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LOWER CENTRAL SERIES OF A FREE ASSOCIATIVE ALGEBRA OVER THE INTEGERS AND FINITE FIELDS

SURYA BHUPATIRAJU, PAVEL ETINGOF, DAVID JORDAN, WILLIAM KUSZMAUL, JASON LI

Abstract. Consider the free algebra A_n generated over $\mathbb{Q}$ by n generators $x_1, \ldots, x_n$. Interesting objects attached to $A = A_n$ are members of its lower central series, $L_i = L_i(A)$, defined inductively by $L_1 = A$, $L_{i+1} = [A, L_i]$, and their associated graded components $B_i = B_i(A)$ defined as $B_i = L_i/L_{i+1}$. These quotients B_i for $i \geq 2$, as well as the reduced quotient $\bar{B}_1 = A/(L_2 + AL_3)$, exhibit a rich geometric structure, as shown by Feigin and Shoikhet [FS] and later authors, ([DKM],[DE],[AJ],[BJ]).

We study the same problem over the integers $\mathbb{Z}$ and finite fields $\mathbb{F}_p$. New phenomena arise, namely, torsion in B_i over $\mathbb{Z}$, and jumps in dimension over $\mathbb{F}_p$. We describe the torsion in the reduced quotient $\bar{B}_1$ and B_2 geometrically in terms of the De Rham cohomology of $\mathbb{Z}^n$. As a corollary we obtain a complete description of $\bar{B}_1(A_n(\mathbb{Z}))$ and $\bar{B}_1(A_n(\mathbb{F}_p))$, as well as of $B_2(A_n(\mathbb{Z}[1/2]))$ and $B_2(A_n(\mathbb{F}_p))$, $p > 2$. We also give theoretical and experimental results for B_i with $i > 2$, formulating a number of conjectures and questions on their basis. Finally, we discuss the supercase, when some of the generators are odd and some are even, and provide some theoretical results and experimental data in this case.

1. Introduction

The lower central series $\{L_i(A), i \geq 1\}$, of a noncommutative algebra A is defined inductively by $L_1(A) = A$, and $L_{i+1}(A) = [A, L_i(A)]$. In other words, $L_i(A)$ is spanned by "i-commutators" $[a_1, [a_2, ...[a_{i-1}, a_i]...]]$, where $a_1, ..., a_i \in A$. It is interesting to consider the successive quotients $B_i(A) = L_i(A)/L_{i+1}(A)$, i.e., i-commutators modulo $i+1$-commutators. The role of the $B_i(A)$ is, roughly, that they characterize, step by step, the deviation of A from being commutative; this is somewhat similar to the quasiclassical perturbation expansion in quantum mechanics.

The quotients $B_i(A)$ can be put together in a direct sum $B(A) = \oplus_{i\geq 1} B_i(A)$. This is a graded Lie algebra: we have $[B_i(A), B_j(A)] \subset B_{i+j}(A)$. It turns out that this Lie algebra has a big central part (i.e., a part commuting with everything) in degree 1, namely the image I of $AL_3(A)$ in B_1. Denote $B_1(A)/I$ by $\bar{B}_1(A)$. Then we have a graded Lie algebra $\bar{B}(A) = \bar{B}_1(A) \bigoplus \oplus_{i\geq 2} B_i(A)$, generated by degree 1 (i.e., $\bar{B}_1(A)$).

The lower central series of the free algebra $A_n = A_n(\mathbb{Q})$ over the field of rational numbers $\mathbb{Q}$ was studied in a number of papers [DE], [DKM], [AJ], [BJ], following the pioneering paper [FS]. In this paper, Feigin and Shoikhet observed a surprising fact: the graded spaces $\bar{B}_1(A_n)$ and $B_i(A_n)$, $i \geq 2$, have polynomial, rather than exponential growth, even though the dimensions of the homogeneous components of A_n itself grow exponentially. This stems from the fact that, even though A_n is "the most noncommutative" of all algebras, the spaces $\bar{B}_1(A_n)$ and $B_i(A_n)$, $i \geq 2$, can be described in terms of usual, "commutative" geometry, namely in terms of tensor fields on the n-dimensional space. This allows one to completely describe $\bar{B}_1(A_n)$ and $B_2(A_n)$ in terms of differential forms [FS] and $B_3(A_n)$ as well as $B_i(A_n)$ for some n and $i > 3$ in terms of more complicated tensor fields [DE], [DKM], [AJ], [BJ]. However, the problem of describing of $B_i(A_n(\mathbb{Q}))$ for all i, n, and in particular of computation of the Hilbert series of these spaces still remains open.

This paper is dedicated to the study of $\bar{B}_1(A_n)$ and $B_i(A_n)$ over the integers $\mathbb{Z}$ and over a finite field $\mathbb{F}_p$. In this case, rich new structures emerge. Namely, in the case of $\mathbb{Z}$, the groups $\bar{B}_1(A_n)$, $B_i(A_n)$ develop torsion, and it is interesting to study the pattern of this torsion. In the case of $\mathbb{F}_p$

the dimensions of certain homogeneous subspaces of $\bar{B}_1(A_n)$, $B_i(A_n)$ differ from those over $\mathbb{Q}$, and the patterns of these discrepancies are of interest.

Our main results are as follows. First of all, we give a complete description of $\bar{B}_1(A_n(\mathbb{Z}))$ in terms of differential forms, and of the torsion in $\bar{B}_1(A_n(\mathbb{Z}))$ in terms of the De Rham cohomology of the n-dimensional space over $\mathbb{Z}$. Since $\bar{B}_1(A_n(\mathbb{F}_p)) = \bar{B}_1(A_n(\mathbb{Z})) \otimes \mathbb{F}_p$, this completely describes $\bar{B}_1(A_n(\mathbb{F}_p))$ in terms of differential forms. Since the Lie algebra $\bar{B}$ is generated by $\bar{B}_1$, this description implies a uniform bound for dimensions of the homogeneous subspaces, which depends only on i and n ("polynomial growth" for $B_i(A_n(\mathbb{F}_p))$). Also, we give a description of the torsion in $B_2(A_n(\mathbb{Z}))$, in terms of the De Rham cohomology over the integers (conjectural for 2^r-torsion). Since $B_2(A_n(\mathbb{F}_p)) = B_2(A_n(\mathbb{Z})) \otimes \mathbb{F}_p$, this describes $B_2(A_n(\mathbb{F}_p))$ in terms of differential forms. Further, we give an explicit formula for some torsion elements in $B_2(A_3(\mathbb{Z}))$, and show that they are a basis of torsion.

We also present some theoretical and computer results (using the MAGMA package, [BCP]) for torsion in $B_i(A_n(\mathbb{Z}))$ for $i > 2$, and propose a number of conjectures based on them. In particular, we conjecture that the degree of any torsion element with respect to each variable is divisible by the order of the torsion. Finally, we provide some computer data and theoretical results for torsion in B_i in the supercase, i.e. for free algebras $A_{n,k}$ in $n+k$ generators, where the first n generators are even and the last k generators are odd, and formulate a number of open questions based on this data. However, the pattern of torsion in higher B_i remains mysterious even at the conjectural level, and will be the subject of further investigation.

The organization of the paper is as follows. In Section 2, we discuss preliminaries. In Section 3, we describe the structure of $\bar{B}_1(A_n)$. In Section 4, we describe the structure of $B_2(A_n)$. In Section 5, we discuss the experimental data and conjectures for the structure of $B_i(A_n(\mathbb{Z}))$ for $i > 2$. In Section 6, we discuss the theoretical results, experimental data and open problems in the supercase. Finally, in Section 7, we outline some directions of further research.

Acknowledgments. The work of P.E. was partially supported by the NSF grant DMS-1000113. The work of D.J. was supported by the NSF grant DMS-1103778. S.B., W.K., and J.L. are grateful to the PRIMES program at the MIT Mathematics Department, where this research was done.

2. Preliminaries

2.1. **The lower central series.** Let A be an algebra over a commutative ring R. In this paper, we will consider algebras over $R = \mathbb{Q}, \mathbb{F}_p, \mathbb{Z}$.

Definition 2.1. *The lower central series filtration of* A *is defined as* $L_1(A) := A, L_{k+1}(A) := [A, L_k(A)]$ *for* $k \in \mathbb{N}$.

In other words, we have

$$L_k = \mathrm{Span}_R\{[a_1, [a_2, \dots, [a_{k-1}, a_k]] \dots] | a_1, \dots, a_k \in A\}.$$

Definition 2.2. *The associated graded Lie algebra of* A *is*

$$B(A) := \bigoplus_{k \geq 1} B_k(A)$$

where $B_k(A) := L_k/L_{k+1}$ *for all* $k \in \mathbb{N}$.

Definition 2.3. *The two-sided ideals* $M_k(A)$ *are defined as* $M_k(A) := AL_k$ *for* $k \in \mathbb{N}$.

Lemma 2.4. $[A, M_3] \subset L_3$.

Proof. Lemma 2.2.1 from [FS] asserts this for algebras over $\mathbb{C}$. However, the formulas in the proof only involve ± 1 coefficients, so the proof applies for an arbitrary ring R. □

Definition 2.5. *The abelian group $\bar{B}_1$ is the quotient $\bar{B}_1 := A/(L_2 + M_3)$.*

Thus, $\bar{B}_1$ is the quotient of B_1 by the image I of M_3 in B_1, which by Lemma 2.4 is central in $B(A)$.

Corollary 2.6. *The graded module*

$$\bar{B}(A) := \bar{B}_1 \oplus \bigoplus_{k\geq 2} B_k$$

is a graded Lie algebra generated in degree 1.

Definition 2.7. *The free algebra $A_n(R)$ of rank n over R is the free R-module generated by all finite words in the letters $x_1, \dots, x_n$; the multiplication is given by concatenation of words.*

2.2. **De Rham cohomology over the integers.** We will see that torsion in $\bar{B}_1(A_n(\mathbb{Z}))$ and $B_2(A_n(\mathbb{Z}))$ is related to the De Rham cohomology of the n-dimensional space over the integers. So let us discuss how to compute this cohomology.

Denote by $\Omega(R^n)$ the module of differential forms in n variables with coefficients in R. We have a rank decomposition $\Omega(R^n) = \oplus_{i=0}^n \Omega^i(R^n)$, and the De Rham differential $d : \Omega^i(R^n) \to \Omega^{i+1}(R^n)$, which defines the De Rham complex of the n-dimensional space over R. We will use the notation Ω_{cl} for closed forms, Ω_{ex} for exact forms, Ω^{ev} for even forms, Ω^{odd} for odd forms, Ω^+ for positive rank forms, $\Omega^{\geq k}$ for forms of rank $\geq k$, etc.

If R is a $\mathbb{Q}$-algebra then by the Poincaré lemma, the De Rham complex is acyclic in degrees $i > 0$, and its zeroth cohomology H^0 consists of constants (i.e., equals R). However, this fails for general rings. In this section we compute the cohomology of this complex for $R = \mathbb{Z}$.

Lemma 2.8. *We have:*

$$H^k(\Omega(\mathbb{Z}))[m] \cong \begin{cases} \mathbb{Z}, & k = 0,\ m = 0 \\ \mathbb{Z}/m, & k = 1,\ m > 0, \\ 0, & \text{otherwise} \end{cases}$$

(Here $[m]$ denotes the degree m part).

Proof. Straightforward computation. □

Corollary 2.9. *Let $H^r_n := H^r(\Omega(\mathbb{Z}^n))$. We have a (noncanonically) split short exact sequence in cohomology,*

$$0 \to H^r_{n-1} \oplus H^{r-1}_{n-1} \otimes H^1_1 \to H^r_n \to \mathrm{Tor}(H^r_{n-1}, H^1_1) \to 0.$$

Moreoever, this is a short exact sequence of graded abelian groups, with respect to the natural $\mathbb{Z}^n$-grading on each term.

