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Macroscopic Limits of Microscopic Models

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Abstract

Many physical systems are comprised of several discrete elements, the equations of motion of each element being known. If the system has a large number of degrees of freedom, it may be possible to treat it as a continuous system. In this event, one might wish to derive the equations of motion of the continuous (macroscopic) system by taking a suitable limit of the equations governing the discrete (microscopic) system. The classical example of this involves a row of particles with each particle connected to its nearest neighbor by a linear spring, its continuum counterpart being a linearly elastic bar; see Figure 1.

In a typical undergraduate engineering subject on, say Dynamics, the transition from a discrete system to a continuous system is usually carried out through a formal Taylor expansion of the terms of the discrete model about some reference configuration. The aim of this paper is to draw attention to the fact that a macroscopic model derived in this way should be examined critically in order to confirm that it provides a faithful representation of the underlying microscopic model. We use a specific (striking) example to make this point. In this example, a simple solution of the discrete model can be stable or unstable depending on the state of the system. However the corresponding solution of the continuous system is always unstable! We go on to show how the dispersion relations of the two models can be used to identify the source of the discrepancy and to suggest how one might modify the continuous model.

Key words: discrete models, continuous models, dispersion relations, stability

1 Introduction

There are numerous examples of physical systems that are modeled as the continuum limit of a system with a large number of discrete elements. For example (Figure 1),

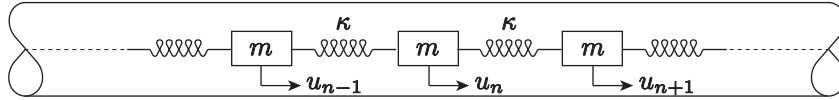


Figure 1: An elastic bar as a model for a row of masses connected by springs.

the partial differential equation

$$\frac{\partial}{\partial x} \left(E \frac{\partial u}{\partial x} \right) = \rho \frac{\partial^2 u}{\partial t^2} \quad (a)$$

governing the longitudinal motion of an elastic bar (elastic modulus E , mass density ρ) can be derived from the system of coupled ordinary differential equations

$$\kappa(u_{n+1} - u_n) - \kappa(u_n - u_{n-1}) = m\ddot{u}_n \quad (b)$$

describing the motion of a row of mass points (mass m) connected by linear springs (stiffness κ), e.g. see Section 12.1 of Goldstein [3]. A relatively more complex discrete system is one consisting of a row of rigid blocks, each with both translational and rotational inertia, connected to each other by both bending and shearing springs. This discrete model can be used to derive the Timoshenko theory of beams, e.g. see Example 7.3 of Crandall et al. [1]. A somewhat different physical setting comes from materials science and concerns the motion of a dislocation in a lattice. The appropriate continuum model can be established by studying the motion of a row of particles moving in a periodic energy potential, e.g. see Rosenau [8]. Typically, the continuous model involves partial differential equations while the discrete system is described by a set of coupled ordinary differential equations.

There are two ways in which to view discrete and continuous models. In one, the continuous model is taken to be “exact”, and the discrete model might, for example, be a discretization of it for purposes of numerical solution. The alternative is where the discrete model is “exact”, such as for example in an atomistic model of materials, and the continuous model is a suitable approximation. In the example studied in this paper we take the latter point of view.

The growing interest in physical phenomena at small length scales has led to a corresponding increase in the need for continuum models that are accurate at such length scales. The additional terms arising in such macroscopic models typically involve a length scale related to the small scale behavior that the model seeks to capture. For example, if additional terms are retained in the derivation of (a) from (b), one might be led to an enhanced continuum model for longitudinal motions of an elastic bar such as

$$\frac{\partial}{\partial x} \left(E \frac{\partial u}{\partial x} + M \frac{\partial^3 u}{\partial x^3} \right) = \rho \frac{\partial^2 u}{\partial t^2}, \quad (c)$$

where the new term captures the effects of strain gradients and M is another material parameter. Dimensional considerations imply that $\sqrt{M/E}$ has the dimension of length.

