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Thermodynamic Entropy in One Higher Dimension*

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## Entanglement Entropy of Dispersive Media from Thermodynamic Entropy in One Higher Dimension

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A dispersive medium becomes entangled with zero-point fluctuations in the vacuum. We consider an arbitrary array of material bodies weakly interacting with a quantum field and compute the quantum mutual information between them. It is shown that the mutual information in  $D$  dimensions can be mapped to classical thermodynamic entropy in  $D + 1$  dimensions. As a specific example, we compute the mutual information both analytically and numerically for a range of separation distances between two bodies in  $D = 2$  dimensions and find a logarithmic correction to the area law at short separations. A key advantage of our method is that it allows the strong subadditivity property to be easily verified.

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Entanglement as a purely quantum phenomenon has become a central subject in many fields, ranging from high-energy physics to many-body systems and quantum information. In particular, the characteristic entropy associated with entanglement has proved a powerful tool for describing the quantum nature of a system [1–4] as well as for quantifying resources available for quantum computation [5]. Inspired by formal similarities with thermodynamics, parallels between quantum and classical notions of entropy have been suggested in the literature [6].

While many studies of entanglement entropy have considered geometrical regions in vacuum [4] or domains in many-body systems [7], rapid advances in quantum computation and atomic and optical physics have prompted studies of entanglement between atoms and light [8,9]. Although most studies have addressed the case of one or a few bodies, recent attention has turned to the problem of quantifying the entanglement of a medium interacting with a quantum field, both described by (infinitely) many modes. In particular, Ref. [10] took key first steps in this direction, but simple conceptual pictures and efficient computational tools have remained elusive.

In this Letter, we consider two or more dispersive media in the presence of a fluctuating field and compute their quantum mutual information, a characteristic measure of quantum correlations between parts of a system, and closely related to the entanglement entropy. Our model is a simplified electromagnetism in which a free scalar field couples weakly to “dielectric” bodies. We relate the mutual information in  $D$  dimensions to classical thermodynamic entropy in  $D + 1$  dimensions, where the dispersive media become planar disklike regions (Fig. 1). Inspired by an electrostatic analogy, our formalism relates quantum information to classical scattering theory, thus allowing practical computations by any number of techniques including

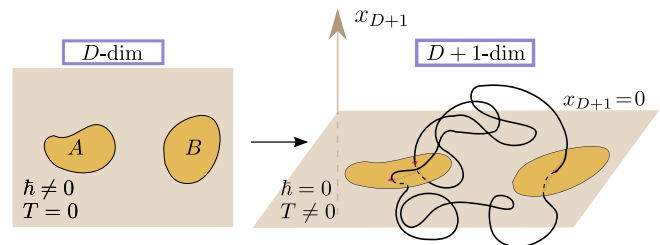


FIG. 1 (color online). Two dispersive media in  $D$  dimensions coupled to a quantum field in vacuum. The mutual information between the two is related to the thermodynamic entropy of fluctuating fields in  $D + 1$  dimensions. The material bodies  $A, B$  are reinterpreted as subregions in the plane  $x_{D+1} = 0$  in  $(D + 1)$ -dimensional space, and impose a boundary condition on the fluctuating fields. This analogy finds a geometrical picture in terms of phantom chain polymers intersecting both regions.

multipole expansion or numerical methods. As an example, we compute the mutual information between two bodies at all separations in  $D = 2$  dimensions and observe area-law-violating logarithmic behavior at short separations.