Proof. Apply the Künneth Theorem and Lemma 2.8 to the decomposition

$$\Omega(\mathbb{Z}^n) = \Omega(\mathbb{Z}^{n-1}) \otimes \Omega(\mathbb{Z}).$$

□

Corollary 2.10. *(see also* [Fr]*) Suppose that all $m_i > 0$. Then we have an isomorphism:*

$$H^i(\Omega(\mathbb{Z}^n))[m_1, \dots, m_n] \cong (\mathbb{Z}/gcd(m_1, \dots, m_n))^{\binom{n-1}{i-1}},$$

where $[m_1, ..., m_n]$ denotes the part of multidegree $m_1, ..., m_n$.

Remark 2.11. Note that this describes the cohomology completely, since the case when $m_i = 0$ for some i reduces to a smaller number of variables.

Proof. We choose a splitting of the short exact sequence of Corollary 2.9, and apply Lemma 2.8 to obtain:

$$H_n^r \cong H_{n-1}^r \oplus \bigoplus_{m\geq 1} \left(H_{n-1}^{r-1} \otimes \mathbb{Z}/m \oplus \mathrm{Tor}(H_{n-1}^r, \mathbb{Z}/m)\right).$$

Note that each $\mathbb{Z}/m$ in the sum above is in multi-degree $(0,\ldots,0,m)$. Thus the $(m_1,\ldots,m_n)$-graded part of the above short exact sequence reads:

$$H_n^i[m_1,...,m_n] \cong H_{n-1}^i[m_1,...,m_{n-1}]/m_n \oplus H_{n-1}^{i-1}[m_1,...,m_{n-1}]/m_n.$$

The formula now follows by induction on n and i, and the identity $\binom{r+1}{i+1} = \binom{r}{i} + \binom{r}{i+1}$. □

2.3. **Universal coefficient formulas for $\bar{B}_1$ and B_2.** Let p be a prime and $\mathbb{Z}^N \supset A \supset B$ be abelian groups. For an abelian group A, let $\mathrm{tor}_p(A)$ denote the p-torsion of A.

Lemma 2.12. *Let A_p and B_p be the images of A and B in $\mathbb{F}_p^N$. If $\mathrm{tor}_p(\mathbb{Z}^N/A) = 0$ (in particular, if $\mathbb{Z}^N/A$ is free) then the natural map*

$$A_p/B_p \to (A/B) \otimes \mathbb{F}_p$$

is an isomorphism.

Proof. We have $\mathbb{Z}^N/A = \dfrac{\mathbb{Z}^N/B}{A/B}$. Since $\mathrm{tor}_p(\mathbb{Z}^N/A) = 0$, this implies that we have an isomorphism of p-local parts $(\mathbb{Z}^N/B) \otimes \mathbb{Z}_{(p)} = (A/B \oplus \mathbb{Z}^N/A) \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the ring of rational numbers whose denominator is coprime to p. When we tensor the equality with $\mathbb{F}_p$, we see that $(A/B) \otimes \mathbb{F}_p$ is the kernel of the projection map $(\mathbb{Z}^N/B) \otimes \mathbb{F}_p \to (\mathbb{Z}^N/A) \otimes \mathbb{F}_p$, which is A_p/B_p, as desired. □

Corollary 2.13. *We have*

1) $\bar{B}_1(A_n(\mathbb{F}_p)) = \bar{B}_1(A_n(\mathbb{Z})) \otimes \mathbb{F}_p$;
2) $B_2(A_n(\mathbb{F}_p)) = B_2(A_n(\mathbb{Z})) \otimes \mathbb{F}_p$.

Proof.

1) Apply Lemma 2.12 for $\mathbb{Z}^N = A = A_n[m]$ and $B = (L_2 + M_3)[m]$, where $[m]$ means total degree m.
2) Apply Lemma 2.12 for $\mathbb{Z}^N = A_n[m]$, $A = L_2(A_n)[m]$, and $B = L_3(A_n)[m]$. $\mathbb{Z}^N/A = B_1[m]$ is freely spanned by cyclic words of length m, so it is free, and the lemma applies.

□

Remark 2.14. This corollary is false for higher B_i. For instance, one can show that

$$\dim(B_4(A_3(\mathbb{Z}))[2,2,2] \otimes \mathbb{F}_2) \neq \dim B_4(A_3(\mathbb{F}_2))[2,2,2].$$

(see Subsection 5.1.1).

3. The Structure of $\bar{B}_1(A_n)$

It will be useful to adapt the presentation of A_n/M_3 given in [FS] to general base ring R. This is accomplished by the following proposition.

Proposition 3.1. *The algebra A_n/M_3 is the algebra generated by $x_1, \ldots x_n$, and u_{ij} for $i,j \in [1\ldots n]$ and $i \neq j$, with the following relations:*

1) $u_{ij} = [x_i, x_j]$, *and so* $u_{ij} + u_{ji} = 0$;
2) $[x_i, u_{jk}] = 0$ *for all* i, j, k;
3) u_{ij} *commute with each other:* $[u_{ij}, u_{kl}] = 0$;
4) $u_{ij}u_{kl} = 0$ *if any of the* i, j, k, l *contain repetitions;*
5) $u_{ij}u_{kl} = -u_{ik}u_{jl}$ *if* i, j, k, l *are all distinct.*

Proof. Relation (1) is the definition of u_{ij}, and relations (2),(3) are obvious, so we will prove only relations (4),(5).

4) We want to show that $[x, y][x, z]$ is in M_3. But,

$$[x, y][x, z] = [[x, y]x, z] - [[x, y], z]x = [[x, yx], z] - [[x, y], z]x,$$

which is in M_3.

5) We want to show that $[x, y][z, t] + [x, z][y, t]$ is in M_3. Modulo M_3, this is

$$[x, y[z, t] + z[y, t]] = [x, [yz, t]] + [x, z[y, t] - [y, t]z],$$

which is obviously in L_3 (not just M_3).

Also, it is easy to show using relations (1)-(5) that in the quotient of the algebra A_n by these relations, we have $[[a, b], c] = 0$ for all a, b, c. Indeed, using the Jacobi identity and the Leibniz rule, it suffices to show that $[x_i, [x_j, c]] = 0$. This is straightforward, substituting $c = x_{i_1} ... x_{i_m}$. The proposition is proved. □

For $I \subset \{1, \dots, n\}$, of even cardinality k, $I = \{i_1, ..., i_k\}$, $i_1 < ... < i_k$, define $u_I := u_{i_1,i_2} \cdots u_{i_{k-1},i_k}$.

Proposition 3.2. *A_n/M_3 is a free abelian group with basis $x_1^{k_1} \cdots x_n^{k_n} u_I$, for $k_i \geq 0$, and I as above.*

Proof. It is clear that the given elements are a spanning set. Let us show that they are a basis. It is enough to show it over $R = \mathbb{Z}$, hence over $R = \mathbb{Q}$, which is known from [FS]. □

Here is a coordinate-free description of A/M_3. Let V be a finitely generated free R-module, and $A = T_R V$ be the tensor algebra of V. Define the algebra $\widetilde{\Omega}(V)$ to be the quotient of the tensor algebra $T_R(V \oplus V)$ with generators x_v, y_v, $v \in V$ (R-linearly depending on v) by the relations

$$[x_v, x_u] = y_v y_u, [x_v, y_u] = 0, v, u \in V.$$

Note that this implies that $y_v y_u = -y_u y_v$ and $y_v^2 = 0$. We have a decomposition $\widetilde{\Omega}(V) = \widetilde{\Omega}^{odd}(V) \oplus \widetilde{\Omega}^{ev}(V)$ into parts of odd and even degree with respect to the y-grading. Also we have a differential on $\widetilde{\Omega}(V)$ given by $dx_v = y_v$.

Also, let $\Omega(V)$ be the algebra of differential forms on V^*, generated by x_v and y_v with relations

$$[x_v, x_u] = 0, \ [x_v, y_u] = 0, \ y_v y_u = -y_u y_v, \ y_v^2 = 0$$

and differential $dx_v = y_v$. If R contains $1/2$, we can define the Fedosov $*$-product on $\Omega(V)$ by $a * b = ab + \frac{1}{2} da \wedge db$. Denote $\Omega(V)$ equipped with this product by $\Omega(V)_*$.

Proposition 3.3. *(i) If R contains $1/2$ then there is a unique differential graded algebra homomorphism $\phi : \widetilde{\Omega}(V) \to \Omega(V)_*$ such that $\phi(x_v) = x_v$, and it is an isomorphism.*

(ii) Over any R, one has $\mathrm{gr}(\widetilde{\Omega}(V)) = \Omega(V)$, where the associated graded is taken with respect to the descending filtration by degree in the y-variables.

(iii) There exists a unique algebra homomorphism $\zeta : A/M_3 \to \widetilde{\Omega}^{ev}(V)$, such that $\phi(v) = x_v$. Moreover, ϕ is an isomorphism.

Proof. (i) It is easy to check that the relations $[x_v, x_u] = y_v y_u$ and $[x_v, y_u] = 0$ are satisfied in $\Omega(V)_*$, so, there is a unique homomorphism ϕ, which is surjective. Also, it is clear that the Hilbert series of $\widetilde{\Omega}(V)$ is dominated by the Hilbert series of $\Omega(V)$, so ϕ is injective, hence an isomorphism.

(ii) It is easy to see that there is a surjective homomorphism $\theta : \Omega(V) \to \mathrm{gr}\widetilde{\Omega}(V)$. So it suffices to show that it is an isomorphism over $\mathbb{Q}$, which follows from (i) by comparison of Hilbert series.

(iii) It is shown similarly to the proof of Proposition 3.1 that in the algebra $\widetilde{\Omega}(V)$, one has $[[a, b], c] = 0$, so that ζ exists and is surjective. But it follows from (ii) and Proposition 3.1 that the Hilbert series of the two algebras are the same, so ζ is injective, i.e. an isomorphism. □

Thus, $\widetilde{\Omega}(V)$ is a quantization of the DG algebra of differential forms $\Omega(V)$. Nevertheless, over a ring R not containing $1/2$, in particular $R = \mathbb{F}_2$, they are not isomorphic even as graded $GL(V)$-modules. Namely, the degree 2 component of $\widetilde{\Omega}(V)$ is $V \otimes V$ (the corresponding isomorphism is defined by $v \otimes u \mapsto x_v x_u$, $v, u \in V$), while for $\Omega(V)$ it is $S^2V \oplus \wedge^2 V$, which is not the same thing as $V \otimes V$ in characteristic 2. However, we will need to work with rings not containing $1/2$ (such as $\mathbb{Z}$ and $\mathbb{F}_2$). For this reason we will use a poor man's version of ϕ, the map φ, introduced in the following proposition. Note that it is not an algebra map and is not $GL(n)$-invariant.

Proposition 3.4. *There exists a unique isomorphism of R-modules $\varphi : A_n/M_3 \to \Omega^{ev}$ such that*

$$\varphi(x_1^{k_1} \cdots x_n^{k_n} u_I) = x_1^{k_1} \cdots x_n^{k_n} dx_I,$$

where $dx_I = d_{x_{i_1}} \wedge d_{x_{i_2}} \wedge \ldots \wedge d_{x_{i_{k-1}}} \wedge d_{x_{i_k}}$. Moreover, one has $\varphi([a, x_i]) = d\varphi(a) \wedge dx_i$.

Proof. The first statement is a direct consequence of Proposition 3.2, and the second one follows by an easy direct computation. □

Theorem 3.5. *The map φ induces an isomorphism $\varphi : L_2(A/M_3) \to \Omega^{ev}_{ex}$.*

Proof. We will need the following lemma.

