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On a Conjecture of Thomassen

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Abstract

In 1989, Thomassen asked whether there is an integer-valued function $f(k)$ such that every $f(k)$ -connected graph admits a spanning, bipartite k -connected subgraph. In this paper we take a first, humble approach, showing the conjecture is true up to a $\log n$ factor.

1 Introduction

Erdős noticed [4] that any graph G with minimum degree $\delta(G)$ at least $2k - 1$ contains a spanning, bipartite subgraph H with $\delta(H) \geq k$. The proof for this fact is obtained by taking a maximal edge-cut, a partition of $V(G)$ into two sets A and B , such that the number of edges with one endpoint in A and one in B , denoted $|E(A, B)|$, is maximal. Observe that if some vertex v in A does not have degree at least k in $G[B]$, then by moving v to B , one would increase $|E(A, B)|$, contrary to maximality. The same argument holds for vertices in B . In fact this proves that for each vertex $v \in V(G)$, by taking such a subgraph H , the degree of v in H , denoted $d_H(v)$, is at least $d_G(v)/2$. This will be used throughout the paper.

Thomassen observed that the same proof shows the following stronger statement. Given a graph G which is at least $(2k - 1)$ *edge-connected* (that is one must remove at least $2k - 1$ edges in order to disconnect the graph), then G contains a bipartite subgraph H for which H is k edge-connected. In fact, each edge-cut keeps at least half of its edges. This observation led Thomassen to conjecture that a similar phenomena also holds for *vertex-connectivity*.

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Before proceeding to the statement of Thomassen's conjecture, we remind the reader that a graph G is said to be k *vertex-connected* or k -*connected* if one must remove at least k vertices from $V(G)$ in order to disconnect the graph (or to remain with one single vertex). We also let $\kappa(G)$ denote the minimum integer k for which G is k -connected. Roughly speaking, Thomassen conjectured that any graph with high enough connectivity also should contain a k -connected spanning, bipartite subgraph. The following appears as Conjecture 7 in [3].

Conjecture 1. *For all k , there exists a function $f(k)$ such that for all graphs G , if $\kappa(G) \geq f(k)$, then there exists a spanning, bipartite $H \subseteq G$ such that $\kappa(H) \geq k$.*

In this paper we prove that Conjecture 1 holds up to a $\log n$ factor by showing the following:

Theorem 1. *For all k and n , and for every graph G on n vertices the following holds. If $\kappa(G) \geq 10^{10}k^3 \log n$, then there exists a spanning, bipartite subgraph $H \subseteq G$ such that $\kappa(H) \geq k$.*

Because of the $\log n$ factor, we did not try to optimize the dependency on k in Theorem 1. However, it looks like our proof could be modified to give slightly better bounds.

2 Preliminary Tools

In this section, we introduce a number of preliminary results.

2.1 Mader's Theorem

The first tool is the following useful theorem due to Mader [2].

Theorem 2. *Every graph of average degree at least 4ℓ has an ℓ -connected subgraph.*

Because we are interested in finding bipartite subgraphs with high connectivity, the following corollary will be helpful.

Corollary 1. *Every graph G with average degree at least 8ℓ contains a (not necessarily spanning) bipartite subgraph H which is at least ℓ -connected.*

Proof. Let G be such a graph and let $V(G) = A \cup B$ be a partition of $V(G)$ such that $|E(A, B)|$ is maximal. Observe that $|E(A, B)| \geq |E(G)|/2$, and therefore, the bipartite graph G' with parts A and B has average degree at least 4ℓ . Now, by applying Theorem 2 to G' we obtain the desired subgraph H . \square

2.2 Merging k -connected Graphs

We will also make use of the following easy expansion lemma.

Lemma 1. *Let H_1 and H_2 be two vertex-disjoint graphs, each of which is k -connected. Let H be a graph obtained by adding k independent edges between these two graphs. Then, $\kappa(H) \geq k$.*

Proof. Note first that by construction, one cannot remove all the edges between H_1 and H_2 by deleting fewer than k vertices. Moreover, because H_1 and H_2 are both k -connected, each will remain connected after deleting less than k vertices. From here, the proof follows easily. \square

Next we will show how to merge a collection of a few k -connected components and single vertices into one k -connected component. Before stating the next lemma formally, we will need to introduce some notation. Let G_1, \dots, G_t be t vertex-disjoint k -connected graphs, let $U = \{u_{t+1}, \dots, u_{t+s}\}$ be a set consisting of s vertices which are disjoint to $V(G_i)$ for $1 \leq i \leq t$, and let R be a k -connected graph on the vertex set $\{1, \dots, t+s\}$. Also let $X = (G_1, \dots, G_t, u_{t+1}, \dots, u_{t+s})$ be a $(t+s)$ -tuple and X_i denote the i th element of X . Finally, let $\mathcal{F}_R := \mathcal{F}_R(X)$ denote the family consisting of all graphs G which satisfy the following:

- (i) the disjoint union of the elements of X is a spanning subgraph of G , and
- (ii) for every distinct $i, j \in V(R)$ if $ij \in E(R)$, then there exists an edge in G between X_i and X_j , and
- (iii) for every $1 \leq i \leq t$, there is a set of k independent edges between $V(G_i)$ and k distinct vertex sets $\{V(X_{j_1}), \dots, V(X_{j_k})\}$, where $V(u_i) = \{u_i\}$.