*Model.*—Our starting point is the action describing a free scalar field in  $D$  spatial dimensions in the presence of dispersive material bodies

$$I_{\text{di}}[\phi] = \int_0^\infty \frac{d\omega}{2\pi} \int d\mathbf{x} \phi_\omega^*(\mathbf{x}) [\epsilon(\omega, \mathbf{x}) \omega^2 + \nabla^2] \phi_\omega(\mathbf{x}), \quad (1)$$

where  $\epsilon = 1 + 4\pi\chi$  is a frequency- and position-dependent dielectric function. This model offers a simplified theoretical framework for exploring quantum electrodynamics and has proven useful in studies of Casimir [11,12] and entanglement [10] phenomena. Specifically, the limit  $\epsilon \rightarrow 1$  is the basis of a vast literature on entanglement entropy. We

take the susceptibility  $\chi$  to be nonvanishing in a region  $\Omega$  (the material bodies), within which it assumes the spatially constant form  $\chi(\omega) = \omega_p^2/(\omega_0^2 - \omega^2)$  with  $\omega_0$  and  $\omega_p$  the resonant and plasma frequencies of the dispersive material, respectively. Equation (1) may be viewed as an effective action arising upon integrating out the matter degrees of freedom from a parent action describing both vacuum and matter fields [10], which (upon Wick rotation  $\omega \rightarrow i\xi$ ) reads

$$I_E[\phi, \psi] = \int_0^\infty \frac{d\xi}{2\pi} \left\{ \int d\mathbf{x} \phi_\xi(\mathbf{x})(\xi^2 - \nabla^2)\phi_\xi(\mathbf{x}) + \int d\mathbf{x} \left[ \frac{\xi^2 + \omega_0^2}{4\pi} \psi_\xi^2(\mathbf{x}) + 2\omega_p \xi \phi_\xi(\mathbf{x}) \psi_\xi(\mathbf{x}) \right] \right\}, \quad (2)$$

where  $\psi$  represents the degrees of freedom within the material bodies. Note that the action in the first line is that of a free scalar field, while the first term in the second line describes harmonic oscillator modes localized in  $\Omega$ , and the last term is assumed to be of the form  $\psi d\phi/dt$  [13]. Integrating out the field  $\psi$  from Eq. (2) recovers the action (1). On the other hand, we are interested in finding an effective action for the matter field  $\psi$ . Integrating out the scalar field  $\phi$  yields

$$I_{\text{eff}}[\psi] = \int_0^\infty \frac{d\xi}{2\pi} \left\{ \int_\Omega d\mathbf{x} \frac{\xi^2 + \omega_0^2}{4\pi} \psi_\xi^2(\mathbf{x}) + \omega_p^2 \xi^2 \int \int_\Omega d\mathbf{x} d\mathbf{x}' \psi_\xi(\mathbf{x}) G_\xi(\mathbf{x}, \mathbf{x}') \psi_\xi(\mathbf{x}') \right\}, \quad (3)$$

where  $G_\xi = (\xi^2 - \nabla^2)^{-1}$  is the Green's function of the Euclidean Helmholtz operator. The above effective action will allow us to study the entanglement between different parts of a medium (or between two media). To this end, we briefly introduce the main tools to compute the entanglement entropy.

*Entanglement from covariance matrices.*—For a mixed state  $\rho$ , the von Neumann entropy, a widely used measure of entanglement, is defined as  $-\text{Tr}[\rho \log \rho]$ . Let us consider the local field  $\{\psi(t, \mathbf{x})\}$  and its conjugate momentum  $\{\pi(t, \mathbf{x})\}$  describing the local degrees of freedom in the region  $\Omega$ . For a quadratic action, the ground state is a Gaussian functional of  $\{\psi\}$  (or  $\{\pi\}$ ). Therefore, the quantum state can be fully represented by equal-time two-point functions  $\Xi(\mathbf{x}, \mathbf{x}') = \langle \psi(t, \mathbf{x}) \psi(t, \mathbf{x}') \rangle$  and  $\Pi(\mathbf{x}, \mathbf{x}') = \langle \pi(t, \mathbf{x}) \pi(t, \mathbf{x}') \rangle$ . The von Neumann entropy is directly related to the two-point functions via  $\mathcal{S}_\Omega = \text{Tr}[(\Delta_\Omega + \mathbf{I}) \log(\Delta_\Omega + \mathbf{I}) - \Delta_\Omega \log \Delta_\Omega]$ , where  $\mathbf{I}$  is the identity matrix and  $\Delta_\Omega \equiv \sqrt{\Pi \Xi} - \mathbf{I}/2$  is the covariance matrix [14,15] with the subscript  $\Omega$  emphasizing its zero support outside  $\Omega$ . We find it more convenient to recast the von Neumann entropy as

