The nonlinear future stability of the FLRW family of solutions to the irrotational Euler–Einstein system with a positive cosmological constant

The Nonlinear Future Stability of the FLRW Family of Solutions to the Irrotational Euler-Einstein System with a Positive Cosmological Constant

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Abstract

In this article, we study small perturbations of the family of Friedmann-Lemaître-Robertson-Walker cosmological background solutions to the coupled Euler-Einstein system with a positive cosmological constant in $1 + 3$ spacetime dimensions. The background solutions model an initially uniform quiet fluid of positive energy density evolving in a spacetime undergoing exponentially accelerated expansion. Our nonlinear analysis shows that under the equation of state $p = c_s^2\rho$, $0 < c_s < \sqrt{1/3}$, the background metric + fluid solutions are globally future-stable under small irrotational perturbations of their initial data. In particular, we prove that the perturbed spacetime solutions, which have the topological structure $[0, \infty) \times \mathbb{T}^3$, are future causally geodesically complete. Our analysis is based on a combination of energy estimates and pointwise decay estimates for quasilinear wave equations featuring dissipative inhomogeneous terms. Our main new contribution is showing that when $0 < c_s < \sqrt{1/3}$, exponential spacetime expansion is strong enough to suppress the formation of fluid shocks. This contrasts against a well-known result of Christodoulou, who showed that in Minkowski spacetime, the corresponding constant-state irrotational fluid solutions are unstable.

Keywords. cosmological constant, energy dissipation, expanding spacetime, geodesically complete, global existence, irrotational fluid, relativistic fluid, wave coordinates

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1 Introduction

The irrotational Euler-Einstein system models the evolution of a dynamic spacetime $(\mathcal{M}, g)$ containing a perfect fluid with vanishing vorticity. By spacetime, we mean a $4-$dimensional time-orientable Lorentzian manifold $\mathcal{M}$ together with a *spacetime metric* $g_{\mu\nu}$ on $\mathcal{M}$ of signature $(-,+,+,+)$. In this article, we endow this system with a positive cosmological constant Λ and consider the equation of state $p = c_s^2 \rho$, where p is the fluid *pressure*, ρ is the *proper energy density*, and the non-negative constant c_s is the *speed of sound*. As is fully discussed in Section 3, under these assumptions, the irrotational Euler-Einstein system comprises the equations (here and throughout, we use units with $8\pi G = c = 1$, where c is the speed of light propagation in Maxwell's theory of electromagnetism, and G is Newton's universal gravitational constant)

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}^{(scalar)}, \qquad (\mu, \nu = 0,1,2,3), \tag{1.1a}$$

$$D_\alpha(\sigma^s g^{\alpha\beta} D_\beta \Phi) = 0, \tag{1.1b}$$

where D is the Levi-Civita connection corresponding to $g_{\mu\nu}$, $\mathrm{Ric}_{\mu\nu}$ is the *Ricci curvature tensor*, $R = g^{\alpha\beta}\mathrm{Ric}_{\alpha\beta}$ is the *scalar curvature*, Φ is the *fluid potential* (see Remark 1.1), $T_{\mu\nu}^{(scalar)} = 2\sigma^s(\partial_\mu \Phi)(\partial_\nu \Phi) + g_{\mu\nu}(s+1)^{-1}\sigma^{s+1}$ is the energy-momentum tensor of an irrotational fluid, $\sigma = -g^{\alpha\beta}(\partial_\alpha \Phi)(\partial_\beta \Phi)$ is the square of the *enthalpy per particle*, and $s = (1-c_s^2)/(2c_s^2)$. The fundamental unknowns are $(\mathcal{M}, g, \partial\Phi)$, while the pressure and proper energy density can be expressed as $p = \frac{1}{s+1}\sigma^{s+1}$, $\rho = \frac{2s+1}{s+1}\sigma^{s+1}$. In this article, we will mainly restrict our attention to the case $s > 1$, which is equivalent to $0 < c_s < \sqrt{1/3}$. Although we limit our discussion to the physically relevant case of $1+3$ dimensions, we expect that our work can be easily generalized to apply to the case of $1+n$ dimensions, $n \geq 3$.

Remark 1.1. Due to possible topological obstructions arising in the application of Poincaré's lemma on the spacetime slab $[0,T] \times \mathbb{T}^3$ (see Section 3.1), the function Φ may only be defined locally (even though the one-form $\partial\Phi$, which is the physically relevant fluid variable, does not suffer from this problem). For simplicity, we only give complete details in this article in the case that Φ can be globally defined. Equivalently, we only give complete details in the case that the spacetime one-form β_μ defined in equation (3.12) is exact. We remark that the exactness condition is preserved by the flow of the relativistic Euler equations if it is verified by the ($3-$dimensional) initial data one-form $\underline{\mathring{\beta}}_j$ (which is discussed in more detail below). Under our exactness assumption, in any spacetime slab $[0,T] \times \mathbb{T}^3$ where β_μ exists, there exists a function Φ such that $\partial_\mu \Phi = \beta_\mu$. For a general irrotational fluid, β_μ is closed (i.e. $d\beta = 0$) but not exact (see Section 3.1 for more details). In this general case, equation (1.1b) would be viewed as an equation for the components β_μ.

Furthermore, one would have to supplement (1.1b) with the equations $d\beta = 0$ (the corresponding evolution equations are $\partial_t \beta_j - \partial_j \beta_0 = 0$ relative to the wave coordinate system we use throughout our analysis). We chose to provide full details only in the exact case because exactness simplifies the presentation and the derivation of the fluid energy estimates. However, we stress that the estimates that we derive for $\partial_\mu \Phi$ in the exact case are precisely the same as those that could be derived for β_μ in the general irrotational case; all of our proofs in the exact case could be slightly altered in a very straightforward fashion to apply to the general irrotational case.

As we explain in Section 3, the specification of an equation of state is sufficient to close the relativistic Euler equations. Our choice of $p = c_s^2 \rho$ is often made in the mathematics and cosmology literature. As is explained in Section 4, under such equations of state, there exists a family of Friedmann-Lemaître-Robertson-Walker (FLRW) solutions to (1.1a)-(1.1b) that are frequently used to model a fluid-filled universe undergoing accelerated expansion; these are the solutions that we investigate in detail in this article. The cases $p = 0$ and $p = (1/3)\rho$, which are known as the "dust" and "radiation" equations of state, are of special significance in the cosmology literature. The latter is often used as a simple model for a "radiation-dominated" universe, while the former for a "matter-dominated" universe. Unfortunately, as we will see, these two equations of state lie just outside of the scope of our main theorem. Our results can be summarized as follows. We state them roughly here; they are stated more precisely as theorems in Sections 11 and 12.

Main Results. If $0 < c_s < \sqrt{1/3}$ (i.e. $s > 1$), then the FLRW background solution $([0, \infty) \times \mathbb{T}^3, \widetilde{g}, \partial\widetilde{\Phi})$ to (1.1a)-(1.1b), which describes an initially uniform quiet fluid of constant *positive* proper energy density evolving in a spacetime undergoing exponentially accelerated expansion, is globally future-stable under small perturbations. In particular, small perturbations of the initial data corresponding to the background solution have maximal globally hyperbolic developments that are future causally geodesically complete. We remark that throughout this article, ∂_t is future-directed. Above, $\widetilde{g} = -dt^2 + e^{2\Omega(t)} \sum_{i=1}^3 (dx^i)^2$, and $\partial\widetilde{\Phi} \overset{\text{def}}{=} (\partial_t \widetilde{\Phi}, \partial_1 \widetilde{\Phi}, \partial_2 \widetilde{\Phi}, \partial_3 \widetilde{\Phi}) = (\bar{\Psi} e^{-\varkappa \Omega(t)}, 0, 0, 0)$, where $\bar{\Psi}$ is a positive constant, $\varkappa = 3/(2s+1) = 3c_s^2$, $\Omega(t) \sim (\sqrt{\Lambda/3})t$ is defined in (4.15), and $\mathbb{T}^3 \overset{\text{def}}{=} [-\pi, \pi]^3$ with the ends identified. Furthermore, in the wave coordinate system introduced in Section 5.1, suitably time-rescaled versions of the components $g_{\mu\nu}$ of the perturbed metric, its inverse $g^{\mu\nu}$, the fluid potential one-form $\partial\Phi$, and various coordinate derivatives of these quantities each converge to functions of the spatial coordinates (x^1, x^2, x^3) as $t \to \infty$. The limiting functions are close to time-rescaled components of the FLRW solution, which are constant in t and (x^1, x^2, x^3).

Remark 1.2. In future work, using alternate techniques, we plan to extend the results as follows: **i)** by removing the assumption of irrotationality, and **ii)** by proving future stability in the case $c_s = 0$. The case $c_s = \sqrt{1/3}$ has recently been addressed [LV11] via Friedrich's conformal method (see Section 1.1). Furthermore, we note that Rendall [Ren04] found (using formal power series expansions) evidence *suggesting* instability when $c_s > \sqrt{1/3}$.

Remark 1.3. In this article, we do not address the issue of whether or not the perturbed solutions are decaying towards the exact FLRW background solution. Note also that our results only address perturbations of fluids featuring a strictly positive proper energy density ρ. We have thus avoided certain technical difficulties, such as dealing with a free boundary, that arise when ρ vanishes. Furthermore, note that we have only shown stability in the "expanding" direction $(t \to \infty)$.

We would now like to make a few remarks about the cosmological constant. We do not attempt to give a detailed account of the rich history of Λ, but instead defer to the discussion in [Car01]; we offer only a brief introduction. While the cosmological constant was originally introduced by Einstein [Ein17] to allow for static solutions to the Einstein equations in the presence of matter, the present day motivation for introducing $\Lambda > 0$ is entirely different. The story behind the modern motivation begins in 1929, when Hubble discovered the expansion of the universe [Hub29]. In brief, Hubble's "law," which was formulated based on measurements of *redshift*, states that the velocities at which distant galaxies are receding from Earth are proportional to their distance from Earth. Furthermore, the present day explanation is that the cause of these velocity shifts is the expansion of spacetime itself. For example, a metric of the form $g = -dt^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2$, with $\frac{d}{dt} a > 0$, creates a redshift effect. Now in the 1990's, experimental evidence derived from sources such as type Ia supernovae and the cosmic microwave background led to a surprising conclusion: the universe is in fact undergoing *accelerated* expansion. Our main motivation for introducing the positive cosmological constant is that it allows for spacetime solutions of (1.1a)-(1.1b) that feature this effect. A simple example of a solution to the Einstein-vacuum equations that features such accelerated expansion is the metric $g = -dt^2 + e^{2Ht} \sum_{i=1}^3 (dx^i)^2$ on the manifold $(-\infty, \infty) \times \mathbb{T}^3$, where $H = \sqrt{\Lambda/3}$.

The introduction of a positive cosmological constant is not the only known mechanism for generating solutions to Einstein's equations with accelerated expansion. In particular, Ringström's work [Rin08], which is the main precursor to this article, shows that the Einstein-nonlinear scalar field system, with a suitably chosen nonlinearity $V(\Phi)$, has an open family of future-global solutions undergoing accelerated expansion. More specifically, the system studied by Ringström can be obtained by replacing (1.1b) with $g^{\alpha\beta} D_\alpha D_\beta \Phi = V'(\Phi)$ and setting $T_{\mu\nu}^{(scalar)} = (\partial_\mu \Phi)(\partial_\nu \Phi) - \big[\frac{1}{2} g^{\alpha\beta}(\partial_\alpha \Phi)(\partial_\beta \Phi) + V(\Phi)\big] g_{\mu\nu}$ in equation (1.1a). V is required to satisfy $V(0) > 0, V'(0) = 0, V''(0) > 0$, so that in effect, the influence of the cosmological constant is emulated by $V(\Phi)$ when Φ is small. Ringström's main result, which is analogous to our main result, is a proof of the future-global stability of a large class of spacetimes featuring accelerated expansion.

The **Main Results** stated above allude to both the existence of an *initial value problem* formulation of the Einstein equations, and the existence of a "maximal" solution. These notions are fleshed out in Section 3.2, but we offer a brief description here. One of the principal difficulties in analyzing the Einstein equations is the lack of a canonical coordinate system. Intimately connected to this difficulty is the fact that due to the diffeomorphism invariance of the equations, their hyperbolic nature is not readily apparent until one makes some kind of gauge choice. One way of resolving these difficulties is to work in a special coordinate system known as *wave coordinates* (also known as *harmonic gauge* or *de Donder gauge*), in which the Einstein equations become a system of *quasilinear wave equations*. One advantage of such a formulation is that local-in-time existence for a system of wave equations is immediate, because a standard hyperbolic theory based on energy estimates has been developed (consult e.g. [Hör97, Ch. VI], [Tay97, Ch. 16], [SS98], [Sog08]). Although the use of wave coordinates is often attributed solely to Choquet-Bruhat, it should be emphasized that use of wave coordinates in the context of the Einstein equations goes back to at least 1921, where it is featured in the work of de Donder [dD21]. However, the completion of the initial value problem formulation of the Einstein equations is in fact due to Choquet-Bruhat [CB52]; her main contribution was a proof that the wave coordinate condition is preserved during the evolution of solutions to a modified version of the equations if it is initially satisfied and the constraint equations (1.2a)-(1.2b) are satisfied (see the remarks below).

The initial data for the irrotational Euler-Einstein system consist of a $3-$dimensional Riemannian manifold $\mathring{\Sigma}$ and the following fields defined on $\mathring{\Sigma}$: a Riemannian metric $\mathring{\underline{g}}$, a symmetric two-tensor $\mathring{\underline{K}}$, a function $\mathring{\Psi}$, and a closed one-form $\mathring{\underline{\beta}}$ (i.e., $d\mathring{\underline{\beta}} = 0$, where d denotes the exterior derivative operator on $\mathring{\Sigma}$). A solution consists of a $4-$dimensional manifold $\mathcal{M}$, a Lorentzian metric g, and a closed one-form $\partial\Phi$ (see Remark 1.1) on $\mathcal{M}$ satisfying (1.1a)-(1.1b), together with an embedding $\mathring{\Sigma} \hookrightarrow \mathcal{M}$ such that $\mathring{\underline{g}}$ is the first fundamental form of $\mathring{\Sigma}$ [see definition (3.33)], $\mathring{\underline{K}}$ is the second fundamental form of $\mathring{\Sigma}$ [see definition (3.34)], the restriction of $\partial_{\hat{N}}\Phi$ to $\mathring{\Sigma}$ is $\mathring{\Psi}$, and the restriction of $\partial\Phi$ to vectors tangent to $\mathring{\Sigma}$ is $\mathring{\underline{\beta}}$. Here $\partial_{\hat{N}}\Phi$ denotes the duality pairing of the one-form $\partial\Phi$ with the vectorfield $\hat{N}$, where $\hat{N}$ is the future-directed unit normal $\hat{N}$ to $\mathring{\Sigma}$ (i.e., $\partial_{\hat{N}}\Phi$ is the normal derivative of Φ); see Section 2.3 for a summary of the conventions we use for identifying tensors inherent to $\mathring{\Sigma}$ with spacetime tensors. It is important to note that the initial value problem is overdetermined, and that the data are subject to the *Gauss* and *Codazzi* constraints. The constraints can be expressed as follows relative to an arbitrary local coordinate system (x^1, x^2, x^3) on $\mathring{\Sigma}$:

$$\underline{\mathring{R}} - \underline{\mathring{K}}_{ab}\underline{\mathring{K}}^{ab} + (\underline{\mathring{g}}^{ab}\underline{\mathring{K}}_{ab})^2 - 2\Lambda = 2T^{(scalar)}(\hat{N}, \hat{N})|_{\mathring{\Sigma}}, \tag{1.2a}$$

$$\underline{\mathring{D}}^a \underline{\mathring{K}}_{aj} - \underline{\mathring{g}}^{ab}\underline{\mathring{D}}_j\underline{\mathring{K}}_{ab} = T^{(scalar)}(\hat{N}, \frac{\partial}{\partial x^j})|_{\mathring{\Sigma}}, \qquad (j = 1, 2, 3), \tag{1.2b}$$

where $\underline{\mathring{R}}$ is the scalar curvature of $\underline{\mathring{g}}$, $\underline{\mathring{D}}$ is the Levi-Civita connection corresponding to $\underline{\mathring{g}}$, and $\hat{N}$ is the future-directed normal to $\mathring{\Sigma}$. We remark that when $p = c_s^2\rho$, the results of Section 3.3 imply that $T^{(scalar)}(\hat{N}, \hat{N})|_{\mathring{\Sigma}} = 2\mathring{\sigma}^s\mathring{\Psi}^2 - (s+1)^{-1}\mathring{\sigma}^{s+1}$ and $T^{(scalar)}(\hat{N}, \frac{\partial}{\partial x^j})|_{\mathring{\Sigma}} = 2\mathring{\sigma}^s\mathring{\Psi}\underline{\mathring{\beta}}_j$, where $\mathring{\sigma} = \mathring{\Psi}^2 - \underline{\mathring{g}}^{ab}\underline{\mathring{\beta}}_a\underline{\mathring{\beta}}_b = \sigma|_{\mathring{\Sigma}}$.

Remark 1.4. In this article, we do not address the issue of solving the constraint equations for the system (1.1a) - (1.1b).

17 years after the initial value problem formulation was understood, Choquet-Bruhat and Geroch showed [CBG69] that every sufficiently smooth initial data set [satisfying the constraints (1.2a)-(1.2b)] launches a unique *maximal globally hyperbolic development.* Roughly speaking, this is the largest spacetime solution to the Einstein equations that is uniquely determined by the data. This result is still a local well-posedness result in the sense that it allows for the possibility that the spacetime might contain singularities. In particular, future-directed, causal geodesics may terminate, which in physics terminology means that an observer (light ray in the case of null geodesics) may run into the end of spacetime in finite affine parameter. For spacetimes launched by initial data near that of the FLRW solution, our main result rules out the possibility of these singularities for observers (light rays) traveling in the "future direction."

We offer a few additional remarks concerning wave coordinates. The classic wave coordinate condition is the algebraic relation $\Gamma^\mu = 0$, where the Γ^μ are the contracted Christoffel symbols of the spacetime metric. In this article, we use a version of the wave coordinate condition that is closer in spirit to the one used by Ringström in [Rin08], which was itself inspired by the ideas in [FR00]. Specifically, we set $\Gamma^\mu = \widetilde{\Gamma}^\mu$, where $\widetilde{\Gamma}^\mu$ is the contracted Christoffel symbol of the background solution metric. Simple computations imply that $\widetilde{\Gamma}^\mu = 3\omega\delta_0^\mu$, where $\omega(t) \sim \sqrt{\Lambda/3}$, which is uniquely determined by the parameters $\Lambda > 0$, $\bar{\rho} > 0$, and $\varsigma = 3(1+c_s^2)$, is the function from (4.21). Here, $\bar{\rho}$ denotes the initial proper energy density of the FLRW solution. It follows that in our wave coordinate system, the (geometric) wave equation $g^{\alpha\beta}D_\alpha D_\beta v = 0$ for the function v is equivalent to the modified (also known as the "reduced") wave equation $g^{\alpha\beta}\partial_\alpha\partial_\beta v = 3\omega\partial_t v$, which features the dissipative source term $\omega\partial_t v$. We provide a more detailed discussion of this modified scalar equation in Section 1.2. Furthermore, in Section 5, we modify the irrotational Euler-Einstein system in an analogous fashion, arriving at an equivalent hyperbolic system featuring dissipative terms. More precisely, the modified system is equivalent to the Einstein equations if the data satisfy the Einstein constraint equations (1.2a)-(1.2b) and the wave coordinate condition.

1.1 Comparison with previous work

First, it should be emphasized that the behavior of solutions the fluid equation (1.1b) on exponentially expanding backgrounds is quite different than it is in flat spacetime. In particular, if one fixes a background metric on $[0, \infty) \times \mathbb{T}^3$ near the FLRW metric $\widetilde{g}_{\mu\nu}$, then our proof shows that the fluid equation (1.1b) on this background with $0 < c_s < \sqrt{1/3}$ has global solutions arising from data that are close to that of an initial uniform quiet fluid state, which is represented by $\partial\widetilde{\Phi}$. This is arguably the most interesting aspect of our main result. In contrast, Christodoulou's monograph [Chr07b] shows that on the Minkowski spacetime background, shock singularities can form in solutions to the irrotational fluid equation arising from smooth data that are arbitrarily close to that of a uniform quiet fluid state. Our original intuition for this article was that rapid spacetime expansion should pull apart the fluid and discourage the formation of shocks.

In addition to Christodoulou's nonlinear instability result in the case of flat spacetime, we also mention the well-known *linear instability* result of Sachs and Wolfe [SW67], which features slowly expanding spacetimes. In this work, they consider the Euler-Einstein system with $\Lambda = 0$ under the equations of state $p = 0$ and $p = (1/3)\rho$. Sachs-Wolfe then consider a family of background solutions to this system on the manifold $(-\infty, \infty) \times \mathbb{R}^3$. We remark that these well-known background solutions are of FLRW type, and can be obtained as special cases of the solutions that we present in Section 4 (modulo the fact that the Sachs-Wolfe solutions have spatial slices diffeomorphic to $\mathbb{R}^3$, while our solutions have spatial slices diffeomorphic to $\mathbb{T}^3$). When $\Lambda = 0$, the background metric is $\widetilde{g} = -dt^2 + t^{2Q}\sum_{i=1}^3 (dx^i)^2$, where $Q = \frac{2}{3(1+c_s^2)}$. In particular, the expansion is not accelerated. For the purposes of the present article, the most important result of [SW67] is that the linearization of the Euler-Einstein system with $\Lambda = 0$ around the FLRW solutions is *unstable*. In particular, the linearized system features solutions whose relative density perturbations can grow like t^C, where $C > 0$ depends on c_s. Combining this Sachs-Wolfe result with the results of the present paper and those of [LV11], one reaches the following moral conclusion: *rapid spacetime expansion can suppress the formation of fluid instabilities, while slow spacetime expansion does not.* As a side remark, we mention the most well-known aspect of [SW67]: Sachs-Wolfe showed that the growing density perturbations couple back into the metric. The resulting variations in the metric lead to anisotropies in the cosmic microwave background. In particular, the amount by which photons are gravitationally shifted varies with direction in the sky. The theoretical predictions of this effect, which is known was the Sachs-Wolfe effect, are consistent with the variations in the cosmic microwave background detected by the Mather-Smoot team's COBE satellite in 1992 [BMW$^+$92].

Next, we note that Brauer, Rendall, and Reula have shown [BRR94] a Newtonian analogue of our main result. More specifically, they studied Newtonian cosmological models with a positive cosmological constant and with perfect fluid sources under the equation of state $p = C\rho_{Newt}^{\gamma}$, where $\rho_{Newt} \geq 0$ is the Newtonian mass density, $C > 0$ is a constant, and $\gamma > 1$ is a constant. These models were based on Newton-Cartan theory, which is a slight generalization of ordinary Newtonian gravitational theory that can be endowed with a highly geometric interpretation. The authors showed that small perturbations of a uniform quiet fluid state of constant positive density lead to a future-global solution. It is of particular interest to note that they do not require the fluid to be irrotational. This suggests that our main result can be extended to allow for (small) non-vanishing vorticity. As discussed in Remark 1.2 we will address this issue in an upcoming article.

We also note a curious anti-correlation between our results and some well-known stability arguments for the Euler-Poisson system (a non-relativistic system with vanishing cosmological constant) which may be found e.g. in Chapter XIII of Chandrasekhar's book [Cha61]. Chandrasekhar considers a simple model for an isolated body in equilibrium, namely a static compactly supported solution to the Euler-Poisson equations under an equation of state equivalent to $p = Cn^{\gamma}$, where n denotes the fluid element *number density*, $C > 0$ is a constant, and $\gamma > 0$. He uses virial identity arguments to suggest that such a configuration is stable if $\gamma > 4/3$. However, since (3.7) implies that the equation of state $p = c_s^2\rho$ (here ρ denotes the proper energy density, a relativistic quantity) is equivalent to $p = Cn^{1+c_s^2}$, our main results show that our background solution is stable under irrotational perturbations if $1 < 1+c_s^2 < 4/3$; i.e., our results seem to anti-correlate with the aforementioned non-relativistic one. We temper this observation by noting that our problem differs in several key ways from that of Chandrasekhar; Chandrasekhar studied compactly supported data for a non-relativistic system on a flat background, while here we study relativistic fluids of everywhere positive energy density on an expanding background.

In addition to the previously mentioned work of Ringström, we would also like to mention some other contributions related to the issue of global nonlinear stability for solutions to the Einstein equations with a positive cosmological constant. The first author to obtain global stability results in this direction was Helmut Friedrich, first in vacuum spacetime [Fri86] in $1+3$ dimensions, and then later for the Einstein-Maxwell and Einstein-Yang-Mills equations [Fri91]. Anderson then extended the vacuum result to cover the case of $1+n$ dimensions, n odd [And05]. Their work was based on the *conformal method*, which reduces the question of global stability for the Einstein equations to the much simpler question of local-in-time stability for the *conformal field equations*, which were developed by Friedrich. We remark that the conformal field equations are symmetric hyperbolic, and for such systems, local-in-time stability is a standard result. Unfortunately, the conformal method does not seem to be easily applicable to all matter models that arise in general relativity. In particular, Ringström has stated that one of his main motivations for his wave coordinate approach in [Rin08] is that the conformal method cannot necessarily be easily adapted to handle matter models other than Maxwell and Yang-Mills fields. Our work can be viewed as an example of the robustness of the wave coordinate approach when $\Lambda > 0$. We also note that future-stability in the case $c_s = \sqrt{1/3}$ was shown [LV11] via the conformal method. We remark that the key structural property that allows one to apply the conformal method is the vanishing of the trace of the energy-momentum tensor; for a perfect fluid, the vanishing occurs only for the equation of state $p = (1/3)\rho$.

Finally, we compare our work here to the body of work on the stability of Minkowski space, which is the most well-known solution to the Einstein-vacuum equations in the case $\Lambda = 0$. This groundbreaking work, which was initiated by Christodoulou and Klainerman [CK93], covered the case of the Einstein-vacuum equations in $1+3$ dimensions. Their proof, which is manifestly covariant, relied upon several geometric foliations of spacetime, including maximal $t = const$ slices and also a family outgoing null cones. In particular, it was believed that wave coordinates were unstable in this setting and therefore were unsuitable for proving the global stability of Minkowski spacetime. However, Lindblad and Rodnianski have recently devised yet another proof for the Einstein-vacuum and Einstein-scalar field systems [LR05], [LR10], which is much shorter but less precise, and which proves the global stability of Minkowski spacetime in the wave coordinate gauge $\Gamma^{\mu} = 0$. In particular, Lindblad and Rodnianski were the first authors to show that a wave coordinate system can be used to prove global stability results for the Einstein equations. As we will explain in the next section, our result was technically simpler to achieve than either of these results. More specifically, in $1+3$ dimensions with $\Lambda = 0$, the Einstein-vacuum equations contain nonlinear terms that are on the border of what can be expected to allow for global existence. More precisely, the equations contain nonlinear terms that, on the basis of their order alone, might be expected to produce finite-time blow-up (even for small data). However, in wave coordinates, the Einstein equations were shown to satisfy the *weak null condition* [LR03], which means that they have a special geo-algebraic structure that allows for small-data global existence. As we will see, the addition of $\Lambda > 0$ to the Einstein equations, together with our previously mentioned wave coordinate choice, will lead to the presence of energy dissipation terms. Consequently in the parameter range $0 < c_s < \sqrt{1/3}$, we do not have to contend with the difficult "borderline integrals" that appear in the proofs of the stability of Minkowski spacetime. A more thorough comparison of the proofs of the stability of Minkowski spacetime to the proofs of the stability of exponentially expanding solutions can be found in the introduction of [Rin08]. Moreover, we remark that readers interested in results related to those of Christodoulou-Klainerman and Lindblad-Rodnianski can consult [BZ09], [KN03], [Loi08], [Spe10].

1.2 Overview of the analysis

Our working version of the modified irrotational Euler-Einstein system is provided by equations (5.15a)-(5.15d) below. These equations form a coupled system of quasilinear wave equations containing dissipative inhomogeneous terms and "error" inhomogeneous terms. The system has a diagonal principal part and features two distinct inverse Lorentzian metrics: **i)** the inverse spacetime metric g^{-1}, and **ii)** the *reciprocal acoustical metric* m^{-1} [see (5.17a)-(5.17c)], which is an inverse Lorentzian metric corresponding to the irrotational fluid equation (1.1b). We remark that m^{-1} depends on both g and $\partial\Phi$. More precisely, each equation in the system (5.15a)-(5.15d) can be written in the form

$$e^{\kappa\lambda}\partial_\kappa\partial_\lambda v = \alpha H\partial_t v + \beta H^2 v + F, \tag{1.3}$$

where $v \in \{\Phi, g_{00}+1, g_{0j}, h_{jk} = e^{-2\Omega}g_{jk}\}_{j,k=1,2,3}$, e is one of the two aforementioned inverse Lorentzian metrics, $\alpha > 0$ and $\beta \geq 0$ are constants, $H = \sqrt{\Lambda/3}$, and F is a nonlinear inhomogeneous error term. We remark that strictly speaking, equation (5.15d) below is not written in the form (1.3). However, since the function $\omega(t)$ [see equation (4.21)] rapidly converges to the constant H, equation (5.15d) can be massaged into this form with $\alpha = \varkappa H$ by viewing the difference $\varkappa(H-\omega(t))\partial_t\Phi$ as an additional error term. Now as we will see, our main future stability theorem is driven by the dissipative terms $\alpha H\partial_t v$ and $\beta H^2 v$. Although the system (5.15a)-(5.15d) is quasilinear, our basic strategy for analyzing (1.3) can readily be seen by studying a model semilinear wave equation for a single unknown. For simplicity, we will only address the case $\beta = 0$ in this section. The model equation is $g^{\alpha\beta}D_\alpha D_\beta v = F(v,\partial v)$ for the pre-specified metric $g_{(model)} = -dt^2 + e^{2t}\sum_{i=1}^3(dx^i)^2$ on the manifold-with-boundary $\mathcal{M} = [0,\infty)\times\mathbb{T}^3$. Here, we are using a standard local coordinate system (x^1,x^2,x^3) on $\mathbb{T}^3$. An omitted computation implies that relative to this coordinate system, this model equation can be expressed as follows (where δ^{jk} is the standard Kronecker delta):

$$-\partial_t^2 v + e^{-2t}\delta^{ab}\partial_a\partial_b v = 3\partial_t v + F. \tag{1.4}$$

A standard strategy for proving future-global existence for wave equations such as (1.4) is to derive a priori estimates showing that suitably strong norms of the solution cannot blow-up in finite time. The proof that global existence follows from the finiteness of the norms is known as a *continuation principle*. We remark that the precise details of the continuation principle used in this article are provided in Theorem 5.4. Roughly speaking, in order to apply the continuation principle, it suffices to control the $H^3(\mathbb{T}^3)$ norm of the perturbations. In order to dynamically control these norms, we use an $L^2 - L^\infty$ framework based on energy estimates + Sobolev embedding. To derive energy estimates for solutions to (1.4), one can define the "usual" energy $E^2(t) = \int_{\mathbb{T}^3}\left\{(\partial_t v)^2 + e^{-2t}\delta^{ab}(\partial_a v)(\partial_b v)\right\}d^3x$, and a standard integration by parts argument together with a Cauchy-Schwarz estimate leads to the estimates

$$\frac{d}{dt}E^2 = -2\int_{\mathbb{T}^3}\Big\{3(\partial_t v)^2 + e^{-2t}\delta^{ab}(\partial_a v)(\partial_b v) + (\partial_t v)F\Big\}\, d^3x \leq -2E^2 + 2E\|F\|_{L^2}. \tag{1.5}$$

Note that the energy dissipative term $-2E^2$ on the right-hand side of (1.5) arises from two sources: **i)** the dissipative term $3\partial_t v$ on the right-hand side of (1.4); **ii)** the fact that the spatial part of g^{-1} decays at the rate e^{-2t}. It is clear from (1.5) that sufficient estimates of $\|F\|_{L^2}$ in terms of E would lead to energy decay as $t \to \infty$, which is the main step in establishing future-global existence.

Our estimates for the modified irrotational Euler-Einstein system are in the spirit of the above argument. The corresponding energies are defined in Section 6 (we remark that in our work below, we work with rescaled energies that are approximately constant in time), and by using integration by parts, one can derive analogous versions of (1.5) for the quasilinear equations (5.15a)-(5.15d); see Lemma 6.4 and Lemma 6.7. However, in (5.15a)-(5.15d) [and hence also in (1.3)], the metric is not pre-specified, but instead depends on the solution itself. Consequently, the coerciveness of the energies also depends on the solution. In order to handle this difficulty, we introduce Sobolev norms (also in Section 6) that are independent of the solution. The Sobolev norms are strong enough to control (by Sobolev embedding) the norms appearing in the continuation principle. We then compare the strength of the energies to the strength of the norms. This comparative analysis, which is carried out in Proposition 10.1, shows that (under viable bootstrap assumptions) the energies and norms are equivalent up to factors bounded by a constant.

The bulk of the work in this article goes towards estimating the inhomogeneous error terms [analogous to F in (1.4)] and towards ensuring that the perturbed solution remains close to the background solution. This analysis is carried out in Section 9. We remark that the main tools used for estimating the inhomogeneous terms are Sobolev-Moser type estimates, which we have placed in the Appendix for convenience. Our analysis of the spacetime metric components closely parallels the work [Rin08] of Ringström. In particular, based on Ringström's work, we provide a *Counting Principle* estimate [see (9.33)] based on the *net number of spatial indices* in a product of metric and fluid quantities. This tool can be used to quickly (and roughly, but good enough to prove small-data future-global existence) determine the rates of decay/growth of products of such terms. We remark that the Counting Principle is not precise enough to detect the improved decay estimates derived in Section 12.

Although Ringström's framework is useful for analyzing the metric components, our analysis of the fluid variables $\partial_\mu \Phi$, $(\mu = 0, 1, 2, 3)$, involves additional complications beyond those encountered in his analysis. We would now like to briefly discuss these complications, and also to indicate why we make the assumption $0 < c_s < \sqrt{1/3}$. We believe that the breakdown of our proof in the case $c_s = 0$ is merely an artifact of our methods; we cannot address the equation of state $p = 0$ here because under the present framework, the Lagrangian for Φ would vanish [see equation (3.17)]. In a future article, we will investigate the Euler-Einstein equations with $p = 0$ using an alternate framework. In addition, as mentioned above, future stability in the case $c_s = \sqrt{1/3}$ has been shown in the recent article [LV11]. In contrast, the question of how perturbed solutions behave when $c_s > \sqrt{1/3}$ is open. However, as noted above, Rendall [Ren04] found heuristic evidence (in the form of formal series expansions) suggesting instability when $c_s > \sqrt{1/3}$. In addition, in Section 11.3, we indicate why our future stability proof breaks down when $c_s \geq \sqrt{1/3}$.

The principal difficulty we encounter in our analysis of $\partial\Phi$ is that the background fluid one-form $\partial\widetilde{\Phi} = (\bar{\Psi}e^{-\varkappa\Omega}, 0, 0, 0)$ and the associated quantity $\widetilde{\sigma} = -\widetilde{g}^{\alpha\beta}(\partial_\alpha\widetilde{\Phi})(\partial_\beta\widetilde{\Phi})$ (here $\widetilde{g} = -dt^2 + e^{2\Omega(t)}\sum_{i=1}^3(dx^i)^2$ is the background FLRW metric discussed in the **Main Results** above) both exponentially decay to 0 as $t \to \infty$. Therefore, the quantity $\sigma = -g^{\alpha\beta}(\partial_\alpha\Phi)(\partial_\beta\Phi)$ corresponding to a slightly perturbed solution will also decay. Furthermore, by examining the fluid equation (1.1b), we see that $\sigma = 0$ corresponds to a possible degeneracy. In particular, our proof (Lemma 7.3) that the reciprocal acoustical metric $(m^{-1})^{\mu\nu}$ is well-defined and Lorentzian can fail when $\sigma = 0$. In order to avoid this degeneracy, and in order to close our bootstrap argument for global existence, we prove that σ doesn't decay too quickly, and that it never becomes 0 in finite time. In fact, for the future-global solutions that we construct, we show that the perturbed σ decays at the same rate as $\widetilde{\sigma}$. Since this fact plays a key role in our estimates for the fluid equation, we now outline our approach to its proof; our approach is intimately connected to the assumption $0 < c_s < \sqrt{1/3}$, which is heavily used throughout the paper.

We begin by noting that our main Sobolev norm bootstrap assumption is that $\mathbf{S}_N(t) \leq \epsilon$ on an interval $t \in [0, T)$, where $\mathbf{S}_N(t)$ is defined in (6.2f), $N \geq 3$ is an integer, and ϵ is a sufficiently small positive number. We remark that our main future-global existence theorem (Theorem 11.5) shows that when the data are small, $\mathbf{S}_N(\cdot)$ is future globally bounded by a multiple of $\mathbf{S}_N(0)$. Now assuming that ϵ is sufficiently small, that the perturbed $g_{\mu\nu}$ is near $\widetilde{g}_{\mu\nu}$, and that $0 < c_s < \sqrt{1/3}$, the results of Proposition 9.1 [see (9.7)] imply that σ is strictly positive and decays at the same rate as $\widetilde{\sigma}$. A key ingredient in the proof of these estimates is suitable L^∞ estimates for the ratios $Z_j \stackrel{\text{def}}{=} \partial_j\Phi/\partial_t\Phi$, $(j = 1, 2, 3)$. Let us explain how these ratios enter into the analysis. First, assuming for simplicity of the discussion that $g_{\mu\nu}$ is near $\widetilde{g}_{\mu\nu}$, we deduce that $\sigma \approx (\partial_t\Phi^2)\left[1 - e^{-2\Omega}\delta^{ab}Z_aZ_b\right]$. Now our bootstrap assumption $\mathbf{S}_N(t) \leq \epsilon$ implies via Sobolev embedding that $\partial_t\Phi \approx \bar{\Psi}e^{-\varkappa\Omega}$, where $\varkappa = 3c_s^2$. Thus, in order to show that σ decays at the same rate as $\widetilde{\sigma}$, it suffices to show that $e^{-2\Omega}\delta^{ab}Z_aZ_b$ decays to 0. The main point is that our proof only allows us to deduce such an estimate when $0 < c_s < \sqrt{1/3}$. Specifically, our bootstrap assumption $\mathbf{S}_N(t) \leq \epsilon$ implies via Sobolev embedding that $\|\partial_t\partial_j\Phi\|_{L^\infty} \leq C\epsilon e^{-\varkappa\Omega(t)} \leq C\epsilon e^{-Ht}$. Integrating this estimate from $t = 0$, assuming that $\partial_j\Phi$ is initially of size ϵ, and using the fact that e^{-Ht} is integrable over the interval $t \in [0, \infty)$, we deduce that $\|\partial_j\Phi\|_{L^\infty} \leq C\epsilon$ on $[0, T)$ (where "C" does not depend on T). Also using $\partial_t\Phi \approx \bar{\Psi}e^{-\varkappa\Omega}$, we deduce that $\|Z_j\|_{L^\infty} \leq C\epsilon e^{\varkappa\Omega}$ on $[0, T)$. Therefore, we conclude that $\|e^{-2\Omega}\delta^{ab}Z_aZ_b\|_{L^\infty} \leq C\epsilon^2 e^{2(\varkappa-1)\Omega}$. Thus, if $0 < \varkappa < 1$, then the estimates we have just outlined strongly suggest that indeed σ decays like $\widetilde{\sigma}$. In contrast, if $\varkappa > 1$, then the estimate $\|e^{-2\Omega}\delta^{ab}Z_aZ_b\|_{L^\infty} \leq C\epsilon^2 e^{2(\varkappa-1)\Omega}$ allows for the possibility that the spatial derivatives $e^{-2\Omega}\delta^{ab}(\partial_a\Phi)(\partial_b\Phi)$ become large in magnitude relative to $(\partial_t\Phi)^2$. In turn, this allows for the possibility that σ becomes 0 in finite time; we expect that instability may be present in these cases. Hence, throughout the article, we make the assumption $0 < \varkappa < 1$. This is equivalent to $1 < s < \infty$ and to $0 < c_s < \sqrt{1/3}$; this parameter range of stability is precisely the one mentioned in the **Main Results** stated above.

The above mathematical conditions have a physical interpretation. To elaborate, we first note that the four-velocity u of the fluid is connected to the fluid one-form via equations (3.12) and (3.14), which imply that $u_\mu = -\sigma^{-1/2}\partial_\mu\Phi$. From this relation, the bootstrap assumption $\mathbf{S}_N(t) \leq \epsilon$, and the Sobolev estimates of Proposition 9.1, it is straightforward to verify (using arguments as in the previous paragraph) that $u^0 - 1$ and $g_{ab}u^au^b$ both decay towards 0 whenever $0 < c_s < \sqrt{1/3}$. Hence, in this parameter regime, the perturbed four-velocity decays towards that of the "quiet FLRW fluid" state $\widetilde{u}^\mu \equiv (1, 0, 0, 0)$. In contrast, when $c_s > \sqrt{1/3}$, our estimates allow for the possibility that $g_{ab}u^au^b$ grows without bound; again, for this reason, even though we currently have no rigorous proof, we suspect that the background solutions may be unstable when $c_s > \sqrt{1/3}$.

1.3 Applications to spatial topologies other than $\mathbb{T}^3$

The model metric $g_{(model)} = -dt^2 + e^{2t}\sum_{i=1}^3(dx^i)^2$ has another feature that is of crucial relevance for possible extensions of our work. To illustrate our point, let us consider $g_{(model)}$ to be a metric on $[0, \infty) \times \mathbb{R}^3$, a Lorentzian manifold-with-boundary that has the Cauchy hypersurface $\mathring{\Sigma} \stackrel{\text{def}}{=} \{t = 0\}$. Simple computations imply that the causal past of the causal future of a point intersects $\mathring{\Sigma}$ in a *precompact* set. For example, in this model spacetime, the causal past of the causal future of the origin is contained in the set $\{(t, x^1, x^2, x^3) \mid t \geq 0, \sum_{i=1}^3(x^i)^2 \leq 4\}$. This is in stark contrast to the situation encountered in Minkowski spacetime, where the causal future of a point is the solid forward null cone emanating from that point, and the causal past of this solid null cone contains the *entire* Cauchy hypersurface $\{t = 0\}$. One consequence of this fact is that the study of solutions to wave equations on exponentially expanding spacetimes is a "very" local problem; i.e., if we makes assumptions about the data in a large enough ball $\mathring{B} \subset \mathring{\Sigma}$, then we can control the solution in a non-compact region of spacetime that includes a cylinder of the form $[0, \infty) \times B$, where $B \subset \mathring{B}$ is a suitably chosen spatial-coordinate ball.

Using these observations, Ringström was able to prove the future stability of various solutions to the Einstein-nonlinear scalar field system for many spatial topologies in addition to $\mathbb{T}^3$ [Rin08]. The main idea of the proof is to choose local coordinate patches on the spatial slices on which the problem is quantitatively close to the case $\mathring{\Sigma} = \mathbb{T}^3$, and to piece together the future development of these patches into a global spacetime. The most difficult part of his argument is the global existence theorem on $\mathbb{T}^3$. However, his patching argument requires that one use cut-off functions, which introduces regions in which the Einstein constraint equations are not satisfied. To deal with this difficulty, he constructs his modified system of equations in such a manner that one can still conclude future-global existence, even if the constraint equations are not satisfied in the cut-off regions. Finally, after patching, these artificially-introduced regions are of course "discarded" and are not part of the spacetime.

The modified system (5.15a)-(5.15d) that we study is similar to Ringström's modified equations in that our global existence argument depends only on a smallness condition on the data, and not on whether or not the constraint equations are satisfied. As noted above, this is the main step in Ringström's work. For these reasons, it is very likely that his patching arguments can be used to extend our result to other spatial topologies. However, for sake of brevity, we do not explore this issue in this article.

1.4 Outline of the structure of the paper

- In Section 2, we describe our conventions for indices and introduce some notation for differential operators and Sobolev norms.
- In Section 3, we introduce the irrotational Euler-Einstein system.
- In Section 4, we use a standard ODE ansatz to derive a well-known family of background Friedmann-Lemaître-Robertson-Walker (FLRW) solutions to the irrotational Euler-Einstein system.
- In Section 5 we introduce wave coordinates and use algebraic identities valid in such a coordinate system to construct a modified version of the irrotational Euler-Einstein system. We then discuss how to construct data for the modified system from data for the unmodified system in a manner that is compatible with the wave coordinate condition. Finally, we discuss classical local well-posedness for the modified system and the continuation principle that is used in Section 11.
- In Section 6, we introduce the relevant norms and the related energies for the modified system that we use in our future-global existence argument. We also provide a

preliminary analysis of the time derivatives of the energies, but the inhomogeneous terms are not estimated until Section 9.

- In Section 7, we introduce some bootstrap assumptions on the spacetime metric $g_{\mu\nu}$. We then use these assumptions to provide some linear-algebraic lemmas that are useful for analyzing $g_{\mu\nu}$, the inverse metric $g^{\mu\nu}$, and the reciprocal acoustical metric $(m^{-1})^{\mu\nu}$, which is the effective inverse metric for the irrotational fluid equation (1.1b).

- In Section 8, we introduce our main bootstrap assumption, which is a smallness condition on $\mathbf{S}_N$, a norm of a difference between the perturbed solution and the FLRW solution. We also define the positive constants q and η_{min}, which play a fundamental role in the technical estimates of the following sections.

- Section 9 contains most of the technical estimates. We assume the bootstrap assumptions from the previous sections and use them to deduce estimates for $g_{\mu\nu}$, $g^{\mu\nu}$, $(m^{-1})^{\mu\nu}$, and for the nonlinearities appearing in the modified equations (5.15a)-(5.15d).

- Section 10 is a very short section in which we show that the Sobolev norms we have defined are equivalent to the energies.

- In Section 11, we use the estimates from the previous sections to prove our main theorem, which is a small-data future-global existence result for the modified equations (where "small" means close to the FLRW background solution). We then discuss the breakdown of our proof in the case $c_s \geq \sqrt{1/3}$. Finally, we use the global existence theorem to prove a related theorem, which states that initial data satisfying the irrotational Euler-Einstein constraints, the wave coordinate condition, and a smallness condition lead to a future geodesically complete solution of the irrotational Euler-Einstein system.

- In Section 12, we prove that the global solution from the main theorem converges (in a certain sense) as $t \to \infty$. The main idea is that once we have a global small solution to the modified system, we can revisit the modified equations and upgrade some of the estimates proved in Section 11.

2 Notation

In this section, we briefly introduce some notation that we use in this article.

2.1 Index conventions

Greek indices $\alpha, \beta, \cdots$ take on the values $0, 1, 2, 3$, while Latin indices $a, b, \cdots$ (which we sometimes call "spatial indices") take on the values $1, 2, 3$. Pairs of repeated indices, with one raised and one lowered, are summed (from 0 to 3 if they are Greek, and from 1 to 3 if they are Latin). We lower and raise indices with the spacetime metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. Some exceptions to this rule include the constraint equations (1.2a)-(1.2b) and (3.32a)-(3.32b), in which we use the $3-$metric $\mathring{\underline{g}}_{jk}$ and its inverse $\mathring{\underline{g}}^{jk}$ to lower and raise indices, and in Section 12, in which all indices are lowered and raised with $g_{\mu\nu}$ and $g^{\mu\nu}$ except for the $3-$metric $g_{jk}^{(\infty)}$, which has $g_{(\infty)}^{jk}$ as its corresponding inverse metric.

