PFC/JA-82-23

Three-Dimensional Theory of Waveguide-Plasma Coupling

A. Bers, K.S. Theilhaber

Plasma Fusion Center
Massachusetts Institute of Technology,
Cambridge, MA 02139

PFC/JA-82-23

THREE-DIMENSIONAL THEORY OF WAVEGUIDE-PLASMA COUPLING*

A. BERS, K.S. THEILHABER
Plasma Fusion Center,
Massachusetts Institute of Technology,
Cambridge, Massachusetts,
United States of America

ABSTRACT. A general, linear theory of coupling electromagnetic fields from free-space waveguide arrays to an inhomogeneous plasma in a magnetic field is presented. In contrast to previous analyses, which assumed parallel-plate waveguides, waveguides having closed cross-sections of arbitrary shape are considered; full account is taken of all the waveguide modes. Far away from the coupling region, the plasma is assumed to be absorptive and the waveguides to propagate only their dominant modes. The formulation is geared to obtain the reflection coefficients in the waveguides and the excitation of waves in the plasma. The results are applicable to RF heating of plasmas whose size is large compared to the waveguide array, as would be the case in a fusion reactor, and valid for the frequency regimes of either ion-cyclotron (harmonic), or lower-hybrid, or electron-cyclotron (harmonic) heating.

1. INTRODUCTION

In what is generically known as 'RF heating and current generation' of plasmas, electromagnetic (EM) energy from high-power sources external to the plasma is used to excite electromagnetic fields that propagate into the plasma and eventually dissipate their energy and momentum on the charged particles of the plasma, thus achieving the desired heating and/or current generation. One important aspect of such interactions is the coupling of power from the EM sources in free-space to the EM fields in the plasma. In magnetically confined plasmas with appreciable bulk temperatures, which are of interest in achieving fusion temperatures in a quasi steady-state, coupling structures cannot be inserted into the plasma and such coupling must take place at the plasma edge. Under these circumstances, and for technological reasons which we shall not go into here, it is useful to consider EM sources at high frequencies whose energy can be brought conveniently to the plasma edge by waveguides. Specific heating and current generation schemes require that the excited fields have a prescribed spatial spectrum. This then requires that the waveguides be fed with specific relative amplitudes and phases, thus forming waveguide arrays.

* Work supported by DoE Contract DE-AC02-78ET-51013 and NSF Grant No.ENG 79-07047.

In this paper, we formulate a general linear theory to describe the coupling of EM power from waveguides to a plasma in a magnetic field. The coupling region (Fig. 1) is taken as the inhomogeneous plasma extending from the free-space waveguide openings in the plasma wall to where the desired plasma modes are excited. The excited plasma modes are assumed to be dissipated beyond the coupling region, further into the plasma core. The plasma is taken to be of sufficiently large (reactor) size so that the limited extent of the coupling region can be modelled in slab geometry (Figs 2 and 3). For definiteness, the plasma in the coupling region is described by the cold-plasma model with inhomogeneous density and magnetic field in one direction, into the plasma. The walls of the plasma and waveguides are assumed perfectly conducting, and *the waveguides are allowed to be of arbitrary cross-section;* the medium in the waveguides is taken as free-space. Far from the plasma wall the waveguides are assumed to propagate only their dominant mode. Given the amplitudes and phases of the incident fields in each of the waveguides and the unperturbed characteristics of the plasma in the coupling region, we show how to determine the reflection coefficient in each of the waveguides and the excited fields and power flow into the plasma modes in and beyond the coupling region.

The plasma modes amenable to waveguide excitation for heating a reactor-type plasma fall into three frequency ranges: (a) ion-cyclotron range of frequencies (ICRF), from the ion-cyclotron frequency ($\Omega_i/2\pi$) to

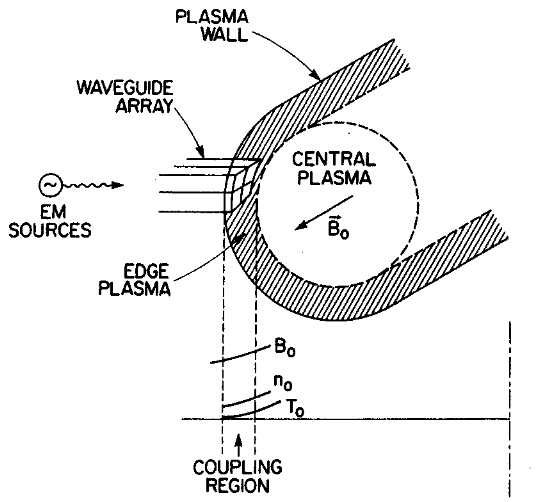

FIG. 1. Waveguide-plasma coupling geometry. The coupling region extends from the plasma wall into the edge plasma; it does not include the hot, central plasma where the excited waves are assumed to be dissipated. For a large plasma, the coupling region can be modelled in slab geometry.