Lemma 3.6. *We have $L_2(A_n) = \sum_{i=1}^n [A_n, x_i]$.*

Proof. This follows from the identity $[a, bc] + [b, ca] + [c, ab] = 0$. □

By this lemma, $L_2(A_n)$ is spanned by $[a, x_i]$, $a \in A_n$. But by Proposition 3.4, $\varphi([a, x_i]) = d\varphi(a) \wedge dx_i$. Thus, $\varphi(L_2)$ is contained in Ω^{ev}_{ex}. On the other hand, Ω^{ev}_{ex} is spanned by elements of the form $\omega = df \wedge dx_{i_1} \wedge ... \wedge dx_{i_{2r+1}}$. So if $a = \varphi^{-1}(f dx_{i_1} \wedge ... \wedge dx_{i_{2r}})$, then $\varphi([a, x_{i_{2r+1}}]) = \omega$, implying the opposite inclusion. □

Corollary 3.7. *Over any base ring R, one has*

$$\bar{B}_1(A_n) = \Omega^{ev}(R^n)/\Omega^{ev}_{ex}(R^n).$$

In particular, for $R = \mathbb{Z}$, one has

$$\bar{B}_1(A_n) = H^{ev,+}(\Omega(\mathbb{Z}^n)) \oplus \Omega^{ev}(\mathbb{Z}^n)/\Omega^{ev,+}_{cl}(\mathbb{Z}^n),$$

where $H^{ev,+} := H^2 \oplus H^4 \oplus ...$ is the even De Rham cohomology of $\mathbb{Z}^n$.

Proof. It is clear that for any algebra A, $\bar{B}_1(A) \cong B_1(A/M_3)$. Therefore, by Theorem 3.5, $\bar{B}_1(A_n) = \Omega^{ev}/\Omega^{ev}_{ex}$. So we have a short exact sequence

$$0 \to H^{ev,+}(\Omega(\mathbb{Z}^n)) \to \bar{B}_1(A_n) \to \Omega^{ev}(\mathbb{Z}^n)/\Omega^{ev,+}_{cl}(\mathbb{Z}^n) \to 0,$$

But it is easy to see that the quotient is a free group (indeed, if $m\omega$ is a closed form for an integer m, then ω is closed as well). This implies the statement. □

Now, the Poincare lemma over $\mathbb{Q}$ implies that $H^{ev,+}(\Omega(\mathbb{Z}^n))$ is torsion. Thus, we have

Theorem 3.8. *The torsion in $\bar{B}_1(A_n\mathbb{Z}))$ is given by the equality*

$$\mathrm{tor}\bar{B}_1(A_n(\mathbb{Z})) = H^{ev,+}(\Omega(\mathbb{Z}^n)) = H^2(\Omega(\mathbb{Z}^n)) \oplus H^4(\Omega(\mathbb{Z}^n)) \oplus \ldots = \bigoplus_{k=1}^{\lfloor \frac{n}{2} \rfloor} H^{2k}(\Omega(\mathbb{Z}^n)).$$

Combining with Corollary 2.10, we have a complete description of the torsion part of $\bar{B}_1(A_n(\mathbb{Z}))$.

Proposition 3.9. *If $m_1, ..., m_n > 0$ then the torsion in $\bar{B}_1(A_n(\mathbb{Z}))[m_1, ..., m_n]$ is isomorphic to $(\mathbb{Z}/\gcd(m_1, ..., m_n))^{2^{n-2}}$.*

Corollary 3.10. *For $q, r > 0$, the torsion in $\bar{B}_1(A_2(\mathbb{Z}))[q,r]$ is isomorphic to $\mathbb{Z}/\gcd(q,r)$ and spanned by the element $x^{q-1}y^{r-1}[x,y]$.*

Proposition 3.9 allows us to explicitly compute $\bar{B}_1(A_n(\mathbb{F}_p))$ for all p.

Corollary 3.11. *If all m_i are positive, then $\dim \bar{B}_1(A_n(\mathbb{F}_p))[m_1, ..., m_n]$ is 2^{n-2} if there exists i such that m_i is not divisible by p, and 2^{n-1} otherwise (i.e., if all m_i are divisible by p).*

Proof. By Corollary 2.13, $\bar{B}_1(A_n(\mathbb{F}_p)) = \bar{B}_1(A_n(\mathbb{Z})) \otimes \mathbb{F}_p$. It follows from the results of [FS] that the rank of the free part of $\bar{B}_1(A_n(\mathbb{Z}))[\mathbf{m}]$ is 2^{n-2}. So the result follows from Corollary 3.9. □

4. The Structure of $B_2(A_n)$

4.1. **Torsion elements in $B_2(A_n(\mathbb{Z}))$.** In this section we study the torsion in $B_2(A_n(\mathbb{Z}))$. We will show below that for $n = 2$ there is no torsion, so the first interesting case is $n = 3$. In this subsection we describe the torsion in the case $n = 3$. Later we will give a more general result which applies to any n; however, first we explicitly work out the cases $n = 3$ and $n = 2$ for the reader's convenience.

Let us denote the generators of $A_3(\mathbb{Z})$ by x, y, z. Let s, q, r be positive integers, and $T(s,q,r) = [z, z^{s-1}x^{q-1}y^{r-1}[x,y]] \in B_2(A_3(\mathbb{Z}))$.

Theorem 4.1.

1) *The element $T(s,q,r)$ is torsion of order dividing $\gcd(s,q,r)$.*
2) *If $\gcd(s,q,r) = 2$ or 3, the order of $T(s,q,r)$ is exactly equal to $\gcd(s,q,r)$.*

Proof.

1) We start with the identity

$$[z, w[x,y]] = [[w,y], xz] - [z, [y, wx]] + [x, [w, zy]] + x[z,w]y + [w,z]yx \tag{1}$$

which is checked by a direct calculation.

Setting in this identity $w = z^{s-1}$, we get that $[z, z^{s-1}[x,y]] \in L_3(A_3(\mathbb{Z}))$. Now replacing y with $x^{q-1}y^r$, we get that $[z, z^{s-1}[x, x^{q-1}y^r]] \in L_3(A_3(\mathbb{Z}))$. Setting $u := [z, z^{s-1}x^{q-1}y^{r-1}[xy]]$, and using Lemma 2.4, we get that $ru \in L_3(A_3(\mathbb{Z}))$. Similarly, $[z, z^{s-1}[y, x^q y^{r-1}]] \in L_3(A_3(\mathbb{Z}))$, and using Lemma 2.4, $qu \in L_3(A_3(\mathbb{Z}))$.

It remains to show that $su \in L_3(A_3(\mathbb{Z}))$. To this end, we set $m = q-1$, $k = r-1$, and write using Lemma 2.4 (mod $L_3(A_3(\mathbb{Z}))$):

$$[z, x^m y^k[xy]] = [z, x^m[y^k x, y]] = [x^m, z[y^k x, y]] = [x^m, [zy]y^k x]$$
$$= [x^m, [zy^k, y]x] = -[x^m, x[y, zy^k]] = -[x, x^m[y, zy^k]],$$

which is in $L_3(A_3(\mathbb{Z}))$ by identity (1). So, putting z^s instead of z we get that $su = 0$, as desired.

2) We consider first the case when $\gcd(s,q,r) = 2$. Let $i = s-1, j = q-1, k = r-1$, and consider the element $F(x,y,z) := [x, x^i y^j z^k[yz]] = T(s,q,r)$. It suffices to show that this element is nontrivial in $B_2(A_3(\mathbb{F}_2))$.

Let t_x, t_y, t_z be commutative variables (i.e. they commute with x, y, z and each other). Consider $F(x + t_x, y + t_y, z + t_z)$, and take the coefficient of tridegree $(i-1, j-1, k-1)$ in t_x, t_y, t_z. Clearly, it is $ijk[x, xyz[yz]]$. But we know, from computer calculations in MAGMA, that this is nonzero in $B_2(A_3(\mathbb{F}_2))$. We see that for all odd i, j, k, $F(x,y,z)$ is nonzero in $B_2(A_3(\mathbb{F}_2))$.

A similar procedure works in the case where $\gcd(s,q,r) = 3$, with i, j, k of the form $3m-1$. The relevant coefficient (of degree $i-2, j-2, k-2$) would be $\binom{i}{2}\binom{j}{2}\binom{k}{2}[x, x^2y^2z^2[yz]]$, and the element $[x, x^2y^2z^2[yz]]$ is nonzero in $B_2(A_3(\mathbb{F}_3))$ by a computer calculation.

□

Remark 4.2. 1. Actually, the argument in the proof for part 2 would work for any $\gcd(s,q,r)=m$ if we knew that $[x, x^{m-1}y^{m-1}z^{m-1}[yz]]$ has order exactly m in $B_2(A_3(\mathbb{Z}))$.

2. Below we will give another proof that the order of $T(s,q,r)$ is exactly $\gcd(s,q,r)$, which works when $\gcd(s,q,r)$ is odd.

4.2. Torsion in $B_2(A_2(\mathbb{Z}))$.

Theorem 4.3. *$B_2(A_2(\mathbb{Z}))$ has no torsion.*

Proof. We make use of the following:

Lemma 4.4. *We have $B_2(A_n(\mathbb{Z})) = \sum_{i=1}^n [x_i, \bar{B}_1(A_n(\mathbb{Z}))]$.*

Proof. The statement follows from Lemma 3.6. □

Let us denote the generators of A_2 by x, y.

Lemma 4.5. *If T is a torsion element of $\bar{B}_1(A_2(\mathbb{Z}))$, then $[x,T]=[y,T]=0$ in $B_2(A_2(\mathbb{Z}))$.*

Proof. By Corollary 3.10, torsion elements are linear combinations of elements of the form $T = x^{q-1}y^{r-1}[xy]$ (corresponding to the 2-form $x^{q-1}dx \wedge y^{r-1}dy$). Now, $[x,T] = [x, x^{q-1}y^{r-1}[xy]]$. This is the specialization of $T(1,q,r)$ under setting $z=x$ (where by "specialization", we mean that we apply the homomorphism $A_3 \to A_2$ such that x,y,z go to x,y,x, respectively). Since $T(1,q,r)=0$ in B_2, we conclude that the specialization is zero as well, i.e., $[x,T]=0$. Similarly, $[y,T]=0$. □

We showed in Theorem 3.5, that

$$\bar{B}_1(A_2) = \Omega^0 \oplus \Omega^2/\Omega^2_{ex}.$$

Because the last summand is all torsion, by Lemma 4.5, we can strengthen the formula for B_2 in Lemma 4.4 to say that $B_2 = [x,\Omega^0] + [y,\Omega^0]$. Let us denote the $\mathbb{Z}$-span of x and y by V. Since x,y is a basis of V, we see that $B_2 = [V,\Omega_0]$, i.e. B_2 is a quotient of $V \otimes \Omega_0$.

Note that we have an identification $V \otimes \Omega^0 \cong \Omega^1$, via $x \otimes f + y \otimes g \mapsto f dx + g dy$. Let us show that closed 1-forms map to zero in B_2. Let $f,g \in \Omega^0$ be such that $f dx + g dy$ is a closed form, i.e. $f_y = g_x$. We may assume that this form is homogeneous of bidegree (q,r). Then we can set $f = qx^{q-1}y^r/d, g = rx^q y^{r-1}/d$, where $d = \gcd(q,r)$. Define lifts $\hat{f}$ and $\hat{g}$ of f,g to A_2 by the formulas

$$\hat{f} = \sum_{i=0}^{\frac{q}{d}-1} x^i y^{\frac{r}{d}} (x^{\frac{q}{d}} y^{\frac{r}{d}})^{d-1} x^{\frac{q}{d}-1-i},$$

$$\hat{g} = \sum_{i=0}^{\frac{r}{d}-1} y^i (x^{\frac{q}{d}} y^{\frac{r}{d}})^{d-1} x^{\frac{q}{d}} y^{\frac{r}{d}-1-i}.$$

Then it is easy to see that

$$[x,\hat{f}] + [y,\hat{g}] = 0.$$

in A_2. This implies that the closed 1-form $f dx + g dy$ is killed, as desired.