Does (c) properly describe the behavior of the discrete model (b)? Is this always true, or only true under certain conditions? If so what conditions? How does one look into such questions? Observe that if we set $E = 0$, equation (c) looks like the dynamic (Bernoulli-Euler) beam equation except that M would be negative. What does this say about the stability of solutions to (c)?

In an undergraduate engineering subject on, say Dynamics, the transition from a discrete system to a continuous system is typically carried out by identifying a small parameter, using it to scale the problem, and Taylor expanding the terms of the discrete model about some reference configuration. The rigorous proofs needed to show that the results of such formal calculations are meaningful (or not) are mathematically highly technical, and beyond the preparation of the typical engineering undergraduate student, e.g. see Giannoulis and Mielke [2]. This does not however mean that the student should therefore accept the model at face value, without some thought into whether the continuous model provides a reasonable representation of the underlying microscopic model.

In order to have some indication that a particular continuous model is a faithful counterpart of a given discrete model, one can study various initial-boundary value problems using both models and compare their responses. Of course it is not possible to study *all* initial-boundary value problems, and so one needs to approach this indirectly. As noted by Whitham in Section 11.1 of [10], there is a direct correspondence between a linear partial differential equation and the corresponding *dispersion relation* – the dispersion relation completely characterizes the dynamical behavior of a linear mathematical model. Thus we can compare two linearized models by comparing their dispersion relations¹.

In this paper we use a particular example to illustrate this point. The example comes from the mathematical modeling of traffic flow as will be explained in the next section. For now it is sufficient to say that in the discrete model we have two sequences $\{v_0(t), v_1(t), \dots, v_N(t)\}$ and $\{\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t)\}$ that obey the system of coupled ordinary differential equations

$$\left. \begin{aligned} \dot{\lambda}_n &= v_{n-1} - v_n, \\ \dot{v}_n &= \frac{V(\lambda_n) - v_n}{\tau}, \end{aligned} \right\} \quad n = 1, 2, \dots, N, \quad t \geq 0.$$

The constant τ and smooth function V characterize the system being modeled. Its continuum counterpart involves fields $v(x, t)$ and $\lambda(x, t)$ that obey the pair of partial differential equations

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial t} &= \frac{\partial v}{\partial x}, \\ \frac{\partial v}{\partial t} &= \frac{V(\lambda) - v}{\tau}, \end{aligned} \right\} \quad 0 \leq x \leq L, \quad t \geq 0.$$

¹This is of course only a necessary check and does not guarantee that the continuous model is always valid. For example very high frequency vibrations of the particles in the model related to (b) are captured as heat at the continuum model, not by (a).

We will study the stability of steady uniform solutions of both models by linearizing the preceding equations about these special solutions, and asking if the perturbations grow or decay. According to the discrete model we find that this solution can be *stable or unstable* depending on the state of the system, but it is *always unstable* according to the continuous model! In order to understand (and remedy) the cause of this variance, we will examine the dispersion relation of the discrete model more closely. By studying its behavior in the limit of waves with long wavelengths (which should describe the continuous model) we can identify the source of the deficiency and obtain some guidance on how one might modify the continuous model.

This paper is organized as follows: in Section 2 we describe the discrete model and derive its continuum counterpart. Steady uniform motions of each model are considered in Section 3, and their stability is examined in Section 4. In Section 5 we inquire into the source of the deficiency in the continuous model, and use that insight to derive a second continuous model. Steady uniform motions according to this modified macroscopic model are found to be stable under the precise conditions for stability of the discrete model. We close with some concluding remarks in Section 6.

2 Mathematical model.

Perhaps it is worth mentioning at the outset that this is not a paper about the dynamics of traffic flow. We will simply be using an example from that field to discuss the relation between discrete and continuous models.

