PFC/JA-84-28

Cavity Mode Analysis of Plasma Frequency Waves in Inhomogeneous Cylindrical Plasmas

I. H. Hutchinson and R. F. Gandy

Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, MA 02139

July, 1984

This work was supported by the U.S. Department of Energy Contract No. DE-AC02-78ET51013. Reproduction, translation, publication, use and disposal, in whole or in part by or for the United States government is permitted.

CAVITY MODE ANALYSIS OF PLASMA FREQUENCY WAVES IN INHOMOGENEOUS CYLINDRICAL PLASMAS

I. H. Hutchinson and R. F. Gandy[a)]

Massachusetts Institute of Technology

Plasma Fusion Center

Cambridge, MA 02139

Abstract

A linear, electromagnetic analysis is presented of the cavity mode structure and dispersion relations for waves with frequency near the electron plasma frequency in cylindrical plasmas with monotonically decreasing electron density and finite magnetic field. Analytic solutions are obtained which are exact in the limits when either the ratio of electron cyclotron to plasma frequency or the ratio of plasma size to parallel wavelength tends to infinity. Comparison with numerical solutions confirms that the analytic results are highly accurate even for fairly modest values of these ratios. Thermal effects are incorporated, including Landau (damping or) gain, which show the lowest order transverse modes to have greatest gain. These modes are highly localized near the cylinder axis so that the plasma itself acts as a cavity, regardless of edge boundary conditions. The theory thus enables an interpretation to be made of maser action in quasi-cylindrical plasmas such as Tokamaks.

PACS#'s 52.35.Hr, 52.40.Fd, 52.70.Gw

I. Introduction

The characteristics of electromagnetic wave propagation in a bounded plasma have proven to be of recurrent interest. Trivelpiece and Gould[1] showed how the dispersion properties of bounded space charge waves could be explained and Bers[2] presented a rather general formalism for treating the problem and gave analyses of a variety of cases theoretically. In the years since then, various extensions and applications of these anlayses have been given.[3]

This continuing interest arises in part, of course, because of the bounded nature of laboratory plasmas which require this analysis for a proper understanding of their behavior. Theoretically, the interest and difficulty of the problem arises because the analysis can no longer be carried out with Fourier transformed quantities leading to algebraic equations, as in the infinite homogeneous plasma case, but must retain differential equations in the coordinate direction of the plasma inhomogeneity. Moreover, in the general case, the differential system obtained is of fourth order because of mode coupling, rather than the more familiar second order systems which are common in theoretical physics.

The problem has strong analogies with important problems in optical and microwave guides, although the plasma case, when a magnetic field is present, represents much stronger anisotropy (birefringence and optical activity) than typical optical fibre guides.

The present anlaysis is motivated by the experimental observation in toroidal plasmas, such as Tokamaks, of electromagnetic emission at (or near) the electron plasma frequency[4] when energetic non-thermal electrons are present in the plasma. Undoubtedly, in most cases the initial mechanism for excitation within the plasma is via the Cerenkov resonance.[5,6] That is, the waves are slow waves whose phase velocity along the field (which in Tokamaks is also essentially along the symmetry direction) is equal to the velocity of the exciting particles and hence less than c.

To the extent that the toroidal plasma can be approximated by an infinite periodic cylinder (for large aspect ratio) these slow waves retain constant <u>longitudinal</u> k-vector, k, and phase velocity. They are therefore, not propagating but evanescent in the radial direction outside the plasma and in the absence of additional asymmetries would not radiate into free space. Because the plasma is not homogeneous, even within the confinement region, but has a continuous variation with radius of, for example, electron density, there are regions <u>within</u> the plasma too where the waves can be considered evanescent. This designation is rather imprecise, though, because what is really required is not a WKBJ approach which assigns a local radial k-vector (real or imaginary corresponding to propagating or evanescent) but a full wave analysis. This analysis in principle provides us with the complete spatial mode structure of the electromagnetic fields as well as a dispersion relation for each mode when appropriate boundary conditions are applied. The point to be made here is that for the slow waves of interest, the plasma itself (without any nearby dielectric structures) acts as a wave-guide whose energy losses will be negligible if the dielectric losses of the plasma (collisions or Landau damping) are negligible.

The toroidal (periodic cylinder) plasma thus constitutes a high-Q cavity for slow electromagnetic waves. It is the purpose of the present study to calculate the cavity eigenmodes and frequencies for relevant cases, applicable to Tokamaks for example, and to relate the results to the observations of plasma frequency emission.

As far as we can discover, previous analyses are not sufficient for this purpose because they all involve approximations which are inappropriate for the case in mind. For cases in which a magnetic field is present (which is essential to our problem) it is common[7] to take the field to be effectively infinite (electron cyclotron frequency, $\Omega >> $ plasma frequency, ω_p). This allows considerable simplification because it removes the optical activity and causes the modes to be decoupled (and hence second order). Our case of interest, though, is Ω greater than ω_p but not effectively infinite. Another common approximation is that an electrostatic treatment is sufficient,[7] normally requiring refractive index much greater than 1 (phase velocity $<< c$). Again, since we are interested in possibly relativistic nonthermal electron populations, this is inappropriate. A third assumption which is often adopted is that the plasma is uniform with sharp boundaries. This too is not realistic in our case, since smooth variation of plasma density is inherent in most experimental cases and changes the character of the problem.

We address then, the rather general inhomogeneous anisotropic plasma case, in cylindrical geometry. We obtain the full cold-plasma (actually general non-spatially-dispersive medium) coupled wave equations for arbitrary density profile. We solve these for $\omega_p < \Omega$ for the mode corresponding to the E-wave (TM mode) of the uncoupled case. This mode corresponds approximately to the whistler branch of the homogeneous plasma dispersion relation (ordinary mode for $\omega < \omega_p$) and thus represents the extension of the mode obtained with the electrostatic analysis to high phase velocity ($\sim c$). We consider the case where the vacuum wavelength corresponding to the central plasma frequency (ω_{po}) is much smaller than the typical plasma size (a) i.e. $k_p a \gg 1$ where $k_p = \omega_{po}/c$. The modes obtained are strongly localized near the cylinder axis (r=o) for monotonically decreasing density, so it is sufficient and generally appropriate to approximate the density profile as the first two terms of a Taylor series about r=o; i.e. a parabolic profile.

The equations prove to be uncoupled to second order in 1/ka and ω_p/Ω and analytically soluble to this approximation. We obtain the dispersion relation and mode structure as simple algebraic expressions in this approximate treatment. Comparison with a full numerical solution of the unapproximated equations in a typical case demonstrates the excellent accuracy of the analytic treatment.

Warm plasma effects, spatial dispersion (Bohm-Gross), and Landau damping or growth are shown to be readily included in the approximate treatment and the full dispersion relation for realistic plasma parameters is used to calculate the cavity mode spacing for comparison with high resolution experimental observations.

II. Coupled Mode Equations

Our starting point is Maxwell's equations

$$\underline{\nabla} \wedge \underline{E} = \frac{-\partial \underline{B}}{\partial t} \tag{1}$$

$$\underline{\nabla} \wedge \underline{B} = \mu_o \underline{j} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \tag{2}$$

together with the constitutive relation for the current $\underline{j}$ in terms of the fields.