Thus, we see that $B_2(A_2(\mathbb{Z}))$ is a quotient of the free group Ω^1/Ω^1_{cl}. But by the Feigin-Shoikhet theorem [FS], this free group already has the same Hilbert series as $B_2(A_2(\mathbb{Q}))$. So, there is no torsion. □

4.3. **Torsion in $B_2(A_n(\mathbb{Z}))$.**

Conjecture 4.6. *The graded abelian group $B_2(A_n(\mathbb{Z}))$ is isomorphic to*

$$\oplus_{i\geq 1}\Omega^{2i+1}(\mathbb{Z}^n)/\Omega^{2i+1}_{ex}(\mathbb{Z}^n).$$

Thus, the torsion in $B_2(A_n(\mathbb{Z}))$ is isomorphic to

$$H^3(\Omega(\mathbb{Z}^n))\oplus H^5(\Omega(\mathbb{Z}^n))\ldots=\bigoplus_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}H^{2k+1}(\Omega(\mathbb{Z}^n)).$$

In particular, the torsion of $B_2(A_3(\mathbb{Z}))[q,r,s]$ is spanned by the element $T(s,q,r)$, which has order exactly $\gcd(s,q,r)$.

This conjecture implies the following conjecture in characteristic p:

Conjecture 4.7. *If all m_i are positive, then* $\dim B_2(A_n(\mathbb{F}_p))[m_1,...,m_n]$ *is 2^{n-2} if there exists i such that m_i is not divisible by p, and $2^{n-1}-1$ otherwise (i.e., if all m_i are divisible by p).*

The following theorem shows that Conjectures 4.6 and 4.7 hold at least as upper bounds.

Theorem 4.8. *The torsion in $B_2(A_n(\mathbb{Z}))$ is a quotient of*

$$H^3(\Omega(\mathbb{Z}^n))\oplus H^5(\Omega(\mathbb{Z}^n))\ldots=\bigoplus_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}H^{2k+1}(\Omega(\mathbb{Z}^n)).$$

In particular, the torsion of $B_2(A_3(\mathbb{Z}))[q,r,p]$ is spanned by the element $T(p,q,r)$. Also, the numbers in Conjecture 4.7 are upper bounds for $\dim B_2(A_n(\mathbb{F}_p))[m_1,...,m_n]$.

Proof. Let V be the $\mathbb{Z}$-span of the generators $x_1,...,x_n$. By Lemma 4.4 and Corollary 3.7, we have a surjective homomorphism $\xi: V\otimes\Omega^{ev}(\mathbb{Z}^n)/\Omega^{ev}_{ex}(\mathbb{Z}^n)\to B_2(A_n(\mathbb{Z}))$. Moreover, for any $a,b,c,z\in A_n$, we have

$$\begin{aligned}[a,[b,c]z]+[b,[a,c]z]&=[a[b,c],z]-[[b,c],za]+[b[a,c],z]-[[a,c],zb]=\\&=[[ab,c],z]-[[b,c],za]+[[b,[a,c]],z]-[[a,c],zb]\in L_3,\end{aligned}$$

which shows that the image of $[a,[b,c]z]$ in B_2 is skewsymmetric in a,b,c. This implies that ξ is a composition of a homomorphism

$$\eta:\Omega^1(\mathbb{Z}^n)\oplus\Omega^{odd,\geq 3}(\mathbb{Z}^n)/\Omega^{odd,\geq 3}_{ex}(\mathbb{Z}^n)\to B_2(A_n(\mathbb{Z}))$$

and the map $v\otimes\omega\mapsto dv\wedge\omega$. So the theorem follows from the following lemma.

Lemma 4.9. *The image $\tilde{I}$ of $\Omega^1_{cl}(\mathbb{Z}^n)\oplus\Omega^{odd,\geq 3}(\mathbb{Z}^n)/\Omega^{odd,\geq 3}_{ex}(\mathbb{Z}^n)$ under η coincides with the image I of $\Omega^{odd,\geq 3}(\mathbb{Z}^n)/\Omega^{odd,\geq 3}_{ex}(\mathbb{Z}^n)$ under η.*

Indeed, by the results of [FS], the natural map $\gamma:\Omega^1(\mathbb{Z}^n)/\Omega^1_{cl}(\mathbb{Z}^n)\to B_2(A_n(\mathbb{Z}))/\tilde{I}$ is an isomorphism after tensoring with $\mathbb{Q}$. Since the group $\Omega^1(\mathbb{Z}^n)/\Omega^1_{cl}(\mathbb{Z}^n)$ is free, γ is injective. Hence, Lemma 4.9 shows that we have an exact sequence

$$0\to K\to\Omega^{odd,\geq 3}(\mathbb{Z}^n)/\Omega^{odd,\geq 3}_{ex}(\mathbb{Z}^n)\to B_2(A_n(\mathbb{Z}))\to\Omega^1(\mathbb{Z}^n)/\Omega^1_{cl}(\mathbb{Z}^n)\to 0,$$

where the last nontrivial map is induced by γ^{-1}. So, since Ω^1/Ω^1_{cl} is a free group, we have

$$B_2(A_n(\mathbb{Z}))\cong\Omega^1(\mathbb{Z}^n)/\Omega^1_{cl}(\mathbb{Z}^n)\oplus(\Omega^{odd,\geq 3}(\mathbb{Z}^n)/\Omega^{odd,\geq 3}_{ex}(\mathbb{Z}^n))/K.$$

Moreover, as follows from the Feigin-Shoikhet results [FS], this isomorphism holds without K after tensoring with $\mathbb{Q}$, which implies that K is a torsion group. So we get

$$\mathrm{tor}_2 B_2(A_n(\mathbb{Z}))\cong(\Omega^{odd,\geq 3}_{cl}(\mathbb{Z}^n)/\Omega^{odd,\geq 3}_{ex}(\mathbb{Z}^n))/K=(H^3\oplus H^5\oplus...)/K,$$

as desired.

Proof. (of Lemma 4.9) The proof is similar to the argument at the end of the proof of Theorem 4.3. Namely, let $f_1dx_1 + ... + f_ndx_n$ be a closed 1-form, which is homogeneous of multidegree $m_1, ..., m_n$. We may assume that $m_i > 0$, otherwise we are reduced to smaller n. Let $d = \gcd(m_1, ..., m_n)$. Then we can set $f_i = \frac{m_i}{d} x_i^{-1} \prod_j x_j^{m_j}$. Define lifts $\hat{f}_i$ of f_i to elements in A_n by the formulas

$$\hat{f}_i = \sum_{j=0}^{\frac{m_i}{d}-1} x_i^j x_{i+1}^{\frac{m_{i+1}}{d}} \dots x_n^{\frac{m_n}{d}} (x_1^{\frac{m_1}{d}} \dots x_n^{\frac{m_n}{d}})^{d-1} x_1^{\frac{m_1}{d}} \dots x_{i-1}^{\frac{m_{i-1}}{d}} x_i^{\frac{m_i}{d}-j-1}.$$

It is easy to see that $\sum_i [x_i, f_i] = 0$. This implies that $\eta(\sum_i f_i dx_i) \in I$, as desired. □

□

Remark 4.10. Theorem 4.8 implies that Conjecture 4.7 holds for $p = 2, 3$ and $n \leq 4$. The proof is analogous to the proof of Theorem 4.1(2).

4.4. **Proof of 2-localized versions of Conjectures 4.6 and 4.7.**

4.4.1. *The statement.*

Theorem 4.11. *(i) Conjecture 4.6 holds over* $\mathbb{Z}[1/2]$. *So if* $\gcd(q, r, s) = 2^\ell(2k+1)$ *then the order of* $T(q, r, s)$ *is divisible by* $2k + 1$.
(ii) Conjecture 4.7 holds for $p > 2$.

The rest of the section is the proof of Theorem 4.11. In view of Corollary 2.13 (2), it suffices to prove (i). We will apply the theory from the appendix to [DKM], namely Theorem 7.2.

4.4.2. *First cyclic homology of the free algebra over the integers.* Let $A = A_n(\mathbb{Z})$. First of all we compute the first cyclic homology group $HC_1(A)$. This computation is known (due to Loday and Quillen), but we will give it here for the convenience of the reader.

Recall that $HC_1(A)$ is the quotient of the kernel of the commutator map $[,] : \wedge^2 A \to A$ by the elements $ab \wedge c + bc \wedge a + ca \wedge b$, $a, b, c \in A$. Also recall that if $a = x_{i_1} ... x_{i_N}$ is a cyclic word in $A/[A, A]$ (i.e., a word up to cyclic permutation) then we can define its noncommutative partial derivative $\partial_i a \in A$ by the formula

$$\partial_i a := \sum_{k: i_k = i} x_{i_{k+1}} ... x_{i_N} x_{i_1} ... x_{i_{k-1}}$$

Finally, note that for any cyclic word a and any positive integer m, the partial derivative $\partial_i(a^m/m)$, has integer coefficients, even though a^m/m does not.

Theorem 4.12. *(*[LQ], *Proposition 5.4) If a is a non-power word (i.e., a nontrivial word that is not a power of another word) and $m > 1$ is a positive integer, then the noncommutative differential*

$$d(a^m/m) := \sum_i \partial_i(a^m/m) \otimes x_i$$

defines an element of $HC_1(A)$ of order m. Moreover, $HC_1(A)$ is the direct sum of the cyclic subgroups $\mathbb{Z}/m$ spanned by these elements.

Proof. We have the Connes-Loday-Quillen exact sequence ([LQ], Theorem 1.6)

$$HC_0(A) = A/[A, A] \to HH_1(A) \to HC_1(A) \to 0.$$

Now, since $A = TV$ is a free algebra, $HH_1(A)$ is the kernel of the commutator map $A \otimes V \to A$, and the map $d : HC_0(A) \to HH_1(A)$ is the noncommutative differential. In characteristic zero, the kernel of the map d is just constants, and $HC_1 = 0$. This implies that over $\mathbb{Z}$, HC_1 is a torsion group, and in each multidegree $\mathbf{m}$, one has

$$HC_1(A)[\mathbf{m}] = \hat{J}_{\mathbf{m}}/J_{\mathbf{m}},$$

where $J_{\mathbf{m}} := \mathrm{Im}(d)[\mathbf{m}]$, and $\hat{J}_{\mathbf{m}}$ is the saturation of $J_{\mathbf{m}}$ inside $HH_1(A)$. Now, $\hat{J}_{\mathbf{m}}/J_{\mathbf{m}}$ is clearly a direct sum of groups $\mathbb{Z}/m$ generated by elements $d(a^m/m)$, where a is a non-power word, which implies the statement. □

4.4.3. *Conclusion of proof of Theorem 4.11.* Now we are ready to prove Theorem 4.11. We will work over $\mathbb{Z}[1/2]$ and for brevity will not explicitly show it in the notation. Recall from the appendix to [DKM], proof of Theorem 7.2, that we have an exact sequence

$$HC_1(A) \to \wedge^2(A/(L_2 + M_3))/W \to B_2(A) \to 0, \tag{2}$$

where W is spanned by the images of the elements $ab \wedge c + bc \wedge a + ca \wedge b$, $a, b, c \in A$.

Now recall that Ω^{ev} is equipped with the Fedosov product $a * b = ab + \frac{1}{2}da \wedge db$, and analogously to [FS], by Proposition 3.3 we have an algebra isomorphism $\phi : A/M_3 \to \Omega^{ev}_*$, which maps $A/(L_2 + M_3) = \bar{B}_1(A)$ isomorphically onto $\Omega^{ev}/\Omega^{ev}_{ex}$. So, the middle term of the sequence (2) is $\wedge^2(\Omega^{ev}/\Omega^{ev}_{ex})/W$, where W is now spanned by the elements $ab \wedge c + bc \wedge a + ca \wedge b$, $a, b, c \in \Omega^{ev}/\Omega^{ev}_{ex}$ (note that it is not important whether ab is the usual or Fedosov product, as they differ by an exact form, and we are working modulo exact forms).

Now, it is shown as in [DKM], Theorem 7.1, that the algebra of differential forms is pseudoregular in the sense of [DKM], so as a result the map

$$\theta : \wedge^2(\Omega^{ev}/\Omega^{ev}_{ex})/W \to \Omega^{odd}/\Omega^{odd}_{ex}$$

given by $\theta(a \otimes b) = a \wedge db$ is an isomorphism. Thus, we have an exact sequence

$$HC_1(A) \to \Omega^{odd}/\Omega^{odd}_{ex} \to B_2(A) \to 0,$$

where the first map is defined by the formula

$$T = \sum_i f_i \otimes x_i \mapsto \sum_i \phi(f_i) \wedge dx_i.$$

Our job is to show that this map lands in Ω^1/Ω^1_{ex}.