Lemma 2. *Let G_1, \dots, G_t be t vertex-disjoint graphs, each of which is k -connected, and let $U = \{u_{t+1}, \dots, u_{t+s}\}$ be a set of s vertices for which $U \cap V(G_i) = \emptyset$ for every $1 \leq i \leq t$. Let R be a k -connected graph on the vertex-set $\{1, \dots, t+s\}$, and let $X = \{G_1, \dots, G_t, u_{t+1}, \dots, u_{t+s}\}$. Then, any graph $G \in \mathcal{F}_R(X)$ is k -connected.*

Proof. Let $G \in \mathcal{F}_R(X)$, and let $S \subseteq V(G)$ be a subset of size at most $k-1$. We wish to show that the graph $G' := G \setminus S$ is still connected. Let $x, y \in V(G')$ be two distinct vertices in G' ; we show that there exists a path in G' connecting x to y . Towards this end, we first note that if both x and y are in the same G_i , then because each G_i is k -connected, there is nothing to prove. Moreover, if both x and y are in distinct elements of X which are also disjoint from S , then we are also finished, as follows. Because R is k -connected, if we delete all of the vertices in R corresponding to elements of X which intersect S , the resulting graph is still connected. Therefore, one can easily find a path between the elements containing x and y which goes only through “untouched” elements of X , and hence, there exists a path connecting x and y .

The remaining case to deal with is when x and y are in different elements of X , and at least one of them is not disjoint with S . Assume x is in some such X_i (y will be

treated similarly). Using Property (iii) of \mathcal{F}_R , there is at least one edge between X_i and an untouched X_j . Therefore one can find a path between x and some vertex x' in an untouched X_j . This takes us back to the previous case. \square

2.3 Main Technical Lemma

A *directed graph* or *digraph* is a set of vertices and a collection of directed edges; note that bidirectional edges are allowed. For a directed graph D and a vertex $v \in V(D)$ we let $d_D^+(v)$ denote the out-degree of v . We let $U(D)$ denote the *underlying graph* of D , that is the graph obtained by ignoring the directions in D and merging multiple edges. In order to find the desired spanning, bipartite k -connected subgraph in Theorem 1, we look at sub-digraphs in an auxiliary digraph.

The following is our main technical lemma and is the main reason why we have a $\log n$ factor.

Lemma 3. *If D is a finite digraph on at most n vertices with minimum out-degree*

$$\delta^+(D) > (k - 1) \lceil \log n \rceil,$$

then there exists a sub-digraph $D' \subseteq D$ such that

1. *For all $v \in V(D')$ we have $d_{D'}^+(v) \geq d_D^+(v) - (k - 1) \lceil \log n \rceil$, and*
2. *$\kappa(U(D')) \geq k$.*

Proof. If $\kappa(U(D)) \geq k$, then there clearly is nothing to prove. So we may assume that $\kappa(U(D)) \leq k - 1$. Delete a separating set of size at most $k - 1$. The smallest component, say C_1 , has size at most $n/2$ and for any $v \in V(C_1)$, every out-neighbor of v is either in $V(C_1)$ or in the separating set that we removed, and so

$$d_{C_1}^+(v) \geq d_D^+(v) - (k - 1).$$

We continue by repeatedly applying this step, and note that this process must terminate. Otherwise, after at most $\log n$ steps we are left with a component which consists of one single vertex and yet contains at least one edge, a contradiction. \square

3 Highly Connected Graphs

With the preliminaries out of the way, we are now ready to prove Theorem 1.

Proof. Let G be a finite graph on n vertices with

$$\kappa(G) \geq 10^{10} k^3 \log n.$$

In order to find the desired subgraph, we first initiate $G_1 := G$ and start the following process.

As long as G_i contains a bipartite subgraph which is at least k -connected on at least $10^3 k^2 \log n$ vertices, let $H_i = (S_i \cup T_i, E_i)$ be such a subgraph of maximum size, and let $G_{i+1} := G_i \setminus V(H_i)$. Note that H_1 must exist as

$$\delta(G_1) \geq 10^{10} k^3 \log n - 2k \geq 8000 k^2 \log n,$$

and so by Corollary 1, G_1 must contain a k -connected bipartite subgraph of size at least $10^3 k^2 \log n$.

