ON INTEGRAL EQUATIONS
Their Solution by Iteration and Analytic Continuation

by

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Abstract

This thesis is concerned with the linear integral equation with fixed finite limits, known as the Fredholm equation of the second kind. The kernel is not assumed to be symmetric; but, as with all other functions involved, is assumed continuous in the unit square \(0 \leq x, y \leq 1\).

Other than the Fredholm procedure involving infinite determinants, solutions for large \(\lambda\) have not been exhibited, at least in forms not demanding prior explicit knowledge of the characteristic values and characteristic functions. From the standpoint of computing, it is desirable to obtain solutions requiring only evaluation of the kernels and iterated kernels of the equation. This thesis exhibits solutions for all but (i) characteristic values, \(\lambda_i\), of the kernel, and (ii) those \(\lambda\) satisfying simultaneously \(|\lambda| > |\lambda_i|\) and \(\arg \lambda = \arg \lambda_i\) for some \(i = 1, 2, \ldots\). Furthermore, the solutions are in the form of power series; or uniform limits of power series; or as uniformly convergent infinite integrals; or as limits of iterative processes.

In Chapter I, are considered certain iterative processes yielding solutions of the integral equation for certain \(\lambda\). Firstly, we consider the well-known method of successive approximations which is a simple iterative scheme defined by:

\[
U_n(x) = g(x) + \lambda \int_0^1 K(x, y) U_{n-1}(y) \, dy
\]

and show that convergence to the solution (always taken to be uniform) obtains if and only if \(|\lambda/\lambda_i| < 1\) where \(|\lambda_i|\) is the
characteristic value of minimum modules. An upper bound on the error after \( n \) steps is obtained.

A variant of this method, useful where numerical calculation is to involve operations at a fixed number of points, is obtained where sequences \( \{g_n(x)\} \) and \( \{K_n(x,y)\} \) converging uniformly to \( g(x) \) and \( K(x,y) \) are employed. Then the scheme defined by:

\[
V_n(x) = g_0(x) + \int_0^1 K(x,y) V_{n-1}(y) \, dy
\]

converges to the solution if \( |k| \lambda h < 1 \).

For some values of \( \lambda \) outside the circle, \( |\lambda| = |\lambda_1| \). The method of back substitutions yields an iterative procedure converging to the solution. The iterative algorithm is given by: \( u_0(x) \), continuous on \([0,1]\), but otherwise arbitrary, function,

\[
u_n(x) = u_{n-1}(x) + \alpha \lambda n(x)
\]

where \( \alpha \) is a fixed parameter, and

\[
\lambda n_{n-1}(x) = u_{n-1}(x) - q(x) - \lambda \int_0^1 K(x,y) u_{n-1}(y) \, dy
\]

the residue at the \((n-1)^{th}\) step. This procedure is actually Euler summability applied to the series whose partial sums are generated by the method of successive approximations. The necessary and sufficient conditions for convergence of \( \{V_n(x)\} \) to the solution of the equation are

1. \( |1 + \alpha| < 1 \)
2. \( \lambda \in \bigcap \nu_i \) where \( \nu_i \) is the set of all \( \lambda \) satisfying \( |1 + \alpha - \epsilon \lambda \lambda_i| < 1 \).

This method in particular yields solutions for all \( |\lambda| = |\lambda_1| \) except, of course, any \( \lambda_i \) on the circle \( |\lambda| = |\lambda_1| \).

Since the method of successive approximations defines
a power series solution, by successive continuation of this power series in a chain of circles in the λ plane having only regular points in their interiors, a solution is obtained for those λ for which, simultaneously, (i) no |λi| ≤ |λ| and arg λ = arg λi for any i. The solution is obtained by iterating successively by successive approximations a finite set of equations, each a finite number of times to yield a solution accurate to within an error whose maximum is explicitly evaluated.

Finally, Newton's method for finding reciprocals of numbers is applied to the solution of integral equations. The algorithm

\[
R_0(x, y; λ) = 1 - λ R(x, y)
\]

\[
R_{n+1}(x, y; λ) = R_n(x, y; λ) - \sum_0^1 R(t, x; λ) k(x, t) dt
\]

yielding a solution \( \lim_{n \to \infty} R_n(x, y; λ) \) satisfies

\[
f(x) = g(x) + λ \int_0^1 R(x, y; λ) g(y) dy
\]

which is the solution of the integral equation. The necessary and sufficient condition for converging to \( R(x, y; λ) \) is that:

\[
\max_{0 \leq t \leq 1} |2 \int_0^1 k(x, t) g(y) dy - λ \int_0^1 k(x, t) g(y) dy| < 1
\]

\[
\max_{0 \leq t \leq 1} |2 \int_0^1 k(x, t) dt - λ \int_0^1 k(x, t) dt| < 1
\]

This method has the advantage of being of the second order or one whose error at the \( n \)th step is the square of that at the \( (n - 1) \)st step, whereas the previously-mentioned processes were of the first order. Indeed, by combining various "Newton's" methods, trivial processes of arbitrary order may be obtained. These "Newton's" methods are the only "polynomial" iterative forms which converge to the reciprocal kernel.
In Chapter II, a formal treatment of the Fredholm integral operator is introduced as an element of a complete normed ring. This is done to bring forth in a clear way the relationship between analytic functions defined by power series and the resolvent or reciprocal kernel of the integral equation. This procedure allows the use of the apparatus of complex variables in describing the resolvent. Indeed, the resolvent is exhibited as a meromorphic function of $\lambda$ and capable of representation by the Cauchy integral theorem. Further, it has a Taylor's series whose radius of convergence is $|\lambda|$, and, hence, by the uniqueness theorem for such power series is the Neumann series.

In Chapter III, by using the representation of the resolvent by the Cauchy integral formula and the concept of the Mittag-Leffler star, the solution is obtained for large by using the following summability methods:

(i) The Euler method, which yields a solution for interior to the Borel polygon of summability, as does

(ii) The Borel integral method, which expresses the solution as an infinite integral in $\lambda$ converging at every interior point of the summability polygon.

(iii) The method of convergence factors, which yields a solution for all $\lambda$ in every bounded region of the Mittag-Leffler star of the resolvent. These solutions are exhibited as limits of uniformly convergent power series.
INTRODUCTION

1.

This thesis concerns itself with the specific problem of providing algorithms for solving the linear Fredholm integral equation of the second kind. The intent is particularly to obtain solutions for large values of a parameter without specifically employing the characteristic functions and values of the Fredholm kernel. Throughout, the approach is from the standpoint of eventual application as numerical methods.

In the not-too-distant future, the advent of rapid large-scale digital machinery will require quite general methods for handling entire classes of functional equations. At the present stage of coding for these machines, iterative procedures would appear to be the most advantageous for at least two reasons: (1) random errors will not tend to increase without bound in the calculations; and (2) loop or iterative coding, at present, the most widely used, places a premium upon those schemes involving a relatively simple process, or sequences of such processes repeated a large number of times.

This thesis will be restricted to developing algorithms for the solution of the linear Fredholm integral equation of the second kind. These algorithms specifically indicate under what conditions solutions exist, the nature of such solutions, and the truncating error in approximate solutions (the sense of the error to be in the norm of a "uniform topology").
It is pertinent to mention those methods available at present for effecting the solution of these integral equations.