$$\mathcal{S}_\Omega = \int_0^1 d\lambda s_\Omega(\lambda) \quad (4)$$

with  $s_\Omega(\lambda) = \text{Tr} \log [\lambda^{-1} \Delta_\Omega + \mathbf{I}]$ ,

where  $\lambda \in [0, 1]$  is an auxiliary parameter. The last expression is reminiscent of a one-loop effective action. In fact, we shall see that, by using the above identity, the entanglement entropy finds a convenient path-integral form.

Next, we compute the covariance matrix  $\Delta_\Omega$ . The correlation functions in imaginary frequency are given by  $\Xi(\mathbf{x}, \mathbf{x}') = \int (d\xi/2\pi) \langle \psi_\xi^*(\mathbf{x}) \psi_\xi(\mathbf{x}') \rangle$  and  $\Pi(\mathbf{x}, \mathbf{x}') = -\int (d\xi/2\pi) \xi^2 \langle \psi_\xi^*(\mathbf{x}) \psi_\xi(\mathbf{x}') \rangle$  [16]; the two-point function in the integrands can be derived from Eq. (3). In general, this is a difficult task that requires knowledge of the full (off-shell)  $\mathbf{T}$  matrix—an object that appears in the Lippmann-Schwinger equation—of the material bodies in position space [11]. Nevertheless, we consider a weak-coupling limit in which  $\omega_{0,p} L \ll 1$  with  $L$  the linear size of the media. In other words, the bodies are sufficiently small that their coupling to the background scalar field [the second line in Eq. (3)] can be treated perturbatively, a condition well satisfied for sub-micron-scale bodies with typical plasma and resonant frequencies. One can then expand the correlation functions perturbatively:  $\Xi \approx (4\pi/\omega_0)(\mathbf{I}/2 + \delta\Xi)$  and  $\Pi \approx (\omega_0/4\pi)(\mathbf{I}/2 + \delta\Pi)$ . In the absence of interaction with the scalar field, one recovers the standard variances of the field and its conjugate momentum as those of a harmonic oscillator, whereupon  $\Delta_\Omega \rightarrow 0$  and the entanglement vanishes. To leading order, we have  $\Delta_\Omega \approx (\delta\Xi + \delta\Pi)/2$ . Within this approximation, we find that  $\Delta_\Omega = \mathbf{P}_\Omega \Delta \mathbf{P}_\Omega$  with  $\mathbf{P}_\Omega$  the projection operator onto the spatial domain of  $\Omega$ , and

$$\Delta(\mathbf{x}, \mathbf{x}') = \omega_c \mathbf{G}_0^{(D+1)}(\mathbf{x}, \mathbf{x}'), \quad (5)$$

where we have put  $\omega_c \equiv 2\pi\omega_p^2/\omega_0$ . This equation relates the covariance matrix  $\Delta$  to the Green's function for the Laplacian in  $D+1$  dimensions  $\mathbf{G}_0^{(D+1)}(\mathbf{x}, \mathbf{x}') \sim 1/|\mathbf{x} - \mathbf{x}'|^{D-1}$ , see the Supplemental Material [17]. It is perhaps surprising that  $\Delta(\mathbf{x}, \mathbf{x}')$  depends only on  $\mathbf{x}$  and  $\mathbf{x}'$ , and not the specific geometry of  $\Omega$ ; however, this is an approximation, and the geometric dependence appears in higher orders in  $\omega_{0,p} L$ .

