial}{\partial v_1}\sum_\omega C_{11}^\phi\langle \phi_{k\omega}f_{k\omega}^*(2)|t\rangle - \frac{q}{m}i(k-k')\frac{\partial}{\partial v_1}\sum_\omega C_{12}^\phi\langle f_{k\omega}(1)\phi_{k\omega}^*|t\rangle + (1 \leftrightarrow 2)\right)\end{aligned} \tag{75}$$

Here $C_{11}$ is the $C_{k\omega}$ operator of Sec. II, and $S_k$ is the Fourier version of (68) which is (101) with the $\omega$ dependance integrated out. The intrinsic non-Markovian nature of the equation is apparent in (75); we have not, as yet, decoupled the slow and fast time scales. At the end of this section we present an approximation which allows such a simplification.

Eq. (75) is very similar to the product of two one point equations except for the bivariate operators which originate from the iteration of the incoherent terms. These are defined by

$$\begin{aligned}C_{12}^f\langle f_{k\omega}(1)f_{k\omega}^*(2)\rangle \equiv \left(\frac{\partial}{\partial v_1}D_{12} * \frac{\partial}{\partial v_2} - \frac{\partial}{\partial v_1}F_{12}*\right)\langle f_{k-k',\omega-\omega'}(1)f_{k-k',\omega-\omega'}^*\rangle \\ + \frac{\partial}{\partial v_1}\left(\langle (d_{12}^t * + d_{12}^f *)f_{k-k',\omega-\omega'}(1)\rangle\right)\frac{\partial \bar{f}}{\partial v_2}\end{aligned} \tag{76}$$

The "*" represents a convolution of the $\{k', \omega'\}$ sum with the correlation function at $\{k{-}k', \omega{-}\omega'\}$. That is

On a more quantitative basis it is helpful to see the origin of the various terms in the iterative process as they relate to Eqs. (28) and (74). We will use the following notation to differentiate terms which have two components. For example the terms in Eqs. (46), (47), (49), and (53) all consist of two parts. These will be written $F_{11}(1) + F_{11}(2)$, $d^t_{11}(1) + d^t_{11}(2)$, etc., where "(1)" refers to the first term in the parentheses and "(2)" to the second. Then the iteration of $f^{(2)}_{k-k'}$ and $\phi^{(2)}_{k-k'}$ in the first term of (28) yields $D_{11}$, $\beta_{11}$, $F_{11}(1)$, $d^t_{11}(1)$, $d^f_{11}(1)$, and $\gamma_{11}(1)$. The iteration of $\phi^{(2)}_{k-k',\omega-\omega'}$ in the second term of (28) yields $\mathcal{F}$, $\delta_{11}$, $F_{11}(2)$, $d^t_{11}(2)$, and $\gamma_{11}(2)$. Finally $D_{12}$, $F_{12}$, $d^t_{12}$, $d^f_{12}$, $\gamma_{12}$, and $\beta_{12}$ come from the iteration of $f^{(2)}_{k\omega}$ and $\phi^{(2)}_{k\omega}$ in (74). The steps of the iteration are illustrated diagramatically in Fig. **1.0**.

The Fourier series are defined over a finite interval of time $T$ and lenght $L$. For a spatially and temporally homogeneous system we can pass to the integral limit through the following transforms:

$$\lim_{L,T\to\infty} \qquad LT\langle f_{k\omega} f_{-k,-\omega}\rangle \to \langle ff\rangle_{k\omega} \tag{85}$$

and

$$\lim_{L,T\to\infty} \qquad LT\sum_{k',\omega'} |\phi_{k'\omega'}|^2 \langle f_{k-k',\omega-\omega'} f_{k'-k,\omega'-\omega}\rangle \to \int \frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi}\langle\phi^2\rangle_{k'\omega'}\langle ff\rangle_{k-k',\omega-\omega'} \tag{86}$$

In Sec. V. we discuss some of the aspects of proceeding to the integral limit of the $\omega$ transform, while at the same time isolating the secular behaviour arising from the "$t_+$" dependance.

To decouple the equal time and two time equations we will make a Markovian approximation. This is of course consistent with our assumption $\tau_c \ll \tau_{tr}$. We thus assume that for small separations $\langle f(1)f(2)\rangle_{k\omega}$ and $\langle f(1)f(2)\rangle_{k-k',\omega-\omega'}$ are strongly peaked about $(\omega - kv_+)$ and $(\omega-\omega'-(k-k')v_+)$ respectively. Coefficients in the renormalization are transformed in a manner illustrated by

$$\begin{aligned}
&\frac{\partial}{\partial v_1}\int d\omega D_{12} * \frac{\partial}{\partial v_2}\langle f(1)f(2)\rangle_{k-k',\omega-\omega'} = \\
&\quad -\frac{q^2}{m^2}\frac{\partial}{\partial v_1}\int d\omega\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi} g^*_{k\omega}(2)k'k'\frac{\langle\tilde{\phi}^2\rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2}\frac{\partial}{\partial v_2}\langle f(1)f(2)\rangle_{k-k',\omega-\omega'} \to \\
&\qquad -\frac{q^2}{m^2}\frac{\partial}{\partial v_1}\int\frac{dk'}{2\pi}\int\frac{d\omega'}{2\pi} g^*_{k'\omega'}(2)k'k'\frac{\langle\tilde{\phi}^2\rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2}\frac{\partial}{\partial v_2}\langle f(1)f(2)\rangle_{k-k'}
\end{aligned} \tag{87}$$

Notice that in the last expression $g^*_{k\omega}(2) \to g^*_{k'\omega'}(2)$ and that $\int d\omega\langle f(1)f(2)\rangle_{k-k',\omega-\omega'} \to \langle f(1)f(2)\rangle_{k-k'}$. Transformations of the type described by (87) allow us to recast the equal time equation into

$$D_{12} * \langle f_{k-k',\omega-\omega'}(1) f^*_{k-k',\omega-\omega'}(2)\rangle = -\frac{q^2}{m^2}\sum_{k',\omega'} g^*_{k\omega}(2) k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2}\langle f_{k-k',\omega-\omega'}(1) f^*_{k-k',\omega-\omega'}(2)\rangle \quad (77)$$

$$F_{12} * \langle f_{k-k',\omega-\omega'}(1) f^*_{k-k',\omega-\omega'}(2)\rangle = \frac{q}{m}\sum_{k',\omega'} ik' \frac{\tilde{\phi}_{k'\omega'}}{\epsilon_{k'\omega'}} \frac{\tilde{f}^{(2)*}_{k\omega}(2)}{f^*_{k-k',\omega-\omega'}(2)}\langle f_{k-k',\omega-\omega'}(1) f^*_{k-k',\omega-\omega'}(2)\rangle \quad (78)$$

$$\langle d^t_{12} * f_{k-k',\omega-\omega'}(1)\rangle \frac{\partial \bar{f}}{\partial v_2} = -\frac{q^2}{m^2}\sum_{k',\omega'} ik'k \frac{\tilde{\phi}_{k'\omega'}}{\epsilon_{k'\omega'}} \frac{\tilde{\phi}^{(2)*}_{k\omega}}{\epsilon^*_{k\omega}} g^*_{k\omega}(2) f_{k-k',\omega-\omega'}(1) \frac{\partial \overline{F}^*_{k\omega}}{\partial v_2} \quad (79)$$

$\tilde{\phi}^{(2)*}_{k\omega}$ and $\tilde{f}^{(2)*}_{k\omega}(2)$ are given by

$$\begin{aligned} \tilde{f}^{(2)*}_{k\omega} &= -\frac{q}{m} g^*_{k\omega}(2) i(k-k')\phi^*_{k-k',\omega-\omega'} \frac{\partial \tilde{f}^{(1)*}_{k'\omega'}}{\partial v_2} \\ \tilde{\phi}^{(2)*}_{k\omega} &= \frac{4\pi ne}{|k|^2}\int dv_3 \tilde{f}^{(2)*}_{k\omega}(3) \end{aligned} \quad (80)$$

Similarly $d^f_{12} *$ satisfies

$$\begin{aligned} \langle d^f_{12} * f_{k-k',\omega-\omega'}(1)\rangle \frac{\partial \bar{f}}{\partial v_2} &= \frac{q^2}{m^2}\omega_p^2 \sum_{k',\omega'} ik' \frac{k}{|k|^2} \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} g^*_{k\omega}(2) \frac{\partial \overline{F}^*_{k\omega}}{\partial v_2} \\ &\times \int dv_3 \frac{g^*_{k\omega}(3)}{\epsilon^*_{k\omega}} k' \frac{\partial}{\partial v_3}\langle f_{k-k',\omega-\omega'}(1) f^*_{k-k',\omega-\omega'}(3)\rangle \end{aligned} \quad (81)$$