The Fredholm procedure involves the generation of two entire functions of a parameter either through an iterative procedure or by the evaluation of a succession of determinants of ever-increasing degree. \[(2)\]

The Hilbert-Schmidt procedure for symmetric kernels involves the representation of the solution as a generalized Fourier series and requires a previously obtained knowledge of the characteristic values and functions of the kernel. \[(3)\]

The Schmidt method is to approximate the kernel by a degenerate kernel with a remainder, the degree of the polynomial being so chosen that, in norm, the remainder is less than one. Using the degenerate portion of the kernel, the procedure then is to solve a set of simultaneous equations and then, using the remainder, to generate a solution by successive approximations. \[(4)\]

The method of successive approximations may be applied for a small enough value of a parameter in the equation.

The above-mentioned methods are most useful in proving existence of solutions, but have not - with the exception of the latter - been found tractable for numerical work.

The numerical procedure most extensively employed thus far is to reduce the integral equation to a system of simultaneous linear equations. The problem then is an algebraic one and is treated by any of the available methods for inverting matrices. \[(5)\]
The existence theorems and the numerical procedure cited above will be discussed in more detail at the end of the introduction.

The notation to be employed throughout the thesis is as follows:

Lower case letters of the lower alphabet, a, b, etc., represent numerical constants. Those of the middle alphabet represent continuous functions of a real variable; i.e., f, g, h, etc. Those of the upper alphabet represent independent variables, both real and complex.


And, of course, i, j, k, n, m, p represent summation indices.

2.

This section deals with some of the properties of the equation under consideration.

The equation is:

\[ f(x) = g(x) + \lambda \int_{0}^{1} k(x, y) f(y) dy \]  

(1)

and the following assumptions are made:
1) \( g(x) \) is continuous for \( 0 \leq x \leq 1 \)

2) \( K(x,y) \) is continuous for \( 0 \leq x, y \leq 1 \)

3) \( \lambda \) is an arbitrary complex constant.

Corresponding to the inhomogeneous equation (1), there is the homogeneous equation:

\[
\psi(x) = \lambda \int_0^1 K(x,y) \psi(y) \, dy \tag{2}
\]

It follows at once that every bounded or integrable \( f(x) \) satisfying (1) is a continuous function for \( 0 \leq x \leq 1 \), and all bounded solutions of (2) are continuous.

Any finite closed interval \([a, b]\) can be mapped onto \([0,1]\); hence, this restriction imposes no loss in generality. At this point, it might be mentioned that the continuity conditions can be replaced by integrability conditions (say, in the \( L^2 \) sense) throughout and the results correspondingly strengthened. However, it is felt that continuity conditions suffice for the purpose of this thesis, they being more likely to occur in a given numerical problem. Obviously in the kernel, \( K(x,y) \) continuity in \( X \) and integrability in \( Y \) suffice to ensure continuity in \( f(x) \), \( g(x) \) being continuous.

The basic existence theorem is:

**Theorem (Fredholm)** If \( g(x) \) and \( K(x,y) \) are continuous over the closed interval \([0,1]\) and if \( \lambda \) is such that the homogeneous equation (2) has only the solution \( \psi(x) \equiv 0 \), then there exists a unique continuous solution \( f(x) \) defined on \([0,1]\). If, however, the homogeneous equation has \( m \) independent solutions, then there exists a solution \( f(x) \) to the nonhomogeneous equation, if and only if \( g(x) \) is
orthogonal to each of the \( m \) solutions of the transposed homogeneous equation

\[
\phi(x) = \lambda \int_0^1 K(y,x) \phi(y) \, dy
\]  

(3)

This is the basic existence theorem for equations of this type.

Assuming \( \lambda \) is not a characteristic value*, the solution of (1) can be written as

\[
f(x) = g(x) + \lambda \int_0^1 R(x,y; \lambda) g(y) \, dy
\]  

(4)

\( R(x,y; \lambda) \) the solving kernel is to be found.

The Fredholm procedure yields \( R(x,y; \lambda) \) as the quotient of two entire functions of \( \lambda \):

\[
R(x,y; \lambda) = \frac{D(x,y; \lambda)}{D(\lambda)} = \frac{d_n(x,y) \lambda^n}{\sum_{n=0}^{\infty} d_n \lambda^n}
\]  

(5)

The roots of \( D(\lambda) \) are the characteristic values of the kernel \( K(x,y) \) and to each of them corresponds at least one \( \phi(x) \neq 0 \) satisfying (2).

The double iterative process defined by:

\[
d_0(x,y) = K(x,y), \quad d_0 = 1
\]

\[
d_{v+1} = -\frac{1}{(v+1)} \int_0^1 d_v(x,x) \, dx
\]  

(6.1)

and

\[
d_v(x,y) = K(x,y)d_v + \int_0^1 K(x,t)d_{v-1}(t,y) \, dt
\]  

(6.2)

yields the coefficients \( d_v(x,y) \) and \( d_v \) of (5).

* \( \lambda \), a characteristic value, means that there exists a \( \phi(x) \neq 0 \) such that (2) is satisfied for this value of \( \lambda \).
Since $D(o) = d_o = 1$ and the characteristic values are isolated*, there exists a neighborhood about $\lambda = 0$ in which \[
\frac{1}{D(\lambda)}
\] is a holomorphic function of $\lambda$. Thus:
\[
\frac{1}{D(\lambda)} = \sum_{n=0}^{\infty} h_n \lambda^n
\]
and hence:
\[
R(x, y; \lambda) = \left( \sum_{n=0}^{\infty} d_n(x, y) \lambda^n \right) \left( \sum_{m=0}^{\infty} h_m \lambda^m \right)
\]
The product of these two uniformly convergent series (for $|\lambda|$ near enough to zero) yields the Neumann series
\[
R(x, y; \lambda) = \sum_{n=1}^{\infty} K_n(x, y) \lambda^{n-1}
\]
(6.3)

The Hilbert-Schmidt theory for symmetric kernels makes extensive use of the following theorems and definitions:

**Theorem:** For a real, symmetric, continuous kernel $K(x, y)$, there exists at least one characteristic value $\lambda_o$. Furthermore, all such characteristic values are real.

**Definition:** A symmetric kernel is said to be positive definite if all characteristic values are positive or, equivalently
\[
\int_0^1 \int_0^1 K(x, y) \phi(x) \phi(y) \, dx \, dy > 0
\]
for every continuous $\phi(x)$.

**Definition:** A set of continuous functions $\{\phi_i(x)\}$ are said

* The set of characteristic values has no limit point in the finite complex plane. See p.
to be complete if there exists no continuous function, 
\( h(x) \), such that
\[
\int_0^1 \phi_i(x) \ h(x) \, dx = 0 \quad i \in \{1, 2, \ldots\}
\]

For the case of continuous functions defined on \([0,1]\), every set of independent functions not complete may be made complete by the introduction of new members into the set.

**Theorem:** To every real symmetric kernel, there belongs a complete set of real characteristic functions \( \{ \psi_i(x) \} \) satisfying:

1. \( \psi_i(x) = \lambda_i \int_0^1 K(x, \eta) \psi_i(\eta) \, d\eta \)
2. \( \int_0^1 \psi_i(x) \psi_j(x) \, dx = \delta_{ij} \)

**Theorem:** If \( \{ \psi_i(x) \} \) form a complete orthonormal set of characteristic functions for the real symmetric kernel \( K(x, \eta) \) and \( \sum_{n=1}^{\infty} \frac{\psi_n(x) \psi_n(\eta)}{\lambda_n} \) is uniformly convergent on \([0,1]\), then:

\[
K(x, \eta) = \sum_{n=1}^{\infty} \frac{\psi_n(x) \psi_n(\eta)}{\lambda_n} \quad (7)
\]

This is the bilinear formula.