2.1 A discrete model.

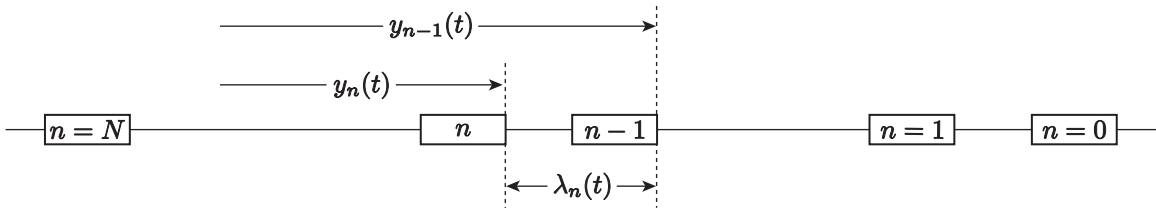


Figure 2: Row of vehicles on a one-lane highway. Current location $y_n(t)$, headway $\lambda_n(t)$.

Consider $N + 1$ identical vehicles moving along a one-lane highway – the x -axis. The position of (say, the front bumper) of vehicle n at time t is $y_n(t)$; its velocity is $v_n(t) = \dot{y}_n(t)$. The vehicles are numbered $n = 0, 1, 2, \dots, N$ such that n increases in the direction of *decreasing* x ; see Figure 2.

The distance between vehicles is of greater interest than the location of each vehicle and so we let

$$\lambda_n(t) = y_{n-1}(t) - y_n(t)$$

denote the *headway*, i.e. the distance between the n th and $(n - 1)$ th vehicles at time t . The λ 's and v 's are the quantities of most interest and so the theory will be formulated in terms of them. Observe that they are related by the compatibility requirement

$$\dot{\lambda}_n(t) = v_{n-1}(t) - v_n(t), \quad (1)$$

where the superior dot denotes the time derivative.

A typical model of traffic flow consists of equation (1), complemented by a second equation that also involves the λ 's and v 's. The classical such model is due to Lighthill, Whitham [5] and Richards [7] – the LWR model. The LWR model is a continuous model whose discrete counterpart consists of equation (1) together with

$$v_n(t) = V(\lambda_n(t)). \quad (2)$$

The empirical function V characterizes the roadway, the driving behavior, the vehicles, etc., an example of which is

$$V(\lambda) = v_{max} \left(1 - e^{-\beta(\lambda - \lambda_{min})} \right) \quad \text{for } \lambda \geq \lambda_{min};$$

here β , λ_{min} and v_{max} are positive parameters. In this example, the value of V (the speed) increases monotonically from zero to the maximum speed v_{max} as the headway increases from its minimum value λ_{min} to infinity; see Figure 3. The results in this paper will *not* rely on any particular choice of V .

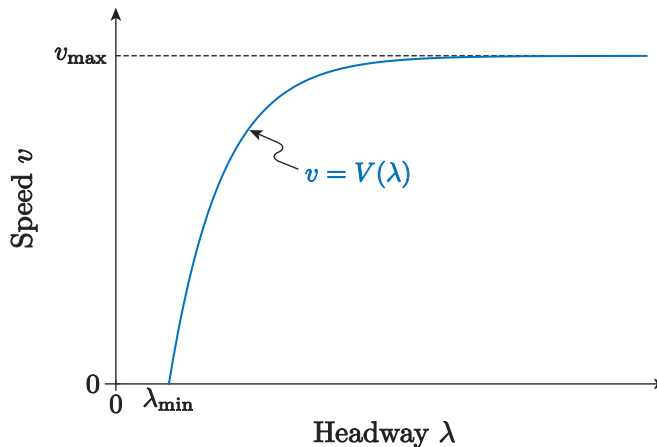


Figure 3: Typical equilibrium velocity function $V(\lambda)$ versus headway λ .

Perhaps the most obvious drawback of the LWR model is that, because equation (2) gives the velocity at time t as a function of the headway at the *same* instant t , any change in the headway is accompanied by an instantaneous change in the velocity, without any timelag between them.