The $C^\phi_{12}$ operator is defined through

$$C^\phi_{12}\langle \phi^*_{k\omega} f_{k\omega}(1)\rangle \equiv (\beta_{12} * + \gamma_{12} *)\langle f_{k-k',\omega-\omega'}(1)\phi^*_{k-k',\omega-\omega'}\rangle \quad (82)$$

where

$$\beta_{12} * = \frac{q^2}{m^2}\sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \frac{\partial}{\partial v_2} g^*_{k\omega}(2) \frac{\partial}{\partial v_2} \overline{F}^*_{k\omega} \quad (83)$$

$$\gamma_{12} * = -\frac{q^2}{m^2}\sum_{k',\omega'} k'k' \frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2}\omega_p^2 \frac{ik}{|k|^2} g^*_{k\omega}(2)\frac{\partial}{\partial v_2}\overline{F}^*_{k\omega}\int dv_3 \frac{g^*_{k\omega}(3)}{\epsilon^*_{k\omega}}\frac{\partial}{\partial v_3} g^*_{k\omega}(3)\frac{\partial}{\partial v_3}\overline{F}^*_{k\omega} \quad (84)$$

$$\left(\frac{\partial}{\partial t} + ikv_1 + C^f_{11}(k) + C^f_{12}(k) + (1 \leftrightarrow 2)\right)\langle f(1)f(2)\rangle_k = S(k)$$
$$+\left(\frac{q}{m}ik\frac{\partial}{\partial v_1}C^\phi_{11}(k)\langle\phi f(2)\rangle_k - \frac{q}{m}i(k-k')\frac{\partial}{\partial v_1}C^\phi_{12}(k)\langle f(1)\phi\rangle_k + (1 \leftrightarrow 2)\right) \tag{88}$$

We have explicitly indicated that the $C$ operators are a function of $k$ only. $S(k)$ is the integral limit of $S_k$ given by (103).

## D. Properties of the Two Point Equation

The first property we would like to investigate is that of phase space conservation. If we sum equation (75) over $k$ and take the $v_1 \to v_2$ limit (neglecting "+" dependances) one can trivially show that

$$\begin{aligned}
&\sum_{k\omega}[D_{11} + D_{12} + D_{21} + D_{22}] = 0\,, \qquad \sum_{k\omega}[F_{11}(1) + F_{12} + F_{21} + F_{22}(1)] = 0 \\
&\sum_{k\omega}[\beta_{11} + \beta_{12} + \beta_{21} + \beta_{22}] = 0\,, \qquad \sum_{k\omega}\left[d^t_{11}(1) + d^t_{12} + d^t_{21} + d^t_{22}(1)\right] = 0 \\
&\sum_{k\omega}\left[d^f_{11}(1) + d^f_{12} + d^f_{21} + d^f_{22}(1)\right] = 0\,, \qquad \sum_{k\omega}[\gamma_{11}(1) + \gamma_{12} + \gamma_{21} + \gamma_{22}(1)] = 0 \\
&\sum_{k\omega}[F_{11}(2) + F_{22}(2)] = 0\,, \qquad \sum_{k\omega}[\mathcal{F}_{11} + \mathcal{F}_{22}] = 0 \\
&\sum_{k\omega}[d^t_{11}(2) + d^t_{22}(2)] = 0\,, \qquad \sum_{k\omega}\left[d^f_{11}(2) + d^f_{22}(2)\right] = 0 \\
&\sum_{k\omega}[\delta_{11} + \delta_{22}] = 0\,, \qquad \sum_{k\omega}[\gamma_{11}(2) + \gamma_{22}(2)] = 0
\end{aligned} \tag{89}$$

Note that the summation over $\{k, \omega\}$ is equivalent to taking the limit $x_-$, $t_- \to 0$ and that (89) is a symbolic representation of terms operating on $\langle\delta f \delta f\rangle$ or $\langle\delta f \phi\rangle$. In affecting the cancellations of (89) the following trends appear. If a term contains two velocity derivatives in the minus coordinate the "11" term will cancel with its "21" counterpart and vice versa. If the term contains only one $v_-$ derivative, "11" will cancel with "22" and "21" (if any) will cancel with "12". Referring to Fig. (1.0), we note that the renormalization originates from three groups. The second group (which comes from allowing the perturbed electric field, $\phi^{(2)}$, to act back on the fluctuations) produces the elements which do not have a bivariate counterpart. These, as we see in (89), cancel "11" with "22".

The second property we which to examine is the behaviour of the bivariate terms for large separation in phase space. We use the the Markovian approximation. Consider, for example, the diffusion coefficients $D_{ij}$. If $v_- \simeq 0$ we can inverse Fourier transform these terms to get

$$\frac{\partial}{\partial v_-} D_-(x_-) \frac{\partial}{\partial v_-} \langle \delta f \delta f | x_-, v_-, t \rangle; \qquad D_-(x_-) = D_{11} + D_{22} - D_{12}(x_-) - D_{21}(x_-) \tag{90}$$

where for example

$$D_{12}(x_-) = \frac{q^2}{m^2} \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} \frac{ik'k' \langle \phi \rangle^2_{k'\omega'}}{(\omega' - k'v_2 - iC^f)} \exp(ik'x_-) \tag{91}$$

$D_-$ is a diffusion coefficient in the relative coordinate. From (91) the properties of $D_-$ become

$$\begin{aligned} D_- &\to 0, \qquad & x_- &\to 0; \\ D_- &\to 2D_{11}, \qquad & x_- &\to \infty \end{aligned} \tag{92}$$

One can understand (92) on the following physical grounds. Two particles which are close together in phase-space experience roughly the same forces and therfore move together even though their average coordinates $x_+$ and $v_+$ may change significantly. On the other hand if $|k_0 x_-| \gg 1$ (where $k_0$ is a measure of the spectrum width), then $D_{12}$ and $D_{21}$ are small and the particles diffuse independantly. In general one expects all the bivariate operators to exhibit a strong dependance on $x_-$ and possibly $v_-$. The latter appears as a Doppler shift in the Green function. The $x_-$ dependance appears through the convolution of the $\{k'\}$ sum with the correlation function at $\{k-k'\}$ since coefficients which have an $x_-$ dependance will transform through

$$\int dx_- e^{ikx_-} A_{12}(x_-) \langle \delta f \delta f | x_- \rangle \equiv \int \frac{dk'}{2\pi} A_{12}(k') \langle ff | k-k' \rangle \tag{93}$$

The cross operators describe the correlated motion between points 1 and 2. This may take the form of a drag, diffusion or other non-linear process. On the other hand it is physically clear that this correlated motion will dissapear for sufficiently large ($|k_0 x_-| > 1$) spatial distances.

## IV. SOURCE TERM

### A. General Properties of a Source Term

We wish to analyze some generic properties of the source as defined through (68). Equation (72) immediately shows that the source term is related to the rate of change of the average distribution function.The underlying mechanism is one of an increase in the level of fluctuations at the expense of the average distribution (and vice versa). For example if the average distribution is unstable it can relax by changing its shape to a more stable configuration. This new configuration is produced through a mixing of fluids of different density. In the process granulations are generated since these different densities cannot interpenetrate.

We can examine, more closely, the division implied by (72) in the case of a one dimensional plasma where normal mode interactions are neglected. The latter is an important constraint because it leads to

$$\frac{\partial}{\partial t}\overline{\langle \delta f \rangle^2} \simeq 0 \tag{94}$$

(The "—— " represents the average over velocity space.) This comes about since an unrenormalized collision operator of the Lenard-Balescu type goes to zero in one dimension, so that (72) reduces to (94). We already identified this property in Sec. II as the result of momentum constraints. However we can make an even stronger statement than (94). Let $\langle f \rangle = \langle f(v,0)\rangle + \Delta\langle f(v,t)\rangle$; $\langle f(v,0)\rangle$ is the initial value of the average distribution function while $\Delta\langle f(v,t)\rangle$ is the change in $\langle f \rangle$. Eq. (72) can be rewritten as

$$\int dv \Big(\frac{\partial}{\partial t}\Big(\langle \delta f^2\rangle + (\Delta\langle f \rangle)^2\Big) + \langle f(v,0)\rangle \frac{\partial}{\partial t}\Delta\langle f \rangle\Big) = 0 \tag{95}$$

If the fluid particles can only transfer momentum locally we can approximate the initial distribution by a Taylor series centered about some average coordinate

$$\langle f(v,0)\rangle \simeq a + bv \tag{96}$$

and equation (95) becomes

$$\int dv \big(\frac{\partial}{\partial t}(\langle \delta f^2\rangle + \Delta\langle f \rangle^2)\big) = 0 \tag{97}$$

We have used

$$a\frac{\partial}{\partial t}\overline{\Delta\langle f\rangle} = 0\,; \qquad b\frac{\partial}{\partial t}\overline{v\Delta\langle f\rangle} = 0 \tag{98}$$

which represent number and momentum conservation, to obtain (97).