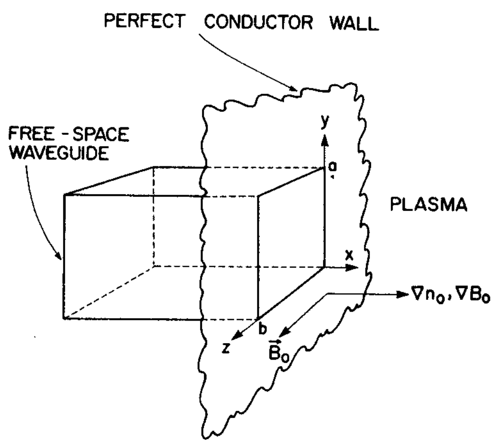

FIG. 2. Geometry used in the analysis of coupling from a single waveguide to a plasma.

a few times $(\Omega_i/2\pi)$; (b) lower-hybrid range of frequencies (LHRF) which is around and above the ion plasma frequency $(\omega_{pi}/2\pi)$. (c) electron cyclotron range of frequencies (ECRF) from the the electron cyclotron frequency $(\Omega_e/2\pi)$ to a few times $(\Omega_e/2\pi)$. For magnetically confined fusion plasmas, in externally applied magnetic fields in the range of 4–6 T, the ICRF would involve waveguides whose cross-sectional dimensions would be comparable to a reactor plasma radius; for plasma densities in the range of $10^{20}-10^{21}$ m^{-3}, the LHRF fall in the usual microwave regime and the waveguide dimensions become smaller than a reactor plasma radius; finally, in the ECRF the waveguides are much smaller than the plasma radius. In all cases, the finite size of all the cross-sectional dimensions of the waveguide relative to the plasma dimensions must be accounted for in the coupling problem.

Previous analyses of the waveguide-plasma coupling problem have concentrated on the LHRF [1–4]. All these analyses were carried out in detail in only two dimensions: the waveguides were assumed to be perfectly conducting parallel-plates of infinite extent in the direction perpendicular to the confining magnetic field $\bar{B}_0$, and the excited plasma modes were thus forced to have no variation in that direction. Most analyses were also concerned mainly with the dominant mode in the waveguides and neglected all of the higher-order modes that are required for a proper description of the fields at the plasma wall where the waveguide ends. In the present analysis, these restrictions are removed; full account is taken of the finite dimensions of the waveguides in both the direction along $\bar{B}_0$ and the direction perpendicular to $\bar{B}_0$, and all the higher-order waveguide modes are accounted for in the vicinity of the waveguide opening in the plasma wall. In addition, we do not restrict ourselves to any particular frequency regime. Instead, the differential equations describing the fields in the inhomogeneous plasma, in the coupling region, are derived to apply to any of the frequency regimes. In general, these are four coupled first-order differential equations with non-constant coefficients, and without further approximations their solutions must be arrived at by numerical techniques. Such numerical integrations are not carried out in this paper, but as a guide to work

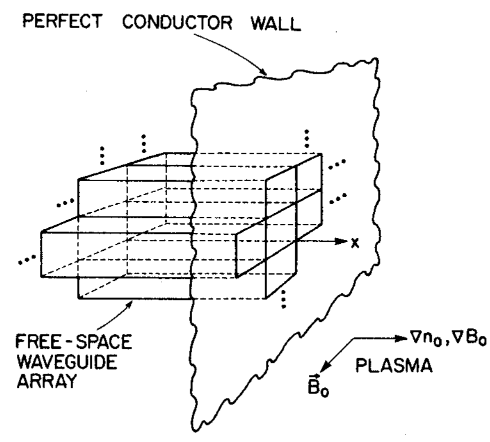

FIG. 3. Geometry used in the analysis of coupling from multiple waveguides – waveguide arrays – to a plasma.

in the future some approximate solutions are outlined. This together with the mode expansion for the fields in the waveguides is shown to lead to a complete solution of the linear coupling problem.

Section 2 describes the mode expansion of the waveguide fields. Section 3 describes the plasma fields and their Fourier transforms in the two directions of assumed plasma homogeneity. Section 4 gives the solution of the coupling problem from a single waveguide. Section 5 generalizes the solution of the coupling problem to multiple waveguides (waveguide arrays) independently fed in amplitude and phase. Section 6 gives some approximate solutions relevant to the LHRF. Section 7 summarizes the results and discusses the limitations of the assumed model and analysis.

2. THE WAVEGUIDE FIELDS

We consider an arbitrary cylindrical waveguide, i.e. one which is uniform in the x-direction and of arbitrary, closed cross-section transverse to x. Figure 2 illustrates a waveguide of rectangular cross-section, but our description is equally applicable to any other cross-section, e.g. circular or elliptical. For simplicity, we shall assume that the waveguide walls are perfectly conducting, and that the plasma does not penetrate into the waveguide so that the medium enclosed by the waveguide walls is taken as free-space (ϵ_0, μ_0). The solution of Maxwell's equations in such cylindrical waveguides is well known [5]. The electromagnetic fields can be expressed as the superposition of an infinite set of orthogonal E (or TM) modes and H (or TE) modes which are complete. For our purposes, we assume that, at the frequency of interest, the waveguide propagates only its dominant (lowest-order) mode, and all other (higher-order) modes are below cut-off. The transverse fields in the waveguide can then be written as follows:

$$\vec{E}_T^w = \frac{V_{+0}}{1+R}\left(e^{ik_d x} + R e^{-ik_d x}\right)\vec{e}_d(y,z) + \sum_H V_{-H} e^{\alpha_H x}\vec{e}_H(y,z) \quad (1)$$

$$\vec{H}_T^w = \frac{Y_d^w V_{+0}}{1+R}\left(e^{ik_d x} - R e^{-ik_d x}\right)\vec{h}_d(y,z) + \sum_H - Y_H^w V_{-H} e^{\alpha_H x}\vec{h}_H(y,z) \quad (2)$$