Now we are in a position to write Eq. (4) in a familiar form. First, note that, with  $\Delta_\Omega = \mathbf{P}_\Omega \Delta \mathbf{P}_\Omega$ , the second line of Eq. (4) can be written as  $\text{Tr} \log [(\lambda^{-1} \mathbf{P}_\Omega + \Delta^{-1})/\Delta^{-1}]$ . The inverse of  $\Delta$  finds an awkward nonlocal form in  $D$  dimensions, while it is simply proportional to the Laplacian (hence, local) operator in  $D+1$  dimensions. Therefore, we can cast the above expression as a functional integral over a real-valued field  $\theta(\mathbf{x}, x_{D+1})$  living in  $D+1$  dimensions  $[(\mathbf{x}, x_{D+1}) \in \mathbb{R}^{D+1}]$

$$s_{\Omega}(\lambda) = -2 \log \frac{\int D\theta \exp[-\omega_c^{-1} \int_{D+1} (\nabla\theta)^2 - \lambda^{-1} \int_{\Omega} \theta^2]}{\int D\theta \exp[-\omega_c^{-1} \int_{D+1} (\nabla\theta)^2]},$$

which is easily seen to reproduce Eq. (4) upon functional integration. We emphasize that  $\theta$  is an auxiliary field, and is not related to the original fields in the model. The exponent in the numerator inside the logarithm can be interpreted as a Hamiltonian, which is a sum of the free and potential terms. The first term is the usual gradient term in  $D + 1$  dimensions, while the latter is the local potential felt only in the  $D$ -dimensional region  $\Omega$  where  $x_{D+1} = 0$  (see Fig. 1 with  $\Omega = A \cup B$ ). Thus, the terms inside the logarithm take the form of a partition function of the classical field  $\theta$  at a finite temperature; we choose  $k_B T = 1$  for convenience. The above expression then characterizes the change of the classical free energy in the presence of the potential term, that is,  $-F_{\Omega}(\lambda) = \log \langle \exp[-\lambda^{-1} \int_{\Omega} \theta^2] \rangle_0$ , where  $\langle \cdots \rangle_0$  denotes the average with respect to the Gaussian fluctuations of the gradient term.

We will argue later that the change of the free energy is purely entropic, and make the connection to the thermodynamic entropy. For now, we point out another simplification:  $\lambda$  can be scaled by  $\omega_c^{-1}$  to eliminate the latter in the gradient term. This gives rise to an overall factor of  $\omega_c$  to the integral in Eq. (4) and changes its upper bound to  $\omega_c^{-1}$ , which we extend to infinity as  $\omega_c L \ll 1$ . The von Neumann entropy is then obtained from the free energy of the thermodynamic model described above as

$$S_{\Omega} = 2\omega_c \int_0^{\infty} d\lambda F_{\Omega}(\lambda). \quad (6)$$

The degrees of freedom in  $\Omega$  are uncountably infinite and thus the entropy depends on an UV cutoff [24]; such a dependence may exhibit some form of universality in one dimension [4,25,26]. We, on the other hand, consider the quantum mutual information (QMI) between two subsystems  $A$  and  $B$  defined as  $\mathcal{I}_{A,B} = S_A + S_B - S_{A \cup B}$ , which is inherently cutoff independent, and measures quantum correlations between  $A$  and  $B$ . Equation (6) then relates the QMI to the change of the free energy with the local potential turned on in one or both regions,  $\Delta F_{A,B}(\lambda) = F_{A \cup B}(\lambda) - F_A(\lambda) - F_B(\lambda)$ ,

$$\mathcal{I}_{A,B} = -2\omega_c \int_0^{\infty} d\lambda \Delta F_{A,B}(\lambda). \quad (7)$$