The model we adopt initially for the plasma response is that of a plasma of cold electrons and stationary ions. The importance of this approximation is that thermal corrections to the plasma response give rise to spatial dispersion. For infinite homogeneous plasmas this causes only a modest increase of complexity, but in our finite plasma, where we cannot Fourier transform the equations in all three spatial dimensions, the effect of spatial dispersion is more serious. In our case, the response of a plasma with spatial dispersion cannot be included fully in an algebraic (non-differential) conductivity ($\underline{\underline{\sigma}}$) and hence dielectric constant. For now, therefore, we shall ignore thermal corrections. Following our analysis of the cold plasma case it will become clear how to include the thermal corrections accurate to the approximation of interest. For time dependence $\propto \exp(-i\omega t)$ the plasma may be expressed by writing Eq.(2) as

$$\underline{\nabla} \wedge \underline{B} = \frac{-i\omega}{c^2} \underline{\underline{\varepsilon}} \cdot \underline{E} \tag{3}$$

with the dielectric tensor $\underline{\underline{\varepsilon}} = (\underline{\underline{1}} - (\mu_0 c^2/i\omega)\underline{\underline{\sigma}})$ given by the standard cold plasma result which for $\underline{B}$ in the z-direction can be written

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_\wedge & \varepsilon_x & 0 \\ -\varepsilon_x & \varepsilon_\wedge & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{X}{Y^2-1} & -\dfrac{iXY}{Y^2-1} & 0 \\ +\dfrac{iXY}{Y^2-1} & 1+\dfrac{X}{Y^2-1} & 0 \\ 0 & 0 & 1-X \end{bmatrix} \tag{4}$$

where we use the standard magnetoionic nomenclature:

$$X = \frac{\omega_p^2}{\omega^2} \quad , \quad Y = \frac{\Omega}{\omega} \quad . \tag{5}$$

We now write out Maxwell's equations for a cylindrically symmetric situation with background magnetic field in the z-direction and seek solutions proportional to exp $i(kz + m\theta)$. The equations obtained constitute six homogeneous equations for the six unknowns $\underline{E}$, $\underline{B}$:

$$i \frac{m}{r} E_z - ikE_\theta = i\omega B_r \tag{6}$$

$$i\, k\, E_r - \frac{dE_z}{dr} = i\omega B_\theta \tag{7}$$

$$\frac{1}{r}\frac{d}{dr} rE_\theta - \frac{im}{r} E_r = i\omega B_z \tag{8}$$

$$\frac{im}{r} B_z - ikB_\theta = \frac{-i\omega}{c^2} \varepsilon_\Lambda E_r - \frac{i\omega}{c^2}\varepsilon_x E_\theta \tag{9}$$

$$ikB_r - \frac{dB_z}{dr} = \frac{i\omega}{c^2}\varepsilon_x E_r - \frac{i\omega}{c^2}\varepsilon_\Lambda E_\theta \tag{10}$$

$$\frac{1}{r}\frac{d}{dr} rB_\theta - \frac{im}{r} B_r = \frac{-i\omega}{c^2} \varepsilon E_z \quad . \tag{11}$$

Two of these equations are algebraic so the system is fourth order. This system can be expressed as two coupled-mode equations by using (6) and (9) to eliminate r components then solving the new versions of (7) and (10) for the θ components in terms of the z-components. Substituting back into (8) and (11) one obtains after some algebra:

$$\left\{\frac{1}{r}\frac{d}{dr}\frac{r}{D}\left[N^2-\left(\varepsilon_\Lambda+\varepsilon_x^2/\varepsilon_\Lambda\right)\right]\frac{d}{dr}-\frac{m^2}{r^2D}\left[N^2-\left(\varepsilon_\Lambda+\varepsilon_x^2/\varepsilon_\Lambda\right)\right]+\frac{1}{r}\left(\frac{d}{dr}\frac{mN^2 i\,\varepsilon_x}{D\varepsilon_\Lambda}\right)-k_o^2\varepsilon\right\}\frac{E_z}{c}$$

$$=\left\{\frac{1}{r}\frac{d}{dr}\frac{Nr\,\varepsilon_x}{D\varepsilon_\Lambda}\frac{d}{dr}-\frac{m^2N\varepsilon_x}{r^2D\varepsilon_\Lambda}-\frac{i}{r}\left(\frac{d}{dr}\frac{mN}{D}\left[N^2/\varepsilon_\Lambda-1\right]\right)\right\}B_z \tag{12}$$

$$\left\{\frac{1}{r}\frac{d}{dr}\frac{r}{D}\left(N^2/\varepsilon_\Lambda-1\right)\frac{d}{dr}-\frac{m^2}{r^2D}\left(N^2/\varepsilon_\Lambda-1\right)+\frac{1}{r}\left(\frac{d}{dr}\frac{mi\,\varepsilon_x}{D\varepsilon_\Lambda}\right)-k_o^2\right\}B_z$$

$$=-\left\{\frac{1}{r}\frac{d}{dr}\frac{Nr\,\varepsilon_x}{D\varepsilon_\Lambda}\frac{d}{dr}-\frac{m^2N\varepsilon_x}{r^2D\varepsilon_\Lambda}-\frac{i}{r}\left(\frac{d}{dr}\frac{mN}{D}\left[N^2/\varepsilon_\Lambda-1\right]\right)\right\}\frac{E_z}{c} \tag{13}$$

where N is the <u>longitudinal</u> refractive index, kc/ω , k_o the vacuum k-vector, ω/c, and

$$D = N^2\left(\frac{\varepsilon_x}{\varepsilon_\Lambda}\right)^2+\left(N^2/\varepsilon_\Lambda-1\right)\left(N^2-\left[\varepsilon_\Lambda+\varepsilon_x^2/\varepsilon_\Lambda\right]\right)-\left(N^2-\varepsilon_\Lambda\right)^2/\varepsilon_\Lambda\cdot\varepsilon_x^2/\varepsilon_\Lambda$$

is the determinant of the equation for θ components in terms of z-components.

In these equations, we regard the left hand sides as the dominant part and the right hand sides as the coupling terms. The reason for this identification is that in the limiting cases $Y \to \infty$ or $N \to \infty$ the right hand sides $\to$ o so that we then obtain uncoupled E- and B-modes (TM and TE respectively). Note that in general the coupling can arise from two effects: (1) The optical activity $\varepsilon_x \neq o$, (from the first and second terms) regardless of nonuniformity of the plasma profile, (2) Inhomogeneity, $d\varepsilon_\Lambda/dr \neq o$, (from the third term) even if there is no optical activity,

ε_x=o. The second effect is zero for m=o so the symmetric modes are not coupled by inhomogeneity as such.

The general dispersion problem consists of solving these equations with given X and Y profiles subject to boundary conditions: that E, B should be regular at r=o and an appropriate condition outside the plasma (e.g. E, B → o as r →∞). Such solutions will exist only for certain eigenvalues of the equation, i.e. only when ω, N satisfy certain possible relationships. These are the dispersion relations of the different eigen-modes. We shall take Y(∝ B) to be uniform but X(∝ n_e) to vary with r.