The dominant mode is characterized by wavenumber $k_d = (2\pi/\lambda_g)$, normalized transverse field patterns $\vec{e}_d$ and $\vec{h}_d$, waveguide admittance Y_d^w, incident-field amplitude V_{+0} (referenced to $x=0$, or any place an integer multiple of $\lambda_g/2$ to the left of $x=0$), and complex reflection coefficient R. The higher-order modes are all evanescent from $x=0$ toward $x<0$ with spatial decay rate α_H and field amplitude V_{-H}; the summation subscript H stands for all the TE and TM modes in this category [5].

3. THE PLASMA FIELDS

In the coupling region, near the waveguide openings in the perfectly conducting wall (Figs 2 and 3), at a given radian frequency ω, the plasma will be assumed to be described by its cold, inhomogeneous dielectric tensor. For simplicity, the inhomogeneity in the unperturbed density n_0 and applied magnetic field $\vec{B}_0 = \hat{z}B_0$ will be taken in the x-direction. The electromagnetic fields in the plasma can then be Fourier-analysed in y and z, and the complete solution for the fields, through Maxwell's equations, is determined by the solution of four coupled first-order, ordinary differential equations (o.d.e.'s) in x, representing the coupling of 'slow' and 'fast' waves [6, 7]:

$$\frac{d\mathcal{E}_y}{d\xi} = -n_y\frac{K_X}{K_\perp}\mathcal{E}_y + i\frac{n_y n_z}{K_\perp}Z_0\mathcal{H}_y + i\left(1-\frac{n_y^2}{K_\perp}\right)Z_0\mathcal{H}_z \quad (3)$$

$$\frac{d\mathcal{E}_z}{d\xi} = -n_z\frac{K_X}{K_\perp}\mathcal{E}_y - i\left(1-\frac{n_z^2}{K_\perp}\right)Z_0\mathcal{H}_y - i\frac{n_y n_z}{K_\perp}Z_0\mathcal{H}_z \quad (4)$$

$$\frac{dZ_0\mathcal{H}_y}{d\xi} = -in_y n_z\mathcal{E}_y + i(n_y^2 - K_\parallel)\mathcal{E}_z \quad (5)$$

$$\frac{dZ_0\mathcal{H}_z}{d\xi} = -i\left(n_z^2 - K_\perp + \frac{K_X^2}{K_\perp}\right)\mathcal{E}_y + in_y n_z\mathcal{E}_z - \frac{K_X}{K_\perp}n_z Z_0\mathcal{H}_y + \frac{K_X}{K_\perp}n_y Z_0\mathcal{H}_z \quad (6)$$

where we normalized co-ordinate distances to (c/ω), e.g. $(\omega x/c) \equiv \xi$, and take the Fourier transform of the fields y and z: $\vec{\mathcal{E}}(\xi, n_y, n_z) \leftrightarrow \vec{E}$ and $\vec{\mathcal{H}}(\xi, n_y, n_z) \leftrightarrow \vec{H}$, where $n_y = (ck_y/\omega)$ and $n_z = (ck_z/\omega)$ are the indices of refraction in y and z, respectively, and k_y and k_z are the wavenumbers in these directions. In the above equations $K_\perp$, K_X and $K_\parallel$, the cold-plasma dielectric tensor elements, are functions of ξ, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$. These equations are all first-order and are well suited for a numerical treatment. The only singularity in the coefficients occurs when $K_\perp \to 0$, which is simply the hybrid resonance (lower or upper hybrid). If we impose the values of $\mathcal{E}_y$ and $\mathcal{E}_z$ at $\xi = 0$, and require that only outgoing or evanescent waves exist as $\xi \to \infty$, then these boundary conditions determine completely the solution of Eqs (3–6). Without further approximations, these equations must be solved numerically as follows. At the end of the coupling region into the plasma, the 'fast-' and 'slow'-wave fields will be assumed uncoupled and described by their WKB form. Furthermore, assuming that the plasma beyond the coupling region is absorptive, we can take the WKB form for each as outgoing waves of, say, unit amplitudes. The differential equations can then be integrated backwards in x to obtain $\vec{\mathcal{E}}_{T0} \equiv \vec{\mathcal{E}}_T(\xi = 0, k_y, k_z)$ and $\vec{\mathcal{H}}_{T0} \equiv \vec{\mathcal{H}}_T(\xi = 0, k_y, k_z)$ for all k_y and k_z of interest. The integration of the coupled differential equations for $\vec{\mathcal{E}}_T$ and $\vec{\mathcal{H}}_T$ must be carried out with care so that resonant (absorptive) or cut-off (evanescent) regions are properly taken into account for each pair of k_y and k_z. Further details of this are discussed in Section 7. The integration of the coupled differential equations can be considered to determine the matrix $\overleftrightarrow{Y}(\xi)$ that gives