The latter, however, is a familiar quantity in the context of the thermal Casimir effect that gives rise to an interaction between two objects due to thermal, rather than quantum, fluctuations in the space between them [27]. The relation in Eq. (7) enables us to use the powerful tools available to study the (thermal or quantum) Casimir effect. It is well known that one can compute the Casimir energy between two objects from the knowledge of their classical scattering

matrices, i.e., their response to incoming classical waves [11,28]. In the case of the thermal Casimir effect, one should compute generalized capacitance matrices, or the multipole (monopole, dipole, etc.) response of the objects to an external electrostatic potential. The interested reader can find more details in Refs. [11,28]; we quote the final expression for  $\Delta F_{A,B}(\lambda)$

$$\Delta F_{A,B}(\lambda) = \frac{1}{2} \text{Tr}_{(D+1)} \log [\mathbf{I} - \mathbf{C}_A(\lambda) \mathbf{G}_0 \mathbf{C}_B(\lambda) \mathbf{G}_0], \quad (8)$$

where  $\mathbf{G}_0$  and  $\mathbf{C}_i$  denote the  $(D + 1)$ -dimensional Green's function and capacitance matrix elements, respectively. To understand the  $\lambda$ -dependent  $\mathbf{C}_A$ , we note that, in the absence of fluctuations, the field  $\theta$  satisfies  $[-\nabla^2 + V_A]\theta = 0$ ; the derivatives in this equation act in  $D + 1$  dimensions, and the potential is defined as  $V_A(\mathbf{x}, x_{D+1}) = \lambda^{-1} \delta(x_{D+1})$  when  $\mathbf{x} \in A$ , and 0 otherwise. As the potential is confined to the plane  $x_{D+1} = 0$ , its effect is simply to impose the boundary condition  $[\zeta \equiv x_{D+1}]$

$$-\partial_{\zeta} \theta|_{0^+} + \lambda^{-1} \theta(\mathbf{x}, \zeta = 0) = 0, \quad \mathbf{x} \in A, \quad (9)$$

on the otherwise free electrostatic field satisfying  $\nabla^2 \theta = 0$ . For example,  $\lambda = 0$  corresponds to a perfect conductor, while in the limit  $\lambda \rightarrow \infty$  the object is completely transparent. The generalized capacitance matrix elements in  $\mathbf{C}_A(\lambda)$  then characterize the electrostatic response of the object due to the  $\lambda$ -dependent boundary condition in Eq. (9). For the sake of completeness, we mention that the capacitance can be expressed as  $\mathbf{C}_A = \mathbf{V}_A / (\mathbf{I} + \mathbf{G}_0 \mathbf{V}_A)$ , where  $\mathbf{V}_A$  is the operator-valued potential corresponding to  $V_A$ . This equation is reminiscent of the Lippmann-Schwinger equation, which is yet another route to scattering theory.

In short, Eqs. (7) and (8) are the central results of this Letter, and relate quantum mutual information in  $D$  dimensions to classical (generalized) capacitance elements in  $D + 1$  dimensions, which allows exploitation of myriad analytical and numerical methods for computing  $\mathbf{C}$ -matrix elements. Next, we consider specific examples to showcase the efficiency of our method.

We start with two material bodies (Fig. 1) separated by a large distance  $d$  compared to their sizes (yet small compared to  $\omega_c^{-1}$ ). The  $\mathbf{C}$ -matrix elements may be computed analytically for simple shapes via techniques reminiscent of electrostatics. For large  $d$ , we can keep only the monopole coefficient in Eq. (8), combined with Eq. (7), to find

$$\mathcal{I}_{A,B} \approx \frac{\omega_c}{A_{D+1} d^{2(D-1)}} \int_0^{\infty} d\lambda C_A^0(\lambda) C_B^0(\lambda), \quad (10)$$

where  $A_D = [\Omega_D (D - 2)]^2$  with  $\Omega_D$  the area of the unit  $D$  sphere, and  $C^0$  is the monopole element of the generalized capacitance matrix. Cardy [29] also finds the same power law and a similar expression for the QMI in terms of the

