PFC/JA-81-24

PHASE SPACE DYNAMICS

D.J. Tetreault

Massachusetts Institute of Technology
Plasma Fusion Center
Cambridge, MA 02139

October, 1981

ABSTRACT

The Vlasov equation describes the conservation of phase space density $f(x,v,t)$. The incompressible flow of $f(x,v,t)$ determines the behavior of the phase space shear $q=\partial f/\partial v$ and $p =- \partial f/\partial x$. It is shown that the equations of motion for p and q can be written in canonical form. The motion of well trapped particles is considered and a Virial theorem is derived from the equations governing p and q.

The collisionless behavior of a classical many body system that interacts through its self-consistent fields is relevant to both plasma and gravitational physics.[1-3] Considered as a one dimensional phase space fluid, the system of particles can be described by the Vlasov equation for phase space density conservation

$$\frac{d}{dt} f = \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{e}{m} E \frac{\partial}{\partial v}\right) f = 0 \qquad (1)$$

where x, v and t are position, velocity, and time respectively. Other quantities in Eq. (1) are charge (e) and electric field (E) for the plasma case, and the particle mass (m). For self-consistent fields, E is a functional of the phase space density (f) through Poisson's equation. The equilibrium solution to Eq. (1) describes stationary flows of the phase space density. In these states, groups of particles can form self-binding configurations under the action of their self-consistent fields. The plasma analogue of this familiar gravitational situation is the BGK mode.[4] These states are characterized by particle trapping and vortex structure of the particle phase space.

Particle trapping causes a phase space mixing of the particle distribution function, f. For example, it has been shown that when a delta-function particle beam in velocity space is trapped in a wave, it rotates in phase space and when the rotation angle reaches 90°

the velocity delta-function is transformed into a spatial delta function.[5] We report here on the dynamics of this phase space mixing. Below we consider the plasma case, but the gravitational dual is straight forward.[1]

It has been noted that the phase space vortex structure of trapped particle motion is remarkably similar to the familiar eddy structure of two dimensional fluid motion.[1] This similarity is formally suggested by expressing Eq. (1) in two dimensional phase space coordinates $\vec{x}$ and $\vec{v}$, where $\vec{x} = (x, \omega_p^{-1} v)$ and $\vec{v} = (v, \omega_p^{-1} \dot{v})$. Here $\dot{v} = dv/dt$ and $\omega_p^2 = 4\pi n_o e^2/m$ is the plasma frequency where n_o as the particle density in space. We define the particle energy variable

$$W = \frac{1}{2} mv^2 + e\phi(x,t) \tag{2}$$

where ϕ is the potential $(E = -\partial\phi/\partial x)$. Then the incompressible fluid flow velocity in phase space can be written as

$$\vec{v} = (mw_p)^{-1} \hat{z}\times\vec{\nabla}W \tag{3}$$

where the unit vector $\hat{z}$ is perpendicular to the x-v plane. Eq. (1) then becomes

$$(\frac{\partial}{\partial t} + \vec{v}.\vec{\nabla}) f = 0 \tag{4}$$

Poisson's equation,

$$\frac{\partial^2 \phi}{\partial x^2} = - 4\pi e \int dvf \tag{5}$$

can be written as

$$\nabla^2 W = - \frac{m}{n_o} \omega_p^2 [\int dvf - n_o] \tag{6}$$

where $\rho = \hat{z} \cdot (\vec{\nabla}\times\vec{v}) = (m\omega_p)^{-1} \nabla^2 W$ plays the role of vorticity.[6] Because of the property $\vec{v}\cdot\vec{\nabla}W = 0$, the total fluid energy is conserved between Eq. (4) and Eq. (6) since

$$\frac{\partial}{\partial t} \int d\vec{x}\, \frac{1}{2} m\vec{v}\cdot\vec{v}f = - \int d\vec{x}\; W\; \vec{v}\cdot\vec{\nabla}f = 0 \tag{7}$$

and

$$\frac{1}{2} mn_o\; \vec{v}\cdot\vec{v} = E^2/8\pi + \frac{1}{2} mn_o v^2 \tag{8}$$