If we integrate (97) over time, we get

$$\langle\overline{\delta f^2}\rangle = \langle\overline{\delta f^2(0)}\rangle - (\Delta\langle\overline{f}\rangle)^2 \tag{99}$$

$\delta f(0)$ represents the initial level of fluctuations. This last equation shows that not only does the level of fluctuations stay constant (94), but in one dimension it will *decrease* since the last term is positive definite.

The same arguments can be used to show that if there is an energy source, coupled to a situation where momentum constraints do not imply $\partial/\partial t\langle f\rangle = 0$, the fluctuations can increase. Consider for example a two species problem in an ion-acoustic regime. The energy source is the drifting electron maxwellian. In that case

$$b\frac{\partial}{\partial t}\overline{v\Delta\langle f_{ion}\rangle} = -b'\frac{\partial}{\partial t}\overline{v\Delta\langle f_{elec}\rangle}$$

since momentum can now be exchanged between the electrons and ions. This implies that

$$\langle\overline{\delta f_{ion}^2}\rangle = \langle\overline{\delta f_{ion}^2(0)}\rangle + b'\overline{v\Delta\langle f_{elec}\rangle} - \overline{\Delta\langle f_{ion}\rangle}^2 \tag{100}$$

If $b'$ is positive (i. e. the average distributions have *opposite* slopes) the velocity gradients can be used to generate a turbulent state where the level of fluctuations increases. Of course this state may also contain eigenmodes of the plasma and one has yet to demonstrate that these are any less efficient transport agents.

It is worth emphasizing that our discussion relies on the "localness" of the interaction. This assumption need not be true if the turbulent spectrum contains waves, since these can transport momentum through non-resonant interactions and expansion (96) would not be valid.

## B. One Species Source Term

We can obtain an expression for the source term by using (37) in the Fourier version of (68). The result is

$$S_{k'\omega'} = \Big(\frac{q^2}{m^2} k'k' \frac{\langle \tilde{\phi}^*_{k'\omega'} \tilde{\phi}_{k'\omega'} \rangle}{|\epsilon_{k'\omega'}|^2} g^*_{k'\omega'} \frac{\partial}{\partial v_2} \overline{F}_{k'\omega'} + ik' \frac{\langle \tilde{\phi}_{k'\omega'} \tilde{f}^*_{k'\omega'} \rangle}{|\epsilon_{k'\omega'}|^2} \epsilon^*_{k'\omega'} \Big) \frac{\partial}{\partial v_1} \langle f \rangle + (1 \leftrightarrow 2) \qquad (101)$$

we pass to the Fourier integral limit and write

$$S(k,\omega) = \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} \delta(\omega - \omega') \delta(k - k') S(k', \omega') \qquad (102)$$

where $S(k', \omega')$ is identical to (101) except that the spectrums are expressed in terms of the integral transforms, i.e. $\langle f_{k'\omega'} f_{k'\omega'} \rangle \rightarrow \langle ff \rangle_{k'\omega'}$.

If we take $\overline{F}_{k'\omega'} \simeq \langle f \rangle$, we can write the equal time version of (102) in the more symmetric form

$$S(k) = (D^0_{12}(k) + D^0_{21}(k)) \frac{\partial^2}{\partial v_1 \partial v_2} \langle f(1) \rangle \langle f(2) \rangle - (F^0_{12}(k) \frac{\partial}{\partial v_1} + F^0_{21}(k) \frac{\partial}{\partial v_2}) \langle f(1) \rangle \langle f(2) \rangle \qquad (103)$$

The zero superscripts mean that the terms are the Markovian version of the cross operators. For example $F_{12}$ can, in that limit, be written as

$$F^0_{12}(x_-) = \frac{q}{m} \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} ik' \frac{\langle \tilde{\phi} \tilde{f} \rangle_{k'\omega'}}{|\epsilon^*_{k'\omega'}|^2} \frac{\epsilon_{k'\omega'}}{\langle f \rangle} e^{ik'x_-} \equiv \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} R_{k'\omega'} e^{ik'x_-}$$

with

$$F^0_{12}(k) = \int \frac{dk'}{2\pi} \int \frac{d\omega'}{2\pi} \delta(k - k') R_{k'\omega'} \qquad (104)$$

We see that the single time version of the source term $S$ contains additional terms (in comparison to (13)) which originate from the inclusion of the incoherent fluctuation. These terms are important, since in one dimension and in the limit of weak turbulence (so that $Re\, g_{k\omega} \rightarrow \delta(\omega - kv)$) they cancel the source as defined through (4.3). This cancellation occurs for small separation (we need $F_{12}$ and $D_{12} \rightarrow F_{11}$ and $D_{11}$) and is directly related to the cancellation of these same terms in a Lenard-Balescu collision integral. This is striaght-forward to see by considering the expression for the current driving the average distribution, Eq (56). Our source, in the limit of small separation, is this same expression without the integral over $dk'$ and multiplied by $\partial / \partial v \langle f \rangle$. If we take $\overline{F} \simeq \langle f \rangle$ and $Re\, g_{k\omega} \simeq \delta(\omega - kv)$ then (56) is identically zero for *every* mode $k'$ (i.e. we do not need to sum over all modes) and the source likewise dissapears. We have already discussed the momentum considerations which lead to such a result. Moreover this is in agreement with (94), where the exact equations predicted that such a term should dissapear, since there is no relaxation of the background distribution. We must emphasize the neglect of any wave-like modes since these can lead to a non-zero relaxation of $\langle f \rangle$ even in one dimension.

## C. Two Species Source Term

We consider a two component plasma made of electrons and ions. The ions are no longer stationary, and participate in the mixing process. We want to obtain an expression for the source term in the equation for the "$i'th$ " species, and demonstrate how the source remains finite in that case. Once again one will clearly see the influence of momentum constraints on the problem.

Let $\phi^e$ and $\phi^i$ be the electron and ion potentials. The total, self consistent, plasma potential is $\phi$ $(= \phi^e + \phi^i)$. Through a simple extension of the procedure in Sec. II, the fields of the dressed ions and electrons can be calculated as

$$\begin{aligned} \phi^e_{k\omega} &= \tilde{\phi}^e_{k\omega} - \phi_{k\omega}\chi^e_{k\omega} \\ \phi^i_{k\omega} &= \tilde{\phi}^i_{k\omega} - \phi_{k\omega}\chi^i_{k\omega} \end{aligned} \tag{105}$$

$\tilde{\phi}^i_{k\omega}$ and $\tilde{\phi}^e_{k\omega}$ are the incoherent ion and electron fluctuations, while $\chi$ is the standard susceptibility defined through (62).

We redefine a dielectric

$$\epsilon_{k\omega} = 1 + \chi^e_{k\omega} + \chi^i_{k\omega} \tag{106}$$

through which we can solve (105) to obtain

$$\phi_{k\omega} = \frac{\tilde{\phi}_{k\omega}}{\epsilon_{k\omega}}; \qquad \tilde{\phi}_{k\omega} = \tilde{\phi}^i_{k\omega} + \tilde{\phi}^e_{k\omega} \tag{107}$$

and

$$\begin{aligned} \phi^e_{k\omega} &= (1 - \frac{\chi^e_{k\omega}}{\epsilon_{k\omega}})\tilde{\phi}^e_{k\omega} - \frac{\chi^e_{k\omega}}{\epsilon_{k\omega}}\tilde{\phi}^i_{k\omega} \\ \phi^i_{k\omega} &= (1 - \frac{\chi^i_{k\omega}}{\epsilon_{k\omega}})\tilde{\phi}^i_{k\omega} - \frac{\chi^i_{k\omega}}{\epsilon_{k\omega}}\tilde{\phi}^e_{k\omega} \end{aligned} \tag{108}$$