III. E-modes

In order to make progress analytically, we shall seek an approximate form of the equations which is tractable. We regard the quantities

$$\varepsilon = 1 - X \tag{15}$$

and

$$\varepsilon_x = \frac{-iXY}{Y^2-1} = \frac{-i(1-\varepsilon)Y}{Y^2-1} \left(\sim \frac{1}{Y}\right) \tag{16}$$

as small expansion parameters and hence find approximations for the equations for X ≃ 1 and large Y.

First, to lowest order, Eq. (12) becomes

$$\frac{1}{r}\frac{d}{dr} r \frac{dE}{dr} - \left(\frac{m^2}{r^2} + (N^2-1)k_o^2\varepsilon\right) E=o \tag{17}$$

(we drop the z suffix for brevity), showing that the nature of the solution is such that $\frac{1}{r} \sim \frac{1}{E}\frac{dE}{dr} \sim \varepsilon^{1/2}k$. (18)

Next, let us consider the relative magnitude of the additional terms compared to these leading terms. Noting that

$$\frac{d\varepsilon_x}{dr} \sim \varepsilon_x \frac{d\varepsilon}{dr}, \qquad \frac{d\varepsilon_\Lambda}{dr} \sim \varepsilon_x^2 \frac{d\varepsilon}{dr} \tag{19}$$

we find that

$$\frac{D}{N^2-(\varepsilon_\Lambda + \varepsilon_x^2/\varepsilon_\Lambda)} \quad r\frac{d}{dr} \left[\frac{N^2-(\varepsilon_\Lambda + \varepsilon_x^2/\varepsilon_\Lambda)}{D}\right] \sim \varepsilon_x^2\, \varepsilon\left(\frac{1}{N^2-\varepsilon_\Lambda} + 1\right) \tag{20}$$

so that provided $N^2-\varepsilon_\Lambda$ is not small the differential of the inner coefficient of the first term of Eq. (12) is of relative order $\varepsilon_x^2\ \varepsilon$. The third term is of relative order $\varepsilon_x\ \varepsilon$.

To estimate the order of the coupling terms we use the second equation to obtain the order of B relative to E then couple back in the first. The highest relative order of the coupling terms, arising from optical acitvity, is $\varepsilon_x^2\ \varepsilon$. The pure inhomogeneity 3rd coupling term is higher order. Thus, if we ignore terms of order $\varepsilon_x^2\ \varepsilon$ and higher the equations once again become uncoupled.

Now we adopt a specific choice of density profile, parabolic such that

$$X = X_o\left(1-r^2/a^2\right) \tag{21}$$

which, as noted earlier can be regarded as an approximation for any monotonically decreasing profile in the vicinity of $r=o$. Small ε corresponds to large ka and small r/a; therefore, the modes are localized within radius $r \ll a$. This allows us to adopt as boundary condition $E \to o$ as $r \to \infty$ and to suppose that the parabolic profile Eq.(21) applies to $r = \infty$ with negligible effect on the eigenvalue. The equation may thus be cast as a Sturm-Liouville problem ignoring $\varepsilon_x^2 \varepsilon$ terms:

$$\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{m^2}{r^2} + \mu^2(\lambda^2 - r^2)\right] E = o \tag{22}$$

with boundary conditions: E regular at $r=o$, zero at $r=\infty$. The parameters λ, μ, in terms of the plasma parameters, are to appropriate order:

$$\mu^2 = \frac{X_o k_o^2}{a^2} \, \frac{(N^2-1)^2 - N^4/Y^2}{N^2 - 1} \tag{23}$$

$$\lambda^2 = \frac{a^2(X_o - 1)}{X_o} - \frac{2m\, N^2}{k_o^2 Y (N^2-1)^2} \quad . \tag{24}$$

The solutions to this equation are

$$E = L_n^m(\mu r^2)(\mu r^2)^{m/2} \exp(-\mu r^2/2) \tag{25}$$

where L_n^m is the associated Laguerre polynomial[8] and $n = (\lambda^2\mu - 2m - 2)/4$. The requirement that n be integral, to ensure $E \to o$ at $r \to \infty$, then immediately provides the dispersion relation:

$$\lambda^2 \mu = 2(2n + m + 1) \tag{26}$$

($n = 0, 1, \ldots$ $\quad m = 0, 1, \ldots$) or, written out in full:

$$\lambda^4\mu^2 = \left[2(2n + m + 1)\right]^2 = \frac{(X_o-1)^2}{X_o} k_o^2a^2 \frac{(N^2-1)^2 - N^4/Y^2}{N^2-1}\left[1 - \frac{4mN^2 X_o}{(N^2-1)^2Yk_o^2a^2(X_o-1)}\right] \tag{27}$$

To transform this to a convenient form we write $k_o^2 = k_p^2/X_o$ where $k_p = \omega_{po}/c$ and note that to sufficient order we can take Y to be Ω/ω_{po} (rather than Ω/ω) and in the final term take $X_o-1 = \lambda^2\mu/k_pa(N^2-1)^{1/2}$ so that the equation may be written

$$1 - \frac{1}{X_o} = b = \frac{2(2n+m+1)}{k_pa}\left[\frac{N^2-1}{(N^2-1)^2 - N^4/Y^2)}\right]^{1/2}\left[1- \frac{2mN^2}{(2n+m+1)(N^2-1)^{3/2}Yk_pa}\right]^{-1/2}. \tag{28}$$

The right hand side, b, is now independent of ω so that the explicit solution for ω in terms of N is

$$\omega = \omega_{po}(1-b)^{1/2} . \tag{29}$$

This solution is accurate up to (and including) terms of order $\varepsilon\varepsilon_x$ (i.e. $1/kaY$) in X_o-1 and hence to order $1/k^2a^2Y$ in ω/ω_{po}. It becomes exact in either limit $ka \to \infty$ or $Y \to \infty$.

The analytic solution is particularly convenient because it allows us to obtain the (longitudinal) group velocity by direct differentiation. We use

$$v_g = \frac{d\omega}{dk} = \frac{c}{N}\frac{1}{1 + \dfrac{\omega}{N}/\dfrac{d\omega}{dN}} \tag{30}$$

and find, after some algebra

$$\frac{v_g}{c} = \frac{1}{N}\left[1 + \left(\frac{1}{b} - \frac{5}{4}\right)\frac{2(N^2-1)}{N^2}\Big/\left\{1 + \frac{2N^2/Y^2}{(N^2-1)^2 - N^4/Y^2} - \frac{m(N^2+2)}{(2n+m+1)k_p aY(N^2-1)^{3/2}}\right\}\right]^{-1} \tag{31}$$

correct to order $\varepsilon\varepsilon_x$.

IV. Thermal Corrections

In an infinite homogeneous plasma the plasma response may be calculated for specified ω and $\underline{k}$ from the linearized Vlasov equation and the result expressed as a dielectric tensor whose components are integrals of functionals of the distribution function (f) over velocity space. A convenient form of these integrals[7] expresses them as sums over cyclotron harmonics of terms like

$$\int d^3\underline{v} \quad \frac{v_z J_\ell^{\,2}\,(k_\perp v_\perp/\Omega)}{\omega - \ell\Omega - k_z v_z}\left[\frac{\partial}{\partial v_z} + \frac{\ell\Omega}{\omega}\left(\frac{v_z}{v_\perp}\frac{\partial}{\partial v_\perp} - \frac{\partial}{\partial v_z}\right)\right] f \tag{32}$$

where the integral in v_z must be taken along the Landau contour. The hermitian part of the dielectric tensor, which governs the real part of ω (for real $\underline{k}$) arises from the principal part of the integral while the contribution from the poles at $\omega = \ell\Omega + kv_z$ governs the antihermitian part of $\underline{\underline{\varepsilon}}$ and hence the imaginary part of ω. The result is Landau and cyclotron damping (or growth) as a result of these resonances.