**Definition:** The iterated kernels of \( K(x, y) \) are defined by:

\[
K^{[1]}(x, y) = K(x, y)
\]

\[
K^{[n]}(x, y) = \int_0^1 K(x, t) K^{[n-1]}(t, y) \, dt \quad (8)
\]

from which follows:

1. If \( \{ \lambda_n \} \) is the set of characteristic values
K(x, y), then \( \{ \lambda_n^m \} \) is the set of characteristic values for 
\( K^{[n]}(x, y) \) and the symmetry of \( K(x, y) \) implies that of \( K^{[n]}(x, y) \), and

\[ \text{(ii) The series } \sum_{n=1}^{\infty} \frac{\psi_m(x) \psi_m(y)}{\lambda_n^m}, n \geq 2 \text{ is uniformly convergent on } [0,1] \text{ and equals } K^{[n]}(x, y). \]

Then, the Hilbert-Schmidt representation of the solution of (1) for \( \lambda \) is not a characteristic value

\[ f(x) = g(x) + \lambda \sum_{n=1}^{\infty} \frac{\int_0^1 g(t) \psi_n(t) dt}{\lambda_n - \lambda} \psi_n(x) \]  

(9)

and the series is absolutely and uniformly convergent on 
\( [0,1] \). The representation for \( R(x, \gamma; \lambda) \) is then

\[ R(x, \gamma; \lambda) = \sum_{n=1}^{\infty} \frac{\psi_k(x) \psi_n(\gamma)}{\lambda_n - \lambda} \]  

(10)

Consider now the case where the kernel is degenerate; 
i.e., is of the form \( \sum_{k=1}^{N} h_k(x) p_k(y) \) with the \( \{ h_k(x) \} \)
and \( \{ p_k(y) \} \) linearly independent. Then (1) becomes

\[ f(x) = g(x) + \lambda \sum_{k=1}^{N} h_k(x) \int_0^1 p_k(y) f(y) dy \]  

(11)

Multiplying by \( p_j(x) \), integrating, and introducing obvious notation yields

\[ \langle p_j, f \rangle = \langle p_j, g \rangle + \lambda \sum_{k=1}^{N} (p_j, h_k)(h_k, f) \]  

(12)

a set of n linear equations, whose matrix is

\[ \begin{bmatrix} \lambda (p_j, h_k) - \delta_{j,k} \end{bmatrix} \]

Thus for \( \lambda \) other than a root of the characteristic polynomial of the matrix, the equation (12) can be solved for
(\psi, \delta)$. Then substitution in (11) yields the solution of (1) for $K(x,y)$ a degenerate kernel.

Suppose now that $K(x,y)$ is not of the degenerate form. However, being uniformly continuous in $[0,1]$, there exists an integer $N$ such that on an interior interval of length $\frac{2}{N}$ and any $y$, the oscillation of $K(x,y) < \delta$.

Dividing $[0,1]$ into $N$ intervals $I_n$, let $x_n$ be the midpoint of $I_n$, then set

$$h_n(y) = K(x_n, y)$$

$h_n(x)$ is constructed in the following fashion:

$$h_n(x) = \begin{cases} 
0 & 0 \leq x \leq \frac{(n-1)\frac{2}{N} - \frac{1}{N^2}}{N} \\
\frac{N}{2}(x - \{n-1\frac{2}{N} + \frac{1}{N^2}\}) & \frac{(n-1)\frac{2}{N} - \frac{1}{N^2}}{N} \leq x \leq \frac{(n-1)\frac{2}{N} + \frac{1}{N^2}}{N} \\
1 & \frac{(n-1)\frac{2}{N} + \frac{1}{N^2}}{N} \leq x \leq \frac{n\frac{2}{N} - \frac{1}{N^2}}{N} \\
\frac{N}{2}(n\frac{2}{N} + \frac{1}{N^2} - x) & \frac{n\frac{2}{N} - \frac{1}{N^2}}{N} \leq x \leq \frac{n\frac{2}{N} + \frac{1}{N^2}}{N} \\
0 & \frac{n\frac{2}{N} + \frac{1}{N^2}}{N} \leq x \leq 1 
\end{cases}$$

which is a continuous function on $I_n$.

So for $x$ in that part of $I_n$ where $h_n(x) = 1$:

$$|K(x, y) - \sum_{n=1}^{N} h_n(x) h_n(y)| = |K(x, y) - K(x, y)| \leq \delta$$

For $x$ in that associated interval where $h_n(x)$ is linear increasing:

$$|K(x, y) - \sum_{n=1}^{N} h_n(x) h_n(y)| \leq |K(x, y) - h_n(x) K(x_n, t) - h_{n+1}(x) K(x_{n+1}, t)|$$
Since $h_n(X) + h_{n+1}(X) = 1$, the above can be made $\leq 2\delta$.

Hence:

$$\sum_{i, j} |k(x, y) - \sum_{n=1}^{N} h_n(x) p_n(y)| dx \, dy \leq 2\delta$$

and for $N$ large enough, the difference may be made as small as desired.

So now choose an $N$ such that

$$|\lambda| \sum_{i, j} |k(x, y) - \sum_{n=1}^{N} h_n(x) p_n(y)| dx \, dy < \frac{1}{2}$$

then put

$$H(x, y) = k(x, y) - \sum_{n=1}^{N} h_n(x) p_n(y)$$

and write (1) as:

$$f(x) = g(x) + \lambda \sum_{n=1}^{N} \int_{0}^{1} h_n(x) p_n(y) f(y) \, dy$$

$$+ \lambda \int_{0}^{1} H(x, y) f(y) \, dy$$

Now for $H(x, y)$, the method of successive approximations for example, (see p. 13) provides an $R(x, y; \lambda)$ such that

$$f(x) = g(x) + \lambda \int_{0}^{1} R(x, y; \lambda) g^*(y) \, dy$$

where $g^*(y)$ is given by the first two terms on the right hand side of (15).

Hence:

$$f(x) = G(x) + \lambda \int_{0}^{1} \left[ \sum_{n=1}^{N} A_n(x, \lambda) p_n(y) f(y) \right] dy$$

where

$$G(x) = g(x) + \lambda \int_{0}^{1} R(x, y; \lambda) g(y) \, dy$$
and

\[ A_n(x, \gamma) = h_n(x) + \lambda \int_0^1 R(x, \gamma; \alpha) \, h_n(\eta) \, d\eta \quad (17.2) \]

Multiplication by \( p_n(x) \) and integrating on \( X \) over \([0,1]\) reduces the equation to the form \((13)\). This is the method devised by Schmidt to resolve the equation. It demands a combined solution by the inversion of a matrix and the method of successive approximations by constructing in a straightforward fashion the approximate degenerate kernel.