$$\vec{\mathcal{H}}_T(\xi) = \overleftrightarrow{Y}_p(\xi) \cdot \vec{\mathcal{E}}_T(\xi) \tag{7}$$

where $\overleftrightarrow{Y}_p = \overleftrightarrow{Y}_p(\xi, k_y, k_z)$ is the plasma admittance matrix. The electromagnetic fields transverse to x in the plasma at x = 0 are then obtained by computing the inverse Fourier transform of $\vec{\mathcal{E}}_T(\xi = 0) \equiv \vec{\mathcal{E}}_{T0}$ and $\vec{\mathcal{H}}_T(\xi = 0) \equiv \vec{\mathcal{H}}_{T0}$

$$\vec{E}_T^P(x = 0) = \int \frac{d^2k_T}{(2\pi)^2} \vec{\mathcal{E}}_{T0} e^{i\vec{k}_T \cdot \vec{r}_T} \tag{8}$$

$$\vec{H}_T^P(x = 0) = \int \frac{d^2k_T}{(2\pi)^2} \vec{\mathcal{H}}_{T0} e^{i\vec{k}_T \cdot \vec{r}_T} \tag{9}$$

where $\vec{k}_T = \hat{y}k_y + \hat{z}k_z$ and $\vec{r}_T = \hat{y}y + \hat{z}z$, and, using Eq.(7),

$$\vec{\mathcal{H}}_{T0} = \overleftrightarrow{Y}_p(\xi = 0) \cdot \vec{\mathcal{E}}_{T0} \equiv \overleftrightarrow{Y}_{po} \cdot \vec{\mathcal{E}}_{T0} \tag{10}$$

4. THE WAVEGUIDE-PLASMA COUPLING

For coupling with a single waveguide, as shown in Fig.2, the waveguide transverse electric and magnetic fields at x = 0 must match onto the plasma fields, i.e.

$$\vec{E}_T^w(x = 0) = \vec{E}_T^P(x = 0) \tag{11}$$

and

$$\vec{H}_T^w(x = 0) = \vec{H}_T^P(x = 0) \tag{12}$$

Using Eqs (1), (2), (8), (9), and (10), we can write expressions (11) and (12) as

$$V_{+0}\vec{e}_d + \sum_H V_{-H}\vec{e}_H = \int \frac{d^2k_T}{(2\pi)^2} \vec{\mathcal{E}}_{T0} e^{i\vec{k}_T \cdot \vec{r}_T} \tag{13}$$

$$Y_d^w \frac{1-R}{1+R} V_{+0}\vec{h}_d - \sum_H Y_H^w V_{-H}\vec{h}_H = \int \frac{d^2k_T}{(2\pi)^2} \overleftrightarrow{Y}_{po} \cdot \vec{\mathcal{E}}_{T0} e^{i\vec{k}_T \cdot \vec{r}_T} \tag{14}$$

where we note that

$$Y_d^w \frac{1-R}{1+R} \equiv Y_d^{Pw} \tag{15}$$

is the admittance of the plasma presented to the dominant mode of the waveguide. Equations (13) and (14) determine, in principle, this admittance (and hence R) as well as the amplitudes of the higher-order modes per unit of incident-field amplitude, i.e. $(V_{-H}/V_{+0}) \equiv \hat{V}_{-H}$. To write this out explicitly, we take the Fourier transform in y and z of Eq.(13) to obtain

$$\vec{\mathcal{E}}_{T0} = V_{+0}\,\tilde{\vec{e}}_d + \sum_H V_{-H}\tilde{\vec{e}}_H \tag{16}$$

where $\tilde{\vec{e}}_i = \int d^2r_T \vec{e}_i \exp(-i\vec{k}_T \cdot \vec{r}_T)$ is the Fourier transform of $\vec{e}_i$, and substitute Eq.(16) into expression (14) to find

$$Y_d^{Pw}\vec{h}_d - \sum_H Y_H^w \hat{V}_{-H}\vec{h}_H = \int \frac{d^2k_T}{(2\pi)^2} \overleftrightarrow{Y}_{po} \cdot \left[\tilde{\vec{e}}_d + \sum_H \hat{V}_{-H}\tilde{\vec{e}}_H\right] e^{i\vec{k}_T \cdot \vec{r}_T} \tag{17}$$

Now, using the orthogonality properties of the waveguide field patterns [5], and introducing the *coupling admittances*

$$Y_{ij} \equiv \int \frac{d^2k_T}{(2\pi)^2} \tilde{\vec{\eta}}_i^* \cdot \overleftrightarrow{Y}_{po} \cdot \tilde{\vec{e}}_j \tag{18}$$

where $\vec{\eta}_i = \int d^2 r_T \vec{h}_i \exp(-i\vec{k}_T \cdot \vec{r}_T)$ in the Fourier transform of $\vec{h}_i$, we extract from Eq.(17)

$$Y_d^{Pw} - \sum_H Y_{dH} \hat{V}_{-H} = Y_{dd} \tag{19}$$

and

$$\sum_H (Y_{H'}^w \delta_{H'H} + Y_{H'H}) \hat{V}_{-H} = -Y_{H'd} \tag{20}$$

Equations (19) and (20) can be solved for the normalized higher-order mode field amplitudes $\hat{V}_{-H}$ and Y_d^{Pw} which by relation (12) determine the complex reflection coefficient R:

$$R = \frac{1-\hat{Y}_d^{Pw}}{1+\hat{Y}_d^{Pw}} \tag{21}$$

where

$$\hat{Y}_d^{Pw} \equiv \frac{Y_d^{Pw}}{Y_d^w}$$

Note that Eq.(19) represents the coupling of the dominant mode to the higher-order modes, and Eq.(20) gives the coupling of the higher-order modes among themselves.