2.2 Coordinate systems and differential operators

Throughout this article, we work in a standard local coordinate system (x^1, x^2, x^3) on $\mathbb{T}^3$. Although strictly speaking this coordinate system is not globally well-defined, the vectorfields $\partial_j \stackrel{\text{def}}{=} \frac{\partial}{\partial x^j}$ are globally well-defined. This coordinate system extends to a local coordinate system (x^0, x^1, x^2, x^3) on manifolds-with-boundary of the form $\mathcal{M} = [0, T) \times \mathbb{T}^3$, and we often write t instead of x^0. In this local coordinate system, the background FLRW metric $\widetilde{g}$ is of the form (4.1). We write ∂_μ to denote the coordinate derivative $\frac{\partial}{\partial x^\mu}$, and we often write ∂_t instead of ∂_0. Throughout the article, we will perform all of our computations with respect to the fixed frame $\{\partial_\mu\}_{\mu=0,1,2,3}$.

If $\vec{\alpha} = (n_1, n_2, n_3)$ is a triplet of non-negative integers, then we define the spatial multi-index coordinate differential operator $\partial_{\vec{\alpha}}$ by $\partial_{\vec{\alpha}} \stackrel{\text{def}}{=} \partial_1^{n_1}\partial_2^{n_2}\partial_3^{n_3}$. We denote the order of $\vec{\alpha}$ by $|\vec{\alpha}|$, where $|\vec{\alpha}| \stackrel{\text{def}}{=} n_1 + n_2 + n_3$.

We write

$$D_\mu T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_s} = \partial_\mu T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_s} + \sum_{a=1}^{r} \Gamma_{\mu\ \alpha}^{\ \nu_a} T^{\nu_1\cdots\nu_{a-1}\alpha\nu_{a+1}\nu_r}_{\mu_1\cdots\mu_s} - \sum_{a=1}^{s} \Gamma_{\mu\ \mu_a}^{\ \alpha} T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_{a-1}\alpha\mu_{a+1}\mu_s} \tag{2.1}$$

[where the Christoffel symbol $\Gamma_{\mu\ \nu}^{\ \alpha}$ is defined in (3.2d)] to denote the components of the covariant derivative of a tensorfield on $\mathcal{M}$ with components $T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_s}$.

We write $\partial^{(N)} T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_s}$ to denote the array containing all of the N^{th} order *spacetime* coordinate derivatives (including time derivatives) of the component $T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_s}$. Similarly, we write $\underline{\partial}^{(N)} T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_s}$ to denote the array containing all of the N^{th} order *spatial coordinate* derivatives of the component $T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_s}$. We omit the superscript $^{(N)}$ when $N = 1$.

2.3 Identification of spacetime tensors and spatial tensors

We will often view $\mathbb{T}^3$ as an embedded submanifold of the spacetime $\mathcal{M}$ under an embedding ι_t of the form $\iota_t : \mathbb{T}^3 \hookrightarrow \{t\} \times \mathbb{T}^3 \subset \mathcal{M}$, $\iota_t(x^1, x^2, x^3) \stackrel{\text{def}}{=} (t, x^1, x^2, x^3)$. Note that the embedding is a diffeomorphism between $\mathbb{T}^3$ and $\{t\} \times \mathbb{T}^3$. We will often suppress the embedding by identifying $\mathbb{T}^3$ with its image $\iota_t(\mathbb{T}^3)$. Furthermore, if $T_{k_1 \cdots k_s}^{j_1 \cdots j_r}$ is a $\mathbb{T}^3-$inherent "spatial" tensorfield, then there is a unique "spacetime" tensorfield $T'^{\nu_1 \cdots \nu_r}_{\mu_1 \cdots \mu_s}$ defined along $\iota_t(\mathbb{T}^3) \simeq \mathbb{T}^3$ such that $\iota_t^* T' = T$ and such that T' is tangent to $\iota_t(\mathbb{T}^3)$. Here ι_t^* denotes the pullback by ι_t. Recall that $T'^{\nu_1 \cdots \nu_r}_{\mu_1 \cdots \mu_s}$ is tangent to $\iota_t(\mathbb{T}^3)$ if any contraction of any upstairs (downstairs) index with the unit normal covector $\hat{N}_\mu$ (unit normal vector $\hat{N}^\mu$) results in 0; for downstairs indices, this notion depends on the spacetime metric $g_{\mu\nu}$. We will sometimes identify T with T' (especially along the initial data Cauchy hypersurface $\mathring{\Sigma} \simeq \mathbb{T}^3$), and use the same symbol to denote both, e.g. $T_{k_1 \cdots k_s}^{j_1 \cdots j_r} \simeq T^{\nu_1 \cdots \nu_r}_{\mu_1 \cdots \mu_s}$. For example, we shift back and forth between viewing $\mathring{\underline{g}}$ as a $\mathring{\Sigma}-$inherent Riemannian metric $\mathring{\underline{g}}_{jk}$, and as a spacetime tensorfield $\mathring{\underline{g}}_{\mu\nu}$ defined along the embedded hypersurface $\mathring{\Sigma} \subset \mathcal{M}$ [i.e., viewing $\mathring{\underline{g}}_{\mu\nu}$ as the first fundamental form of $\mathring{\Sigma}$ relative to $(\mathcal{M}, g)$]. All of these standard identifications should be clear in context.

2.4 Norms

All of the Sobolev norms we use are defined relative to the local coordinate system (x^1, x^2, x^3) on $\mathbb{T}^3$ introduced above. We remark that our norms are not coordinate invariant quantities, since we work with the norms of the *components* of tensorfields relative to this coordinate system. If f is a function defined on the hypersurface $\{x \in \mathcal{M} \mid x^0 = t\} \simeq \mathbb{T}^3$, then relative to this coordinate system, we define the standard Sobolev norm $\|f\|_{H^N}$ as follows, where $d^3x \stackrel{\text{def}}{=} dx^1 dx^2 dx^3$:

$$\|f\|_{H^N} \stackrel{\text{def}}{=} \Big(\sum_{|\vec{\alpha}| \leq N} \int_{\mathbb{T}^3} \big| \partial_{\vec{\alpha}} f(t, x^1, x^2, x^3) \big|^2 d^3x \Big)^{1/2}. \tag{2.2}$$

The symbol d^3x represents a slight abuse of notation since the coordinate system (x^1, x^2, x^3) is not globally well-defined on $\mathbb{T}^3$. More precisely, by "$\int_{\mathbb{T}^3} f \, d^3x$," we mean the integral of f over $\mathbb{T}^3$ with respect to the measure corresponding to the volume form of the standard Euclidean metric on $\mathbb{T}^3$.

Using the above notation, we can write the N^{th} order homogeneous Sobolev norm of f as

$$\|\underline{\partial}^{(N)} f\|_{L^2} \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}| = N} \|\partial_{\vec{\alpha}} f\|_{L^2}. \tag{2.3}$$

If $\mathfrak{K} \subset \mathbb{R}^n$ or $\mathfrak{K} \subset \mathbb{T}^n$, then $C_b^N(\mathfrak{K})$ denotes the set of $N-$times continuously differentiable functions (either scalar or array-valued, depending on context) on the interior of $\mathfrak{K}$ with bounded derivatives up to order N that extend continuously to the closure of $\mathfrak{K}$. The norm of a function $F \in C_b^N(\mathfrak{K})$ is defined by

$$|F|_{N,\mathfrak{K}} \stackrel{\text{def}}{=} \sum_{|\vec{I}|\leq N} \sup_{\cdot\in\mathfrak{K}} |\partial_{\vec{I}} F(\cdot)|, \tag{2.4}$$

where $\partial_{\vec{I}}$ is a multi-indexed operator representing repeated partial differentiation with respect to the arguments $\cdot$ of F, which may be either spacetime coordinates or metric/fluid potential one-form components depending on context. When $N = 0$, we also use the notation

$$|F|_{\mathfrak{K}} \stackrel{\text{def}}{=} \sup_{\cdot\in\mathfrak{K}} |F(\cdot)|. \tag{2.5}$$

Furthermore, we use the notation

$$|F^{(N)}|_{\mathfrak{K}} \stackrel{\text{def}}{=} \sum_{|\vec{I}|=N} |\partial_{\vec{I}} F|_{\mathfrak{K}}. \tag{2.6}$$

In the case that $\mathfrak{K} = \mathbb{T}^3$, we sometimes use the more familiar notation

$$\|F\|_{L^\infty} \stackrel{\text{def}}{=} \text{ess}\sup_{x\in\mathbb{T}^3} |F(x)|, \tag{2.7}$$

$$\|F\|_{C_b^N} \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N} \|\partial_{\vec{\alpha}} F\|_{L^\infty}. \tag{2.8}$$

If $I \subset \mathbb{R}$ is an interval and X is a normed function space, then we use the notation $C^N(I, X)$ to denote the set of N-times continuously differentiable maps from I into X.

2.5 Running constants

We use C to denote a running constant that is free to vary from line to line. In general, it can depend on N [see (8.1)], c_s, and Λ, but can be chosen to be independent of all functions $(g_{\mu\nu}, \partial_\mu\Phi)$, $(\mu, \nu = 0, 1, 2, 3)$, that are sufficiently close to the background solution $(\widetilde{g}_{\mu\nu}, \partial_\mu\widetilde{\Phi})$ of Section 4. We sometimes use notation such as $C(N)$ to indicate the dependence of C on quantities that are peripheral to the main argument. Occasionally, we use c, C_*, K_1, etc., to denote a constant that plays a distinguished role in the analysis. We remark that many of the constants blow-up as $\Lambda \to 0^+$.

2.6 A warning on the sign of $\hat{\square}_g$

Although we often choose notation that agrees with the notation used by Ringström in [Rin08], our reduced wave operator $\hat{\square}_g \stackrel{\text{def}}{=} g^{\alpha\beta}\partial_\alpha\partial_\beta$ has the opposite sign of the one in [Rin08].

3 The Irrotational Euler-Einstein System

The Einstein equations connect the *Einstein tensor* $\mathrm{Ric}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, which contains information about the curvature of the spacetime $(\mathcal{M}, g)$, to the *energy-momentum-stress-density tensor* (energy-momentum tensor for short) $T_{\mu\nu}$, which contains information about the matter content of spacetime. In $1+3$ dimensions, they can be expressed as

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \qquad (\mu, \nu = 0,1,2,3), \tag{3.1}$$

where $\mathrm{Ric}_{\mu\nu}$ is the *Ricci curvature tensor*, R is the *scalar curvature*, and Λ is the *cosmological constant*. We stress that the stability results proved in this article heavily depend upon the assumption $\Lambda > 0$. Recall that the Ricci curvature tensor and scalar curvature are defined in terms of the *Riemann curvature tensor* $\mathrm{Riem}_{\mu\alpha\nu}^{\ \ \ \ \beta}$, which can be expressed in terms of the *Christoffel symbols* $\Gamma_{\mu\ \nu}^{\ \alpha}$ of the metric. In a local coordinate system, these quantities can be expressed as follows $(\alpha, \beta, \mu, \nu = 0,1,2,3)$:

$$\mathrm{Riem}_{\mu\alpha\nu}^{\ \ \ \ \beta} \stackrel{\text{def}}{=} \partial_\alpha \Gamma_{\mu\ \nu}^{\ \beta} - \partial_\mu \Gamma_{\alpha\ \nu}^{\ \beta} + \Gamma_{\alpha\ \lambda}^{\ \beta}\Gamma_{\mu\ \nu}^{\ \lambda} - \Gamma_{\mu\ \lambda}^{\ \beta}\Gamma_{\alpha\ \nu}^{\ \lambda}, \tag{3.2a}$$

$$\mathrm{Ric}_{\mu\nu} \stackrel{\text{def}}{=} \mathrm{Riem}_{\mu\alpha\nu}^{\ \ \ \ \alpha} = \partial_\alpha \Gamma_{\mu\ \nu}^{\ \alpha} - \partial_\mu \Gamma_{\alpha\ \nu}^{\ \alpha} + \Gamma_{\alpha\ \lambda}^{\ \alpha}\Gamma_{\mu\ \nu}^{\ \lambda} - \Gamma_{\mu\ \lambda}^{\ \alpha}\Gamma_{\alpha\ \nu}^{\ \lambda}, \tag{3.2b}$$

$$R \stackrel{\text{def}}{=} g^{\alpha\beta}\mathrm{Ric}_{\alpha\beta}, \tag{3.2c}$$

$$\Gamma_{\mu\ \nu}^{\ \alpha} \stackrel{\text{def}}{=} \frac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{3.2d}$$

We remark that under our sign convention, $D_\mu D_\nu X_\alpha - D_\nu D_\mu X_\alpha = \mathrm{Riem}_{\mu\nu\alpha}^{\ \ \ \ \beta}X_\beta$.

The Bianchi identities (see e.g. [Wal84]) imply that the left-hand side of (3.1) is divergence free, which leads to the following equations being satisfied by $T^{\mu\nu}$:

$$D_\nu T^{\mu\nu} = 0, \qquad (\mu = 0,1,2,3). \tag{3.3}$$

By contracting each side of (3.1) with $g^{\mu\nu}$, we deduce that $R = 4\Lambda - T$, where $T \stackrel{\text{def}}{=} g^{\alpha\beta}T_{\alpha\beta}$ is the trace of $T_{\mu\nu}$. From this fact, it easily follows that (3.1) is equivalent to

$$\mathrm{Ric}_{\mu\nu} = \Lambda g_{\mu\nu} + T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}, \qquad (\mu, \nu = 0,1,2,3). \tag{3.4}$$

It is not our aim to give a complete discussion of the notion of a perfect fluid. A thorough introduction to the subject, including a discussion of its history, can be found in Christodoulou's survey article [Chr07a]. Here, we only provide a brief introduction. In general, the energy-momentum tensor for a perfect fluid is

$$T_{(fluid)}^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \qquad (\mu, \nu = 0,1,2,3), \tag{3.5}$$

where $\rho \geq 0$ is the *proper energy density*, $p \geq 0$ is the *pressure*, and u is the *four-velocity*, a unit-length (i.e., $u_\alpha u^\alpha = -1$) future-directed vectorfield on $\mathcal{M}$. The *relativistic*

Euler equations, which are the laws of motion for a perfect relativistic fluid, are the four equations (3.3) together with a conservation law (3.6b) for the number of fluid elements. In a local coordinate system, they can be expressed as follows:

$$D_\nu T^{\mu\nu}_{(fluid)} = 0, \qquad (\mu = 0, 1, 2, 3), \tag{3.6a}$$

$$D_\nu (n u^\nu) = 0, \tag{3.6b}$$

where n is the *proper number density* of the fluid elements. We also introduce the thermodynamic variable η, the *entropy per fluid element*, which we will discuss below.

Unfortunately, even in a prescribed spacetime $(\mathcal{M}, g)$, the equations (3.6a)-(3.6b) do not form a closed system. The standard means of closing the equations is to appeal to the laws of thermodynamics, which imply the following relationships between the fluid variables (see e.g. [GTZ99]):

1. $\rho \geq 0$ is a function of $n \geq 0$ and $\eta \geq 0$.
2. $p \geq 0$ is defined by

$$p = n \left. \frac{\partial \rho}{\partial n} \right|_\eta - \rho, \tag{3.7}$$

   where the notation $|_\cdot$ indicates partial differentiation with $\cdot$ held constant.

3. A perfect fluid satisfies

$$\left. \frac{\partial \rho}{\partial n} \right|_\eta > 0, \left. \frac{\partial p}{\partial n} \right|_\eta > 0, \left. \frac{\partial \rho}{\partial \eta} \right|_n \geq 0 \text{ with } \text{“} = \text{”} \iff \eta = 0. \tag{3.8}$$

   As a consequence, we have that ζ, the *speed of sound* in the fluid, is always real for $\eta > 0$:

$$\zeta^2 \stackrel{def}{=} \left. \frac{\partial p}{\partial \rho} \right|_\eta = \frac{\partial p / \partial n|_\eta}{\partial \rho / \partial n|_\eta} > 0. \tag{3.9}$$

   In general, ζ is not constant. However, for the equations of state we study in this article, ζ is equal to the constant c_s.

4. We also demand that the speed of sound is positive and less than the speed of light whenever $n > 0$ and $\eta > 0$:

$$n > 0 \text{ and } \eta > 0 \implies 0 < \zeta < 1. \tag{3.10}$$

The relationships 1 - 3 express the laws of thermodynamics and are fundamental thermodynamic assumptions, while the relationship 4 ensures that at each $x \in \mathcal{M}$, vectors that are causal with respect to the *sound cone* in $T_x\mathcal{M}$ are necessarily causal with respect to the gravitational null cone in $T_x\mathcal{M}$. The sound cone is defined to be the subset of tangent vectors $X \in T_x\mathcal{M}$ such that $m_{\alpha\beta}X^\alpha X^\beta = 0$, where $m_{\mu\nu}$ is the *acoustical metric*. The matrix $m_{\mu\nu}$ is the inverse of the *reciprocal acoustical metric* $(m^{-1})^{\mu\nu}$, which is introduced in Section 5.4; i.e., $m_{\mu\nu}$ is not obtained by lowering the indices of $(m^{-1})^{\mu\nu}$ with $g_{\mu\nu}$. The gravitational null cone is the subset of tangent vectors $X \in T_x\mathcal{M}$ such that $g_{\alpha\beta}X^\alpha X^\beta = 0$. The physical interpretation of relationship 4 is that the speed of sound is less than the speed of propagation of gravitational waves. See [Spe09b] for a more detailed analysis of the geometry of the sound cone and the gravitational null cone.

We note that the assumptions $\rho \geq 0, p \geq 0$ together imply that the energy-momentum tensor (3.5) satisfies both the *weak energy condition* ($T_{\alpha\beta}^{(fluid)}X^\alpha X^\beta \geq 0$ holds whenever X is timelike and future-directed with respect to the gravitational null cone) and the *strong energy condition* ($[T_{\mu\nu}^{(fluid)} - 1/2g^{\alpha\beta}T_{\alpha\beta}^{(fluid)}g_{\mu\nu}]X^\mu X^\nu \geq 0$ holds whenever X is timelike and future-directed with respect to the gravitational null cone). Furthermore, if we assume that the equation of state is such that $p = 0$ when $\rho = 0$, then (3.9) and (3.10) guarantee that $p \leq \rho$. It is then easy to check that $0 \leq p \leq \rho$ implies that the *dominant energy condition* holds ($-g^{\mu\alpha}T_{\alpha\nu}^{(fluid)}X^\nu$ is causal and future-directed whenever X is causal and future-directed with respect to the gravitational null cone).

Under the remaining relationships, relationship 1 is equivalent to making a choice of an *equation of state*, which is a function that expresses p in terms of η and ρ. An equation of state is not necessarily a fundamental law of nature, but can instead be an empirical relationship between the fluid variables. In this article, we consider the case of an *irrotational, barotropic* fluid under the equation of state $p = c_s^2\rho$, where $0 < c_s < \sqrt{1/3}$, and according to (3.9), the constant c_s is the *speed of sound*. A barotropic fluid is one for which p is a function of ρ alone. Because η plays no role in the analysis of such fluids, this quantity is absent from the remainder of our article. As discussed in Section 3.1, these assumptions imply that the tensor $T_{(fluid)}^{\mu\nu}$ defined in (3.5) is equal to the tensor $T_{(scalar)}^{\mu\nu}$ defined in (3.44), and that the equations (3.3) are equivalent to (3.43), a single quasilinear wave equation for a scalar function Φ (see Remark 1.1). As a consequence, it follows that $T_{(scalar)}^{\mu\nu}$ also satisfies the weak, strong, and dominant energy conditions.

3.1 Irrotational fluids

In this section, we introduce the notion of an irrotational fluid. Our main goal is to show that for an irrotational fluid, *the entire content of the relativistic Euler equations is contained in a single scalar wave equation for the fluid potential* [equation (3.20)]. We assume that the fluid is barotropic, but we do not yet impose the particular equation of state $p = c_s^2\rho$. The fluid potential description of an irrotational fluid in a curved spacetime goes back to at least 1937 [Syn02]. However, in this article, we use modern terminology and notation found e.g. in [Chr07b]. We begin by introducing an important thermodynamic quantity $\sigma \geq 0$, which is the square of the *enthalpy per particle* $\sqrt{\sigma} \geq 0$:

$$\sqrt{\sigma} \stackrel{\text{def}}{=} \frac{\rho + p}{n} = \frac{d\rho}{dn}, \tag{3.11}$$

where we have used (3.7).

We also introduce the following one-form:

$$\beta_\mu \stackrel{\text{def}}{=} -\sqrt{\sigma} u_\mu, \qquad (\mu = 0, 1, 2, 3). \tag{3.12}$$

The *fluid vorticity* v is then defined to be $d\beta$, where d denotes the exterior derivative operator. In local coordinates, we have that

$$v_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, \qquad (\mu, \nu = 0, 1, 2, 3). \tag{3.13}$$

An *irrotational fluid* is defined to be a fluid for which $v_{\mu\nu}$ vanishes everywhere. In this case, by Poincaré's lemma, there *locally* exists (see Remark 1.1) a scalar function Φ, known as the *fluid potential*, such that

$$\beta_\mu = \partial_\mu \Phi, \qquad (\mu = 0, 1, 2, 3), \tag{3.14}$$

which implies that the four-velocity is connected to $\partial\Phi$ via the equation

$$u_\mu = -\frac{\partial_\mu \Phi}{\sqrt{\sigma}}, \qquad (\mu = 0, 1, 2, 3), \tag{3.15}$$

and the square of the enthalpy per particle is connected to $\partial\Phi$ via

$$\sigma = -g^{\alpha\beta}(\partial_\alpha \Phi)(\partial_\beta \Phi). \tag{3.16}$$

We now show that under the assumption of irrotationality, the equations (3.6a) reduce to a single quasilinear wave equation for Φ. We begin by postulating that the Lagrangian for the purported wave equation is equal to p :

$$p = \mathscr{L} = \mathscr{L}(\sigma). \tag{3.17}$$

We note for future use that we can differentiate (3.11) with respect to $\sqrt{\sigma}$ and use the chain rule to conclude that $dp = n d\sqrt{\sigma} + dn(\sqrt{\sigma} - \frac{d\rho}{dn}) = n d\sqrt{\sigma}$, i.e.,

$$\frac{dp}{d\sqrt{\sigma}} = n. \tag{3.18}$$

In equation (3.18), we are viewing p as a function of σ. We also note that from (3.17) and (3.18), it follows that

$$2\frac{\partial \mathscr{L}}{\partial \sigma} = \sigma^{-1/2} \frac{\partial \mathscr{L}}{\partial \sqrt{\sigma}} = \frac{n}{\sqrt{\sigma}} = \frac{n^2}{\rho + p}. \tag{3.19}$$

We now recall a standard fact: that the Euler-Lagrange equation corresponding to the Lagrangian (3.17) is

$$D_\alpha \left[\frac{\partial \mathscr{L}}{\partial \sigma} g^{\alpha\beta} D_\beta \Phi \right] = 0. \tag{3.20}$$

Using (3.15) and (3.19), we conclude that for an irrotational fluid, (3.20) is equivalent to the continuity equation $D_\nu(nu^\nu) = 0$ from (3.6b).

To show that the remaining fluid equations (3.6a) follow from (3.20), we first recall that the energy-momentum tensor for a Lagrangian scalar-field theory can be expressed as

$$T_{\mu\nu}^{(scalar)} = -2\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathscr{L}, \tag{3.21}$$

and that if $\partial\Phi$ is a solution to (3.20), then $T_{\mu\nu}^{(scalar)}$ is symmetric and divergence free:

$$T^{\mu\nu} = T^{\nu\mu}, \qquad (\mu, \nu = 0, 1, 2, 3), \tag{3.22}$$

$$D_\nu T^{\mu\nu}_{(scalar)} = 0, \qquad (\mu = 0, 1, 2, 3). \tag{3.23}$$

For future use, we remark that if $\partial\Phi$ is a solution to the inhomogeneous equation

$$D_\alpha \left[\frac{\partial \mathscr{L}}{\partial \sigma} g^{\alpha\beta} D_\beta \Phi \right] + I_{\partial\Phi} = 0, \tag{3.24}$$

then

$$D_\nu T^{\mu\nu}_{(scalar)} = -2 I_{\partial\Phi} D^\mu \Phi, \qquad (\mu = 0, 1, 2, 3). \tag{3.25}$$

In the case of the Lagrangian (3.17), one can check that (3.21) implies that

$$T_{\mu\nu}^{(scalar)} = 2\frac{\partial \mathscr{L}}{\partial \sigma}(\partial_\mu \Phi)(\partial_\nu \Phi) + g_{\mu\nu}\mathscr{L}. \tag{3.26}$$

Using (3.11), (3.15), (3.17), and (3.26), we compute that

$$T_{\mu\nu}^{(scalar)} = \frac{n}{\sqrt{\sigma}}(\partial_\mu \Phi)(\partial_\nu \Phi) + g_{\mu\nu} p = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}. \tag{3.27}$$

By examining (3.5) and (3.27), we observe that $T_{\mu\nu}^{(scalar)} = T_{\mu\nu}^{(fluid)}$. To summarize, we have shown that if $\partial\Phi$ is a solution to (3.20), then (3.23) necessarily holds. Furthermore, we have shown that if $p, n, u,$ and ρ are defined through $\partial\Phi$ via equations (3.17), (3.18), (3.15), and (3.19) respectively, then it follows that all 5 equations from (3.6a)-(3.6b) are necessarily satisfied. Thus, it follows that for an irrotational fluid, the entire content of the Euler equations is contained in the single scalar equation (3.20) (see Remark 1.1).

We conclude with a summary of the constraints that $\partial\Phi$ and $\mathscr{L}(\sigma)$ must satisfy in order to have an irrotational fluid interpretation. We first summarize the following relationships between various fluid quantities, $\partial\Phi$, and $\mathscr{L}(\sigma)$:

$$\sqrt{\sigma} = \frac{\rho + p}{n} = \frac{d\rho}{dn} = \big[- g^{\alpha\beta}(\partial_\alpha \Phi)(\partial_\beta \Phi)\big]^{1/2}, \tag{3.28a}$$

$$p = \mathscr{L}(\sigma), \tag{3.28b}$$

$$n = \frac{d\mathscr{L}}{d\sqrt{\sigma}} = 2\sqrt{\sigma}\frac{d\mathscr{L}}{d\sigma}, \tag{3.28c}$$

$$\rho = 2\sigma\frac{d\mathscr{L}}{d\sigma} - \mathscr{L} = \sigma\frac{d}{d\sqrt{\sigma}}\left(\frac{\mathscr{L}}{\sqrt{\sigma}}\right) = 2\sigma^{3/2}\frac{d}{d\sigma}\left(\frac{\mathscr{L}}{\sqrt{\sigma}}\right), \tag{3.28d}$$

$$\frac{dp}{dn} = \frac{\sqrt{\sigma}\frac{d\mathscr{L}}{d\sigma}}{2\sigma\frac{d^2\mathscr{L}}{d\sigma^2} + \frac{d\mathscr{L}}{d\sigma}}, \tag{3.28e}$$

$$\frac{dp}{d\rho} = \frac{\frac{d\mathscr{L}}{d\sigma}}{2\sigma\frac{d^2\mathscr{L}}{d\sigma^2} + \frac{d\mathscr{L}}{d\sigma}}. \tag{3.28f}$$

Let us quickly discuss how to derive the relations (3.28a)-(3.28f). Equation (3.28a) follows from (3.7), (3.11) and (3.16). Equation (3.28b) is a restatement of the postulate (3.17). Equation (3.28c) follows from (3.18) and (3.28b). Equation (3.28d) follows from the thermodynamic relation $\rho = n\sqrt{\sigma} - p$, (3.28b), and (3.28c). Equation (3.28e) follows from the chain rule relation $\frac{dp}{dn} = \frac{dp}{d\sigma}\big[\frac{dn}{d\sigma}\big]^{-1}$, (3.28b), and (3.28c). Equation (3.28f) follows from the chain rule relation $\frac{dp}{d\rho} = \frac{dp}{dn}\big[\frac{d\rho}{dn}\big]^{-1}$, (3.28a), and (3.28e).

As we discussed at the beginning of Section 3, physical considerations lead to constraints on the fluid variables. For simplicity, we assume that all of the scalar-valued fluid variables are strictly non-zero; this assumption holds for the fluid solutions considered in this article. Then the following physical constraints hold:

$$\sigma > 0, \tag{3.29a}$$

$$p > 0, \tag{3.29b}$$

$$n > 0, \tag{3.29c}$$

$$\rho > 0, \tag{3.29d}$$

$$\frac{dp}{dn} > 0, \qquad \text{[see equation (3.8)]}, \tag{3.29e}$$

$$0 < \frac{dp}{d\rho} < 1, \qquad \text{[see equations (3.9) and (3.10)]}. \tag{3.29f}$$

With the help of the relations (3.28a)-(3.28f), it is straightforward to verify that (3.29a)-

(3.29f) are collectively equivalent to the following inequalities regarding $\partial\Phi$ and $\mathscr{L}$:

$$g^{\alpha\beta}(\partial_\alpha\Phi)(\partial_\beta\Phi) < 0, \tag{3.30a}$$
$$\mathscr{L}(\sigma) > 0, \tag{3.30b}$$
$$\frac{d\mathscr{L}}{d\sigma} > 0, \tag{3.30c}$$
$$\frac{d}{d\sigma}\left(\mathscr{L}/\sqrt{\sigma}\right) > 0, \tag{3.30d}$$
$$\frac{d^2\mathscr{L}}{d\sigma^2} > 0. \tag{3.30e}$$

We remark that the Lagrangians $\mathscr{L}(\sigma)$ corresponding to the equations of state $p = c_s^2\rho$ [see (3.40)] verify the above assumptions when $0 < c_s < 1$; this claim is verified in Section 3.3.

3.2 The initial value problem for the irrotational Euler-Einstein system

In this section, we discuss various aspects of the initial value problem for the Einstein equations, including the initial data and the notion of the *maximal globally hyperbolic development* of the data. We assume that we are given a Lagrangian $\mathscr{L} = \mathscr{L}(\sigma)$ and a fluid one-form $\partial\Phi$ that are subject to the constraints (3.30a)-(3.30e). We remark that the discussion in this section is very standard, and we provide it only for convenience.

3.2.1 Summary of the irrotational Euler-Einstein system

We first summarize the results of the previous sections by stating that the irrotational Euler-Einstein system is the following system of equations:

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}^{(scalar)}, \qquad (\mu,\nu = 0,1,2,3), \tag{3.31a}$$
$$D_\alpha\left[\frac{\partial\mathscr{L}}{\partial\sigma}g^{\alpha\beta}D_\beta\Phi\right] = 0, \tag{3.31b}$$

where $\mathscr{L} = \mathscr{L}(\sigma)$, $\sigma = -g^{\alpha\beta}(\partial_\alpha\Phi)(\beta_\alpha\Phi)$, and $T_{\mu\nu}^{(scalar)} = 2\frac{\partial\mathscr{L}}{\partial\sigma}(\partial_\mu\Phi)(\partial_\nu\Phi) + g_{\mu\nu}\mathscr{L}$.

3.2.2 Initial data for the irrotational Euler-Einstein system

The initial value problem formulation of the Einstein equations goes back to the seminal work [CB52] by Choquet-Bruhat. Initial data for the system (3.31a)-(3.31b) consist of a $3-$dimensional manifold $\mathring{\Sigma}$ together with the following fields on $\mathring{\Sigma}$: a Riemannian metric $\mathring{\underline{g}}$, a symmetric covariant two-tensor $\mathring{\underline{K}}$, a function $\mathring{\Psi}$, and a closed one-form $\mathring{\underline{\beta}}$ (i.e., $\partial_i\mathring{\underline{\beta}}_j - \partial_j\mathring{\underline{\beta}}_i = 0$).

It is well-known that one cannot consider arbitrary data for the Einstein equations. The data are in fact subject to the following constraints, which can be expressed as follows relative to an arbitrary local coordinate system (x^1, x^2, x^3) on $\mathring{\Sigma}$:

$$\underline{\mathring{R}} - \underline{\mathring{K}}_{ab}\underline{\mathring{K}}^{ab} + (\underline{\mathring{g}}^{ab}\underline{\mathring{K}}_{ab})^2 - 2\Lambda = 2T^{(scalar)}(\hat{N}, \hat{N})|_{\mathring{\Sigma}}, \tag{3.32a}$$

$$\underline{\mathring{D}}^a \underline{\mathring{K}}_{aj} - \underline{\mathring{g}}^{ab}\underline{\mathring{D}}_j\underline{\mathring{K}}_{ab} = T^{(scalar)}(\hat{N}, \frac{\partial}{\partial x^j})|_{\mathring{\Sigma}}, \qquad (j = 1, 2, 3). \tag{3.32b}$$

Above, $\underline{\mathring{R}}$ is the scalar curvature of $\underline{\mathring{g}}$, $\underline{\mathring{D}}$ is the Levi-Civita connection corresponding to $\underline{\mathring{g}}$, and $\hat{N}$ is the future-directed normal to $\mathring{\Sigma}$. We remark that when $p = c_s^2\rho$, the results of Section 3.3 imply that $T^{(scalar)}(\hat{N}, \hat{N})|_{\mathring{\Sigma}} = 2\mathring{\sigma}^s\mathring{\Psi}^2 - (s+1)^{-1}\mathring{\sigma}^{s+1}$ and $T^{(scalar)}(\hat{N}, \frac{\partial}{\partial x^j})|_{\mathring{\Sigma}} = 2\mathring{\sigma}^s\mathring{\Psi}\underline{\mathring{\beta}}_j$, where $\mathring{\sigma} = \mathring{\Psi}^2 - \underline{\mathring{g}}^{ab}\underline{\mathring{\beta}}_a\underline{\mathring{\beta}}_b$.

The constraints (3.32a)-(3.32b) are manifestations of the *Gauss* and *Codazzi* equations respectively. These equations relate the geometry of the ambient Lorentzian spacetime $(\mathcal{M}, g)$ + matter field $\partial\Phi$ (which have to be constructed in the problem at hand) to the geometry + matter field inherited by an embedded Riemannian hypersurface (which will be $(\mathring{\Sigma}, \underline{\mathring{g}})$ + $(\mathring{\Psi}, \underline{\mathring{\beta}})$ after construction). Without providing the rather standard details (see e.g. [Chr08]), we remark that they can be derived as consequences of the following assumptions:

- $\mathring{\Sigma}$ is a submanifold of the spacetime manifold $\mathcal{M}$
- $\underline{\mathring{g}}$ is the first fundamental form of $\mathring{\Sigma}$
- $\underline{\mathring{K}}$ is the second fundamental form of $\mathring{\Sigma}$
- $\partial_{\hat{N}}\Phi = \mathring{\Psi}$ and $\underline{\partial}\Phi|_{\mathring{\Sigma}} = \underline{\mathring{\beta}}$ (see Remark 1.1), where $\hat{N}$ is the future-directed normal to $\mathring{\Sigma}$ and $\underline{\partial}\Phi|_{\mathring{\Sigma}}$ denotes the restriction of $\partial\Phi$ to $\mathring{\Sigma}$
- The irrotational Euler-Einstein system is satisfied along $\mathring{\Sigma}$

We recall that $\underline{\mathring{g}}$ is the Riemann metric on $\mathring{\Sigma}$ defined by

$$\underline{\mathring{g}}|_x(X, Y) = g|_x(X, Y), \qquad \forall X, Y \in T_x\mathring{\Sigma}, \tag{3.33}$$

and that $\underline{\mathring{K}}$ is the symmetric tensorfield on $\mathring{\Sigma}$ defined by

$$\underline{\mathring{K}}|_x(X, Y) = g|_x(D_X\hat{N}, Y) = g|_x(D_Y\hat{N}, X), \qquad \forall X, Y \in T_x\mathring{\Sigma}, \tag{3.34}$$

where D is the Levi-Civita connection corresponding to g.

3.2.3 The definition of a solution to the irrotational Euler-Einstein system

In this section, we define the notion of a solution to the irrotational Euler-Einstein system launched by a given initial data set. We begin with the following definition, which describes the maximal possible region of causal influence associated to a set $S \subset \mathcal{M}$, where $(\mathcal{M}, g)$ is a spacetime.

Definition 3.1 (Cauchy developments). Given any set $S \subset \mathcal{M}$, we define $\mathcal{D}(S)$, the Cauchy development of S, to be the union $\mathcal{D}(S) = \mathcal{D}^+(S) \cup \mathcal{D}^-(S)$, where $\mathcal{D}^+(S)$ is the set of all points $p \in \mathcal{M}$ such that every past-inextendible causal curve through p intersects S, and $\mathcal{D}^-(S)$ is the set of all points $p \in \mathcal{M}$ such that every future-inextendible causal curve through p intersects S. Recall that a curve $\gamma : [s_0, s_{max}) \to \mathcal{M}$ is said to be *future-inextendible* if there does not exist an immersed future-directed curve $\widetilde{\gamma} : I = [s_0, s_1) \to \mathcal{M}$ with $s_1 > s_{max}$ and $\widetilde{\gamma}|_{[s_0, s_{max})} = \gamma$. Past-inextendibility is defined in an analogous manner. $\mathcal{D}^+(S)$ is called the *future Cauchy development of* S, while $\mathcal{D}^-(S)$ is called the *past Cauchy development of* S.

We also rigorously define a *Cauchy hypersurface*.

Definition 3.2 (Cauchy hypersurface). A Cauchy hypersurface $\mathring{\Sigma}$ in a Lorentzian manifold $\mathcal{M}$ is a hypersurface that is intersected exactly once by every inextendible timelike curve in $\mathcal{M}$.

It is well-known that if $\mathring{\Sigma} \subset \mathcal{M}$ is a Cauchy hypersurface, then $\mathcal{D}(\mathring{\Sigma}) = \mathcal{M}$ (see e.g. [O'N83]).

Definition 3.3 (A solution). Given sufficiently smooth initial data $(\mathring{\Sigma}, \mathring{\underline{g}}_{jk}, \mathring{\underline{K}}_{jk}, \mathring{\Psi}, \mathring{\underline{\beta}}_j)$, $(j, k = 1, 2, 3)$, as described in Section 3.2.2, a (classical) solution to the irrotational Euler-Einstein system (3.31a)-(3.31b) is a $4-$dimensional manifold $\mathcal{M}$, a Lorentzian metric $g_{\mu\nu}$, a closed one-form $\partial_\mu \Phi$ (corresponding to a locally defined function Φ - see Remark 1.1), $(\mu, \nu = 0, 1, 2, 3)$, and an embedding $\mathring{\Sigma} \hookrightarrow \mathcal{M}$ subject to the following conditions:

- g is a C^2 tensorfield and $\partial\Phi$ is a C^1 tensorfield
- Equations (3.31a)-(3.31b) are verified in $\mathcal{M}$ by the components of g and $\partial\Phi$
- $\mathring{\Sigma}$ is a Cauchy hypersurface in $(\mathcal{M}, g)$
- $\mathring{\underline{g}}$ is the first fundamental form of $\mathring{\Sigma}$
- $\mathring{\underline{K}}$ is the second fundamental form of $\mathring{\Sigma}$
- $\partial_{\hat{N}} \Phi = \mathring{\Psi}$ and $\underline{\partial}\Phi|_{\mathring{\Sigma}} = \mathring{\underline{\beta}}$ (see Remark 1.1), where $\hat{N}$ is the future-directed normal to $\mathring{\Sigma}$ and $\underline{\partial}\Phi|_{\mathring{\Sigma}}$ denotes the restriction of $\partial\Phi$ to $\mathring{\Sigma}$

The triple $(\mathcal{M}, g, \partial\Phi)$ is called a *globally hyperbolic development* of the initial data.

3.2.4 The maximal globally hyperbolic development

We now recall a fundamental abstract existence result of Choquet-Bruhat and Geroch [CBG69], which states that for initial data of sufficient regularity, there is a unique "largest" spacetime determined by it. The following definition captures the notion of this "largest" spacetime.

Definition 3.4 (Maximal globally hyperbolic development). Given sufficiently smooth initial data for the irrotational Euler-Einstein system (3.31a)-(3.31b) [which by definition satisfy the constraints (3.32a)-(3.32b) and $d\underline{\mathring{\beta}} = 0$], a *maximal globally hyperbolic development* of the data is a globally hyperbolic development $(\mathcal{M}, g, \partial\Phi)$ together with an embedding $\iota : \mathring{\Sigma} \hookrightarrow \mathcal{M}$ with the following property: if $(\mathcal{M}', g', \partial\Phi')$ is any other globally hyperbolic development of the same data with an associated embedding $\iota' : \mathring{\Sigma} \hookrightarrow \mathcal{M}'$, then there is a map $\psi : \mathcal{M}' \to \mathcal{M}$ that is a diffeomorphism onto its image such that $\psi^* g = g', \psi^* \partial\Phi = \partial\Phi'$, and $\psi \circ \iota' = \iota$. Here, ψ^* denotes the pullback by ψ.

Before we can state the theorem, we also need the following definition, which captures the notion of having two different representations of the same spacetime.

Definition 3.5 (Isometrically isomorphic developments). The developments $(\mathcal{M}, g, \partial\Phi)$ and $(\mathcal{M}', g', \partial\Phi')$ are said to be *isometrically isomorphic* if the map ψ from the previous definition is a diffeomorphism from $\mathcal{M}$ to $\mathcal{M}'$.

We now state the theorem. The first conclusion is from [CBG69], and the second from [Ger70].

Theorem 3.6 (Existence and topological structure of a maximal globally hyperbolic development). *Given sufficiently smooth initial data for the irrotational Euler-Einstein system* (3.31a)-(3.31b) *[which by definition satisfy the constraints* (3.32a)-(3.32b) *and* $d\underline{\mathring{\beta}} = 0$*], there exists a maximal globally hyperbolic development of the data which is unique up to isometric isomorphism. If* $\mathring{\Sigma}$ *is a Cauchy hypersurface in* $\mathcal{M}$, *then the maximal globally hyperbolic development is homeomorphic to* $\mathbb{R} \times \mathring{\Sigma}$.

We remark that the article [CBG69] only discusses the case of smooth data. However, as discussed in [CGP10, Section 6], the regularity assumptions on the data stated in Theorem 5.2 are sufficient for the conclusions of Theorem 3.6 to be valid.

Most of the remainder of this article is dedicated towards addressing the properties of the maximal globally hyperbolic developments of sufficiently smooth data near those corresponding to the FLRW background solutions of Section 4. The following proposition gives a simple criterion for identifying the portion of the maximal globally hyperbolic development manifold that lies to the future of a Cauchy hypersurface $\mathring{\Sigma}$.

Proposition 3.7 (Identification of $\mathcal{D}^+(\mathring{\Sigma})$). *Let $(\mathcal{M}, g, \partial\Phi)$ be the maximal globally hyperbolic development of initial data given on the Cauchy hypersurface $\mathring{\Sigma}$, and let $\mathcal{M} = \mathcal{D}^+(\mathring{\Sigma}) \cup \mathcal{D}^-(\mathring{\Sigma})$ be the splitting of $\mathcal{M}$ into the future and past of $\mathring{\Sigma}$. Assume that $\mathcal{F} \subset \mathcal{D}^+(\mathring{\Sigma})$ has the following properties:* ***i)*** *$\mathring{\Sigma} \subset \mathcal{F}$, and* ***ii)*** *$(\mathcal{F}, g|_{\mathcal{F}})$ is future-causally geodesically complete. By "future-causally geodesically complete," we mean that all future-directed causal geodesics can be extended indefinitely in affine parameter. Furthermore, $g|_{\mathcal{F}}$ denotes the restriction of g to $\mathcal{F}$. Then*

$$\mathcal{F} = \mathcal{D}^+(\mathring{\Sigma}). \tag{3.35}$$

Proof. Any point $x \in \mathcal{D}^+(\mathring{\Sigma})$ can be joined to $\mathring{\Sigma}$ by an affinely parameterized, past-directed timelike geodesic γ. If we reverse the orientation of γ, we have an affinely parameterized future-directed timelike geodesic initiating from $\mathring{\Sigma}$ and passing through x. By assumption, all such curves are contained in $\mathcal{F}$. □

3.3 Calculations for the equation of state $p = c_s^2\rho$

For the remainder of this article, we restrict our attention to equations of state of the form

$$p = c_s^2\rho, \tag{3.36}$$

where by equation (3.9), c_s is the *speed of sound*. As mentioned in the introduction to this article, our stability results are limited to the following parameter range:

$$0 < c_s < \sqrt{\frac{1}{3}}. \tag{3.37}$$

Equations (3.7), (3.17), (3.19), and (3.36) imply that there exist constants $C > 0$ and $\widetilde{C} > 0$ such that

$$p = Cn^{1+c_s^2} = \widetilde{C}\sigma^{s+1}, \tag{3.38}$$

where

$$s = \frac{1-c_s^2}{2c_s^2}, \qquad c_s^2 = \frac{1}{2s+1}. \tag{3.39}$$

Choosing a convenient normalization constant, we conclude that under the equation of state (3.36), the Lagrangian (3.17) is given by

$$\mathscr{L} = \frac{\sigma^{s+1}}{s+1}. \tag{3.40}$$

Recall that in order for the Lagrangian (3.40) to have a fluid interpretation, we must verify that (3.30a)-(3.30e) hold. The following computations confirm that the Lagrangian (3.40) in fact has a fluid interpretation whenever $\sigma > 0$:

$$\frac{d\mathscr{L}}{d\sigma} = \sigma^s > 0, \tag{3.41a}$$

$$\frac{d}{d\sigma}\left(\mathscr{L}/\sqrt{\sigma}\right) = \left(\frac{2s+1}{2(s+1)}\right)\sigma^{s-1/2} > 0, \tag{3.41b}$$

$$\frac{d^2\mathscr{L}}{d\sigma^2} = s\sigma^{s-1} > 0. \tag{3.41c}$$

In particular, (3.28b)-(3.28f) imply that for the Lagrangian (3.40), we have

$$p = (s+1)^{-1}\sigma^{s+1} > 0, \tag{3.42a}$$

$$n = 2\sigma^{s+1/2} > 0, \tag{3.42b}$$

$$\rho = \frac{2s+1}{s+1}\sigma^{s+1} > 0, \tag{3.42c}$$

$$\frac{dp}{dn} = \frac{1}{2s+1}\sqrt{\sigma} > 0, \tag{3.42d}$$

$$\frac{dp}{d\rho} = \frac{1}{2s+1}. \tag{3.42e}$$

Furthermore, in the case of the Lagrangian (3.40), the Euler-Lagrange equation (3.20) is

$$D_\alpha(\sigma^s g^{\alpha\beta} D_\beta \Phi) = 0, \tag{3.43}$$

while the energy-momentum tensor (3.26) is easily calculated to be

$$T_{\mu\nu}^{(scalar)} = 2\sigma^s(\partial_\mu\Phi)(\partial_\nu\Phi) + g_{\mu\nu}(s+1)^{-1}\sigma^{s+1}. \tag{3.44}$$

For future reference, we record here the following two identities, which follow easily from (3.44):

$$T^{(scalar)} \stackrel{\text{def}}{=} g^{\alpha\beta}T_{\alpha\beta}^{(scalar)} = \left[\frac{2(1-s)}{s+1}\right]\sigma^{s+1}, \tag{3.45}$$

$$T_{\mu\nu}^{(scalar)} - \frac{1}{2}T^{(scalar)}g_{\mu\nu} = 2\sigma^s(\partial_\mu\Phi)(\partial_\nu\Phi) + \left(\frac{s}{s+1}\right)\sigma^{s+1}g_{\mu\nu}, \quad (\mu,\nu = 0,1,2,3). \tag{3.46}$$

3.3.1 Summary of the irrotational Euler-Einstein system under the equation of state $p = c_s^2\rho$

To summarize, we note that under the equation of state $p = c_s^2\rho$, the irrotational Euler-Einstein system comprises the equations

$$\mathrm{Ric}_{\mu\nu} - \Lambda g_{\mu\nu} - T_{\mu\nu}^{(scalar)} + \frac{1}{2}T^{(scalar)}g_{\mu\nu} = 0, \qquad (\mu,\nu = 0,1,2,3), \tag{3.47a}$$

where $T_{\mu\nu}^{(scalar)} = 2\sigma^s(\partial_\mu\Phi)(\partial_\nu\Phi) + g_{\mu\nu}(s+1)^{-1}\sigma^{s+1}$ is as in (3.44), together with (3.43), the equation of motion for an irrotational fluid:

$$D_\alpha(\sigma^s g^{\alpha\beta} D_\beta\Phi) = 0. \tag{3.47b}$$

4 FLRW Background Solutions

Our main results concern the future stability (with respect to irrotational perturbations) of a class of background solutions $([0,\infty) \times \mathbb{T}^3, \widetilde{g}, \partial\widetilde{\Phi})$ to the system (3.47a)-(3.47b). These background solutions, which are of FLRW type, physically model the evolution of an initially *uniform quiet* fluid in a spacetime that is undergoing rapid expansion. We remark that strictly speaking, the terminology "FLRW" is usually reserved for a class of solutions that have spatial slices diffeomorphic to $\mathbb{S}^3$, $\mathbb{R}^3$, or hyperbolic space (see e.g. [Wal84]). To find our FLRW-type solutions of interest, we follow a procedure outlined in [Wal84, Chapter 5] which, under appropriate ansatzes, reduces the Euler-Einstein equations to ODEs. Although our goal is to find ODE solutions to the irrotational equations (3.47a)-(3.47b), the procedure we follow will produce ODE solutions to the full Euler-Einstein system (3.1) + (3.5). However, these ODE solutions will turn out to be irrotational. Thus, as discussed in Section 3.1, these solutions can also be interpreted as solutions to the irrotational system. We remark that the derivation of these solutions is very well-known, and that we have provided it only for convenience.