It is pertinent to observe at this point that the difficulty in effecting a solution by the method of successive approximations depends critically on the value of \(|\lambda|\) at least in proportion to the "norm" or "size" of the kernel (a concept to be defined more vigorously at a later point).
I. Iteration Procedures

In this chapter are exhibited five methods for obtaining the solution of (1) by iterative procedures. Throughout, it is assumed that \( \lambda \) is not a characteristic value of the kernel. Otherwise, restrictions on its value are stated where pertinent. Of these methods, the first is well known and yields solutions for small enough \( \lambda \). The second is a rather obvious variant of the first and will have value in those calculations where it is pointless to provide more accuracy in the functions than the algorithm permits in the solution at any stage of iteration. The third method, long known for inversion of matrices, was only recently and independently of this work applied to integral equations. It was this method that led to the interpretation and formulation expressed in Chapter III. The fourth method extends to much larger values of \( \lambda \) a solution of the integral equation by iteration and as applied to integral equations is here presented for the first time. The fifth method is one of great power and is based on Newton's method for finding reciprocals of numbers. This has been applied to the inversion matrices and is here applied to integral equations. Herein is introduced the concept of the order of an iterative procedure and from Newton's method are derived admittedly trivial variants of any arbitrary order.
1. The Method of Successive Approximations

Select an arbitrary continuous function \( u_0(x) \) defined over \([0,1]\). Then construct a sequence \( \{ u_n(x) \} \) defined by:

\[
U_{n+1}(x) = g(x) + \lambda \int_0^1 K(x,y) u_n(y) \, dy
\]  

(18)

Corresponding to the sequence \( \{ u_n(x) \} \) is the sequence \( \{ v_n(x) \} \) defined by:

\[
v_n(x) = u_{n+1}(x) - u_n(x).
\]

From the definition of the iterated kernels, the sequence satisfies:

\[
v_{n+1}(x) = \lambda \int_0^1 K(x,y) v_n(y) \, dy
\]

(19.1)

\[
= \lambda^n \int_0^1 K^n(x,y) v_0(y) \, dy
\]

(19.2)

If the sequence \( \{ u_n(x) \} \) converges uniformly to a limit function \( u(x) \), it then follows that

(i) Since \( u_n(x) \) is a continuous function for every \( n \), then \( u(x) \) is a continuous function, and from

\[
\lim_{n \to \infty} u_{n+1}(x) = g(x) + \lim_{n \to \infty} \lambda \int_0^1 K(x,y) u_n(y) \, dy
\]

follows

(ii) \( u(x) = \lim_{n \to \infty} u_n(x) \) is a solution.

The limit function \( u(x) \) is independent of \( u_0(x) \).

Hence the solution so obtained is unique since if \( w(x) \) were a solution different from \( u(x) \), putting \( u_0(x) = w(x) \) yields as solution \( w(x) \), hence equal identically to \( u(x) \).

A necessary and sufficient condition that \( \{ u_n(x) \} \) converge uniformly to \( u(x) \) is that \( \{ v_n(x) \} \) converge uniformly
to zero.

Since the solution is independent of the initial function \( V_0(x) = u_0(x) \), put \( V_0(x) = \phi_k(x) \) where \( \phi_k(x) \) is a characteristic function associated with \( \lambda_k \), a characteristic value. Since \( \phi_k(x) \) satisfies

\[
\phi_k(x) = \lambda_k \int_0^1 K(x, y) \phi_k(y) dy
\]

then

\[
V_{n+1}(x) = \left( \frac{\lambda}{\lambda_k} \right)^n \phi_k(x)
\]

From which two necessary conditions for convergence follow:

(i) \( \lambda = \lambda_k \) for no \( k \) since then

\[
V_{n+1}(x) = V_0(x) = \phi_k(x) \quad n = 1, 2, \ldots
\]

and there is no convergence.

(ii) \( \left| \frac{\lambda}{\lambda_k} \right| < 1 \), all \( k \), which implies

(iiia) \( \left| \frac{\lambda}{\lambda_k} \right| < 1 \)

(iiib) \( \left| \frac{\lambda}{\lambda_1} \right| < 1 \)

Furthermore, (iiib) is sufficient for uniform convergence since

\[
\lim_{n \to \infty} \left| V_{n+1}(x) \right| = \left| \lambda \right|^n \left| \frac{\lambda}{\lambda_k} \right| \int_0^1 K(x, y) V_0(y) dy
\]

\[
\leq \max_{0 \leq x \leq 1} \left| V_0(x) \right| \left| \lambda \right|^n \left| \lambda \right| \int_0^1 \left| K(x, y) \right| dy
\]

by the Cauchy-Hadamard theorem, since \( \left| \lambda_1 \right| \) is the radius of convergence of the series (Neumann)

\[
U_n(x) = \sum_{k=0}^{\infty} \phi_k(x)
\]

generated by (18).
Hence for \(|\lambda/\lambda_1| < 1\), \(\lim_{n \to \infty} \left| V_{n+1}(x) \right| = 0\) which proves convergence.

If the kernel has no characteristic value other than \(\infty\), the above method converges for all finite \(\lambda\).

Summing (18) yields:

\[
    u_n(x) = g(x) + \lambda \sum_{k=1}^{n} \lambda^{k-1} k^{[n]}(x, y) g(y) dy
\]

with \(u_0(x) = g(x)\),

\[
    u_n(x) - g(x) - \lambda \sum_{k=0}^{n-1} k(x, y) u_n(y) dy = -\lambda \sum_{k=0}^{n-1} k(x, y) g(y) dy
\]

Hence, the error \(\varepsilon_n\) at the nth step is:

\[
    \varepsilon_n = \max_{0 \leq x, y \leq 1} |\lambda^{n+1} \sum_{k=0}^{n} K^{[n+1]}(x, y) g(y) dy|
\]

and if

\[
    M = \max_{0 \leq x, y \leq 1} |K(x, y)|, \quad L = \max_{0 \leq x \leq 1} |g(x)|
\]

then

\[
    \varepsilon_n \leq |\lambda|^{n+1} M^{n+1} L.
\]

2. A Variant on the Method of Successive Approximations

With the error at each stage in the process of 1., there will exist a truncation error endowed by the arithmetic approximation to the integral. This being so, the following theorem is of interest:

Theorem: Given any sequences of continuous functions \(\{g_n(x)\}\) and \(\{K_n(x, y)\}\) converging, over \(0 \leq x, y \leq 1\) uniformly to \(g(x)\) and \(K(x, y)\) respectively; then the sufficient condition that the sequence \([u_n(x)]\) defined by
\[ u_0(x) = q(x) \]

and

\[ u_{n+1}(x) = q_n(x) + \lambda \int_0^1 K_n(x, y) u_n(y) \, dy \]

converges uniformly to the solution of (1) if

\[ \max_{0 \leq x \leq 1} |K_n(x, y)| < 1 \]

for all but, say, a finite number of \( n \) with \( M \) satisfying

\[ |\lambda| M < 1 \]

Certainly if \( u_n(x) \to u(x) \) uniformly then \( u(x) \) is the solution of (1) for

\[ \lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} q_n(x) + \lim_{n \to \infty} \lambda \int_0^1 K_n(x, y) u_n(y) \, dy \]

The interchange of limits being permissible. The above yields by the hypotheses on \( \{q_n(x)\} \) and \( \{K_n(x, y)\} \) the solution of (1).