If we neglect any correlations between incoherent fluctuations of different species, and follow the procedure of the previous section, we get for the ion source term

$$S^i(k) = \sum_{j=i,e} \left(D^{ij}_{12}(k) + D^{ij}_{21}(k)\right)\frac{\partial^2}{\partial v_1 \partial v_2}\langle f^i(1)\rangle\langle f^i(2)\rangle - \left(F^{ij}_{12}(k)\frac{\partial}{\partial v_1} + F^{ij}_{21}(k)\frac{\partial}{\partial v_2}\right)\langle f^i(1)\rangle\langle f^i(2)\rangle \tag{109}$$

Four new terms appear which consist of diffusion and drag on the ion distribution driven by the gradients of the electron distribution. For example

$$D^{ie}_{12}(k)\frac{\partial^2}{\partial v_1 \partial v_2}\langle f^i(1)\rangle\langle f^i(2)\rangle = \frac{q^2}{m^2}\int \frac{dk'}{2\pi}\int \frac{d\omega'}{2\pi}\frac{\langle \tilde{\phi}^{e2}\rangle_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} g^*_{k'\omega'}\delta(k-k')\frac{\partial^2}{\partial v_1 \partial v_2}\langle f^i(1)\rangle\langle f^i(2)\rangle \tag{110}$$

represents the generation of fluctuations from the mixing of the gradients of the ion distribution, the mixing process being generated by the turbulent spectrum of electron fluctuations. Similarly

$$F^{ie}_{12}(k)\frac{\partial}{\partial v_1}\langle f^i(1)\rangle\langle f^i(2)\rangle = \frac{q}{m}\int \frac{dk'}{2\pi}\int \frac{d\omega'}{2\pi} ik' \frac{\langle \tilde{\phi}^i \tilde{f}^i\rangle_{k'\omega'}}{|\epsilon^*_{k'\omega'}|^2}\frac{\chi^e_{k'\omega'}}{\langle f(2)^i\rangle}\delta(k-k')\frac{\partial}{\partial v_1}\langle f^i(1)\rangle\langle f^i(2)\rangle \tag{1.11}$$

describes the mixing of the ion distribution through the dynamical drag driven by the gradients of the average electron distribution.

For the one dimensional problem, we note that $D^{ii}$ and $F^{ii}$ approximately cancel so that the ion source reduces to

$$S^i(k) \simeq \left[D^{ie}_{12}(k) + D^{ie}_{21}(k)\right]\frac{\partial^2}{\partial v_1 \partial v_2}\langle f^i(1)\rangle\langle f^i(2)\rangle - \left[F^{ie}_{12}\frac{\partial}{\partial v_2} + F^{ie}_{21}\frac{\partial}{\partial v_1}\right]\langle f^i(1)\rangle\langle f^i(2)\rangle \tag{112}$$

This is an important result since for low frequency turbulence the two terms can reinforce rather than subtract. For example if the average electron distribution has a bulk drift there will be regions where the gradient of the ion distribution has opposite sign to that of the electron. Thus the two terms in (112) will add since $F^{ie}_{12}$ is proportional to $\partial/\partial\langle f^e\rangle$. This is in contrast to the one species case where the terms in the source cancelled for small separation. Once again we can reconcile this behaviour in terms of momentum conservation arguments. For the two species problem the ion distribution can relax independantly of ion-ion "collisions" (in fact these terms, $F_{ii}$ and $D_{ii}$, cancel) since it can redistribute its average density in velocity space by exchanging momentum with the electron distribution.

# V. SOLUTION

## A. Equal Time Equation

Of the number of equations we have developed, the one time (or equal time) two point equation describes the more involved interactions in a plasma. It is only for $t_1 = t_2$ that the cross operators become important in the evolution of two neighbouring points. The enhancement of the correlation between such points is in part due to these terms, and allows the existence of a "clumping" mechansim. The one time equation dwells on the creation and destruction of these "fluid" elements. For example if the phase space volume of such elements is sufficiently small then all the particles within that fluctuation will move together since they feel approximately the same forces. The period for which the fluctuations exist, as independant discrete elements, is determined through the "time constant" ($T_{12}^{-1}$) of the governing equation. This section analyzes a method by which one can "solve" for the singular portion ($\tilde{G}$) of the correlation function. Such a quantity describes the structure (in an ensemble averaged sense) of these fluid elements or "macro particles". Our governing equation is Eq. (88).

We define (symbolically) the following operators

$$E_1 = E_1^0(k) + C_1^\phi(k) \tag{113}$$

$E_1$ is a renormalized Coulomb operator. That is $E_1^0 G_k \equiv \langle \delta E \delta f(2) \rangle_k \partial/\partial v_1 \langle f(1) \rangle$ and $C_1^\phi G_k \equiv \langle \delta E \delta f(2) \rangle_k \partial/\partial v_1 C_{11}^\phi(k)$: note that $[E_1^0 + E_2^0] G_k \equiv S_k$. We also write

$$\begin{aligned} T_1 &= ikv_1 + C_{11}^f(k) \\ \Delta T_{12}* &= C_{12}^f(k) * + C_{12}^\phi(k) * \end{aligned} \tag{114}$$

(The "*" is a reminder that the "12" terms are in fact convolutions of a $k'$ sum with functions at $k$–$k'$.) Eq. (88) can be written in terms of these operators as

$$(\frac{\partial}{\partial t} + T_1 + T_2) G_k(1,2,t) = [\Delta T_{12} * + \Delta T_{21} *] G_k(1,2,t) + [E_1 + E_2] G_k(1,2,t) \tag{115}$$

In a manner analogous to the test particle picture we will assume that $G_k(1,2,t)$ consists of two parts

$$G_k(1,2) = \overline{G}_k(1,2) + \tilde{G}_k(1,2) \tag{116}$$

$\tilde{G}(1,2)$ represents that part of the correlation function which describes the singular behaviour for small separation (in the case of discrete particles this would be the self correlation (58)), while $\overline{G}_k(1,2,t)$ will be associated with the shielding properties of the plasma.

We *define* the equation for $\overline{G}$ through

$$(\frac{\partial}{\partial t} + T_1 + T_2)\overline{G}_k(1,2,t) = [E_1 + E_2][\overline{G}_k(1,2,t) + \tilde{G}_k(1,2,t)] \tag{117}$$

This immediately defines $\tilde{G}_k(1,2)$ since $\tilde{G}_k(1,2) = G_k(1,2) - \overline{G}_k(1,2)$. We recognize that $(E_1 + E_2)\tilde{G}_k(1,2)$ acts as a source in that equation. This format is very reminiscent of the second equation in the BBGKY hierarchy with discretness effects *included.* In fact in the absence of any renormalization we would identify $(E_1 + E_2)\tilde{G}_k(1,2)$ as the exact discrete particle source. This is straightforward to see if we use $\tilde{G}_k(1,2) = n^{-1}\delta(v_1 - v_2)\langle f(1)\rangle$. In that case we also know that $\overline{G}_k(1,2)$ will describe the shielding of the discrete particles by the collective interactions of the plasma. Indeed, it is this analogy which motivated this particular choice in the first place. Time asymptotically, one can solve (115) and (117) to get

$$G_k(1,2) = [T_1 + T_2 - E_1 - E_2]^{-1}[T_1 + T_2]\tilde{G}_k(1,2) \tag{118}$$

$\tilde{G}_k$ is defined as the difference between the exact solution $G_k$, and $\overline{G}_k(1,2)$. From (115) we have

$$(\frac{\partial}{\partial t} + T_{12}(k)*)G_k(1,2) = S_k \tag{119}$$

In this formulation $T_{12}$ $(= T_1 + T_2 - (\Delta T_{12} * + \Delta T_{21} *))$ contains all the $v_-$ dependance while $S_k$ is assumed *given* and the solution (103) is used to explicitly evaluate that term. This is in contrast to the way we treated that term when evaluating $\overline{G}_k(1,2)$. There we took $S_k$ $(= (E_0^1 + E_0^2)G_k(1,2))$ to be part of a homogeneous equation for $\overline{G}_k(1,2)$. $\tilde{G}_k(1,2,t)$ is then obtained from

$$\tilde{G}_k(1,2,t) = \int dt' g_{12}(k,t,t')S_k(t') - \overline{G}_k(1,2,t) \tag{120}$$

where $g_{12}(k,t,t')$ is the Greens function which solves (119), with the right-hand side set equal to $\delta(t)$. The incoherent self correlation can be expressed in terms of the $T$ and $E$ operators as

$$\tilde{G}_k(1,2,t) = \left(1 - [T_1 + T_2]^{-1}[E_1 + E_2]\right) \int dt' g_{12} S_k \tag{121}$$