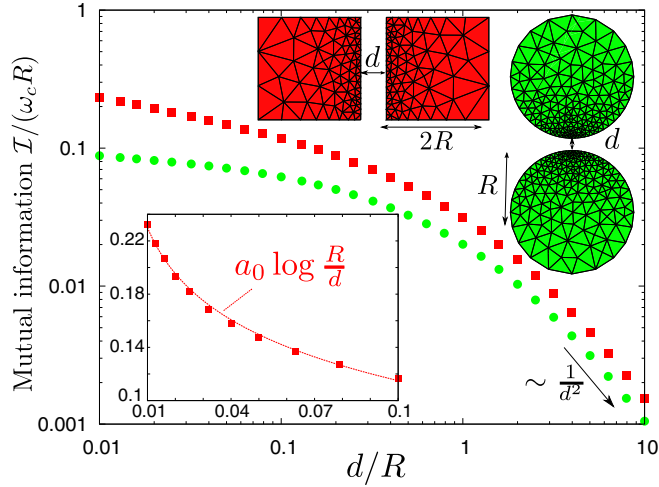


FIG. 2 (color online). QMI versus separation for circular (green circles) or square (red squares) material bodies in  $D = 2$  dimensions, as computed from Eqs. (7) and (8) using the numerical method outlined in the text with the bodies discretized into small triangles (upper insets). At large separations, the QMI obeys a power law, while at short separations the square-square data suggest a logarithmic dependence on  $d$ . The lower inset shows an enlarged view of the short-distance data for squares, together with a logarithmic fit ( $a_0 = 0.050$ ).

capacitance for a scalar field theory, albeit in the absence of matter, see also Refs. [30,31]. This suggests that we can view material bodies coupled to a free field as a probe of quantum information shared between different regions in space.

As a specific example, we consider two disk-shaped bodies in  $D = 2$  dimensions of radii  $R_{A,B}$ . For a single disk of radius  $R_i$ , one computes  $C_i^0(\lambda) = R_i/(4\lambda/R_i + \pi/2)$ , see the Supplemental Material [17]. The QMI is then given by

$$\mathcal{I}_{A,B} = \frac{\omega_c R_A^2 R_B^2 \log R_B/R_A}{32\pi^3 (R_B - R_A) d^2} \xrightarrow{R_A \rightarrow R_B = R} \frac{\omega_c R^3}{32\pi^3 d^2}. \quad (11)$$

Note that, even at large separations, the mutual information depends nontrivially on the radii, reflecting the difficulty of computations in general.

At short separations  $d \lesssim R$ , on the other hand, the multipole expansion becomes inconvenient due to the need to retain large numbers of multipoles. In this case it is useful to note that, while the  $\mathbf{C}$  operator itself is unwieldy to describe in real space, the inverse of this operator has a simple real-space representation [32], namely,  $\mathbf{C}^{-1}(\mathbf{x}, \mathbf{x}') = \lambda \delta(\mathbf{x} - \mathbf{x}') + \mathbf{G}_0(\mathbf{x}, \mathbf{x}')$ . It is then tempting to discretize regions  $A$  and  $B$  into small finite elements (Fig. 2) and introduce a basis of  $N$  expansion functions  $\{b_\alpha(\mathbf{x})\}$  localized on these elements; in this basis, we easily compute matrix elements of  $\mathbf{C}^{-1}$  directly according to  $C_{\alpha\beta}^{-1}(\lambda) \equiv \langle b_\alpha | \mathbf{C}^{-1}(\lambda) | b_\beta \rangle$ , then simply invert the matrix  $\mathbf{C}^{-1}$  numerically to yield the  $\mathbf{C}$  matrix for arbitrary  $\lambda$  (with