As a matter of fact the particular sequences \( q_n(x) \equiv q(x) \) and \( K_n(x, y) \equiv K(x, y) \) show that

\[ |\lambda| M < 1 \]

Now to prove uniform convergence of the \( \{u_n(x)\} \). By applying the iterative process on the \( n \)th and \( n+1 \)st step one obtains

\[ u_{n+1}(x) = q_{n+1}(x) + \lambda \int_0^1 K_{n+1}(x, y) u_n(y) \, dy \]

\[ u_{n+2}(x) = q_{n+2}(x) + \lambda \int_0^1 K_{n+2}(x, y) u_{n+1}(y) \, dy \]

Take \( N \) large enough so that for \( n > N \), one has

\[ \max_{0 \leq x \leq 1} |q_n(x) - q_{n+1}(x)| < \varepsilon \]

Let now

\[ v_n(x) = u_{n+1}(x) - u_n(x), \]

\[ |v_n(x)| \leq |q_{n+1}(x) - q_n(x)| + |\lambda| \int_0^1 K_{n+1}(x, y) u_n(y) - K_n(x, y) u_n(y) \, dy| \]

\[ \leq \varepsilon + M |\lambda| |v_{n-1}(x)| \]
Then
\[ |v_{n+1}(x)| \leq \frac{\varepsilon}{1 - |\lambda|} \frac{\varepsilon}{1 - |\lambda|} q^{n-1} \] (25)
where
\[ q = \max |v_{n-1}(x)| \]
Hence the series for \( v_n(x) \) converges uniformly, so that for \( u_n(x) \) converges uniformly and so must converge to the solution \( u(x) \).

3. The Method of Back Substitutions

This iteration procedure has been used for the inversion of matrices and has also been treated by Buckner for the Fredholm integral equations though his method of proof differs from that given here for convergence, and he makes no evaluation of the truncation error.

A sequence \( \{u_n(x)\} \) is constructed as follows:
\[ u_0(x) \]
is continuous on \( [0,1] \), but otherwise arbitrary.
\[ u_{n+1}(x) = u_n(x) + \alpha r_n(x) \] (26)
where
\[ r_n(x) = u_n(x) - g(x) - \lambda \int_0^1 K(x,y) u_n(y) dy \] (27)
and \( \alpha \) is a constant to be chosen so that the iteration procedure converges. \( r_n(x) \) is the residue at the \( n \)th iteration. Hence \( E_n = \max_{x\in[0,1]} |r_n(x)| \) is the error after \( n \) iterations.

It follows that
(i) Every \( u_n(x) \) is continuous
(ii) If, and only if, \( \lim_{n \to \infty} E_n = 0 \), does \( \{u_n(x)\} \)
converge to a limit function \( u(x) \) and satisfy equation (1). Furthermore, the convergence will be uniform and \( u(x) \) will be continuous.

(ii) \( \alpha = -1 \) gives the method of successive approximations and \( \alpha = 0 \) yields only the trivially convergent sequence \( u_0(x) = f(x) \). Hence, we eliminate at least these two values.

Consider, then, the sequence \( \{ \alpha r_n(x) \} \). From (26) and (27), it follows that

\[
\eta_{n+1}(x) = (1 + \alpha) \eta_n(x) - \alpha \sum_{\kappa=1}^{\eta} \kappa(x,\eta) \eta_n(\eta) d\eta
\]

and hence, defining

\[
\sum_{\kappa=0}^{\eta} \kappa(x,\eta) \eta_n(\eta) d\eta = \eta_n(x),
\]

\[
\eta_{n+1}(x) = \sum_{\kappa=0}^{\eta} (1 + \alpha)^{-\kappa} (-\alpha)^\kappa \lambda^\kappa (\eta_n(x)) \sum_{\kappa=1}^{\eta} \kappa(x,\eta) \eta_n(\eta) d\eta
\]

where \( \eta_n(\eta) \) is an arbitrary continuous function defined by (27).

Again, since convergence is to be independent of an initial function, let

\[
\eta_0(\eta) = \phi_i(\eta)
\]

where \( \phi_i(\eta) \) satisfies:

\[
\phi_i(x) = \lambda_i \sum_{\kappa=1}^{\eta} \kappa(x,\eta) \phi_i(\eta) d\eta
\]

(29) now yields:

\[
\eta_{n+1}(x) = (1 + \alpha - \alpha \lambda \lambda^{-1})^{n+1} \phi_i(x)
\]

From this, necessary conditions for convergence follow:

(1) \( \lambda \neq \lambda_j \) for any \( j = 1, 2, \ldots \)
And, since $\lambda = \infty$ is a limit point for the $\{\lambda_j\}$ or a singular value if the kernel has no $\lambda_j$ values, then from (31.2) follows a third condition:

\[(iii) \quad |1 + \alpha| < 1\]  

(31.3)

Ordering the $\{\lambda_j\}$ for (30) such that:

\[|\lambda_j| < |\lambda_{j+1}| \quad j=1,2,...\]

then (iii) and

\[(ii) \quad |1+\alpha - \alpha \lambda \lambda_j^i| < 1\]  

(31.4)

are sufficient for convergence. The proof of this we leave until a later section (p. 56).

4.

**The Method of Stepwise Continuation**

Thus far, the values of $\lambda$, for which solutions by iterations have been obtained, have been quite restricted. This method removes the restriction somewhat. Indeed, it all but solves the problem of obtaining an iterative form of the solution for any $\lambda$ not a characteristic value. Later, the restriction imposed here will be lifted completely and the problem will be completely solved.

Indeed, we prove the following:

**Theorem:** If for a given value of $\lambda, \lambda^*$ in equation (1), no characteristic value of (1), $\lambda_j$, satisfying $|\lambda_j| < |\lambda^*|$ lies on the line $\phi = \arg \lambda^*$ in the complex $\lambda$ plane; then, for an $E \gamma$, there exists an integer $p$ and a set of integers $n_1, n_2, ..., n_p$ such that the iteration by the
method of successive approximations of a set of $p$ Fredholm equations, the first $n_1$ times, the second $n_2$ times, ... and the $p$th $n_p$ times, yields a function $u_p(x)$ such that

$$
\max_{x \in X} |u_p(x) - f(x)| < \varepsilon
$$

Firstly, observe that if there exists an $R(x, y; \lambda)$ such that

$$
f(x) = g(x) + \lambda \int_0^1 R(x, y; \lambda) g(y) \, dy \quad (34)
$$

and

$$
R(x, y; \lambda) - K(x, y) = \lambda \int_0^1 K(x, t) R(t, y; \lambda) \, dt \quad (35)
$$

are satisfied, then (34) is the solution of (1). For, multiplying (1) by $R(x, t; \lambda)$ and integrating on $t$ over $[0, 1]$ yields:

$$
\int_0^1 f(t) R(x, t; \lambda) \, dt = \int_0^1 g(t) R(x, t; \lambda) \, dt \\
+ \lambda \int_0^1 K(t, y) \left( \int_0^1 R(x, y; \lambda) \, dt \right) f(y) \, dy. \quad (36)
$$

Inverting the integration:

$$
\int_0^1 f(t) \left[ R(x, t; \lambda) - \lambda \int_0^1 R(x, y; \lambda) K(y, t) \right] \, dt \\
= \int_0^1 R(x, t; \lambda) \\
= \frac{f(x) - g(x)}{\lambda} \quad \text{if (34) holds.}
$$

By (35) the left hand side is

$$
\int_0^1 f(t) K(x, t) \, dt.
$$

Hence, if (35) is satisfied, an $f(x)$ given by (34) satisfies (1).
Suppose that $R(x, t; \lambda)$ is a function satisfying
\[ f(x) = q(x) + \mu \int_0^1 R(x, t; \lambda) g(t) \, dt \tag{37} \]
and $f(x)$ is a solution of (1) for $\lambda = \mu$.