Let H_1, \dots, H_t be the sequence obtained in this manner, and note that all the H_i 's are vertex disjoint with $\kappa(H_i) \geq k$ and $|V(H_i)| \geq 10^3 k^2 \log n$. Observe that if H_1 is spanning, then there is nothing to prove. Therefore, suppose for a contradiction that H_1 is not spanning. Let $V_0 := V(G_{t+1}) = \{v_1, \dots, v_s\}$ be the subset of $V(G)$ remaining after this process; note that it might be the case that $V_0 = \emptyset$. Because each H_i is a bipartite, k -connected subgraph of G_i of maximum size and G is $10^{10} k^3 \log n$ connected, we show that the following are true:

- (a) For every $1 \leq i < j \leq t$, there are less than $4k$ independent edges between H_i and H_j , and
- (b) for every $j > i$ and $v \in V(G_j)$, the number of edges in G between v and H_i , denoted by $d_G(v, V(H_i))$, is less than $2k$, and
- (c) for every $1 \leq i \leq t$, there exists a set M_i consisting of exactly $10^3 k^2 \log n$ independent edges, each of which has exactly one endpoint in H_i .

Indeed, for showing (a), note that if there are at least $4k$ independent edges between H_i to H_j , by pigeonhole principle, at least k of them are between the same part of H_i (say S_i) and the same part of H_j (say S_j). Therefore, the graph obtained by joining H_i to H_j with this set of at least k edges is a k -connected (by Lemma 1), bipartite graph and is larger than H_i , contrary to the maximality of H_i .

For showing (b), note that if there are at least $2k$ between v and H_i then there are at least k edges incident with v touch the same part of H_i , and let F be a set of k such edges. Second, we mention that joining a vertex of degree at least k to a k -connected graph trivially yields a k -connected graph. Next, since all the edges in F are touching the same part, the graph obtained by adding v to $V(H_i)$ and F to $E(H_i)$, will also be bipartite. This contradicts the maximality of H_i .

For (c), note first that since H_1 is not spanning, using (b) we conclude that in the construction of the bipartite subgraphs H_1, \dots, H_t in the process above,

$$\delta(G_2) \geq 10^{10} k^3 \log n - 2k \geq 8000 k^2 \log n.$$

Therefore, using Corollary 1, it follows that G_2 contains a bipartite subgraph of size at least $10^3 k^2 \log n$ which is also k -connected.

Therefore, the process does not terminate at this point, and H_2 exists (that is, $t \geq 2$). It also follows that for each $1 \leq i \leq t$ we have $|V(G) \setminus V(H_i)| \geq 10^3 k^2 \log n$. Next, note that G is $10^{10} k^3 \log n$ connected, and that each H_i is of size at least $10^3 k^2 \log n$. For each

i , consider the bipartite graph with parts $V(H_i)$ and $V(G) \setminus V(H_i)$ and with the edge-set consisting of all the edges of G which touch both of these parts. Using König's Theorem (see [5], p. 112), it follows that if there is no such M_i of size $10^3 k^2 \log n$, then there exists a set of strictly fewer than $10^3 k^2 \log n$ vertices that touch all the edges in this bipartite graph (a vertex cover). By deleting these vertices, one can separate what is left from H_i and its complement, contrary to the fact that G is $10^{10} k^3 \log n$ connected.

In order to complete the proof, we wish to reach a contradiction by showing that one can either merge few members of $\{H_1, \dots, H_t\}$ with vertices of V_0 into a k -connected component or find a k -connected component of size at least $10^3 k^2 \log n$ which is contained in V_0 . In order to do so, we define an auxiliary digraph, using a special subgraph $G' \subseteq G$, and use Lemmas 3 and 2 to achieve the desired contradiction. We first describe how to find G' .

First, we partition V_0 into two sets, say A and B , where

$$A = \left\{ v \in V_0 : d_G \left(v, \bigcup_{i=1}^t V(H_i) \right) \geq 10^4 k^3 \log n \right\},$$

and observe that, using (b), since $A \subseteq V_0$, any vertex $a \in A$ must send edges to more than

$$10^4 k^3 \log n / (2k) = 5000 k^2 \log n$$

distinct elements in $X := \{H_1, \dots, H_t, v_1, \dots, v_s\}$. For each $1 \leq i \leq t$, let M_i be a set as described in (c). Observe that, using (b), each such M_i touches more than

$$10^3 k^2 \log n / (4k) = 250 k \log n$$

distinct elements of $X \setminus \{H_i\}$. Let $M'_i \subseteq M_i$ be a subset of size exactly $250 k \log n$ such that each pair of edges in M'_i touches two distinct elements of $X \setminus \{H_i\}$, which of course are distinct from G_i . Recall that $H_i = (S_i \cup T_i, E_i)$ for every $1 \leq i \leq t$.