In more realistic models of traffic flow, the algebraic equation (2) is replaced by an evolution equation such as, for example,

$$\dot{v}_n(t) = \frac{V(\lambda_n(t)) - v_n(t)}{\tau}. \quad (3)$$

Observe that if at some instant the current velocity v_n is smaller than the value $V(\lambda_n)$ the vehicle will accelerate² and v_n will increase towards $V(\lambda_n)$; if v_n is larger than $V(\lambda_n)$ the vehicle will decelerate and v_n will decrease towards $V(\lambda_n)$. Moreover, in a steady motion where all the \dot{v}_n 's vanish, (3) specializes to (2). For these reasons the function V is referred to as the “*equilibrium velocity function*”. Next, suppose that instead of (3), we use the following alternative generalization of (2): $v_n(t + \tau) = V(\lambda_n(t))$. This says that the driver responds with a time lag τ , i.e. the velocity at time $t + \tau$ is determined by the headway at a slightly earlier time t . Thus the parameter τ here represents the driver’s reaction time. Quick reaction corresponds to small values of τ , and vice versa. For small τ , one might Taylor expand the left hand side of $v_n(t + \tau) = V(\lambda_n(t))$ and write $v_n(t) + \tau\dot{v}_n(t) = V(\lambda_n(t))$. This is identical to (3). This suggests that we can view the parameter τ in (3) also as the *driver’s reaction time*.

The *discrete dynamical model* that we consider in this paper comprises of (1) and (3). It involves the equilibrium velocity function $V(\lambda)$ and the reaction time τ .

2.2 A continuous model.

We now turn to a continuous model of traffic flow. Since we shall work throughout within a *Lagrangian framework*, we identify each vehicle by its position x_n in a reference configuration. The vehicles need not occupy the reference configuration during the motion. It is simply a conveniently chosen configuration that they could occupy. If the $N + 1$ vehicles occupy a total length L of the roadway in the reference configuration, and they are uniformly spaced, then

$$x_n = -n\ell, \tag{4}$$

where $\ell = L/N$ and we have taken $x_0 = 0$ with no loss of generality. In this section we seek to replace the discrete model (1), (3) by a continuous model when N is large (i.e. ℓ/L is small) at fixed L .

In order to develop the continuous model, let $\lambda(x, t)$ and $v(x, t)$ be smooth functions of the continuous variables $0 \leq x \leq L, t \geq 0$ such that

$$\lambda_n(t) = \lambda\left(\frac{x_{n-1} + x_n}{2}, t\right), \quad v_n(t) = v(x_n, t).$$

Thus $\lambda(x, t)$ and $v(x, t)$ are, respectively, the headway and velocity at time t of the vehicle that is at x in the reference configuration. Formal Taylor expansions give

$$\begin{aligned} v_{n-1}(t) - v_n(t) &= v(x_{n-1}, t) - v(x_n, t) = v(x_n + \ell, t) - v(x_n, t) = \ell v_x + \dots, \\ \lambda_n(t) &= \lambda\left(\frac{x_{n-1} + x_n}{2}, t\right) = \lambda(x_n + \ell/2, t) = \lambda + \dots, \end{aligned} \tag{5}$$

where we have noted from (4) that $x_{n-1} = x_n + \ell$. Subscripts x and t denote partial differentiation with respect to those subscripts. Thus we have

$$\lambda_n(t) \sim \lambda(x, t), \quad v_n(t) \sim v(x, t), \quad \dot{\lambda}_n(t) \sim \lambda_t, \quad v_{n-1}(t) - v_n(t) \sim \ell v_x(x, t),$$

²This assumes that $\tau > 0$.

and so we may replace the system of equations (1), (3) of the discrete model by the pair of partial differential equations

$$\lambda_t = \ell v_x, \quad v_t = \frac{V(\lambda) - v}{\tau}. \quad (6)$$

The presence of the parameter ℓ here is simply a reflection of the fact that we have adopted a Lagrangian formulation; ℓ is a characteristic of the reference configuration. Whenever it appears, it does so in the form $\ell \frac{\partial}{\partial x}$.

3 Steady uniform motion.

3.1 Steady uniform motion in discrete model.

Now consider a steady uniform motion of the row of vehicles in which each vehicle travels at a velocity v_* and the spacing between each pair of adjacent vehicles is λ_* . In such a motion $y_n(t) = -n\lambda_* + v_*t$ whence

$$\lambda_n(t) = \lambda_*, \quad v_n(t) = v_*, \quad (7)$$

for all n . The compatibility equation (1) is satisfied automatically whereas the equation of motion (3) yields

$$v_* = V(\lambda_*). \quad (8)$$

Thus the λ 's and v 's in a steady uniform motion are *not independent*. They are related through the equilibrium velocity function V .