The approximation we have adopted, $k_p a \gg 1$, $\omega/\Omega \ll 1$ in our treatment of the finite plasma leads to a situation in which the perpendicular derivatives of E are much smaller than the longitudinal derivatives (for the lower order modes). This leads us to the conclusion that the effective $k_\perp$ is small and we should consider the limit $k_\perp v_\perp/\Omega \to 0$. This approximation will again be exact in the limits: $Y \to \infty$ or $k_p a \to \infty$ as well as the limit $v_\perp \to o$ (cold plasma).

In this approximation the dielectric tensor becomes independent of transverse derivatives and hence is equally applicable to the finite plasma case. That is, we have removed all the transverse spatial dispersion. Moreover, we are interested in waves excited by the Cerenkov (ℓ=o) resonance so we shall choose to ignore all contributions from cyclotron resonance ($\ell \neq$o) to wave growth or decay.

In this case the only modification to the cold plasma dielectric is to replace the previous expression for ε_{zz} (= ε) by

$$\varepsilon = 1 + \frac{\omega_p^2}{k^2} \int \frac{k \ \partial f/\partial v}{\omega - kv} dv \tag{33}$$

where k is now the longitudinal k-vector, f is the (normalized) one-dimensional distribution function (integrated over perpendicular velocities), and v refers to longitudinal velocity only. This is the usual expression for the dielectric constant for longitudinal waves in a field free plasma (It should be noted, however, that Eq. 33 is a non-relativistic expression and so the imaginary part, dependent on relativistic particles for N~1, is only approximate. The relativistic corrections do not change qualitatively our result.) For $\omega/k \gg v_t$ (assuming the distribution to be nearly Maxwellian) it can be written for real ω, k as real and imaginary parts

$$\varepsilon_r(\omega_r) \simeq 1 - \frac{\omega_p^2}{\omega_r^2} \left(1 + 3 N^2 v_t^2/c^2\right) \tag{34}$$

$$\varepsilon_i(\omega_r) = - \pi \frac{\omega_p^2}{k^2} \left.\frac{\partial f}{\partial v}\right|_{\omega_r/k} \tag{35}$$

where v_t is the electron thermal velocity.

Recalling now that the dispersion relation in the cold plasma case is

$$\frac{1-X_o}{X_o} = \left(\frac{\varepsilon}{1-\varepsilon}\right)_{r=o} = -b \tag{36}$$

we seek a solution for complex ω with real k in the case where the imaginary part of ω is small so that

$$\varepsilon(\omega_r + i\omega_i) \simeq \varepsilon_r(\omega_r) + i\left[\varepsilon_i(\omega_r) + \omega_i \frac{\partial \varepsilon_r}{\partial \omega_r}\right] . \tag{37}$$

Provided the shape of the distribution function can be taken independent of position, all spatial dependence being in ω_p, we can immediately generalize the cold plasma result by substituting this value of ε into the dispersion relation Eq.(36) so that:

$$\varepsilon(\omega_r + i\omega_i) = -b/(1-b) \tag{38}$$

where we now mean the value of ε at r=o.

Then the real and imaginary parts of this equation provide the relations

$$\varepsilon_r(\omega_r) = \frac{-b(\omega_r)}{1-b(\omega_r)} \tag{39}$$

$$\varepsilon_i(\omega_r) + \omega_i \frac{\partial \varepsilon_r}{\partial \omega_r} = \omega_i \frac{\partial}{\partial \omega}\left(\frac{-b(\omega_r)}{1-b(\omega_r)}\right) \tag{40}$$

Substituting our values for ε_r and ε_i, the first equation gives the real part of the frequency as

$$\omega_r = \omega_{po}\,(1 + 3N_r^2\,v_t^2/c^2)^{1/2}\quad(1-b)^{1/2} \tag{41}$$

where $N_r = kc/\omega_r$. The second equation becomes

$$-\pi\,\frac{\omega_p^2}{k^2}\,\frac{\partial f}{\partial v}\bigg|_{\omega_r/k} + \frac{2\,\omega_p^2\,\omega_i}{\omega_r^3}\,(1+6N_r^2\,v_t^2/c^2) \simeq \frac{-\omega_i}{\omega_r}b(\omega_r)\,\frac{N_r^2}{(N_r^2-1)} \tag{42}$$

to first order in b, which can be solved in the same approximation to give:

$$\omega_i = \frac{\pi}{2}\,\frac{c^2}{{N_r}^2}\,\frac{\partial f}{\partial v}\bigg|_{N_r c}\,\omega_r/\left[1 + 6\,N_r^2\,v_t^2/c^2 + bN_r^2/2(N_r-1)\right]. \tag{43}$$

We thus have obtained the appropriate generalization of the Bohm-Gross dispersion and Landau damping or growth, applicable to this finite plasma case. The modification to the previous, cold plasma, real frequency is simply to multiply it by $(1 + 3N^2v_t^2/c^2)^{1/2}$ (We now drop the r suffix on N for brevity).

In calculating the group velocity it is sufficient to add to the cold plasma result (Eq. 31) the differential of $\omega_{po}3N^2v_t^2/2c^2$ so that

$$\frac{d\omega}{dk} = v_{g\ cold} + (3\,N\,v_t^2/c^2)c. \tag{44}$$

V. Examples and Applications

It is of interest, first, to note the relationship of the dispersion relation we have obtained with what would be obtained by the various more restrictive approximations mentioned earlier. To take the density to be uniform would totally change the character of the solution, because, as our eigenmode solution indicates, the smoothly varying density causes the

E-mode to be highly localized near the peak density (e.g. if n=m=o a Gaussian: $E \propto \exp(-\mu r^2/2)$).

The electrostatic approximation would give a value for b ($= 1- \omega^2/\omega_{po}^2$)

$$b_{es} = \frac{2(2n+m+1)}{k_p a} \quad \frac{1}{N(1-1/Y^2)^{1/2}} \tag{45}$$

instead of that given in Eq. (28). The infinite field case gives

$$b_\infty = \frac{2(2n+m+1)}{k_p a} \quad \frac{1}{(N^2-1)^{1/2}} \quad . \tag{46}$$

Either of these additional approximations may give rise to noticeable discrepancies in cases of interest.

The extent to which the present solution provides a more accurate result is shown by Fig. 1 where we plot the cold plasma dispersion relation for the lowest mode n=m=o, together with the other approximate results and some points obtained from a numerical solution of the full coupled wave equations. The parameters adopted here ($k_p a$ = 20, Y=5) present a fairly challenging case for all the analytic approximations, since the expansion parameters are not particularly small. Nevertheless, rather good results are obtained from our present treatment, provided ω is not too much less than ω_{po}.

In Fig. 2 we show the eigenmode structure for the modes n=m=o, n=1 m=o and n=o m=1 for $k_p a$ = 20, Y=5 and N=1.2 . For comparison, the numerical solution of the full coupled equations for n=m=o is also plotted; it shows only slight deviations from the approximate analytic solution, despite the fact that this is a rather extreme case.