For small separation, equation (121) can be written in the physically appealing form

$$\tilde{G}_k(1,2,t) = (\langle \tau_{cl} \rangle - \tau_{tr}) S_k \tag{122}$$

(122) derives from noting that as $x_-, v_- \rightarrow 0$, $E_1 + E_2 \rightarrow E_1^0 + E_2^0$ and $(T_1 + T_2)^{-1} \simeq \tau_{tr}$. Thus the clump portion of the correlation function is the difference between the total solution $(\langle \tau_{cl} \rangle S)$ and the shielding solution $(\tau_{tr} S)$. $\langle \tau_{cl} \rangle$ is some e-folding time characteristic of the solution[2] to (119). For example in the case where (in real space) we approximate $T_{12}$ by

$$T_{12}(x_-, v_-) \simeq v_- \frac{\partial}{\partial x_-} - \frac{\partial}{\partial v_-} D_- \frac{\partial}{\partial v_-} \tag{123}$$

with

$$\begin{aligned} D_- =& D_{11} + D_{22} - D_{12} - D_{21} \\ \simeq& \frac{q^2}{m^2} \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} k^2 \langle \phi^2 \rangle_{k\omega} 2 Re g_{k\omega}(v_+)(1 - \cos k x_-) \end{aligned} \tag{124}$$

one can obtain the expression

$$\begin{aligned} \langle \tau_{cl}(x_-, v_-) \rangle =& \tau_0 \ln \frac{3}{k_0^2 [x_-^2 - 2x_- v_- \tau_0 + 2 v_-^2 \tau_0^2]} \quad \arg \ln > 1 \\ =& 0 \qquad \text{otherwise} \end{aligned} \tag{125}$$

$$\tau_0 = (4k_0^2 D)^{-\frac{1}{3}} = (12)^{-\frac{1}{3}} \tau_{tr}$$

by calculating the length of time during which particles that are initially separated by $x_-$, $v_-$, will move together before they separate by $k_0^{-1}$. This is be achieved by computing the moments

$$\langle x_-^n(t) v_-^m(t) \rangle \equiv \int dx_- \int dv_- x_-^n v_-^m g_{12} \tag{126}$$

setting $k_0^2 \langle x_-^2(\tau_{cl}) \rangle \simeq 1$, and solving the resulting equation.

In general the following observations can be made from the simple form (122). First as $x_-, v_-$ approach zero $\tau_{cl} \gg \tau_{tr}$ since the first is singular while the second is not. Thus $\tilde{G}_k$ approaches $\langle \tau_{cl} \rangle S_k$ which is equal to the total response $G_k$. Second for large separation $(k_0 x_- > 1)$, $\tau_{cl} \simeq \tau_{tr}$ so that $\tilde{G}_k \rightarrow 0$.

## B. Two Time Equation

We now wish to show that this particular choice of $\overline{G}$ and $\tilde{G}$ leads to the shielded test particle picture where $\tilde{G}$ obeys a ballistic equation of motion (with Fokker-Planck renormalization). The assumption of time stationarity allows us to take Eq. (73) of Sec. III. Neglecting the cross terms (see Sec. III C.) we have

$$\frac{\partial}{\partial t_1}\langle f(t_1)f(0)\rangle_k + ikv_1\langle f(t_1)f(0)\rangle_k + \int_0^{t_1} dt' C^f(t_1-t')\langle f(t')f(0)\rangle_k = -\frac{q}{m}\int_0^{t_1} dt'\langle E(t')f(0)\rangle_k \frac{\partial}{\partial v_1}\overline{F}_k(t_1-t') \tag{127}$$

The term containig $\partial/\partial v\langle f\rangle$ has been included with $C_k^\phi$ to produce $\overline{F}_k$ (see (34c)). This equation has to be solved with $\langle f(t_1)f(0)\rangle_k = \langle f(0)f(0)\rangle_k = G_k(1,2,t)$ as an initial condition. (127) propagates point 1 keeping point 2 fixed and is therfore valid for $t_1 \geq t_2 = 0$. For $t_2 \geq t_1 = 0$ the operator is changed form coordinate 1 to 2.

The solution to (127) is given by

$$\langle f(1)f(2)\rangle_{k\omega} = \{\overline{P}(1,k,\omega) + \overline{P}^*(2,k,\omega)\}\langle f(1)f(2)|t_1=t_2\rangle_k \tag{128}$$

where the $\overline{P}$ propagator is the solution to

$$-i\left(\omega - kv_1 + iC^f_{11}(k) - \frac{\omega_p^2}{|k|^2}k\frac{\partial\overline{F}_k}{\partial v_1}\int dv_1\right)\overline{P}(1,k,\omega) = 1 \tag{129}$$

and can be written as

$$\overline{P}_{k\omega} = g_{k\omega}(1)\left\{1 - \frac{\omega_p^2}{|k|^2}k\frac{\partial\overline{F}_k}{\partial v_1}\frac{1}{\epsilon_{k\omega}}\int dv_1 g_{k\omega}(1)\right\} \tag{130}$$

To obtain (128) we have used

$$\begin{aligned}\langle f(1)f(2)\rangle_{k\omega} &\equiv \int_{-\infty}^{\infty} dx\_ \int_{-\infty}^{\infty} dt\_ e^{ikx-} e^{-i\omega t-}\langle f(v_1,x_1,t_1)f(v_2,x_2,t_2)\rangle \\ &= \int_0^{\infty} dt\_ e^{-i\omega t-}P(1,k,t_1-t_2)\langle f(1)f(2)|t_1=t_2\rangle_k \qquad t_1>t_2 \\ &+ \int_{-\infty}^{0} dt\_ e^{-i\omega t-}P(2,k,t_2-t_1)\langle f(1)f(2)|t_1=t_2\rangle_k \qquad t_2>t_1\end{aligned} \tag{131}$$

From Eq. (31) we know that on the fast time scale $\langle\tilde{f}(1,t_1)\tilde{f}(2,0)\rangle$ satisfies

$$\frac{\partial}{\partial t_1}\langle\tilde{f}(1,t_1)\tilde{f}(2,0)\rangle + ikv_1\langle\tilde{f}(1,t_1)\tilde{f}(2,0)\rangle + \int_0^{t_1} dt' C^f_{11}(k,t_1-t')\langle\tilde{f}(1,t')\tilde{f}(2,0)\rangle = 0 \tag{132}$$

with solution

$$\langle\tilde{f}(1)\tilde{f}(2)\rangle_{k\omega} = \tilde{G}_{k\omega}(1,2) = [g_{k\omega}(1) + g^*_{k\omega}(2)]\tilde{G}_k(1,2) \tag{133}$$

Applying the $P$ propagators we find that, *given* (133), $\overline{G}_{k\omega}(1,2)$ satisfies

$$\begin{aligned}\overline{G}_{k\omega}(1,2) = &\left(g_{k\omega}(1)\left\{1 - \frac{\omega_p^2}{|k|^2}k\frac{\partial\overline{F}_k}{\partial v_1}\frac{1}{\epsilon_{k\omega}}\int dv_1 g_{k\omega}(1)\right\} + (1\leftrightarrow 2)\right)G_k(1,2) \\ &- [g_{k\omega}(1) + g^*_{k\omega}(2)]\tilde{G}_k(1,2)\end{aligned} \tag{134}$$

We substitute the expression for $G_k(1,2)$ in (134) to get

$$\overline{G}_{k\omega} = \Big(\frac{i}{(\omega + iT_1 - iE_1)} - \frac{i}{(\omega - iT_2 + iE_2)}\Big)\frac{T_1 + T_2}{T_1 + T_2 - E_1 - E_2}\tilde{G}_k(1,2) - \tilde{G}_{k\omega}(1,2) \tag{135}$$

we have expressed the $P$ operators as $i/(\omega + i[T - E])$. To be consistent with our earlier assumptions on the nature of the time integral in the collision operator we have to set $D_{k\omega}$ etc. equal to $D(k)$ in the $P$ operators. (135) can then be simplified to

$$\begin{aligned}\overline{G}_{k\omega}(1,2) &= \frac{T_1 + T_2}{[\omega + iT_1 - iE_1][\omega - iT_2 + iE_2]}\tilde{G}_k(1,2) - \tilde{G}_{k\omega}(1,2) \\ &= -\tilde{G}_{k\omega}(1,2) + \left(1 + i\frac{\omega_p^2}{|k|^2}g_{k\omega}(1)\frac{k}{\epsilon_{k\omega}}\frac{\partial\overline{F}_k}{\partial v_1}\int dv_1\right) \\ &\qquad \times\left(1 - i\frac{\omega_p^2}{|k|^2}g^*_{k\omega}(2)\frac{k}{\epsilon^*_{k\omega}}\frac{\partial\overline{F}^*_k}{\partial v_2}\int dv_2\right)[g_{k\omega}(1) + g^*_{k\omega}(2)]\tilde{G}_k(1,2)\end{aligned} \tag{136}$$

which is identical to $\langle f^c(1)f^{c*}(2)\rangle_{k\omega} + \langle f^c(1)\tilde{f}^*(2)\rangle_{k\omega} + \langle\tilde{f}(1)f^{c*}(2)\rangle_{k\omega}$. This is of course the result we set out to prove. As previously advertised, we can also identify $\overline{G}_{k\omega}(1,2)$ as the shielding response to the incoherent spectrum $\int dv_1 \int dv_2 \tilde{G}_{k\omega}(1,2)$, since (136) can be integrated over $v_1$ and $v_2$ to yield

$$\langle\phi^2\rangle_{k\omega} = \frac{\langle\tilde{\phi}^2\rangle_{k\omega}}{|\epsilon_{k\omega}|^2} \qquad (137)$$