Then
\[ R(x, y; \lambda) - R(x, y; \mu) = (\lambda - \mu) \int_0^1 R(x, t; \lambda) R(t, y; \mu) \, dt \tag{38} \]
This result will be proved later. We assume it now.

The method of solution is as follows:

From the set of kernels $R(x, y; j \frac{\lambda^*}{p})$ are calculated and the following iterative scheme is employed:

\[
R_n(x, y; j \frac{\lambda^*}{p}) = R(x, y; (j-1) \frac{\lambda^*}{p}) + \left( \frac{\lambda}{p} \right) \int_0^1 R(x, t; j \frac{\lambda^*}{p}) R_n(t, y; j \frac{\lambda^*}{p}) \, dt \tag{39}
\]

The kernels $R(x, y; j \frac{\lambda^*}{p})$ are defined by
\[
R_0(x, y; j \frac{\lambda^*}{p}) = R(x, y; 0) = K(x, y) 	ag{40}
\]
by (38) and $R(x, y; j \frac{\lambda^*}{p})$ is defined by:

\[
R(x, y; j \frac{\lambda^*}{p}) - R(x, y; (j-1) \frac{\lambda^*}{p}) = \left( \frac{\lambda}{p} \right) \int_0^1 R(x, t; j \frac{\lambda^*}{p}) R(t, y; j \frac{\lambda^*}{p}) \, dt \tag{41}
\]
for $1 \leq j \leq p$.
The function $U_p(x)$ defined by

$$U_p(x) = g(x) + \lambda^x \int_0^1 R(x, t; \lambda^x) g(y) \, dy$$

will then be such that

$$\max_{0 \leq x \leq 1} |U_p(x) - f(x)| < \varepsilon$$

is satisfied.

To prove this, we need the following lemmas which are proved later (p. 52):

**Lemma 1** The resolvent $R(x, y; j \lambda^x)$ commutes with $R(x, y; i \lambda^x)$ for $j = 1, 2, \ldots \ldots$ wherever they both exist.

**Lemma 2** If $R(x, y; j \lambda^x)$ has as resolvent $R(x, y; i \lambda^x)$, then the expansion of the $k$th iterate of $R(x, y; j \lambda^x)$ takes the form

$$R^k(x, y; j \lambda^x) = \frac{1}{k!} \sum_{n=0}^{\infty} R^{(k+n)}(x, y; j \lambda^x) \frac{(j \lambda^x)^n}{n!}$$

and all iterates exist in the same domains as do their respective resolvents.

**Lemma 3** If (1) has a solution given by (34) for a particular value of $\lambda, \lambda'$, it has solutions for all $\lambda$ in a sufficiently small neighborhood (domain) of $\lambda'$.

We now prove the theorem:

Since there exists a solution in a neighborhood of $\lambda = 0$, the Neumann expansion yields:

$$R(x, y; j \lambda^x) = \sum_{n=0}^{\infty} R^{(n)}(x, y; 0)(j \lambda^x)^n$$

---

*We have incorporated the term of $\left(\frac{\lambda^x}{p}\right)^n$ in the sums.*
a uniformly convergent series in $\lambda$, for a suitably chosen integer $p$. Each term is dominated by:

$$\max_{0 \leq x, y \leq 1} \left| R_{C_k^j}(x, y; j \frac{A^*}{p}) (\frac{A^*}{p})^{n_1} \right| \leq \frac{C}{|\lambda_1|^{n_1}} \left| \frac{A^*}{p} \right|^{n_1}, \quad j = 1, 2, \ldots$$

where $C$ is the maximum value attained by $R(x, y; j \frac{A^*}{p})$ in the domain consisting of the intersection of the $p$ circles of radius $|\frac{A^*}{p}|$ containing the origin, $\lambda^*$ and the straight line connecting them. We need estimates of the error induced in $R_{C_k^j}(x, y; j \frac{A^*}{p})$ by the finite number of iterations employed in approximating the $R(x, y; j \frac{1}{p} \lambda^*)$ for $\lambda = 1, 2, \ldots, j$. We use the relation:

$$R_{C_k^j}(x, y; j \frac{A^*}{p}) = \frac{1}{k!} \sum_{n_1=0}^{\infty} \frac{(k + n_1)!}{n_1!} R^{(k+n_1)}_{C_k^j}(x, y; j \frac{1}{p} \lambda^*) (\frac{A^*}{p})^{n_1}$$

Consider the above equation for $j = 1$. Then each term on the right is dominated by:

$$\frac{(k + n_1)!}{n_1!} \frac{C}{|\lambda_1|^{n_1}} \left| \frac{A^*}{p} \right|^{n_1} \leq (k + n_1)^k \frac{C}{|\lambda_1|^{n_1}} \left| \frac{A^*}{p} \right|^{n_1}$$

Now pick $p$, $n_1$, and $k$ such that for $n_1 > m_1$, $k \leq k_0$

$$\left| \frac{A^*}{p \lambda_1} \right| < \frac{1}{2}, \quad (k_0 + n_1)^{k_0} \left| \frac{A^*}{p \lambda_1} \right|^{n_1} < \left( \frac{1}{2} \right)^{n_1}$$

That this can be done we show later.
Hence:
\[
R_{\nu_p}^{\text{CK}}(x, y; \lambda^*) = \frac{1}{k!} \sum_{n_1=0}^{n_1} \frac{R_{\nu_p}^{\text{CK}}(x, y; \nu^*)(k+n_1)!}{n_1!} \left( \frac{\lambda^*}{p} \right)^{n_1} + \varepsilon_{i, k}
\]

where
\[
|\varepsilon_{i, k}| \leq \frac{1}{k!} \frac{C}{|\lambda|^k} \frac{1}{2^{n_1}} k \leq k_0
\]

Accordingly:
\[
R_{\nu_p}^{\text{CK}}(x, y; \frac{\lambda^*}{p}) = \frac{1}{k!} \sum_{n_1=0}^{n_1} \frac{R_{\nu_p}^{\text{CK}}(x, y; \frac{\lambda^*}{p})(k+n_1)!}{n_1!} \left( \frac{\lambda^*}{p} \right)^{n_1} + \varepsilon_{i, k}'
\]

where
\[
|\varepsilon_{i, k}'| \leq \frac{1}{k!} \frac{C}{|\lambda|^k} \frac{1}{2^{n_1}} k \leq k_0, n_2 \leq n_1
\]

Combining:
\[
R_{\nu_p}^{\text{CK}}(x, y; \frac{\lambda^*}{p}) = \frac{1}{k!} \sum_{n_1=0}^{n_1} \frac{(k+n_1)!}{n_1! k!} \sum_{n_2=0}^{n_2} \frac{R_{\nu_p}^{\text{CK}}(x, y; \nu^*)(k+n_2)!}{n_2!} \left( \frac{\lambda^*}{p} \right)^{n_2} + \varepsilon_{i, k}''
\]

where
\[
|\varepsilon_{i, k}''| \leq \frac{1}{k!} \sum_{n_1=0}^{n_1} \frac{(k+n_1)!}{n_1!} \left( \frac{\lambda^*}{p} \right)^{n_1} |\varepsilon_{i, k+n_1}| \leq \frac{1}{k!} \sum_{n_2=0}^{n_2} \frac{1}{n_2!} \left( \frac{\lambda^*}{p} \right)^{n_2} \frac{C}{|\lambda|^k} \leq \frac{1}{k!} \frac{C}{|\lambda|^k} e^{\lambda^*}
\]

where \( p \) is so chosen such that \( \frac{\lambda^*}{|\lambda|} < \frac{1}{3} \).