For $Y := \{S_1, \dots, S_t, T_1, \dots, T_t, v_1, \dots, v_s\}$, let

$$\Phi : Y \rightarrow \{L, R\}$$

be a mapping, generated according to the following random process:

Let $X_1, \dots, X_t, Y_1, \dots, Y_s \sim \text{Bernoulli}(1/2)$ be mutually independent random variables. For each $1 \leq i \leq t$, if $X_i = 1$, then let $\Phi(S_i) = L$ and $\Phi(T_i) = R$. Otherwise, let $\Phi(S_i) = R$ and $\Phi(T_i) = L$. For every $1 \leq j \leq s$, if $Y_j = 1$, then let $\Phi(v_j) = L$, and otherwise $\Phi(v_j) = R$. Now, delete all of the edges between two distinct elements of Y which receive the same label according to Φ .

Finally, define G' as the spanning bipartite graph of G obtained by deleting all of the edges within A and for distinct i and j , the edges between H_i and H_j which are not contained in $M'_i \cup M'_j$.

Recall by construction, using Φ we generated labels at random; therefore, by using Chernoff bounds (for instance see [1]), one can easily check that with high probability the following hold:

- (i) For every $1 \leq i \leq t$, each set $M'_i \cap E(G')$ touches at least (say) $120k \log n$ other elements of X , and
- (ii) for each $b \in B$, the degree of b into $A \cup B$ is at least (say) $d_{G'}(b, A \cup B) \geq 10^5 k^3 \log n$, and
- (iii) for each vertex $a \in A$, there exist edges between a and $\cup_{i=1}^t V(H_i)$ that touch at least (say) $2000k^2 \log n$ distinct members of $\{H_1, \dots, H_t\}$.

Note that here we relied on the luxury of losing the $\log n$ factor for using Chernoff bounds, but it seems like we could easily handle this “cleaning process” completely by hand.

Now we are ready to define our auxiliary digraph D . To this end, we first orient edges (again, bidirectional edges are allowed, and un-oriented edges are considered as bidirectional) of G' in the following way:

For every $1 \leq i \leq t$, we orient all of the edges in $E(G') \cap M'_i$ out of H_i . We orient all of the edges between A and $\cup_{i=1}^t V(H_i)$ out of A . We orient edges between B and $\cup_{i=1}^t V(H_i)$ arbitrarily, and we orient the remaining edges within $A \cup B$ in both directions.

Now, we define D to be the digraph with vertex set $V(D) = X$, and $\vec{xy} \in E(D)$ if and only if there exists an edge between x and y in G' which is oriented from x to y .

In order to complete the proof, we first note that with high probability D is a digraph on at most n vertices with out-degree $\delta^+(D) > (k - 1) \lceil \log n \rceil$. This follows immediately from Properties (i)-(iii) as well as the way we oriented the edges. Therefore, one can apply Lemma 3 to find a sub-digraph $D' \subseteq D$ such that

1. For all $v \in V(D')$ we have $d_{D'}^+(v) \geq d_D^+(v) - (k - 1) \lceil \log n \rceil$, and
2. $\kappa(U(D')) \geq k$.

In fact, with high probability, $\delta^+(D) \geq 120k \log n \geq k + (k - 1) \lceil \log n \rceil$. Note that by construction, every pair of edges which are oriented out of some H_i must be independent and go to different components. Using Property 1. above combined with the fact that $\delta^+(D') \geq \delta^+(D) - (k - 1) \lceil \log n \rceil \geq k$, we may conclude that the subgraph $G'' \subseteq G'$ induced by the union of all the components in $V(D')$ satisfies $G'' \in \mathcal{F}_{U(D')}(V(D'))$. Applying Lemma 2 with $X = V(D')$ and $R = U(D')$, it follows that $\kappa(G'') \geq k$.

In order to obtain the desired contradiction, we consider the following two cases:

Case 1: $V(G'')$ contains $V(H_i)$ for some i . We note that this case is actually impossible because it would contradict the maximality of H_i for the minimal index i such that $V(H_i) \subseteq V(G'')$.

Case 2: $V(G'') \subseteq A \cup B$. We note that in this case, there must be at least one vertex $b \in B \cap V(G'')$. Indeed, G'' is k -connected, and there are no edges within A . Now, it follows from Properties 1. and (ii) above that

$$d_{D'}^+(b) \geq d_D^+(b) - (k - 1) \lceil \log n \rceil \geq 10^4 k^3 \log n.$$

Thus, it follows that $|V(G'')| \geq 10^4 k^3 \log n$. Combining this observation with the facts that G'' is k -connected and $V(G'') \subseteq A \cup B$, we obtain a contradiction. This case can not arise because G'' should have been included as one of the bipartite subgraphs $\{H_1, \dots, H_t\}$.

This completes the proof. \square

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