3.2 Steady uniform motion in continuous model.

In the continuous model, a steady uniform motion where each vehicle travels at a velocity v_* and the headway between each pair of vehicles is λ_* is characterized by

$$\lambda(x, t) = \lambda_*, \quad v(x, t) = v_*. \quad (9)$$

This satisfies the compatibility equation (6)₁ automatically, and the equation of motion (6)₂ requires that (8) hold, just as in the discrete model.

4 Stability of a steady uniform motion.

4.1 Stability of a steady uniform motion in discrete model

In order to examine the stability of a steady uniform motion, we now consider the behavior of a perturbed motion close to it. Thus we now consider a motion

$$y_n(t) = -n\lambda_* + v_*t + u_n(t), \quad v_* = V(\lambda_*),$$

where u_n represents the departure from the steady uniform motion. It is assumed to be suitably small. The headway $\lambda_n = y_{n-1} - y_n$ and velocity $v_n = \dot{y}_n$ associated with this neighboring motion are

$$\lambda_n(t) = \lambda_* + u_{n-1}(t) - u_n(t), \quad v_n = v_* + \dot{u}_n(t). \quad (10)$$

The compatibility equation (1) is satisfied automatically. Substituting (10) into (3), linearizing and using (8) leads to the system of linear equations

$$\ddot{u}_n = a_1(u_{n-1} - u_n) - a_2\dot{u}_n \quad (11)$$

where we have set

$$a_1 = V'(\lambda_*)/\tau, \quad a_2 = 1/\tau; \quad (12)$$

a_1 and a_2 are constants.

Consider solutions of (11) in the form

$$u_n(t) = e^{i(-kn\ell + \omega t)}. \quad (13)$$

Here k is real and ω may be complex. This can be viewed as one term in a Fourier expansion of a more general motion; k is the wave number of the motion (the wave length is $\sim 1/k\ell$), the real part of ω is the frequency of oscillation, and its imaginary part is the growth/decay rate of the amplitude. For stability, the imaginary part of ω must be positive for all wave numbers so that the amplitude of oscillation then decays with time.

Substituting (13) into (11) leads to the following *dispersion relation*, a (quadratic) equation for ω in terms of k :

$$\omega^2 + (2b_1 + i2b_2)\omega + (d_1 + id_2) = 0 \quad (14)$$

where we have set

$$2b_1 = 0, \quad 2b_2 = -a_2, \quad d_1 = -2a_1 \sin^2 k\ell/2, \quad d_2 = 2a_1 \sin k\ell/2 \cos k\ell/2. \quad (15)$$

From Appendix 2 we know that both roots ω of this quadratic equation have positive imaginary parts if and only if $b_2 < 0$ and $4b_1b_2d_2 - 4d_1b_2^2 > d_2^2$. These two inequalities specialize, on using (15), to

$$a_2 > 0, \quad \sin^2 \frac{k\ell}{2} + \frac{a_2^2}{2a_1} - 1 > 0.$$

The latter inequality must hold for all wave numbers k and for this it is necessary and sufficient that

$$\frac{a_2^2}{2a_1} > 1.$$

Note from this that a_1 necessarily has to be positive. On using this, the preceding inequality can be written as $a_2^2 > 2a_1 > 0$.

Thus in *summary*, a steady uniform motion (7) of the discrete model (1), (3) is stable if and only if $a_2^2 > 2a_1 > 0, a_2 > 0$, which can be written equivalently, in terms of the equilibrium velocity function and reaction time by using (12), as

$$\frac{1}{2\tau} > V'(\lambda_*) > 0. \quad (16)$$

Thus stability requires the slope of the equilibrium velocity function at the relevant headway and the reaction time, to both be positive: $V'(\lambda_*) > 0, \tau > 0$. Equation (16) then states that for a given headway λ_* , the steady uniform motion is stable if the driver has a fast response, i.e. if τ is sufficiently small, specifically if $\tau < 1/[2V'(\lambda_*)]$. Instability occurs if the driver's response is too slow, i.e. if $\tau > 1/[2V'(\lambda_*)]$.