4.1 Derivation of the FLRW solution

To proceed, we first make the ansatz that the background metric $\widetilde{g}$ has the *warped product* structure (see e.g. [O'N83])

$$\widetilde{g} = -dt^2 + a^2(t)\sum_{i=1}^{3}(dx^i)^2, \tag{4.1}$$

from which it follows that the only corresponding non-zero Christoffel symbols are

$$\widetilde{\Gamma}_{j\ k}^{\ 0} = \omega g_{jk}, \qquad \widetilde{\Gamma}_{j\ 0}^{\ k} = \omega\delta_j^k, \qquad (j,k = 1,2,3), \tag{4.2}$$

where

$$\omega(t) \stackrel{\text{def}}{=} a^{-1}(t)\frac{d}{dt}a(t). \tag{4.3}$$

Using definitions (3.2b) and (3.2c), together with (4.2), we compute that

$$\widetilde{\mathrm{Ric}}_{00} - \frac{1}{2}\widetilde{R}\widetilde{g}_{00} = 3\omega^2, \tag{4.4a}$$

$$\widetilde{\mathrm{Ric}}_{0j} - \frac{1}{2}\widetilde{R}\widetilde{g}_{0j} = 0, \qquad (j = 1,2,3), \tag{4.4b}$$

$$\widetilde{\mathrm{Ric}}_{jk} - \frac{1}{2}\widetilde{R}\widetilde{g}_{jk} = -\left\{2a^{-1}\frac{d^2}{dt^2}a + \omega^2\right\} g_{jk}, \qquad (j,k = 1,2,3). \tag{4.4c}$$

We then assume that $\widetilde{\rho} = \widetilde{\rho}(t), \widetilde{p} = \widetilde{p}(t)$, and $\widetilde{u}^\mu \equiv (1,0,0,0)$, which implies that $\partial\widetilde{\Phi} = (\partial_t\widetilde{\Phi}(t), 0, 0, 0)$. We also assume that the equation of state (3.36) holds. Inserting these ansatzes into the Bianchi identity (3.3) with $\mu = 0$ and using (3.5), we compute that

$$\frac{d}{dt}\ln\widetilde{\rho} = -3\omega(1 + c_s^2) = -\frac{d}{dt}\ln\left([a(t)]^{3(1+c_s^2)}\right). \tag{4.5}$$

Integrating (4.5), we deduce that $\widetilde{\rho}(t)[a(t)]^{3(1+c_s^2)}$ is constant:

$$\widetilde{\rho}a^{3(1+c_s^2)} \equiv \bar{\rho}\mathring{a}^{3(1+c_s^2)} \stackrel{\text{def}}{=} \mathring{\kappa}, \tag{4.6}$$

where the *positive* constant $\bar{\rho}$ denotes the initial (uniform) energy density, and the *positive* constant $\mathring{a}$ is defined by $\mathring{a} \stackrel{\text{def}}{=} a(0)$.

Similarly, inserting the ansatzes into Einstein's equations (3.1) + (3.5), equating 00 components, and using (4.4a) + (4.6) , we deduce (as in e.g. [Wal84]) the following ODE:

$$\frac{d}{dt}a = a\sqrt{\frac{\Lambda}{3} + \frac{\widetilde{\rho}}{3}} = a\sqrt{\frac{\Lambda}{3} + \frac{\mathring{\kappa}}{3a^{3(1+c_s^2)}}}. \tag{4.7}$$

Equations (4.6) and (4.7) are known as the *Friedmann equations* in the cosmology literature. We observe that the rapid expansion of the background spacetime can be easily deduced from the ODE (4.7), which suggests that the asymptotic behavior is $a(t) \sim e^{Ht}$, where $H \stackrel{\text{def}}{=} \sqrt{\Lambda/3}$. A more detailed analysis of $a(t)$ is provided in Lemma 4.2.

Let us make a few remarks about the remaining $0j$ and jk components of Einstein's equations (3.1) + (3.5). Clearly, (4.4b) and (3.5) imply that the $0j$ components of Einstein's equations are satisfied by $(\widetilde{g}, \widetilde{\rho}, \widetilde{u})$ since both sides of Einstein's equations are equal to 0 in this case. In contrast, using (4.4c) and (3.5), we deduce that the quantities $(\widetilde{g}, \widetilde{\rho}, \widetilde{u})$ verify the jk components of (3.1) + (3.5) if and only if the following ODE is verified by $a(t)$:

$$2a\frac{d^2}{dt^2}a + \left(\frac{d}{dt}a\right)^2 - \Lambda a^2 = -c_s^2 a^2\widetilde{\rho}. \tag{4.8}$$

It is straightforward to verify that the relation (4.8) in fact follows as an *automatic consequence* of (4.6)-(4.7). We conclude that if $a(t)$ verifies (4.7), $\widetilde{g}_{\mu\nu}$ is defined by (4.1), $\widetilde{\rho}$

is implicitly defined by (4.6), and $\widetilde{u}^\mu = (1,0,0,0)$, then the quantities $\big(\widetilde{g}_{\mu\nu}, \widetilde{\rho}, \widetilde{u}^\mu\big)$ do in fact solve Einstein's equations:

$$\widetilde{\mathrm{Ric}}_{\mu\nu} - \frac{1}{2}\widetilde{R}\widetilde{g}_{\mu\nu} + \Lambda \widetilde{g}_{\mu\nu} = \widetilde{T}^{(fluid)}_{\mu\nu}, \qquad (\mu,\nu = 0,1,2,3). \tag{4.9}$$

In addition, the fluid equations

$$D_\nu \widetilde{T}^{\mu\nu}_{(fluid)} = 0, \qquad (\mu = 0,1,2,3), \tag{4.10}$$

follow as a consequence of (4.9) and the Bianchi identities. Finally, for barotropic fluids, it is straightforward to verify that (3.6b) follows as an automatic consequence of (4.10) and the thermodynamic relation (3.7):

$$D_\nu(\widetilde{n}\widetilde{u}^\nu) = 0. \tag{4.11}$$

In summary, we have shown that the FLRW background variables verify the full Euler-Einstein system (3.1) + (3.5) + (3.6a)-(3.6b).

We now use the above results to calculate the background one-form $\partial\widetilde{\Phi}$. With $\widetilde{\sigma} \stackrel{\text{def}}{=} -\widetilde{g}^{\alpha\beta}(\partial_\alpha\widetilde{\Phi})(\partial_\beta\widetilde{\Phi}) = (\partial_t\widetilde{\Phi})^2$, we use (3.39), (3.42a), and (4.6) to compute that

$$c_s^2 \mathring{\kappa} a^{-3(1+c_s^2)} = c_s^2\widetilde{\rho} = \widetilde{p} = \frac{2c_s^2}{1+c_s^2}\widetilde{\sigma}^{(1+c_s^2)/(2c_s^2)} = \frac{2c_s^2}{1+c_s^2}(\partial_t\widetilde{\Phi})^{(1+c_s^2)/(c_s^2)}. \tag{4.12}$$

The equalities in (4.12) imply that

$$\partial_t\widetilde{\Phi} = \bar{\Psi} a^{-3/(2s+1)} = \bar{\Psi} e^{-\varkappa\Omega}, \tag{4.13}$$

where

$$\bar{\Psi} \stackrel{\text{def}}{=} \Big(\bar{\rho}\frac{s+1}{2s+1}\Big)^{1/(2s+2)} \mathring{a}^{\varkappa}, \tag{4.14}$$

$$\Omega(t) \stackrel{\text{def}}{=} \ln\Big(a(t)\Big), \tag{4.15}$$

$$\varkappa \stackrel{\text{def}}{=} \frac{3}{2s+1} = 3c_s^2. \tag{4.16}$$

Remark 4.1. $\Omega(t)$ has been introduced solely for cosmetic purposes.

For future use, we also note the following consequences of the above discussion:

$$3\omega^2 - \Lambda = \widetilde{\rho} = \mathring{\kappa} e^{-2\varkappa(s+1)\Omega} = \Big(\frac{2s+1}{s+1}\Big)\widetilde{\sigma}^{s+1}, \tag{4.17a}$$

$$-3\frac{d}{dt}\omega = 3\omega^2 - \Lambda + \mathring{\kappa}\Big(\frac{s+2}{2s+1}\Big)e^{-2\varkappa(s+1)\Omega} = 3\mathring{\kappa}\Big(\frac{s+1}{2s+1}\Big)e^{-2\varkappa(s+1)\Omega} = 3\widetilde{\sigma}^{s+1}. \tag{4.17b}$$

4.2 Analysis of Friedmann's equation

In the following lemma, we analyze the asymptotic behavior of solutions to the ODE (4.7).

Lemma 4.2 (Analysis of Friedmann's equation). *Let $\mathring{a}, \mathring{\kappa}, \varsigma > 0$ be constants, and let $a(t)$ be the solution to the following ODE:*

$$\frac{d}{dt}a(t) = a(t)\sqrt{\frac{\Lambda}{3} + \frac{\mathring{\kappa}}{3[a(t)]^{\varsigma}}}, \qquad a(0) = \mathring{a}. \tag{4.18}$$

Then with $H \stackrel{\mathit{def}}{=} \sqrt{\Lambda/3}$, the solution $a(t)$ is given by

$$a(t) = \left\{ \sinh\left(\frac{\varsigma H t}{2}\right)\sqrt{\frac{\mathring{\kappa}}{3H^2} + \mathring{a}^{\varsigma}} + \mathring{a}^{\varsigma/2}\cosh\left(\frac{\varsigma H t}{2}\right)\right\}^{2/\varsigma}, \tag{4.19}$$

and for all integers $N \geq 0$, there exists a constant $C_N > 0$ such that for all $t \geq 0$, with $A \stackrel{\mathit{def}}{=} \left\{ \frac{1}{2}\left(\sqrt{\frac{\mathring{\kappa}}{3H^2} + \mathring{a}^{\varsigma}} + \mathring{a}^{\varsigma/2}\right)\right\}^{2/\varsigma}$, we have that

$$(1/2)^{2/\varsigma}\mathring{a}e^{Ht} \leq a(t) \leq Ae^{Ht}, \tag{4.20a}$$

$$\left| e^{-Ht}\frac{d^N}{dt^N}a(t) - AH^N \right| \leq C_N e^{-\varsigma H t}. \tag{4.20b}$$

Furthermore, for all integers $N \geq 0$, there exists a constant $\widetilde{C}_N > 0$ such that for all $t \geq 0$, with

$$\omega \stackrel{\mathit{def}}{=} a^{-1}\frac{d}{dt}a, \tag{4.21}$$

we have that

$$H \leq \omega(t) \leq \sqrt{H^2 + \frac{\mathring{\kappa}}{3\mathring{a}^{\varsigma}}}, \tag{4.22a}$$

$$\left| \frac{d^N}{dt^N}\big(\omega(t) - H\big)\right| \leq \widetilde{C}_N e^{-\varsigma H t}. \tag{4.22b}$$

Remark 4.3. Because of equation (4.7), we will assume for the remainder of the article that $\varsigma = 3(1 + c_s^2)$.

Proof. We leave the elementary analysis of this ODE to the reader. □

5 The Modified Irrotational Euler Einstein System

In this section, we introduce our version of wave coordinates, which is based on the framework developed in [Rin08]. We then use algebraic identities that are valid in wave coordinates to construct a modified version of the irrotational Einstein equations, which is a system of quasilinear wave equations containing energy-dissipative terms. Next, to facilitate our analysis in later sections, we algebraically decompose the modified system into principal terms and error terms. Finally, we show that solutions to the modified system also verify the unmodified system if the Einstein constraint equations and the wave coordinate condition are both satisfied along the Cauchy hypersurface $\mathring{\Sigma}$.

5.1 Wave coordinates

To hyperbolize the Einstein equations, we use a version of the well-known family of *wave coordinate* systems. More specifically, we use a coordinate system in which the contracted Christoffel symbols $\Gamma^{\mu} \stackrel{\text{def}}{=} g^{\alpha\beta}\Gamma_{\alpha\ \beta}^{\ \mu}$ of the spacetime metric g are equal to the contracted Christoffel symbols $\widetilde{\Gamma}^{\mu} \stackrel{\text{def}}{=} \widetilde{g}^{\alpha\beta}\widetilde{\Gamma}_{\alpha\ \beta}^{\ \mu}$ of the background metric $\widetilde{g}$. This condition is known as a *wave coordinate* condition since $\Gamma^{\mu} \equiv \widetilde{\Gamma}^{\mu}$ if and only if the coordinate functions (which are scalar-valued) are solutions to the wave equation $g^{\alpha\beta}D_{\alpha}D_{\beta}x^{\mu} + F^{\mu} = 0$. In the wave coordinate system, the four scalar-valued functions F^{μ} can be expressed as $F^{\mu} = \widetilde{\Gamma}^{\mu} = 3\omega(t)\delta_0^{\mu}$. Using (4.1) and (4.2), we compute that in wave coordinates, we have

$$\Gamma^{\mu} = \widetilde{\Gamma}^{\mu} = 3\omega\delta_0^{\mu}, \qquad \Gamma_{\mu} = g_{\mu\alpha}\Gamma^{\alpha} = 3\omega g_{0\mu}, \tag{5.1}$$

where $\omega(t)$, which is uniquely determined by the parameters $\Lambda > 0$, $\bar{\rho} > 0$, and $\varsigma = 3(1 + c_s^2)$, is the function from (4.21).

We now define the quantities

$$Q^{\mu} \stackrel{\text{def}}{=} F^{\mu} - \Gamma^{\mu}, \qquad Q_{\mu} = F_{\mu} - \Gamma_{\mu}. \tag{5.2}$$

We will treat Q_{μ}, Γ_{μ}, and F_{μ} as one-forms when we compute their covariant derivatives. However, one should note that this is an abuse of notation; for example, the F_{μ} do not have the transformation properties of a one-form under changes of coordinates.

The idea behind wave coordinates is to work in a coordinate system in which $Q^{\mu} \equiv 0$, so that whenever it is expedient, we may replace Γ^{μ} with $3\omega\delta_0^{\mu}$ (and vice-versa) without altering the content of the Einstein equations. The existence of such a coordinate system is nontrivial, and it was only in 1952 that Choquet-Bruhat [CB52] first showed that they exist in general. With this idea in mind, we define (as in [Rin08, Equation (47)]) the *modified Ricci tensor* $\widehat{\text{Ric}}_{\mu\nu}$ by

$$\begin{aligned}\widehat{\text{Ric}}_{\mu\nu} &\stackrel{\text{def}}{=} \text{Ric}_{\mu\nu} + \frac{1}{2}\big(D_{\mu}Q_{\nu} + D_{\nu}Q_{\mu}\big) \\ &= -\frac{1}{2}\hat{\square}_g g_{\mu\nu} + \frac{1}{2}\big(D_{\mu}F_{\nu} + D_{\nu}F_{\mu}\big) + g^{\alpha\beta}g^{\gamma\delta}(\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}),\end{aligned} \tag{5.3}$$

where

$$\hat{\square}_g \stackrel{\text{def}}{=} g^{\alpha\beta}\partial_{\alpha}\partial_{\beta} \tag{5.4}$$

is the *reduced wave operator* corresponding to the metric g.

We now replace the $\mathrm{Ric}_{\mu\nu}$ with $\widehat{\mathrm{Ric}}_{\mu\nu}$ in (3.47a), expand the covariant differentiation in (3.47b), and add additional inhomogeneous terms $I_{\mu\nu}$ and $I_{\partial\Phi}$ to the left-hand sides of (3.47a) and (3.47b) respectively, thus arriving at the *modified irrotational Euler-Einstein system* $(\mu, \nu = 0, 1, 2, 3)$:

$$\widehat{\mathrm{Ric}}_{\mu\nu} - \Lambda g_{\mu\nu} - T_{\mu\nu}^{(scalar)} + \frac{1}{2}T^{(scalar)} g_{\mu\nu} + I_{\mu\nu} = 0, \tag{5.5a}$$

$$\begin{aligned}\big[\sigma g^{\alpha\beta} - 2s(g^{\alpha\kappa}\partial_\kappa\Phi)(g^{\beta\lambda}\partial_\lambda\Phi)\big]\partial_\alpha\partial_\beta\Phi - \sigma\Gamma^\alpha\partial_\alpha\Phi \\ +2s\Gamma^{\alpha\lambda\beta}(\partial_\alpha\Phi)(\partial_\lambda\Phi)(\partial_\beta\Phi) + I_{\partial\Phi} = 0.\end{aligned} \tag{5.5b}$$

Here, the additional terms are defined to be

$$I_{00} \overset{\text{def}}{=} -2\omega Q^0 = 2\omega(\Gamma^0 - 3\omega), \tag{5.6a}$$

$$I_{0j} = I_{j0} \overset{\text{def}}{=} 2\omega Q_j = 2\omega(3\omega g_{0j} - \Gamma_j), \qquad (j = 1, 2, 3), \tag{5.6b}$$

$$I_{jk} = I_{jk} \overset{\text{def}}{=} 0, \qquad (j, k = 1, 2, 3), \tag{5.6c}$$

$$I_{\partial\Phi} \overset{\text{def}}{=} -\sigma g^{\alpha\beta} Q_\alpha \partial_\beta \Phi = \sigma\Gamma^\alpha\partial_\alpha\Phi - 3\omega\sigma\partial_t\Phi. \tag{5.6d}$$

We have several important remarks to make concerning the modified system (5.5a)-(5.5b). First, because the principal term on the left-hand side of (5.5a) is $-\frac{1}{2}\hat{\Box}_g g_{\mu\nu}$, the modified system comprises a quasilinear system of wave equations and is of hyperbolic character. Second, the gauge terms $I_{\mu\nu}, I_{\partial\Phi}$ have been added to the system in order to produce an energy dissipation effect that is analogous to the effect created by the $3\partial_t v$ term on the right-hand side of the model equation (1.4). These dissipation-inducing terms play a key role in the future-global existence theorem of Section 11. Finally, in Section 5.5, we will show that if the initial data satisfy the Gauss and Codazzi constraints (3.32a)-(3.32b), and if the wave coordinate condition $Q_\mu|_{t=0} = 0$ is satisfied, then $Q_\mu, I_{\mu\nu}, I_{\partial\Phi} \equiv 0$, and $\widehat{\mathrm{Ric}}_{\mu\nu} \equiv \mathrm{Ric}_{\mu\nu}$; i.e., under these conditions, the solution to (5.5a)-(5.5b) is also a solution to the irrotational Euler-Einstein system (3.47a)-(3.47b).

5.2 Summary of the modified irrotational Euler-Einstein system for the equation of state $p = c_s^2 \rho$

For convenience, we summarize [with the help of (3.46)] the results of the previous section by listing the modified irrotational Euler-Einstein system $(j, k = 1, 2, 3)$:

$$\widehat{\mathrm{Ric}}_{00} + 2\omega\Gamma^0 - 6\omega^2 - \Lambda g_{00} - 2\sigma^s(\partial_t\Phi)^2 - \Big(\frac{s}{s+1}\Big)\sigma^{s+1} g_{00} = 0, \tag{5.7a}$$

$$\widehat{\mathrm{Ric}}_{0j} + 2\omega(3\omega g_{0j} - \Gamma_j) - \Lambda g_{0j} - 2\sigma^s(\partial_t\Phi)(\partial_j\Phi) - \Big(\frac{s}{s+1}\Big)\sigma^{s+1} g_{0j} = 0, \tag{5.7b}$$

$$\widehat{\mathrm{Ric}}_{jk} - \Lambda g_{jk} - 2\sigma^s(\partial_j\Phi)(\partial_k\Phi) - \Big(\frac{s}{s+1}\Big)\sigma^{s+1} g_{jk} = 0, \tag{5.7c}$$

$$\big[\sigma g^{\alpha\beta} - 2s(g^{\alpha\kappa}\partial_\kappa\Phi)(g^{\beta\lambda}\partial_\lambda\Phi)\big]\partial_\alpha\partial_\beta\Phi - 3\omega\sigma\partial_t\Phi + 2s\Gamma^{\alpha\lambda\beta}(\partial_\alpha\Phi)(\partial_\lambda\Phi)(\partial_\beta\Phi) = 0. \tag{5.7d}$$

5.3 Construction of initial data for the modified system

In this section, we assume that we are given initial data $(\mathring{\Sigma}, \underline{\mathring{g}}, \underline{\mathring{K}}, \mathring{\Psi}, \underline{\mathring{\beta}})$ for the irrotational Euler-Einstein equations (3.47a)-(3.47b) as described in Section 3.2.2 [which by definition satisfy the constraints (3.32a)-(3.32b) and $d\underline{\mathring{\beta}} = 0$]. We will use these data to construct initial data for the modified equations that lead to a solution $(\mathcal{M}, g, \partial\Phi)$ of both the modified system and the unmodified irrotational Euler-Einstein equations; recall that a solution solves both systems $\iff$ $Q_\mu \equiv 0$, where Q_μ is defined in (5.2). We remark that in general, we may consider arbitrary data for the modified equations (5.7a)-(5.7d). However, if the solution of the modified system is also to be a solution of the Einstein equations (3.31a), then we cannot choose the data arbitrarily.

To supply data for the modified equations, we must specify along $\mathring{\Sigma} = \{t = 0\}$ the full spacetime metric components $g_{\mu\nu}|_{t=0}$, their transversal derivatives $\partial_t g_{\mu\nu}|_{t=0}$, $(\mu, \nu = 0, 1, 2, 3)$, the transversal derivative $\partial_t\Phi|_{t=0}$ of the fluid potential, and the tangential (i.e., spatial) derivatives $\underline{\partial}\Phi|_{t=0}$ of the fluid potential (see Remark 1.1). To satisfy the requirements

- $\mathring{\Sigma} = \{t = 0\}$
- $\underline{\mathring{g}}$ is the first fundamental form of $\mathring{\Sigma}$
- ∂_t is transversal to $\mathring{\Sigma}$
- $\underline{\mathring{K}}$ is the second fundamental form of $\mathring{\Sigma}$

- $\partial_{\hat{N}}\Phi|_{\mathring{\Sigma}} = \partial_t\Phi|_{t=0} = \mathring{\Psi}$, where $\partial_{\hat{N}} \stackrel{\text{def}}{=}$ differentiation in the direction of the future-directed normal to $\mathring{\Sigma}$,
- $\underline{\partial}\Phi|_{\mathring{\Sigma}} = \underline{\mathring{\beta}}$

we set (for $j,k = 1,2,3$)

$$g_{00}|_{t=0} = -1, \qquad g_{0j}|_{t=0} = 0, \qquad g_{jk}|_{t=0} = \underline{\mathring{g}}_{jk}, \tag{5.8}$$

$$\partial_t g_{jk}|_{t=0} = 2\underline{\mathring{K}}_{jk}, \qquad \partial_t\Phi|_{t=0} = \mathring{\Psi}, \qquad \partial_j\Phi|_{t=0} = \underline{\mathring{\beta}}_j. \tag{5.9}$$

Furthermore, we need to satisfy the wave coordinate condition $Q_\mu|_{t=0} = 0$, $(\mu = 0,1,2,3)$. To meet this need, we first calculate that

$$\Gamma_0|_{t=0} = -\frac{1}{2}(\partial_t g_{00})|_{t=0} - \underline{\mathring{g}}^{ab}\underline{\mathring{K}}_{ab}, \tag{5.10}$$

$$\Gamma_j|_{t=0} = -\partial_t g_{0j}|_{t=0} + \frac{1}{2}\underline{\mathring{g}}^{ab}(2\partial_a\underline{\mathring{g}}_{bj} - \partial_j\underline{\mathring{g}}_{ab}), \qquad (j = 1,2,3). \tag{5.11}$$

With the help of (5.10) and (5.11), the condition $Q_\mu|_{t=0} = 0$ is easily seen to be equivalent to the following relations, where $\omega(t)$, which is uniquely determined by the parameters $\Lambda > 0$, $\bar{\rho} > 0$, and $\varsigma = 3(1 + c_s^2)$, is the function from (4.21) (and $j = 1,2,3$):

$$\partial_t g_{00}|_{t=0} = 2(-3\omega|_{t=0}\underbrace{g_{00}|_{t=0}}_{-1} - \underline{\mathring{g}}^{ab}\underline{\mathring{K}}_{ab}) = 2\big(3\omega(0) - \underline{\mathring{g}}^{ab}\underline{\mathring{K}}_{ab}\big), \tag{5.12}$$

$$\partial_t g_{0j}|_{t=0} = -3\omega|_{t=0}\underbrace{g_{0j}|_{t=0}}_{0} + \frac{1}{2}\underline{\mathring{g}}^{ab}(2\partial_a\underline{\mathring{g}}_{bj} - \partial_j\underline{\mathring{g}}_{ab}) = \underline{\mathring{g}}^{ab}\Big(\partial_a\underline{\mathring{g}}_{bj} - \frac{1}{2}\partial_j\underline{\mathring{g}}_{ab}\Big). \tag{5.13}$$

We remark that in the above expressions, $\underline{\mathring{g}}^{jk}$ denotes a component of $\underline{\mathring{g}}^{-1}$. This completes our specification of the data for the modified equations.

5.4 Decomposition of the modified irrotational Euler-Einstein system in wave coordinates

Naturally, the key step in our proof of our global existence theorem is our careful analysis of the nonlinear terms. In order to better see their structure, we dedicate this section to a decomposition of the modified system (5.7a)-(5.7d) into principal terms and error terms, which we denote by variations of the symbol $\triangle$. The estimates of Section 9 will justify the claim that the $\triangle$ terms are in fact error terms. We begin by recalling the previously mentioned rescaling h_{jk} of the spatial indices of the metric:

$$h_{jk} \stackrel{\text{def}}{=} e^{-2\Omega}g_{jk}, \qquad (j,k = 1,2,3). \tag{5.14}$$

The decomposition is captured in the next proposition. Additional details are provided in Appendix A.

Proposition 5.1 (Decomposition of the modified equations). *The equations* (5.7a)-(5.7d) *can be written as follows (for $j, k = 1, 2, 3$):*

$$\hat{\Box}_g(g_{00} + 1) = 5H\partial_t g_{00} + 6H^2(g_{00} + 1) + \triangle_{00}, \tag{5.15a}$$
$$\hat{\Box}_g g_{0j} = 3H\partial_t g_{0j} + 2H^2 g_{0j} - 2Hg^{ab}\Gamma_{ajb} + \triangle_{0j}, \tag{5.15b}$$
$$\hat{\Box}_g h_{jk} = 3H\partial_t h_{jk} + \triangle_{jk}, \tag{5.15c}$$
$$\hat{\Box}_m \Phi = \varkappa\omega\partial_t\Phi + \triangle_{\partial\Phi}, \tag{5.15d}$$

where $H \overset{\text{def}}{=} \sqrt{\Lambda/3}$, $\omega(t)$, which is uniquely determined by the parameters $\Lambda > 0$, $\bar{\rho} > 0$, and $\varsigma = 3(1 + c_s^2)$, is the function from (4.21)*, $\varkappa \overset{\text{def}}{=} \frac{3}{1+2s} = 3c_s^2$,*

$$\hat{\Box}_m \overset{\text{def}}{=} -\partial_t^2 + 2(m^{-1})^{0a}\partial_t\partial_a + (m^{-1})^{ab}\partial_a\partial_b \tag{5.16}$$

is the reduced wave operator corresponding to the ***reciprocal acoustical metric*** $(m^{-1})^{\mu\nu}$, *($\mu, \nu = 0, 1, 2, 3$), and the components of m^{-1} are given by*

$$(m^{-1})^{00} = -1, \tag{5.17a}$$
$$(m^{-1})^{0j} = -\frac{\triangle_{(m)}^{0j}}{(1 + 2s) + \triangle_{(m)}}, \tag{5.17b}$$
$$(m^{-1})^{jk} = \frac{g^{jk} - \triangle_{(m)}^{jk}}{(1 + 2s) + \triangle_{(m)}}. \tag{5.17c}$$

The error terms $\triangle_{\mu\nu}$, $\triangle_{\partial\Phi}$, $\triangle_{(m)}$, $\triangle_{(m)}^{0j}$, and $\triangle_{(m)}^{jk}$, from above are provided in equations (A.4a)-(A.4d) *and* (A.6a)-(A.6c).

5.5 Classical local well-posedness

In this section, we discuss classical local well-posedness for the modified system of PDEs (5.15a)-(5.15d). The theorems in this section are stated without proof; we instead provide references for the rather standard techniques that can be used to prove them.

Theorem 5.2 (Local well-posedness). *Let $N \geq 3$ be an integer, and assume that $0 < c_s < 1$. Let $\mathring{g}_{\mu\nu} = g_{\mu\nu}|_{t=0}$, $2\mathring{K}_{\mu\nu} = (\partial_t g_{\mu\nu})|_{t=0}$, $(\mu, \nu = 0, 1, 2, 3)$, $\mathring{\Psi} = (\partial_t\Phi)|_{t=0}$, and $\partial_j\mathring{\Phi} = \underline{\beta}_j = \partial_j\Phi|_{t=0}$ (see Remark 1.1) be initial data (not necessarily satisfying the Einstein constraints) on the manifold $\mathring{\Sigma} = \mathbb{T}^3$ for the modified irrotational equations* (5.15a) - (5.15d) *satisfying (for $i, j, k = 1, 2, 3$)*

$$\mathring{g}_{00} + 1 \in H^{N+1}, \qquad \mathring{g}_{0j} \in H^{N+1}, \qquad \partial_i\mathring{g}_{jk} \in H^N, \tag{5.18a}$$
$$\mathring{K}_{00} \in H^N, \qquad \mathring{K}_{0j} \in H^N, \qquad \mathring{K}_{jk} - \omega(0)e^{2\Omega(0)}\mathring{g}_{jk} \in H^N, \tag{5.18b}$$
$$\mathring{\Psi} - \bar{\Psi} \in H^N, \qquad \partial_j\mathring{\Phi} \in H^N, \tag{5.18c}$$

where $\bar{\Psi} > 0$ *is a constant. Assume further that there are constants* $C_1 > 1$ *and* $C_2, C_3 > 0$ *such that*

$$C_1^{-1}\delta_{ab}X^aX^b \leq \mathring{g}_{ab}X^aX^b \leq C_1\delta_{ab}X^aX^b, \qquad \forall(X^1,X^2,X^3) \in \mathbb{R}^3, \tag{5.19a}$$

$$\mathring{g}_{00} \leq -C_2, \tag{5.19b}$$

$$\mathring{\sigma} \geq C_3, \tag{5.19c}$$

where $\mathring{\sigma} = -g^{\alpha\beta}(\partial_\alpha\Phi)(\partial_\beta\Phi)|_{t=0}$. *Then these data launch a unique classical solution* $(g_{\mu\nu}, \partial_\mu\Phi)$, $(\mu,\nu = 0,1,2,3)$, *to the modified system existing on a spacetime slab* $(T_-,T_+)\times\mathbb{T}^3$, *with* $T_- < 0 < T_+$, *such that*

$$g_{\mu\nu} \in C_b^{N-1}((T_-,T_+)\times\mathbb{T}^3), \qquad \partial_\mu\Phi \in C_b^{N-2}((T_-,T_+)\times\mathbb{T}^3), \tag{5.20}$$

such that $g_{00} < 0$, $\sigma > 0$, *and such that the eigenvalues of the* 3×3 *matrix* g_{jk} *are uniformly bounded from below strictly away from* 0 *and from above.*

The solution has the following regularity properties:

$$g_{00}+1,\ g_{0j} \in C^0((T_-,T_+),H^{N+1}), \tag{5.21a}$$

$$\partial_i g_{jk} \in C^0((T_-,T_+),H^{N}), \tag{5.21b}$$

$$\partial_t g_{00},\ \partial_t g_{0j},\ \partial_t g_{jk} - 2\omega(t)g_{jk} \in C^0((T_-,T_+),H^{N}), \tag{5.21c}$$

$$e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi},\ \partial_j\Phi \in C^0((T_-,T_+),H^{N}). \tag{5.21d}$$

Furthermore, $g_{\mu\nu}$ *is a Lorentzian metric on* $(T_-,T_+)\times\mathbb{T}^3$, *and the sets* $\{t\}\times\mathbb{T}^3$ *are Cauchy hypersurfaces in the Lorentzian manifold* $(\mathcal{M} \stackrel{def}{=} (T_-,T_+)\times\mathbb{T}^3, g)$ *for* $t \in (T_-,T_+)$. *Similarly, the reciprocal acoustical metric* $(m^{-1})^{\mu\nu}$ *is an inverse Lorentzian metric on* $(T_-,T_+)\times\mathbb{T}^3$.

In addition, there exists an open neighborhood $\mathcal{O}$ *of* $(\mathring{g}_{\mu\nu}, \mathring{K}_{\mu\nu}, \mathring{\Psi}, \partial_j\mathring{\Phi} = \mathring{\underline{\beta}}_j)$ *such that all irrotational data belonging to* $\mathcal{O}$ *launch solutions that also exist on the interval* (T_-,T_+) *and that have the same regularity properties as* $(g_{\mu\nu},\partial_\mu\Phi)$. *Furthermore, on* $\mathcal{O}$, *the map from the initial data to the solution is continuous. By continuous, we mean continuous relative to the norms on the data and the norms on the solution that are stated in the hypotheses and conclusions of this theorem.*

Finally, if, as described in Section 5.3, the data for the modified system are constructed from data for the irrotational Euler-Einstein system (which by definition satisfy the constraints (3.32a)-(3.32b) *and* $d\mathring{\underline{\beta}} = 0$ *on the Cauchy hypersurface* $\{0\}\times\mathbb{T}^3$*), and if the wave coordinate condition* $Q_\mu|_{\{0\}\times\mathbb{T}^3} = 0$ *holds, then* $(g_{\mu\nu},\partial_\mu\Phi)$ *is also a solution to the unmodified equations* (3.47a)-(3.47b) *on* $(T_-,T_+)\times\mathbb{T}^3$.

Remark 5.3. The hypotheses in Theorem 5.2 have been stated in a manner that allows us to apply it to sufficiently smooth initial data near that of the background solution of

Section 4. Furthermore, we remark that the assumptions and conclusions concerning the metric components g_{jk} would appear more natural if expressed in terms of the variables $h_{jk} \stackrel{\text{def}}{=} e^{-2\Omega} g_{jk}$; these rescaled quantities are the ones that we use in our global existence proof.

Proof. Theorem 5.2 can be proved using standard methods that follow from energy estimates in the spirit of the ones proved below in Sections 6.2, 6.3, and 10. See e.g. [Hör97, Ch. VI], [Maj84, Ch. 2], [SS98, Ch. 5], [Sog08, Ch. 1], [Spe09b], and [Tay97, Ch. 16] for details on how to prove local well-posedness as a consequence of the availability of these kinds of energy estimates. Also see [Rin08, Proposition 1]. We remark that the Lorentzian nature of $(m^{-1})^{\mu\nu}$ follows from that of $g_{\mu\nu}$ and the inequality $\sigma > 0$; see Lemma 7.3. The fact that $(g_{\mu\nu}, \partial_\mu \Phi)$ is also a solution to the unmodified equations if the constraints and the wave coordinate condition $Q_\mu|_{\mathring{\Sigma}} = 0$ are satisfied is addressed in Section 5.6. $\square$

In our proof of Theorem 11.5, we will use the following continuation principle, which provides criteria that are sufficient to ensure that a solution to the modified equations exists globally in time.

Theorem 5.4 (Continuation principle). *Assume the hypotheses of Theorem 5.2. Let T_{max} be the supremum over all times T_+ such that the solution $(g_{\mu\nu}, \partial_\mu \Phi)$, $(\mu, \nu = 0, 1, 2, 3)$, exists on the slab $[0, T_+) \times \mathbb{T}^3$ and has the properties stated in the conclusions of Theorem 5.2. Then if $T_{max} < \infty$, one of the following four possibilities must occur:*

1) *There is a sequence $(t_n, x_n) \in [0, T_{max}) \times \mathbb{T}^3$ such that $\lim_{n\to\infty} g_{00}(t_n, x_n) = 0$.*

2) *There is a sequence $(t_n, x_n) \in [0, T_{max}) \times \mathbb{T}^3$ such that the smallest eigenvalue of the 3×3 matrix $g_{jk}(t_n, x_n)$ converges to 0 as $n \to \infty$.*

3) *There is a sequence $(t_n, x_n) \in [0, T_{max}) \times \mathbb{T}^3$ such that $\lim_{n\to\infty} \sigma(t_n, x_n) = 0$, where $\sigma = -g^{\alpha\beta}(\partial_\alpha \Phi)(\partial_\beta \Phi)$.*

4) $\lim_{t\to T_{max}^-} \sup_{0\le\tau\le t} \left\{ \|\partial\Phi(\tau,\cdot)\|_{C_b^1} + \sum_{\mu,\nu=0}^3 \Big(\|g_{\mu\nu}(\tau,\cdot)\|_{C_b^2} + \|\partial_t g_{\mu\nu}(\tau,\cdot)\|_{C_b^1} \Big) \right\} = \infty$.

Similar results hold for an interval of the form $(T_{min}, 0]$.

Remark 5.5. If **1)** or **2)** occurs, then the hyperbolicity of equations (5.15a)-(5.15c) breaks down. Similarly, if **3)** occurs, then either the finiteness or the Lorentzian nature of the reciprocal acoustical metric $(m^{-1})^{\mu\nu}$ can break down (see Lemma 7.3).

Proof. See e.g. [Hör97, Ch. VI], [Sog08, Ch. 1], [Spe09a] for the ideas behind a proof. $\square$

5.6 Preservation of the wave coordinate condition

In Section 5.3, from given initial data for the Einstein equations, we constructed initial data for the modified equations that in particular satisfy the wave coordinate condition along the Cauchy hypersurface $\mathring{\Sigma}$; i.e., $Q_\mu|_{t=0} = 0$. As mentioned in the statement of Theorem 5.2, these data launch a solution of both the modified equations and the Einstein equations. As we have discussed previously, this fact would follow from the condition $Q_\mu \equiv 0$. In the next proposition, we sketch a proof of the fact that this condition holds.

Proposition 5.6 (Preservation of the wave coordinate condition). *Let* $\big(\mathring{\underline{g}}_{jk}, \mathring{\underline{K}}_{jk}, \mathring{\Psi}, \partial_j\mathring{\Phi} = \mathring{\underline{\beta}}_j\big)$, $(j,k = 1,2,3)$, *be initial data (see Remark 1.1) for the unmodified irrotational Euler-Einstein system* (3.47a)-(3.47b) *[which by definition satisfy the constraints* (3.32a)-(3.32b) *and* $d\mathring{\underline{\beta}} = 0$*]. Let* $\big(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0}, \mathring{\Psi}, \partial_j\mathring{\Phi}\big)$, $(\mu,\nu = 0,1,2,3)$, *be the initial data for the modified equations* (5.15a)-(5.15d) *that are constructed from the data for the unmodified irrotational Euler-Einstein system as described in Section 5.3. In particular, we recall that the construction of Section 5.3 leads the fact that* $Q_\mu|_{t=0} = 0$ *where* Q_μ *is defined in* (5.2). *Assume that the data for the modified system verify the hypotheses of Theorem 5.2, and let* $((T_-,T_+)\times\mathbb{T}^3, g_{\mu\nu}, \partial_\mu\Phi)$ *be the corresponding solution to the modified equations provided by the theorem. Then* $Q_\mu = 0$ *in* $(T_-,T_+)\times\mathbb{T}^3$.

Proof. First, using definition (5.3), we compute that for a solution of the modified equation (5.5a), the following identity holds:

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - T^{(scalar)}_{\mu\nu} = -\frac{1}{2}\big(D_\mu Q_\nu + D_\nu Q_\mu\big) + \frac{1}{2}(D^\alpha Q_\alpha)g_{\mu\nu} - I_{\mu\nu} + \frac{1}{2}g^{\alpha\beta}I_{\alpha\beta}g_{\mu\nu}. \tag{5.22}$$

Note that the left-hand side of (5.22) is the difference of the left and right sides of the unmodified Einstein equations (3.31a). We then apply D^ν to each side of equation (5.22) and use the Bianchi identity $D^\nu(\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0$, equation (3.25), and the curvature relation $D_\mu D^\alpha Q_\alpha = D^\alpha D_\mu Q_\alpha - \mathrm{Ric}_\mu^{\ \alpha}Q_\alpha$ to deduce that Q_μ verifies the following *hyperbolic* system:

$$g^{\alpha\beta}D_\alpha D_\beta Q_\mu + \mathrm{Ric}_\mu^{\ \alpha}Q_\alpha + 2g^{\alpha\beta}D_\alpha I_{\mu\beta} - g^{\alpha\beta}D_\mu I_{\alpha\beta} = -4I_{\partial\Phi}D_\mu\Phi, \tag{5.23}$$

where $I_{\mu\nu}$ and $I_{\partial\Phi}$, which depend linearly on the Q_μ, are defined in (5.6a)-(5.6d).

Since (5.23) is a system of wave equations and is of hyperbolic character, the fact that $Q_\mu = 0$ in $(T_-,T_+)\times\mathbb{T}^3$ would follow from a standard uniqueness theorem for such systems (see e.g. [Hör97, Ch. VI], [Tay97, Ch.16]), together with the knowledge that *both* $Q_\mu|_{\mathring{\Sigma}} = 0$ *and* $\partial_t Q_\mu|_{\mathring{\Sigma}} = 0$ hold. However, in constructing the data for the modified equations, we have already exhausted our gauge freedom. Although the construction of Section 5.3 has led to the condition $Q_\mu|_{\mathring{\Sigma}} = 0$, it seems that we have no way to enforce the condition $\partial_t Q_\mu|_{\mathring{\Sigma}} = 0$. The remarkable fact, first exploited by Choquet-Bruhat in [CB52], is that the construction of the modified data carried out in Section 5.3 *necessarily implies* that $\partial_t Q_\mu|_{\mathring{\Sigma}} = 0$. The remainder of the proof is dedicated to proving this fact.

Since the left-hand side of (5.22) is the difference of the left and right sides of the Einstein equations (3.31a), and since the initial data $(\mathring{\Sigma}, \underline{\mathring{g}}, \underline{\mathring{K}}, \mathring{\Psi}, \underline{\mathring{\beta}})$ for the Einstein equations are assumed to satisfy the constraints (3.32a)-(3.32b), it follows that the left-hand side of (5.22) is equal to 0 at $t = 0$ after contracting against $\hat{N}^\mu \hat{N}^\nu$ or $\hat{N}^\mu X^\nu$. Here, $\hat{N}^\mu$ is the future-directed unit normal to $\mathring{\Sigma}$ and X^μ is any vector tangent to $\mathring{\Sigma}$ (in fact, one derives the constraint equations by assuming that these contractions are 0 at $t = 0$). Furthermore, since $Q_\mu|_{t=0} = 0$, it follows from definitions (5.6a)-(5.6d) that $I_{\mu\nu}|_{t=0} = 0$, $(\mu, \nu = 0, 1, 2, 3)$, and $I_{\partial\Phi}|_{t=0} = 0$. Using these facts and (5.22), we conclude that the following equations hold:

$$\Big\{ -\frac{1}{2}\big(D_\mu Q_\nu + D_\nu Q_\mu\big) + \frac{1}{2}(D^\alpha Q_\alpha) g_{\mu\nu} \Big\}\big|_{t=0} \hat{N}^\mu \hat{N}^\nu = 0, \tag{5.24a}$$

$$\Big\{ -\frac{1}{2}\big(D_\mu Q_\nu + D_\nu Q_\mu\big) + \frac{1}{2}(D^\alpha Q_\alpha) g_{\mu\nu} \Big\}\big|_{t=0} \hat{N}^\mu X^\nu = 0. \tag{5.24b}$$

Recalling that $\hat{N}^\mu = \delta_0^\mu$ is the future-directed unit normal to $\mathring{\Sigma}$, setting $X_{(j)}^\nu = \delta_j^\nu$, and using the facts that $Q_\mu|_{t=0} = 0$, $(X_{(j)}^\nu \partial_\nu Q_\mu)|_{t=0} = 0$, and $g_{0j}|_{t=0} = 0$, we deduce from (5.24b) that

$$0 = -\frac{1}{2}\hat{N}^\mu X_{(j)}^\nu (\partial_\mu Q_\nu + \partial_\nu Q_\mu)|_{t=0} = -\frac{1}{2}\partial_t Q_j|_{t=0}, \qquad (j = 1, 2, 3). \tag{5.25}$$

Similarly, we use (5.25), the fact that $Q_\mu|_{t=0} = 0$, the fact that $g_{00}|_{t=0} = -1$, the fact that $g^{0\mu}|_{t=0} = -\delta_0^\mu$, and (5.24a) to conclude that

$$\partial_t Q_0|_{t=0} = 0. \tag{5.26}$$

From (5.25) and (5.26), we conclude that the data for the system (5.23) are trivial.

□

6 Norms and Energies

In this section, we define the Sobolev norms and energies that will play a central role in our global existence theorem of Section 11. Let us make a few comments on them. First, we remark that in Section 10, we will show that if the norms are sufficiently small, then they are equivalent to the energies; i.e., the energies can be used to control Sobolev norms of solutions. The reason that we introduce the energies is that their time derivatives can be estimated with the help of integration by parts. Next, we recall that the background solution fluid one-form $\partial\widetilde{\Phi}$ satisfies $\partial_t\widetilde{\Phi} = \bar{\Psi}e^{-\varkappa\Omega}$, $\underline{\partial}\widetilde{\Phi} = 0$, where $\bar{\Psi} > 0$ is the constant defined in (4.14). The quantity $S_{\partial\Phi;N}$, which is introduced below in (6.2e), measures the difference between the perturbed variable $(\partial_t\Phi, \underline{\partial}\Phi)$ and the background $(\partial_t\widetilde{\Phi}, 0)$. We also follow Ringström [Rin08] by introducing scalings by $e^{\alpha\Omega}$, where α is a number, in the definitions of the norms and energies. The effect of these scalings is that in our proof of global existence, a convenient and viable bootstrap assumption to make for these quantities is that they are $\leq \epsilon$, where ϵ is sufficiently small. Finally, we remark that the small positive constant q that appears in this section and throughout this article is defined in (8.5) below, and we remind the reader that $h_{jk} \stackrel{\text{def}}{=} e^{-2\Omega} g_{jk}$, $(j, k = 1, 2, 3)$.