The importance of higher-order modes can be determined by solving various truncated forms of Eqs (19) and (20). If we ignore all the higher-order modes, then from Eq.(19) we simply have

$$Y_d^{Pw} = Y_{dd} \tag{22}$$

If account is taken of one higher-order mode, say H = 1, Eqs (19) and (20) give a two-by-two matrix of equations which solves readily to give

$$Y_d^{Pw} = Y_{dd} - Y_{d1} \frac{Y_{1d}}{Y_1^w + Y_{11}} \tag{23}$$

and

$$\hat{V}_{-1} = \frac{-Y_{1d}}{Y_1^w + Y_{11}} \tag{24}$$

In general, Eqs (19) and (20) form a square matrix of equations of order equal to the number of higher-order modes plus one. For computational purposes proper ordering and truncation can be determined from estimates of expression (18) for different pairs of modes.

Having determined Y_d^{Pw} and the $\hat{V}_{-H}$, Eqs (13) and (14) give

$$\vec{\mathcal{E}}_{T0} = V_{+0}\left[\vec{e}_d + \sum_H \hat{V}_{-H}\vec{e}_H\right] \tag{25}$$

$$\vec{\mathcal{H}}_{T0} = V_{+0}\left[Y_d^{Pw}\vec{\eta}_d - \sum_H Y_H^w \hat{V}_{-H}\vec{\eta}_H\right] \tag{26}$$

Using these we can then obtain the electromagnetic field spectra in the plasma as excited by the dominant-mode incident-field amplitude (V_{+0}) in the waveguide, and the power flow into the plasma.

5. COUPLING FROM WAVEGUIDE ARRAYS TO THE PLASMA

To have more control on the excited spectrum of fields in the plasma it is common to use multiple waveguides appropriately phased, i.e. waveguide arrays, as shown for example in Fig.3. The coupling analysis of the preceding section is easily generalized to this case. The matching of the tangential $\vec{E}$ and $\vec{H}$ fields, Eqs (11) and (12), take on the more general form

$$\sum_w \left[V_{+0}^w \vec{e}_d^{\,w} + \sum_H V_{-H}^w \vec{e}_H^{\,w}\right] = \int \frac{d^2 k_T}{(2\pi)^2} \vec{\mathcal{E}}_{T0}\, e^{i\vec{k}_T \cdot \vec{r}_T} \tag{27}$$

$$\sum_w \left[Y_d^{Pw} V_{+0}^w \vec{h}_d^{\,w} - \sum_H Y_H^w V_{-H}^w \vec{h}_H^{\,w}\right]$$

$$= \int \frac{d^2 k_T}{(2\pi)^2} \overleftrightarrow{Y}_{P0} \cdot \vec{\mathcal{E}}_{T0}\, e^{i\vec{k}_T \cdot \vec{r}_T} \tag{28}$$

where the summation over the superscript w is over all the waveguides in the array. The Fourier transform of Eq.(27) gives

$$\vec{\mathcal{E}}_{T0} = \sum_w \left[V_{+0}^w \vec{\epsilon}_d^{\,w} + \sum_H V_{-H}^w \vec{\epsilon}_H^{\,w}\right] \tag{29}$$

where $\vec{\epsilon}_i^{\,w}(k_y, k_z) \leftrightarrow \vec{e}_i^{\,w}(y, z)$ is a Fourier transform pair, and using expression (29) in Eq.(28) we obtain

$$\sum_w \left[Y_d^{Pw} V_{+0}^w \vec{h}_d^{\,w} - \sum_H Y_H^w V_{-H}^w \vec{h}_H^{\,w}\right]$$

$$= \int \frac{d^2 k_T}{(2\pi)^2} \overleftrightarrow{Y}_{p0} \cdot \sum_w \left[V_{+0}^w \vec{\epsilon}_d^{\,w} + \sum_H V_{-H}^w \vec{\epsilon}_H^{\,w}\right] e^{i\vec{k}_T \cdot \vec{r}_T} \tag{30}$$