$\mathbf{G}$  computed similarly). The operator products and the trace in Eq. (8) become  $(N \times N)$ -dimensional matrix operations and the  $\lambda$  integral is evaluated by numerical quadrature. As two examples, Fig. 2 plots the predictions of Eq. (11) for disk-shaped and square-shaped bodies evaluated numerically for a range of separations  $d$ . In the long-distance regime, the QMI follows the power law in Eq. (11) and increases monotonically as the separation becomes shorter. For small separations  $d \ll R$ , the numerical data suggest a logarithmic dependence on  $d$

$$\mathcal{I}_{A,B} \sim \omega_c R \log \frac{R}{d}. \quad (12)$$

Note that the mutual information is proportional to the “area” of the common boundary of the two media up to a logarithm, in this case the area being the common edge of the two-dimensional regions. The area law can be proved under certain conditions [33–35]; for a review, see Ref. [24]. Logarithmic violations of the area law also appear in several contexts [25,26,36–39].

*Mutual information from thermodynamic entropy.*— Finally, we provide a geometrical interpretation of the thermodynamic model described above in terms of the random-chain polymer ensemble [40]; this approach was inspired by the worldline formalism [41]. We first note that the functional integral of the gradient term  $\int D\theta \exp[-\int (\nabla\theta)^2]$  has an entropic origin as it sums over configurations of phantom chain polymers in free space [42]. In the presence of boundaries [Eq. (9)], this sum is further weighted when a polymer intersects a boundary. In fact, to compute  $\Delta F_{A,B}(\lambda)$ , one must sum over chain polymers that intersect both the  $A$  and  $B$  regions, weighted by their size and intersections, the latter through  $\lambda$  (see Fig. 1). In this sense, the change in free energy is nothing but the change of the entropy ( $k_B T = 1$ ), that is,  $-\Delta F = \Delta S_{\text{th}}$ , where  $S_{\text{th}}$  denotes the purely thermodynamic entropy of a fluctuating polymer. We thus arrive at the fundamental conclusion of this Letter: The QMI in  $D$  dimensions is obtained from the thermodynamic entropy in  $D + 1$  dimensions as

$$\mathcal{I}^{(D)} = 2\omega_c \int_0^\infty d\lambda \Delta S_{\text{th}}^{(D+1)}(\lambda), \quad (13)$$

where we dropped the dependence on  $A, B$ , but made the dependence on the dimension explicit. We stress that Eq. (13) and the above geometrical picture is easily generalized to three or more regions. For example, the quantum tripartite information is due to polymer chains that intersect all three regions. In fact, the strong subadditivity property for quantum systems [43] expressed in terms of mutual and tripartite information becomes rather trivial once formulated in the geometrical picture above, see the Supplemental Material [17] for details. The thermodynamic analogy (sum over classical configurations) is crucial here.

*Discussion and outlook.*—We have studied the entanglement of dispersive media, and shown that their mutual (and tripartite, etc.) information maps to the thermodynamic entropy in a higher dimension. Both analytical and numerical computations are illustrated based on an electrostatic analogy. Extending the results of this Letter to realistic models such as electromagnetism is worthwhile. The response function can also be generalized to a sum of Lorentzians, and thus a general  $\epsilon(\omega)$ , at the expense of introducing several copies of the matter field for each term in the sum. Including finite temperature should be of interest from both fundamental and practical perspectives. Finally, lifting the assumption of weak coupling as well as generalizing to nonlinear and conformal field theories are worthwhile but more challenging.