By Theorem 4.12, for this it suffices to show that for every cyclic word a of multidegree $(m_1, ..., m_n)$,

$$\sum_i \phi(\partial_i(a^m/m)) \wedge dx_i \in \Omega^1 + \Omega^{odd,\geq 3}_{ex}.$$

This is equivalent to saying that the form

$$\omega(a, m) := \sum_i (\phi(\partial_i(a^m/m)) - \partial_i(a^m/m)) \wedge dx_i$$

belongs to $\Omega^{odd,\geq 3}_{ex}$. Since this form is in $\Omega^{\geq 3}$ and is clearly closed, it suffices to show that it represents the trivial class in the De Rham cohomology. But by Corollary 2.10, for this it suffices to show that $\omega(a, m)$ is divisible by mD, where $D = \gcd(m_1, ..., m_n)$.

To this end, let us compute $\omega(a, m)$ explicitly. Suppose that w be a cyclic word. By a shuffle subword of w we will mean a cyclic word obtained by crossing out some letters from w. Let $1 \leq i_1 < i_2 < ... < i_{2k+1} \leq n$. Let $N_{i_1,...,i_{2k+1}}(w)$ be the number of shuffle subwords of w which are even permutations of $x_{i_1}, ..., x_{i_{2k+1}}$ minus the number of shuffle subwords which are odd permutations. Note that since the length of the shuffle subword is odd, it makes sense to talk about its parity, as it does not change under cyclic permutation.

Lemma 4.13. *One has*

$$\sum_i \phi(\partial_i w)\wedge dx_i = \sum_{k\geq 0}\frac{1}{2^k}\sum_{i_1<...<i_{2k+1}} N_{i_1,...,i_{2k+1}}(w)dx_{i_1}\wedge ...\wedge dx_{i_{2k+1}}.$$

Proof. This follows by direct calculation using the formula for the Fedosov product. □

Corollary 4.14. *One has*

$$\omega(a,m)=\frac{1}{m}\sum_{k\geq 1}\frac{1}{2^k}\sum_{i_1<...<i_{2k+1}} N_{i_1,...,i_{2k+1}}(a^m)dx_{i_1}\wedge ...\wedge dx_{i_{2k+1}}.$$

Therefore, the theorem follows from the following combinatorial lemma.

Lemma 4.15. *Let a be a cyclic word of degrees m_i with respect to x_i, and $D=\gcd(m_1,...,m_n)$. Then for $k\geq 1$, the number $N_{i_1,...,i_{2k+1}}(a^m)$ is divisible by $m^{k+1}D$.*

Proof. Let y_i, $i=1,...,n$ be anticommuting variables (i.e., $y_iy_j=-y_jy_i$ and $y_i^2=0$). Suppose that $a=x_{j_1}...x_{j_M}$ (where $M=\sum m_i$), and consider the product $Y(a,m)=((1+y_{j_1})...(1+y_{j_M}))^m$. It is easy to see that $N_{i_1,...,i_{2k+1}}(a^m)$ is the coefficient of $y_{i_1}...y_{i_{2k+1}}$ in $Y(a,m)$. However, it is easily shown by induction that

$$(1+y_{j_1})...(1+y_{j_M})=(1+m_1y_1+...+m_ny_n)\prod_{1\leq r<s\leq n}(1+y_{j_r}y_{j_s}).$$

This implies that

$$Y(a,m)=(1+mm_1y_1+...+mm_ny_n)\prod_{1\leq r<s\leq n}(1+my_{j_r}y_{j_s})$$

and the statement follows. □

The theorem is proved.

5. Experimental data and conjectures

5.1. **Experimental data.** In this subsection we summarize the experimental data obtained by direct computation in MAGMA.

In the tables below, multi-degree components are greater than or equal to 1 since cases with one degree being 0 reduce to a smaller number of variables. Due to the S_n action permuting generators, we list only multi-degrees with weakly descending entries. Note that we are asserting the torsion subgroups are trivial for omitted degrees in each specified range.

Table 1. Torsion in $B_\ell(A_2(\mathbb{Z}))$ in degrees (i,j), with $i\geq j$, such that $2\leq \ell\leq i+j\leq 12$.

$(i,j)\backslash\ell$:	5	6	7	9
(4,4)	$\mathbb{Z}/2$			
(6,4)	$\mathbb{Z}/2$		$\mathbb{Z}/2$	
(6,6)	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
(8,4)	$\mathbb{Z}/2$		$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$

Remark 5.1. Note that we did not discover torsion in $B_2(A_2(\mathbb{Z}))$. This lack of torsion is proved in Theorem 4.3.

TABLE 2. Torsion in $B_\ell(A_3(\mathbb{Z}))$ in degrees (i,j,k), with $i \geq j \geq k$, such that $2 \leq \ell \leq i+j+k$ and either $i,j,k \leq 3$ or $j,k \leq 4$, $i \leq 2$.

$(i,j,k)\backslash\ell$:	2	3	5	7
(2,2,2)	$\mathbb{Z}/2$			
(3,3,3)	$\mathbb{Z}/3$	$\mathbb{Z}/3$		
(4,2,2)	$\mathbb{Z}/2$		$(\mathbb{Z}/2)^2$	
(4,4,2)	$\mathbb{Z}/2$		$(\mathbb{Z}/2)^5$	$(\mathbb{Z}/2)^5$

TABLE 3. Torsion in $B_\ell(A_4(\mathbb{Z}))$ in degrees (i,j,k,l), with $i \geq j \geq k \geq l$, such that $2 \leq \ell \leq i+j+k+l$ and $i,j,k,l \leq 2$.

$(i,j,k,l)\backslash\ell$	2	5
(2,2,2,2)	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^5$

5.1.1. *Discrepancies between ranks of B_ℓ in characteristic zero and positive characteristic.* The difference $D = \dim B_\ell(A_3(\mathbb{F}_2)) - \dim B_\ell(A_3(\mathbb{Q}))$ in degrees (i,j,k) was calculated using MAGMA in degrees up to $(3,3,3)$ and $(4,2,2)$ and was found to be nonzero in the following cases:

ℓ	(i,j,k)	D
2	$(2,2,2)$	1
2	$(4,2,2)$	1
4	$(2,2,2)$	-1
4	$(4,2,2)$	-1
5	$(4,2,2)$	2
6	$(4,2,2)$	-2

The difference $D = \dim B_\ell(A_3(\mathbb{F}_3)) - \dim B_\ell(A_3(\mathbb{Q}))$ in degrees (i,j,k) calculated using MAGMA in degrees up to $(3,3,3)$ and $(4,2,2)$ and was found to be nonzero in the following cases:

ℓ	(i,j,k)	D
2	$(3,3,3)$	1
5	$(3,3,3)$	-1

5.2. **Conjectures.** The theoretical results of this paper and the computational results in the previous subsection motivate the following conjectures.

Conjecture 5.2. *If a torsion element T in $B_\ell(A_n(\mathbb{Z}))$ has degree m with respect to some generator x_j, then the order of T divides m.*

Conjecture 5.3. *There is no torsion in $B_\ell(A_n(\mathbb{Z}))[m_1,\ldots,m_n]$ unless $\ell \leq m_1+\cdots+m_n-3$.*

Remark 5.4. One can prove a weaker statement that there is no torsion unless $\ell \leq m_1+\cdots+m_n-2$. Indeed, it is clear that $B_\ell[m_1,...,m_n]$ has no torsion if $m_1+...+m_n = \ell$ because $L_{\ell+1}[m_1,...,m_n] = 0$ in this case. Also, in this case $L_\ell[m_1,...,m_n]$ is a saturated subgroup in the free algebra $A_n(\mathbb{Z})$ (i.e., if $jx \in L_\ell$ for a positive integer j, then $x \in L_\ell$) since this is a graded component of the free Lie algebra in n generators over $\mathbb{Z}$ inside its universal enveloping algebra, which is $A_n(\mathbb{Z})$. This means that $B_{\ell-1}[m_1,...,m_n]$ is also free in this case.

Also we propose the following questions:

Question 1. Suppose that $\ell > 2$ and n are fixed. Can there exist p-torsion in $B_\ell(A_n(\mathbb{Z}))$ for arbitrarily large p? What if ℓ, n are allowed to vary? (In the computer search, we only found 2-torsion and 3-torsion).

Question 2. Does 2-torsion in $B_\ell(A_2(\mathbb{Z}))$ where $\ell > 2$ occur only for odd ℓ?

6. The supercase

In this section we consider the super-extension of the preceding results, in the style of the paper [BJ]. Namely, let $A_{n,k}(R)$ be the free algebra generated over a commutative ring R by n even generators $x_1, ..., x_n$ and k odd generators, $y_1, ..., y_k$. This is really the same algebra as $A_{n+k}(R)$; the only thing that changes is the notion of the commutator. Namely, for two words $a, b \in A_{n,k}$, we define $[a, b] = ab - (-1)^{d_a d_b} ba$, where d_a is the number of odd generators y_j in the word a. Then we define the modules L_i and B_i in the same way as in the usual case.

The structure of $\bar{B}_1(A_{n,k})$ and $B_i(A_{n,k})$ over $\mathbb{Q}$ was studied in [BJ]. In particular, the structure of $\bar{B}_1$, B_2, and B_3 was completely computed. Here we provide some results and data and formulate a number of questions regarding the structure of $\bar{B}_1(A_{n,k})$ and $B_i(A_{n,k})$ over $\mathbb{Z}$.

6.1. **Experimental data.** Tables 4-8 list the torsion subgroup of $B_\ell(A_{m,n})(\mathbb{Z})$ for small values of ℓ, m, n. Note that, as in Section 5.1, we are asserting these subgroups to be trivial for all omitted degrees in the described range.

6.2. **General statements and questions.**

6.2.1. *Torsion in $\bar{B}_1(A_{n,k}(\mathbb{Z}))$.*

Theorem 6.1. *Let $k \geq 1$, $n + k \geq 2$, and $m_1, ..., m_{n+k} > 0$. Then the torsion in $\bar{B}_1(A_{n,k}(\mathbb{Z}))[\mathbf{m}]$ is $(\mathbb{Z}/2)^{2^{n+k-2}}$ if all m_i are even, and zero otherwise.*

Remark 6.2. We see that the behavior of torsion is different from the case $k = 0$, when there is torsion of all orders.

Proof. Let $x_1, ..., x_n, y_1, ..., y_k$ be the even and odd generators, respectively, and let $\Omega(\mathbb{Z}^{n|k})$ be the algebra of differential forms on these variables over $\mathbb{Z}$ (so the elements x_i and dy_j are even, while dx_i and y_j are odd). Then similarly to Section 3, we have a linear isomorphism $\varphi : A_{n,k}(\mathbb{Z})/M_3 \to \Omega(\mathbb{Z}^{n|k})^{ev}$, which maps L_2 into exact forms of positive degree. However, the image is not the entire set of exact positive forms; for instance, if y is an odd variable, then $(dy)^{2r} = d(y(dy)^{2r-1})$ is an exact form but is not in the image of L_2. On the other hand, $2(dy)^{2r}$ is $\varphi([y, y^{2r-1}]) = \varphi(2y^r)$. More generally, it is easy to show as in Section 3 that if ω is a positive exact form then $2\omega \in \varphi(L_2)$. This shows that we have an exact sequence

$$0 \to K \to \bar{B}_1(A_{n,k}(\mathbb{Z})) \to \Omega^{ev}/\Omega^{ev}_{ex} \to 0,$$

where K is a vector space over $\mathbb{F}_2$.