Let the waveguides be designated by superscripts $w = \alpha, \beta, \gamma ...$, and the modes in each waveguide by the vertically aligned subscript $(i, j) = d$ or H. Using the orthogonality properties [5] of the α-waveguide field patterns, and introducing the *coupling admittances*

$$Y_{ij}^{\alpha w} \equiv \int \frac{d^2 k_T}{(2\pi)^2} \vec{\eta}_i^{\alpha *} \cdot \overleftrightarrow{Y}_{po} \cdot \vec{e}_j^{w} \tag{31}$$

where $\vec{\eta}_i^{\alpha}(k_y, k_z) \leftrightarrow \vec{h}_i^{\alpha}(y, z)$ is a Fourier transform pair, we find from Eq.(30):

$$Y_d^{P\alpha} - \sum_H Y_{dH}^{\alpha\alpha} \hat{V}_{-H}^{\alpha\alpha} - \sum_{\substack{w \neq \alpha \\ H}} Y_{dH}^{\alpha w} \hat{V}_{-H}^{w\alpha}$$

$$= Y_{dd}^{\alpha\alpha} + \sum_{w \neq \alpha} Y_{dd}^{\alpha w} \hat{V}_{+0}^{w\alpha} \tag{32}$$

and

$$\sum_H (Y_{H'}^{\alpha} \delta_{H'H} + Y_{H'H}^{\alpha\alpha}) \hat{V}_{-H}^{\alpha\alpha} + \sum_{\substack{w \neq \alpha \\ H}} Y_{H'H}^{\alpha w} \hat{V}_{-H}^{w\alpha}$$

$$= - Y_{H'd}^{\alpha\alpha} - \sum_{w \neq \alpha} Y_{H'd}^{\alpha w} \hat{V}_{+0}^{w\alpha} \tag{33}$$

where the mode amplitudes have been normalized to the incident dominant mode amplitude of waveguide α,

$$\hat{V}_{+0}^{w\alpha} \equiv \frac{V_{+0}^{w}}{V_{+0}^{\alpha}}; \qquad \hat{V}_{-H}^{\alpha\alpha} \equiv \frac{V_{-H}^{\alpha}}{V_{+0}^{\alpha}}; \text{ and } \hat{V}_{-H}^{w\alpha} \equiv \frac{V_{-H}^{w}}{V_{+0}^{\alpha}} \tag{34}$$

The first two terms on the left-hand sides of Eqs (32) and (33) and the first terms on their right-hand sides are, of course, the same as in Eqs (19) and (20), respectively, and represent the coupling of all the modes in waveguide α. The last term on the left-hand side of Eq.(32) represents the coupling of the dominant mode in waveguide α to the higher-order modes in all the other waveguides, and the last term on the left-hand side of Eq.(33) is due to the coupling of the higher-order modes in waveguide α to the higher-order modes in all the other waveguides. Finally, the second terms on the right-hand sides of Eqs (32) and (33) represent the driving by the incident dominant-mode fields in all the other waveguides of, respectively, the dominant and higher-order modes in the α-waveguide. Note from (34) that the normalized, complex amplitudes $\hat{V}_{+0}^{w\alpha}$ contain the relative phasings in the incident dominant-mode fields of the various waveguides.

Proceeding in similar manner to apply to Eq.(30) the orthogonality properties of the field patterns of the other waveguides $\beta, \gamma, ...$ we obtain a set of Eqs (32) and (33) for each waveguide by simply replacing α with $\beta, \gamma, ...$. The simultaneous solution of all these sets of equations gives the higher-order mode amplitudes and reflection coefficient (through Y_d^{Pw} and (21)) in each of the waveguides. Again, proper ordering and truncation of all these equations can be arrived at from estimates of the coupling admittances (31). Thus using the results in expression (29), we can finally also find the electromagnetic field spectra in the plasma, as excited by a given set of dominant-mode incident-field, complex amplitudes (V_{+0}^{w}) in the waveguide array, and the power flow into the plasma.

6. COUPLING AT LOWER-HYBRID FREQUENCIES

The evaluation of the admittance matrix $\overleftrightarrow{Y}_{p0}$ defined in Eq.(10), and which is essential in the calculation of the coupling admittances, (31), will in general require a numerical computation. However, in the lower-hybrid frequency range, an approximate decoupling of Eqs (3–6) can be made. The result is an analytic expression valid in most (but not all) of the Fourier space (n_y, n_z).

For the fast wave, decoupling comes from assuming a shorted parallel electric field, $\mathcal{E}_z = 0$, and the resulting equation for $\mathcal{E}_y$ is:

$$\frac{d^2 \mathcal{E}_F}{d\xi^2} + \left(\frac{K_X^2}{n_z^2 - K_\perp} + K_\perp - n_y^2 - n_z^2 + \frac{n_y K_X'}{n_z^2 - K_\perp} \right) \mathcal{E}_F = 0 \tag{35}$$

so that we have: $[\mathcal{E}_y, \mathcal{E}_z]_F = [\mathcal{E}_F(\xi), 0]$. For the slow wave, we assume the local polarization, $\mathcal{E}_y \cong [n_y n_z / (n_z^2 - K_\perp)] \mathcal{E}_z$, and we find:

$$\frac{d^2 \mathcal{E}_S}{d\xi^2} + \left(\frac{-K_\parallel}{K_\perp} (n_z^2 - K_\perp) - n_y^2 \right) \mathcal{E}_S = 0 \tag{36}$$

with $[\mathcal{E}_y, \mathcal{E}_z]_S = [\mathcal{E}_S, (n_y n_z / (n_z^2 - K_\perp)) \mathcal{E}_S]$. Let us assume a linear density profile near the plasma edge, writing $K_\perp \cong 1$, $K_\perp = 1 - \alpha\xi$, $K_X = \beta\xi$ where