It is evident from Eq. (4) that f=f(W) is an equilibrium solution. This distribution is a bound state that satisfies the Virial Theorem[7], since

$$\int d\vec{x} \;\; xv\; \vec{v}\cdot\vec{\nabla}f = 0 \tag{9}$$

implies that

$$\int d\vec{x}\; mv^2 f = - \int d\vec{x}\; eExf \tag{10}$$

where $\int d\vec{x} = \int dx \int dv$. For this state, the bounce frequency $\omega^2 = -\frac{e}{m}\,\partial E/\partial x > 0$ is constant along the orbit of a well trapped particle. Therefore, $\phi = \frac{1}{2}\frac{m}{e}\,\omega^2 x^2$ so that the contours of constant $f(W) = f(v^2 + \omega^2 x^2)$ are the familiar vortices in phase space. An interesting viewpoint of these constant contours follows from Eq. (1). We derive

$$\frac{d}{dt}\left(\frac{\partial f}{\partial v^2}\right) = -\frac{1}{2v^2}\frac{\partial f}{\partial t} \qquad (11)$$

and conclude that $\partial f/\partial W$ is constant along the orbit in the equilibrium state. Since $v=0$ is the turning point of the trapped particles, $\partial/\partial t = 0$ ensures that the trapping potential is perfectly reflecting. There is no particle loss from the well, so that $\partial f/\partial v^2 = m\ \ \partial f/\partial W$ has no discontinuity at $v=0$.

As the trapped particles move along the contours of constant $f(W)$, they do so in such a way as to conserve phase space density along the orbits. This incompressible property of the phase space determines the dynamics of the distribution f. We pursue the viewpoint of Eq. (11) further to illustrate this by deriving the following "equations of motion" from Eq. (1)

$$\frac{d}{dt}\left(\frac{\partial f}{\partial v}\right) = -\frac{\partial f}{\partial x} \qquad (12)$$

and

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) = -\frac{e}{m}\frac{\partial E}{\partial x}\frac{\partial f}{\partial v}\,. \tag{13}$$

These relations express the incompressibility of the phase space fluid since an x-shear in f causes a counter v-shear in f (and vice-versa). This "restoring force" tendency is evident by combining Eq. (12) and Eq. (13). We find that

$$\frac{d^2}{dt^2}\left(\frac{\partial f}{\partial v}\right) = \frac{e}{m}\frac{\partial E}{\partial x}\frac{\partial f}{\partial v} \tag{14}$$

For well trapped particles ($\omega^2 = -\frac{e}{m}\,\partial E/\partial x$ = constant), the $\partial f/\partial v$ shear undergoes simple harmonic oscillation in the particle's frame of reference. This motion is also true of $\partial f/\partial x$. As a result, $\partial f/\partial v$ and $\partial f/\partial x$ will phase mix into each other, since Eq. (12) and Eq. (13) imply

$$\frac{d}{dt}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \omega^2\left(\frac{\partial f}{\partial v}\right)^2\right] = 0 \tag{15}$$

if $d\omega^2/dt = 0$. Therefore, a distribution $f=\delta(x)$ will be changed into the distribution $f = \delta(v)$ because of the conservation of phase space density along well trapped particle orbits.

The equations of motion (12) and (13) can be put in canonical form by defining the canonical "position" (q) and "momentum" (p) variables as

$$q = \partial f/\partial v \quad \text{and} \quad p = -\, \partial f/\partial x \tag{16}$$

so that

$$\frac{dq}{dt} = p \quad \text{and} \quad \frac{dp}{dt} = \frac{e}{m}\,\frac{\partial E}{\partial x}\, q \tag{17}$$

Eq. (14) then resembles an "acceleration" equation satisfying a Hookes law restoring force for $\omega^2 = -\frac{e}{m}\,\partial E/\partial x > 0$. We define the "energy density"

$$2H = \left(\frac{\partial f}{\partial x}\right)^2 - \frac{e}{m}\,\frac{\partial E}{\partial x}\,\left(\frac{\partial f}{\partial v}\right)^2 \tag{18}$$

so that Eq. (15) becomes the conservation of total energy. The quantity H is proportional to a particle's total energy W. We write

$$2H = \left(\frac{\partial f}{\partial W}\right)^2 \left[\left(\frac{\partial W}{\partial x}\right)^2 - \frac{e}{m}\,\frac{\partial E}{\partial x}\,\left(\frac{\partial W}{\partial v}\right)^2\right] \tag{19}$$