If we neglect the incoherent fluctuation then the solution becomes $\langle f^c f^c\rangle$. The equations revert to the more common weak turbulence expansions (including renormalizations). These solutions are ultimately concerned with wave, and mode coupling type of interactions, since the driving mechanism is the zeroes of the dielectric function.

## C. Breakdown of λ Expansion

The equal time two point equation exhibits a singular behaviour as $\{x_-, v_- \to 0\}$. It is instructive to examine the breakdown of the ordering as one approachs that region. This change in ordering is particulary relevant to the conclusion reached by Dubois and Espedal[11]: namely that the contribution from "clumps" is down by a factor of $|\phi|^2$ compared to the coherent response $\langle f^c f^c\rangle$. This result is arrived at by observing that no source term of the form described in Sec. IV appears in the equation for $\langle\tilde{f}\tilde{f}\rangle$. In fact the only source is proportional to $|\phi|^4$. In this section we indicate, schematically, how to recover the correct source and ordering for $\tilde{G}(1,2)$. A detailed proof is given in Ref. [7].

Consider Eqs. (115) and (117) suplemented by the equation for $\tilde{G}$

$$(\frac{\partial}{\partial t} + T_1 + T_2)\tilde{G}_k(1,2,t) = -[\Delta T_{12} * + \Delta T_{21} *][\overline{G}_k(1,2,t) + \tilde{G}_k(1,2,t)] \qquad (138)$$

(138) plus (117) add up to the original equation (115). A cursory examination of (138) and (117) leads to an intriguing conclusion: the information on the singular behaviour seems to have been lost. There is no doubt that equation (115), for small separation, is singular with a "second" order source ($T_{12} \to 0$ while the Coulomb operator does not). The equation for $\overline{G}$ does not contain that information since the operator on the left-hand side remains finite for small separation. Furthermore the equation for $\tilde{G}$ seems to have a fourth order source, since $\overline{G}$ is nominally of second order (proportional to $\langle\delta E \delta f\rangle\partial/\partial v\langle f\rangle$) and $\Delta T_{12}$ is also of second order. This seeming discrepancy can easily be reconciled when one realizes that the ordering is measured relative to the *linear* operator $\partial/\partial t + v_-\partial/\partial x_-$. In a steady state, as $\{x_-, v_- \to 0\}$, this ordering breaks down. From (89) one can show that as $\{x_-, v_- \to 0\}$ $\Delta T_{12} \to T_1$; but from (117) we know that $[T_1 + T_2]\overline{G} = [E_1 + E_2]G$ so that (138) becomes

$$\lim_{x_-, v_- \to 0} \quad \frac{\partial}{\partial t}\tilde{G}_k(1,2,t) \to [T_1 + T_2]\overline{G}_k \to S \qquad (139)$$

The ordering of the source for $\tilde{G}$ is back to $|\phi|^2$! The partition used in Ref. [11], for $\overline{G}$ and $\tilde{G}$, is slightly different, but the same conclusions hold (see Ref. [7]). It is also worth mentioning that in Ref. [11] the one point equation is obtained by iterating $f = f^c$ rather than $f = f^c + \tilde{f}$. In other words the renormalized collision operators do not contain terms such as $F, \mathcal{G}$ etc., which means that none of the conservation properties described in Sec. II. can be obtained.

In the time domain, the important point to note is that the ordering of the $\Delta T_{12}$ terms will be of fourth order *unless* $t_1 = t_2$ and $(t_1 + t_2) \to \infty$ (steady state). These constraints imply that the "clump" contribution derives from a *secular* interaction which only occurs when two phase space points see the same forces (i.e. equal time) so that their relative orbit is correlated. This immediately leads to our solution technique which neglects the $T_{12}$ terms (being of fourth order) in the two time equation $(t_1 \neq t_2,\ t_1 + t_2 < \tau_{tr})$ but incorporates the macro-particle spectrum as an initial condition. The latter is obtained by solving the *equal time* equation $(t_1 = t_2,\ t_1 + t_2 \gg \tau_{tr})$ at which point the ordering is meaningless and we can set $\lambda$ equal to 1.

## D. One Point Review

We have presented an approximate technique for solving the two time and equal time two point equatons. This approach has relied on the presence of two time scales which allow us to decouple these equations and treat them independantly. Starting from the two point equation for $G_k$ the partition of $G_k$, defined through (117), into $\overline{G}_k$ and $\tilde{G}_k$ has led quite naturally to a "test-clump" picture. It is interesting to see how this partition is related to the one point equations. Contrary to one's first inclination, our $\overline{G}$ and $\tilde{G}$ are *not* consistent with

$$\left(\frac{\partial}{\partial t} + T_1\right) f_k^c(1) = E_k \frac{\partial}{\partial v_1} \overline{F}_k \tag{140a}$$

$$\left(\frac{\partial}{\partial t} + T_1\right) \tilde{f}_k(1) = \mathcal{I}_k(1); \qquad \mathcal{I}_k(1) = -\frac{q}{m}\frac{\partial}{\partial v_1} \sum_{k',\omega'} E_{k'}^{(1)}(t) f_{k-k'}^{(1)}(1,t) \tag{140b}$$

but rather with

$$\left(\frac{\partial}{\partial t} + T_1\right) f_k^c(1) \tilde{f}_k^*(2) = E_k \tilde{f}_k^*(2) \frac{\partial}{\partial v} \overline{F}_k - \mathcal{I}_k(1) f_k^{c*}(2) \tag{141a}$$

$$\left(\frac{\partial}{\partial t} + T_1\right) \tilde{f}_k(1) \tilde{f}_k^*(2) = \mathcal{I}_k(1) \tilde{f}_k^*(2) + \mathcal{I}_k(1) f^{c*}(2) \tag{141b}$$

As evidenced by (141a) and (141b), there is no *one point* version of our division of $G$ into $\tilde{G}$ and $\overline{G}$. The set (140) and (141) are, however, equivalent since they add up to the same equation: $(\partial/\partial t + T_1)f_k(1) = \mathcal{I}_k(1) + E_k\partial/\partial v\overline{F}_k$. On the fast or ballistic time scale *both* these set of equations reduce to (30) and (31) (since the $\mathcal{I}$ contribution generates the $\Delta T_{12}$ terms and may be neglected):

$$\left(\frac{\partial}{\partial t} + T_1\right)f_k^c(1) \simeq E_k\frac{\partial}{\partial v}\overline{F}_k \tag{142a}$$

$$\left(\frac{\partial}{\partial t} + T_1\right)\tilde{f}_k(1) \simeq 0 \tag{142b}$$

From which we recover the one point shielding results of Sec. II.

We have shown that the set (141) will yield the shielded clump picture when the "slow" (equal time) version (141) is used as an initial condition for the "fast" (two time) version (142). The following question arises: what is the effect of using the same procedure with, instead of (141), the set (140). In fact one might worry that an inconsistency is generated since we will be propagating a *different* incoherent response ($\tilde{G}'$) through (142b). This paradox is easily resolved by noting that the initial condition "$\overline{G}$" will also be different ($\overline{G}'$). In fact it is simple to show that this different initial condition produces the missing part (as it clearly should) of the incoherent response. In other words the total potential will be given by

$$\langle\phi^2\rangle_{k\omega} = \frac{\langle\tilde{\phi}'^2\rangle_{k\omega}}{|\epsilon_{k\omega}|^2} + R_{k\omega} \tag{143}$$

$R_{k\omega}$ is the remainder generated by the different $\overline{G}'$ condition and $\tilde{\phi}'$ is the potential generated by the $\tilde{f}$ defined through (140b).This can also be written as

$$\langle\phi^2\rangle_{k\omega} = \frac{\langle\tilde{\phi}^2\rangle_{k\omega}}{|\epsilon_{k\omega}|^2} \tag{144}$$

where now $\tilde{\phi}$ is the potential generated by the $\tilde{f}$ defined through (141b). Thus both partitions yield the same *total* potential on the other hand (141) is emminently more useful since it leads to the concept of a shielded "macro-particle" which can be treated in much the same way as a shielded "test-particle".