$$\alpha = \frac{d}{d\xi}\,(\omega_{pe}^2(\xi)/\omega^2)\Big|_{\xi=0}$$

$$\beta = \frac{d}{d\xi}\,(\omega_{pe}^2(\xi)/\omega\Omega_e)\Big|_{\xi=0}$$

and Eq.(36) can then be solved in terms of parabolic cylinder functions and Airy functions, respectively. The final result is:

$$\overleftrightarrow{Y}_{po}(n_y, n_z) = Y_0\,\frac{ig_S(n_y, n_z)}{n_z^2-1}\,\hat{y}\hat{z} + Y_0\,\frac{ig_F(n_y, n_z)}{n_z^2-1}\Big[n_y n_z(-\hat{y}\hat{y} + \hat{z}\hat{z}) + \hat{y}\hat{z}\,\frac{n_y^2 n_z^2}{n_z^2-1} - \hat{z}\hat{y}\,(n_z^2-1)\Big] \quad (37)$$

where the admittance terms g_S and g_F are:

$$g_S(n_y, n_z) = \alpha^{1/3} e^{-i\pi/3} (n_z^2-1)^{1/3}\,\frac{Ai'(0)}{Ai(0)} \quad (38)$$

$$g_F(n_y, n_z) = -\,2\beta^{1/2}(1-n_z^2)^{-1/4}\,\frac{\Gamma(3/4 + a/2)}{\Gamma(1/4 + a/2)} \quad (39)$$

and

$$a = -\,\frac{(1-n_z^2)^{1/2}}{2\beta}\left(\frac{\beta n_y}{1-n_z^2} + 1 - n_{yz}^2\right)$$

with $n_{yz}^2 = n_y^2 + n_z^2$. The expression (37) for $\overleftrightarrow{Y}_{po}$ can then be substituted into Eq.(31) for an evaluation of the $Y_{ij}^{\alpha w}$. The limitations on the use of this approximation arise because of the resonant denominators $(n_{yz}^2-1)^{-1}$ and $(n_z^2-1)^{-1}$. When $|n_{yz}|\to 1$ or $|n_z|\to 1$, the assumptions which led to Eqs (35) and (36) break down and Eqs (5) and (6) cannot be decoupled. Furthermore, the resonant terms cannot simply be ignored because they lead to a divergent integral in expression (31). Thus, expression (37) is useful only if the excitation spectrum has little energy near the lines $n_z = \pm 1$ and the circle $n_y^2 + n_z^2 = 1$ in the (n_y, n_z) space; then the singularities in the integral (31) can be avoided by setting the spectrum to zero at some arbitrary distance from the resonance. In effect, this limits the application of expression (37) to phased arrays which create a spectrum in n_z which is well above $n_z = \pm 1$.

We now qualitatively consider excitations designed to couple predominantly to slow or fast waves. In both cases, a TE mode is incident from the waveguide, the difference being in the waveguide aperture's orientation with respect to the static magnetic field. The main result is that because, in general, $g_S/g_F \sim \alpha^{1/3}/\beta^{1/2} \gg 1$, the incident polarization remains dominant in the waveguide and most of the energy is reflected in a TE mode.

For the slow-wave excitation, $\vec{E}_{inc} \cong \hat{z}E_z$ and we can estimate the resulting magnetic field at the plasma edge from expression (37). The result is $H_y/H_z \sim g_S/g_F \gg 1$ and the polarization is approximately TE. The reflection coefficient is found as in Brambilla's theory, but with an additional geometrical factor accounting for the finite extent in y. Specifically, we calculate:

$$Y_{00} = Y_0 \int \frac{dn_y dn_z\, i g_S(n_y, n_z)}{(2\pi)^2 (n_z^2-1)}\,|\epsilon_{z,TE}(n_y, n_z)|^2 \quad (40)$$

and the reflection coefficient is $R = (Y_{00} - Y_{TE}^{w})/(Y_{00} + Y_{TE}^{w})$.

For the fast-wave excitation we find a similar result. The incident field is now $\vec{E} = \hat{y}E_y$ and the resulting polarization from expression (37) has $H_z \sim H_y$. However a small E_z component

$$E_z/E_y \sim \frac{\beta^{1/2}}{\alpha^{1/3}} \ll 1$$

will cancel the H_z, with almost all the energy reflected in the TE mode. We find $R = (Y_{00} - Y_{TE}^{w})/(Y_{00} + Y_{TE}^{w})$ where:

$$Y_{00} = -\,Y_0 \int \frac{dn_y dn_z}{(2\pi)^2}\, i g_F(n_y, n_z)\,\frac{n_z^2-1}{n_{yz}^2-1}\,|\epsilon_{y,TE}(n_y, n_z)|^2 \quad (41)$$

In both cases, scattering into the other polarization results in the excitation of a wave with amplitude $\beta^{1/2}/\alpha^{1/3} \ll 1$ that of the dominant wave. The finite extent in y results in the inclusion of the shape of the n_y spectrum in the calculation of Y_{00} (the $|\epsilon_{z,yTE}(n_y, n_z)|^2$ terms) and also possibly more evanescence (reactance) for large values of n_y.