Now, as in the even case, we have

$$(\Omega^{ev}/\Omega^{ev}_{ex})[m_1, ..., m_{n+k}] = (\Omega^{ev}/\Omega^{ev,+}_{cl})[m_1, ..., m_{n+k}] \oplus H^{ev,+}(\mathbb{Z}^{n|k})[m_1, ..., m_{n+k}],$$

where the first summand is a free group and the second summand is torsion. Furthermore, it is easy to show that for positive m, $H^i(\mathbb{Z}^{0|1})[m] = 0$ (as the Poincaré lemma for odd variables holds over $\mathbb{Z}$), so using the Künneth formula as in the even case, we see that for $k > 0$ we have $H^{ev,+}(\mathbb{Z}^{n|k})[m_1, ..., m_{n+k}] = 0$. This means that the group $(\Omega^{ev}/\Omega^{ev}_{ex})[m_1, ..., m_{n+k}]$ is free, and hence the above exact sequence splits.

It remains to compute the dimension of K. To do so, note that $\bar{B}_1(A_{n,k}(\mathbb{Z})) \otimes \mathbb{F}_2 = \bar{B}_1(A_{n,k}(\mathbb{F}_2))$. On the other hand, $\bar{B}_1(A_{n,k}(\mathbb{F}_2)) = \bar{B}_1(A_{n+k}(\mathbb{F}_2))$ (the super-structure does not matter in characteristic 2), so the dimension of the right hand side in multidegree $(m_1, ..., m_{n+k})$ is known from

TABLE 4. Torsion in $B_\ell(A_{1,1}(\mathbb{Z}))$ in degrees (r,s), where r is the degree of the even generator and s is the degree of the odd generator, such that $2 \le \ell \le r+s \le 11$, $r,s \le 9$, excepting $(r,s) = (8,3)$.

$(r,s)\backslash\ell$	2	3	4	5	6	7	8	9
$(2,2)$	$\mathbb{Z}/2$							
$(3,3)$			$\mathbb{Z}/2$					
$(4,2)$	$\mathbb{Z}/2$		$\mathbb{Z}/2$					
$(3,4)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$				
$(4,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$				
$(2,6)$	$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$			
$(3,5)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$			
$(4,4)$	$\mathbb{Z}/2$		$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$			
$(5,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$			
$(6,2)$	$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$			
$(3,6)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$		
$(4,5)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^7$	$(\mathbb{Z}/2)^4$		
$(5,4)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^6$	$(\mathbb{Z}/2)^4$		
$(6,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$		
$(3,7)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	
$(4,6)$	$\mathbb{Z}/2$		$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^{10}$	$(\mathbb{Z}/2)^{10}$	$(\mathbb{Z}/2)^8$	
$(5,5)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^{11}$	$(\mathbb{Z}/2)^{13}$	$(\mathbb{Z}/2)^9$	
$(6,4)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^6$	$(\mathbb{Z}/2)^8$	$(\mathbb{Z}/2)^6$	
$(7,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	
$(8,2)$	$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	
$(3,8)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^5$	$(\mathbb{Z}/2)^3$
$(4,7)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^8$	$(\mathbb{Z}/2)^{11}$	$(\mathbb{Z}/2)^{19}$	$(\mathbb{Z}/2)^9$
$(5,6)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^{12}$	$(\mathbb{Z}/2)^{20}$	$(\mathbb{Z}/2)^{31}$	$(\mathbb{Z}/2)^{14}$
$(6,5)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^{11}$	$(\mathbb{Z}/2)^{18}$	$(\mathbb{Z}/2)^{27}$	$(\mathbb{Z}/2)^{13}$
$(7,4)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^6$	$(\mathbb{Z}/2)^8$	$(\mathbb{Z}/2)^{13}$	$(\mathbb{Z}/2)^8$

Section 3. Finally, the rank of the free part $\bar{B}_1(A_{n,k}(\mathbb{Z}))[\mathbf{m}]$ is known from [BJ]. Subtracting the two, we get the statement. $\square$

6.2.2. *Torsion in $B_1(A_{n,k}(\mathbb{Z}))$.* In the case $k > 0$, unlike the purely even case $k = 0$, the group $B_1(A_{n,k}(\mathbb{Z}))$ has torsion. Namely, recall that $B_1(A_n(\mathbb{Z}))$ has a basis consisting of cyclic words, i.e., words in the generators up to cyclic permutation. This remains true for $A_{n,k}$, except that some words turn out to be equal to minus themselves, thus being 2-torsion. It is clear that such words are exactly even powers of odd words. This implies the following result.

Proposition 6.3. *The group* $\mathrm{tor}B_1(A_{n,k}(\mathbb{Z}))$ *is a vector space over* $\mathbb{F}_2$*, whose multivariable Hilbert series is given by the formula*

$$h_{n,k}(t_1, ..., t_n, u_1, ..., u_k) =$$

$$\sum_{\mathbf{m}: m_{n+1}+...+m_{n+k} \text{ is odd}} a_{m_1,...,m_{n+k}} \frac{t_1^{2m_1}...t_n^{2m_n} u_1^{2m_{n+1}}...u_k^{2m_{n+k}}}{1 - t_1^{2m_1}...t_n^{2m_n} u_1^{2m_{n+1}}...u_k^{2m_{n+k}}},$$

TABLE 5. Torsion in $B_\ell(A_{0,2}(\mathbb{Z}))$ in degrees (s_1, s_2), $s_1 \geq s_2$, such that $2 \leq \ell \leq s_1 + s_2 \leq 11$, $2 \leq s_1, s_2 \leq 9$.

$(s_1, s_2)\backslash\ell$	2	3	4	5	6	7	8	9
$(3,3)$			$\mathbb{Z}/2$					
$(4,2)$	$\mathbb{Z}/2$		$\mathbb{Z}/2$					
$(4,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$				
$(4,4)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$			
$(5,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$			
$(5,4)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^7$	$(\mathbb{Z}/2)^4$		
$(6,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$		
$(5,5)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^{12}$	$(\mathbb{Z}/2)^{14}$	$(\mathbb{Z}/2)^{10}$	
$(6,4)$	$\mathbb{Z}/2$		$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^{10}$	$(\mathbb{Z}/2)^{10}$	$(\mathbb{Z}/2)^8$	
$(7,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	
$(8,2)$	$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	
$(6,5)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^{12}$	$(\mathbb{Z}/2)^{21}$	$(\mathbb{Z}/2)^{33}$	$(\mathbb{Z}/2)^{15}$
$(7,4)$			$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^8$	$(\mathbb{Z}/2)^{11}$	$(\mathbb{Z}/2)^{19}$	$(\mathbb{Z}/2)^9$
$(8,3)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^5$	$(\mathbb{Z}/2)^3$

TABLE 6. Torsion in $B_\ell(A_{2,1}(\mathbb{Z}))$ in degrees (r_1, r_2, s), $r_1 \geq r_2$, such that either $r_1, r_2, s \leq 3$ or $r_1 \leq 4, r_2 \leq 3, s \leq 2$.

$(r_1, r_2, s)\backslash\ell$	2	3	4	5	6	7
$(2,1,3)$			$\mathbb{Z}/2$			
$(2,2,2)$	$(\mathbb{Z}/2)^2$		$(\mathbb{Z}/2)^2$			
$(3,1,2)$			$\mathbb{Z}/2$			
$(2,2,3)$			$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^2$		
$(3,1,3)$			$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$		
$(3,2,2)$			$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$		
$(4,1,2)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$		
$(3,2,3)$			$(\mathbb{Z}/2)^7$	$(\mathbb{Z}/2)^8$	$(\mathbb{Z}/2)^{10}$	
$(3,3,2)$			$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^5$	
$(4,2,2)$	$(\mathbb{Z}/2)^2$		$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^5$	
$(3,3,3)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^{10}$	$(\mathbb{Z}/2)^{18}$	$(\mathbb{Z}/2)^{40}$	$(\mathbb{Z}/2)^{20} + \mathbb{Z}/3$
$(4,3,2)$			$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^6$	$(\mathbb{Z}/2)^{12}$	$(\mathbb{Z}/2)^7$

where $f(z) := \sum a_{m_1,\dots,m_{n+k}} z_1^{m_1} \dots z_{n+k}^{m_{n+k}}$ *is the Hilbert series of the free Lie algebra on* $n+k$ *generators:*

$$f(z) = \sum_{d \geq 1} \frac{\mu(d)}{d} \log(1 - z_1^d - \dots - z_{n+k}^d),$$

where $\mu(d)$ *is the Möbius function.*

In particular, torsion in multidegree $\mathbf{m}$ *is nonzero if and only if all* m_i *are even, and* $(m_{n+1} + \dots + m_{n+k})/\gcd(\mathbf{m})$ *is odd.*

Example 6.4. Let $\nu(m)$ be the maximal power of 2 dividing m. For $n = 0$, $k = 2$, the torsion in B_1 in bidegree (m_1, m_2) is nonzero iff m_1 and m_2 are even, and $\nu(m_1) \neq \nu(m_2)$. For $n = 1$ and $k = 1$, the torsion in B_1 in bidegree (m_1, m_2) is nonzero iff m_1 and m_2 are even, and $\nu(m_1) \geq \nu(m_2)$.

Table 7. torsion in $B_\ell(A_{1,2}(\mathbb{Z}))$ in degrees (r, s_1, s_2), $s_1 \geq s_2$, such that either $r, s_1, s_2 \leq 3$ or $r \leq 4, s_1 \leq 3, s_2 \leq 2$.

$(r, s_1, s_2)\backslash\ell$	2	3	4	5	6	7
$(1,2,2)$		$\mathbb{Z}/3$				
$(1,3,2)$		$\mathbb{Z}/3$	$\mathbb{Z}/2+\mathbb{Z}/3$			
$(2,2,2)$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/2+\mathbb{Z}/3$			
$(2,3,1)$			$\mathbb{Z}/2$			
$(3,2,1)$			$\mathbb{Z}/2$			
$(1,3,3)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^3+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^2+\mathbb{Z}/3$		
$(2,3,2)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^4+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^2+\mathbb{Z}/3$		
$(3,2,2)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^3+\mathbb{Z}/3$	$(\mathbb{Z}/2)^2+\mathbb{Z}/3$		
$(3,3,1)$			$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$		
$(4,2,1)$			$\mathbb{Z}/2$	$\mathbb{Z}/2$		
$(2,3,3)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^8+(\mathbb{Z}/3)^3$	$(\mathbb{Z}/2)^{11}+(\mathbb{Z}/3)^3$	$(\mathbb{Z}/2)^{12}+(\mathbb{Z}/3)^3$	
$(3,3,2)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^7+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^{10}+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^{11}+(\mathbb{Z}/3)^2$	
$(4,2,2)$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^3+\mathbb{Z}/3$	$(\mathbb{Z}/2)^5+\mathbb{Z}/3$	$(\mathbb{Z}/2)^6+(\mathbb{Z}/3)^2$	
$(4,3,1)$			$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^5$	$(\mathbb{Z}/2)^6$	
$(3,3,3)$			$(\mathbb{Z}/2)^{11}+(\mathbb{Z}/3)^3$	$(\mathbb{Z}/2)^{25}+(\mathbb{Z}/3)^4$	$(\mathbb{Z}/2)^{60}+(\mathbb{Z}/3)^7$	$(\mathbb{Z}/2)^{28}+(\mathbb{Z}/3)^3$
$(4,3,2)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^7+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^{15}+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^{38}+(\mathbb{Z}/3)^4$	$(\mathbb{Z}/2)^{19}+(\mathbb{Z}/3)^2$

Table 8. Torsion in $B_\ell(A_{0,3}(\mathbb{Z}))$ in degrees $\mathbf{s} = (s_1, s_2, s_3)$, , such that either $2 \leq \ell \leq s_1 + s_2 + s_3 \leq 10$ and $s_1, s_2, s_3 \leq 8$, or $\mathbf{s} = (7, 3, 1)$.