The cold plasma dispersion relation for various different modes is plotted in Fig 3. (for Y=5, $k_p a$ = 160) showing the mode spacing of different transverse modes of the cavity. The degeneracy between, for example, n=1 m=o and n=o m=2, is broken slightly by the additional term of order $\varepsilon\varepsilon_x \sim (1/kaY)$ which we retained in the expression for b, although this effect is too small to show in the graph for these parameters.

Turning now to a practical example, recent measurements on the Alcator Tokamak [4,9] have shown highly coherent emission at 60 GHz near the central plasma frequency at 10 T magnetic field, bulk electron temperature about 500 eV. This appears to be produced by Maser action caused by a positive slope on the distribution function in which the high Q cavity formed by the plasma allows the fastest growing mode to become dominant in some circumstances. Modelling the plasma as a periodic cylinder the modes of this Maser cavity are defined by the radial (n) and azimuthal (m) numbers already discussed and in addition a longitudinal mode number (ℓ) given by

$$kR = N\,\omega\,R/c = \ell \tag{47}$$

where R is the major radius.

The spacing of different transverse modes in this plasma cavity (for which Y=5, $k_p a$ = 160) is essentially given by Fig. 3 since the thermal corrections are the same for all modes. The exact numerical value of the spacing depends on the parallel refractive index which itself is determined by the resonant velocity(v_r) of the driving electrons: $N = c/v_r$. For example at N=1.5 the mode spacing is $\Delta\omega/2\pi$ = 0.37 GHz. This spacing is much larger than the width of many of the bursts of emission observed, although spectral features this widely spaced sometimes appear. On some occasions abrupt jumps in frequency occur, consistent with changes in transverse mode number.

In order to calculate the longitudinal mode spacing it is necessary to account properly for the thermal contribution to the dispersion. Fig. 4a shows the dispersion curve including the Bohm-Gross correction Eq. (41) for this plasma (and n=m=o) as well as the cold plasma result. Clearly the thermal corrections can be appreciable.

The longitudinal mode spacing is given by

$$\Delta\omega = \frac{d\omega}{dk}\,\Delta k = \frac{d\omega}{dk}\,\frac{1}{R} \tag{48}$$

for which we require the group velocity Eq. (44). This is plotted in Fig. 4b. The right hand scale indicates the theoretical mode spacing for the particular major radius (R = 0.64m) of Alcator C. The interest and potential usefulness of this result is that observation of mode spacing enables one to deduce the refractive index N. For example under some circumstances a multiplet structure to the coherent emission has been observed, in which the spacing of the peaks is of the order of 2 MHz. If this is interpreted as representing the spacing of adjacent modes then one can deduce N $\simeq$ 1.3 and hence that the electrons responsible for the mode amplification have longitudinal energy ~ 300 keV.

Our calculation of the imaginary part of the frequency Eq.(43) shows that for any given N (and hence set of resonant electrons) the mode with the greatest gain (or damping if $\partial f/\partial v < o$) is the lowest order mode n=m=o. Of course, the differences in the growth rate are small, ~ $1-\omega/\omega_{po}$, so that excitation of higher modes is not completely ruled out, particularly if the shape of f has significant spatial variation. However, it does show that we expect low order modes in general to be the most unstable. This fact, and the calculations indicating frequency extremely close to ω_{po} except for a very narrow range of N close to the cutoff for this wave, resolves quantitatively a long standing question about how close to the central plasma frequency the emission is. The answer: very close indeed, as given by Figs. 3, 4(a) Eqs. (29, 41). The emission therefore provides a very accurate measurement of the central plasma density.

The mechanism for coupling the radiation out of the cavity into the free-space propagating radiation observed remains a matter for investigation. The present results allow us to estimate the amount of radiation which could arise by virtue of the scattering of the evanescent field from longitudinal asymmetries (e.g. limiters or density fluctuations) at the plasma edge. The electric field at the plasma edge is given (approximately) by Eq.(25). Since the exponential term is dominant we consider it alone and putting r=a its magnitude is

$$\exp - \left[\frac{1}{2} k_p a \left(\frac{(N^2-1)^2 - N^4/Y^2}{N^2-1} \right)^{1/2} \right] . \tag{49}$$

Except for N values which make $(N^2-1)^2 - N^4/Y^2 << 1$ this is of order exp(-80) for $k_p a = 160$. Obviously, this number is so small that only a totally negligible coupling of radiation out of the plasma could possibly occur via edge effects. This compells us to invoke some internal scattering mechanism from density perturbations near the plasma center, such as has previously been suggested[10,6]. (Note however, that the extremely narrow linewidth must apply also to the frequency spread of the internal scatterer, which probably rules out ion acoustic fluctuations as a candidate).

VI. Conclusions

Our analysis of the cavity mode structure and eigenfrequencies of waves near the plasma frequency in a cylindical plasma with monotonically decreasing density profile has obtained simple analytic forms for the dispersion relation and mode structure, valid for values of the magnetic field and of the ratio of plasma size to vacuum wavelength large but finite. These modes correspond to the whistler branch (slow ordinary wave) of the homogeneous plasma dispersion relation and to the E-modes (TM-modes) of a vacuum waveguide but they are guided by the plasma itself regardless of plasma edge boundary conditions.

The results are directly applicable to the interpretation of masing action in bounded plasmas, such as Tokamaks, and provide the mode frequencies and spacings for the plasma cavity formed. When the shape of the electron distribution function, whose positive slope provides the gain of the medium, is uniform in space the lowest transverse modes have greatest gain. For typical experimental parameters, these modes have frequency within ~ 1% of the central plasma frequency and negligible field at the plasma edge. Thus, coupling of energy out of the cavity must rely on scattering from toroidally asymmetric density perturbations. Our analysis allows observation of radiation to be used as diagnostic of a central electron density and nonthermal electron distributions.

The modelling of the Tokamak case is only approximate because effects of poloidal magnetic field and toroidicity have not been included. However, because E-modes are highly localized near the axis, the effective aspect ratio is very large and these effects should be small. The B-modes which we have not analyzed, corresponding to the other dispersion branch, the slow extraordinary wave in the vicinity of ω_p, are much less localized. These waves are more sensitive, therefore, to density profile effects and toroidicity. In some cases, they are able to radiate directly into free space [6, 11]. Thus a realistic analysis of this second branch would have to be more elaborate than our present treatment.

Our results may also have relevance to astrophysical plasma problems such as coherent radiation from planetary magnetospheres and pulsars, although these situations may be further from our idealized geometry except when quasi-cylindrical filaments of plasma density exist, aligned along the magnetic field.

Acknowledgements

This work was supported under DOE contract #DE-AC02-78ET51013.

References

a) Present address: Physics Dept., Auburn University, Auburn, Alabama 36849.

1 A. W. Trivelpiece and R. W. Gould, J. Appl. Phys 11, 1784 (1959).

2 W. P. Allis, S. J. Buchsbaum and A. Bers, "Waves in Anisotropic Plasmas," MIT, Cambridge MA (1963).

3 See e.g. P. Vandenplas, "Electron Waves and Resonances in Bounded Plasma", Wiley, N.Y. (1968); D. L. Book, Phys. Fluids 10, 198 (1967); V. Bevc, J. Appl. Phys. 41, 2408 (1970); J. Basu and S. K. Das, J. Appl. Phys. 51, 6134 (1980).