6.1 Norms for g and $\partial\Phi$

In this section, we introduce the weighted Sobolev norms that will be used in Section 9 to estimate the terms appearing in the modified equations. The weights are designed in order to make the bootstrap argument of Section 11 easy to close.

Definition 6.1. Let $N \geq 1$ be an integer. We define the norms $\underline{S}_{g_{00}+1;N}(t)$, $\underline{S}_{g_{0*};N}(t)$, $\underline{S}_{h_{**};N}(t)$, $\underline{S}_{g;N}(t)$, $\underline{S}_{\partial\Phi;N}(t)$, $S_{g_{00}+1;N}(t)$, $S_{g_{0*};N}(t)$, $S_{h_{**};N}(t)$, $S_{g;N}(t)$, $S_{\partial\Phi;N}(t)$, and $\mathbf{S}_N(t)$ as follows:

$$\underline{S}_{g_{00}+1;N} \stackrel{\text{def}}{=} e^{q\Omega}\|\partial_t g_{00}\|_{H^N} + e^{q\Omega}\|g_{00}+1\|_{H^N} + \sum_{i=1}^{3} e^{(q-1)\Omega}\|\partial_i g_{00}\|_{H^N}, \tag{6.1a}$$

$$\underline{S}_{g_{0*};N} \stackrel{\text{def}}{=} \sum_{j=1}^{3}\Big(e^{(q-1)\Omega}\|\partial_t g_{0j}\|_{H^N} + e^{(q-1)\Omega}\|g_{0j}\|_{H^N}\Big) + \sum_{i,j=1}^{3} e^{(q-2)\Omega}\|\partial_i g_{0j}\|_{H^N}, \tag{6.1b}$$

$$\underline{S}_{h_{**};N} \stackrel{\text{def}}{=} \sum_{j,k=1}^{3} e^{q\Omega}\|\partial_t h_{jk}\|_{H^N} + \sum_{i,j,k=1}^{3}\Big(\|\partial_i h_{jk}\|_{H^{N-1}} + e^{(q-1)\Omega}\|\partial_i h_{jk}\|_{H^N}\Big), \tag{6.1c}$$

$$\underline{S}_{\partial\Phi;N} \stackrel{\text{def}}{=} \|e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi}\|_{H^N} + e^{(\varkappa-1)\Omega}\sum_{i=1}^{3}\|\partial_i\Phi\|_{H^N}, \tag{6.1d}$$

$$S_{g_{00}+1;N}(t) \stackrel{\text{def}}{=} \sup_{0\leq\tau\leq t} \underline{S}_{g_{00}+1;N}(\tau), \tag{6.2a}$$

$$S_{g_{0*};N}(t) \stackrel{\text{def}}{=} \sup_{0\leq\tau\leq t} \underline{S}_{g_{0*};N}(\tau), \tag{6.2b}$$

$$S_{h_{**};N}(t) \stackrel{\text{def}}{=} \sup_{0\leq\tau\leq t} \underline{S}_{h_{**};N}(\tau), \tag{6.2c}$$

$$S_{g;N} \stackrel{\text{def}}{=} S_{g_{00}+1;N} + S_{g_{0*};N} + S_{h_{**};N}, \tag{6.2d}$$

$$S_{\partial\Phi;N}(t) \stackrel{\text{def}}{=} \sup_{0\leq\tau\leq t} \underline{S}_{\partial\Phi;N}(\tau), \tag{6.2e}$$

$$\mathbf{S}_N \stackrel{\text{def}}{=} S_{g;N} + S_{\partial\Phi;N}. \tag{6.2f}$$

6.2 Energies for the metric g

6.2.1 The building block energy for g

The energies for the metric components will be built from the quantities defined in the following lemma. They are designed with equations (5.15a)-(5.15c) in mind.

Lemma 6.2 (**Properties of the building blocks of energies for the metric**)**.** *[Rin08, Lemma 15] Let v be a solution to the scalar equation*

$$\hat{\square}_g v = \alpha H \partial_t v + \beta H^2 v + F, \tag{6.3}$$

where $\hat{\square}_g = g^{\lambda\kappa}\partial_\lambda\partial_\kappa$, $\alpha > 0$ and $\beta \geq 0$ are constants, and define $\mathcal{E}_{(\gamma,\delta)}[v, \partial v] \geq 0$ by

$$\mathcal{E}^2_{(\gamma,\delta)}[v, \partial v] \overset{def}{=} \frac{1}{2}\int_{\mathbb{T}^3} \Big\{ - g^{00}(\partial_t v)^2 + g^{ab}(\partial_a v)(\partial_b v) - 2\gamma H g^{00} v \partial_t v + \delta H^2 v^2 \Big\}\, d^3x. \tag{6.4}$$

Then there exist constants $\eta > 0$, $C > 0$, $C_{(\beta)} \geq 0$, $\gamma \geq 0$, and $\delta \geq 0$ such that

$$|g^{00} + 1| \leq \eta \tag{6.5}$$

implies that

$$\mathcal{E}^2_{(\gamma,\delta)}[v, \partial v] \geq C \int_{\mathbb{T}^3} \Big\{ (\partial_t v)^2 + g^{ab}(\partial_a v)(\partial_b v) + C_{(\beta)} v^2 \Big\}\, d^3x. \tag{6.6}$$

The constants γ and δ depend on α and β, while η, C, and $C_{(\beta)}$ depend on α, β, γ and δ. Furthermore, $C_{(\beta)} = 0$ if $\beta = 0$ and $C_{(\beta)} = 1$ if $\beta > 0$. In addition, if $\beta = 0$, then $\gamma = \delta = 0$, while if $\beta > 0$, then we can arrange for $\gamma > 0$ and $\delta > 0$. Finally, we have that

$$\frac{d}{dt}(\mathcal{E}^2_{(\gamma,\delta)}[v, \partial v]) \leq -\eta H \mathcal{E}^2_{(\gamma,\delta)}[v, \partial v] + \int_{\mathbb{T}^3} \Big\{ - (\partial_t v + \gamma H v)F + \triangle_{\mathcal{E};(\gamma,\delta)}[v, \partial v] \Big\}\, d^3x, \tag{6.7}$$

where

$$\begin{aligned} \triangle_{\mathcal{E};(\gamma,\delta)}[v, \partial v] = & -\gamma H (\partial_a g^{ab}) v \partial_b v - 2\gamma H(\partial_a g^{0a}) v \partial_t v - 2\gamma H g^{0a}(\partial_a v)(\partial_t v) \\ & - (\partial_a g^{0a})(\partial_t v)^2 - (\partial_a g^{ab})(\partial_b v)(\partial_t v) - \frac{1}{2}(\partial_t g^{00})(\partial_t v)^2 \\ & + \left(\frac{1}{2}\partial_t g^{ab} + H g^{ab}\right)(\partial_a v)(\partial_b v) - \gamma H (\partial_t g^{00}) v \partial_t v \\ & - \gamma H (g^{00} + 1)(\partial_t v)^2. \end{aligned} \tag{6.8}$$

Proof. The proof is a standard integration by parts argument that begins with the multiplication of both sides of equation (6.3) by $-(\partial_t v + \gamma H v)$; see Lemma 15 of [Rin08] for the details. For later use, we quote the following identity from Ringström's proof:

$$\begin{aligned}
&\frac{d}{dt}(\mathcal{E}^2_{(\gamma,\delta)}[v,\partial v]) \qquad (6.9)\\
&\quad = \int_{\mathbb{T}^3}\Big\{-(\alpha-\gamma)H(\partial_t v)^2 + (\delta-\beta-\gamma\alpha)H^2 v\partial_t v - \beta\gamma H^3 v^2\\
&\qquad - (1+\gamma)Hg^{ab}(\partial_a v)(\partial_b v) - (\partial_t v + \gamma H v)F + \triangle_{\mathcal{E};(\gamma,\delta)}[v,\partial v]\Big\}\, d^3x.
\end{aligned}$$

□

6.2.2 Energies for the components of g

In this section, we will use rescaled versions of energies of the form (6.4) to construct energies for the components of g.

Definition 6.3. We define the *non-negative* energies $\underline{E}_{g_{00}+1;N}(t)$, $\underline{E}_{g_{0*};N}(t)$, $\underline{E}_{h_{**};N}(t)$, $\underline{E}_{g;N}(t)$, $E_{g_{00}+1;N}(t)$, $E_{g_{0*};N}(t)$, $E_{h_{**};N}(t)$, and $E_{g;N}(t)$ as follows:

$$\underline{E}^2_{g_{00}+1;N} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N} e^{2q\Omega}\mathcal{E}^2_{(\gamma_{00},\delta_{00})}[\partial_{\vec{\alpha}}(g_{00}+1), \partial(\partial_{\vec{\alpha}}g_{00})], \qquad (6.10a)$$

$$\underline{E}^2_{g_{0*};N} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N}\sum_{j=1}^{3} e^{2(q-1)\Omega}\mathcal{E}^2_{(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}g_{0j}, \partial(\partial_{\vec{\alpha}}g_{0j})], \qquad (6.10b)$$

$$\underline{E}^2_{h_{**};N} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N}\Big\{\sum_{j,k=1}^{3} e^{2q\Omega}\mathcal{E}^2_{(0,0)}[0, \partial(\partial_{\vec{\alpha}}h_{jk})] + \frac{1}{2}\int_{\mathbb{T}^3} c_{\vec{\alpha}}H^2\big(\partial_{\vec{\alpha}}h_{jk}\big)^2\, d^3x\Big\}, \qquad (6.10c)$$

$$\underline{E}_{g;N} \overset{\text{def}}{=} \underline{E}_{g_{00}+1;N} + \underline{E}_{g_{0*};N} + \underline{E}_{h_{**};N}, \qquad (6.10d)$$

$$E_{g_{00}+1;N}(t) \overset{\text{def}}{=} \sup_{0\leq\tau\leq t}\underline{E}_{g_{00}+1;N}(\tau), \qquad (6.11a)$$

$$E_{g_{0*};N}(t) \overset{\text{def}}{=} \sup_{0\leq\tau\leq t}\underline{E}_{g_{0*};N}(\tau), \qquad (6.11b)$$

$$E_{h_{**};N}(t) \overset{\text{def}}{=} \sup_{0\leq\tau\leq t}\underline{E}_{h_{**};N}(\tau), \qquad (6.11c)$$

$$E_{g;N} \overset{\text{def}}{=} E_{g_{00}+1;N} + E_{g_{0*};N} + E_{h_{**};N}, \qquad (6.11d)$$

where

$$h_{jk} \overset{\text{def}}{=} e^{-2\Omega}g_{jk}, \qquad (j,k=1,2,3), \qquad (6.12a)$$

$$c_{\vec{\alpha}} \overset{\text{def}}{=} 0, \qquad \text{if } |\vec{\alpha}|=0, \qquad (6.12b)$$

$$c_{\vec{\alpha}} \overset{\text{def}}{=} 1, \qquad \text{if } |\vec{\alpha}|>0, \qquad (6.12c)$$

and $(\gamma_{00}, \delta_{00}), (\gamma_{0*}, \delta_{0*})$, and $(\gamma_{**}, \delta_{**})$ are the constants generated by applying Lemma 6.2 to equations (5.15a)-(5.15c) respectively [note that $(\gamma_{**}, \delta_{**}) = (0, 0)$ in definition (6.10c)].

In the next lemma, we provide a preliminary estimate of the time derivative of these energies.

Lemma 6.4 (A first differential inequality for the metric energies). *Assume that* $(g_{\mu\nu}, \partial_\mu \Phi)$, $(\mu, \nu = 0, 1, 2, 3)$, *is a solution to the modified equations* (5.15a)-(5.15c)*, and let* $\underline{E}_{g_{00}+1;N}$, $\underline{E}_{g_{0*};N}$, *and* $\underline{E}_{h_{**};N}$ *be as in Definition 6.3. Let* $[\hat{\square}_g, \partial_{\vec{\alpha}}]$ *denote the commutator of the operators* $\hat{\square}_g$ *and* $\partial_{\vec{\alpha}}$*. Then under the assumptions of Lemma 6.2, the following differential inequalities are satisfied, where* $\triangle_{\mathcal{E};(\gamma,\delta)}[\cdot, \partial(\cdot)]$ *is defined in* (6.8)*, the constants* $(\gamma_{00}, \delta_{00}), (\gamma_{0*}, \delta_{0*})$, *and* $(\gamma_{**}, \delta_{**})$ *[note that* $(\gamma_{**}, \delta_{**}) = (0, 0)$*] are the constants from Definition 6.3, and* $\eta_{00}, \eta_{0*}, \eta_{**}$ *are the positive constants "*η*" produced by applying Lemma 6.2 to each of* *the equations* (5.15a)-(5.15c) *respectively:*

$$\begin{aligned}
\frac{d}{dt}(\underline{E}^2_{g_{00}+1;N}) \leq & (2q - \eta_{00})H\underline{E}^2_{g_{00}+1;N} + 2q(\omega - H)\underline{E}^2_{g_{00}+1;N} \\
& - \sum_{|\vec{\alpha}|\leq N} \int_{\mathbb{T}^3} e^{2q\Omega}\big\{\partial_t\partial_{\vec{\alpha}}(g_{00}+1) + \gamma_{00}H\partial_{\vec{\alpha}}(g_{00}+1)\big\} \\
& \qquad \times \big\{\partial_{\vec{\alpha}}\triangle_{00} + [\hat{\Box}_g, \partial_{\vec{\alpha}}](g_{00}+1)\big\}\, d^3x \\
& + \sum_{|\vec{\alpha}|\leq N} \int_{\mathbb{T}^3} e^{2q\Omega}\triangle_{\mathcal{E};(\gamma_{00},\delta_{00})}[\partial_{\vec{\alpha}}(g_{00}+1), \partial(\partial_{\vec{\alpha}}g_{00})]\, d^3x,
\end{aligned} \tag{6.13a}$$

$$\begin{aligned}
\frac{d}{dt}(\underline{E}^2_{g_{0*};N}) \leq & [2(q-1) - \eta_{0*}]H\underline{E}^2_{g_{0*};N} + 2(q-1)(\omega - H)\underline{E}^2_{g_{0*};N} \\
& - \sum_{|\vec{\alpha}|\leq N}\sum_{j=1}^{3} \int_{\mathbb{T}^3} e^{2(q-1)\Omega}\big\{\partial_t\partial_{\vec{\alpha}}g_{0j} + \gamma_{0*}H\partial_{\vec{\alpha}}g_{0j}\big\} \\
& \qquad \times \big\{ -2H\partial_{\vec{\alpha}}(g^{ab}\Gamma_{ajb}) + \partial_{\vec{\alpha}}\triangle_{0j} + [\hat{\Box}_g, \partial_{\vec{\alpha}}]g_{0j}\big\}\, d^3x \\
& + \sum_{|\vec{\alpha}|\leq N}\sum_{j=1}^{3} \int_{\mathbb{T}^3} e^{2(q-1)\Omega}\triangle_{\mathcal{E};(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}g_{0j}, \partial(\partial_{\vec{\alpha}}g_{0j})]\, d^3x,
\end{aligned} \tag{6.13b}$$

$$\begin{aligned}
\frac{d}{dt}(\underline{E}^2_{h_{**};N}) \leq & (2q - \eta_{**})H \sum_{|\vec{\alpha}|\leq N}\sum_{j,k=1}^{3} e^{2q\Omega}\mathcal{E}^2_{(0,0)}[0, \partial(\partial_{\vec{\alpha}}h_{jk})] \\
& + 2q(\omega - H)\sum_{|\vec{\alpha}|\leq N}\sum_{j,k=1}^{3} e^{2q\Omega}\mathcal{E}^2_{(0,0)}[0, \partial(\partial_{\vec{\alpha}}h_{jk})] \\
& - \sum_{|\vec{\alpha}|\leq N}\sum_{j,k=1}^{3} \int_{\mathbb{T}^3} e^{2q\Omega}(\partial_t\partial_{\vec{\alpha}}h_{jk})\big\{\partial_{\vec{\alpha}}\triangle_{jk} + [\hat{\Box}_g, \partial_{\vec{\alpha}}]h_{jk}\big\}\, d^3x \\
& + \sum_{|\vec{\alpha}|\leq N}\sum_{j,k=1}^{3} \int_{\mathbb{T}^3} e^{2q\Omega}\triangle_{\mathcal{E};(0,0)}[0, \partial(\partial_{\vec{\alpha}}h_{jk})]\, d^3x \\
& + \sum_{1\leq|\vec{\alpha}|\leq N}\sum_{j,k=1}^{3} \int_{\mathbb{T}^3} H^2(\partial_{\vec{\alpha}}\partial_t h_{jk})(\partial_{\vec{\alpha}}h_{jk})\, d^3x.
\end{aligned} \tag{6.13c}$$

Proof. Lemma 6.4 follows easily from definitions (6.10a)-(6.11d), and from (6.7). □

The following corollary follows easily from Lemma 6.4, definitions (6.10a)-(6.10c), and the Cauchy-Schwarz inequality for integrals.

Corollary 6.5. *There exists a constant $C > 0$ such that under the assumptions of Lemma*

6.4, we have that

$$\begin{aligned}
\frac{d}{dt}(\underline{E}^2_{g_{00}+1;N}) \leq{}& (2q - \eta_{00})H\underline{E}^2_{g_{00}+1;N} + 2q(\omega - H)\underline{E}^2_{g_{00}+1;N} && (6.14a)\\
&+ C\underline{S}_{g_{00}+1;N}e^{q\Omega}\|\triangle_{00}\|_{H^N}\\
&+ C\underline{S}_{g_{00}+1;N}\sum_{|\vec{\alpha}|\leq N} e^{q\Omega}\|[\hat{\square}_g, \partial_{\vec{\alpha}}](g_{00}+1)\|_{L^2}\\
&+ \sum_{|\vec{\alpha}|\leq N} e^{2q\Omega}\|\triangle_{\mathcal{E};(\gamma_{00},\delta_{00})}[\partial_{\vec{\alpha}}(g_{00}+1), \partial(\partial_{\vec{\alpha}}g_{00})]\|_{L^1},\\
\frac{d}{dt}(\underline{E}^2_{g_{0*};N}) \leq{}& [2(q-1) - \eta_{0*}]H\underline{E}^2_{g_{0*};N} + 2(q-1)(\omega - H)\underline{E}^2_{g_{0*};N} && (6.14b)\\
&+ C\underline{S}_{g_{0*};N}\sum_{j=1}^{3} e^{(q-1)\Omega}\|g^{ab}\Gamma_{ajb}\|_{H^N}\\
&+ C\underline{S}_{g_{0*};N}\sum_{j=1}^{3} e^{(q-1)\Omega}\|\triangle_{0j}\|_{H^N}\\
&+ C\underline{S}_{g_{0*};N}\sum_{|\vec{\alpha}|\leq N}\sum_{j=1}^{3} e^{(q-1)\Omega}\|[\hat{\square}_g, \partial_{\vec{\alpha}}]g_{0j}\|_{L^2}\\
&+ \sum_{|\vec{\alpha}|\leq N}\sum_{j=1}^{3} e^{2(q-1)\Omega}\|\triangle_{\mathcal{E};(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}g_{0j}, \partial(\partial_{\vec{\alpha}}g_{0j})]\|_{L^1},\\
\frac{d}{dt}(\underline{E}^2_{h_{**};N}) \leq{}& (2q - \eta_{**})H\sum_{|\vec{\alpha}|\leq N}\sum_{j,k=1}^{3} e^{2q\Omega}\mathcal{E}^2_{(0,0)}[0, \partial(\partial_{\vec{\alpha}}h_{jk})] && (6.14c)\\
&+ 2q(\omega - H)\underline{E}^2_{h_{**};N}\\
&+ C\underline{S}_{h_{**};N}\sum_{j,k=1}^{3} e^{q\Omega}\|\triangle_{jk}\|_{H^N}\\
&+ C\underline{S}_{h_{**};N}\sum_{|\vec{\alpha}|\leq N}\sum_{j,k=1}^{3} e^{q\Omega}\|[\hat{\square}_g, \partial_{\vec{\alpha}}]h_{jk}\|_{L^2}\\
&+ \sum_{|\vec{\alpha}|\leq N}\sum_{j,k=1}^{3} e^{2q\Omega}\|\triangle_{\mathcal{E};(0,0)}[0, \partial(\partial_{\vec{\alpha}}h_{jk})\|_{L^1} + Ce^{-q\Omega}\underline{S}^2_{h_{**};N},
\end{aligned}$$

where the norms $\underline{S}_{g_{00}+1;N}, \underline{S}_{g_{0*};N}, \underline{S}_{h_{**};N}$ *are defined in Definition 6.1.*

6.3 Energies for the fluid one-form $\partial\Phi$

In this section, we define the energies that we will use to analyze solutions to the irrotational fluid equation (5.15d). We begin by stating their definitions.

Definition 6.6. Let $\bar{\Psi}$ be the positive constant defined in (4.14). We define the *non-*

negative energies $\underline{E}_{\partial\Phi;N}(t), E_{\partial\Phi;N}(t)$ for $\partial\Phi$ as follows:

$$\underline{E}^2_{\partial\Phi;N} \overset{\text{def}}{=} \frac{1}{2}\int_{\mathbb{T}^3}\Big\{(e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi})^2 + e^{2\varkappa\Omega}(m^{-1})^{ab}(\partial_a\Phi)(\partial_b\Phi)\Big\}\,d^3x \tag{6.15a}$$
$$+ \frac{1}{2}\sum_{1\leq|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3}\Big\{e^{2\varkappa\Omega}(\partial_t\partial_{\vec{\alpha}}\Phi)^2 + e^{2\varkappa\Omega}(m^{-1})^{ab}(\partial_a\partial_{\vec{\alpha}}\Phi)(\partial_b\partial_{\vec{\alpha}}\Phi)\Big\}\,d^3x,$$
$$E_{\partial\Phi;N}(t) \overset{\text{def}}{=} \sup_{0\leq\tau\leq t}\underline{E}_{\partial\Phi;N}(\tau). \tag{6.15b}$$

In the next lemma, we provide a preliminary estimate of $\frac{d}{dt}\left(\underline{E}^2_{\partial\Phi;N}(t)\right)$.

Lemma 6.7 (A first differential inequality for the fluid energy). *Assume that $\partial\Phi$ verifies*

$$\hat{\square}_m\Phi = \varkappa\omega(t)\partial_t\Phi + \triangle_{\partial\Phi}, \tag{6.16}$$

where $\hat{\square}_m \overset{def}{=} -\partial_t^2 + 2(m^{-1})^{0a}\partial_t\partial_a + (m^{-1})^{ab}\partial_a\partial_b$ is the reduced wave operator corresponding to the reciprocal acoustical metric $(m^{-1})^{\mu\nu}$ [see (5.17a)-(5.17c)*], and $\varkappa = \frac{3}{1+2s} = 3c_s^2$. Let $[\hat{\square}_m, \partial_{\vec{\alpha}}]$ denote the commutator of the operators $\hat{\square}_m$ and $\partial_{\vec{\alpha}}$. Then*

$$\begin{aligned}\frac{d}{dt}(\underline{E}^2_{\partial\Phi;N}) = &-\int_{\mathbb{T}^3}\big(\partial_a(m^{-1})^{0a}\big)(e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi})^2\,d^3x \\ &- \sum_{1\leq|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}\big(\partial_a(m^{-1})^{0a}\big)(\partial_t\partial_{\vec{\alpha}}\Phi)^2\,d^3x \\ &- \int_{\mathbb{T}^3} e^{\varkappa\Omega}\big(\partial_a(m^{-1})^{ab}\big)(e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi})(\partial_b\Phi)\,d^3x \\ &- \sum_{1\leq|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}\big(\partial_b(m^{-1})^{ab}\big)(\partial_t\partial_{\vec{\alpha}}\Phi)(\partial_b\partial_{\vec{\alpha}}\Phi)\,d^3x \\ &- \int_{\mathbb{T}^3} e^{\varkappa\Omega}(\triangle_{\partial\Phi})(e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi})\,d^3x \\ &- \sum_{1\leq|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}(\partial_{\vec{\alpha}}\triangle_{\partial\Phi})\partial_t\partial_{\vec{\alpha}}\Phi\,d^3x \\ &- \sum_{1\leq|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}([\hat{\square}_m, \partial_{\vec{\alpha}}]\Phi)\partial_t\partial_{\vec{\alpha}}\Phi\,d^3x \\ &+ \frac{1}{2}\sum_{|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}\big\{\partial_t(m^{-1})^{ab} + 2\varkappa\omega(m^{-1})^{ab}\big\}(\partial_a\partial_{\vec{\alpha}}\Phi)(\partial_b\partial_{\vec{\alpha}}\Phi)\,d^3x.\end{aligned} \tag{6.17}$$

Proof. This is a standard integration by parts lemma that can be proved using the ideas of Lemma 6.2. We provide a sketch of the proof. We begin by differentiating under the integral in the definition of $\underline{E}^2_{\partial\Phi;N}$ to conclude that

$$\begin{aligned}
\frac{d}{dt}(\underline{E}^2_{\partial\Phi;N}) &= \sum_{|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3}\big[\partial_{\vec{\alpha}}(e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi})\big]\partial_t\partial_{\vec{\alpha}}(e^{\varkappa\Omega}\partial_t\Phi)\,d^3x \qquad (6.18)\\
&+ \sum_{|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}(m^{-1})^{ab}(\partial_t\partial_a\partial_{\vec{\alpha}}\Phi)(\partial_b\partial_{\vec{\alpha}}\Phi)\,d^3x\\
&+ \frac{1}{2}\sum_{|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}\left\{\partial_t(m^{-1})^{ab} + 2\varkappa\omega(m^{-1})^{ab}\right\}(\partial_a\partial_{\vec{\alpha}}\Phi)(\partial_b\partial_{\vec{\alpha}}\Phi)\,d^3x.
\end{aligned}$$

For each fixed $\vec{\alpha}$, we will now eliminate the highest derivatives of Φ in (6.18) (i.e., the derivatives of order $|\vec{\alpha}| + 2$). To this end, we first differentiate equation (6.16) with $\partial_{\vec{\alpha}}$ and multiply both sides of the equation by $e^{\varkappa\Omega}$, which allows us to express the resulting equality as

$$\begin{aligned}
-\partial_t\partial_{\vec{\alpha}}(e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi}) + e^{\varkappa\Omega}(m^{-1})^{ab}\partial_a\partial_b\partial_{\vec{\alpha}}\Phi &= e^{\varkappa\Omega}\partial_{\vec{\alpha}}\triangle_{\partial\Phi} + e^{\varkappa\Omega}[\hat{\Box}_m, \partial_{\vec{\alpha}}]\Phi \qquad (6.19)\\
&\quad - 2e^{\varkappa\Omega}(m^{-1})^{0a}\partial_t\partial_a\partial_{\vec{\alpha}}\Phi.
\end{aligned}$$

We then multiply both sides of (6.19) by $-\partial_{\vec{\alpha}}(e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi})$, integrate over $\mathbb{T}^3$, and integrate by parts. Inserting the resulting identity into (6.18), we arrive at (6.17). $\square$

We now state the following corollary, which follows easily from definition (6.1d), Lemma 6.7, and the Cauchy-Schwarz inequality for integrals.

Corollary 6.8. *Under the hypotheses of Lemma 6.7, we have that*

$$\begin{aligned}
\frac{d}{dt}(\underline{E}^2_{\partial\Phi;N}) &\leq \underline{S}^2_{\partial\Phi;N}\|\partial_a(m^{-1})^{0a}\|_{L^\infty} + \underline{S}^2_{\partial\Phi;N}\sum_{b=1}^{3} e^{\Omega}\|\partial_a(m^{-1})^{ab}\|_{L^\infty} \qquad (6.20)\\
&+ \underline{S}_{\partial\Phi;N}e^{\varkappa\Omega}\|\triangle_{\partial\Phi}\|_{H^N} + \underline{S}_{\partial\Phi;N}\sum_{1\leq|\vec{\alpha}|\leq N} e^{\varkappa\Omega}\|[\hat{\Box}_m, \partial_{\vec{\alpha}}]\Phi\|_{L^2}\\
&+ \frac{1}{2}\underline{S}^2_{\partial\Phi;N}\sum_{a,b=1}^{3} e^{2\varkappa\Omega}\|\partial_t(m^{-1})^{ab} + 2\omega(m^{-1})^{ab}\|_{L^\infty}\\
&+ (\varkappa - 1)\omega\sum_{|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}(m^{-1})^{ab}(\partial_a\partial_{\vec{\alpha}}\Phi)(\partial_b\partial_{\vec{\alpha}}\Phi)\,d^3x,
\end{aligned}$$

where $\underline{S}_{\partial\Phi;N}$ *is defined in* (6.1d).

6.4 The total energy $\mathbf{E}_N$

Definition 6.9. Let $E_{g;N}$ and $E_{\partial\Phi;N}$ be the metric and fluid energies defined in (6.11d) and (6.15b) respectively. We define $\mathbf{E}_N$, the total energy associated to $(g_{\mu\nu}, \partial_\mu\Phi)$, $(\mu, \nu = 0, 1, 2, 3)$, as follows:

$$\mathbf{E}_N \overset{\text{def}}{=} E_{g;N} + E_{\partial\Phi;N}. \qquad (6.21)$$

7 Linear-Algebraic Estimates of $g_{\mu\nu}$, $g^{\mu\nu}$, and $(m^{-1})^{\mu\nu}$

In this section, we provide some linear-algebraic estimates of $g_{\mu\nu}$, $g^{\mu\nu}$, and $(m^{-1})^{\mu\nu}$ In addition to providing some rough L^{∞} estimates that we will use in Sections 9 and 10, the lemmas will guarantee that $g_{\mu\nu}$ is a Lorentzian metric and that $(m^{-1})^{\mu\nu}$ is an inverse Lorentzian metric. We remark that we already made use of these facts in our statement of the conclusions of Theorem 5.2.

Lemma 7.1 (The Lorentzian nature of $g^{\mu\nu}$). *[Rin08, Lemmas 1 and 2] Let $g_{\mu\nu}$ be a symmetric 4×4 matrix of real numbers. Let $(g_\flat)_{jk}$ be the 3×3 matrix defined by $(g_\flat)_{jk} = g_{jk}$, and let $(g_\flat^{-1})^{jk}$ be the 3×3 inverse of $(g_\flat)_{jk}$. Assume that $g_{00} < 0$ and that $(g_\flat)_{jk}$ is positive definite. Then $g_{\mu\nu}$ is a Lorentzian metric with inverse $g^{\mu\nu}$, $g^{00} < 0$, and the 3×3 matrix $(g^{\#})^{jk}$ defined by $(g^{\#})^{jk} \overset{def}{=} g^{jk}$ is positive definite. Furthermore, the following relations hold:*

$$g^{00} = \frac{1}{g_{00} - d^2}, \tag{7.1a}$$

$$\frac{g_{00}}{g_{00} - d^2}(g_\flat^{-1})^{ab}X_aX_b \leq (g^{\#})^{ab}X_aX_b \leq (g_\flat^{-1})^{ab}X_aX_b, \quad \forall (X_1, X_2, X_3) \in \mathbb{R}^3, \tag{7.1b}$$

$$g^{0j} = \frac{1}{d^2 - g_{00}}(g_\flat^{-1})^{aj}g_{0a}, \qquad (j = 1, 2, 3), \tag{7.1c}$$

where

$$d^2 = (g_\flat^{-1})^{ab}g_{0a}g_{0b}. \tag{7.2}$$

□

The next lemma requires the following rough bootstrap assumptions, which we will improve during our global existence argument.

Rough Bootstrap Assumptions for $g_{\mu\nu}$:

We assume that there are constants $\eta > 0$ and $K_1 \geq 1$ such that

$$|g_{00} + 1| \leq \eta, \tag{7.3a}$$

$$K_1^{-1}\delta_{ab}X^aX^b \leq e^{-2\Omega}g_{ab}X^aX^b \leq K_1\delta_{ab}X^aX^b, \qquad \forall (X^1, X^2, X^3) \in \mathbb{R}^3, \tag{7.3b}$$

$$\sum_{a=1}^{3} |g_{0a}|^2 \leq \eta K_1^{-1}e^{2(1-q)\Omega}. \tag{7.3c}$$

For our global existence argument, we will assume that $\eta = \eta_{min}$, where η_{min} is defined in Section 8.2.

Lemma 7.2 (First estimates of $g^{\mu\nu}$). *[Rin08, Lemma 7] Let $g_{\mu\nu}$ be a symmetric 4×4 matrix of real numbers satisfying* (7.3a) - (7.3c)*, where $\Omega \geq 0$ and $0 \leq q < 1$. Then $g_{\mu\nu}$ is a Lorentzian metric, and there exists a constant $\eta_0 > 0$ such that $0 \leq \eta \leq \eta_0$ implies that the following estimates hold for its inverse $g^{\mu\nu}$:*

$$|g^{00} + 1| \leq 4\eta, \tag{7.4a}$$

$$\sqrt{\sum_{a=1}^{3} |g^{0a}|^2} \leq 2K_1 e^{-2\Omega} \sqrt{\sum_{a=1}^{3} |g_{0a}|^2}, \tag{7.4b}$$

$$|g^{0a} g_{0a}| \leq 2K_1 e^{-2\Omega} \sum_{a=1}^{3} |g_{0a}|^2, \tag{7.4c}$$

$$\frac{2}{3K_1}\delta^{ab} X_a X_b \leq e^{2\Omega} g^{ab} X_a X_b \leq \frac{3K_1}{2}\delta^{ab} X_a X_b, \qquad \forall (X_1, X_2, X_3) \in \mathbb{R}^3. \tag{7.4d}$$

□

The next lemma provides criteria that are sufficient to ensure that the reciprocal acoustical metric $(m^{-1})^{\mu\nu}$ is finite and Lorentzian. It is needed to fully justify the conclusions of the continuation principle (Theorem 5.4).

Lemma 7.3 (The Lorentzian nature of $(m^{-1})^{\mu\nu}$). *Let $g_{\mu\nu}$ be a symmetric 4×4 matrix of real numbers satisfying the assumptions of Lemma 7.1. Assume further that $\sigma > 0$. Let $(m^{-1})^{\mu\nu}$ denote the reciprocal acoustical metric defined in* (5.17a)-(5.17c)*. Then $(m^{-1})^{\mu\nu}$ is an inverse Lorentzian metric of signature $(-,+,+,+)$. Furthermore, if ξ is any timelike covector (i.e., $g^{\alpha\beta}\xi_\alpha\xi_\beta < 0$), then $(m^{-1})^{\alpha\beta}\xi_\alpha\xi_\beta < 0$.*

Proof. It is straightforward to verify using (5.7d) and (5.17a)-(5.17c) that the assumptions of the lemma and the conclusions of Lemma 7.1 imply that

$$(m^{-1})^{\mu\nu} = P(n^{-1})^{\mu\nu}, \tag{7.5}$$

$$(n^{-1})^{\mu\nu} = g^{\mu\nu} - 2s\sigma^{-1}(g^{\mu\alpha}\partial_\alpha\Phi)(g^{\nu\beta}\partial_\beta\Phi), \tag{7.6}$$

where $P > 0$. The fact that $(m^{-1})^{\alpha\beta}\xi_\alpha\xi_\beta < 0$ whenever $g^{\alpha\beta}\xi_\alpha\xi_\beta < 0$ now follows trivially from the expression (7.6). To show that $(m^{-1})^{\mu\nu}$ is Lorentzian, we set $T^\mu = g^{\mu\alpha}\partial_\alpha\Phi$. The assumptions of the lemma guarantee that T^μ is timelike relative to g (i.e., $g_{\alpha\beta}T^\alpha T^\beta = -\sigma < 0$), and Lemma 7.1 ensures that $g_{\mu\nu}$ and $g^{\mu\nu}$ are Lorentzian. Set $\hat{T}^\mu = \sigma^{-1/2}T^\mu$, so that $g_{\alpha\beta}\hat{T}^\alpha\hat{T}^\beta = -1$. It follows that we can choose spacelike (relative to g) vectors $X^\mu_{(1)}$, $X^\mu_{(2)}$, $X^\mu_{(3)}$ such that $\{\hat{T}, X_{(1)}, X_{(2)}, X_{(3)}\}$ is a $g-$orthonormal basis. Let $X^{(j)}_\mu \overset{\text{def}}{=} g_{\mu\alpha}X^\alpha_{(j)}$, $(j = 1, 2, 3)$. It follows from (7.6) that

$$(n^{-1})^{\alpha\beta}\hat{T}_\alpha\hat{T}_\beta = -(1 + 2s), \tag{7.7}$$

$$(n^{-1})^{\alpha\beta}\hat{T}_\alpha X^{(j)}_\beta = 0, \qquad (j = 1, 2, 3), \tag{7.8}$$

$$(n^{-1})^{\alpha\beta}X^{(j)}_\alpha X^{(k)}_\beta = \delta^{jk}, \qquad (j, k = 1, 2, 3), \tag{7.9}$$

where δ^{jk} is the standard Kronecker delta. Thus, $(n^{-1})^{\mu\nu}$ is an inverse Lorentzian metric of signature $(-,+,+,+)$. Since $P > 0$, it follows that $(m^{-1})^{\mu\nu}$ is also an inverse Lorentzian metric of signature $(-,+,+,+)$.

□

8 The Bootstrap Assumption for S_N and the Definition of N, η_{min} and q

In this short section, we define the quantities N, η_{min}, and q. We then introduce some bootstrap assumptions that will be used in our derivation of the estimates of Sections 9 and 10.

8.1 The definition of N and the assumption $\mathrm{S}_N \leq \epsilon$

For the remainder of the article, we will assume that N is an integer subject to one of the following requirements:

$$N \geq 3, \qquad \text{(this is large enough for the validity of all of our results except for some of the conclusions of Theorem 12.1)}, \tag{8.1}$$

$$N \geq 5, \qquad \text{(this is large enough for all of our results to be valid)}. \tag{8.2}$$

We require N to be of this size to ensure that various Sobolev embedding results are valid; see also Remark 12.2.

In our global existence argument, we will make the following bootstrap assumption:

$$\mathrm{S}_N \leq \epsilon, \tag{8.3}$$

where S_N is defined in (6.2f), and ϵ is a sufficiently small positive number. Observe that S_N measures how much (g, p, u) differs from the FLRW solution $(\widetilde{g}, \widetilde{p}, \widetilde{u})$ derived in Section 4. In particular, $\mathrm{S}_N \equiv 0$ for the FLRW solution.

8.2 The definitions of η_{min} and q

Definition 8.1. Let η_{00}, η_{0*}, η_{**} be the positive constants appearing in the conclusions of Lemma 6.4. Furthermore, let η_0 be the constant from Lemma 7.2. We now define the positive quantities (recalling that $0 < \varkappa < 1$ when $0 < c_s < \sqrt{1/3}$) η_{min} and q by

$$\eta_{min} \stackrel{\text{def}}{=} \frac{1}{8}\min\{1, \eta_0, \eta_{00}, \eta_{0*}, \eta_{**}\}, \tag{8.4}$$

$$q \stackrel{\text{def}}{=} \frac{2}{3}\min\{\eta_{min}, \varkappa, 1 - \varkappa\}. \tag{8.5}$$

We remark that $\upeta_{min}$ and q have been chosen to be small enough so that the bootstrap argument for global existence given in Section 11.2 will close. In particular, inequality (7.4a), with $\upeta \leq \upeta_{min}$, guarantees that the energies $\mathcal{E}_{(\upgamma,\updelta)}[\cdot, \partial(\cdot)]$ for solutions to (5.15a)-(5.15c) have the coerciveness property (6.6).

Remark 8.2. If ϵ is sufficiently small, then the inequalities (7.3a) and (7.3c) (for $\upeta \leq \upeta_{min}$) are implied by the definition of $\mathbf{S}_N$, the bootstrap assumption $\mathbf{S}_N \leq \epsilon$, and Sobolev embedding.

9 Sobolev Estimates

In this section, we use the bootstrap assumptions of Sections 7 and 8 to derive estimates of all of the terms appearing in the modified equations (5.15a) - (5.15d) in terms of the norms defined in Section 6.1. The main goal is to show that the error terms are small compared to the principal terms, which is the main step in closing the bootstrap argument in our proof of future-global existence (Theorem 11.5). More specifically, in Section 11.1, the estimates of this section will be coupled with the energy inequalities of Corollary 6.5 and Corollary 6.8 in order to derive a system of energy integral inequalities for the solution. We divide the analysis into two propositions: Proposition 9.1 provides basic estimates for g and $\partial\Phi$, while Proposition 9.3 provides estimates for the nonlinearities and error terms. In particular, Proposition 9.1 provides estimates for the ratio $Z_j \overset{\text{def}}{=} \frac{\partial_j \Phi}{\partial_t \Phi}$, $(j = 1, 2, 3)$, that are crucial for closing the bootstrap argument of Theorem 11.5. The main tools for proving the propositions are standard Sobolev-Moser product-type estimates, which we have collected together in the Appendix for convenience.

9.1 Estimates of the basic metric and fluid variables

In this section, we state and prove the first proposition that will be used to deduce the energy inequalities of Section 11.1.

Proposition 9.1 (Estimates of the basic metric and fluid variables). *Let $N \geq 3$ be an integer and assume that the bootstrap assumption* (7.3b) *holds on the spacetime slab* $[0, T) \times \mathbb{T}^3$ *for some constant $K_1 \geq 1$. Assume further that $0 < c_s < \sqrt{1/3}$. Then there exist a constant $\epsilon' > 0$ and a constant $C > 0$, where C depends on N and K_1, such that if $\mathbf{S}_N(t) \leq \epsilon'$ on $[0, T)$, then the following estimates also hold on $[0, T)$, where*

$h_{jk} = e^{-2\Omega} g_{jk}$:

$$\|g^{00} + 1\|_{H^N} \leq C e^{-q\Omega} S_{g;N}, \tag{9.1a}$$
$$\|g^{jk}\|_{L^\infty} \leq C e^{-2\Omega}, \tag{9.1b}$$
$$\|\underline{\partial} g^{jk}\|_{H^{N-1}} \leq C e^{-2\Omega} S_{g;N}, \tag{9.1c}$$
$$\|g^{0j}\|_{H^N} \leq C e^{-(1+q)\Omega} S_{g;N}, \tag{9.1d}$$

$$\|\partial_t g_{jk} - 2\omega g_{jk}\|_{H^N} \leq C e^{(2-q)\Omega} S_{h_{**};N}, \tag{9.2a}$$
$$\|\partial_t g_{jk}\|_{C_b^1} \leq C e^{2\Omega}, \tag{9.2b}$$

$$\|g^{aj} \partial_t g_{ak} - 2\omega \delta_k^j\|_{H^N} \leq C e^{-q\Omega} S_{g;N}, \tag{9.3}$$

$$\|\partial_t g^{jk} + 2\omega g^{jk}\|_{H^N} \leq C e^{-(2+q)\Omega} S_{g;N}, \tag{9.4a}$$
$$\|\partial_t g^{00}\|_{H^N} \leq C e^{-q\Omega} S_{g;N}, \tag{9.4b}$$
$$\|\partial_t g^{0j}\|_{H^N} \leq C e^{-(1+q)\Omega} S_{g;N}, \tag{9.4c}$$
$$\|\partial_t g^{jk}\|_{L^\infty} \leq C e^{-2\Omega}. \tag{9.4d}$$

The following estimates for the term $g^{ab}\Gamma_{ajb}$ from the right-hand side of equation (5.15b) *hold on $[0,T)$:*

$$\|g^{ab}\Gamma_{ajb}\|_{H^N} \leq C e^{(1-q)\Omega} \underline{S}_{h_{**};N}, \tag{9.5a}$$
$$\|g^{ab}\Gamma_{ajb}\|_{H^{N-1}} \leq C \underline{S}_{h_{**};N}. \tag{9.5b}$$

The following estimates for $Z_j = \frac{\partial_j \Phi}{\partial_t \Phi}$ hold on $[0,T)$, where $\varkappa = 3c_s^2$:

$$\|Z_j\|_{H^N} \leq C e^{\Omega} S_{\partial\Phi;N}, \tag{9.6a}$$
$$\|Z_j\|_{H^{N-1}} \leq C e^{\varkappa\Omega} S_{\partial\Phi;N}. \tag{9.6b}$$

The following estimates for $\sigma = -g^{\alpha\beta}(\partial_\alpha \Phi)(\partial_\beta \Phi)$ and for the FLRW quantity $\widetilde{\sigma} \overset{def}{=} -\widetilde{g}^{\alpha\beta}(\partial_\alpha \widetilde{\Phi})(\partial_\beta \widetilde{\Phi}) = e^{-2\varkappa\Omega}\bar{\Psi}^2$ hold on $[0,T)$:

$$\|e^{2\varkappa\Omega}\sigma - \overbrace{e^{2\varkappa\Omega}\widetilde{\sigma}}^{\bar{\Psi}^2}\|_{H^N} \leq C\mathbf{S}_N. \tag{9.7}$$

In the above estimates, the norms $S_{h_{**};N}$, $S_{g;N}$, $S_{\partial\Phi;N}$, and $\mathbf{S}_N$ are defined in Definition 6.1.

Proof. Most of these estimates can be found in the statements and proofs of Lemmas 9, 11, 18, and 20 of [Rin08]. The exceptions are (9.4c), (9.4d), and (9.6a)-(9.7). For brevity, we do not repeat all of the details of the estimates that are proved in [Rin08].

Remark 9.2. Throughout all of the remaining proofs in this article, we freely use the results of Lemma 4.2, the definitions of the norms from Section 6.1, the definitions (8.4), (8.5) of η_{min} and q, and the Sobolev embedding result $H^{M+2}(\mathbb{T}^3) \hookrightarrow C_b^M(\mathbb{T}^3)$, $(M \geq 0)$. **We also freely use the assumption that $\mathbf{S}_N$, which is defined in (6.2f), is sufficiently small without explicitly mentioning it every time. Furthermore, the smallness is adjusted as necessary at each step in the proof.** To avoid overburdening the paper with details, we don't give explicit estimates for how small $\mathbf{S}_N$ must be. We also remark that as discussed in Section 2.5, the constants c, C, C_* that appear throughout the article can be chosen uniformly (however, they may depend on N, c_s, and Λ) as long as $\mathbf{S}_N$ is sufficiently small. Finally, we prove statements in logical order, rather than the order in which they are stated in the proposition.

Proofs of (9.1b) *and the preliminary estimates* $\|g^{00}+1\|_{L^\infty} + \|g^{00}+1\|_{L^2} \leq Ce^{-q\Omega}S_{g;N}$ *and* $\|g^{0j}\|_{L^\infty} + \|g^{0j}\|_{L^2} \leq Ce^{-(1+q)\Omega}S_{g;N}$*:* The L^∞ estimate (9.1b), as well as the aforementioned preliminary estimates, which we will need shortly, follow from the definition (6.2d) of $S_{g;N}$, the assumption (7.3b), Lemma 7.1, and Lemma 7.2.