Finally we can briefly pursue the following tempting (but erroneous) procedure. Consider the set (140) which we Fourier series transform according to (22) and (33). Using Poisson's equation one can easily obtain $\phi_{k\omega} = \tilde{\phi}'_{k\omega}/\epsilon_{k\omega}$ where

$$\tilde{\phi}'_{k\omega} = (4\pi ne/|k|^2) \int dv g_{k\omega} \mathcal{f}_{k\omega}$$

Multiplying this expression by its complex conjugate and proceeding (according to (35)) to the integral limit we would get

$$\langle\phi^2\rangle_{k\omega} = \frac{\langle\tilde{\phi}'^2\rangle_{k\omega}}{|\epsilon_{k\omega}|^2} \tag{145}$$

which contradicts (143) and (144)! The error in deriving (145) results from the application of the $T \to \infty$ limit to the Fourier series expansion. Eq. (33) is useful for the analysis of non-secular time dependances (stationary turbulence), and fails to treat the secular aspect inherent to (140a) and (140b). In fact the $T \to \infty$ limit is ill-defined (singular) and cannot lead to the spectral (integral) transforms. The latter can only be obtained by treating the secular (slow) and non-secular (fast) contributions *separately.*

In conclusion we add that if by the direct interaction approximation (as applied to plasma turbulence) we understand a scheme which iterates the coherent response only then this procedure will break down. For small separation the incoherent response is certainly of the same "order", if not larger than $f^c$. This is in agreement with the physical models behind the coherent and incoherent response. The former represents a weak coupling, sufficient to describe shielding and other non local phenomena. The latter is concerned with the much more violent interactions at wave particle resonances: there results a strong distortion and modulation of resonant velocity streams of the distribution function. This is a strong coupling problem where the stream develops a complicated or "incoherent" phase dependance due to the highly non-linear interaction at the resonance.

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## Appendix A. Perturbation of the Lenard-Balescu Equation

In Sec. II, we interpreted some of the terms in the collision operator $C^f$ by considering an expansion of the discrete particle Lenard-Balescu collision integral

$$\left.\frac{\partial f}{\partial t}\right]_{LB} = -\frac{\partial}{\partial v}Ff + \frac{\partial}{\partial v}D\frac{\partial}{\partial v}f \tag{A.1}$$

$F$ and $D$ are the drag and diffusion coefficients given by

$$Ff = \frac{q}{m}\sum_{k',\omega'} k' \frac{\epsilon^i_{k'\omega'}}{|\epsilon_{k'\omega'}|^2}\langle\tilde{\phi}^*_{k'\omega'}\tilde{f}_{k'\omega'}(1)\rangle \qquad D = \sum_{k',\omega'}\frac{q^2}{m^2}\pi\delta(\omega' - k'v)\frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon_{k'\omega'}|^2} \tag{A.2}$$

The spectrum of fluctuations is given by

$$\langle\tilde{f}(1)\tilde{f}(2)\rangle_{k'\omega'} = \frac{2\pi}{n}\delta(\omega' - k'v)\delta(v_1 - v_2)\langle f\rangle \tag{A.3}$$

where $n$ is the average density of particles. Using (A.3) and (A.2) in (A.1) we can write the collision integral as

$$\left.\frac{\partial f(1)}{\partial t}\right]_{LB} = \pi\frac{\omega_p^4}{n}\sum_{k'}\frac{k'k'}{|k'|^4}\frac{\partial}{\partial v_1}\int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon_{k',k'v_1}|^2}\left[\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_3}\right]f(1)f(3) \tag{A.4}$$

We consider the response to a *wave*, of a plasma described by the equation

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{q}{m}E\frac{\partial}{\partial v}\right)f = \left.\frac{\partial f}{\partial t}\right]_{LB} \tag{A.5}$$

We will treat the Fokker-Planck coefficients as a perturbation acting on the correlated motion described by the Vlasov operator. Thus the wave field is present both as the smooth macrofield $E$ in the Vlasov operator and in its effects on the drift and diffusion coefficients. Following the procedure in Sec. II we linearize $f$ and $\epsilon_{k'\omega'}$ according to (60) and (61). This yields

$$-i(\omega - kv)f_{k\omega} - i\frac{q}{m}k\frac{\partial f_0^0}{\partial v}\phi_{k\omega} = \left.\frac{\partial f_{k\omega}}{\partial t}\right]_{LB} \tag{A.6}$$

If we write the perturbed collision operator as $-Cf_{k\omega}$ we have

$$f_{k\omega} = -\frac{q}{m}k\frac{\partial f_0^0}{\partial v}\phi_{k\omega}(\omega - kv + iC)^{-1} \tag{A.7}$$

and

$$f_0^1 = \lim_{k,\omega \to 0} f_{k\omega} \tag{A.8}$$

The perturbed collision operator becomes

$$\left.\frac{\partial f_0^1(1)}{\partial t}\right]_{LB} = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \frac{\partial}{\partial v_1} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \left[\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_3}\right]\left[f_0^1(1) f_0^0(3) + f_0^0(1) f_0^1(3)\right]$$

$$- \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \frac{\partial}{\partial v_1} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \left[\frac{\chi^1_{k'\omega'}}{\epsilon^0_{k'\omega'}} + \frac{\chi^{1*}_{k'\omega'}}{\epsilon^{0*}_{k'\omega'}}\right]\left[\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_3}\right] f_0^0(1) f_0^0(3) \tag{A.9}$$

which can be rewritten as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial v} F - \frac{\partial}{\partial v} D \frac{\partial}{\partial v}\right) f_0^1 = \left(-\frac{\partial}{\partial v}(\mathfrak{F}^t + \mathfrak{F}^f) + \frac{\partial}{\partial v}(d^t + d^f)\frac{\partial}{\partial v}\right) f_0^0 \tag{A.10}$$

where

$$D = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} f_0^0(3)$$

$$F = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \frac{\partial}{\partial v_3} f_0^0(3) \tag{A.11}$$

are the diffusion and drag coefficients in the absence of the wave field $E_{k\omega}$

$$\mathfrak{F}^t = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \frac{\partial}{\partial v_3} f_0^1(3)$$

$$\mathfrak{F}^f = -\pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \left[\frac{\chi^1_{k'\omega'}}{\epsilon^0_{k'\omega'}} + \frac{\chi^{1*}_{k'\omega'}}{\epsilon^{0*}_{k'\omega'}}\right] \frac{\partial}{\partial v_3} f_0^0(3) \tag{A.12}$$

and

$$d^t = \pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} f_0^1(3)$$

$$d^f = -\pi \frac{\omega_p^4}{n} \sum_{k'} \frac{k'k'}{|k'|^4} \int dv_3 \frac{\delta(k'v_1 - k'v_3)}{|\epsilon^0_{k',k'v_1}|^2} \left[ \frac{\chi^1_{k'\omega'}}{\epsilon^0_{k'\omega'}} + \frac{\chi^{1*}_{k'\omega'}}{\epsilon^{0*}_{k'\omega'}} \right] f_0^0(3) \tag{A.13}$$

are the modifications to the Fokker-Planck coefficients due to the wave induced distortion of the distribution function.