4 E.g. I. H. Hutchinson and S. E. Kissel, Phys. Fluids 26, 310 (1983).

5 H. P. Freund, L. C. Lee and C. S. Wu, Phys. Rev. Lett. 40, 1563 (1981).

6 K. Swartz, I. H. Hutchinson and K. Molvig, Phys. Fluids 24, 1689 (1981).

7 N. A. Krall and A. W. Trivelpiece, "Principles of Plasma Physics", McGraw Hill New York (1973).

8 P. M. Morse and H. Feshbach "Methods of Theoretical Physics", p. 784 McGraw Hill, New York (1953).

9 R. F. Gandy and D. H. Yates, MIT report PFC/RR84-4 (1984), submitted for publication.

10 I. H. Hutchinson, K. Molvig, S. Y. Yuen, Phys. Rev. Lett. 40, 1091 (1978).

11 M. Bornatici and F. Engelmann, Phys. Fluids 22, 1409 (1979).

Figure Captions

Fig. 1 Cold plasma dispersion relaton (frequency versus parallel refractive index) of the lowest order (n=o, m=o) mode for $\omega_{po}a/c=20$, $\Omega/\omega=5$. The present analytic solution is compared with that obtained by the electrostatic and infinite B approximations. Points are numerical solutions of the full coupled wave equations.

Fig. 2 Mode structures for the lowest order modes for $\omega_{po}a/c=20$, $\Omega/\omega=5$ N=1.2. Also shown is the numerical solution for the (o, o) mode, which corresponds to x on Fig. 1 .

Fig. 3 Cold plasma dispersion relations for the first few modes for typical tokamak parameters: $\omega_{pa}/c=160$, $\Omega/\omega=5$.

Fig. 4(a) Dispersion relation of (o, o) mode including thermal corrections for $T_e=500$ eV.

(b) Corresponding group velocity, and hence longitudinal mode spacing for a cavity of length 2π (.64) m. The parallel energy of electrons with velocity c/N is also indicated.