$\mathbf{s}\backslash\ell$	2	3	4	5	6	7	8	9
$(2,2,1)$		$\mathbb{Z}/3$						
$(2,2,2)$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^2+(\mathbb{Z}/3)^2$					
$(3,2,1)$		$\mathbb{Z}/3$	$\mathbb{Z}/2+\mathbb{Z}/3$					
$(3,2,2)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^4+(\mathbb{Z}/3)^4$	$(\mathbb{Z}/2)^2+(\mathbb{Z}/3)^2$				
$(3,3,1)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^3+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^2+\mathbb{Z}/3$				
$(4,2,1)$			$(\mathbb{Z}/2)^2+\mathbb{Z}/3$	$\mathbb{Z}/2$				
$(3,3,2)$			$(\mathbb{Z}/2)^8+(\mathbb{Z}/3)^4$	$(\mathbb{Z}/2)^{10}+(\mathbb{Z}/3)^6$	$(\mathbb{Z}/2)^{11}+(\mathbb{Z}/3)^4$			
$(4,2,2)$	$\mathbb{Z}/2$		$(\mathbb{Z}/2)^6+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^7+(\mathbb{Z}/3)^3$	$(\mathbb{Z}/2)^8+(\mathbb{Z}/3)^2$			
$(4,3,1)$			$(\mathbb{Z}/2)^4+\mathbb{Z}/3$	$(\mathbb{Z}/2)^6+\mathbb{Z}/3$	$(\mathbb{Z}/2)^7+\mathbb{Z}/3$			
$(5,2,1)$			$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$			
$(3,3,3)$		$\mathbb{Z}/3$	$(\mathbb{Z}/2)^{12}+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{25}+(\mathbb{Z}/3)^{10}$	$(\mathbb{Z}/2)^{64}+(\mathbb{Z}/3)^{18}$	$(\mathbb{Z}/2)^{28}+(\mathbb{Z}/3)^6$		
$(4,3,2)$			$(\mathbb{Z}/2)^9+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{19}+(\mathbb{Z}/3)^5$	$(\mathbb{Z}/2)^{50}+(\mathbb{Z}/3)^{10}$	$(\mathbb{Z}/2)^{22}+(\mathbb{Z}/3)^3$		
$(4,4,1)$			$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^{10}+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{29}+(\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^{14}+\mathbb{Z}/3$		
$(5,2,2)$			$(\mathbb{Z}/2)^6$	$(\mathbb{Z}/2)^{10}+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{28}+(\mathbb{Z}/3)^3$	$(\mathbb{Z}/2)^{12}$		
$(5,3,1)$			$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^7$	$(\mathbb{Z}/2)^{21}+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{10}$		
$(6,2,1)$			$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^6$	$(\mathbb{Z}/2)^3$		
$(5,3,2)$			$(\mathbb{Z}/2)^9$	$(\mathbb{Z}/2)^{21}+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{90}+(\mathbb{Z}/3)^7$	$(\mathbb{Z}/2)^{94}+(\mathbb{Z}/3)^4$	$(\mathbb{Z}/2)^{62}+(\mathbb{Z}/3)^3$	
$(5,4,1)$			$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^{10}$	$(\mathbb{Z}/2)^{48}+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{55}+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{36}+\mathbb{Z}/3$	
$(6,2,2)$	$(\mathbb{Z}/2)^2$		$(\mathbb{Z}/2)^8$	$(\mathbb{Z}/2)^{10}$	$(\mathbb{Z}/2)^{41}+\mathbb{Z}/3$	$(\mathbb{Z}/2)^{39}$	$(\mathbb{Z}/2)^{30}$	
$(6,3,1)$			$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^7$	$(\mathbb{Z}/2)^{27}$	$(\mathbb{Z}/2)^{29}$	$(\mathbb{Z}/2)^{21}$	
$(7,2,1)$			$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^6$	$(\mathbb{Z}/2)^6$	$(\mathbb{Z}/2)^6$	
$(7,3,1)$			$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^7$	$(\mathbb{Z}/2)^{27}$	$(\mathbb{Z}/2)^{36}$	$(\mathbb{Z}/2)^{63}$	$(\mathbb{Z}/2)^{27}$

Proof. It is clear that any nontrivial word is a power of a non-power word (i.e. a word which is not a power of another word) in a unique way, and words generating 2-torsion are exactly the even powers of non-power words of odd total degree with respect to the odd generators (if z is such a non-power word, then $2z^{2l} = [z, z^{2l-1}]$).

Also, it is well known that the generating function for non-power words modulo cyclic permutation is the same as the Hilbert series of the free Lie algebra. Indeed, if the generating function for non-power words modulo cyclic permutation is $f(v)=\sum a_{\mathbf{m}}v^{\mathbf{m}}$, where $v^{\mathbf{m}}=v_1^{m_1}...v_{n+k}^{m_{n+k}}$, then

$$\sum_{\mathbf{m}} a_{\mathbf{m}}(\sum m_j)\frac{v^{\mathbf{m}}}{1-v^{\mathbf{m}}}=\frac{\sum v_j}{1-\sum v_j},$$

since any nontrivial word is uniquely a power of a non-power word. (The factor $\sum m_j$ appears because non-power words are considered up to cyclic permutations, which act freely on them). Applying the inverse of the Euler operator, E^{-1}, which divides terms of total degree N by N, we get

$$\sum_{\mathbf{m}} a_{\mathbf{m}}\log(1-v^{\mathbf{m}})=\log(1-\sum v_j),$$

or

$$\prod_{\mathbf{m}}(1-v^{\mathbf{m}})^{a_{\mathbf{m}}}=1-\sum v_j,$$

which implies that $a_{\mathbf{m}}$ are the dimensions of the components of the free Lie algebra.

These two statements imply the proposition. □

6.2.3. *Torsion in $B_2(A_{n,k}(\mathbb{Z}))$.*

Proposition 6.5. *If $k\geq 1$ and $m_i>0$ then torsion in $B_2(A_{n,k}(\mathbb{Z}))[m_1,...,m_{n+k}]$ is a vector space over $\mathbb{F}_2$, which is a quotient of $\Omega^{odd}_{ex}(\mathbb{Z}^{n|k})[\mathbf{m}]\otimes\mathbb{F}_2$. In particular, there is no 4-torsion, and $B_2(A_{n,k}(\mathbb{Z}[1/2]))[m_1,...,m_{n+k}]$ is a free abelian group.*

Proof. It is shown in the proof of Theorem 6.1 that $\bar{B}_1(A_{n,k}(\mathbb{Z}))[\mathbf{m}]$ is a quotient of $\Omega^{ev}(\mathbb{Z}^{n|k})[\mathbf{m}]/2\Omega^{ev}_{ex}(\mathbb{Z}^{n|k})[\mathbf{m}]$. Therefore, analogously to the proof of Theorem 4.8 one shows that $B_2(A_{n,k}(\mathbb{Z}))[\mathbf{m}]$ is a quotient of $\Omega^{odd}(\mathbb{Z}^{n|k})[\mathbf{m}]/2\Omega^{odd}_{ex}(\mathbb{Z}^{n|k})[\mathbf{m}]$. But as explained in the proof of Theorem 6.1, the De Rham cohomology groups vanish in positive multidegrees. Hence, the subgroup $\Omega^{odd}_{ex}(\mathbb{Z}^{n|k})[\mathbf{m}]$ is saturated. So the torsion in $B_2(A_{n,k}(\mathbb{Z}))[\mathbf{m}]$ is a quotient of $\Omega^{odd}_{ex}(\mathbb{Z}^{n|k})[\mathbf{m}]/2$, as desired. □

Let us now discuss the 2-torsion in $B_2(A_{n,k}(\mathbb{Z}))$ (which by Proposition 6.5 is all the torsion in positive degrees for $k>0$). Arguing as in Lemma 2.12, we see that for any abelian groups $\mathbb{Z}^N\supset A\supset B$, we have an exact sequence

$$\mathrm{tor}_p(\mathbb{Z}^N/A)\to(A/B)\otimes\mathbb{F}_p\to A_p/B_p\to 0,$$

which implies that

$$\dim(A_p/B_p)\leq\dim((A/B)\otimes\mathbb{F}_p)\leq\dim(A_p/B_p)+\dim\mathrm{tor}_p(\mathbb{Z}^N/A).$$

Setting $p=2$, $\mathbb{Z}^N=L_1[\mathbf{m}]$, $A=L_2[\mathbf{m}]$, and $B=L_3[\mathbf{m}]$, this becomes

$$\dim B_2(A_{n+k}(\mathbb{F}_2))[\mathbf{m}]\leq\dim B_2(A_{n,k}(\mathbb{Z}))[\mathbf{m}]\otimes\mathbb{F}_2\leq\dim B_2(A_{n+k}(\mathbb{F}_2))[\mathbf{m}]+\dim\mathrm{tor}_2(B_1(A_{n,k}(\mathbb{Z}))[\mathbf{m}].$$

Subtracting $\dim B_2(A_{n+k}(\mathbb{Q}))$, we get

Proposition 6.6.

$$\dim\mathrm{tor}_2 B_2(A_{n+k}(\mathbb{Z}))[\mathbf{m}]\leq\dim\mathrm{tor}_2(B_2(A_{n,k}(\mathbb{Z})))[\mathbf{m}]\leq$$
$$\leq\dim\mathrm{tor}_2 B_2(A_{n+k}(\mathbb{Z}))[\mathbf{m}]+\dim\mathrm{tor}_2(B_1(A_{n,k}(\mathbb{Z}))[\mathbf{m}].$$

Remark 6.7. 1. Both inequalities in Proposition 6.6 can be strict. For instance, for $B_2(A_{2,1}(\mathbb{Z}))[2,2,2]$, they look like $1\leq 2\leq 3$.

2. The upper bound of Proposition 6.6 is asymptotically bad – it grows exponentially with degrees, while the bound of Proposition 6.5 is uniform in degrees. However, in particular degrees the bound of Proposition 6.6 may be better. For example, for $n=k=1$ and $\mathbf{m}=(2,4)$, then there is a unique odd exact form up to scaling, namely $xdx(dy)^4=d(xydx(dy)^3)$, so Proposition 6.5 gives the bound of 1, while Proposition 6.6 implies that $\mathrm{tor}B_2(A_{1,1}(\mathbb{Z}))[2,4]=0$.

Proposition 6.6 together with Proposition 6.3 implies the following corollary.

Corollary 6.8. *If the group $B_2(A_{n,k}(\mathbb{Z}))[\mathbf{m}]$ has nontrivial 2-torsion, then all m_i are even. Moreover, unless $(m_{n+1} + ... + m_{n+k})/\gcd(\mathbf{m})$ is odd, this 2-torsion depends only on $n+k$ (i.e., is the same as the 2-torsion in $B_2(A_{n+k}(\mathbb{Z}))$).*

Corollary 6.9. *If $n+k=2$ then necessary conditions for a nontrivial 2-torsion are the conditions of Example 6.4.*

Note that this agrees with the results of computer calculations.

6.2.4. *Results and questions on torsion in higher B_ℓ.*

Proposition 6.10. *Nontrivial torsion in $B_\ell(A_{n,k}(\mathbb{Z}[1/2]))[m_1, ..., m_{n+k}]$ implies $\ell \le m_1 + \ldots + m_{n+k} - 2$.*

Proof. Same as in Remark 5.4 (but using the free Lie superalgebra). □

Remark 6.11. 1. Proposition 6.10 seems to be true over $\mathbb{Z}$ but it seems that the proof would need to be modified, since Lie superalgebras don't behave well in characteristic 2.

2. We see from the tables that there *can* be torsion if $\ell = m_1 + \ldots + m_{n+k} - 2$ (this does not seem to occur in the even case).

Proposition 6.12. *If $n+k=2$, there is no torsion in $B_\ell[m_1, m_2]$ if either m_1 or m_2 equals* 1.

Proof. Denote the generators of $A_{n,k}(\mathbb{Z})$ by x, y, and consider $A_{n,k}(\mathbb{Z})[m,1]$. A basis of this group is the collection of elements $x^i y x^j$, where $i+j=m$.