We wish to compare the above to the $k, \omega \to 0$ limit of the $C^f_{k\omega}$ operator

$$-C^f_{k\omega} f_{k\omega} \equiv \frac{\partial}{\partial v}\left(D_{k\omega}\frac{\partial}{\partial v} - F_{k\omega}\right) f_{k\omega} + \frac{\partial}{\partial v}\left((d^f + d^t)\frac{\partial}{\partial v} - (\mathcal{F}^t + \mathcal{F}^f)\right)\bar{f} \tag{A.14}$$

For the spectrum given by (A.3) it is clear that setting $\{k, \omega \to 0\}$ in (45) will reduce the diffusion $D_{k\omega}$ to the "zero" order Lenard-Balescu coefficient. Furthermore in the long wavelength limit, and time asymptotically we have

$$\lim_{k,\omega \to 0} (\mathcal{F}^t + \mathcal{F}^f) f_{k\omega} = -\frac{q}{m}\omega_p^2 \sum_{k',\omega'} \frac{k'k'}{|k'|^2} \frac{\langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'}(1)\rangle}{\epsilon^*_{k'\omega'}\epsilon^*_{k'\omega'}} \int dv_3 \frac{i}{\omega' - k'v_3 - i\delta} \frac{\partial}{\partial v_3} f_0^1(3)$$

$$= -\frac{q}{m} \sum_{k',\omega'} k' \langle \tilde{\phi}^*_{k'\omega'} \tilde{f}_{k'\omega'}(1)\rangle i \frac{\chi^*_{k'\omega'}}{\epsilon^*_{k'\omega'}\epsilon^*_{k'\omega'}} \tag{A.15}$$

If we use

$$Re\left[i\frac{\chi^*_{k'\omega'}}{\epsilon^*_{k'\omega'}\epsilon^*_{k'\omega'}}\right] = \frac{Im\chi_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} - \left[\frac{\chi_{k'\omega'}}{\epsilon_{k'\omega'}} + \frac{\chi^*_{k'\omega'}}{\epsilon^*_{k'\omega'}}\right]\frac{Im\epsilon_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} \tag{A.16}$$

in (A.15) and perform the $\omega'$ integral we recover the coefficients obtained from the linearization (A.12).

At the end of this appendix we show that

$$\left[\langle\tilde{\phi}^{(1)}\tilde{f}^{(2)}\rangle_{k'\omega'} + \langle\tilde{f}^{(1)}\tilde{\phi}^{(2)}\rangle_{k'\omega'}\right] = \frac{(4\pi ne)}{|k'|^2}\frac{2\pi}{n}\delta(\omega' - k'v) f_0^1$$

$$\left[\langle\tilde{\phi}^{(1)}\tilde{\phi}^{(2)}\rangle_{k'\omega'} + \langle\tilde{\phi}^{(1)}\tilde{\phi}^{(2)}\rangle_{k'\omega'}\right] = \frac{(4\pi ne)^2}{|k'|^4}\int dv \frac{2\pi}{n}\delta(\omega' - k'v) f_0^1 \tag{A.17}$$

where the expressions in (A.17) are the limit $k, \omega \to 0$, of

$$\tilde{f}^{(2)}_{k-k',\omega-\omega'}\tilde{\phi}^{(1)}_{k'\omega'} + \tilde{f}^{(1)*}_{k'\omega'}\phi^{(2)}_{k+k',\omega+\omega'}$$

when one goes from the discrete to the continuous limit of the Fourier transform.

We also have

$$\lim_{k,\omega\to 0} \quad d^t\frac{\partial \bar{f}}{\partial v_1} = -\frac{q^2}{m^2}\sum_{k',\omega'} k'k'\frac{\left[\tilde{\phi}^{(1)}_{k'\omega'}\tilde{\phi}^{(2)*}_{k'\omega'} + \tilde{\phi}^{(1)*}_{k'\omega'}\tilde{\phi}^{(2)}_{k'\omega'}\right]}{[\omega' - k'v_1 - i\delta]}\frac{i}{|\epsilon^0_{k'\omega'}|^2}\frac{\partial}{\partial v_1}f^0_0(1)$$

$$\lim_{k,\omega\to 0} \quad d^f\frac{\partial \bar{f}}{\partial v_1} = -\frac{q^2}{m^2}\sum_{k',\omega'} k'k'\frac{|\tilde{\phi}_{k'\omega'}|^2}{|\epsilon^0_{k'\omega'}|^2}\frac{-i}{[\omega' - k'v_1 - i\delta]}\left[\frac{\chi_{k'\omega'}}{\epsilon_{k'\omega'}} + \frac{\chi^*_{k'\omega'}}{\epsilon^*_{k'\omega'}}\right]\frac{\partial}{\partial v_1}f^0_0(1) \qquad (A.18)$$

and

$$\lim_{k,\omega\to 0} \quad F_{k\omega}f_{k\omega} = \frac{q}{m}k'\sum_{k',\omega'} i\left[\tilde{f}^{(2)*}_{k'\omega'}\tilde{\phi}^{(1)}_{k'\omega'} + \tilde{f}^{(1)*}_{k'\omega'}\tilde{\phi}^{(2)}_{k'\omega'}\right]\frac{\epsilon^*_{k'\omega'}}{|\epsilon_{k'\omega'}|^2} \qquad (A.19)$$

Using (A.17) in (A.18) and (A.19), and retaining the real part of these terms it is easy to see that they reduce to their counterparts (A.11) and (A.13).

To show (A.17) we take (48) multiply respective terms by $\phi_{k'\omega'}$ and $f^*_{k'\omega'}$ to get

$$\tilde{f}^{(2)}_{k-k',\omega-\omega'}\tilde{\phi}^{(1)}_{k'\omega'} + \tilde{f}^{(1)*}_{k'\omega'}\phi^{(2)}_{k+k',\omega+\omega'} = \frac{q}{m}g_{k-k',\omega-\omega'}(1)ik\phi_{k\omega}\frac{\partial}{\partial v_1}\langle\tilde{f}^{(1)*}_{k'\omega'}(1)\tilde{\phi}^{(1)}_{k'\omega'}\rangle$$
$$+i\frac{\omega_p^2}{|k+k'|^2}\int dv_3 g_{k+k',\omega+\omega'}(3)\frac{\partial}{\partial v_3}\langle\tilde{f}^{(1)*}_{k'\omega'}(1)\tilde{f}^{(1)}_{k'\omega'}(3)\rangle k\phi_{k\omega} \qquad (A.20)$$

Since these expression are going to be summed over $k'$ and $\omega'$ we can set $\langle\tilde{f}^*_{k'\omega'}\tilde{f}_{k'\omega'}\rangle$ equal to $\langle ff\rangle_{k'\omega'}$ by changing the summation to an integration. Using (A.3) we get for the right hand side of (A.20)

$$\frac{(4\pi ne)}{|k'|^2}\frac{2\pi}{n}\delta(\omega' - k'v)g_{k-k',\omega-\omega'}\frac{q}{m}ik\phi_{k\omega}\frac{\partial f^0_0}{\partial v}$$
$$+\frac{(4\pi ne)}{|k'|^2}\frac{2\pi}{n}\frac{q}{m}k\phi_{k\omega}f^0_0\left[\frac{(k+k')\delta(\omega' - k'v)}{[\omega + \omega' - (k+k')v + i\delta]^2} - \frac{\partial/\partial v\delta(\omega' - k'v)}{[\omega - \omega' - (k-k')v + i\delta]}\right] \qquad (A.21)$$

The term on the first line is the desired answer since we can use (A.7) and (A.8) to reduce it to (A.17). We thus want to show that the remaining terms are zero in the limit of $\boldsymbol{k}, \omega \rightarrow \mathbf{0}$. Using

$$\delta(x) = \lim_{\delta \to 0} \frac{\delta}{(x^2 + \delta^2)} \tag{A.22}$$

the term in square brackets can be written as

$$\lim_{\Delta, \delta \to 0} \quad k' \frac{\delta}{(\Delta'^2 + \delta^2)} \left[ \frac{(\Delta + \Delta' - i\delta)^2}{[(\Delta + \Delta')^2 + \delta^2]^2} - \frac{2\Delta'[\Delta - \Delta' - i\delta]}{[\Delta'^2 + \delta^2][(\Delta - \Delta')^2 + \delta^2]} \right] \tag{A.23}$$

where $\Delta = \omega - kv$ and $\Delta' = \omega' - k'v$. As $\Delta$ approaches zero the imaginary parts exactly cancel and we are left with an expression that is entirely real. Furthermore since any $k'$ dependance in $\Delta'$ will get eliminated by the $\omega'$ inetgral the only $k'$ element to survive is the one outside the square brackets in (A.23). For $F$ and $d^t$, the final result has to be real. Since $|\epsilon_{k'\omega'}|^2$ is an even function of $k'$ we are left with $k'$ integrals which integrate odd functions. In the case of $F_{k\omega}$ this is $\approx \int dk'k'^2 Im\, \epsilon_{k',k'v}$ while for the diffusion, $d^t$, it is $\approx \int dk'k'^3$. Both integrals are identically equal to zero. Thus the only term to survive in the long wavelength limit is the first one in equation (A.20).

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