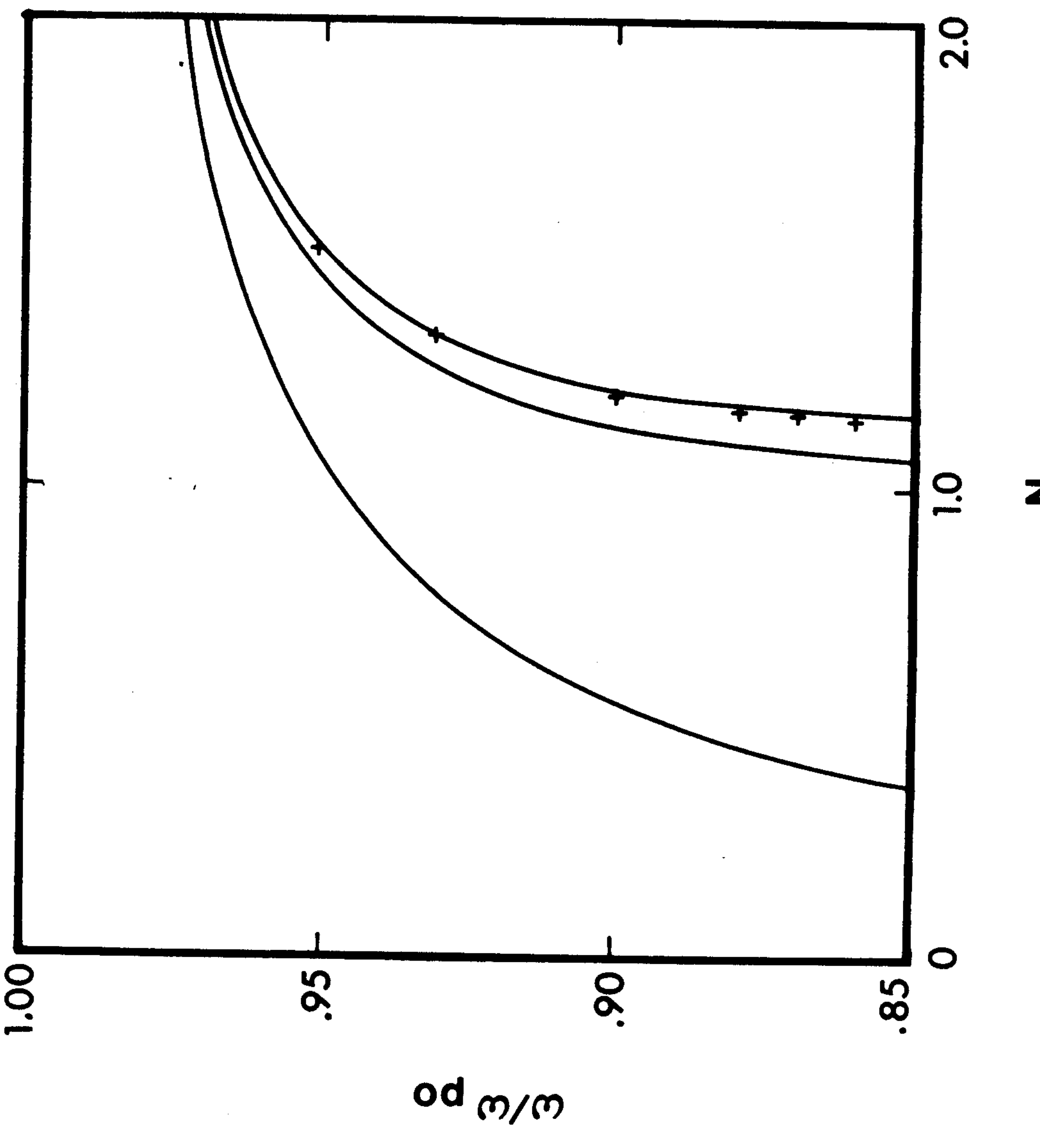

Figure 1

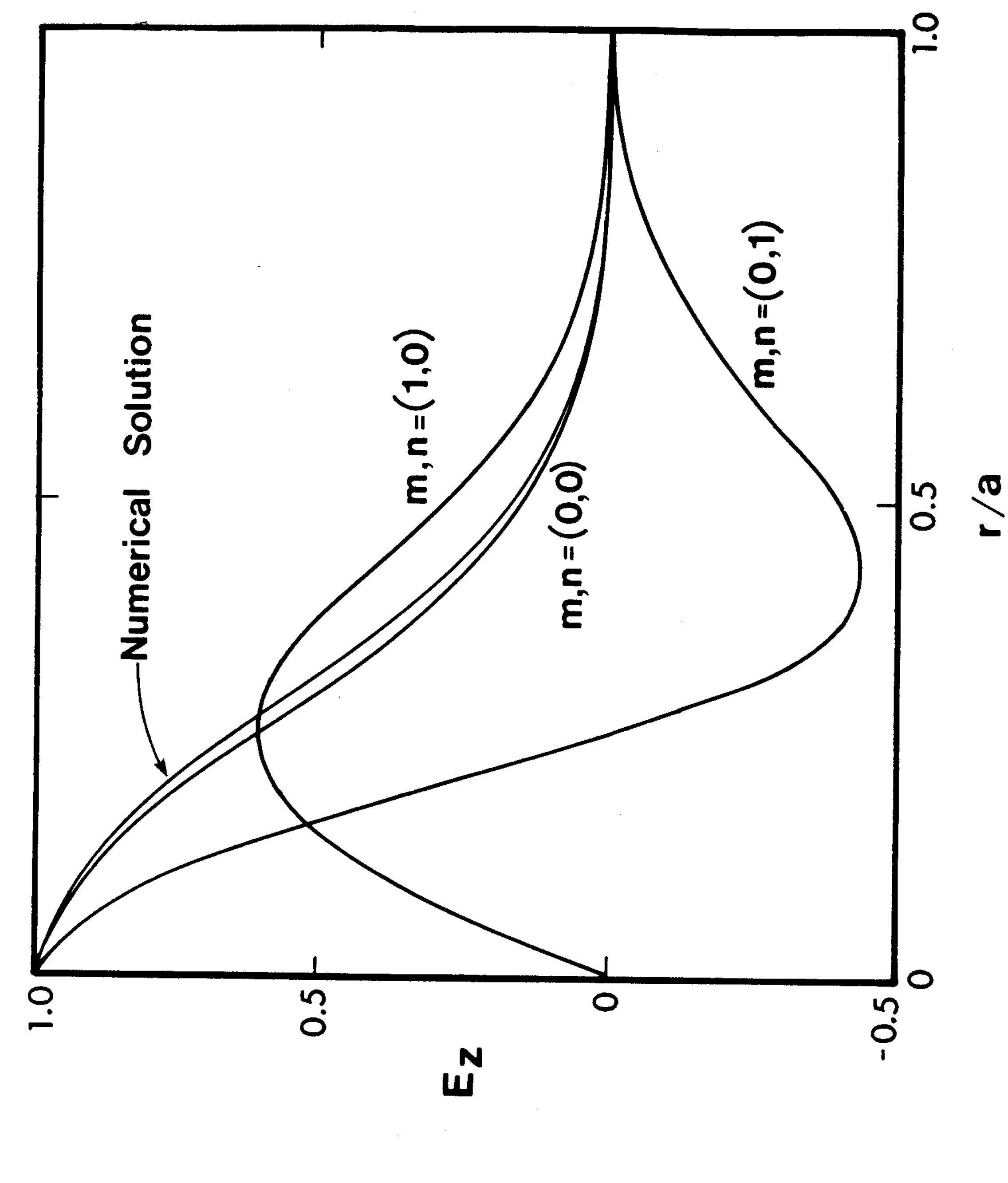

Figure 2

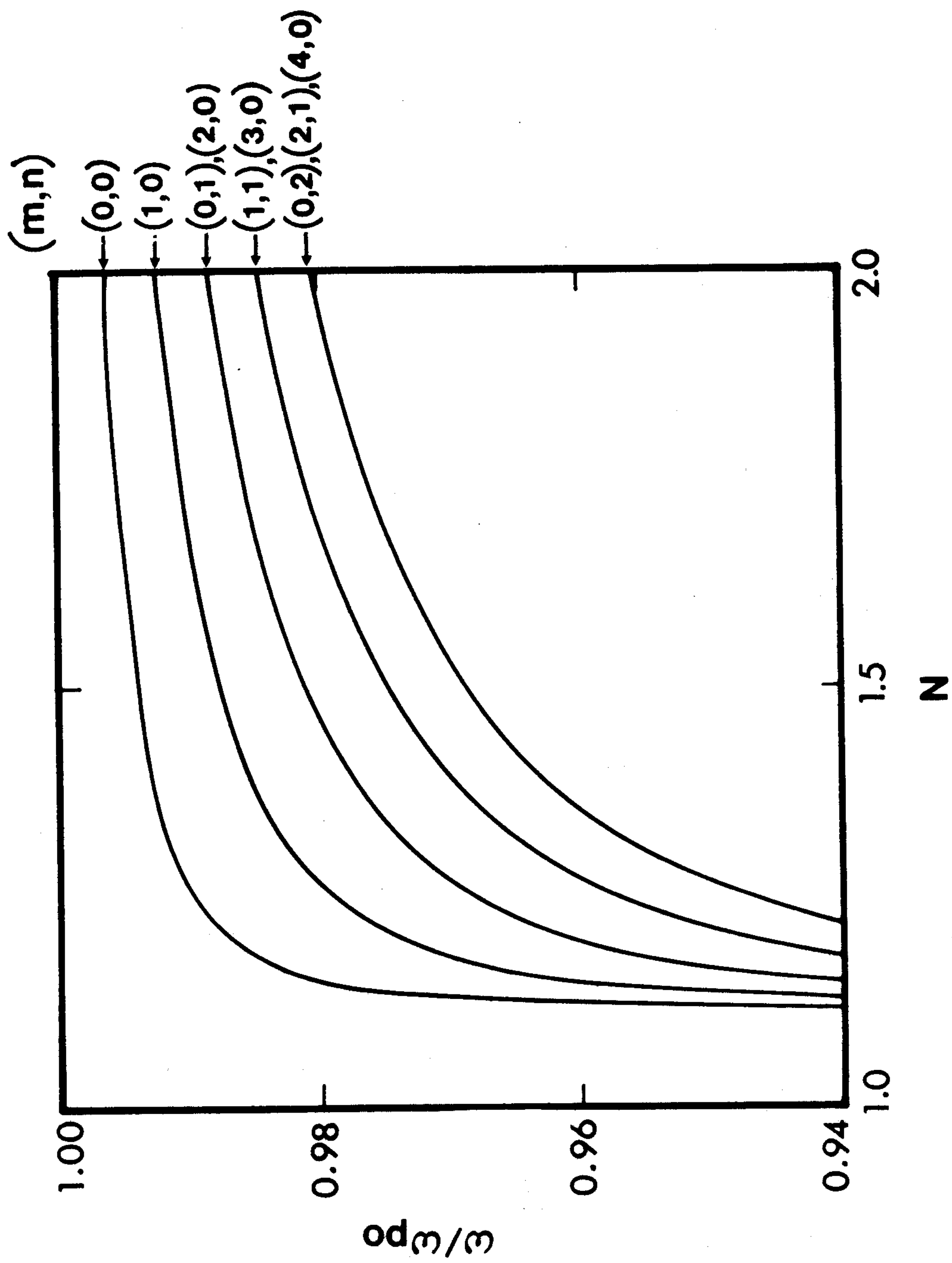
(m,n)
(0,0)
(1,0)
(0,1),(2,0)
(1,1),(3,0)
(0,2),(2,1),(4,0)
1.00
0.98
0.96
0.94
1.0
1.5
2.0
N
ω/ω_{po}