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- [1] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, *Phys. Rev. Lett.* **90**, 227902 (2003).
- [2] M. Levin and X.-G. Wen, *Phys. Rev. Lett.* **96**, 110405 (2006).
- [3] A. Kitaev and J. Preskill, *Phys. Rev. Lett.* **96**, 110404 (2006).
- [4] P. Calabrese and J. Cardy, *J. Stat. Mech.: Theory Exp.* (2004), P06002.
- [5] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996).
- [6] S. Popescu and D. Rohrlich, *Phys. Rev. A* **56**, R3319 (1997).
- [7] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, *Rev. Mod. Phys.* **80**, 517 (2008).
- [8] N. Lambert, C. Emary, and T. Brandes, *Phys. Rev. Lett.* **92**, 073602 (2004).
- [9] K. L. Hur, *Ann. Phys. (Amsterdam)* **323**, 2208 (2008).
- [10] I. Klich, *Phys. Rev. Lett.* **109**, 061601 (2012).
- [11] O. Kenneth and I. Klich, *Phys. Rev. Lett.* **97**, 160401 (2006).
- [12] M. F. Maghrebi, R. L. Jaffe, and M. Kardar, *Phys. Rev. A* **90**, 012515 (2014).
- [13] Here we have closely followed Klich [10], see also his discussion on the last term in Eq. (2).
- [14] A. Botero and B. Reznik, *Phys. Rev. A* **67**, 052311 (2003).
- [15] I. Peschel and V. Eisler, *J. Phys. A* **42**, 504003 (2009).
- [16] The two-point function of conjugate momenta should be regularized by subtracting an unimportant divergent term.
- [17] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.114.151602>, which includes Refs. [18–23], for the details of analytical and numerical methods used in the text.
- [18] U. Weiss, *Quantum Dissipative Systems*, Vol. 10 (World Scientific, Singapore, 1999).
- [19] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part II (McGraw-Hill Book Company, New York, 1953).
- [20] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).
- [21] M. Krüger, G. Bimonte, T. Emig, and M. Kardar, *Phys. Rev. B* **86**, 115423 (2012).
- [22] M. T. Homer Reid, J. White, and S. G. Johnson, *Phys. Rev. A* **88**, 022514 (2013).
- [23] A. W. Rodriguez, M. T. H. Reid, and S. G. Johnson, *Phys. Rev. B* **88**, 054305 (2013).
- [24] J. Eisert, M. Cramer, and M. B. Plenio, *Rev. Mod. Phys.* **82**, 277 (2010).
- [25] C. Callan and F. Wilczek, *Phys. Lett. B* **333**, 55 (1994).
- [26] F. W. C. Holzhey and F. Larsen, *Nucl. Phys.* **B424**, 443 (1994).
- [27] H. Li and M. Kardar, *Phys. Rev. Lett.* **67**, 3275 (1991).
- [28] T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, *Phys. Rev. D* **77**, 025005 (2008).
- [29] J. Cardy, *J. Phys. A* **46**, 285402 (2013).
- [30] H. Casini and M. Huerta, *J. High Energy Phys.* 03 (2009) 048.
- [31] N. Shiba, *J. High Energy Phys.* 07 (2012) 100.
- [32] S. J. Rahi, T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, *Phys. Rev. D* **80**, 085021 (2009).
- [33] M. B. Hastings, *J. Stat. Mech.: Theory Exp.* (2007), P08024; *J. Stat. Mech.* (2007) P08024.
- [34] M. M. Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac, *Phys. Rev. Lett.* **100**, 070502 (2008).
- [35] L. Masanes, *Phys. Rev. A* **80**, 052104 (2009).
- [36] M. M. Wolf, *Phys. Rev. Lett.* **96**, 010404 (2006).
- [37] D. Gioev and I. Klich, *Phys. Rev. Lett.* **96**, 100503 (2006).
- [38] B. Swingle, *Phys. Rev. Lett.* **105**, 050502 (2010).
- [39] H. Leschke, A. V. Sobolev, and W. Spitzer, *Phys. Rev. Lett.* **112**, 160403 (2014).
- [40] H. Gies and K. Klingmüller, *Phys. Rev. D* **74**, 045002 (2006).
- [41] C. Schubert, *Phys. Rep.* **355**, 73 (2001).
- [42] The mapping between phantom or self-avoiding polymers and field theory is discussed in P. G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell University Press, Ithaca, 1979). See also Ref. [40].
- [43] E. H. Lieb and M. B. Ruskai, *J. Math. Phys. (N.Y.)* **14**, 1938 (1973).