for the equilibrium state, so that

$$H = \frac{1}{2}\, m^2 \omega^2 \left(\frac{\partial f}{\partial W}\right)^2 (v^2 + \omega^2 x^2) \tag{20}$$

where we've used $\omega^2 = -\frac{e}{m}\,\partial E/\partial x$ and $W = \frac{1}{2}\, mv^2 + e\phi = \frac{1}{2}m(v^2+\omega^2 x^2)$ for the well trapped case. The "Lagrangian density"

$$2L = \left(\frac{\partial f}{\partial x}\right)^2 + \frac{e}{m}\frac{\partial E}{\partial x}\left(\frac{\partial f}{\partial v}\right)^2 \tag{21}$$

gives $p = \partial L/\partial \dot{q} = \partial L/\partial p$ since $\dot{q} = dq/dt = p$ from Eq. (17). We note also that $\dot{q}\partial L/\partial \dot{q} - L = H$ as expected. It is now straightforward to show that the equations of motion follow in the usual way from Hamilton's equatons or the Lagrangian Variational principle.[7]

These considerations suggest that the equations of motion (12) and (13) satisfy their own Virial Theorem in the equilibrium state. Let T be the kinetic energy density, and U be the potential energy density that is an n^{th} degree polynomial of the position. The Virial theorem is $2\overline{T} = n\overline{U}$ where bar denotes a time average. For our case, $U = -\frac{e}{m}\,\partial E/\partial x(\partial f/\partial v)^2 = \omega^2 q^2$ so that n=2. For well trapped particles, $p = -\partial f/\partial x$ and $q = \partial f/\partial v$ undergo simple harmonic oscillation and are thus 90° out of phase. Then

$$\overline{\frac{d}{dt}(pq)} = -\overline{\frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial v}\right)} = 0 \tag{22}$$

But from Eq. (1) we derive

$$\frac{1}{2}\frac{d^2}{dt^2}q^2 = \frac{d}{dt}(pq) = p^2 - \omega^2 q^2 \tag{23}$$

where $\omega^2 = -\frac{e}{m}\,\partial E/\partial x$. The Virial Theorem thus gives, $\overline{L} = \overline{p}^2 - \overline{\omega^2 q^2} = 0$ or $\overline{T} = \overline{U}$ as expected. This is equivalent to Eq. (10) if we assume that an ensemble average is equivalent to a time average.

It has been argued that the results of this Letter can describe some of the particle trapping properties of nonlinear fluctuations in a turbulent plasma.[8] In particular, the use of Eq. (17) in evaluating the plasma response[9,10] allows the integration of $\partial f[x(t), v(t), t]/\partial v(t)$ along trapped particle orbits. The result is a kinetic equation that describes the tendency for particle trapping as well as diffusion of turbulent fluctuations.[8]

Finally, we note that since Eq. (1) is merely a statement of the particle orbits, relations analogous to Eq. (17) can be derived to describe the particle orbits $x(t)$ and $v(t)$ explicitly.[11,12]

This work has been supported by the National Science Foundation and United States Department of Energy.

REFERENCES:

1. H.L. Berk, C.E. Nielsen, K.V. Roberts, Phys. Fluids 13 980 (1970).

2. F. Hohl and M. Feix, Astrophys. J. 147, 1164 (1967)

3. T.H. Dupree, "Theory of Phase Space Holes", to be published.

4. I.B. Bernstein, J.M. Green, M.D. Kruskal, Phys. Rev. 108, 546 (1957).

5. T.M. O'Neil, J.H. Winfrey, and J.H. Malmberg, Phys. Fluids 14, 1204 (1971).

6. T.H. Dupree, Phys. Fluids, 17, 100 (1974)

7. H. Goldstein, Classical Mechanics, Addision Wesley 1980.

8. D.J. Tetreault, "Phase Space Dynamics", Bull. Amer. Phys. Soc., 26, 1040 (1981), to be published.

9. T.H. Dupree, Phys. Fluids 9, 1773 (1966).

10. T.H. Dupree, Phys. Fluids 15, 334 (1972).

11. D.J. Tetreault, "Renormalization of the Wave Particle Resonance in Turbulent Plasma", Ph.D. Thesis (MIT) April 1976, unpublished.

12. It has been brought to the author's attention that a related investigation occurs in P. Koch, Phys. Fluids 16, 651 (1973).