Hence
\[
|\varepsilon_{i, k}| = |\varepsilon_{i, k}' + \varepsilon_{i, k}''| \leq \frac{C}{k! |\lambda|^k} \frac{2}{2^{n_1}}
\]

Since \( n_2 \leq n_1 \).

Now again at the \( p \)th stage:
\[
R_{\nu_p}^{\text{CK}}(x, y; \lambda^*) = \frac{1}{k!} \sum_{n_1=0}^{n_1} \frac{1}{n_1!} (k+n_1)! R_{\nu_p}^{\text{CK}(k+n_1)}(x, y; \frac{\lambda^*}{p}) \left( \frac{\lambda^*}{p} \right)^{n_1} + \varepsilon_{i, k}'
\]

Since \( n_2 \leq n_1 \).
and so:

\[ R_{np}^{\varepsilon_{k, p}}(x, y; \lambda^*) = \frac{1}{k!} \sum_{n_p=0}^{n_k} \frac{1}{n_p!} \left( \frac{\lambda^*}{\rho} \right)^{n_p} \sum_{n_{p-1}=0}^{n_k-1} \frac{1}{n_{p-1}!} \left( \frac{\lambda^*}{\rho} \right)^{n_{p-1}} \ldots 
\]

\[ \ldots \sum_{n_1=0}^{n_k} \frac{1}{k!} (k+n_1)! R^{\varepsilon_{k+n_1+\ldots+n_p}}(x, y; 0) \left( \frac{\lambda^*}{\rho} \right)^{n_k} + \varepsilon_{k, p} \]

where

\[ |\varepsilon_{k, p}| \leq \frac{1}{k!} \left| \frac{c}{|\lambda|} \right| \frac{1}{2^{n_k}} e^{y_j} \]

and

\[ |\varepsilon_{k, p}| = |\varepsilon_{k, p}^\prime + \varepsilon_{k, p}^\prime\prime| \leq \frac{1}{k!} \left| \frac{2c}{|\lambda|} \right| \frac{1}{2^{n_k}} \]

We now use these inequalities. At the first iteration:

\[ R_{n_1}(x, y; \frac{\lambda^*}{\rho}) = \sum_{n_1=0}^{n_k} R^{\varepsilon_{n_1}}(x, y; 0) \left( \frac{\lambda^*}{\rho} \right)^{n_1} + \varepsilon_{1, 1} \]

At the second iteration:

\[ R_{n_1}(x, y; \frac{\lambda^*}{\rho}) = \sum_{n_2=0}^{n_k} R^{\varepsilon_{n_2}}(x, y; \frac{\lambda^*}{\rho}) \left( \frac{\lambda^*}{\rho} \right)^{n_2} + \varepsilon_{1, 2} \]

Since the iterates are approximate, the error induced is obtained from:

\[ R_{n_1}(x, y; \frac{\lambda^*}{\rho}) = \sum_{n_2=0}^{n_k} \frac{1}{n_1!} \left[ \sum_{n_3=0}^{n_2} R^{\varepsilon_{n_3}}(x, y; 0) \left( \frac{\lambda^*}{\rho} \right)^{n_3} \frac{(n_1+n_3)!}{n_1!} \right] \left( \frac{\lambda^*}{\rho} \right)^{n_2} \]

\[ + \sum_{n_2=0}^{n_k} \frac{1}{n_1!} \left( \frac{\lambda^*}{\rho} \right)^{n_2} \varepsilon_{n_2, 1} \]
So the total error is less than:

\[ |E_{i,\nu}^1| + \frac{C}{\lambda_{n_1}} e^{\frac{1}{\lambda_{n_1}}} \leq \frac{2C}{2^{n_1}} \]

Similarly, after \((p-1)\) steps:

\[ R_{p}^{\lambda}(x, y; \lambda) = \sum_{n_1=0}^{\infty} \frac{1}{n_1!} (\frac{\lambda}{p})^{n_1} \sum_{n_2=0}^{\infty} \frac{1}{n_2!} (\frac{\lambda}{p})^{n_2-1} R_{n_1+n_2+...+n_p}^{\lambda}(x, y; \lambda) \]

\[ + \varepsilon \]

and

\[ |\varepsilon| < \frac{2C}{2^{n_p}} \]

Hence write:

\[ R(x, y; \lambda) = R_{p}^{\lambda}(x, y; \lambda) + \varepsilon \]

and if

\[ U_p(x) = g(x) + \lambda^* \int_0^1 R_{p}^{\lambda}(x, y; \lambda^*) g(y) dy \]

and

\[ f(x) = g(x) + \lambda^* \int_0^1 R'(x, y; \lambda^*) g(y) dy \]

then \(U_p(x)\) and \(f(x)\) differ at most by:

\[ \max_{0 \leq x \leq 1} |U_p(x) - f(x)| < |\lambda^*| \frac{2C}{2^{n_1}} \max_{0 \leq x \leq 1} |g(x)| \]

where the primes are introduced because the resolvents were calculated with the \(\lambda^*\) already absorbed in the resolvent.

We have now to show that, with \(p\) such that

\[ |\frac{\lambda^*}{p} \frac{1}{n_1}| < \frac{1}{3} \]

we can find a sequence \(\{ n_j \}\) \(j = 1, 2, ..., p\) and a \(k_0\) such
that the inequalities are satisfied. The basic inequality is
\[ k_o \log (k_o + n_1) < n_1 \left[ \log^3 - \log^2 \right] \]
and if true for \( k_o \) is certainly true for \( k < k_o \). We pick a descending sequence based on \( n_1 \). For if \( n_1 = p^{2p} \) and \( k = 2p^{2p} - 2 \), the inequality is satisfied for \( p > 10 \).

And then if the values are assigned as follows:
\[ n_j = p^{1p-1(j-1)} \]
\[ j = 1, 2, \ldots \]
and
\[ k_j = 2p^{1p-1(j+1)} \]
the inequality is maintained.

Admittedly, these inequalities are poor for numerical analysis; however, they can be improved. One means of doing so is by improving the basic inequality for determining the \( \{ n_j \} \). Another is by improving the iterative method.

This we now proceed to do by introducing Newton's method which allows improvement in both lessening the number of steps in the continuation process and in decreasing the error for a given number of iterations.