Figure 3

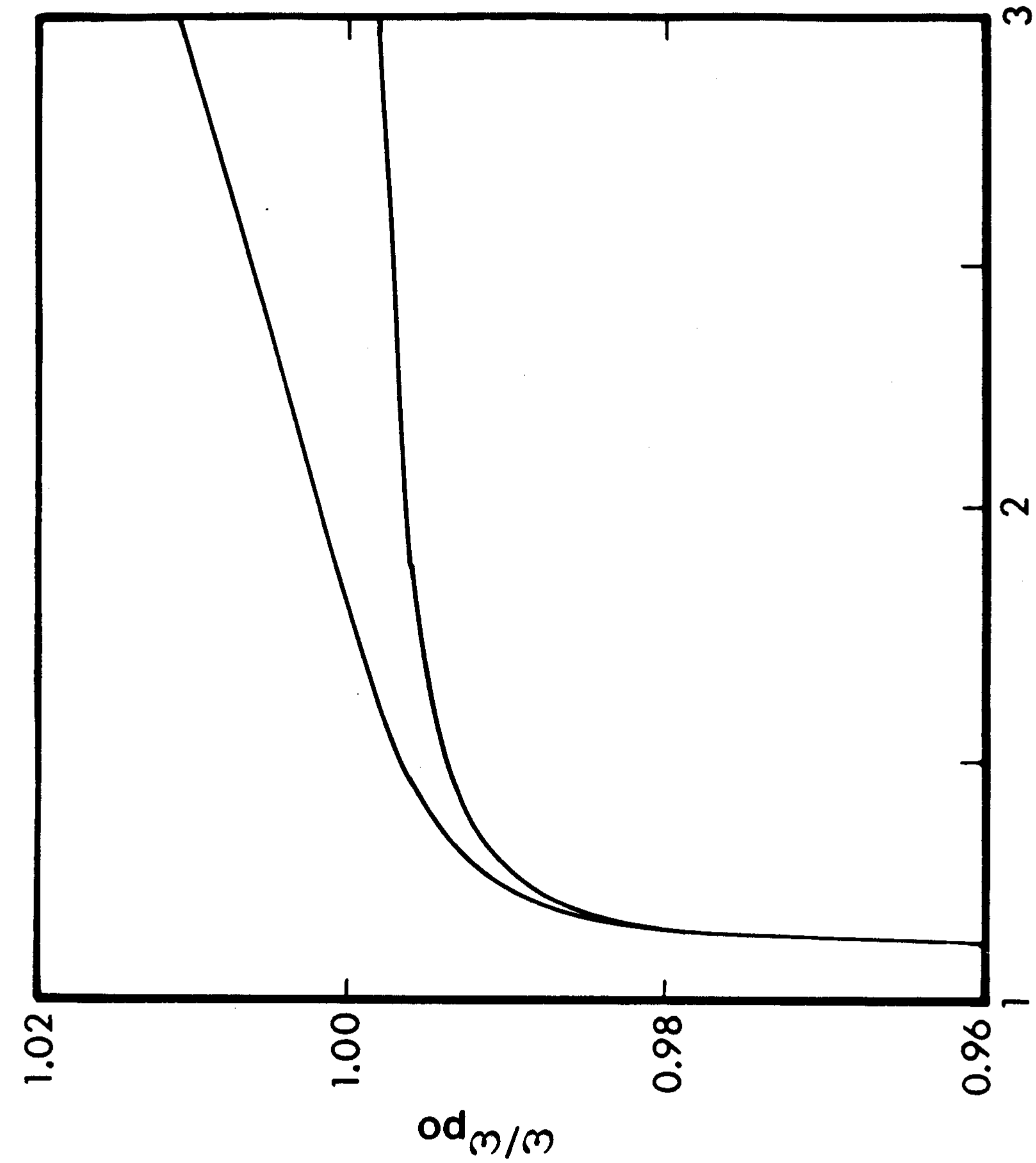

Figure 4a

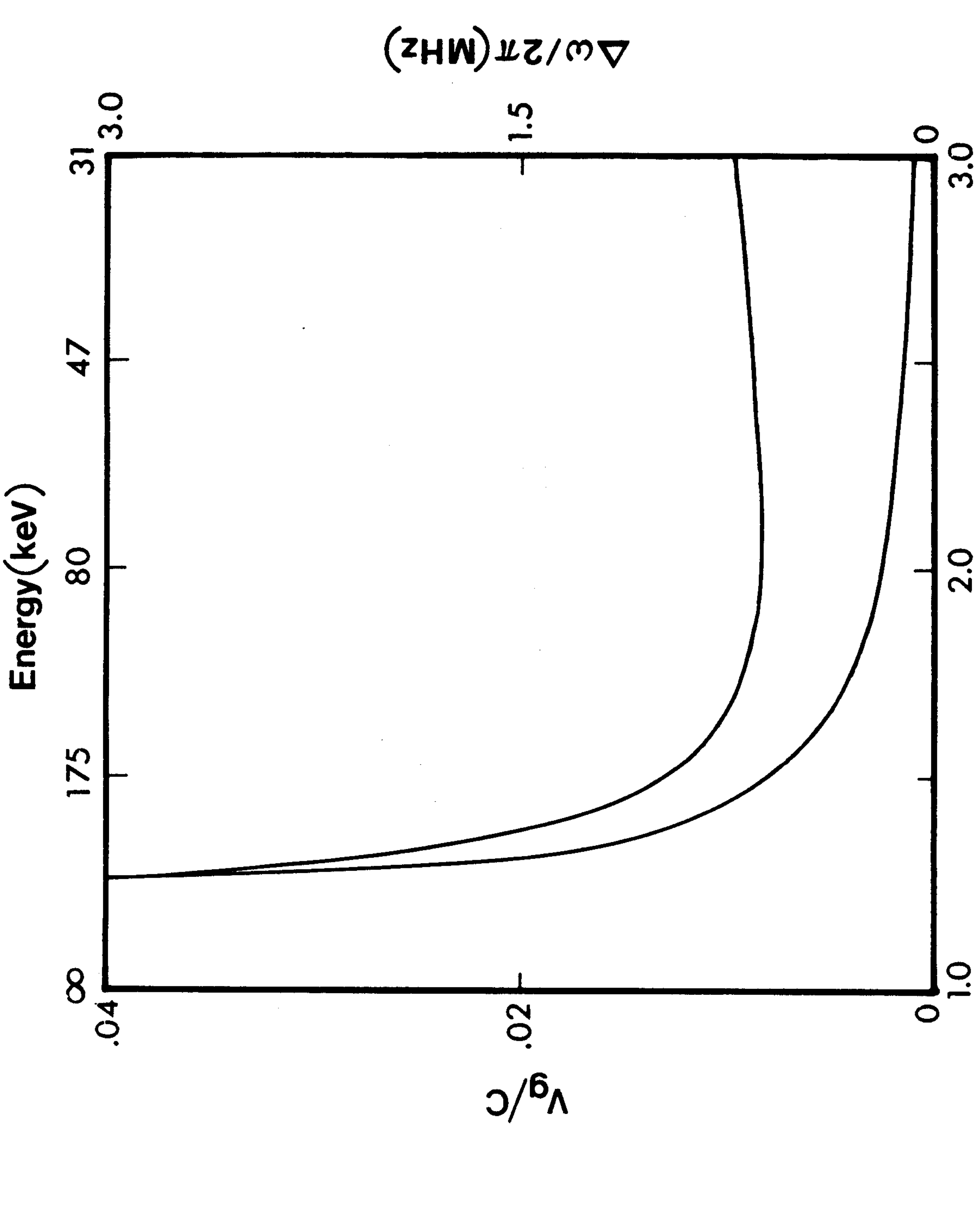

Figure 4b