Consider first the case when x is even. Let us identify $A_{n,k}(\mathbb{Z})[m,1]$ with the space of polynomials of a variable z with integer coefficients and degree $\le m$, by $x^i y x^j \mapsto z^i$. In this case, the operation of bracketing with x^l acting from $A_{n,k}(\mathbb{Z})[m-l,1]$ to $A_{n,k}(\mathbb{Z})[m,1]$ corresponds to the operator $P \mapsto (z^l - 1)P$ on polynomials. This implies that $L_{i+1}(A_{n,k}(\mathbb{Z}))[m,1]$ is identified with polynomials of degree $\le m$ which are divisible by $(z-1)^i$. So we see that $B_{i+1}(A_{n,k}(\mathbb{Z}))[m,1] = \mathbb{Z}$ for $i = 0, 1., , , m$, and zero for larger i, so there is no torsion.

If x is odd, the argument is similar. Namely, if y is even, we should use the assignment $x^i y x^j \mapsto (-1)^{ij} z^i$, while if y is odd, we should use the assignment $x^i y x^j \mapsto (-1)^{i(j+1)} z^i$. □

Theorem 6.13. *The dimensions of $B_i(A_{n,k}(\mathbb{F}_p))[\mathbf{m}]$ for $i \ge 2$ are bounded from above by a constant K_i independent of $\mathbf{m}$ and p (but depending on n, k).*

Proof. Since $[ab, c] + [bc, a] + [ca, b] = 0$, we have

$$[ab, [c, d]] + [bc, [a, d]] + [ca, [b, d]] = [c, [ab, d]] + [a, [bc, d]] + [b, [ca, d]].$$

Let $A = A_{n,k}(\mathbb{F}_p)$. Suppose we know that $L_i = [A_{\le r}, L_{i-1}]$, where $A_{\le r}$ is the part of A of degree $\le r$. Let u be a word of degree $\ge 2r+2$. Then we can write u as ab, where the degrees of a and b are $\ge r+1$. Since L_i is the sum of the spaces $[c, L_{i-1}]$, where c has degree $\le r$, we find that $[u, L_i]$ is contained in the sum of spaces $[v, L_i]$, where v has degree $\le 2r+1$. This implies that $L_{i+1} = [A_{\le 2r+1}, L_i]$. Since $L_2 = [A_{\le 1}, L_1]$ by Lemma 3.6, we obtain by induction in i that $L_{i+1} = [A_{\le 2^i - 1}, L_i]$ and hence $B_{i+1} = [A_{\le 2^i - 1}, B_i]$. This implies that if K_i is defined, then one can take $K_{i+1} = K_i \dim A_{\le 2^i - 1}$.

It remains show that K_2 is defined. To show this, recall that $B_2 = \sum [x_i, \bar{B}_1]$. On the other hand, by the results of [BJ] and Theorem 6.1, the dimensions of the homogeneous components of $\bar{B}_1$ are bounded by some constant K. So we can take $K_2 = (n+k)K$, and we are done. □

Remark 6.14. The bound K_i given by this proof is very poor (it grows like $2^{i^2/2}$). For $p \geq 3$, one can get a much better bound, which grows exponentially with i. Namely, since $L_\ell(A_{n,k}(\mathbb{Z}[1/2]))[\ell]$ is a saturated subgroup of $A_{n,k}(\mathbb{Z}[1/2])[\ell]$, it follows from Lemma 3.1 in [AJ] that over any ring R containing $1/2$, one has
$$[(ab+ba)c, B_i] \subset [Q, B_i],$$
where Q is the span of at most quadratic monomials in a, b, c (where $a, b, c \in A_{n,k}$). This implies that if E is a complement to the image in $\bar{B}_1$ of 1 and all elements $(ab+ba)c$ where a, b, c are of positive degree, then $B_{i+1} = [E, B_i]$. But for E we can take the span of all words of degrees $1, 2$ in the generators. The number of such words is $\frac{(n+k)^2}{2} + \frac{3n+k}{2}$. Thus, for $p \geq 3$ we can take
$$K_i = (\frac{(n+k)^2}{2} + \frac{3n+k}{2})^{i-2}(n+k)K.$$
for $i \geq 2$.

We expect that by a suitable strengthening of Lemma 3.1 in [AJ], this argument can be adapted to get an exponentially growing bound K_i also in the case $p = 2$.

Let us now state some questions that are motivated by the data in the tables.

Question 3. Suppose $k \geq 1, n+k \geq 2$. Can $B_i(A_{n,k}(\mathbb{Z}))$ have p-torsion with $p > n+k$? For example, if $n+k = 2$ (i.e., $(n,k) = (0,2)$ or $(1,1)$), can B_i have p-torsion for odd p? Is the torsion in B_i in this case a vector space over $\mathbb{F}_2$?

Question 4. Is there torsion in $B_3(A_{0,2}(\mathbb{Z}))$?

Question 5. If $n+k = 3$, can there be torsion in $B_3(A_{n,k}(\mathbb{Z}))[m_1, m_2, m_3]$ $(m_1, m_2, m_3 > 0)$ other than 3-torsion?

Remark 6.15. We note that, as follows from the above tables, it is not true in the supercase that if a torsion element T in B_i has degree m with respect to some generator, then the order of T divides m. For example, for $n = 1, k = 2$, we have a 3-torsion element in multidegree $(1, 2, 2)$.

6.3. **Lower central series and modular representations.** Let us discuss the connection of the theory of lower central series with modular representation theory. It is clear that the algebraic supergroup scheme $GL(n|k)(\mathbb{Z})$ acts on each $B_i(A_{n,k}(\mathbb{Z}))[m]$ (elements of total degree m), with decomposition into multidegrees being the weight decomposition. Moreover, for each prime p, the p-torsion in this representation is a subrepresentation, which actually factors through $GL(n|k)(\mathbb{F}_p)$. Moreover, the tensor product of this representation with the algebraic closure $\overline{\mathbb{F}_p}$ naturally extends to a representation of $GL(n|k)(\overline{\mathbb{F}_p})$.

In the purely even case, from computer data it appears that this modular representation is always trivial at the Lie superalgebra level. However, in the supercase we see nontrivial Lie superalgebra representations.

For instance, for $n = 1$ and $k = 2$, we have a 3-torsion element in B_3 in multidegree $(1, 2, 2)$, which is unique up to scaling in degree 5. Its degrees are not divisible by 3, so it defines a 1-dimensional representation of $GL(1|2)$ in characteristic 3 which is nontrivial at the Lie superalgebra level. It is easy to see that at the Lie superalgebra level, this is in fact the well-known Berezinian representation with weight $(1, -1, -1)$ (which is the same as $(1, 2, 2)$ in characteristic 3).

Now consider the 3-torsion in B_3 for $n = 0$ and $k = 3$, which gives representations of $GL(3)$ (see the last table in Subsection 6.1). In this case, in degree 5 we have three independent torsion elements, in multidegrees $(1, 2, 2)$, $(2, 1, 2)$ and $(2, 2, 1)$, respectively. Clearly, they generate a copy of the representation $V^* \otimes (\wedge^3 V)^{\otimes 2}$, where V is the 3-dimensional vector representation spanned by the generators of $A_{0,3}$.

In degree 6, there is an 8-dimensional representation, with weights being all the permutations of $(1, 2, 3)$, as well as $(2, 2, 2)$ (with multiplicity 2). This representation has the same character as

$\mathfrak{sl}(V)\otimes(\wedge^3V)^{\otimes 2}$. Note that we cannot claim without additional argument that they are isomorphic, since $\mathfrak{sl}(V)$ is reducible ($1\in\mathfrak{sl}(V)$ in characteristic 3 as V is 3-dimensional). All we can say is that its composition series consists of a 7-dimensional irreducible representation and the 1-dimensional representation $(\wedge^3V)^{\otimes 2}$.

In degree 7, we have a 6-dimensional representation (whose weights are permutations of $(1,3,3)$ and $(2,2,3)$, with 1-dimensional weight spaces), which is clearly $S^2V^*\otimes(\wedge^2V)^{\otimes 3}$. Finally, we have a representation $(\wedge^3V)^{\otimes 3}$ (trivial at the Lie algebra level) sitting in degree 9.

Actually, by analogy with [FS], for $p>2$ the p-torsion in B_i is a representation of the Lie superalgebra $W_{n|k}$ of supervector fields in characteristic p (this action is compatible with the action of $GL(n|k)$). In particular, we have an action of the Lie superalgebra $\mathfrak{pgl}(n+1|k)$ which sits inside $W(n|k)$ as the Lie algebra of the supergroup of projective transformations. Looking at the action of this subalgebra, we get more information.

For instance, consider the case $n=0$, $k=3$. In this case, the 3-torsion in B_3 tensored with the algebraic closure $\overline{\mathbb{F}}_p$ carries an action of the Harish-Chandra pair $(\mathfrak{sl}(1|3),GL(3))$ (which is equivalent to an action of the supergroup $SL(1|3)$). Note that the trivial 1-dimensional representation of $\mathfrak{sl}(1|3)$ has a family of lifts $\chi^{\otimes r}$ to $SL(1|3)$, parametrized by integers r; namely $\chi|_{GL(3)}=(\wedge^3V)^{\otimes 3}$.

Now consider the 3-torsion X in degrees 5,6,7 in $B_3(A_{0,3}(\mathbb{Z}))$ tensored with the algebraic closure (so $\dim(X)=17$). Let us describe the composition series of X as a representation of $SL(1|3)$. It is easy to see that the irreducible module for $SL(1|3)$ with lowest degree component $V^*\otimes(\wedge^3V)^{\otimes 2}$ is $\tilde{V}^*\otimes\chi$, where $\tilde{V}$ is the 4-dimensional defining representation of $SL(1,3)$. As a $GL(3)$-module, this representation is $V^*\otimes(\wedge^2V)^{\otimes 2}\oplus(\wedge^3V)^{\otimes 2}$. Thus, we see that $\tilde{V}^*\otimes\chi$ is contained in the composition series of X. The remaining $GL(3)$-modules are the 6-dimensional and 7-dimensional irreducible representations of $GL(3)$, and they have to comprise an irreducible $SL(1|3)$-module, since any $SL(1|3)$ module with the trivial action of the odd generators of the Lie algebra must be trivial for the whole Lie algebra. Thus, we see that the composition series of X has length 2 and consist of a 4-dimensional and a 13-dimensional irreducible representations, living n degrees 5,6 and 6,7, respectively. They have Hilbert series $3t^5+t^6$ and $7t^6+6t^7$.

In a similar way one should be able to study the 3-torsion in $B_i(A_{n,k})$ for $i>3$ (as well as higher p-torsion).

7. Further work

Besides attempting to answer the questions stated above and demystify the enigmatic pattern of torsion in $B_i(A_n)$ and more generally $B_i(A_{n,k})$, we can point out the following interesting directions of future research.

1. Over $\mathbb{Z}[1/2]$, the Lie algebra W_n of vector fields on the n-dimensional space acts on $\bar{B}_1(A_n)$ and $B_i(A_n)$ for $i\geq 2$, similarly to [FS] (in characteristic 2, this can be generalized but a modification is needed). It would be interesting to study this action and to apply representation theory of W_n to the study of the torsion in $B_i(A_n)$. In particular, we conjecture that the Lie algebra W_n acts trivially on the torsion in $B_i(A_n)$ (by analogy with the fact that vector fields on a manifold act trivially on its cohomology). Note that this conjecture implies the conjecture that the order of a torsion element divides its degree with respect to each variable x_i, by considering the vector field $x_i\frac{\partial}{\partial x_i}$.

We expect that in this study over $\mathbb{F}_p$, the restricted structure of the Lie algebra W_n will play a role.

Also it would be interesting to extend this approach to the supercase, where there is an action of the Lie superalgebra $W_{n|k}$.

2. It would be interesting to generalize the results of this paper from the quotients B_i to the quotients $N_i := M_i/M_{i+1}$. (We note that a search in low degrees found no torsion in these quotients).

3. It would be interesting to extend the results of this paper to algebras with relations, along the lines of the appendix to [DKM] and of [BB].

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