4.2 Stability of steady uniform motion in continuous model

In order to study the stability of the steady uniform motion (9) according to the continuous model (6) we again consider the response of a perturbed motion that is close to it. Thus, consider a motion $\lambda(x, t) = \lambda_* + f(x, t), v(x, t) = v_* + g(x, t)$ where f and g are suitably small. Substituting this into (6)₁ yields $f_t = \ell g_x$. It can be readily verified by substitution that, for any smooth function $u(x, t), f = \ell u_x(x, t), g = u_t(x, t)$ is a solution of this partial differential equation $f_t = \ell g_x$. In fact, this can be shown to be its general solution³. Thus in the continuous model, a perturbed motion can be expressed as

$$\lambda(x, t) = \lambda_* + \ell u_x(x, t), \quad v(x, t) = v_* + u_t(x, t), \quad (17)$$

where $u(x, t)$ denotes the departure from the steady uniform motion.

Equation (17) automatically satisfies the compatibility equation (6)₁. Substituting (17) into (6)₂, linearizing, and using (8) leads to

$$u_{tt} = a_1 \ell u_x - a_2 u_t \quad (18)$$

where the constants a_1 and a_2 are given by (12).

We again seek solutions in the form

$$u = e^{i(kx + \omega t)} \quad (19)$$

which when substituted into (18) leads to the dispersion relation $\omega^2 + (2b_1 + i2b_2)\omega + (d_1 + id_2) = 0$ where now we have set

$$2b_1 = 0, \quad 2b_2 = -a_2, \quad d_1 = 0, \quad d_2 = k\ell a_1. \quad (20)$$

For stability, both roots ω of this quadratic equation must have positive imaginary parts. From Appendix 2 we know that the requirement for this is that $b_2 < 0$ and $4b_1 b_2 d_2 - 4d_1 b_2^2 > d_2^2$. These two inequalities specialize on using (20) to

$$a_2 > 0, \quad 0 > a_1^2.$$

³This requires the domain of the x, t -plane on which the various fields are defined to be simply connected which it is.

The latter inequality cannot hold and so we conclude that at least one root ω of the dispersion relation must have a negative imaginary part, implying that in general, perturbations will grow. Thus the steady uniform motion according to this continuous model is always unstable.

5 A second continuous model

In order to understand why the discrete and continuous models led to such different conclusions, it is illuminating to look at the long wavelength limit of the discrete problem since this should correspond to the continuous problem. The wavelength of the motion (13) is $\sim 1/(k\ell)$ and so we are interested in small $k\ell$. For small $k\ell$ (15) yields

$$d_1 = -a_1 k^2 \ell^2 / 2 + O((k\ell)^4), \quad d_2 = a_1 k\ell + O((k\ell)^3). \quad (21)$$

It should be pointed out that, though d_1 and d_2 in (21) have different orders of magnitude (quadratic and linear respectively in $k\ell$), it can be readily verified that when they are substituted into the stability inequality $4b_1 b_2 d_2 - 4d_1 b_2^2 > d_2^2$ they contribute equally to it. On comparing (21) with the corresponding expressions (20)_{3,4} of the continuous model, we see that the expression for d_1 in (20) is deficient. The deficiency involves a k^2 term. When the exponential solution (19) is substituted into a linear differential equation, each derivative $\partial/\partial x$ leads to a term ik in the dispersion relation. Therefore the deficiency in a k^2 term suggests that the linearized equation (18) is missing a u_{xx} term. This in turn suggests that the nonlinear equation (6)₂ is missing a λ_x term.