Proofs of (9.1c)-(9.1d): These proofs begin with the fact that when $1 \leq |\vec{\alpha}| \leq N$, $\partial_{\vec{\alpha}} g^{\mu\nu}$ is a linear combination of terms of the form

$$g^{\lambda_1\mu} g^{\lambda_2\kappa_1} \cdots g^{\lambda_n\kappa_{n-1}} g^{\nu\kappa_n} (\partial_{\vec{\alpha}_1} g_{\lambda_1\kappa_1}) \cdots (\partial_{\vec{\alpha}_n} g_{\lambda_n\kappa_n}), \tag{9.8}$$

where $\vec{\alpha}_1 + \cdots + \vec{\alpha}_n = \vec{\alpha}$ and each $|\vec{\alpha}_i| > 0$. We remark that (9.8) can be shown inductively via the identity $\partial_\alpha g^{\mu\nu} = -g^{\mu\kappa} g^{\nu\lambda} \partial_\alpha g_{\kappa\lambda}$.

To prove (9.1d), we first recall that the bound $\|g^{0j}\|_{L^2} \leq Ce^{-(1+q)\Omega}S_{g;N}$ was shown above. Therefore, it remains to estimate (9.8) in L^2 for $1 \leq |\vec{\alpha}| \leq N$, with μ, ν in (9.8) equal to $0, j$. To this end, we first bound the terms $g^{\lambda_1 0} g^{\lambda_2\kappa_1} \cdots g^{\lambda_n\kappa_{n-1}} g^{j\kappa_n}$ in L^∞, and then estimate the remaining product $(\partial_{\vec{\alpha}_1} g_{\lambda_1\kappa_1}) \cdots (\partial_{\vec{\alpha}_n} g_{\lambda_n\kappa_n})$ in L^2 using Proposition B.2. The L^∞ terms are bounded using (9.1b) and the preliminary estimates $\|g^{00}+1\|_{L^\infty} \leq Ce^{-q\Omega}S_{g;N}$ and $\|g^{0j}\|_{L^\infty} \leq Ce^{-(1+q)\Omega}S_{g;N}$ shown above. The L^2 norm of the product is controlled by Proposition B.2 and the definition (6.2d) of $S_{g;N}$. The only difficulty is keeping track of the powers of e^Ω, which Ringström accomplishes inductively through a counting argument that is analogous to the Counting Principle estimate (9.33) that we provide below; the details can be found in the proof of Lemma 9 of [Rin08]. The proof of (9.1c) is similar.

Proof of (9.1a)*:* The estimate (9.1a) follows from the identity $g^{00} + 1 = \frac{1}{g_{00}}[(g_{00}+1) - g^{0a}g_{0a}]$, Corollary B.3 with $v = g_{00}$ and $F(g_{00}) = \frac{1}{g_{00}}$ in the corollary, Proposition B.5, the definition (6.2d) of $S_{g;N}$, and (9.1d).

Proofs of (9.2a)-(9.2b): The estimate (9.2a) follows directly from the definition (6.2c) of $S_{h_{**};N}$ and the observation that $\partial_t h_{jk} = e^{-2\Omega}(\partial_t g_{jk} - 2\omega g_{jk})$. Inequality (9.2b) then follows from the assumption (7.3b) and (9.2a).

Proof of (9.3): Note the identity $\delta_k^j = g^{0j}g_{0k} + g^{aj}g_{ak}$. From this fact, the estimate (9.3) follows from Proposition B.5, the definition (6.2d) of $S_{g;N}$, and (9.1a)-(9.2a).

Proofs of (9.4a)-(9.4d): To prove (9.4a), we first note the identity $\partial_t g^{jk} = -g^{\alpha j}g^{\beta k}\partial_t g_{\alpha\beta}$. We then use Proposition B.5, the definition (6.2d) of $S_{g;N}$, (9.1a), (9.1b), (9.1c), (9.1d), and (9.3) to conclude that

$$\begin{aligned}\|\partial_t g^{jk} + 2\omega g^{jk}\|_{H^N} &\leq \|g^{bk}(g^{aj}\partial_t g_{ab} - 2\omega\delta_b^j)\|_{H^N} + \|g^{0j}g^{bk}\partial_t g_{0b}\|_{H^N} \\ &\quad + \|g^{aj}g^{0k}\partial_t g_{0a}\|_{H^N} + \|g^{0j}g^{0k}\partial_t g_{00}\|_{H^N} \\ &\leq Ce^{-(2+q)\Omega}S_{g;N},\end{aligned} \tag{9.9}$$

which gives (9.4a). Inequality (9.4d) then follows from (9.1b) and (9.4a). The proofs of (9.4b) and (9.4c) are similar, and we omit the details.

Proofs of (9.5a)-(9.5b): To prove (9.5a), we first use Proposition B.5 to conclude that

$$\|g^{ab}\Gamma_{ajb}\|_{H^N} \leq C\big\{\|g^{ab}\|_{L^\infty} + \|\underline{\partial} g^{ab}\|_{H^{N-1}}\big\}\|\Gamma_{ajb}\|_{H^N}. \tag{9.10}$$

Recalling that $\Gamma_{ajb} = \frac{1}{2}\big(\partial_a g_{bj} + \partial_b g_{aj} - \partial_j g_{ab}\big)$ and that $g_{jk} = e^{2\Omega}h_{jk}$, and using (9.1b), (9.1c), the definitions (6.2c) and (6.2f) of $S_{h_{**};N}$ and $\mathbf{S}_N$, and Sobolev embedding, we deduce that the right-hand side of (9.10) is bounded from above by $Ce^{(1-q)\Omega}S_{h_{**};N}$. This proves (9.5a). The proof of (9.5b) follows similarly.

Proofs of (9.6a)-(9.6b): To prove (9.6a), we first express

$$Z_j = \frac{e^{\varkappa\Omega}\partial_j\Phi}{e^{\varkappa\Omega}\partial_t\Phi}. \tag{9.11}$$

Then applying Corollary B.3 (with $v = e^{\varkappa\Omega}\partial_t\Phi$ and $F(v) = v^{-1}$) and Proposition B.5 to the right-hand side of (9.11), and using the definition of $S_{\partial\Phi;N}$, we deduce that

$$\|Z_j\|_{H^N} \leq C\|e^{\varkappa\Omega}\partial_j\Phi\|_{H^N}\left(\left\|\frac{1}{e^{\varkappa\Omega}\partial_t\Phi}\right\|_{L^\infty} + \left\|\underline{\partial}\left(\frac{1}{e^{\varkappa\Omega}\partial_t\Phi}\right)\right\|_{H^{N-1}}\right) \leq Ce^{\Omega}S_{\partial\Phi;N} \tag{9.12}$$

as desired.

To prove (9.6b), we first note that by the definition of $\underline{S}_{\partial\Phi;N}$, we have

$$\|\partial_t\partial_{\vec{\alpha}}\Phi\|_{L^2} \leq e^{-\varkappa\Omega}\underline{S}_{\partial\Phi;N}, \qquad (1 \leq |\vec{\alpha}| \leq N). \tag{9.13}$$

Integrating (9.13) from 0 to t, using the fact that $e^{-\varkappa\Omega(t)}$ is integrable over the interval $t \in [0,\infty)$, using the initial condition $\|\partial_t\partial_{\vec{\alpha}}\Phi\|_{L^2}|_{t=0} \leq S_{\partial\Phi;N}(0)$, and using the fact that $S_{\partial\Phi;N}(t)$ is increasing, we deduce

$$\|\partial_{\vec{\alpha}}\Phi\|_{L^2} \leq CS_{\partial\Phi;N}(t), \qquad (1 \leq |\vec{\alpha}| \leq N). \tag{9.14}$$

We now revisit the proof of (9.12) and use the estimate (9.14) to deduce the desired bound

$$\|Z_j\|_{H^{N-1}} \leq Ce^{\varkappa\Omega}S_{\partial\Phi;N}. \tag{9.15}$$

Proof of (9.7): To prove (9.7), we first decompose $e^{2\varkappa\Omega}\sigma - e^{2\varkappa\Omega}\widetilde{\sigma}$ as follows:

$$\begin{aligned} e^{2\varkappa\Omega}\sigma - e^{2\varkappa\Omega}\widetilde{\sigma} &= (e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi})(e^{\varkappa\Omega}\partial_t\Phi + \bar{\Psi}) - (g^{00}+1)(e^{\varkappa\Omega}\partial_t\Phi)^2 \\ &\quad - 2g^{0a}(e^{\varkappa\Omega}\partial_t\Phi)^2 Z_a - g^{ab}(e^{\varkappa\Omega}\partial_t\Phi)^2 Z_a Z_b. \end{aligned} \tag{9.16}$$

Inequality (9.7) now follows from Proposition B.5, the definition (6.2f) of $\mathbf{S}_N$, Sobolev embedding, (9.1a), (9.1b), (9.1c), (9.1d), (9.6a), and the fact that $e^{2\varkappa\Omega}\widetilde{\sigma} = \bar{\Psi}^2$.

□

9.2 Estimates of the nonlinearities and error terms

In this section, we state and prove the second proposition that will be used to deduce the energy inequalities of Section 11.1.

Proposition 9.3 (Estimates of the nonlinearities and error terms). *Let $N \geq 3$ be an integer, and assume that $0 < c_s < \sqrt{1/3}$. Let $(g_{\mu\nu}, \partial_\mu\Phi)$, $(\mu,\nu = 0,1,2,3)$, be a solution to the modified equations* (5.15a)-(5.15d) *on the spacetime slab $[0,T) \times \mathbb{T}^3$, and assume that the bootstrap assumption* (7.3b) *holds on the same slab for some constant $K_1 \geq 1$. Then there exist a constant $\epsilon'' > 0$ and a constant $C > 0$, where C depends on N and K_1, such that if $\mathbf{S}_N(t) \leq \epsilon''$ on $[0,T)$, then the following estimates also hold on $[0,T)$ for the quantities $\triangle_{A,\mu\nu}$, $\triangle_{C,00}$, and $\triangle_{C,0j}$, defined in* (A.13a) - (A.13c) *and* (A.15a)-(A.15b)*:*

$$\|\triangle_{A,00}\|_{H^N} \leq Ce^{-2q\Omega}\mathbf{S}_N^2, \tag{9.17a}$$
$$\|\triangle_{A,0j}\|_{H^N} \leq Ce^{(1-2q)\Omega}\mathbf{S}_N^2, \tag{9.17b}$$
$$\|\triangle_{A,jk}\|_{H^N} \leq Ce^{(2-2q)\Omega}\mathbf{S}_N^2, \tag{9.17c}$$
$$\|\triangle_{C,00}\|_{H^N} \leq Ce^{-2q\Omega}\mathbf{S}_N^2, \tag{9.17d}$$
$$\|\triangle_{C,0j}\|_{H^N} \leq Ce^{(1-2q)\Omega}\mathbf{S}_N^2. \tag{9.17e}$$

Additionally, for the quantities $\triangle_{Rapid,\mu\nu}$ *defined in* (A.7a) - (A.7c)*, we have the following estimates on* $[0,T)$:

$$\|\triangle_{Rapid,00}\|_{H^N} \leq Ce^{-(3+3c_s^2)\Omega}\mathbf{S}_N, \tag{9.18a}$$
$$\|\triangle_{Rapid,0j}\|_{H^N} \leq Ce^{-(2+3c_s^2)\Omega}\mathbf{S}_N, \tag{9.18b}$$
$$\|\triangle_{Rapid,jk}\|_{H^N} \leq Ce^{-(3+3c_s^2)\Omega}\mathbf{S}_N. \tag{9.18c}$$

For the quantities $\triangle_{\mu\nu}$ *defined in* (A.4a)-(A.4c)*, we have the following estimates on* $[0,T)$:

$$\|\triangle_{00}\|_{H^N} \leq Ce^{-2q\Omega}\mathbf{S}_N, \tag{9.19a}$$
$$\|\triangle_{0j}\|_{H^N} \leq Ce^{(1-2q)\Omega}\mathbf{S}_N, \tag{9.19b}$$
$$\|\triangle_{jk}\|_{H^N} \leq Ce^{-2q\Omega}\mathbf{S}_N. \tag{9.19c}$$

For the commutator terms from Corollary 6.5, we have the following estimates on $[0,T)$:

$$\|[\hat{\Box}_g, \partial_{\vec{\alpha}}](g_{00}+1)\|_{L^2} \leq Ce^{-2q\Omega}\mathbf{S}_N, \qquad (|\vec{\alpha}| \leq N), \tag{9.20a}$$
$$\|[\hat{\Box}_g, \partial_{\vec{\alpha}}]g_{0j}\|_{L^2} \leq Ce^{(1-2q)\Omega}\mathbf{S}_N, \qquad (|\vec{\alpha}| \leq N), \tag{9.20b}$$
$$\|[\hat{\Box}_g, \partial_{\vec{\alpha}}]h_{jk}\|_{L^2} \leq Ce^{-2q\Omega}\mathbf{S}_N, \qquad (|\vec{\alpha}| \leq N). \tag{9.20c}$$

For the terms from Corollary 6.5, where $\triangle_{\mathcal{E};(\gamma,\delta)}[v,\partial v]$ *is defined in* (6.8)*, we have the following estimates on* $[0,T)$:

$$e^{2q\Omega}\|\triangle_{\mathcal{E};(\gamma_{00},\delta_{00})}[\partial_{\vec{\alpha}}(g_{00}+1), \partial(\partial_{\vec{\alpha}}g_{00})]\|_{L^1} \leq Ce^{-q\Omega}\underline{S}_{g_{00}+1;N}\mathbf{S}_N, \quad (|\vec{\alpha}| \leq N), \tag{9.21a}$$
$$e^{2(q-1)\Omega}\|\triangle_{\mathcal{E};(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}g_{0j}, \partial(\partial_{\vec{\alpha}}g_{0j})]\|_{L^1} \leq Ce^{-q\Omega}\underline{S}_{g_{0*};N}\mathbf{S}_N, \quad (|\vec{\alpha}| \leq N), \tag{9.21b}$$
$$e^{2q\Omega}\|\triangle_{\mathcal{E};(0,0)}[0, \partial(\partial_{\vec{\alpha}}h_{jk})]\|_{L^1} \leq Ce^{-q\Omega}\underline{S}_{h_{**};N}\mathbf{S}_N, \quad (|\vec{\alpha}| \leq N). \tag{9.21c}$$

For the error terms (A.17a)-(A.17f) *corresponding to the fully raised Christoffel symbols of Lemma A.7, we have the following estimates on* $[0,T)$:

$$\|\triangle_{(\Gamma)}^{000}\|_{H^N} \leq Ce^{-q\Omega}\mathbf{S}_N, \tag{9.22a}$$
$$\|\triangle_{(\Gamma)}^{j00}\|_{H^N} \leq Ce^{-(1+q)\Omega}\mathbf{S}_N, \tag{9.22b}$$
$$\|\triangle_{(\Gamma)}^{0j0}\|_{H^N} \leq Ce^{-(1+q)\Omega}\mathbf{S}_N, \tag{9.22c}$$
$$\|\triangle_{(\Gamma)}^{0jk}\|_{H^N} \leq Ce^{-(2+q)\Omega}\mathbf{S}_N, \tag{9.22d}$$
$$\|\triangle_{(\Gamma)}^{j0k}\|_{H^N} \leq Ce^{-(2+q)\Omega}\mathbf{S}_N, \tag{9.22e}$$
$$\|\triangle_{(\Gamma)}^{ijk}\|_{H^N} \leq Ce^{-(3+q)\Omega}\mathbf{S}_N. \tag{9.22f}$$

For the fluid wave equation error terms Θ, *and* $\triangle_{\partial\Phi}$ *defined in* (A.6d) *and* (A.4d) *respectively, we have the following estimates on* $[0,T)$:

$$\|\Theta\|_{H^N} \leq Ce^{-q\Omega}\mathbf{S}_N, \tag{9.23a}$$
$$e^{\varkappa\Omega}\|\triangle_{\partial\Phi}\|_{H^N} \leq Ce^{-q\Omega}\mathbf{S}_N. \tag{9.23b}$$

For the reciprocal acoustical metric error terms $\triangle_{(m)}$, $\triangle_{(m)}^{0j}$, *and* $\triangle_{(m)}^{jk}$, *defined in* (A.6a), (A.6b), *and* (A.6c) *respectively, we have the following estimates on* $[0,T)$:

$$\|\triangle_{(m)}\|_{H^N} \leq Ce^{-q\Omega}\mathbf{S}_N, \tag{9.24a}$$
$$\|\triangle_{(m)}^{0j}\|_{H^N} \leq Ce^{-\Omega}\mathbf{S}_N, \tag{9.24b}$$
$$\|\triangle_{(m)}^{jk}\|_{H^N} \leq Ce^{-(2+q)\Omega}\mathbf{S}_N. \tag{9.24c}$$

For the reciprocal acoustical metric components $(m^{-1})^{0j}$ *and* $(m^{-1})^{jk}$ *defined in* (5.17b) *and* (5.17c), *we have the following estimates on* $[0,T)$:

$$\|(m^{-1})^{0j}\|_{H^N} \leq Ce^{-\Omega}\mathbf{S}_N, \tag{9.25a}$$
$$\Big\|(m^{-1})^{jk} - \frac{1}{2s+1}g^{jk}\Big\|_{H^N} \leq Ce^{-(2+q)\Omega}\mathbf{S}_N, \tag{9.25b}$$
$$\|(m^{-1})^{jk}\|_{L^\infty} \leq Ce^{-2\Omega}, \tag{9.25c}$$
$$C^{-1}\delta^{ab}X_aX_b \leq e^{2\Omega}(m^{-1})^{ab}X_aX_b \leq C\delta^{ab}X_aX_b, \qquad \forall (X_1,X_2,X_3) \in \mathbb{R}^3, \tag{9.25d}$$
$$\|\underline{\partial}(m^{-1})^{jk}\|_{H^{N-1}} \leq Ce^{-2\Omega}\mathbf{S}_N. \tag{9.25e}$$

For the commutator terms from Corollary 6.8, we have the following estimates on $[0,T)$, *where* $\varkappa = 3c_s^2$:

$$\|[\hat{\square}_m, \partial_{\vec{\alpha}}]\Phi\|_{L^2} \leq Ce^{-(1+\varkappa)\Omega}\mathbf{S}_N, \qquad (|\vec{\alpha}| \leq N). \tag{9.26}$$

For the time derivatives of the fluid-related and reciprocal acoustical metric-related quantities, we have the following estimates on $[0,T)$:

$$\|\partial_t[e^{\varkappa\Omega}\partial_t\Phi]\|_{H^{N-1}} \leq Ce^{-q\Omega}\mathbf{S}_N, \tag{9.27a}$$
$$\|\partial_t Z_j\|_{H^{N-1}} \leq Ce^{\varkappa\Omega}\mathbf{S}_N, \tag{9.27b}$$
$$\|\partial_t\triangle_{(m)}\|_{H^{N-1}} \leq Ce^{-q\Omega}\mathbf{S}_N, \tag{9.27c}$$
$$\|\partial_t\triangle_{(m)}^{0j}\|_{H^{N-1}} \leq Ce^{-(1+q)\Omega}\mathbf{S}_N, \tag{9.27d}$$
$$\|\partial_t\triangle_{(m)}^{jk}\|_{H^{N-1}} \leq Ce^{-(2+q)\Omega}\mathbf{S}_N, \tag{9.27e}$$
$$\|\partial_t(m^{-1})^{0j}\|_{H^{N-1}} \leq Ce^{-(1+q)\Omega}\mathbf{S}_N, \tag{9.27f}$$
$$\|\partial_t(m^{-1})^{jk} + 2\omega(m^{-1})^{jk}\|_{H^{N-1}} \leq Ce^{-(2+q)\Omega}\mathbf{S}_N, \tag{9.27g}$$
$$\|\partial_t(m^{-1})^{jk}\|_{L^\infty} \leq Ce^{-2\Omega}. \tag{9.27h}$$

In the above estimates, the norms $\underline{S}_{g_{00}+1;N}$, $\underline{S}_{g_{0*};N}$, $\underline{S}_{h_{**};N}$, *and* $\mathbf{S}_N$ *are defined in Definition 6.1.*

The proof of Proposition 9.3 is located in the next section. As we will see, many of the estimates in the proposition can be essentially reduced to counting spatial indices. This motivates the following definition, in which we introduce three classes of quantities that have slightly different properties with regard to spatial indices.

Definition 9.4 (**The sets $\mathcal{G}_M$, $\mathcal{H}_M$, and $\mathcal{Z}_N$**)**.** Let $N \geq 3$ be an integer and assume that $M = N - 1$ or $M = N$. Let v be a function on $\mathbb{T}^3$, and let A denote its number of downstairs spatial indices minus its number of upstairs spatial indices (e.g. $A = -1$ when $v = \partial_i g^{jk}$). We write $v \in \mathcal{G}_M$ if there exists a constant $C > 0$ such that

$$\|v\|_{H^M} \leq Ce^{-q\Omega}e^{A\Omega}\mathbf{S}_N \tag{9.28}$$

holds for all $t \geq 0$ whenever $\mathbf{S}_N$ is sufficiently small. Above, q is the small positive constant that is defined in Section 8.2 and that appears in the definition of the norms (i.e., Definition 6.1).

We write $v \in \mathcal{H}_M$ if there exists a constant $C > 0$ such that either

$$\|v\|_{H^M} \leq Ce^{A\Omega}\mathbf{S}_N \tag{9.29}$$

holds for all $t \geq 0$ whenever $\mathbf{S}_N$ is sufficiently small, or

$$\|v\|_{L^\infty} \leq Ce^{A\Omega}, \tag{9.30}$$

$$\text{and } \|\underline{\partial} v\|_{H^{M-1}} \leq Ce^{A\Omega}\mathbf{S}_N \tag{9.31}$$

holds for all $t \geq 0$ whenever $\mathbf{S}_N$ is sufficiently small.

In the case $M = N$ only, we write $v \in \mathcal{Z}_N$ if $v = Z_j$, $(j = 1, 2, 3)$.

Remark 9.5. Note that $v \in \mathcal{G}_M \implies v \in \mathcal{H}_M$. Also, by (9.6a) and (9.6b), $v \in \mathcal{Z}_N \implies v \in \mathcal{G}_{N-1} \cap \mathcal{H}_N$ whenever $N \geq 3$.

Remark 9.6. The main idea of the above definition is that if $v \in \mathcal{G}_M \cup \mathcal{H}_M$, then up to correction factors various norms of v can be estimated by counting its net number of spatial indices. This idea is made precise in Lemma 9.7. The point of introducing the sets $\mathcal{G}$ is that their elements are "good" in the sense that they decay by a factor of $e^{-q\Omega}$ faster than the rate predicted by counting spatial indices.

Observe that the definition of $\mathbf{S}_N$ (for $N \geq 3$) and the estimates of Proposition 9.1 imply the following estimates, which will implicitly be used many times in our proof of Proposition 9.3:

$$g_{00} + 1, g_{0j}, g^{00} + 1, g^{0j}, \partial_t g_{00}, \partial_t g_{0j}, \underbrace{\partial_t g_{jk} - 2\omega g_{jk}}_{e^{2\Omega}\partial_t h_{jk}}, \partial_t g^{00}, \partial_t g^{0j}, \partial_t g^{jk} - 2\omega g^{jk}, \tag{9.32a}$$

$$\partial_i g_{00}, \partial_i g_{0j}, \partial_i g_{jk}, \Gamma_{000}, \Gamma_{j00}, \Gamma_{0j0}, \Gamma_{ijk} \in \mathcal{G}_N,$$

$$\underline{\partial} g_{00}, \underline{\partial} g_{0j}, \partial_t \underline{\partial} g_{00}, \partial_t \underline{\partial} g_{0j}, \underbrace{\partial_t \underline{\partial} g_{jk} - 2\omega \underline{\partial} g_{jk}}_{e^{2\Omega}\partial_t \underline{\partial} h_{jk}}, \partial_i \underline{\partial} g_{00}, \partial_i \underline{\partial} g_{0j}, \partial_i \underline{\partial} g_{jk}, Z_j \in \mathcal{G}_{N-1}, \tag{9.32b}$$

$$g_{jk}, g^{jk}, \partial_t g_{jk}, \partial_t g^{jk}, \Gamma_{j0k}, \Gamma_{jk0}, e^{\varkappa\Omega}\partial_t \Phi, Z_j \in \mathcal{H}_N, \tag{9.32c}$$

$$\underline{\partial} g_{jk}, \underline{\partial} g^{jk} \in \mathcal{H}_{N-1}. \tag{9.32d}$$

Note that, for example, $\underline{\partial} g_{jk}$ **in** (9.32d) **is counted as having only** 2 **spatial indices**.

In our proof of Proposition 9.3, we will often use the following lemma.

Lemma 9.7 (**Counting Principle**)**.** *Let $N \geq 3$ be an integer and assume that $M = N-1$ or $M = N$. Let $l \geq 1$ be an integer, and suppose that $v_{(i)} \in \mathcal{G}_M \cup \mathcal{H}_M$ for $1 \leq i \leq l$. Assume further that $v_{(j)}$ verifies* (9.29) *for some j satisfying $1 \leq j \leq l$ (i.e., that at least one of the $v_{(i)}$ is in L^2 and is controlled by $\mathbf{S}_N$). Then there exist constants $\epsilon > 0$ and $C > 0$ such that if $\mathbf{S}_N \leq \epsilon$, then*

$$\left\| \prod_{i=1}^{l} v_{(i)} \right\|_{H^M} \leq C e^{-n(\mathcal{G}_M) q\Omega} e^{-n_*(\mathcal{Z}_M) q\Omega} e^{n_{total}\Omega} \mathbf{S}_N \tag{9.33}$$

holds for all $t \geq 0$. In inequality (9.33)*,*

- *n_{total} is the total number of downstairs spatial indices minus the total number of upstairs spatial indices in the product (e.g. $n_{total} = -1$ for $g^{ja} Z_a$).*
- *$n(\mathcal{G}_M)$ is the number of the $v_{(i)}$ that belong to $\mathcal{G}_M$ (e.g. $n(\mathcal{G}_N) = 0$ and $n(\mathcal{G}_{N-1}) = 1$ for $g^{ja} Z_a$).*
- *In the case $M = N-1$, $n_*(\mathcal{Z}_{N-1}) \overset{def}{=} 0$, and the factor $e^{-n_*(\mathcal{Z}_{N-1})q\Omega}$ is therefore absent; in this case, the quantities Z_j, (j=1,2,3), are counted as elements of $\mathcal{G}_{N-1}$ (see Remark 9.5).*
- *In the case $M = N$, $n_*(\mathcal{Z}_N) \overset{def}{=} 0$ if none of the $v_{(i)}$ belong to $\mathcal{Z}_N$, and $n_*(\mathcal{Z}_N)$ is the number of the $v_{(i)}$ that belong to $\mathcal{Z}_N$* ***minus one*** *if at least one of the $v_{(i)}$ belongs to $\mathcal{Z}_N$ (e.g. $n_*(\mathcal{Z}_N) = 0$ for $g^{ja} Z_a$ and $n_*(\mathcal{Z}_N) = 1$ for $g^{ab} Z_a Z_b$).*

Remark 9.8. Note that since $h_{jk} = e^{-2\Omega} g_{jk}$, estimates involving h_{jk} need to be modified by a factor of $e^{-2\Omega}$. That is, $e^{2\Omega} h_{jk} \in \mathcal{H}_N$, $e^{2\Omega} \partial_t h_{jk} \in \mathcal{G}_N$, and $e^{2\Omega} \partial_i h_{jk} \in \mathcal{G}_N$, etc.

Proof of Lemma 9.7. Without loss of generality, we assume that $v_{(l)} \in L^2$. By Proposition B.5, we have

$$\left\| \prod_{i=1}^{l} v_{(i)} \right\|_{H^M} \leq C \left\{ \| v_{(l)} \|_{H^M} \prod_{i=1}^{l-1} \| v_{(i)} \|_{L^\infty} + \sum_{i=1}^{l-1} \| \underline{\partial} v_{(i)} \|_{H^{M-1}} \prod_{j \neq i} \| v_{(j)} \|_{L^\infty} \right\}. \tag{9.34}$$

The estimate (9.33) follows easily from (9.34), Sobolev embedding, and the definition of $\mathcal{H}_M$, $\mathcal{G}_M$, and $\mathcal{Z}_N$. Note that in the case $M = N$, if more than one element of $\mathcal{Z}_N$ is present in $\prod_{i=1}^{l} v_{(i)}$, then in each product on the right-hand side of (9.33), all but at most one of these elements of $\mathcal{Z}_N$ are bounded in the L^∞ norm. Hence, by (9.6b) and the Sobolev embedding estimate $\|Z_j\|_{L^\infty} \leq C \|Z_j\|_{H^{N-1}} \leq C e^{\varkappa\Omega} \mathbf{S}_N \leq C e^{(1-q)\Omega} \mathbf{S}_N$, these elements contribute the additional decay factor $e^{-n_*(\mathcal{Z}_N) q\Omega}$ to the right-hand side of (9.33). □

9.3 Proof of Proposition 9.3

Proof of Proposition 9.3. We prove statements in logical order, rather than in the order in which they are listed in the conclusions of the proposition. See Remark 9.2 for some conventions that we use throughout the proof.

Proofs of (9.17a)-(9.17e)*: and* (9.22a)-(9.22f): To prove (9.17a)-(9.17e), we apply the Counting Principle estimate (9.33) to the right-hand sides of (A.13a)-(A.13c) and (A.15a)-(A.15b). Note that every product on the right-hand side of the expression for $\triangle_{A,\mu\nu}$ has the same net number of spatial indices as $\triangle_{A,\mu\nu}$. Furthermore, by inspection, we see that these right-hand sides are quadratic in the elements of $\mathcal{G}_N$ (i.e., $n(\mathcal{G}_N) \geq 2$); this results in the presence of the $e^{-2q\Omega}$ factor on the right-hand sides of (9.17a)-(9.17e). The same reasoning allows us to deduce (9.22a)-(9.22f), but in this case, some of the corresponding products on the right-hand sides of (A.17a)-(A.17f) are only linear in elements of $\mathcal{G}_N$ (which results in a single $e^{-q\Omega}$ decay factor).

Proofs of (9.18a)-(9.18c): All products on the right-hand sides of (A.7a)-(A.7c) except two are of the form $F(Q_{(1)}, Q_{(2)})\theta_{\mu\nu}$, where F is a smooth function of its arguments, $Q_{(1)}, Q_{(2)} \in \{e^{3(1+c_s^2)\Omega}(\omega - H), \omega + H, \omega, e^{\varkappa\Omega}\partial_t\Phi, e^{2\varkappa\Omega}\sigma\}$, $\theta_{\mu\nu}$ is a product of elements of $\mathcal{G}_N \cup \mathcal{H}_N$, and each product contains at least one element of $\mathcal{G}_N \cup \mathcal{Z}_N$. Hence, $\theta_{\mu\nu}$ can be bounded in H^N by counting spatial indices and using (9.33):

$$\|\theta_{\mu\nu}\|_{H^N} \leq Ce^{n_{total}\Omega}\mathbf{S}_N. \tag{9.35}$$

Above, n_{total} refers to $\theta_{\mu\nu}$, i.e., $n_{total} = 0$ for θ_{00}, $n_{total} = 1$ for θ_{0j}, and $n_{total} = 0$ for θ_{jk} (we made the adjustment involving $e^{-2\Omega}$ mentioned in Remark 9.8 when computing n_{total} for θ_{jk}). The remaining two terms [the last term on the right-hand side of (A.7a) and the next-to-last term on the right-hand side of (A.7c)] are, up to constants, of the form $\big(F(Q_{(1)}, Q_{(2)}) - F(\widetilde{Q}_{(1)}, \widetilde{Q}_{(2)})\big)\xi_{\mu\nu}$, where F and the $Q_{(i)}$ are as before, $\widetilde{Q}_{(i)}$ is equal to $Q_{(i)}$ evaluated at the FLRW background (and thus $\widetilde{Q}_{(i)}$ is either constant or depends only on t), and $\xi_{\mu\nu} \in \mathcal{H}_N$ (specifically, $\xi_{00} = 1$, and $\xi_{jk} = h_{jk}$). Hence by (9.33), we have

$$\|\xi_{\mu\nu}\|_{L^\infty} \leq Ce^{n_{total}\Omega}, \tag{9.36}$$

$$\|\underline{\partial}\xi_{\mu\nu}\|_{H^{N-1}} \leq Ce^{n_{total}\Omega}\mathbf{S}_N, \tag{9.37}$$

where n_{total} is as above. We claim that: **i)** $\|Q_{(i)}\|_{L^\infty} \leq C$ and $\|\underline{\partial}Q_{(i)}\|_{H^{N-1}} \leq C\mathbf{S}_N$; and **ii)** $\|Q_{(i)} - \widetilde{Q}_{(i)}\|_{H^N} \leq C\mathbf{S}_N$. When $Q_{(i)} = e^{3(1+c_s^2)\Omega}(\omega - H)$ or $Q_{(i)} = \omega + H$ or ω, **i)** and **ii)** follow from Lemma 4.2. When $Q_{(i)} = e^{\varkappa\Omega}\partial_t\Phi$, **i)** and **ii)** follow from the definition of $\mathbf{S}_N$. When $Q_{(i)} = e^{2\varkappa\Omega}\sigma$, **i)** and **ii)** follow from (9.7). Consequently, we can invoke Corollary B.4 to deduce that

$$\|F(Q_{(1)}, Q_{(2)})\|_{L^\infty} \leq C, \qquad \|\underline{\partial}[F(Q_{(1)}, Q_{(2)})]\|_{H^{N-1}} \leq C\mathbf{S}_N, \tag{9.38}$$

$$\|F(Q_{(1)}, Q_{(2)}) - F(\widetilde{Q}_{(1)}, \widetilde{Q}_{(2)})\|_{H^N} \leq C\mathbf{S}_N. \tag{9.39}$$

Now by Proposition B.5, all of the terms on the right-hand sides of (A.7a)-(A.7c) can be respectively bounded in H^N by one of

$$C\|\theta_{\mu\nu}\|_{H^N}\left(\|F(Q_{(1)},Q_{(2)})\|_{L^\infty}+\|\underline{\partial}[F(Q_{(1)},Q_{(2)})]\|_{H^{N-1}}\right), \tag{9.40}$$

$$C\left(\|\xi_{\mu\nu}\|_{L^\infty}+\|\underline{\partial}\xi_{\mu\nu}\|_{H^{N-1}}\right)\|F(Q_{(1)},Q_{(2)})-F(\widetilde{Q}_{(1)},\widetilde{Q}_{(2)})\|_{H^N}. \tag{9.41}$$

Thus, using (9.35)-(9.37), (9.38)-(9.39), and (9.40)-(9.41), we conclude that all products on the right-hand sides of (A.7a)-(A.7c) are bounded in H^N by

$$Ce^{n_{total}\Omega}\mathbf{S}_N. \tag{9.42}$$

This yields the desired estimates (9.18a)-(9.18c).

Proofs of (9.19a)-(9.19c): The estimates (9.19a)-(9.19b) follow trivially from the definitions (A.4a)-(A.4b) and the estimates (9.17a)-(9.17e), (9.18a)-(9.18c). The estimate (9.19c) follows similarly, but we also have to estimate the $-2\omega g^{0a}\partial_a h_{jk}=-2e^{-2\Omega}\omega g^{0a}\partial_a g_{jk}$ term from (A.4c); the Counting Principle estimate (9.33) with $n(\mathcal{G}_N)=2$ immediately yields $e^{-2\Omega}\omega\|g^{0a}\partial_a g_{jk}\|_{H^N}\leq Ce^{-2q\Omega}\mathbf{S}_N$ as desired.

Proofs of (9.21a)-(9.21c): We first rewrite equation (6.8) as follows:

$$\begin{aligned}\triangle_{\mathcal{E};(\gamma,\delta)}[v,\partial v]=&-\gamma H(\partial_a g^{ab})v\partial_b v-2\gamma H(\partial_a g^{0a})v\partial_t v-2\gamma Hg^{0a}(\partial_a v)(\partial_t v)\\&-(\partial_a g^{0a})(\partial_t v)^2-(\partial_a g^{ab})(\partial_b v)(\partial_t v)-\frac{1}{2}(\partial_t g^{00})(\partial_t v)^2\\&+\Big(\frac{1}{2}\partial_t g^{ab}+\omega g^{ab}\Big)(\partial_a v)(\partial_b v)+(H-\omega)g^{ab}(\partial_a v)(\partial_b v)\\&-\gamma H(\partial_t g^{00})v\partial_t v-\gamma H(g^{00}+1)(\partial_t v)^2.\end{aligned} \tag{9.43}$$

We now claim that the following inequality holds for any function v for which the right-hand side is finite:

$$\|\triangle_{\mathcal{E};(\gamma,\delta)}[v,\partial v]\|_{L^1}\leq Ce^{-q\Omega}\Big\{\|\partial_t v\|_{L^2}^2+e^{-2\Omega}\|\underline{\partial}v\|_{L^2}^2+C_{(\beta)}\|v\|_{L^2}^2\Big\}, \tag{9.44}$$

where $C_{(\beta)}$ is defined in (6.6) (recall that $C_{(\beta)}=\gamma=0$ when $\beta=0$). To obtain (9.44), we use the Cauchy-Schwarz inequality for integrals, (9.1a), (9.1b), (9.1c), (9.1d), (9.4a), and (9.4b). Inequalities (9.21a)-(9.21c) now easily follow from definitions (6.1a)-(6.2f) and (9.44).

Proofs of (9.24a)-(9.24c): These estimates (9.24a)-(9.24c) all follow from the Counting Principle estimate (9.33). Note that all products on the right-hand side of (A.6a) and (A.6c) either contain a factor belonging to $\mathcal{G}_N$ (i.e, $n(\mathcal{G}_N)\geq 1$) or are quadratic in $\mathcal{Z}_N$ (i.e., $n_*(\mathcal{Z}_N)\geq 2-1=1$); hence, the estimates (9.24a) and (9.24c) feature an $e^{-q\Omega}$

factor. In contrast, the right-hand side of (A.6b) features the term $g^{00}g^{aj}Z_a$ (for which $n(\mathcal{G}_N) = n_*(\mathcal{Z}_N) = 0$), which results in the lack of an $e^{-q\Omega}$ factor on the right-hand side of (9.24b).

Proof of (9.23a): We first note that (9.22a)-(9.22f) show that $\triangle_{(\Gamma)}^{\mu\nu\lambda} \in \mathcal{G}_N$. Consequently, it follows by inspection that all products on the right-hand side of the expression (A.6d) for Θ either contain an element of $\mathcal{G}_N$ as a factor (i.e, $n(\mathcal{G}_N) \geq 1$) or are quadratic in $\mathcal{Z}_N$ (i.e., $n_*(\mathcal{Z}_N) \geq 2 - 1 = 1$). By the Counting Principle estimate (9.33), the estimate (9.23a) thus follows.

Proof of (9.23b): We first multiply each side of equation (A.4d) by $e^{\varkappa\Omega}$ to deduce that

$$e^{\varkappa\Omega}\triangle_{\partial\Phi} = 3\omega[e^{\varkappa\Omega}\partial_t\Phi]\left\{F(\triangle_{(m)}) - F(0)\right\} - [e^{\varkappa\Omega}\partial_t\Phi]\left\{F(\triangle_{(m)}) - F(0)\right\}\Theta \quad (9.45)$$
$$- F(0)[e^{\varkappa\Omega}\partial_t\Phi]\Theta,$$

where

$$F(\triangle_{(m)}) \overset{\text{def}}{=} \left[(1 + 2s) + \triangle_{(m)}\right]^{-1}, \quad (9.46)$$

and $\triangle_{(m)}$ and Θ are defined in (A.6a) and (A.6d). By Corollary B.4, with $v = \triangle_{(m)}$ and $\bar{v} = 0$ in the corollary, and (9.24a), it follows that

$$\left\|F(\triangle_{(m)}) - F(0)\right\|_{H^N} + \|\underline{\partial}\left[F(\triangle_{(m)})\right]\|_{H^{N-1}} \leq C\|\triangle_{(m)}\|_{H^N} \leq Ce^{-q\Omega}\mathbf{S}_N. \quad (9.47)$$

Therefore, by definition, $F(\triangle_{(m)}) - F(0) \in \mathcal{G}_N$ and $F(\triangle_{(m)}) \in \mathcal{H}_N$. Note that $e^{\varkappa\Omega}\partial_t\Phi \in \mathcal{H}_N$ and that (9.23a) implies $\Theta \in \mathcal{G}_N$. Thus, all terms on the right-hand side of (9.45) are products of elements of $\mathcal{G}_N \cup \mathcal{H}_N$ and each product contains an element of $\mathcal{G}_N$ (i.e., $n(\mathcal{G}_N) \geq 1$). By the Counting Principle estimate (9.33), the estimate (9.23b) thus follows.

Proofs of (9.25a)-(9.25e): To prove (9.25b)-(9.25e), we first use (5.17c) to decompose

$$(m^{-1})^{jk} - \frac{1}{2s+1}g^{jk} = -\frac{1}{2s+1}\triangle_{(m)}^{jk} + \left(F(\triangle_{(m)}) - \overbrace{F(0)}^{\frac{1}{2s+1}}\right)\left(g^{jk} - \triangle_{(m)}^{jk}\right), \quad (9.48)$$

where $F(\cdot)$ is defined in (9.46). The estimates (9.1b), (9.1c), (9.24c), and (9.47) show that $g^{jk} \in \mathcal{H}_N$ and $\triangle_{(m)}^{jk}, F(\triangle_{(m)}) - F(0) \in \mathcal{G}_N$. Thus, both terms on the right-hand side of (9.48) contain an element of $\mathcal{G}_N$ (i.e, $n(\mathcal{G}_N) \geq 1$) and have two upstairs spatial indices. Therefore, the Counting Principle estimate (9.33) implies that

$$\left\|(m^{-1})^{jk} - \frac{1}{2s+1}g^{jk}\right\|_{H^N} \leq Ce^{-(2+q)\Omega}\mathbf{S}_N. \quad (9.49)$$

Furthermore, it easily follows from (9.1b), (9.1c), and (9.49) that

$$\|(m^{-1})^{jk}\|_{L^\infty} \leq Ce^{-2\Omega}, \tag{9.50}$$

$$\|\underline{\partial}(m^{-1})^{jk}\|_{H^{N-1}} \leq Ce^{-2\Omega}\mathbf{S}_N. \tag{9.51}$$

We have thus proved (9.25b), (9.25c), and (9.25e). Inequality (9.25d) now follows from (7.4d) and (9.25b).

Inequality (9.25a) can be proved using ideas similar to the ones we used to prove (9.25b)-(9.25e); we omit the details.

Proofs of (9.20a)-(9.20c): Let $\chi_{00} = g_{00} + 1$, $\chi_{0j} = g_{0j}$, $\chi_{jk} = h_{jk}$. We rewrite equations (5.15a)-(5.15c) in the form

$$\partial_t^2 \chi_{\mu\nu} = (g^{00})^{-1}\left(g^{ab}\partial_a\partial_b\chi_{\mu\nu} + 2g^{0a}\partial_t\partial_a\chi_{\mu\nu} - f_{\mu\nu}\right), \tag{9.52}$$

where the $f_{\mu\nu}$ are the terms on the right-hand sides of (5.15a)-(5.15c). By (9.1a) and Corollary B.4, with $v = g^{00}$, $\bar{v} = -1$, and $F(v) = v^{-1}$ in the corollary, it follows that

$$\|(g^{00})^{-1}\|_{L^\infty} \leq C, \tag{9.53}$$

$$\|\underline{\partial}\big[(g^{00})^{-1}\big]\|_{H^{N-1}} \leq Ce^{-q\Omega}\mathbf{S}_N. \tag{9.54}$$

Hence, by definition, $(g^{00})^{-1} \in \mathcal{H}_N \subset \mathcal{H}_{N-1}$; we will use these estimates below.

Let us first consider the cases $\chi_{00} = g_{00} + 1$, $\chi_{0j} = g_{0j}$. In these cases, we see by inspection of the right-hand sides of (5.15a)-(5.15c) that $(g^{00})^{-1}f_{\mu\nu}$ is a sum of products of elements of $\mathcal{H}_{N-1}$ and an element of $\mathcal{G}_{N-1}$. To justify this claim, we are also using (9.19a)-(9.19b), which show that $\triangle_{\mu\nu} \in \mathcal{G}_{N-1}$ in these cases. Furthermore, the same claim is true for the term $(g^{00})^{-1}\left(g^{ab}\partial_a\partial_b\chi_{\mu\nu} + 2g^{0a}\partial_t\partial_a\chi_{\mu\nu}\right)$ from (9.52) in these cases; this claim relies on (9.32b), which shows that $\partial_a\partial_b\chi_{\mu\nu}, \partial_t\partial_a\chi_{\mu\nu} \in \mathcal{G}_{N-1}$ (more precisely, (9.32b) implies that $e^{\Omega}\partial_a\partial_b\chi_{\mu\nu}, e^{\Omega}\partial_t\partial_a\chi_{\mu\nu} \in \mathcal{G}_{N-1}$, but we do not make use of the "extra" factor e^{Ω} in this argument). In total, we have shown the right-hand side of (9.52) is a sum of products of elements of $\mathcal{H}_{N-1}$ and $\mathcal{G}_{N-1}$, and that all products contain an element of $\mathcal{G}_{N-1}$. Since each product has the same net number of spatial indices as $\chi_{\mu\nu}$, the Counting Principle estimate (9.33) (with $n(\mathcal{G}_{N-1}) \geq 1$ for all products) implies that

$$\|\partial_t^2 g_{00}\|_{H^{N-1}} \leq Ce^{-q\Omega}\mathbf{S}_N, \tag{9.55a}$$

$$\|\partial_t^2 g_{0j}\|_{H^{N-1}} \leq Ce^{(1-q)\Omega}\mathbf{S}_N. \tag{9.55b}$$

In the case $\chi_{jk} = h_{jk}$, we can apply a similar strategy. However, in this case, we take into account the adjustment mentioned in Remark 9.8. That is, the counting estimate (9.33) is modified by a factor of $e^{-2\Omega}$:

$$\|\partial_t^2 h_{jk}\|_{H^{N-1}} \leq Ce^{-q\Omega}\mathbf{S}_N \tag{9.55c}$$

(note that this adjustment has also been accounted for in the estimates (9.32a) and (9.19c), which show that $e^{2\Omega}f_{jk} = 3He^{2\Omega}\partial_t h_{jk} + e^{2\Omega}\triangle_{jk} \in \mathcal{G}_{N-1}$).

Now by Proposition B.7, the following commutator estimate holds for $|\vec{\alpha}| \leq N$:

$$\|[\hat{\square}_g, \partial_{\vec{\alpha}}]\chi_{\mu\nu}\|_{L^2} \leq C\|g^{00}+1\|_{H^N}\|\partial_t^2\chi_{\mu\nu}\|_{H^{N-1}} + C\|g^{0a}\|_{H^N}\|\partial_t\chi_{\mu\nu}\|_{H^N} + C\sum_{a,b,c=1}^{3}\|\underline{\partial} g^{ab}\|_{H^{N-1}}\|\partial_c\chi_{\mu\nu}\|_{H^N}. \tag{9.56}$$

Let us first discuss the cases of χ_{00} and χ_{0j}. We define $n_{total} = 0$ for χ_{00} and $n_{total} = 1$ for χ_{0j}. In these cases, our previous estimates have shown that $\underline{\partial} g^{ab} \in \mathcal{H}_{N-1}$ and $g^{00}+1, g^{0a}, \partial_t\chi_{\mu\nu}, \partial_c\chi_{\mu\nu} \in \mathcal{G}_N$, while (9.55a)-(9.55b) show that $\partial_t^2\chi_{\mu\nu} \in \mathcal{G}_{N-1}$. Thus, the $\|g^{00}+1\|_{H^N}\|\partial_t^2\chi_{\mu\nu}\|_{H^{N-1}}$ term is quadratic in norms elements of $\mathcal{G}_N$ and $\mathcal{G}_{N-1}$ and can therefore be bounded from above [via the Counting Principle estimate (9.33)] by $Ce^{-2q\Omega}e^{n_{total}\Omega}\mathbf{S}_N$. The $\|g^{0a}\|_{H^N}\|\partial_t\chi_{\mu\nu}\|_{H^N}$ term is also quadratic in norms of elements of $\mathcal{G}_N$, but it has an *additional upstairs spatial index* compared to the first product, and it can therefore be bounded from above by $Ce^{-(1+2q)\Omega}e^{n_{total}\Omega}\mathbf{S}_N \leq Ce^{-2q\Omega}e^{n_{total}\Omega}\mathbf{S}_N$ as desired. The $\|\underline{\partial} g^{ab}\|_{H^{N-1}}\|\partial_c\chi_{\mu\nu}\|_{H^N}$ term is only linear in $\partial_c\chi_{\mu\nu} \in \mathcal{G}_N$, but like the second product, it has an additional upstairs spatial index compared to the first product. Therefore, it can be bounded from above by $Ce^{-(1+q)\Omega}e^{n_{total}\Omega}\mathbf{S}_N \leq Ce^{-2q\Omega}e^{n_{total}\Omega}\mathbf{S}_N$ as desired.