7. SUMMARY AND DISCUSSION

We have derived a general formalism for the treatment of waveguide coupling in any frequency range, provided the slab geometry adopted for the plasma edge is valid. In particular, coupling in the ICRF and ECRF frequency ranges as well as LHRF can be treated by this method. Furthermore, the formalism accounts for finite waveguide dimensions both along and across $\vec{B}_0$, a feature neglected in all previous treatments of the problem, and allowance is made for the interaction of all possible waveguide modes.

The plasma is characterized by four coupled first-order differential equations which have to be solved for a complete spectrum of wavenumbers. In certain domains of wavenumber space, these equations can be uncoupled and solved analytically. This is illustrated in Section 6 for fast- and slow-wave coupling in the LHRF. In general, however, solutions must be found by numerical integration, and this was assumed in Sections 4 and 5. We can proceed as follows. First, we specify a density profile which joins smoothly to a region of constant density far from the waveguides. In this region of constant density, plane waves propagate, and by choosing only outgoing waves we satisfy the radiation conditions. Starting with arbitrary amplitudes, we integrate back to the waveguides at $x = 0$. This yields linear transfer matrices for $\vec{\mathcal{E}}_T(x)$ and $\vec{\mathcal{H}}_T(x)$ in terms of the amplitudes of the plane waves at large x. Eliminating these amplitudes yields the admittance matrix $\vec{\vec{Y}}_p(x)$ defined in Eq.(7), and with this admittance known, the problem is reduced to solving a set of linear equations in the waveguide mode amplitudes, as outlined in Sections 4 and 5, respectively, Eqs (19) and (20) for a single waveguide and Eqs (32) and (33) for multiple waveguides, i.e. waveguide arrays. In general, the unknown waveguide mode amplitudes consist of the reflected wave of the dominant mode and all the higher-order modes. The coupling of these amplitudes in the linear equations, just mentioned, is given by expressions (18) and (31), respectively, for the single waveguide and the waveguide arrays.

We note that the backward, numerical integration of the plasma equations (3–6) assumes that the resonance $K_\perp \to 0$ does not occur in the coupling region. This is so in the LHRF where the LHR is chosen to be in the central part of the plasma. However, in the ICRF it may be that $K_\perp \to 0$ in the edge plasma region, indicating that coupling can occur to the slow wave. In such cases Eqs (3–6) may have to be supplemented with thermal correction terms.

Finally, we should like to point out two limitations of the model and analysis presented. The first relates to the assumption that the plasma does not extend into the waveguides. Waveguides containing a plasma in a magnetic field can have mode structures [7] very different from empty waveguides, and this may modify the linear coupling problem in an important way. Secondly, non-linear effects in the edge plasma, due to pondermotive forces [9], can modify the plasma dynamics and hence the coupling problem. Much attention has been given to this recently for coupling in the LHRF [10–13].

ACKNOWLEDGEMENTS

This work was supported in part by the US-DoE Contract DE-AC02-78ET-51013, and in part by the US-NSF Grant No.ENG 79-07047. The authors also wish to thank J. Freidberg and D. Hewett for several useful discussions.

REFERENCES

[1] BRAMBILLA, M., Nucl. Fusion **16** (1976) 47.
[2] KRAPCHEV, V., BERS, A., Nucl. Fusion **18** (1978) 519.
[3] BRAMBILLA, M., Nucl. Fusion **19** (1979) 1343.
[4] THEILHABER, K., BERS, A., Nucl. Fusion **20** (1980) 547.
[5] MARCUVITZ, N., Waveguide Handbook, M.I.T. Radiation Laboratory Sciences, Vol.10, McGraw-Hill Book Co., Inc., New York (1951); COLLIN, R.E., Field Theory of Guided Waves, McGraw-Hill Book Co., Inc., New York (1960); HARRINGTON, R.F., Time Harmonic Electromagnetic Fields, McGraw-Hill Book Co., Inc., New York (1961).
[6] In a local sense, the solution of the fourth-order dispersion relation for a cold plasma, with given ω, k_y and k_z, results in two values of k_x^2[7]; we call the larger one the 'slow' wave and the smaller one the 'fast' wave.
[7] ALLIS, W.P., BUCHSBAUM, S.J., BERS, A., Waves in Anisotropic Plasmas, MIT Press, Cambridge, Mass. (1963).
[8] ABRAMOWITZ, M., STEGUN, I.A., Handbook of Mathematical Functions, National Bureau of Standards, 10th Printing (1970), pp.435, 685.
[9] KRAPCHEV, V., BERS, A., in RF Plasma Heating (Proc. 3rd Top. Conf. California Institute of Technology, Pasadena, California 1978) G5.1–G5.4.
[10] CHAN, V., CHIU, S., Phys. Fluids **20** (1977) 1164.
[11] FUKUYAMA, A., MORISHITA, T., FURUTANI, Y., Plasma Phys. **22** (1980) 565.
[12] KRAPCHEV, V., THEILHABER, K., KO, K., BERS, A., Phys. Rev. Lett. **46** (1981) 1398.
[13] THEILHABER, K., Nonlinear Coupling to Lower Hybrid Waves in a Tokamak Plasma, Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1981; also Nucl. Fusion **22** (1982) 363.

(Manuscript received 17 November 1981
Final manuscript received 5 October 1982)