Motivated by this we now return to the analysis in Section 2.2 and retain a higher order term in the Taylor expansion of $\lambda(x_n + \ell/2, t)$ in (5)₂. Thus we now write

$$\lambda_n(t) = \lambda\left(\frac{x_{n-1} + x_n}{2}, t\right) = \lambda(x_n + \ell/2, t) = \lambda + \frac{1}{2}\ell\lambda_x + \dots$$

and

$$V(\lambda_n) = V(\lambda + \ell\lambda_x/2 + \dots) = V(\lambda) + V'(\lambda)\frac{1}{2}\ell\lambda_x + \dots$$

Using this approximation for $V(\lambda_n)$ in (3) leads to

$$v_t = \frac{V(\lambda) - v}{\tau} + \frac{\ell}{2\tau} V'(\lambda)\lambda_x. \quad (22)$$

The modified continuous model is therefore comprised of (22) and (6)₁. Note the presence of the λ_x term above which will lead to a u_{xx} term in the linearized equation and therefore an additional k^2 term in the dispersion relation.

5.1 Stability of steady uniform motion in second continuous model

The stability of a steady uniform motion based on the modified continuous model (6)₁, (22) can be examined as before. For the reasons described at the beginning of Section

4.2, the perturbed motion can be expressed as

$$\lambda(x, t) = \lambda_* + \ell u_x(x, t), \quad v(x, t) = v_* + u_t(x, t).$$

This automatically satisfies the compatibility equation (6)₁. Substituting this into the equation of motion (22) and linearizing leads to

$$u_{tt} = a_1 \ell u_x + \frac{1}{2} a_1 \ell^2 u_{xx} - a_2 u_t$$

where a_1 and a_2 are again given by (12). On seeking exponential solutions in the form (19) we are led to the dispersion relation $\omega^2 + (2b_1 + i2b_2)\omega + (d_1 + id_2) = 0$ where now

$$2b_1 = 0, \quad 2b_2 = -a_2, \quad d_1 = -\frac{1}{2}k^2\ell^2 a_1, \quad d_2 = k\ell a_1.$$

Both roots ω of the dispersion relation have positive imaginary parts provided $b_2 < 0$ and $4b_1b_2d_2 - 4d_1b_2^2 > d_2^2$, which specialize to $a_2 > 0$, $a_2^2 > 2a_1 > 0$. These can be written in terms of the equilibrium velocity function and reaction time by using (12) as

$$\frac{1}{2\tau} > V'(\lambda_*) > 0.$$

This is identical to the requirements for stability according to the discrete model.

Thus again, stability requires the equilibrium velocity function to be monotonically increasing, as well as the reaction time to be positive. These are both quite reasonable requirements. However they are not sufficient for stability. For a given headway λ_* , the steady uniform motion will be unstable if the driver has a slow response, i.e. if $\tau > 1/[2V'(\lambda_*)]$. Otherwise it is stable.

6 Concluding remark.

In summary, in this paper we have used an explicit example to illustrate how a continuum model can behave differently to a discrete model even if the former was nominally “derived” from the latter. The steps involved in the typical derivations can be quite subtle and so it is important to not blindly accept the resulting macroscopic model. Similar investigations can of course be carried out on various other examples including the one mentioned in the Introduction: the relation between the microscopic model (b) and the macroscopic model (c).

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7 APPENDIX 1: Comments on traffic flow

This is not of course a paper about the dynamics of traffic flow. However, since this field is not particularly familiar to mechanical engineers, it may be interesting to the reader to connect some aspects of this paper to the literature on traffic flow.

- (i) In the mathematical modeling of traffic, discrete models of the type discussed in Section 2.1 are referred to as Car Following Models. The particular discrete model (1), (3) is referred to as the Optimum Velocity Model.
- (ii) Despite its simplicity, the LWR model – the continuous version of (1), (2) – is remarkably successful in describing many (but not all) phenomena observed in traffic flow, e.g. see Chapter 8 of Treiber and Kesting [9].
- (iii) The modified macroscopic equation of motion (22) is precisely (the Lagrangian version of) the Payne-Whitham model, see Payne [6] and Whitham, Section 3.1 of [10].
- (iv) Many models of traffic flow correspond to various generalizations of (3) of the form