In the case $\chi_{jk} = h_{jk}$, we can apply a similar strategy. However, in this case, we take into account the adjustment mentioned in Remark 9.8, i.e., all counting estimates are modified by a factor of $e^{-2\Omega}$ (this is possible since we have already shown that $e^{2\Omega}\partial_t^2 h_{jk} \in \mathcal{G}_{N-1}$, and $e^{2\Omega}\partial_t h_{jk}, e^{2\Omega}\partial_i h_{jk} \in \mathcal{G}_N$).

Proof of (9.26): We first note the identity

$$e^{\varkappa\Omega}\partial_a\Phi = [e^{\varkappa\Omega}\partial_t\Phi]Z_a. \tag{9.57}$$

We then use Propositions B.5 and B.7 and the fact that $(m^{-1})^{00} = -1$ to deduce

$$e^{\varkappa\Omega}\|[\hat{\square}_m, \partial_{\vec{\alpha}}]\Phi\|_{L^2} \leq C\sum_{a=1}^{3}\|(m^{-1})^{0a}\|_{H^N}\|[e^{\varkappa\Omega}\partial_t\Phi] - \bar{\Psi}\|_{H^N} + C\|\underline{\partial}(m^{-1})^{ab}\|_{H^{N-1}}\big(|\bar{\Psi}| + \|[e^{\varkappa\Omega}\partial_t\Phi] - \bar{\Psi}\|_{H^N}\big)\|Z_a\|_{H^N}. \tag{9.58}$$

Using (9.25a), (9.25c), and (9.25e), we see that all terms on the right-hand side of (9.58) are products of norms of elements of $\mathcal{H}_{N-1}$ and $\mathcal{H}_N$ and are quadratic in quantities that are controlled by $\mathbf{S}_N$. Since there is one net upstairs spatial index in each product, the right-hand side of (9.58) can be bounded from above [via the Counting Principle estimate (9.33)] by $Ce^{-\Omega}\mathbf{S}_N$; this yields the desired estimate (9.26).

Proof of (9.27a): To prove (9.27a), we first solve for $\partial_t[e^{\varkappa\Omega}\partial_t\Phi]$ using (5.15d) and (9.57):

$$\partial_t[e^{\varkappa\Omega}\partial_t\Phi] = 2(m^{-1})^{0a}\partial_a[e^{\varkappa\Omega}\partial_t\Phi] + (m^{-1})^{ab}\partial_a\big([e^{\varkappa\Omega}\partial_t\Phi]Z_b\big) - e^{\varkappa\Omega}\triangle_{\partial\Phi}. \tag{9.59}$$

Using Propositions B.5 and B.7, we deduce that

$$\begin{aligned} &\|\partial_t[e^{\varkappa\Omega}\partial_t\Phi]\|_{H^{N-1}} \\ &\leq C\sum_{a=1}^{3}\|(m^{-1})^{0a}\|_{H^N}\|[e^{\varkappa\Omega}\partial_t\Phi] - \bar{\Psi}\|_{H^N} \\ &\quad + C\sum_{b=1}^{3}\big(\|(m^{-1})^{ab}\|_{L^\infty} + \|\underline{\partial}(m^{-1})^{ab}\|_{H^{N-1}}\big)\big(|\bar{\Psi}| + \|[e^{\varkappa\Omega}\partial_t\Phi] - \bar{\Psi}\|_{H^N}\big)\|Z_a\|_{H^N} \\ &\quad + C\|e^{\varkappa\Omega}\triangle_{\partial\Phi}\|_{H^{N-1}}. \end{aligned} \tag{9.60}$$

The last term on the right-hand side of (9.60) was bounded in (9.23b). Using (9.25a), (9.25c), and (9.25e), we see that the terms under the Σ's are quadratic/cubic in norms of elements of $\mathcal{H}_N$, feature one net spatial index upstairs, and are at least linear in quantities that are controlled by $\mathbf{S}_N$. Hence, the Counting Principle estimate (9.33) implies that these terms can be bounded from above by $Ce^{-\Omega}\mathbf{S}_N$. The result (9.27a) thus follows.

Proof of (9.27b): Since $Z_j = \frac{e^{\varkappa\Omega}\partial_j\Phi}{e^{\varkappa\Omega}\partial_t\Phi}$, it follows that

$$\partial_t Z_j = \varkappa\omega Z_j - Z_j\frac{\partial_t[e^{\varkappa\Omega}\partial_t\Phi]}{e^{\varkappa\Omega}\partial_t\Phi}. \tag{9.61}$$

Hence, by Corollary B.4 (with $v = e^{\varkappa\Omega}\partial_t\Phi$, $\bar{v} = \bar{\Psi}$, and $F(v) = v^{-1}$) and Proposition B.5, we have

$$\begin{aligned} \|\partial_t Z_j\|_{H^{N-1}} &\leq C\|Z_j\|_{H^{N-1}} \\ &\quad + C\|Z_j\|_{H^{N-1}}\big(|\bar{\Psi}|^{-1} + \|[e^{\varkappa\Omega}\partial_t\Phi] - \bar{\Psi}\|_{H^N}\big)\|\partial_t[e^{\varkappa\Omega}\partial_t\Phi]\|_{H^{N-1}}. \end{aligned} \tag{9.62}$$

Using (9.6b) and (9.27a), we conclude that the right-hand side of of (9.62) is $\leq Ce^{\varkappa\Omega}\mathbf{S}_N$. This is the desired estimate (9.27b).

Proofs of (9.27c)-(9.27e): The proof of (9.27c) is similar to the proof of (9.24a). More precisely, we note that the estimates (9.6b) and (9.27b) show that $Z_j, \partial_t Z_j \in \mathcal{G}_{N-1}$. Thus, when we differentiate (A.6a) with ∂_t, we observe that all products contain a factor belonging to $\mathcal{G}_{N-1}$. Hence, the Counting Principle estimate (9.33) guarantees the availability of the factor $e^{-q\Omega}$ on the right-hand side of (9.27c). The proofs of (9.27d) and (9.27e) are identical.

Proof of (9.27f): We differentiate equation (5.17b) with ∂_t to deduce

$$\partial_t(m^{-1})^{0j} = -F(0)\partial_t\triangle_{(m)}^{0j} + \big(F(0) - F(\triangle_{(m)})\big)\partial_t\triangle_{(m)}^{0j} - \triangle_{(m)}^{0j}\partial_t\big[F(\triangle_{(m)})\big], \quad (9.63)$$

where $F(\triangle_{(m)})$ is defined in (9.46). We next use Corollary B.4, Proposition B.5, (9.24a), and (9.27c) to deduce that

$$\begin{aligned}\|\partial_t\big[F(\triangle_{(m)})\big]\|_{H^{N-1}} &\leq \|F'(0)\partial_t\triangle_{(m)}\|_{H^{N-1}} + \|\big(F'(0) - F'(\triangle_{(m)})\big)\partial_t\triangle_{(m)}\|_{H^{N-1}} \\ &\leq Ce^{-q\Omega}\mathbf{S}_N.\end{aligned} \quad (9.64)$$

The estimates (9.24a), (9.24b), (9.47), (9.27d), and (9.64) show that $\triangle_{(m)}^{0j} \in \mathcal{H}_{N-1}$ and $\triangle_{(m)}$, $F(\triangle_{(m)}) - F(0)$, $\partial_t\triangle_{(m)}^{0j}$, $\partial_t\big[F(\triangle_{(m)})\big] \in \mathcal{G}_{N-1}$. By the Counting Principle estimate (9.33) [with $n(\mathcal{G}_{N-1}) \geq 1$ for all products in (9.63)], the desired estimate (9.27f) thus follows.

Proofs of (9.27g)-(9.27h): From the decomposition (9.48), we deduce that

$$\begin{aligned}\partial_t(m^{-1})^{jk} + 2\omega(m^{-1})^{jk} &= \frac{1}{2s+1}(\partial_t g^{jk} + 2\omega g^{jk}) \\ &\quad - \frac{2\omega}{2s+1}\triangle_{(m)}^{jk} + 2\omega\big(F(\triangle_{(m)}) - F(0)\big)\big(g^{jk} - \triangle_{(m)}^{jk}\big) \\ &\quad - \frac{1}{2s+1}\partial_t\triangle_{(m)}^{jk} + \big(\partial_t\big[F(\triangle_{(m)})\big]\big)\big(g^{jk} - \triangle_{(m)}^{jk}\big) \\ &\quad + \big(F(\triangle_{(m)}) - F(0)\big)\partial_t g^{jk} - \big(F(\triangle_{(m)}) - F(0)\big)\partial_t\triangle_{(m)}^{jk},\end{aligned} \quad (9.65)$$

where $F(\triangle_{(m)})$ is defined in (9.46). The estimates (9.4a), (9.24c), (9.47), (9.27e), and (9.64) show that $\partial_t g^{jk} + 2\omega g^{jk}$, $\triangle_{(m)}^{jk}$, $F(\triangle_{(m)}) - F(0)$, $\partial_t\triangle_{(m)}^{jk}$, $\partial_t\big[F(\triangle_{(m)})\big] \in \mathcal{G}_{N-1}$. Thus, all terms on the right-hand side of (9.65) are products of elements of $\mathcal{G}_{N-1} \cup \mathcal{H}_{N-1}$, and each product has two upstairs spatial indices and contains an element of $\mathcal{G}_{N-1}$. By the Counting Principle estimate (9.33) [with $n(\mathcal{G}_{N-1}) \geq 1$ for all products in (9.65)], we deduce the estimate (9.27g). The estimate (9.27h) follows from (9.25c) and (9.27g).

This concludes our proof of Proposition 9.3.

□

10 The Equivalence of Sobolev and Energy Norms

As is typical in the theory of nonlinear hyperbolic PDEs, our global existence proof is based on showing that the energies of Section 6 remain finite (they happen to be uniformly bounded for $t \geq 0$ in the problem studied here). However, the boundedness of the energies does not in itself preclude the possibility of blow-up; to show that the blow-up scenarios from the conclusions of Theorem 5.4 do *not* occur, we will control appropriate

Sobolev norms of ∂g and $\partial\Phi$. In this short section, we supply the bridge between the energies and the norms. More specifically, in the following proposition, we prove that under suitable bootstrap assumptions, the Sobolev-type norms and energies defined in Section 6 are equivalent.

Proposition 10.1 (**Equivalence of Sobolev norms and energy norms**). *Let* $N \geq 3$ *be an integer and assume that the bootstrap assumption* (7.3b) *holds on the spacetime slab* $[0,T) \times \mathbb{T}^3$ *for some constant* $K_1 \geq 1$. *Let* $(\upgamma, \updelta)$ *be any of the pairs of constants from Definition 6.3, and let* $C_{(\upbeta)}$ *be the corresponding constant from Lemma 6.2. Then there exist constants* $\epsilon''' > 0$ *and* $C > 0$ *depending on* N, K_1, $\upgamma$, *and* $\updelta$, *such that if* $\mathbf{S}_N \leq \epsilon'''$, *then the following inequalities hold on the interval* $[0,T)$ *for the norms and energies defined in* (6.2a)-(6.2f), (6.4), (6.11a)-(6.11d), (6.15b), *and* (6.21)*:*

$$C^{-1}\big\{\|\partial_t v\|_{L^2} + C_{(\upbeta)}\|v\|_{L^2} + e^{-\Omega}\|\underline{\partial} v\|_{L^2}\big\} \leq \mathcal{E}_{(\upgamma,\updelta)}[v,\partial v] \tag{10.1a}$$
$$\leq C\big\{\|\partial_t v\|_{L^2} + C_{(\upbeta)}\|v\|_{L^2} + e^{-\Omega}\|\underline{\partial} v\|_{L^2}\big\},$$
$$C^{-1}E_{g00+1;N} \leq S_{g00+1;N} \leq CE_{g00+1;N}, \tag{10.1b}$$
$$C^{-1}E_{g0*;N} \leq S_{g0*;N} \leq CE_{g0*;N}, \tag{10.1c}$$
$$C^{-1}E_{h**;N} \leq S_{h**;N} \leq CE_{h**;N}, \tag{10.1d}$$
$$C^{-1}E_{g;N} \leq S_{g;N} \leq CE_{g;N}, \tag{10.1e}$$
$$C^{-1}E_{\partial\Phi;N} \leq S_{\partial\Phi;N} \leq CE_{\partial\Phi;N}, \tag{10.1f}$$
$$C^{-1}\mathbf{E}_N \leq \mathbf{S}_N \leq C\mathbf{E}_N. \tag{10.1g}$$

Analogous inequalities hold if we make the replacements
$(E_{g00+1;N}, E_{g0*;N}, E_{h**;N}, E_{g;N}, E_{\partial\Phi;N}) \rightarrow (\underline{E}_{g00+1;N}, \underline{E}_{g0*;N}, \underline{E}_{h**;N}, \underline{E}_{g;N}, \underline{E}_{\partial\Phi;N})$ *and*
$(S_{g00+1;N}, S_{g0*;N}, S_{h**;N}, S_{g;N}, S_{\partial\Phi;N}) \rightarrow (\underline{S}_{g00+1;N}, \underline{S}_{g0*;N}, \underline{S}_{h**;N}, \underline{S}_{g;N}, \underline{S}_{\partial\Phi;N})$.

Proof. The inequalities in (10.1a) follow from the definition (6.4) of $\mathcal{E}_{(\upgamma,\updelta)}[v,\partial v]$, the definition (6.2f) of $\mathbf{S}_N$, (6.6), and (7.4d). The inequalities in (10.1b) - (10.1d) then follow from definitions (6.2a) - (6.2c), definitions (6.11a)-(6.11c), and (10.1a). The inequalities in (10.1f) follow from definitions (6.2e) and (6.15b), and from (9.25d). Finally, (10.1e) and (10.1g) follow trivially from definitions (6.2d), (6.2f), (6.11d), and (6.21), and from the previous inequalities.

□

11 Future-Global Existence

In this section, we use the estimates derived in Sections 9 and 10 to prove two main theorems. In the first theorem, we show that the modified system (5.15a)-(5.15d) has future-global solutions for initial data near that of the FLRW background solution $(\widetilde{g}, \partial\widetilde{\Phi})$ on

$[0,\infty)\times\mathbb{T}^3$, which was derived in Section 4. As described in Section 5.6, if the Einstein constraint equations and the wave coordinate condition $Q_\mu=0$, $(\mu=0,1,2,3)$, are both satisfied along the Cauchy hypersurface $\mathring{\Sigma}=\{x\in\mathcal{M}\mid t=0\}$, then the solution to the modified equations is also a solution to the irrotational Euler-Einstein system. The main idea of the proof is to show that the energies satisfy a system of integral inequalities that forces them (via Gronwall-type estimates) to remain uniformly small on the time interval of existence. Since Proposition 10.1 shows that the norms of the solution must also remain small, the Continuation Principle (Theorem 5.4) can be applied to conclude that the solution exists globally in time. In the second theorem, we provide for convenience a proof of Propositions 3 and 4 of [Rin08], which provide criteria for the initial data that are sufficient to ensure that the spacetime they launch is a future geodesically complete solution to the irrotational Euler-Einstein system.

11.1 Integral inequalities for the energies

In this section, we derive the system of integral inequalities that was mentioned in the previous paragraph.

Proposition 11.1 (Integral inequalities). *Let $N\geq 3$ be an integer and assume that the bootstrap assumption* (7.3b) *holds on the spacetime slab $[0,T)\times\mathbb{T}^3$ for some constant $K_1\geq 1$. Assume further that $0<c_s<\sqrt{1/3}$. Assume that on the same slab, $(g_{\mu\nu},\partial_\mu\Phi)$, $(\mu,\nu=0,1,2,3)$, is a classical solution to the modified system* (5.15a)-(5.15d)*. Then there exist constants $\epsilon''''>0$ and $C>0$, where C depends on N and K_1, such that if $\mathbf{S}_N(t)\leq\epsilon''''$ on $[0,T)$ and $t_1\in[0,T)$, then the following system of integral inequalities is also satisfied for $t\in[t_1,T)$:*

$$\underline{E}^2_{\partial\Phi;N}(t)\leq\underline{E}^2_{\partial\Phi;N}(t_1)+C\int_{t_1}^t e^{-qH\tau}\mathbf{E}^2_N(\tau)\,d\tau, \tag{11.1a}$$

$$\underline{E}^2_{g_{00}+1;N}(t)\leq\underline{E}^2_{g_{00}+1;N}(t_1)+C\int_{t_1}^t e^{-qH\tau}\mathbf{E}^2_N(\tau)\,d\tau, \tag{11.1b}$$

$$\begin{aligned}\underline{E}^2_{g_{0*};N}(t)\leq\underline{E}^2_{g_{0*};N}(t_1)&+\int_{t_1}^t\Big\{-4qH\underline{E}^2_{g_{0*};N}(\tau)+C\underline{E}_{h_{**};N}(\tau)\underline{E}_{g_{0*};N}(\tau)\Big\}\,d\tau\\&+C\int_{t_1}^t e^{-qH\tau}\mathbf{E}_N(\tau)\underline{E}_{g_{0*};N}(\tau)\,d\tau,\end{aligned} \tag{11.1c}$$

$$\underline{E}^2_{h_{**};N}(t)\leq\underline{E}^2_{h_{**};N}(t_1)+C\int_{t_1}^t e^{-qH\tau}\mathbf{E}^2_N(\tau)\,d\tau. \tag{11.1d}$$

Proof. We apply Corollary 6.8, using (9.23b), (9.25a), (9.25e), (9.26), and (9.27g) to estimate the terms on the right-hand side of (6.20), using Proposition 10.1 to bound the norms with corresponding energies, and dropping the term

$$(\varkappa - 1)\omega \sum_{|\vec{\alpha}| \leq N} \int_{\mathbb{T}^3} e^{2\varkappa\Omega} (m^{-1})^{ab} (\partial_a \partial_{\vec{\alpha}} \Phi)(\partial_b \partial_{\vec{\alpha}} \Phi) \, d^3x \tag{11.2}$$

on the right-hand side of (6.20), which by (9.25d) is non-positive for $\varkappa < 1$, thereby arriving at the following inequality:

$$\frac{d}{dt}\Big(\underline{E}^2_{\partial\Phi;N}(t) \Big) \leq C e^{-qHt} \mathbf{E}^2_N(t). \tag{11.3}$$

Integrating (11.3) from t_1 to t gives (11.1a).

To prove (11.1c), we apply Corollary 6.5, using (9.5a), (9.19b), (9.20b), and (9.21b) to estimate the terms on the right-hand side of (6.14b), using Proposition 10.1 to bound the norms with corresponding energies, and using definition (8.5) to deduce that $2(q-1) - \eta_{0*} \leq -4q$, thereby arriving at the following inequality:

$$\frac{d}{dt}\Big(\underline{E}^2_{g_{0*};N}(t) \Big) \leq -4qH \underline{E}^2_{g_{0*};N}(t) + C \underline{E}_{h_{**};N} \underline{E}_{g_{0*};N}(t) + C e^{-qHt} \mathbf{E}_N(t) \underline{E}_{g_{0*};N}(t). \tag{11.4}$$

Inequality (11.1c) now follows by integrating from t_1 to t. Inequalities (11.1b) and (11.1d) can be proved similarly; we omit the details. □

Remark 11.2. The term $C\underline{E}_{h_{**};N}\underline{E}_{g_{0*};N}$ in inequality (11.1c) arises from the $C\underline{S}_{g_{0*};N} \sum_{j=1}^{3} e^{(q-1)\Omega} \| g^{ab}\Gamma_{ajb} \|_{H^N}$ term on the right-hand side of (6.14b). This term is dangerous in the sense that it does not contain an exponentially decaying factor, and looks like it could lead to the growth of $\underline{E}_{g_{0*};N}$. However, as we shall see in the proof of Theorem 11.5, there is a partial decoupling in the integral inequalities in the sense that the $C\underline{E}_{h_{**};N}$ factor in the dangerous term can be effectively controlled from inequality (11.1d) alone. We will then insert this information into inequality (11.1c), and also make use of the negative term $-4qH\underline{E}^2_{g_{0*};N}$ to obtain a bound for $\underline{E}_{g_{0*};N}$.

For completeness, we state the following version of Gronwall's inequality; we omit the simple proof. We will use it in Section 11.2.

Lemma 11.3 (**Basic Gronwall estimate**). *Let $b(t) \geq 0$ be a continuous function on the interval $[t_1, T]$, and let $B(t)$ be an anti-derivative of $b(t)$. Suppose that $A \geq 0$ and that $y(t) \geq 0$ is a continuous function satisfying the inequality*

$$y(t) \leq A + \int_{t_1}^{t} b(\tau) y(\tau) \, d\tau \tag{11.5}$$

for $t \in [t_1, T]$. Then for $t \in [t_1, T]$, we have

$$y(t) \leq A \exp\Big[B(t) - B(t_1) \Big]. \tag{11.6}$$

In addition, in our proof of Theorem 11.5, we will apply the following integral estimate to inequality (11.1c) in order to estimate the energy $E_{g_{0*};N}(t)$.

Lemma 11.4 (An integral estimate). *Let $b(t) > 0$ be a continuous **non-decreasing** function on the interval $[0,T]$, and let $\epsilon > 0$. Suppose that for each $t_1 \in [0,T]$, $y(t) \geq 0$ is a continuous function satisfying the inequality*

$$y^2(t) \leq y^2(t_1) + \int_{\tau=t_1}^{t} -b(\tau)y^2(\tau) + \epsilon y(\tau)\, d\tau \tag{11.7}$$

for $t \in [t_1,T]$. Then for any $t_1, t \in [0,T]$ with $t_1 \leq t$, we have that

$$y(t) \leq y(t_1) + \frac{\epsilon}{b(t_1)}. \tag{11.8}$$

Proof. Let $\mathcal{C}$ be the "highest" curve in the (t,y) plane on which the integrand in (11.7) vanishes; i.e. $\mathcal{C} = \{(t,y)|y = \frac{\epsilon}{b(t)}\}$. Then by (11.7), above $\mathcal{C}$ (i.e. for larger y values), $y(t)$ is *strictly* decreasing. Let $y(t)$ achieve its maximum at $t_{max} \in [t_1,T]$. We separate the proof of (11.8) into two cases. Case i) assume that $t_{max} = t_1$. Then $y(t) \leq y(t_{max}) = y(t_1)$ for $t \in [t_1,T]$, which implies (11.8). Case ii) assume that $t_{max} \in (t_1,T]$. We claim that $y(t_{max}) \leq \frac{\epsilon}{b(t_{max})}$. For otherwise, the point $\big(t_{max}, y(t_{max})\big)$ lies above $\mathcal{C}$. Since $y(t)$ is then strictly decreasing in a neighborhood of t_{max}, it follows that there are times $t_* < t_{max}$, with $t_* \in (t_1,T)$, at which $y(t_*) > y(t_{max})$. This contradicts the definition of t_{max}. Using also the fact that $\frac{1}{b(t)}$ is non-increasing, we deduce that $y(t) \leq y(t_{max}) \leq \frac{\epsilon}{b(t_{max})} \leq \frac{\epsilon}{b(t_1)}$; this concludes the proof of (11.8). □

11.2 The future-global existence theorem

In this section, we state and prove our main theorem, which provides global existence criteria for the modified system (5.15a)-(5.15d).

Theorem 11.5 (Future-global existence). *Let $N \geq 3$ be an integer, and assume that $0 < c_s < \sqrt{1/3}$, where c_s denotes the speed of sound. Let $(\mathring{g}_{\mu\nu}, \mathring{K}_{\mu\nu}, \mathring{\Psi}, \partial_j\mathring{\Phi} = \underline{\mathring{\beta}}_j)$, $(j = 1,2,3; \mu,\nu = 0,1,2,3)$, be initial data (see Remark 1.1) on the manifold $\mathbb{T}^3$ (not necessarily satisfying the wave coordinate condition or the Einstein constraints but verifying $d\underline{\mathring{\beta}} = 0$) for the modified system* (5.15a)-(5.15d)*, and let $\mathbf{S}_N \overset{def}{=} S_{g_{00}+1;N} + S_{g_{0*};N} + S_{h_{**};N} + S_{\partial\Phi;N}$ be the norm defined in* (6.2f)*. Assume that there is a positive constant $K_1 \geq 2$ such that*

$$\frac{2}{K_1}\delta_{ab}X^aX^b \leq \mathring{g}_{ab}X^aX^b \leq \frac{K_1}{2}\delta_{ab}X^aX^b, \qquad \forall (X^1,X^2,X^3) \in \mathbb{R}^3. \tag{11.9}$$

Then there exist a small constant ϵ_0, with $0 < \epsilon_0 < 1$, and a large constant $C_* > 0$, both depending on K_1 and N, such that if $\epsilon \leq \epsilon_0$ and $\mathbf{S}_N(0) \leq C_*^{-1}\epsilon$, then the classical *solution $(g_{\mu\nu}, \partial_\mu \Phi)$ provided by Theorem 5.2 exists on $[0,\infty) \times \mathbb{T}^3$, and*

$$\mathbf{S}_N(t) \leq \epsilon \tag{11.10}$$

holds for all $t \geq 0$. Furthermore, the time T_{max} from the hypotheses of Theorem 5.4 is infinite.

Proof. See Remark 9.2 for some conventions that we abide by during this proof. To prove future-global existence, we will use a standard bootstrap argument to prove that $\mathbf{S}_N(t)$ remains uniformly small for all time and that the 3×3 matrix g_{jk} remains positive definite; the theorem will then follow from the continuation principle (Theorem 5.4).

To begin, we use Theorem 5.2, which implies that if $\mathbf{S}_N(0) \leq \frac{1}{2}\epsilon$ and $\epsilon > 0$ is sufficiently small, then there is a maximal time $T > 0$ such that a unique local solution $(g_{\mu\nu}, \partial_\mu \Phi)$ exists on the slab $[0,T) \times \mathbb{T}^3$ and such that the following bootstrap assumptions hold (we are using the continuity of $\mathbf{S}_N(t)$ and Sobolev embedding):

$$\mathbf{S}_N(t) \leq \epsilon, \tag{11.11}$$

$$K_1^{-1}\delta_{ab}X^aX^b \leq e^{-2\Omega}g_{ab}X^aX^b \leq K_1\delta_{ab}X^aX^b, \qquad \forall (X^1,X^2,X^3) \in \mathbb{R}^3. \tag{11.12}$$

Note that by Remark 8.2, (11.11) implies that the rough bootstrap assumptions (7.3a) and (7.3c) are satisfied with room to spare (for $\eta \leq \eta_{min}$) if ϵ is sufficiently small. Furthermore, we note that (11.12) is precisely the rough bootstrap assumption (7.3b). By "maximal," we mean that

$$T \overset{\text{def}}{=} \sup\big\{t \geq 0 \mid \text{The solution exists on } [0,t) \times \mathbb{T}^3, \text{ and } (11.11) - (11.12) \text{ hold}\big\}. \tag{11.13}$$

We may assume that $T < \infty$, since otherwise the theorem follows. The remainder of this proof is dedicated to reaching a contradiction if ϵ is small enough and C_* is large enough. For the remainder of the proof, we assume that ϵ is small enough so that Propositions 9.1, 9.3, 10.1, and 11.1 are valid on $[0,T)$. We will make repeated use of Proposition 10.1 throughout this proof without explicitly mentioning it each time. We remark that $q > 0$ [see definition (8.5)] *is essential for many of the estimates we derive*; this is where we use the assumption $0 < c_s < \sqrt{1/3}$.

As a first step toward deriving a contradiction, we will show that inequality (11.12) can be improved. By (11.11) and the definition of $\mathbf{S}_N$, we have that

$$\|\partial_t(e^{-2\Omega}g_{jk})\|_{L^\infty} = \|\partial_t h_{jk}\|_{L^\infty} \leq C\epsilon e^{-qHt}. \tag{11.14}$$

Integrating $\partial_t(e^{-2\Omega}g_{jk})$ in time from $t=0$ and using inequality (11.14), we deduce

$$\|e^{-2\Omega}g_{jk}(t,\cdot) - \mathring{g}_{jk}(\cdot)\|_{L^\infty} \leq C\epsilon. \tag{11.15}$$

By (11.9) and (11.15), we conclude that if ϵ is small enough, then on $[0,T) \times \mathbb{T}^3$, the following improvement of (11.12) holds:

$$\frac{3}{2K_1}\delta_{ab}X^aX^b \leq e^{-2\Omega}g_{ab}X^aX^b \leq \frac{2K_1}{3}\delta_{ab}X^aX^b, \quad \forall (X^1,X^2,X^3) \in \mathbb{R}^3. \tag{11.16}$$

To complete our proof of the theorem, we will show that if ϵ is small enough and C_* is large enough, then the bootstrap assumption (11.11) can be improved; the primary tool for deducing an improvement is Proposition 11.1. For the remainder of the proof, we use the notation $\mathring{\epsilon} \stackrel{\text{def}}{=} \mathbf{S}_N(0)$, and we assume $\mathring{\epsilon} \leq \frac{\epsilon}{C_*}$, where the constant C_* will be chosen near the end of the proof. To begin our proof of an improvement of (11.11), we use a very non-optimal application of Proposition 11.1 with $t_1 = 0$, deducing that on $[0,T)$, we have that

$$\mathbf{E}_N^2(t) \leq \mathbf{E}_N^2(0) + \int_{\tau=0}^{t} c\mathbf{E}_N^2(\tau)\, d\tau. \tag{11.17}$$

Applying Lemma 11.3 (Gronwall's inequality) to (11.17), using $\mathbf{S}_N(0) = \mathring{\epsilon}$, and using Proposition 10.1, we conclude that the following preliminary "Cauchy stability" estimates hold on $[0,T)$:

$$\mathbf{S}_N(t) \leq C\mathring{\epsilon}e^{ct}, \tag{11.18}$$

$$\mathbf{E}_N(t) \leq C\mathring{\epsilon}e^{ct}. \tag{11.19}$$

Remark 11.6. By modifying the argument in the last paragraph of this proof, Theorem 5.4 and inequality (11.18) can be used to deduce that T is at least of order $c^{-1}\ln\left(\frac{C_*}{C}\right)$ if ϵ is sufficiently small and C_* is sufficiently large.

We now fix a time $t_1 \in [0,T)$; t_1 will be adjusted at the end of the proof. Roughly speaking, it will play the role of a time that is large enough so that the exponentially damped terms on the right-hand sides of the inequalities of Proposition 11.1 are of size $\ll \epsilon$. To estimate $E_{\partial\Phi;N}(t)$ for $t \in [t_1,T)$, we simply use (11.11) and (11.19) to estimate the two terms on the right-hand side of (11.1a):

$$\underline{E}_{\partial\Phi;N}^2(t) \leq \underline{E}_{\partial\Phi;N}^2(t_1) + C\epsilon^2\int_{\tau=t_1}^{t} e^{-qH\tau}\, d\tau \leq C\{\mathring{\epsilon}e^{ct_1} + \epsilon e^{-qHt_1/2}\}^2. \tag{11.20}$$

Using (11.19) again to estimate $\underline{E}_{\partial\Phi;N}(t)$ on $[0,t_1]$, and then taking the sup over the interval $[0,t)$, we thus conclude that the following inequality is valid for $t \in [0,T)$:

$$E_{\partial\Phi;N}(t) \leq C\{\mathring{\epsilon}e^{ct_1} + \epsilon e^{-qHt_1/2}\}. \tag{11.21}$$

Applying similar reasoning to inequalities (11.1b) and (11.1d), we also have the following inequalities on $[0,T)$:

$$E_{g_{00}+1;N}(t) \leq C\{\mathring{\epsilon}e^{ct_1} + \epsilon e^{-qHt_1/2}\}, \tag{11.22}$$

$$E_{h_{**};N}(t) \leq C\{\mathring{\epsilon}e^{ct_1} + \epsilon e^{-qHt_1/2}\}. \tag{11.23}$$

To estimate $E_{g_{0*};N}(t)$, we use (11.1c), the bootstrap assumption (11.11), and (11.23) to arrive at the following inequality valid for $t \in [t_1, T)$:

$$\underline{E}^2_{g_{0*};N}(t) \leq \underline{E}^2_{g_{0*};N}(t_1) + \int_{\tau=t_1}^{t} -4qH\underline{E}^2_{g_{0*};N}(\tau) + C\{\mathring{\epsilon}e^{ct_1} + \epsilon e^{-qHt_1/2}\}\underline{E}_{g_{0*};N}(\tau)\,d\tau. \tag{11.24}$$

Applying Lemma 11.4 to (11.24), with $y(\tau) = \underline{E}_{g_{0*};N}(\tau)$ and $b(\tau) = 4qH$ in the lemma, and also using (11.19), we conclude that the following inequality holds on $[0, T)$:

$$E_{g_{0*};N}(t) \leq C\{\mathring{\epsilon}e^{ct_1} + \epsilon e^{-qHt_1/2}\}. \tag{11.25}$$

Adding (11.21), (11.22), (11.23), and (11.25), referring to definitions (6.2f) and (6.21), and using Proposition 10.1, we deduce that the following inequality holds on $[0, T)$:

$$\mathbf{S}_N(t) \leq C\{\mathring{\epsilon}e^{ct_1} + \epsilon e^{-qHt_1/2}\} \tag{11.26}$$

$$\leq \frac{C}{C_*}\epsilon e^{ct_1} + C\epsilon e^{-qHt_1/2}. \tag{11.27}$$

We now choose t_1 sufficiently large such that $Ce^{-qHt_1/2} \leq \frac{1}{4}$, and then C_* sufficiently large such that $\frac{C}{C_*}e^{ct_1} \leq \frac{1}{4}$. Consequently, the following inequality holds on $[0, T)$:

$$\mathbf{S}_N(t) \leq \frac{1}{2}\epsilon. \tag{11.28}$$

We remark that in order to guarantee that the solution exists long enough (i.e. that T is large enough) so that $t_1 \in [0, T)$, we may have to further shrink ϵ and enlarge C_*; see Remark 11.6.

We now claim that $T = \infty$. We argue by contradiction, assuming that $T < \infty$. Then by combining (11.16) and (11.28), it follows that none of the four existence-breakdown scenarios stated in the conclusions of Theorem 5.4 occur: **1)** is ruled out by the Sobolev embedding result $\|g_{00} + 1\|_{L^\infty} \leq Ce^{-q\Omega}\mathbf{S}_N$; **2)** is ruled out by (11.16); **3)** is ruled out by (9.7); and **4)** is ruled out by the Sobolev embedding results $\|g_{00} + 1\|_{C_b^2} + \|\partial_t g_{00}\|_{C_b^1} \leq Ce^{-q\Omega}\mathbf{S}_N$, $\sum_{j=1}^3 \left(\|g_{0j}\|_{C_b^2} + \|\partial_t g_{0j}\|_{C_b^1}\right) \leq Ce^{(1-q)\Omega}\mathbf{S}_N$, $\sum_{j,k=1}^3 \|\underline{\partial} g_{jk}\|_{C_b^1} \leq Ce^{2\Omega}\mathbf{S}_N$, $\|e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi}\|_{L^\infty} \leq C\mathbf{S}_N$, and $\sum_{j=1}^3 \|\partial_j\Phi\|_{L^\infty} \leq Ce^{(1-\varkappa)\Omega}\mathbf{S}_N$, together with inequalities (11.16) and (9.2b). By the continuity of $\mathbf{S}_N(t)$, it thus follows from Theorem 5.4 that there exists a $\delta > 0$ such that the solution can be extended to the interval $[0, T + \delta)$ on which the estimates (11.11)-(11.12) hold. This contradicts the maximality of T; we therefore conclude that $T = \infty$.

□

11.3 On the breakdown of the proof for $c_s \geq \sqrt{1/3}$ (i.e. $s \leq 1, \varkappa \geq 1$)

In this short section, we give a brief example of how our proof breaks down when $c_s \geq \sqrt{1/3}$. If $\varkappa \geq 1$, we cannot use our previous reasoning to bound the following term, which is the last term on the right-hand side of (6.17):

$$\frac{1}{2}\sum_{|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3} e^{2\varkappa\Omega}\big(\partial_t(m^{-1})^{ab} + 2\varkappa\omega(m^{-1})^{ab}\big)(\partial_a\partial_{\vec{\alpha}}\Phi)(\partial_b\partial_{\vec{\alpha}}\Phi)\,d^3x. \tag{11.29}$$

Previously, we had split this term into two pieces, one of which is

$$(\varkappa - 1)\omega\sum_{|\vec{\alpha}|\leq N}\int_{\mathbb{T}^3}(m^{-1})^{ab}(\partial_a\partial_{\vec{\alpha}}\Phi)(\partial_b\partial_{\vec{\alpha}}\Phi)\,d^3x, \tag{11.30}$$

which, as is explained in our proof of Proposition 11.1, could be discarded from the energy inequality (11.1a) because it is non-positive when $\varkappa < 1$. Obviously, we can no longer discard this term when $\varkappa > 1$. Furthermore, even in the case $\varkappa = 1$, inequality (9.6b) is weakened to

$$\|\partial_t(m^{-1})^{jk} + 2\omega(m^{-1})^{jk}\|_{L^\infty} \leq C\{e^{-2\Omega}\mathbf{S}_N^2 + \text{Positive terms}\}. \tag{11.31}$$

Ultimately, this fact can be traced to the fact that the L^∞ estimate for Z_j from (9.6b) must be replaced with $\|Z_j\|_{L^\infty} \leq Ce^{\Omega}\mathbf{S}_N$.

With the help of Proposition 10.1, it can be shown that the net result in both the case $\varkappa = 1$ and the case $\varkappa > 1$ is that the term (11.29) leads to (as in our proof of Proposition 11.1) an integral inequality of the form

$$\underline{E}^2_{\partial\Phi;N}(t) \leq \underline{E}^2_{\partial\Phi;N}(t_1) + \text{positive terms} + C\epsilon\int_{\tau=t_1}^{t}\mathbf{E}_N^2(\tau)\,d\tau. \tag{11.32}$$

Inequality (11.32) allows for the possible growth of $\underline{E}_{\partial\Phi;N}(t)$; i.e., unlike in the cases $0 < c_s < \sqrt{1/3}$, there is no $e^{-qH\tau}$ factor with $q > 0$ in the integrand. Therefore, this inequality does not provide a means of improving the bootstrap assumption $\mathbf{S}_N(t) \leq \epsilon$.

11.4 Future causal geodesic completeness

In this section, we prove our second main theorem, which provides criteria for the initial data under which the global solutions provided by Theorem 11.5 are future causally geodesically complete. The theorem and its proof are based on Propositions 3 and 4 of [Rin08].

Theorem 11.7 (Future causal geodesic completeness). *Let $N \geq 3$ be an integer, and assume that $0 < c_s < \sqrt{1/3}$, where c_s denotes the speed of sound. Let $([0,\infty) \times \mathbb{T}^3, \widetilde{g}_{\mu\nu}, \partial_\mu\widetilde{\Phi})$, $(\mu,\nu = 0,1,2,3)$, be one of the FLRW background solutions derived in Section 4. Let $(\mathring{g}_{\mu\nu}, \mathring{K}_{\mu\nu}, \mathring{\Psi}, \partial_j\mathring{\Phi} = \underline{\mathring{\beta}}_j)$, $(j = 1,2,3; \mu,\nu = 0,1,2,3)$, be initial data for the modified irrotational Euler-Einstein system* (5.15a)-(5.15d) *on the manifold $\mathbb{T}^3$ (see Remark 1.1) that are constructed from initial data $(\mathbb{T}^3, \underline{\mathring{g}}_{jk}, \underline{\mathring{K}}_{jk}, \mathring{\Psi}, \partial_j\mathring{\Phi} = \underline{\mathring{\beta}}_j)$,*

$(j, k = 1, 2, 3)$, *for the unmodified system* (3.47a)-(3.47b) *[which by definition satisfy the constraints* (3.32a)-(3.32b) *and* $d\underline{\mathring{\beta}} = 0$*] as described in Section 3.2.2. Let* $\mathbf{S}_N$ *be the norm defined in* (6.2f)*, and assume that the data for the modified system are near the FLRW data in the sense that* $\mathbf{S}_N(0) \leq C_*^{-1}\epsilon_0$*, where* ϵ_0 *and* C_* *are the constants from the conclusion of Theorem 11.5. Also assume that the perturbed data satisfy the inequality* (11.9)*, so that all of the hypotheses of Theorem 11.5 are satisfied. Let* $([0, \infty) \times \mathbb{T}^3, g_{\mu\nu}, \partial_\mu \Phi)$ *be the future-global solution to both the modified system and the unmodified system guaranteed by Theorem 11.5 and Proposition 5.6, and let* $\gamma(s)$ *be a future-directed causal curve in* $\mathcal{M}$ *with domain* $s \in [s_0, s_{max})$ *such that* $\gamma^0(s_0) = 0$*. Let* γ^μ *denote the coordinates of this curve in the universal covering space of the spacetime (i.e.* $[0, \infty) \times \mathbb{R}^3$*). Then there exist constants* $C > 0$ *and* ϵ_1*, where* $0 < \epsilon_1 \leq \epsilon_0$*, such that if* $\mathbf{S}_N(0) \leq C_*^{-1}\epsilon$ *and* $\epsilon < \epsilon_1$*, then* $\dot{\gamma}^0(s) > 0$ *for* $s \in [s_0, s_{max})$ *and furthermore, the length of the spatial part of the curve as measured by the metric* $\mathring{g}_{jk} = \underline{\mathring{g}}_{jk}$ *satisfies*

$$\int_{s_0}^{s_{max}} \sqrt{\underline{\mathring{g}}_{ab}(\pi \circ \gamma)\dot{\gamma}^a \dot{\gamma}^b}\, ds \leq C, \tag{11.33}$$

where π *denotes projection onto spatial indices; i.e.,* $\pi^j \circ \gamma \stackrel{def}{=} \gamma^j$*. The constants* C *and* ϵ_1 ***can be chosen to be independent of*** γ*. Additionally, if* γ *is future-inextendible, then* $\gamma^0(s) \uparrow \infty$ *as* $s \uparrow s_{max}$*.*

Finally, the spacetime-with-boundary $([0, \infty) \times \mathbb{T}^3, g)$ *is future causally geodesically complete. In particular, if* $(\mathcal{M}, g', \partial\Phi')$ *denotes the maximal globally hyperbolic development of the data, then relative to our wave coordinate system, the portion of* $\mathcal{M}$ *lying to the future of* $\{0\} \times \mathbb{T}^3$ *[i.e.* $\mathcal{D}^+(\{0\} \times \mathbb{T}^3)$*] is exactly* $[0, \infty) \times \mathbb{T}^3$*.*

Remark 11.8. It is possible to restate the stability criteria in terms of quantities that manifestly depend only on the closeness of the initial data $(\mathbb{T}^3, \underline{\mathring{g}}_{jk}, \underline{\mathring{K}}_{jk}, \mathring{\Psi}, \underline{\mathring{\beta}}_j)$ for the unmodified system to the corresponding data for the FLRW background solution $(\widetilde{g}, \partial\widetilde{\Phi})$. For example, a sufficient condition for future-global existence and future causal geodesic completeness would be

$$\sum_{j,k=1}^{3} \|\underline{\mathring{g}}_{jk} - a^2(0)\delta_{jk}\|_{H^{N+1}} + \sum_{j,k=1}^{3} \|\underline{\mathring{K}}_{jk} - \omega(0)a^2(0)\delta_{jk}\|_{H^N} \tag{11.34}$$
$$+ \|a^{\varkappa}(0)\mathring{\Psi} - \bar{\Psi}\|_{H^N} + \sum_{j=1}^{3} \|\underline{\mathring{\beta}}_j\|_{H^N} \leq \epsilon,$$

where ϵ is sufficiently small. This is because the condition (11.34) implies that $\mathbf{S}_N(0) \leq C\epsilon$ and furthermore (by Sobolev embedding) that a condition of the form (11.9) holds (i.e. that the hypotheses of Theorem 11.7 hold).

To see that $\mathbf{S}_N(0) \leq C\epsilon$ follows from (11.34), we first use the definition (6.2f) of $\mathbf{S}_N(0)$, the construction of the modified data described in Section 5.3, and the triangle inequality to deduce that

$$\mathbf{S}_N(0) \leq 2\|3\omega(0) - \mathring{\underline{g}}^{ab}\mathring{\underline{K}}_{ab}\|_{H^N} + \sum_{j,k=1}^{3} \|\underline{\partial}\mathring{\underline{g}}_{jk}\|_{H^N} + 2a^{-2}(0)\sum_{j,k=1}^{3}\|\omega(0)\mathring{\underline{g}}_{jk} - \mathring{\underline{K}}_{jk}\|_{H^N} \tag{11.35}$$

$$\begin{aligned}
&+ \sum_{j=1}^{3}\|\mathring{\underline{g}}^{ab}\big(\partial_a\mathring{\underline{g}}_{bj} - \frac{1}{2}\partial_j\mathring{\underline{g}}_{ab}\big)\|_{H^N} + \|a^{\varkappa}(0)\mathring{\Psi} - \bar{\Psi}\|_{H^N} + \sum_{j=1}^{3}\|\mathring{\underline{\beta}}_j\|_{H^N} \\
&\leq 2\|\big(\mathring{\underline{g}}^{ab} - a^{-2}(0)\delta^{ab}\big)\mathring{\underline{K}}_{ab}\|_{H^N} + 2a^{-2}(0)\|\delta^{ab}\big(\mathring{\underline{K}}_{ab} - \omega(0)a^2(0)\delta_{ab}\big)\|_{H^N} \\
&+ a^{-2}(0)\sum_{j,k=1}^{3}\|\underline{\partial}\mathring{\underline{g}}_{jk}\|_{H^N} + 2\omega(0)a^{-2}(0)\sum_{j,k=1}^{3}\|\mathring{\underline{g}}_{jk} - a^2(0)\delta_{jk}\|_{H^N} \\
&+ 2a^{-2}(0)\sum_{j,k=1}^{3}\|\omega(0)a^2(0)\delta_{jk} - \mathring{\underline{K}}_{jk}\|_{H^N} + \sum_{j=1}^{3}\|\mathring{\underline{g}}^{ab}\big(\partial_a\mathring{\underline{g}}_{bj} - \frac{1}{2}\partial_j\mathring{\underline{g}}_{ab}\big)\|_{H^N} \\
&+ \|a^{\varkappa}(0)\mathring{\Psi} - \bar{\Psi}\|_{H^N} + \sum_{j=1}^{3}\|\mathring{\underline{\beta}}_j\|_{H^N}.
\end{aligned}$$

Now using Corollary B.4, Proposition B.5, and Sobolev embedding, we deduce that if (11.34) holds and if ϵ is sufficiently small, then the right-hand side of (11.35) is $\leq C\epsilon$.

Proof of Theorem 11.7. See Remark 9.2 for some conventions that we use throughout this proof. The conclusions of Theorem 11.5 imply that $\mathbf{S}_N(t) \leq \epsilon$ holds for all $t \geq 0$; we will make repeated use of this estimate throughout the proof.

Let $\gamma(s)$ be a future-directed causal curve in $\mathcal{M}$ with domain $s \in [s_0, s_{max})$ such that $\gamma^0(s_0) = 0$. We first show that $\dot{\gamma}^0(s) > 0$ for $s \in [s_0, s_{max})$, where $\dot{\gamma}^\mu(s) \stackrel{\text{def}}{=} \frac{d}{ds}\gamma^\mu(s)$. To this end, we note that since γ is causal and future-directed, and since ∂_t is future-directed and timelike, we have that

$$g_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta \leq 0, \tag{11.36}$$
$$g_{0\alpha}\dot{\gamma}^\alpha < 0. \tag{11.37}$$

Our first goal is to prove that if ϵ is small enough, then the following estimates hold:

$$g_{ab}\dot{\gamma}^a\dot{\gamma}^b \leq C(\dot{\gamma}^0)^2, \tag{11.38}$$
$$\delta_{ab}\dot{\gamma}^a\dot{\gamma}^b \leq Ce^{-2\Omega}(\dot{\gamma}^0)^2. \tag{11.39}$$

To this end, we use the Cauchy-Schwarz inequality and the estimate (11.16) to deduce

$$\begin{aligned}
|2g_{0a}\dot{\gamma}^0\dot{\gamma}^a| &\leq C\epsilon e^{(1-q)\Omega}|\dot{\gamma}^0|\sum_{a=1}^{3}|\dot{\gamma}^a| \qquad (11.40)\\
&\leq C\epsilon(\dot{\gamma}^0)^2 + C\epsilon e^{-2q\Omega}g_{ab}\dot{\gamma}^a\dot{\gamma}^b.
\end{aligned}$$

Combining (11.36) and (11.40), we have that

$$g_{ab}\dot{\gamma}^a\dot{\gamma}^b \leq -g_{00}(\dot{\gamma}^0)^2 + |2g_{0a}\dot{\gamma}^0\dot{\gamma}^a| \leq (1+C\epsilon)(\dot{\gamma}^0)^2 + C\epsilon e^{-2q\Omega}g_{ab}\dot{\gamma}^a\dot{\gamma}^b. \tag{11.41}$$

From (11.41), it follows that if ϵ is small enough, then there exists a constant $C > 0$ such that (11.38) holds. Inequality (11.39) then follows from the estimate (11.16) and (11.38).

We now claim that $|\dot{\gamma}^0| > 0$. For if $|\dot{\gamma}^0| = 0$, then inequality (11.39) shows that $\sum_{\alpha=0}^{3}|\dot{\gamma}^\alpha| = 0$, which contradicts (11.37). We may therefore divide each side of (11.40) by $|\dot{\gamma}^0|$ and use (11.38) to arrive at the following inequality:

$$|g_{0a}\dot{\gamma}^a| \leq C\epsilon|\dot{\gamma}^0|. \tag{11.42}$$

Using (11.42) and the estimate $|g_{00}| \geq 1 - C\epsilon$, we conclude that if ϵ is sufficiently small, then

$$\text{sgn}(g_{0\alpha}\dot{\gamma}^\alpha) = \text{sgn}(g_{00}\dot{\gamma}^0), \tag{11.43}$$

where $\text{sgn}(y) = 1$ if $y > 0$, and $\text{sgn}(y) = -1$ if $y < 0$. Since (11.37) implies that the left-hand side of (11.43) is negative, and since $g_{00} < 0$, we conclude that $\dot{\gamma}^0 > 0$.

We now show the estimate (11.33). First, from the assumption (11.9), the fact that $\underline{\mathring{g}}_{jk} = \mathring{g}_{jk}$, and the estimate (11.16), it follows that

$$\underline{\mathring{g}}_{ab}\dot{\gamma}^a\dot{\gamma}^b \leq \frac{K_1}{2}\delta_{ab}\dot{\gamma}^a\dot{\gamma}^b \leq Ce^{-2\Omega}g_{ab}\dot{\gamma}^a\dot{\gamma}^b. \tag{11.44}$$

Integrating the square root of each side of (11.44) from s_0 to s_{max}, recalling that $e^{-\Omega(\gamma^0(s))} \leq Ce^{-H\gamma^0(s)}$, using (11.38), using $\dot{\gamma}^0(s) > 0$, and using $\gamma^0(s_0) = 0$, we have that

$$\begin{aligned}\int_{s_0}^{s_{max}} \sqrt{\underline{\mathring{g}}_{ab}(\pi\circ\gamma)\dot{\gamma}^a\dot{\gamma}^b}\,ds &\leq \int_{s_0}^{s_{max}} Ce^{-H\gamma^0(s)}\dot{\gamma}^0(s)\,ds \\ &= -\frac{C}{H}\int_{s_0}^{s_{max}}\left(\frac{d}{ds}e^{-H\gamma^0(s)}\right)ds \leq \frac{C}{H},\end{aligned} \tag{11.45}$$

which proves (11.33).

We now show that the additional assumption that γ is future-inextendible necessarily implies that $\gamma^0(s) \uparrow \infty$ as $s \uparrow s_{max}$. Since $\dot{\gamma}^0 > 0$, it follows that either $\gamma^0(s)$ converges to infinity as $s \uparrow s_{max}$, which is the desired result, or that $\gamma^0(s)$ converges to a finite number. In the latter case, by (11.39), we also conclude that the $\gamma^j(s)$ converge to finite numbers as $s \uparrow s_{max}$. Thus, in this case, the curve γ can be extended towards the future, which contradicts the definition of future-inextendibility.

To show that $([0,\infty)\times\mathbb{T}^3, g_{\mu\nu})$ is future causally geodesically complete, we consider a future-directed causal geodesic γ. We recall that the geodesic equations (for a geodesic

γ parameterized by affine parameter s) are $\ddot{\gamma}^\mu(s) + \Gamma_{\alpha\ \beta}^{\ \mu}\dot{\gamma}^\alpha(s)\dot{\gamma}^\beta(s) = 0$, which in the case of $\mu = 0$ reads

$$\ddot{\gamma}^0 + \Gamma_{\alpha\ \beta}^{\ 0}\dot{\gamma}^\alpha\dot{\gamma}^\beta = 0. \tag{11.46}$$

We assume that $[s_0, s_{max})$ is the maximal interval of existence for γ (i.e., that $\gamma|_{[s_0,s_{max})}$ is future-inextendible). To analyze solutions to equation (11.46), we will use the following estimates for the Christoffel symbols:

$$|\Gamma_{0\ 0}^{\ 0}| \leq C\epsilon e^{-qHt}, \tag{11.47}$$

$$|\Gamma_{0\ j}^{\ 0}| \leq C\epsilon e^{(1-q)Ht}, \tag{11.48}$$

$$|\Gamma_{j\ k}^{\ 0} - \omega g_{jk}| \leq C\epsilon e^{(2-q)Ht}. \tag{11.49}$$

We will prove the estimate (11.49); the estimates (11.47) - (11.48) can be shown similarly. To begin, we use the definition (3.2d) of $\Gamma_{j\ k}^{\ 0}$ and the triangle inequality to obtain the following estimate:

$$\begin{aligned}|\Gamma_{j\ k}^{\ 0} - \omega g_{jk}| \leq \frac{1}{2}|g^{00}||\partial_j g_{0k} + \partial_k g_{0j}| + \frac{1}{2}|g^{00} + 1||\partial_t g_{jk}| + \frac{1}{2}|\partial_t g_{jk} - 2\omega g_{jk}| \\ + \frac{1}{2}|g^{0a}||\partial_j g_{ak} + \partial_k g_{ja} - \partial_a g_{jk}|.\end{aligned} \tag{11.50}$$

Inequality (11.49) now follows easily from (11.50), the estimate $\mathbf{S}_N < \epsilon$, Sobolev embedding, (9.1a), (9.1d), (9.2a), and (9.2b).

We now use (11.38), (11.39), and (11.47)-(11.49) to arrive at the following inequality:

$$|\Gamma_{0\ 0}^{\ 0}|(\dot{\gamma}^0)^2 + 2|\Gamma_{0\ a}^{\ 0}\dot{\gamma}^0\dot{\gamma}^a| + |\Gamma_{a\ b}^{\ 0} - \omega g_{ab}|\dot{\gamma}^a\dot{\gamma}^b \leq C\epsilon e^{-qH\gamma^0(s)}(\dot{\gamma}^0)^2, \tag{11.51}$$

where it is understood that both sides of (11.51) are evaluated along the curve $\gamma(s)$. From (11.46), (11.51), and the negative-definiteness of the 3×3 matrix $-g_{jk}$, it follows that

$$\begin{aligned}\ddot{\gamma}^0 &= -\Gamma_{0\ 0}^{\ 0}(\dot{\gamma}^0)^2 - 2\Gamma_{0\ a}^{\ 0}\dot{\gamma}^0\dot{\gamma}^a - \Gamma_{a\ b}^{\ 0}\dot{\gamma}^a\dot{\gamma}^b \\ &\leq |\Gamma_{0\ 0}^{\ 0}|(\dot{\gamma}^0)^2 + 2|\Gamma_{0\ a}^{\ 0}\dot{\gamma}^0\dot{\gamma}^a| + |\Gamma_{a\ b}^{\ 0} - \omega g_{ab}|\dot{\gamma}^a\dot{\gamma}^b \leq C\epsilon e^{-qH\gamma^0(s)}(\dot{\gamma}^0)^2.\end{aligned} \tag{11.52}$$

Since we have already shown above that $\dot{\gamma}^0 > 0$ if ϵ is small enough, we may divide inequality (11.52) by $\dot{\gamma}^0$ and integrate:

$$\begin{aligned}\ln\left(\frac{\dot{\gamma}^0(s)}{\dot{\gamma}^0(s_0)}\right) &= \int_{s_0}^{s} \frac{\ddot{\gamma}^0(s')}{\dot{\gamma}^0(s')}\,ds' \leq C\epsilon \int_{s_0}^{s} e^{-qH\gamma^0(s')}\dot{\gamma}^0(s')\,ds' \\ &= -\frac{C}{qH}\epsilon \int_{s_0}^{s} \left(\frac{d}{ds'}e^{-qH\gamma^0(s')}\right)\,ds' \leq \frac{C}{qH}e^{-qH\gamma^0(s)}.\end{aligned} \tag{11.53}$$

Using also the fact that $\gamma^0(s_0) \geq 0$ (since $t \geq 0$ in $(0, \infty) \times \mathbb{T}^3$), we therefore conclude that $\dot{\gamma}^0(s)$ is bounded from above for $s \in [s_0, s_{max})$:

$$\dot{\gamma}^0(s) \leq C. \tag{11.54}$$

Integrating (11.54) from s_0 to s, we have that

$$\gamma^0(s) - \gamma^0(s_0) = \int_{s_0}^{s} \dot{\gamma}^0(s')\, ds' \leq C|s - s_0|. \tag{11.55}$$

Since we have already shown that $\gamma^0(s) \uparrow \infty$ as $s \uparrow s_{max}$, it follows from (11.55) that $s_{max} = \infty$.

Finally, thanks to the future-causal geodesic completeness of the spacetime-with-boundary $([0,\infty) \times \mathbb{T}^3, g)$, Proposition 3.7 implies that $\mathcal{D}^+(\{0\} \times \mathbb{T}^3)$, the future Cauchy development of $\{0\} \times \mathbb{T}^3$ in the maximal globally hyperbolic development $(\mathcal{M}, g', \partial\Phi')$, is exactly $[0,\infty) \times \mathbb{T}^3$.

□

12 Asymptotics

In this section, we strengthen the conclusions of Theorem 11.5. More precisely, we show that suitably time-rescaled versions of the components $g_{\mu\nu}, g^{\mu\nu}, \partial_\mu \Phi$, $(\mu, \nu = 0, 1, 2, 3)$, and various coordinate derivatives of the rescaled quantities converge as $t \to \infty$. Because our strategy is to integrate bounds for time derivatives (with spatial derivative terms on the right-hand side of the inequality), we will lose some differentiability in our convergence estimates. Furthermore, we note that although our bootstrap assumptions were sufficient to close the global existence argument, they are far from optimal from the point of view of decay rates. Thus, at the cost of a few more derivatives, we will also revisit the modified equations and derive improved rates of decay compared to what can be directly concluded from the estimate $\mathbf{S}_N \leq \epsilon$. In particular, the Counting Principle estimate (9.33) is not precise enough to detect these refinements. These results should be viewed as an initial investigation of the asymptotics; it is clear that more information could be extracted at the expense of more work. This theorem is analogous to [Rin08, Proposition 2].

Theorem 12.1 (Asymptotics). *Assume that the initial data (see Remark 1.1)* $(\mathring{g}_{\mu\nu}, \mathring{K}_{\mu\nu}, \mathring{\Psi}, \partial_j \mathring{\Phi} = \mathring{\underline{\beta}}_j)$, $(j = 1, 2, 3; \mu, \nu = 0, 1, 2, 3)$, *for the modified system* (5.15a)-(5.15d) *satisfy the assumptions of Theorem 11.5, including the smallness assumption* $\mathbf{S}_N(0) \leq C_*^{-1}\epsilon$, *where* $0 \leq \epsilon \leq \epsilon_0$. *Let* $\mathring{g}^{\mu\nu}$ *denote the inverse of* $\mathring{g}_{\mu\nu}$. *Assume in addition that* $N \geq 5$, *and let* $(g_{\mu\nu}, \partial_\mu \Phi)$ *be the future-global solution launched by the data. Then there exist a constant* ϵ_2 *satisfying* $0 < \epsilon_2 \leq \epsilon_0$ *and a large constant* $C > 0$ *such that if* $\epsilon \leq \epsilon_2$, *then there exist a Riemann metric* $g_{jk}^{(\infty)}$, *with corresponding Christoffel symbols* $\Gamma_{ijk}^{(\infty)}$, $(i, j, k = 1, 2, 3)$, *and inverse* $g^{jk}_{(\infty)}$ *on* $\mathbb{T}^3$, *a function* $\Psi_{(\infty)}$ *on* $\mathbb{T}^3$, *and a one-form* $\underline{\beta}_j^{(\infty)}$ *on* $\mathbb{T}^3$ *such that* $g_{jk}^{(\infty)} - \mathring{g}_{jk} \in H^N$, $g^{jk}_{(\infty)} - \mathring{g}^{jk} \in H^N$, $\Psi_{(\infty)} - \bar{\Psi} \in H^{N-1}$ *[where* $\bar{\Psi}$ *is defined in* (4.14)*]* *and* $\underline{\beta}_j^{(\infty)} \in H^{N-1}$, *and such that the following estimates hold for all* $t \geq 0$:

$$\|g_{jk}^{(\infty)} - \mathring{g}_{jk}\|_{H^N} \leq C\epsilon, \tag{12.1a}$$
$$\|g^{jk}_{(\infty)} - \mathring{g}^{jk}\|_{H^N} \leq C\epsilon, \tag{12.1b}$$

$$\|e^{-2\Omega} g_{jk} - g_{jk}^{(\infty)}\|_{H^N} \leq C\epsilon e^{-qHt}, \tag{12.2a}$$
$$\|e^{-2\Omega} g_{jk} - g_{jk}^{(\infty)}\|_{H^{N-2}} \leq C\epsilon e^{-2Ht}, \tag{12.2b}$$
$$\|e^{2\Omega} g^{jk} - g^{jk}_{(\infty)}\|_{H^N} \leq C\epsilon e^{-qHt}, \tag{12.2c}$$
$$\|e^{2\Omega} g^{jk} - g^{jk}_{(\infty)}\|_{H^{N-2}} \leq C\epsilon e^{-2Ht}, \tag{12.2d}$$
$$\|e^{-2\Omega} \partial_t g_{jk} - 2\omega g_{jk}^{(\infty)}\|_{H^N} \leq C\epsilon e^{-qHt}, \tag{12.2e}$$
$$\|e^{-2\Omega} \partial_t g_{jk} - 2\omega g_{jk}^{(\infty)}\|_{H^{N-2}} \leq C\epsilon e^{-2Ht}, \tag{12.2f}$$
$$\|e^{2\Omega} \partial_t g^{jk} + 2\omega g^{jk}_{(\infty)}\|_{H^N} \leq C\epsilon e^{-qHt}, \tag{12.2g}$$
$$\|e^{2\Omega} \partial_t g^{jk} + 2\omega g^{jk}_{(\infty)}\|_{H^{N-2}} \leq C\epsilon e^{-2Ht}, \tag{12.2h}$$

$$\|g_{0j} - H^{-1} g^{ab}_{(\infty)} \Gamma^{(\infty)}_{ajb}\|_{H^{N-3}} \leq C\epsilon e^{-qHt}, \tag{12.3a}$$
$$\|\partial_t g_{0j}\|_{H^{N-3}} \leq C\epsilon e^{-qHt}, \tag{12.3b}$$

$$\|g_{00} + 1\|_{H^N} \leq C\epsilon e^{-qHt}, \tag{12.4a}$$
$$\|g_{00} + 1\|_{H^{N-2}} \leq C\epsilon(1+t) e^{-2Ht}, \tag{12.4b}$$
$$\|\partial_t g_{00}\|_{H^N} \leq C\epsilon e^{-qHt}, \tag{12.4c}$$
$$\|\partial_t g_{00} + 2\omega(g_{00} + 1)\|_{H^{N-2}} \leq C\epsilon e^{-2Ht}, \tag{12.4d}$$

$$\|e^{-2\Omega} \underline{K}_{jk} - \omega g_{jk}^{(\infty)}\|_{H^{N-1}} \leq C\epsilon e^{-qHt}, \tag{12.5a}$$
$$\|e^{-2\Omega} \underline{K}_{jk} - \omega g_{jk}^{(\infty)}\|_{H^{N-2}} \leq C(1+t)\epsilon e^{-2Ht}. \tag{12.5b}$$

In the above inequalities, $\underline{K}_{jk}$ *is the second fundamental form of the hypersurface* $\{t = const\}$.

Furthermore, we have that

$$\|e^{\varkappa\Omega} \partial_t \Phi - \Psi_{(\infty)}\|_{H^{N-1}} \leq C\epsilon e^{-qHt}, \tag{12.6a}$$
$$\|e^{\varkappa\Omega} \partial_t \Phi - \Psi_{(\infty)}\|_{H^{N-2}} \leq C\epsilon e^{-2(1-\kappa)Ht}, \tag{12.6b}$$
$$\|\Psi_{(\infty)} - \bar{\Psi}\|_{H^{N-1}} \leq C\epsilon, \tag{12.6c}$$
$$\|\partial_j \Phi - \underline{\beta}_j^{(\infty)}\|_{H^{N-1}} \leq C\epsilon e^{-\varkappa Ht}, \tag{12.6d}$$
$$\|\underline{\beta}_j^{(\infty)}\|_{H^{N-1}} \leq C\epsilon. \tag{12.6e}$$

Remark 12.2. We assume that $N \geq 5$ so that we can use standard Sobolev-Moser estimates during our proofs the improved rates of decay.

Proof. See Remark 9.2 for some conventions that we use throughout this proof. In our proofs below, we will introduce new energies, and the differential inequalities that we will derive for them are valid only under the assumption that the energies are sufficiently small; we don't explicitly mention the smallness assumption each time we make it. In the interest of brevity, we will only sketch the proofs of the estimates involving the improved decay rates. We also remind the reader of the conclusion (11.10) of Theorem 11.5, which is that $\mathbf{S}_N \stackrel{\text{def}}{=} S_{g_{00}+1;N} + S_{g_{0*};N} + S_{h_{**};N} + S_{\partial\Phi;N}$ satisfies $\mathbf{S}_N(t) \leq \epsilon$ for $t \geq 0$.

Proofs of (12.1a)*,* (12.1b)*,* (12.2a)*,* (12.2c)*,* (12.2e)*, and* (12.2g): It follows from the definition (6.2f) of $\mathbf{S}_N$ that

$$\|\partial_t h_{jk}\|_{H^N} \leq C\epsilon e^{-qHt}. \tag{12.7}$$

Integrating $\partial_t h_{jk}$ and using (12.7), it follows that for $t_1 \leq t_2$, we have that

$$\|h_{jk}(t_2) - h_{jk}(t_1)\|_{H^N} \leq C\epsilon e^{-qHt_1}. \tag{12.8}$$

From (12.8) and the fact that $h_{jk} = e^{-2\Omega} g_{jk}$, it easily follows that there exist functions $g_{jk}^{(\infty)}(x^1, x^2, x^3)$ such that

$$\|e^{-2\Omega} g_{jk} - g_{jk}^{(\infty)}\|_{H^N} \leq C\epsilon e^{-qHt}, \tag{12.9}$$

and such that

$$\|g_{jk}^{(\infty)} - \mathring{g}_{jk}\|_{H^N} \leq C\epsilon. \tag{12.10}$$

We have thus shown (12.1a) and (12.2a) . Inequality (12.2e) follows from (12.2a) and (12.7).

To obtain the asymptotics for g^{jk}, we use (9.4a), which implies that

$$\|\partial_t\big(e^{2\Omega} g^{jk}\big)\|_{H^N} \leq C\epsilon e^{-qHt}. \tag{12.11}$$

From (12.11) and the estimates (9.1b), (9.1c) at $t = 0$, it follows as in the previous argument that there exist functions $g^{jk}_{(\infty)}(x^1, x^2, x^3)$ such that

$$\|e^{2\Omega} g^{jk} - g^{jk}_{(\infty)}\|_{H^N} \leq C\epsilon e^{-qHt}, \tag{12.12}$$

and such that

$$\|g^{jk}_{(\infty)} - \mathring{g}^{jk}\|_{H^N} \leq C\epsilon, \tag{12.13}$$

$$\|\underline{\partial} g^{jk}_{(\infty)}\|_{H^{N-1}} \leq C\epsilon, \tag{12.14}$$

$$\|g^{jk}_{(\infty)}\|_{L^\infty} \leq C, \tag{12.15}$$

where $\mathring{g}^{jk} \stackrel{\text{def}}{=} g^{jk}|_{t=0}$. This proves (12.1b) and (12.2c). (12.2g) then follows from (12.2c) and (12.11). Furthermore, since the identity $g^{aj}g_{ak} + g^{0j}g_{0k} = \delta_k^j$, Proposition B.5, $\mathbf{S}_N \leq \epsilon$, (9.1b), (9.1c), (9.1d), (12.2a), and (12.2c) imply that

$$\begin{aligned} \|g^{aj}_{(\infty)}g^{(\infty)}_{ak} - \delta^j_k\|_{H^N} &\leq \|(g^{aj}_{(\infty)} - g^{aj})g^{(\infty)}_{ak}\|_{H^N} + \|g^{aj}(g^{(\infty)}_{ak} - g_{ak})\|_{H^N} + \|g^{0j}g_{0k}\|_{H^N} \\ &\leq C\epsilon e^{-2qHt}, \end{aligned} \tag{12.16}$$

it follows that $g^{(\infty)}_{jk}$ are the components of a Riemannian $3-$metric $g^{(\infty)}$ and that $g^{jk}_{(\infty)}$ are the components of its inverse $g^{-1}_{(\infty)}$.

Proofs of (12.4a) *and* (12.4c): The estimates (12.4a) and (12.4c) follow trivially from definition (6.2a).

Proofs of (12.6a)*,* (12.6d)*, and* (12.6e): To prove (12.6a), we first recall equation (5.15d), which can be re-expressed as follows:

$$\partial_t\Big(e^{\varkappa\Omega}\partial_t\Phi - \bar{\Psi}\Big) = e^{\varkappa\Omega}\triangle'_{\partial\Phi}, \tag{12.17}$$

where $\bar{\Psi}$ is defined in (4.14) and

$$\triangle'_{\partial\Phi} = (m^{-1})^{ab}\partial_a\partial_b\Phi + 2(m^{-1})^{0a}\partial_a\partial_t\Phi - \triangle_{\partial\Phi}. \tag{12.18}$$

From Proposition B.5, the definition (6.2f) of $\mathbf{S}_N$, Sobolev embedding, (9.23b), (9.25c), (9.25e), and (9.25a), it follows that

$$\|e^{\varkappa\Omega}\triangle'_{\partial\Phi}\|_{H^{N-1}} \leq C\epsilon e^{-qHt}. \tag{12.19}$$

As in our proof of (12.9), it easily follows from (12.17), (12.19), and the initial condition $\|e^{\varkappa\Omega(0)}\partial_t\Phi(0,\cdot) - \bar{\Psi}\|_{H^N} \leq C\epsilon$ that there exists a function $\Psi_{(\infty)}(x^1,x^2,x^3)$ with $\Psi_{(\infty)} - \bar{\Psi} \in H^{N-1}$ such that

$$\|e^{\varkappa\Omega}\partial_t\Phi(t,\cdot) - \Psi_{(\infty)}\|_{H^{N-1}} \leq C\epsilon e^{-qHt}, \tag{12.20}$$

$$\|\Psi_{(\infty)} - \bar{\Psi}\|_{H^{N-1}} \leq C\epsilon, \tag{12.21}$$

which proves (12.6a) and (12.6c).

Furthermore, the bound $\mathbf{S}_N(t) \leq \epsilon$ implies that

$$\|\partial_t\partial_j\Phi\|_{H^{N-1}} \leq C\epsilon e^{-\varkappa Ht}. \tag{12.22}$$

It follows easily from (12.22) and the initial condition $\|\partial_j\Phi(0,\cdot)\|_{H^N} \leq C\epsilon$ that there exists a one-form $\underline{\beta}^{(\infty)}_j(x^1,x^2,x^3)$ satisfying $\underline{\beta}^{(\infty)}_j \in H^{N-1}$ such that

$$\|\partial_j\Phi(t,\cdot) - \underline{\beta}^{(\infty)}_j\|_{H^{N-1}} \leq C\epsilon e^{-\varkappa Ht}, \tag{12.23}$$

$$\|\underline{\beta}^{(\infty)}_j\|_{H^{N-1}} \leq C\epsilon, \tag{12.24}$$

which proves (12.6d) and (12.6e).

Proof of (12.5a): We first observe that $\hat{N}$, the future-directed normal to the hypersurface $\{t = const\}$, can be expressed in components as

$$\hat{N}^\mu = -(-g^{00})^{-1/2} g^{0\mu}. \tag{12.25}$$

Using the relation $\underline{K}_{jk} = g_{\alpha k}\partial_j \hat{N}^\alpha + \Gamma_{jk\alpha}\hat{N}^\alpha$, the relations $2\Gamma_{jk0} = \partial_t g_{jk} + \partial_j g_{k0} - \partial_k g_{j0}$ and $\partial_t h_{jk} = e^{-2\Omega}(\partial_t g_{jk} - 2\omega g_{jk})$, and Proposition B.5, we have

$$\begin{aligned}
\|\underline{K}_{jk} - \omega g_{jk}\|_{H^{N-1}} \leq {} & \|\underline{\partial}\big[(-g^{00})^{-1/2} g^{0\alpha}\big]\|_{H^{N-1}}\big(\|g_{\alpha k}\|_{L^\infty} + \|\underline{\partial} g_{\alpha k}\|_{H^{N-1}}\big) \\
& + \|(-g^{00})^{-1/2} g^{0a}\Gamma_{jka}\|_{H^{N-1}} + \|\left[g^{00} - 1\right]\Gamma_{jk0}\|_{H^{N-1}} \\
& + \|\left[(-g^{00})^{-1/2} - 1\right] g^{00}\Gamma_{jk0}\|_{H^{N-1}} + \frac{1}{2}\|\partial_j g_{k0} - \partial_k g_{j0}\|_{H^{N-1}} \\
& + \frac{1}{2} e^{2\Omega}\|\partial_t h_{jk}\|_{H^{N-1}}.
\end{aligned} \tag{12.26}$$

By Corollary B.4 (with $v = g^{00}+1$, $\bar{v} = 0$, and $F(v) = (1-v)^{-1/2}-1 = (-g^{00})^{-1/2}-1$ in the Corollary), it follows that $(-g^{00})^{-1/2} - 1 \in \mathcal{G}_N$ (see Definition 9.4). Therefore, every product on the right-hand side of (12.26) is a product of elements of $\mathcal{H}_{N-1}$ and $\mathcal{G}_{N-1}$ and contains at least one element of $\mathcal{G}_{N-1}$. Hence, by the Counting Principle estimate (9.33) [with $n(\mathcal{G}_{N-1}) \geq 1$ for all products on the right-hand side of (12.26)], we have

$$\|e^{-2\Omega}\underline{K}_{jk} - \omega e^{-2\Omega} g_{jk}\|_{H^{N-1}} \leq C\epsilon e^{-qHt}. \tag{12.27}$$

Inequality (12.5a) now follows from combining (12.2a) and (12.27).

□

Proofs of (12.3a)-(12.3b): Our proofs of (12.3a)-(12.3b) are based on two refined versions of the energy inequality (11.1c). The main point is that even though the energy $\underline{E}_{g_{0*};N}$ defined in (6.10b) allows us to efficiently close the bootstrap argument of Theorem 11.5, there is room for improvement. In particular, Theorem 11.5 only allows us to conclude that $\underline{E}_{g_{0*};N} \leq C\mathbf{S}_N \leq C\epsilon$, which implies that $\|\partial_t g_{0j}\|_{H^N} \leq \epsilon e^{(1-q)Ht}$ and $\|g_{0j}\|_{H^N} \leq \epsilon e^{(1-q)Ht}$. As we will see, it is possible to improve these estimates by a factor of $e^{(q-1)Ht}$. This is a preliminary step that we will need in our remaining proofs. This improvement will be based in part on the following simple matrix identity:

$$g^{0j} = -\frac{1}{g_{00}} g^{aj} g_{0a}. \tag{12.28}$$

We begin our proof of the improvement by defining a new energy $\underline{\mathscr{E}}_{g_{0*};N-1}$ for the g_{0j}, $(j=1,2,3)$, by

$$\underline{\mathscr{E}}^2_{g_{0*};N-1} \overset{\mathrm{def}}{=} \sum_{|\vec{\alpha}|\leq N-1}\sum_{j=1}^{3}\mathcal{E}^2_{(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}g_{0j},\partial(\partial_{\vec{\alpha}}g_{0j})], \tag{12.29}$$

where $\mathcal{E}^2_{(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}g_{0j},\partial(\partial_{\vec{\alpha}}g_{0j})]$ is defined in (6.4), and the constants γ_{0*},δ_{0*} are defined in Definition 6.3. Note that the scaling in (12.29) differs from the scaling used in definition (6.10b) by a factor of $e^{(1-q)\Omega}$. Furthermore, we are using an $(N-1)^{st}$ order energy rather than an N^{th} order energy because we will make use of the improved rates of decay for lower derivatives that are already discernible from the fact that $\mathbf{S}_N\leq\epsilon$ [compare e.g. inequalities (9.5a) and (9.5b)]. Now using (10.1a), we have the following comparison estimate:

$$C^{-1}\underline{\mathscr{E}}_{g_{0*};N-1}\leq\sum_{j=1}^{3}\|\partial_t g_{0j}\|_{H^{N-1}}+e^{-\Omega}\|\underline{\partial}g_{0j}\|_{H^{N-1}}+\|g_{0j}\|_{H^{N-1}}\leq C\underline{\mathscr{E}}_{g_{0*};N-1}. \tag{12.30}$$

From the definition of $\underline{\mathscr{E}}_{g_{0*};N-1}$ and the comparison estimate (12.30), it follows that the energy inequality (6.14b) can be replaced with

$$\begin{aligned}\frac{d}{dt}(\underline{\mathscr{E}}^2_{g_{0*};N-1})\leq&-\eta_{0*}H\underline{\mathscr{E}}^2_{g_{0*};N-1}+C\underline{\mathscr{E}}_{g_{0*};N-1}\sum_{j=1}^{3}\|g^{ab}\Gamma_{ajb}\|_{H^{N-1}}\\&+C\underline{\mathscr{E}}_{g_{0*};N-1}+C\underline{\mathscr{E}}_{g_{0*};N-1}\sum_{j=1}^{3}\|\triangle_{0j}\|_{H^{N-1}}\\&+\sum_{|\vec{\alpha}|\leq N-1}\sum_{j=1}^{3}\|[\hat{\square}_g,\partial_{\vec{\alpha}}]g_{0j}\|_{L^2}\\&+\sum_{|\vec{\alpha}|\leq N-1}\sum_{j=1}^{3}\|\triangle_{\mathcal{E};(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}g_{0j},\partial(\partial_{\vec{\alpha}}g_{0j})]\|_{L^1}.\end{aligned} \tag{12.31}$$

We now claim that the following improvements of (9.17b), (9.17e), (9.19b), (9.20b), and (9.21b) hold for $|\vec{\alpha}|\leq N-1$:

$$\|\triangle_{A,0j}\|_{H^{N-1}}\leq C\epsilon e^{-qHt}\underline{\mathscr{E}}_{g_{0*};N-1}+C\epsilon e^{-qHt}, \tag{12.32}$$

$$\|\triangle_{C,0j}\|_{H^{N-1}}\leq C\epsilon e^{-qHt}\underline{\mathscr{E}}_{g_{0*};N-1}, \tag{12.33}$$

$$\|\triangle_{0j}\|_{H^{N-1}}\leq Ce^{-qHt}\underline{\mathscr{E}}_{g_{0*};N-1}+C\epsilon e^{-qHt}, \tag{12.34}$$

$$\|[\hat{\square}_g,\partial_{\vec{\alpha}}]g_{0j}\|_{L^2}\leq C\epsilon e^{-qHt}\underline{\mathscr{E}}_{g_{0*};N-1}+C\epsilon e^{-qHt}, \tag{12.35}$$

$$\|\triangle_{\mathcal{E};(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}g_{0j},\partial(\partial_{\vec{\alpha}}g_{0j})]\|_{L^1}\leq Ce^{-qHt}\underline{\mathscr{E}}^2_{g_{0*};N-1}. \tag{12.36}$$

These improved estimates can be derived using methods similar to the ones we used in our proofs of the original estimates, together with the identity (12.28). More specifically, in our proofs of (9.17b), (9.17e), (9.19b), (9.20b), and (9.21b), we used the Counting Principle-type estimates $\|\partial_t g_{0j}\|_{H^N} \leq e^{(1-q)\Omega}\underline{S}_{g_{0*};N}$, $\|g_{0j}\|_{H^N} \leq e^{(1-q)\Omega}\underline{S}_{g_{0*};N}$, and $\|\underline{\partial} g_{0j}\|_{H^N} \leq e^{(2-q)\Omega}\underline{S}_{g_{0*};N}$, which follow directly from the definition of $\underline{S}_{g_{0*};N}$, together with (9.1d), which reads $\|g^{0j}\|_{H^N} \leq Ce^{-(1+q)\Omega}S_{g;N}$. However, whenever it is convenient, these estimates can be replaced with

$$\|\partial_t g_{0j}\|_{H^{N-1}} \leq C\underline{\mathscr{E}}_{g_{0*};N-1}, \tag{12.37}$$
$$\|g_{0j}\|_{H^{N-1}} \leq C\underline{\mathscr{E}}_{g_{0*};N-1}, \tag{12.38}$$
$$\|\underline{\partial} g_{0j}\|_{H^{N-1}} \leq Ce^{\Omega}\underline{\mathscr{E}}_{g_{0*};N-1}, \tag{12.39}$$
$$\|g^{0j}\|_{H^{N-1}} \leq Ce^{-2\Omega}\underline{\mathscr{E}}_{g_{0*};N-1} \tag{12.40}$$

respectively, where (12.37)-(12.39) follow from (12.30), while (12.40) follows from applying (12.30), Proposition B.5, Sobolev embedding, (9.1b), and (9.1c) to the identity (12.28). We remark that the $Ce^{-qHt}\underline{\mathscr{E}}_{g_{0*};N-1}$ term on the right-hand side of (12.34) arises from from e.g. the $(\omega - H)\partial_t g_{0j}$ term on the right-hand side of (A.4b). Similarly, the $Ce^{-qHt}\underline{\mathscr{E}}^2_{g_{0*};N-1}$ term on the right-hand side of (12.36) arises from the e.g. the $(H - \omega)g^{ab}(\partial_a v)(\partial_b v)$ term on the right-hand side of (6.8). See [Rin08, Section 14] for additional details on these improved estimates.

Now using (9.5b), (12.31) and (12.34)-(12.36), we argue as in our proof of (11.4) to deduce the following inequality:

$$\frac{d}{dt}\Big(\underline{\mathscr{E}}^2_{g_{0*};N-1}\Big) \leq -\eta_{0*}H\underline{\mathscr{E}}^2_{g_{0*};N-1} + Ce^{-qHt}\underline{\mathscr{E}}^2_{g_{0*};N-1} + C\epsilon\underline{\mathscr{E}}_{g_{0*};N-1}, \tag{12.41}$$

where the $C\epsilon\underline{\mathscr{E}}_{g_{0*};N-1}$ term on the right-hand side of (12.41) arises from applying inequality (9.5b) to the second term on the right-hand side of (12.31). Integrating (12.41) from 0 to t, using the smallness condition $\underline{\mathscr{E}}_{g_{0*};N-1}(0) \leq C\epsilon$ for small t, and applying Lemma 11.4 for large t (the $Ce^{-qHt}\underline{\mathscr{E}}^2_{g_{0*};N-1}$ term is dominated by the $-\eta_{0*}H\underline{\mathscr{E}}^2_{g_{0*};N-1}$ term for large t), we conclude that the following bound holds for all $t \geq 0$:

$$\underline{\mathscr{E}}_{g_{0*};N-1} \leq C\epsilon. \tag{12.42}$$

This completes the proof of our preliminary improved estimate.

We are now ready for the proofs of (12.3a) and (12.3b). Defining

$$v_j \stackrel{\text{def}}{=} g_{0j} - H^{-1}g^{ab}_{(\infty)}\Gamma^{(\infty)}_{ajb}, \tag{12.43}$$

and using equation (5.15b), we compute that v_j is a solution to

$$\hat{\Box}_g v_j = 3H\partial_t v_j + 2H^2 v_j + \triangle_j, \tag{12.44}$$

where

$$\triangle_j = \triangle_{0j} + 2H\big(g^{ab}_{(\infty)}\Gamma^{(\infty)}_{ajb} - g^{ab}\Gamma_{ajb}\big) - H^{-1}g^{lm}\partial_l\partial_m\big(g^{ab}_{(\infty)}\Gamma^{(\infty)}_{ajb}\big). \tag{12.45}$$

To estimate $v_j,\ (j = 1, 2, 3)$, we will use the energy

$$\underline{\mathscr{E}}^2_{v_*;N-3} \stackrel{\text{def}}{=} e^{2qHt} \sum_{|\vec{\alpha}|\leq N-3} \sum_{j=1}^{3} \mathcal{E}^2_{(\upgamma_{0*},\updelta_{0*})}[\partial_{\vec{\alpha}} v_j, \partial(\partial_{\vec{\alpha}} v_j)], \tag{12.46}$$

where $\mathcal{E}^2_{(\upgamma_{0*},\updelta_{0*})}[\partial_{\vec{\alpha}} v_j, \partial(\partial_{\vec{\alpha}} v_j)]$ is defined in (6.4), and the constants $\upgamma_{0*}, \updelta_{0*}$ are defined in Definition 6.3. Note that in response to the last term on the right-hand side of (12.45), we have further reduced the number of derivatives in the definition of our energy by two. From (10.1a), it follows that

$$\begin{aligned} C^{-1}\underline{\mathscr{E}}_{v_*;N-3} &\leq \sum_{j=1}^{3} e^{qHt}\|\partial_t v_j\|_{H^{N-3}} + C_{(\upgamma_{0*})} e^{qHt}\|v_j\|_{H^{N-3}} + e^{(q-1)Ht}\|\underline{\partial} v_j\|_{H^{N-3}} \\ &\leq C\underline{\mathscr{E}}_{v_*;N-3}. \end{aligned} \tag{12.47}$$

Arguing as in our proof of (12.31), we have that

$$\begin{aligned} \frac{d}{dt}(\underline{\mathscr{E}}^2_{v_*;N-3}) \leq (2q - \upeta_{0*})H\underline{\mathscr{E}}^2_{v_*;N-3} &+ Ce^{qHt}\underline{\mathscr{E}}_{v_*;N-3} \sum_{j=1}^{3} \|\triangle_j\|_{H^{N-3}} \\ &+ Ce^{qHt}\underline{\mathscr{E}}_{v_*;N-3} \sum_{|\vec{\alpha}|\leq N-3} \sum_{j=1}^{3} \|[\hat{\square}_g, \partial_{\vec{\alpha}}]v_j\|_{L^2} \\ &+ e^{2qHt} \sum_{|\vec{\alpha}|\leq N-3} \sum_{j=1}^{3} \|\triangle_{\mathcal{E};(\upgamma_{0*},\updelta_{0*})}[\partial_{\vec{\alpha}} v_j, \partial(\partial_{\vec{\alpha}} v_j)]\|_{L^1}. \end{aligned} \tag{12.48}$$

Note that the inequality (12.48) does *not* have a term corresponding to the $\|g^{ab}\Gamma_{ajb}\|_{H^{N-1}}$ term in (12.31); the remnants of this term are present in $\|\triangle_j\|_{H^{N-3}}$.

We will now estimate $\|\triangle_j\|_{H^{N-3}}$, our goal being to show that it is bounded by $C\epsilon e^{-qHt}$. We begin by estimating the term $2H\big(g^{ab}_{(\infty)}\Gamma^{(\infty)}_{ajb} - g^{ab}\Gamma_{ajb}\big)$ from the right-hand side of (12.45). Let us first recall the definitions of the lowered Christoffel symbols $\Gamma^{(\infty)}_{ijk}$ and $\Gamma_{\mu\alpha\nu}$ corresponding to $g^{(\infty)}$ and g respectively:

$$\Gamma^{(\infty)}_{ijk} \stackrel{\text{def}}{=} \frac{1}{2}(\partial_i g^{(\infty)}_{jk} + \partial_k g^{(\infty)}_{ij} - \partial_j g^{(\infty)}_{ik}), \tag{12.49}$$

$$\Gamma_{\mu\alpha\nu} \stackrel{\text{def}}{=} \frac{1}{2}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}). \tag{12.50}$$

Using Proposition B.5, the definition (6.2f) of $\mathbf{S}_N$, Sobolev embedding, (12.1a), (12.1b), (12.2a), (12.2c), (12.15), (12.49), and (12.50), we conclude that

$$\begin{aligned}
&\|g^{ab}(t,\cdot)\Gamma_{ajb}(t,\cdot) - g^{ab}_{(\infty)}\Gamma^{(\infty)}_{ajb}\|_{H^{N-1}} \\
&\qquad \le \|e^{2\Omega}g^{ab}(t,\cdot) - g^{ab}_{(\infty)}\|_{H^{N-1}}\|e^{-2\Omega}\Gamma_{ajb}(t,\cdot)\|_{H^{N-1}} \\
&\qquad\quad + \big(\|g^{ab}_{(\infty)}\|_{L^\infty} + \|\underline{\partial} g^{ab}_{(\infty)}\|_{H^{N-2}}\big)\|e^{-2\Omega}\Gamma_{ajb}(t,\cdot) - \Gamma^{(\infty)}_{ajb}\|_{H^{N-1}} \\
&\qquad \le C\epsilon e^{-qHt},
\end{aligned} \tag{12.51}$$

$$\|g^{ab}_{(\infty)}\Gamma^{(\infty)}_{ajb}\|_{H^{N-1}} \le C\epsilon. \tag{12.52}$$

We now estimate the term $H^{-1}g^{lm}\partial_l\partial_m\big(g^{ab}_{(\infty)}\Gamma^{(\infty)}_{ajb}\big)$ from the right-hand side of (12.45). By Proposition B.5, (9.1b), (9.1c), and (12.52), it follows that

$$\|g^{lm}\partial_l\partial_m(g^{ab}_{(\infty)}\Gamma^{(\infty)}_{ajb})\|_{H^{N-3}} \le C\epsilon e^{-2Ht}. \tag{12.53}$$

Applying the estimates (12.34), (12.42), (12.51), and (12.53) to the terms in (12.45), we conclude the desired estimate for $\|\triangle_j\|_{H^{N-3}}:$

$$\|\triangle_j\|_{H^{N-3}} \le C\epsilon e^{-qHt}. \tag{12.54}$$

In addition, we argue as in our proof of (12.35) and (12.36) and in particular make use of the improved estimates (12.37)-(12.40) and (12.42), thus arriving at the following inequalities:

$$\|[\hat{\Box}_g, \partial_{\vec{\alpha}}]v_j\|_{L^2} \le C\epsilon e^{-qHt}\underline{\mathscr{E}}_{v_*;N-3} + C\epsilon e^{-qHt}, \qquad (|\vec{\alpha}| \le N-3), \tag{12.55}$$

$$\|\triangle_{\mathcal{E};(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}}v_j, \partial(\partial_{\vec{\alpha}}v_j)]\|_{L^1} \le Ce^{-3qHt}\underline{\mathscr{E}}^2_{v_*;N-3}, \qquad (|\vec{\alpha}| \le N-3). \tag{12.56}$$

From (12.48), (12.54), (12.55), and (12.56), it follows that

$$\frac{d}{dt}\Big(\underline{\mathscr{E}}^2_{v_*;N-3}\Big) \le (2q - \eta_{0*})H\underline{\mathscr{E}}^2_{v_*;N-3} + Ce^{-qHt}\underline{\mathscr{E}}^2_{v_*;N-3} + C\epsilon\underline{\mathscr{E}}_{v_*;N-3}. \tag{12.57}$$

Using the fact that $2q - \eta_{0*} < 0$, we argue as in our proof of (12.42) to conclude that the following inequality holds for all $t \ge 0:$

$$\underline{\mathscr{E}}_{v_*;N-3} \le C\epsilon. \tag{12.58}$$

Inequalities (12.3a) and (12.3b) now follow from the comparison estimate (12.47) and from (12.58).

Proofs of (12.2b)*,* (12.2d)*,* (12.2f)*,* (12.2h)*,* (12.4b)*, and* (12.4d): We begin by using equations (A.1a) and (A.1c), together with the identity $g^{00}+1 = g^{00}(g_{00}+1)+g^{0a}g_{0a}$ to derive the following equations:

$$\partial_t^2\big[e^{2\Omega}(g_{00}+1)\big] = -\omega\partial_t\big[e^{2\Omega}(g_{00}+1)\big] + 2\left(\frac{d}{dt}\omega\right)e^{2\Omega}(g_{00}+1) + e^{2\Omega}\triangle'_{00}, \tag{12.59}$$

$$\partial_t[e^{2\Omega}\partial_t h_{jk}] = -\omega e^{2\Omega}\partial_t h_{jk} + e^{2\Omega}\triangle'_{jk}, \tag{12.60}$$

where

$$\begin{aligned}\triangle'_{00} = &-(g^{00})^{-1}\left\{2g^{0a}\partial_a\partial_t g_{00} + g^{ab}\partial_a\partial_b g_{00}\right\} \\ &+ 5\omega(g^{00})^{-1}\left\{g^{00}(g_{00}+1) + g^{0a}g_{0a}\right\}\partial_t g_{00} \\ &+ 6\omega^2(g^{00})^{-1}\left\{g^{00}(g_{00}+1)^2 + g^{0a}g_{0a}(g_{00}+1)\right\} \\ &+ 2(g^{00})^{-1}\left\{\triangle_{A,00} + \triangle_{C,00} + \triangle'_{Rapid,00}\right\},\end{aligned} \tag{12.61}$$

$$\begin{aligned}\triangle'_{jk} = &-(g^{00})^{-1}\left\{2g^{0a}\partial_a\partial_t h_{jk} + g^{ab}\partial_a\partial_b h_{jk}\right\} \\ &+ 3\omega(g^{00})^{-1}\left\{g^{00}(g_{00}+1) + g^{0a}g_{0a}\right\}\partial_t h_{jk} \\ &+ 2(g^{00})^{-1}\left\{e^{-2\Omega}\triangle_{A,jk} + \triangle'_{Rapid,jk} - \omega g^{0a}\partial_a h_{jk}\right\},\end{aligned} \tag{12.62}$$

where $\triangle'_{Rapid,00}$ is obtained by setting all terms that explicitly involve the factor $\omega - H$ equal to 0 in the expression (A.7a) for $\triangle_{Rapid,00}$, and similarly for $\triangle'_{Rapid,jk}$.