$$\dot{v}_n = a(\dot{\lambda}_n, \lambda_n, v_n).$$

- (v) As noted at the end of Section 5.1, there is a critical time $\tau_c = 1/[2V'(\lambda_*)]$ such that the steady uniform motion is stable only if the driver's reaction time is smaller than this critical value. Note that if V is a concave function, then $V'' < 0$ and V' decreases monotonically. Thus the larger the headway λ_* , the larger is the value of the critical time τ_c , and so the driver has more time to react in.
- (vi) Much of the literature on traffic flow is formulated in an Eulerian framework. In such a formulation, it is more convenient to work with the traffic density ρ rather than the headway λ . They are related by $\rho = 1/\lambda$. All of the fields in the Eulerian formulation are expressed as functions of the current location of a vehicle y , and time t . The theory is then formulated in terms of $\rho(y, t)$ and $v(y, t)$.

8 APPENDIX 2: An elementary result in algebra.

In control theory, one frequently has to examine the (complex) zeros of a polynomial with real coefficients. The stability or not of the underlying system usually depends on the signs of the real parts of these zeros. Many special methods for examining these signs, without having to explicitly find the zeros, have been developed in that literature. However those special methods appear to be limited to polynomials with *real coefficients*. In this paper we repeatedly encounter a quadratic equation with *complex coefficients* whose roots must have a certain sign for stability. Even though this is an elementary problem, and the result is easily derivable, there does not seem to be a standard reference for the result. Thus in this appendix, we derive necessary and

sufficient conditions for both roots of a quadratic equation with complex coefficients to have the same sign.

Following Hardy [4], consider the quadratic equation

$$z^2 + 2bz + d = 0 \quad (23)$$

where the coefficients b and c are complex:

$$b = b_1 + ib_2, \quad d = d_1 + id_2. \quad (24)$$

First observe that since the sum of the two roots of the quadratic equation (23) equals $-2b$, if both roots have negative imaginary parts then necessarily $b_2 > 0$ while if both roots have positive imaginary parts then necessarily $b_2 < 0$.

It is convenient to write (23) as

$$(z + b)^2 = b^2 - d, \quad (25)$$

and to set

$$z + b = \alpha + i\beta, \quad b^2 - d = h + ik. \quad (26)$$

The quadratic equation can now be written as

$$(\alpha + i\beta)^2 = h + ik, \quad (27)$$

leading to the pair of equations

$$\alpha^2 - \beta^2 = h, \quad 2\alpha\beta = k, \quad (28)$$

that only involve real valued quantities. Solving (28) gives

$$\alpha = \pm \sqrt{\frac{\sqrt{h^2 + k^2} + h}{2}}, \quad \beta = \pm \sqrt{\frac{\sqrt{h^2 + k^2} - h}{2}}. \quad (29)$$

Since $2\alpha\beta = k$, if $k > 0$ we take the same sign for both square roots (i.e. both plus and both minus); if $k < 0$ we take opposite signs (i.e. one plus, the other minus, and the converse).

Since $z = \alpha + i\beta - b = \alpha + i\beta - b_1 - ib_2 = \alpha - b_1 + i(\beta - b_2)$ both roots z have negative imaginary part if $\beta < b_2$, i.e.

$$-b_2 < \sqrt{\frac{\sqrt{h^2 + k^2} - h}{2}} < b_2, \quad (\text{recall that } b_2 > 0 \text{ in this case}). \quad (30)$$

Both roots have positive imaginary parts if $\beta > b_2$, i.e.

$$b_2 < \sqrt{\frac{\sqrt{h^2 + k^2} - h}{2}} < -b_2, \quad (\text{recall that } b_2 < 0 \text{ in this case}). \quad (31)$$

On substituting for h and k from (26)₂ and (24) and simplifying leads to the following explicit conditions: Both roots have *negative imaginary parts* if and only if

$$b_2 > 0, \quad 4b_1b_2d_2 - 4d_1b_2^2 > d_2^2. \quad (32)$$

Both roots have *positive imaginary parts* if and only if

$$b_2 < 0, \quad 4b_1b_2d_2 - 4d_1b_2^2 > d_2^2. \quad (33)$$