Using the fact that $\mathbf{S}_N \leq \epsilon$, the improved estimates (12.37)-(12.40) and (12.42) (whenever they are convenient), and the Sobolev-Moser type inequalities in the Appendix, we derive the following inequalities:

$$\|\triangle'_{00}\|_{H^{N-2}} \leq C\epsilon e^{-2\Omega} + C\epsilon e^{-qHt}\big\{\|g_{00}+1\|_{H^{N-2}} + \|\partial_t g_{00}\|_{H^{N-2}} + \|\partial_t h_{jk}\|_{H^{N-2}}\big\}, \tag{12.63}$$

$$\|\triangle'_{jk}\|_{H^{N-2}} \leq C\epsilon e^{-2\Omega} + C\epsilon e^{-qHt}\|\partial_t h_{jk}\|_{H^{N-2}}. \tag{12.64}$$

Since the derivation of the above inequalities is similar to many other estimates proved in this article, we have left the tedious details up to the reader. However, we remark that the $\epsilon e^{-qHt}\|g_{00}+1\|_{H^{N-2}}$ term on the right-hand side of (12.63) arises from e.g. the first term on the right-hand side of (A.15a), that the $\epsilon e^{-qHt}\|\partial_t g_{00}\|_{H^{N-2}}$ term on the right-hand side of (12.63) arises from e.g. the first term on the right-hand side of (A.13a), that the $\epsilon e^{-qHt}\|\partial_t h_{jk}\|_{H^{N-2}}$ term on the right-hand side of (12.63) arises from e.g. the last term on the right-hand side of (A.13a), and that the $\epsilon e^{-qHt}\|\partial_t h_{jk}\|_{H^{N-2}}$ term on the right-hand side of (12.64) arises from e.g. terms on the fifth, sixth, and seventh lines of (A.13c). Furthermore, we remark that we are using H^{N-2} norms because of the presence of the terms on the right-hand sides of (12.61)-(12.62) that contain second spatial derivatives, e.g. the term $\partial_a\partial_b g_{00}$; by examining e.g. the definition (6.1a), we see that the H^{N-2} norm of such a term has a more favorable rate of decay than its H^{N-1} norm. We need this additional decay to deduce (12.63)-(12.64).

Now in order to derive our desired inequalities, it is convenient to introduce the following non-negative energies $\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2}$, $\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2}$, and $\underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2}$, which are defined by

$$\underline{\mathscr{E}}^2_{e^{2\Omega}(g_{00}+1);N-2} \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N-2} \int_{\mathbb{T}^3} \left(\partial_{\vec{\alpha}}\left[e^{2\Omega}(g_{00}+1)\right]\right)^2 d^3x = \|e^{2\Omega}(g_{00}+1)\|^2_{H^{N-2}}, \tag{12.65a}$$

$$\underline{\mathscr{E}}^2_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2} \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N-2} \int_{\mathbb{T}^3} \left(\partial_{\vec{\alpha}}\partial_t[e^{2\Omega}(g_{00}+1)]\right)^2 d^3x = \|\partial_t[e^{2\Omega}(g_{00}+1)]\|^2_{H^{N-2}}, \tag{12.65b}$$

$$\underline{\mathscr{E}}^2_{e^{2\Omega}\partial_t h_{**};N-2} \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N-2} \sum_{j,k=1}^{3} \int_{\mathbb{T}^3} e^{4\Omega}(\partial_{\vec{\alpha}}\partial_t h_{jk})^2 \, d^3x = \sum_{j,k=1}^{3} \|e^{2\Omega}\partial_t h_{jk}\|^2_{H^{N-2}}. \tag{12.65c}$$

Observe the following simple consequence of the definitions (12.65a) and (12.65b):

$$e^{2\Omega}\|\partial_t g_{00}\|_{H^{N-2}} \leq C\left\{\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2} + \underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2}\right\}. \tag{12.66}$$

Now from (12.63) and (12.64), and (12.66), it follows that

$$e^{2\Omega}\|\triangle'_{00}\|_{H^{N-2}} \leq C\epsilon + C\epsilon e^{-qHt}\left\{\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2} + \underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2} + \underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2}\right\}, \tag{12.67}$$

$$e^{2\Omega}\|\triangle'_{jk}\|_{H^{N-2}} \leq C\epsilon + C\epsilon e^{-qHt}\underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2}. \tag{12.68}$$

We may therefore use equations (12.59)-(12.60) together with (12.67)-(12.68) to derive the following system of differential inequalities, which is valid if $\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2}$, $\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2}$, and $\underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2}$ are sufficiently small:

$$\frac{d}{dt}\left(\underline{\mathscr{E}}^2_{e^{2\Omega}(g_{00}+1);N-2}\right) \leq 2\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2}\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2}, \tag{12.69a}$$

$$\begin{aligned}
\frac{d}{dt}\left(\underline{\mathscr{E}}^2_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2}\right) &\leq -2\omega\underline{\mathscr{E}}^2_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2} + 2e^{2\Omega}\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2}\|\triangle'_{00}\|_{H^{N-2}} \\
&\quad + 4\left|\frac{d}{dt}\omega\right|\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2}\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2} \\
&\leq -2\omega\underline{\mathscr{E}}^2_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2} \\
&\quad + Ce^{-qHt}\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2}\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2} \\
&\quad + C\epsilon\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2} \\
&\quad + C\epsilon e^{-qHt}\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2}\underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2},
\end{aligned} \tag{12.69b}$$

$$\begin{aligned}
\frac{d}{dt}\left(\underline{\mathscr{E}}^2_{e^{2\Omega}\partial_t h_{**};N-2}\right) &\leq -2\omega\underline{\mathscr{E}}^2_{e^{2\Omega}\partial_t h_{**};N-2} + 2e^{2\Omega}\underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2}\|\triangle'_{jk}\|_{H^{N-2}} \\
&\leq -2\omega\underline{\mathscr{E}}^2_{e^{2\Omega}\partial_t h_{**};N-2} + C\epsilon\underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2} \\
&\quad + C\epsilon e^{-qHt}\underline{\mathscr{E}}^2_{e^{2\Omega}\partial_t h_{**};N-2}.
\end{aligned} \tag{12.69c}$$

We remark that (12.69a) follows from a simple application of the Cauchy-Schwarz inequality, after bringing the time derivative under the integral over $\mathbb{T}^3$. Using the initial conditions $\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2}(0) \leq C\epsilon$, $\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2}(0) \leq C\epsilon$, and $\underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2}(0) \leq C\epsilon$, and assuming that ϵ is sufficiently small, we apply a Gronwall-type inequality to the system (12.69a)-(12.69c), concluding that the following inequalities hold for all $t \geq 0$:

$$\underline{\mathscr{E}}_{e^{2\Omega}(g_{00}+1);N-2} \leq C\epsilon(1+t), \tag{12.70a}$$

$$\underline{\mathscr{E}}_{\partial_t[e^{2\Omega}(g_{00}+1)];N-2} \leq C\epsilon, \tag{12.70b}$$

$$\underline{\mathscr{E}}_{e^{2\Omega}\partial_t h_{**};N-2} \leq C\epsilon. \tag{12.70c}$$

Inequalities (12.4b) and (12.4d) now follow from (12.70a) and (12.70b). Similarly, (12.70c) implies that

$$\|\partial_t h_{jk}\|_{H^{N-2}} \leq C\epsilon e^{-2\Omega}, \tag{12.71}$$

from which (12.2b) and (12.2f) easily follow.

To obtain (12.2d), we first use the improved estimates (12.37)-(12.40), (12.42), and (12.71) to modify inequality (9.4a) as follows:

$$\|\partial_t g^{jk} + 2\omega g^{jk}\|_{H^{N-2}} \leq C\epsilon e^{-4\Omega}. \tag{12.72}$$

(12.2d) then follows from (12.72) as in our proof of (12.2c). (12.2h) then follows from (12.2d) and (12.72).

Proof of (12.5b): To prove (12.5b), we use the identity $g^{00}+1 = \frac{1}{g_{00}}[(g_{00}+1) - g^{0a}g_{0a}]$ and the improved estimates (12.4b), (12.4d), (12.37)-(12.40), (12.42), and (12.71) to modify inequalities (12.26)-(12.27) as follows:

$$\|e^{-2\Omega}\underline{K}_{jk} - \omega e^{-2\Omega}g_{jk}\|_{H^{N-2}} \leq C\epsilon(1+t)e^{-2Ht}. \tag{12.73}$$

Combining (12.2b) and (12.73), we deduce (12.5b) [as in our proof of (12.5a)].

Proof of (12.6b): To prove (12.6b), we use the improved estimates (12.4b), (12.4d), (12.37) - (12.40), (12.42), and (12.71) to modify inequality (12.19) as follows:

$$\|e^{\varkappa\Omega}\triangle'_{\partial\Phi}\|_{H^{N-2}} \leq C\epsilon e^{-2(1-\varkappa)Ht}. \tag{12.74}$$

(12.6b) then follows from (12.74), as in our proof of (12.6a).

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Appendices

A Derivation of the Modified System

In Appendix A, we sketch a derivation of the modified system (5.15a)-(5.15d).

Proposition A.1 (Decomposition of the modified equations). *The equations* (5.7a)-(5.7d) *can be written as follows (for* $j,k=1,2,3$*):*

$$\hat{\square}_g(g_{00}+1) = 5H\partial_t g_{00} + 6H^2(g_{00}+1) + \triangle_{00}, \tag{A.1a}$$

$$\hat{\square}_g g_{0j} = 3H\partial_t g_{0j} + 2H^2 g_{0j} - 2Hg^{ab}\Gamma_{ajb} + \triangle_{0j}, \tag{A.1b}$$

$$\hat{\square}_g h_{jk} = 3H\partial_t h_{jk} + \triangle_{jk}, \tag{A.1c}$$

$$\hat{\square}_m \Phi = \varkappa\omega\partial_t\Phi + \triangle_{\partial\Phi}, \tag{A.1d}$$

where $H \overset{def}{=} \sqrt{\Lambda/3}$, $\omega(t)$, *which is uniquely determined by the parameters* $\Lambda > 0$, $\bar{\rho} > 0$, *and* $\varsigma = 3(1+c_s^2)$, *is the function from* (4.21), $\varkappa \overset{def}{=} \frac{3}{1+2s} = 3c_s^2$,

$$\hat{\square}_m \overset{def}{=} -\partial_t^2 + 2(m^{-1})^{0a}\partial_t\partial_a + (m^{-1})^{ab}\partial_a\partial_b \tag{A.2}$$

is the reduced wave operator corresponding to the ***reciprocal acoustical metric*** $(m^{-1})^{\mu\nu}$, *(*$\mu,\nu = 0,1,2,3$*), and the components of* m^{-1} *are given by*

$$(m^{-1})^{00} = -1, \tag{A.3a}$$

$$(m^{-1})^{0j} = -\frac{\triangle_{(m)}^{0j}}{(1+2s)+\triangle_{(m)}}, \tag{A.3b}$$

$$(m^{-1})^{jk} = \frac{g^{jk} - \triangle_{(m)}^{jk}}{(1+2s)+\triangle_{(m)}}. \tag{A.3c}$$

The error terms $\triangle_{\mu\nu}$, $\triangle_{\partial\Phi}$, $\triangle_{(m)}$, $\triangle_{(m)}^{0j}$, *and* $\triangle_{(m)}^{jk}$ *from above can be decomposed as* *follows:*

$$\frac{1}{2}\triangle_{00} = \triangle_{A,00} + \triangle_{C,00} + \triangle_{Rapid,00} \tag{A.4a}$$

$$\frac{1}{2}\triangle_{0j} = \triangle_{A,0j} + \triangle_{C,0j} + \triangle_{Rapid,0j}, \tag{A.4b}$$

$$\frac{1}{2}\triangle_{jk} = e^{-2\Omega}\triangle_{A,jk} + \triangle_{Rapid,jk} - 2\omega g^{0a}\partial_a h_{jk}, \tag{A.4c}$$

$$\triangle_{\partial\Phi} = \omega(\partial_t\Phi)\left\{\frac{3}{(1+2s)+\triangle_{(m)}} - \frac{3}{\underbrace{1+2s}_{\varkappa}}\right\} - (\partial_t\Phi)\left\{\frac{1}{(1+2s)+\triangle_{(m)}}\right\}\Theta, \tag{A.4d}$$

where

$$Z_j \stackrel{def}{=} \frac{\partial_j\Phi}{\partial_t\Phi}, \tag{A.5}$$

$$\triangle_{(m)} = (1+2s)(g^{00}+1)(g^{00}-1) + 2(1+2s)g^{00}g^{0a}Z_a + (g^{00}g^{ab} + 2sg^{0a}g^{0b})Z_aZ_b, \tag{A.6a}$$

$$\triangle_{(m)}^{0j} = g^{00}g^{0j}(1+2s) + 2\big[(s+1)g^{0j}g^{0a} + sg^{00}g^{aj}\big]Z_a + (g^{0j}g^{ab} + 2sg^{aj}g^{0b})Z_aZ_b, \tag{A.6b}$$

$$\begin{aligned}\triangle_{(m)}^{jk} &= (g^{00}+1)g^{jk} + 2sg^{0j}g^{0k} + 2(g^{jk}g^{0a} + sg^{0k}g^{aj} + sg^{0j}g^{ak})Z_a \\ &\quad + (g^{jk}g^{ab} + 2sg^{aj}g^{bk})Z_aZ_b,\end{aligned} \tag{A.6c}$$

$$\begin{aligned}\Theta &= 3\omega(g^{00}+1) + 6\omega g^{0a}Z_a + (3-2s)\omega g^{ab}Z_aZ_b \\ &\quad + 2s\left\{\triangle_{(\Gamma)}^{000} + 6\triangle_{(\Gamma)}^{(a00)}Z_a + 6\triangle_{(\Gamma)}^{(0ab)}Z_aZ_b + \triangle_{(\Gamma)}^{abc}Z_aZ_bZ_c\right\},\end{aligned} \tag{A.6d}$$

$(a00)$ *and* $(0ab)$ *denote symmetrization, the* $\triangle_{(\Gamma)}^{\mu\alpha\nu}$ *(*$\mu, \alpha, \nu = 0, 1, 2, 3$*) are defined below* *in Lemma A.7,*

$$e^{3(1+c_s^2)\Omega}\triangle_{Rapid,00} = 3[e^{3(1+c_s^2)\Omega}(\omega - H)][\omega + H](g_{00}+1) + \frac{5}{2}[e^{3(1+c_s^2)\Omega}(\omega - H)]\partial_t g_{00} \\ - \frac{s}{s+1}[e^{2\varkappa\Omega}\sigma]^{s+1}(g_{00}+1) - [\widetilde{f}](g_{00}+1) + [\widetilde{f} - f], \tag{A.7a}$$

$$e^{3(1+c_s^2)\Omega}\triangle_{Rapid,0j} = [e^{3(1+c_s^2)\Omega}(\omega - H)][\omega + H]g_{0j} + \frac{3}{2}[e^{3(1+c_s^2)\Omega}(\omega - H)]\partial_t g_{0j} \\ - [e^{3(1+c_s^2)\Omega}(\omega - H)]g^{ab}\Gamma_{ajb} + \frac{s-1}{2(s+1)}[e^{2\varkappa\Omega}\widetilde{\sigma}]^{s+1}g_{0j} \\ - \frac{s}{s+1}[e^{2\varkappa\Omega}\sigma]^{s+1}g_{0j} - 2[e^{2\varkappa\Omega}\sigma]^s[e^{\varkappa\Omega}\partial_t\Phi]^2 Z_j, \tag{A.7b}$$

$$e^{3(1+c_s^2)\Omega}\triangle_{Rapid,jk} = \frac{3}{2}[e^{3(1+c_s^2)\Omega}(\omega - H)]\partial_t h_{jk} \\ + [e^{2\varkappa\Omega}\widetilde{\sigma}]^{s+1}(g^{00}+1)h_{jk} + \frac{s}{s+1}\left[[e^{2\varkappa\Omega}\widetilde{\sigma}]^{s+1} - [e^{2\varkappa\Omega}\sigma]^{s+1}\right]h_{jk} \\ - 2e^{-2\Omega}[e^{2\varkappa\Omega}\sigma]^s[e^{\varkappa\Omega}\partial_t\Phi]^2 Z_j Z_k, \tag{A.7c}$$

$$f = 2[e^{2\varkappa\Omega}\sigma]^s[e^{\varkappa\Omega}\partial_t\Phi]^2 - \frac{s}{s+1}[e^{2\varkappa\Omega}\sigma]^{s+1}, \tag{A.7d}$$

$$\widetilde{f} = 2[e^{2\varkappa\Omega}\widetilde{\sigma}]^s[e^{\varkappa\Omega}\partial_t\widetilde{\Phi}]^2 - \frac{s}{s+1}[e^{2\varkappa\Omega}\widetilde{\sigma}]^{s+1} = \frac{s+2}{s+1}[e^{\varkappa\Omega}\partial_t\widetilde{\Phi}]^{2(s+1)}, \tag{A.7e}$$

the $\triangle_{A,\mu\nu}$ *are defined in* (A.13a)-(A.13c)*, and the* $\triangle_{C,00}, \triangle_{C,0j}$ *are defined in* (A.15a)-(A.15b)*. In the above expressions, quantities associated to the FLRW background solution of Section 4 are decorated with the symbol* $\widetilde{\ }$.

Remark A.2. As discussed in Section 1.2, the variable Z_j has been introduced to facilitate our analysis of the ratio of the size of the spatial derivatives of Φ to its time derivative.

Remark A.3. In Section 9, we show that under suitable bootstrap assumptions, various norms of the quantities in brackets $[\cdot]$ in equations (A.7a)-(A.7c) are either $\leq C$ or $\leq C\mathbf{S}_N$.

Proof. The proof involves a series of tedious computations, some of which are contained in Lemmas A.4 - A.7 below. We sketch the proof of (A.1c) and leave the derivation of the remaining equations to the reader. To obtain (A.1c), we first use equation (5.7c), Lemma A.4, and Lemma A.5 to obtain the following equation for $h_{jk} = e^{-2\Omega}g_{jk}$:

$$\hat{\square}_g h_{jk} = 3\omega\partial_t h_{jk} - 2(g^{00}+1)\left(\frac{d}{dt}\omega\right)h_{jk} - 4\omega g^{0a}\partial_a h_{jk} \\ + 2\left[e^{-2\Omega}\triangle_{A,jk} - 2e^{-2\Omega}\sigma^s(\partial_j\Phi)(\partial_k\Phi)\right] \\ + 2\left\{(3\omega^2 - \Lambda + \frac{d}{dt}\omega)h_{jk} - \frac{s}{s+1}\sigma^{s+1}h_{jk}\right\}. \tag{A.8}$$

We now use (4.17a)-(4.17b) to substitute in equation (A.8) for $3\omega^2 - \Lambda$ and $\frac{d}{dt}\omega$ in terms of $\widetilde{\sigma}$, thus arriving at the following equation:

$$\begin{aligned}\hat{\square}_g h_{jk} &= 3\omega\partial_t h_{jk} + 2(g^{00}+1)\widetilde{\sigma}^{s+1}h_{jk} - 4\omega g^{0a}\partial_a h_{jk} \\ &+ 2\left[e^{-2\Omega}\triangle_{A,jk} - 2e^{-2\Omega}\sigma^s(\partial_j\Phi)(\partial_k\Phi)\right] \\ &+ 2\frac{s}{s+1}(\widetilde{\sigma}^{s+1} - \sigma^{s+1})h_{jk}.\end{aligned} \tag{A.9}$$

Equation (A.1c) now easily follows from (A.9) via straightforward algebraic manipulation and the definition $Z_j = \frac{\partial_j\Phi}{\partial_t\Phi}$. We remark that the proofs of (A.1a) and (A.1b) require the use of Lemma A.6, and that the proof of (A.1d) requires the use of Lemma A.7. To obtain (A.1d), it is also helpful to note that $-(\partial_t\Phi)^2\big\{(1+2s) + \triangle_{(m)}\big\}$ is the coefficient of the differential operator ∂_t^2 in equation (5.7d). □

We now state the following four lemmas, which are needed for the proof of Proposition A.1.

Lemma A.4. *[Rin08, Lemma 4] The modified Ricci tensor from* (5.3) *can be decomposed as follows:*

$$\widehat{Ric}_{\mu\nu} = -\frac{1}{2}\hat{\square}_g g_{\mu\nu} + \frac{3}{2}(g_{0\mu}\partial_\nu\omega + g_{0\nu}\partial_\mu\omega) + \frac{3}{2}\omega\partial_t g_{\mu\nu} + A_{\mu\nu}, \quad (\mu,\nu = 0,1,2,3), \tag{A.10}$$

where

$$A_{\mu\nu} = g^{\alpha\beta}g^{\kappa\lambda}\big[(\partial_\alpha g_{\nu\kappa})(\partial_\beta g_{\mu\lambda}) - \Gamma_{\alpha\nu\kappa}\Gamma_{\beta\mu\lambda}\big], \qquad (\mu,\nu = 0,1,2,3). \tag{A.11}$$

□

Lemma A.5. *[Rin08, Lemma 5] The term $A_{\mu\nu}$ ($\mu,\nu = 0,1,2,3$) defined in* (A.11) *can be decomposed into principal terms and error terms $\triangle_{A,\mu\nu}$ as follows:*

$$A_{00} = 3\omega^2 - \omega g^{ab}\partial_t g_{ab} + 2\omega g^{ab}\partial_a g_{0b} + \triangle_{A,00}, \tag{A.12a}$$

$$A_{0j} = 2\omega g^{00}\partial_t g_{0j} - 2\omega^2 g^{00}g_{0j} - \omega g^{00}\partial_j g_{00} + \omega g^{ab}\Gamma_{ajb} + \triangle_{A,0j}, \quad (j = 1,2,3), \tag{A.12b}$$

$$A_{jk} = 2\omega g^{00}\partial_t g_{jk} - 2\omega^2 g^{00}g_{jk} + \triangle_{A,jk}, \qquad (j,k = 1,2,3), \tag{A.12c}$$

where

$$\begin{aligned}\triangle_{A,00} = {} & (g^{00})^2\Big\{(\partial_t g_{00})^2 - (\Gamma_{000})^2\Big\} \\
& + g^{00}g^{0a}\Big\{2(\partial_t g_{00})(\partial_t g_{0a} + \partial_a g_{00}) - 4\Gamma_{000}\Gamma_{00a}\Big\} \\
& + g^{00}g^{ab}\Big\{(\partial_t g_{0a})(\partial_t g_{0b}) + (\partial_a g_{00})(\partial_b g_{00}) - 2\Gamma_{00a}\Gamma_{00b}\Big\} \\
& + g^{0a}g^{0b}\Big\{2(\partial_t g_{00})(\partial_a g_{0b}) + 2(\partial_t g_{0b})(\partial_a g_{00}) - 2\Gamma_{000}\Gamma_{a0b} - 2\Gamma_{00b}\Gamma_{00a}\Big\} \\
& + g^{ab}g^{0l}\Big\{2(\partial_t g_{0a})(\partial_l g_{0b}) + 2(\partial_b g_{00})(\partial_a g_{0l}) - 4\Gamma_{00a}\Gamma_{l0b}\Big\} \\
& + g^{ab}g^{lm}(\partial_a g_{0l})(\partial_b g_{0m}) + \frac{1}{2}g^{lm}(\underbrace{g^{ab}\partial_t g_{al} - 2\omega\delta_l^b}_{e^{2\Omega}g^{ab}\partial_t h_{al} - 2\omega g^{0b}g_{0l}})(\partial_b g_{0m} + \partial_m g_{0b}) \\
& - \frac{1}{4}g^{ab}g^{lm}(\partial_a g_{0l} + \partial_l g_{0a})(\partial_b g_{0m} + \partial_m g_{0b}) \\
& - \frac{1}{4}(\underbrace{g^{ab}\partial_t g_{al} - 2\omega\delta_l^b}_{e^{2\Omega}g^{ab}\partial_t h_{al} - 2\omega g^{0b}g_{0l}})(\underbrace{g^{lm}\partial_t g_{bm} - 2\omega\delta_b^l}_{e^{2\Omega}g^{lm}\partial_t h_{bm} - 2\omega g^{0l}g_{0b}}),
\end{aligned} \tag{A.13a}$$

$$\begin{aligned}\triangle_{A,0j} = {} & (g^{00})^2\Big\{(\partial_t g_{00})(\partial_t g_{0j}) - \Gamma_{000}\Gamma_{0j0}\Big\} \\
& + g^{00}g^{0a}\Big\{(\partial_t g_{00})(\partial_t g_{aj} + \partial_a g_{0j}) + (\partial_t g_{0j})(\partial_t g_{0a} + \partial_a g_{00})\Big\} \\
& - 2g^{00}g^{0a}\Big\{\Gamma_{000}\Gamma_{0ja} + \Gamma_{0j0}\Gamma_{00a}\Big\} \\
& + g^{00}(\underbrace{g^{ab}\partial_t g_{bj} - 2\omega\delta_j^a}_{e^{2\Omega}g^{ab}\partial_t h_{bj} - 2\omega g^{0a}g_{0j}})\Big(\partial_t g_{0a} - \frac{1}{2}\partial_a g_{00}\Big) + \frac{1}{2}g^{00}g^{ab}(\partial_a g_{00})(\partial_b g_{0j} + \partial_j g_{0b}) \\
& + g^{0a}g^{0b}\Big\{(\partial_t g_{00})(\partial_a g_{bj}) + (\partial_t g_{0b})(\partial_a g_{0j}) + (\partial_a g_{00})(\partial_t g_{bj}) + (\partial_a g_{0b})(\partial_t g_{0j})\Big\} \\
& - g^{0a}g^{0b}\Big\{\Gamma_{000}\Gamma_{ajb} + 2\Gamma_{00b}\Gamma_{0ja} + \Gamma_{a0b}\Gamma_{0j0}\Big\} \\
& + g^{ab}g^{0l}\Big\{(\partial_t g_{0a})(\partial_l g_{bj}) + (\partial_l g_{0a})(\partial_t g_{bj}) + (\partial_b g_{00})(\partial_a g_{lj}) + (\partial_b g_{0l})(\partial_a g_{0j})\Big\} \\
& - g^{ab}g^{0l}\Big\{2\Gamma_{00a}\Gamma_{ljb} + (\partial_l g_{0a} + \partial_a g_{0l})\Gamma_{0jb} - \frac{1}{2}(\partial_t g_{la})(\partial_b g_{0j} - \partial_j g_{0b})\Big\} \\
& + \omega g^{0a}(\underbrace{\partial_t g_{aj} - 2\omega g_{aj}}_{e^{2\Omega}\partial_t h_{aj}}) + \frac{1}{2}g^{0l}(\underbrace{g^{ab}\partial_t g_{la} - 2\omega\delta_l^b}_{e^{2\Omega}g^{ab}\partial_t h_{la} - 2\omega g^{0b}g_{0l}})\partial_t g_{bj} \\
& + g^{ab}g^{lm}\Big\{(\partial_a g_{0l})(\partial_b g_{mj}) - \frac{1}{2}(\partial_a g_{0l} + \partial_l g_{0a})\Gamma_{bjm}\Big\} \\
& + \frac{1}{2}g^{ab}(\underbrace{g^{lm}\partial_t g_{la} - 2\omega\delta_a^m}_{e^{2\Omega}g^{lm}\partial_t h_{la} - 2\omega g^{0m}g_{0a}})\Gamma_{bjm},
\end{aligned} \tag{A.13b}$$

$$
\begin{aligned}
\triangle_{A,jk} = {} & (g^{00})^2\Big\{(\partial_t g_{0j})(\partial_t g_{0k}) - \Gamma_{0j0}\Gamma_{0k0}\Big\} && \text{(A.13c)}\\
& + g^{00}g^{0a}\Big\{(\partial_t g_{0j})(\partial_t g_{ak} + \partial_a g_{0k}) + (\partial_t g_{0k})(\partial_t g_{aj} + \partial_a g_{0j})\Big\}\\
& - g^{00}g^{0a}\Big\{2\Gamma_{0j0}\Gamma_{0ka} + 2\Gamma_{0k0}\Gamma_{0ja}\Big\}\\
& + g^{00}g^{ab}\Big\{(\partial_a g_{0j})(\partial_b g_{0k}) - \frac{1}{2}(\partial_a g_{0j} - \partial_j g_{0a})(\partial_b g_{0k} - \partial_k g_{0b})\Big\}\\
& - \frac{1}{2}g^{00}(\underbrace{g^{ab}\partial_t g_{aj} - 2\omega\delta_j^b}_{e^{2\Omega}g^{ab}\partial_t h_{aj} - 2\omega g^{0b}g_{0j}})(\partial_b g_{0k} - \partial_k g_{0b})\\
& - \frac{1}{2}g^{00}(\underbrace{g^{ab}\partial_t g_{bk} - 2\omega\delta_k^a}_{e^{2\Omega}g^{ab}\partial_t h_{bk} - 2\omega g^{0a}g_{0k}})(\partial_a g_{0j} - \partial_j g_{0a})\\
& + \omega g^{00}(\underbrace{g_{bk}g^{ab} - \delta_k^a}_{-g_{0k}g^{0a}})\partial_t g_{aj} + \frac{1}{2}g^{00}(\underbrace{g^{ab}\partial_t g_{aj} - 2\omega\delta_j^b}_{e^{2\Omega}g^{ab}\partial_t h_{aj} - 2\omega g^{0b}g_{0j}})(\underbrace{\partial_t g_{bk} - 2\omega g_{bk}}_{e^{2\Omega}\partial_t h_{bk}})\\
& + g^{0a}g^{0b}\Big\{(\partial_t g_{0j})(\partial_a g_{bk}) + (\partial_t g_{bj})(\partial_a g_{0k}) + (\partial_a g_{0j})(\partial_t g_{bk}) + (\partial_a g_{bj})(\partial_t g_{0k})\Big\}\\
& - g^{0a}g^{0b}\Big\{\Gamma_{0j0}\Gamma_{akb} + 2\Gamma_{0jb}\Gamma_{0ka} + \Gamma_{ajb}\Gamma_{0k0}\Big\}\\
& + g^{ab}g^{0l}\Big\{(\partial_t g_{aj})(\partial_l g_{bk}) + (\partial_l g_{aj})(\partial_t g_{bk}) + (\partial_b g_{0j})(\partial_a g_{lk}) + (\partial_b g_{lj})(\partial_a g_{0k})\Big\}\\
& - g^{ab}g^{0l}\Big\{2\Gamma_{0ja}\Gamma_{lkb} + 2\Gamma_{lja}\Gamma_{0kb}\Big\}\\
& + g^{ab}g^{ml}\Big\{(\partial_a g_{lj})(\partial_b g_{mk}) - \Gamma_{ajl}\Gamma_{bkm}\Big\}.
\end{aligned}
$$

□

Lemma A.6. *[Rin08, Lemma 6] The sums $A_{00} + I_{00}$ and $A_{0j} + I_{0j}$, $(j = 1, 2, 3)$, can be decomposed into principal terms and error terms as follows, where I_{00}, I_{0j} are defined in* (5.6a)-(5.6b)*; A_{00}, A_{0j} are defined in* (A.11)*; and $\triangle_{A,00}, \triangle_{A,0j}$ are defined in* (A.13a) - (A.13b)*:*

$$A_{00} + 2\omega\Gamma^0 - 6\omega^2 = \omega\partial_t g_{00} + 3\omega^2(g_{00} + 1) + 3\omega^2 g_{00} + \triangle_{A,00} + \triangle_{C,00}, \quad \text{(A.14a)}$$

$$A_{0j} + 2\omega(3\omega g_{0j} - \Gamma_j) = 4\omega^2 g_{0j} - \omega g^{ab}\Gamma_{ajb} + \triangle_{A,0j} + \triangle_{C,0j}, \quad \text{(A.14b)}$$

where

$$
\begin{aligned}
\triangle_{C,00} = {} & -6(g_{00})^{-1}\omega^2\left\{(g_{00} + 1)^2 - g^{0a}g_{0a}\right\} && \text{(A.15a)}\\
& - \omega(g^{00} + 1)(\underbrace{g^{ab}\partial_t g_{ab} - 6\omega}_{e^{2\Omega}g^{ab}\partial_t h_{ab} - 2\omega g^{0a}g_{0a}})\\
& + 2\omega(g^{00} + 1)g^{ab}\partial_a g_{0b} + \omega(g^{00} + 1)(g^{00} - 1)\partial_t g_{00}\\
& + 2\omega g^{00}g^{0a}(\Gamma_{0a0} + 2\Gamma_{00a}) + 4\omega g^{0a}g^{0b}\Gamma_{0ab} + 2\omega g^{ab}g^{0l}\Gamma_{alb},\\
\triangle_{C,0j} = {} & 2\omega^2(g^{00} + 1)g_{0j} - 2\omega g^{0a}\Big\{(\underbrace{\partial_t g_{aj} - 2\omega g_{aj}}_{e^{2\Omega}\partial_t h_{aj}}) + \partial_a g_{0j} - \partial_j g_{0a}\Big\}. && \text{(A.15b)}
\end{aligned}
$$

□

Lemma A.7. *The fully raised Christoffel symbols $\Gamma^{\mu\alpha\nu}$ $(\mu, \alpha, \nu = 0,1,2,3)$ can be decomposed into principal terms and error terms $\triangle_{(\Gamma)}^{\mu\alpha\nu}$ as follows:*

$$\Gamma^{000} = \triangle_{(\Gamma)}^{000}, \tag{A.16a}$$

$$\Gamma^{j00} = \Gamma^{00j} = \triangle_{(\Gamma)}^{j00}, \qquad (j = 1,2,3), \tag{A.16b}$$

$$\Gamma^{0j0} = \triangle_{(\Gamma)}^{0j0}, \qquad (j = 1,2,3), \tag{A.16c}$$

$$\Gamma^{0jk} = \Gamma^{kj0} = -\omega g^{jk} + \triangle_{(\Gamma)}^{0jk}, \qquad (j,k = 1,2,3), \tag{A.16d}$$

$$\Gamma^{j0k} = \Gamma^{k0j} = \omega g^{jk} + \triangle_{(\Gamma)}^{j0k}, \qquad (j,k = 1,2,3), \tag{A.16e}$$

$$\Gamma^{ijk} = \triangle_{(\Gamma)}^{ijk}, \qquad (i,j,k = 1,2,3), \tag{A.16f}$$

where

$$\triangle^{000}_{(\Gamma)} = \frac{1}{2}(g^{00})^3\partial_t g_{00} + \frac{1}{2}(g^{00})^2 g^{0a}(2\partial_t g_{0a} + \partial_a g_{00}) + \frac{1}{2}g^{00}g^{0a}g^{0b}(\partial_t g_{ab} + 2\partial_a g_{0b}) + \frac{1}{2}g^{0a}g^{0b}g^{0l}\partial_a g_{bl}, \tag{A.17a}$$

$$\triangle^{j00}_{(\Gamma)} = \frac{1}{2}(g^{00})^2(g^{0j}\partial_t g_{00} + g^{aj}\partial_a g_{00}) + g^{00}g^{0j}g^{0a}\partial_t g_{0a} + \frac{1}{2}g^{0j}g^{0a}g^{0b}\partial_t g_{ab} + g^{00}g^{aj}g^{0b}\partial_a g_{0b} + \frac{1}{2}g^{aj}g^{0b}g^{0l}\partial_a g_{bl}, \tag{A.17b}$$

$$\triangle^{0j0}_{(\Gamma)} = \frac{1}{2}(g^{00})^2(g^{0j}\partial_t g_{00} + 2g^{aj}\partial_t g_{0a} - g^{aj}\partial_a g_{00}) + g^{00}(g^{aj}g^{0b}\partial_t g_{ab} + g^{0a}g^{0j}\partial_a g_{00} + g^{0a}g^{bj}\partial_a g_{0b} - g^{aj}g^{0b}\partial_a g_{0b}) + \frac{1}{2}g^{0a}g^{0j}g^{0b}(2\partial_a g_{0b} - \partial_t g_{ab}) + g^{0a}g^{bj}g^{0l}\partial_a g_{bl} - \frac{1}{2}g^{0a}g^{bj}g^{0l}\partial_b g_{al}, \tag{A.17c}$$

$$\begin{aligned}\triangle^{0jk}_{(\Gamma)} &= \frac{1}{2}g^{00}\Big\{g^{0j}g^{0k}\partial_t g_{00} + g^{0j}g^{ak}\partial_a g_{00}\Big\} \\ &+ \frac{1}{2}g^{00}\Big\{g^{aj}g^{0k}(2\partial_t g_{0a} - \partial_a g_{00}) + g^{aj}g^{kb}(\partial_b g_{0a} - \partial_a g_{0b})\Big\} \\ &+ \frac{1}{2}g^{0a}\Big\{g^{0j}g^{0k}\partial_a g_{00} + g^{0j}g^{bk}(\partial_a g_{0b} + \partial_b g_{0a} - \partial_t g_{ab})\Big\} \\ &+ \frac{1}{2}g^{0a}\Big\{g^{bj}g^{0k}(\partial_t g_{ab} + \partial_a g_{0b} - \partial_b g_{0a}) + g^{bj}g^{lk}(\partial_a g_{bl} + \partial_l g_{ab} - \partial_b g_{al})\Big\} \\ &+ \frac{1}{2}g^{00}\big(\underbrace{g^{aj}g^{bk}\partial_t g_{ab} - 2\omega g^{jk}}_{g^{bk}(e^{2\Omega}g^{aj}\partial_t h_{ab} - 2\omega g^{0j}g_{0b})}\big) + \omega(g^{00}+1)g^{jk},\end{aligned} \tag{A.17d}$$

$$\begin{aligned}\triangle^{j0k}_{(\Gamma)} &= \frac{1}{2}g^{00}\Big\{g^{0j}g^{0k}\partial_t g_{00} + g^{0j}g^{ak}\partial_a g_{00} + g^{aj}g^{0k}\partial_a g_{00} + g^{aj}g^{bk}(\partial_a g_{0b} + \partial_b g_{0a})\Big\} \\ &+ \frac{1}{2}g^{0a}\Big\{g^{0j}g^{0k}(2\partial_t g_{0a} - \partial_a g_{00}) + g^{0j}g^{bk}(\partial_t g_{ab} + \partial_b g_{0a} - \partial_a g_{0b})\Big\} \\ &+ \frac{1}{2}g^{0a}\Big\{g^{bj}g^{0k}(\partial_t g_{ab} + \partial_b g_{0a} - \partial_a g_{0b}) + g^{bj}g^{lk}(\partial_b g_{al} + \partial_l g_{ab} - \partial_a g_{bl})\Big\} \\ &- \frac{1}{2}g^{00}\big(\underbrace{g^{aj}g^{bk}\partial_t g_{ab} - 2\omega g^{jk}}_{g^{bk}(e^{2\Omega}g^{aj}\partial_t h_{ab} - 2\omega g^{0j}g_{0b})}\big) - \omega(g^{00}+1)g^{jk},\end{aligned} \tag{A.17e}$$

$$\begin{aligned}\triangle^{ijk}_{(\Gamma)} &= \frac{1}{2}g^{0i}g^{0j}g^{0k}\partial_t g_{00} + \frac{1}{2}\big(g^{ai}g^{0j}g^{0k} + g^{0i}g^{0j}g^{ak} - g^{0i}g^{aj}g^{0k}\big)\partial_a g_{00} + g^{0i}g^{aj}g^{0k}\partial_t g_{0a} \\ &+ \frac{1}{2}\big(g^{ai}g^{bj}g^{0k} + g^{ai}g^{0j}g^{bk} + g^{bi}g^{0j}g^{ak} + g^{0i}g^{bj}g^{ak} - g^{bi}g^{aj}g^{0k} - g^{0i}g^{aj}g^{bk}\big)\partial_a g_{0b} \\ &+ \frac{1}{2}\big(g^{0i}g^{aj}g^{bk} + g^{ai}g^{bj}g^{0k} - g^{ai}g^{0j}g^{bk}\big)\partial_t g_{ab} \\ &+ \frac{1}{2}\big(g^{ai}g^{bj}g^{lk} + g^{bi}g^{lj}g^{ak} - g^{bi}g^{aj}g^{lk}\big)\partial_a g_{bl}.\end{aligned} \tag{A.17f}$$

Proof. The proof is again a series of tedious computations that follow from the formula $\Gamma^{\mu\alpha\nu} = \frac{1}{2}g^{\mu\kappa}g^{\alpha\lambda}g^{\nu\sigma}(\partial_\kappa g_{\lambda\sigma} + \partial_\sigma g_{\kappa\lambda} - \partial_\lambda g_{\kappa\sigma})$. □

B Sobolev-Moser Inequalities

In Appendix B, we provide some standard Sobolev-Moser type estimates that play a fundamental role in our analysis of the nonlinear terms in our equations. The propositions and corollaries stated below can be proved using methods similar to those used in [Hör97, Chapter 6] and in [KM81]. The proofs given in the literature are commonly based on a version of the Gagliardo-Nirenberg inequality [Nir59], which we state as Lemma B.1, together with repeated use of Hölder's inequality and/or Sobolev embedding. Throughout this appendix, we abbreviate $L^p = L^p(\mathbb{T}^3)$, and $H^M = H^M(\mathbb{T}^3)$.

Lemma B.1. *If M, N are integers such that $0 \leq M \leq N$, and v is a function on $\mathbb{T}^3$ such that $v \in L^\infty$, $\|\underline{\partial}^{(N)}v\|_{L^2} < \infty$, then*

$$\|\underline{\partial}^{(M)}v\|_{L^{2N/M}} \leq C(M,N)\|v\|_{L^\infty}^{1-\frac{M}{N}}\|\underline{\partial}^{(N)}v\|_{L^2}^{\frac{M}{N}}. \tag{B.1}$$

Proposition B.2. *Let $M \geq 0$ be an integer. If $\{v_a\}_{1\leq a\leq l}$ are functions such that $v_a \in L^\infty$, $\|\underline{\partial}^{(M)}v_a\|_{L^2} < \infty$ for $1 \leq a \leq l$, and $\vec{\alpha}_1, \cdots, \vec{\alpha}_l$ are spatial derivative multi-indices with $|\vec{\alpha}_1| + \cdots + |\vec{\alpha}_l| = M$, then*

$$\|(\partial_{\vec{\alpha}_1}v_1)(\partial_{\vec{\alpha}_2}v_2)\cdots(\partial_{\vec{\alpha}_l}v_l)\|_{L^2} \leq C(l,M)\sum_{a=1}^{l}\Big(\|\underline{\partial}^{(M)}v_a\|_{L^2}\prod_{b\neq a}\|v_b\|_{L^\infty}\Big). \tag{B.2}$$

Corollary B.3. *Let $M \geq 1$ be an integer, let $\mathfrak{K}$ be a compact set, and let $F \in C_b^M(\mathfrak{K})$ be a function. Assume that v is a function such that $v(\mathbb{T}^3) \subset \mathfrak{K}$ and $\underline{\partial}v \in H^{M-1}$. Then $\underline{\partial}(F\circ v) \in H^{M-1}$, and*

$$\|\underline{\partial}(F\circ v)\|_{H^{M-1}} \leq C(M)\|\underline{\partial}v\|_{H^{M-1}}\sum_{l=1}^{M}|F^{(l)}|_{\mathfrak{K}}\|v\|_{L^\infty}^{l-1}. \tag{B.3}$$

Corollary B.4. *Let $M \geq 1$ be an integer, let $\mathfrak{K}$ be a compact, convex set, and let $F \in C_b^M(\mathfrak{K})$ be a function. Assume that v is a function such that $v(\mathbb{T}^3) \subset \mathfrak{K}$ and $v - \bar{v} \in H^M$, where $\bar{v} \in \mathfrak{K}$ is a constant. Then $F\circ v - F\circ\bar{v} \in H^M$, and*

$$\|F\circ v - F\circ\bar{v}\|_{H^M} \leq C(M)\Big\{|F^{(1)}|_{\mathfrak{K}}\|v-\bar{v}\|_{L^2} + \|\underline{\partial}v\|_{H^{M-1}}\sum_{l=1}^{M}|F^{(l)}|_{\mathfrak{K}}\|v\|_{L^\infty}^{l-1}\Big\}. \tag{B.4}$$

Proposition B.5. *Let $M \geq 1, l \geq 2$ be integers. Suppose that $\{v_a\}_{1 \leq a \leq l}$ are functions such that $v_a \in L^\infty$ for $1 \leq a \leq l$, that $v_l \in H^M$, and that $\underline{\partial} v_a \in H^{M-1}$ for $1 \leq a \leq l-1$. Then*

$$\|v_1 v_2 \cdots v_l\|_{H^M} \leq C(l,M)\Big\{\|v_l\|_{H^M}\prod_{a=1}^{l-1}\|v_a\|_{L^\infty} + \sum_{a=1}^{l-1}\|\underline{\partial} v_a\|_{H^{M-1}}\prod_{b\neq a}\|v_b\|_{L^\infty}\Big\}. \quad \text{(B.5)}$$

Remark B.6. The significance of this proposition is that only one of the functions, namely v_l, is estimated in L^2.

Proposition B.7. *Let $M \geq 1$ be an integer, let $\mathfrak{K}$ be a compact, convex set, and let $F \in C_b^M(\mathfrak{K})$ be a function. Assume that v_1 is a function such that $v_1(\mathbb{T}^3) \subset \mathfrak{K}$, that $\underline{\partial} v_1 \in L^\infty$, and that $\underline{\partial}^{(M)} v_1 \in L^2$. Assume that $v_2 \in L^\infty$, that $\underline{\partial}^{(M-1)} v_2 \in L^2$, and let $\vec{\alpha}$ be a spatial derivative multi-index with with $|\vec{\alpha}| = M$. Then $\partial_{\vec{\alpha}}\left((F \circ v_1)v_2\right) - (F \circ v_1)\partial_{\vec{\alpha}} v_2 \in L^2$, and*

$$\begin{aligned}
&\|\partial_{\vec{\alpha}}\left((F \circ v_1)v_2\right) - (F \circ v_1)\partial_{\vec{\alpha}} v_2\|_{L^2} \\
&\quad \leq C(M)\Big\{|F^{(1)}|_{\mathfrak{K}}\|\underline{\partial} v_1\|_{L^\infty}\|\underline{\partial}^{(M-1)} v_2\|_{L^2} + \|v_2\|_{L^\infty}\|\underline{\partial} v_1\|_{H^{M-1}}\sum_{l=1}^{M}|F^{(l)}|_{\mathfrak{K}}\|v_1\|_{L^\infty}^{l-1}\Big\}.
\end{aligned} \quad \text{(B.6)}$$

Remark B.8. The significance of this proposition is that the M^{th} order derivatives of v_2 do not play a role in the conclusions.

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