5. Newton's Method

Suppose we consider the equation \( F(y) = 0 \) which we transform to \( y = f(y) \). We seek a \( y = Y \) \( \Rightarrow \) \( F(Y) = 0 \) and \( Y = f(Y) \). Suppose an iterative scheme constructs a sequence of \( \{ y_n \} \) satisfying:
\[ y_{n+1} = \delta(y_n) \]
and let  

\[ \varepsilon_n = Y_n - Y \]

We seek obviously a criterion by which \( \varepsilon_n \), in some norm, converges to zero. But we also seek information about the rate by which \( \varepsilon_n \to 0 \) as a function of \( n \). Now form  

\[ y_{n+1} = f(Y + \varepsilon_n); \]

a Taylor expansion about \( Y \) yields  

\[ y_{n+1} = f(Y) + f'(Y) \varepsilon_n + \frac{f''(Y) \varepsilon_n^2}{2!} + \ldots \]

and  

\[ \varepsilon_{n+1} = f'(Y) \varepsilon_n + \frac{f''(Y) \varepsilon_n^2}{2!} + \ldots \]

assuming the derivatives exist. Now if the iterative scheme converges, for large enough \( n \),  

\[ \varepsilon_{n+1} \sim f'(Y) \varepsilon_n \sim f''(Y) \varepsilon_n^2 \]

providing \( f'(Y) \neq 0 \). Such an iterative process whose error at the \( n \)th stage is proportional to the error at the \( n-1 \)st stage is called a process of the first order.  

Suppose, however, that \( f(Y) = 0 \). Then:  

\[ \varepsilon_{n+1} \sim \frac{f''(Y)}{2} \varepsilon_n^2 \]

for \( n \) large enough, and the process convergent. Hence  

\[ \frac{f''(Y)}{2} \varepsilon_{n+1} \sim \left( \frac{f''(Y)}{2} \varepsilon_n \right)^2 \sim \left( \frac{f''(Y)}{2} \varepsilon_0 \right)^2 \]

Such a process where the error at the \( n \)th stage is roughly the square of that at the \( (n-1) \)th is a second order process.
Hence, a $k^{th}$ order process is one in which the error at the $n^{th}$ step is the $k^{th}$ power of that at the $(n-1)^{th}$ step.

Hartree proves that having an $n^{th}$ order process and a $(n-1)^{th}$ order process for any equation implies the non-uniqueness of the $(n-1)^{th}$ order process. Conversely, given two $(n-1)^{th}$ order processes whose $n^{th}$ derivatives, $f^n(y)$, are different, then there exists an $n^{th}$ order iterative process for solving the equation.

Rather than introduce the concept of Taylor's expansion in operator equations and defining differentiation over certain combinations of operators, we use the definition of order of these processes as given by the definition. Now for $\mathcal{E}_n$ to converge to zero in norm, it is necessary and sufficient that the first non-zero term of the expansion be less than one in norm.

Thus far, the processes considered are all of the first order and are relatively inefficient on that basis. However, these first order processes suffice to provide solutions for almost all $\lambda$ which are not characteristic values of (1).

* The $\lambda$ values for which a solution has not been exhibited are those which originate in the $\lambda$ plane such that a characteristic value is on this ray between $\lambda$ and the origin; i.e., for which a $\lambda_j$ exists and satisfies

$$\arg \lambda_j = \arg \lambda$$

$$|\lambda_j| \leq |\lambda|$$

We call these the exceptional set.
We now exhibit iterative procedures of arbitrary order which converge in such regions as to allow us to apply the methods of 14. to obtain solution for large $\lambda$ and yet use higher order processes to attain this desired result. This, then, effectively solves the iterative problem for the Fredholm integral equation of the second kind, for we exhibit iterative procedures of arbitrary order solving the equation for all $\lambda$ except those lying in the exceptional set.

We consider Newton's formula for obtaining the reciprocal of a real number $x$:

$$y_{n+1} = y_n (1 - y_n x)$$

By choosing $y_0$ properly, this method converges $\{y_n\}$ to $\frac{1}{x}$. Multiplying by $x$ and subtracting the above equation from 1 yields:

$$1 - x y_{n+1} = 1 - 2 y_n x + y_n^2 x^2$$

$$1 - x y_{n+1} = (1 - x y_n)^2$$

Now, letting

$$E_n = (1 - x y_n) = (1 - x y_0)^2$$

the above equation shows Newton's method to be of the second order. Hence, pick $y_0$ so that $xy_0$ satisfies $|1 - xy_0| < 1$ and convergence is attained.

Consider now the equations

$$f(x) = g(x) + \lambda \int_0^1 k(x, y) f(y) dy \quad (1)$$

and

$$f(x) = g(x) + \lambda \int_0^1 k(x, y) g(y) dy \quad (2)$$
Substituting (34) into (1) gives an identity and in this sense the two equations are reciprocal of one another. Thus we desire to obtain an \( R_n(x, \eta; \lambda) \) "reciprocal" to \( K(x, y) \) by Newton's method.

Hence, we generate a sequence \( R_n(x, \eta; \lambda) \) by the algorithm:

\[
R_0(x, \eta; \lambda) = 1 - \lambda K(x, \eta) \tag{62}
\]

and

\[
R_n(x, \eta; \lambda) = R_{n-1}(x, \eta; \lambda) \left[ 2 - R_{n-1}(x, \eta; \lambda) (1 - \lambda K(x, \eta)) \right] \tag{61}
\]

where, whenever multiplication is to take place between any \( R_n(x, \eta; \lambda) \) and an \( R_n(x, \eta; \lambda) \) for any \( n \), or with \( K(x, y) \) integration over the second variable of the first factor and the first of the second is always implied; for example:

\[
R_n(x, \eta; \lambda) \cdot \lambda K(x, \eta) \equiv \lambda \int_0^1 R_n(x, t; \lambda) K(t, \eta) \, dt \tag{63}
\]

With this convention in mind and for the sake of simplicity of notation, we replace (62) and (61) by:

\[
R_0 = 1 - \lambda K
\]

and

\[
R_n = R_{n-1} \left[ 2 - R_{n-1} (1 - \lambda K) \right]
\]

respectively.

The following may be said of commutativity:

1. If \( \lim_{n \to \infty} \max_{0 \leq x, \eta \leq 1} | R_n - R | = 0 \), then every \( R_n(x, \eta; \lambda) \) commutes with \( K(x, y) \) and with \( R(x, \eta; \lambda) \) in the sense of (63).
Firstly, \( R_0(x, y; \lambda) \) commutes with \( K(x, y) \) and commutes with \( R(x, y; \lambda) \).

Assume \( R_n(x, y; \lambda) \) commutes with \( K(x, y) \) and \( R(x, y; \lambda) \). Then:

\[
R_{n+1}(1-\lambda k) = R_n \left(2 - (1-\lambda k) R_n \right)(1-\lambda k)
\]

\[
= R_n \left[2(1-\lambda k) - (1-\lambda k)(1-\lambda k) R_n \right]
\]

\[
= R_n (1-\lambda k) \left[2 - (1-\lambda k) R_n \right]
\]

\[
= (1-\lambda k) R_n \left[2 - R_n(1-\lambda k) \right]
\]

\[
= (1-\lambda k) R_{n+1}.
\]

The same proof holds for \( R \). Hence, the assertion is true by induction.

Consider now the equation:

\[
1 - (1-\lambda k) R_{n+1} = 1 - (1-\lambda k) R_n \left[2 - (1-\lambda k) R_n \right]
\]

\[
= (1 - (1-\lambda k) R_n)^2 \quad \text{(64.1)}
\]

\[
= (1 - (1-\lambda k) R_n)^2. \quad \text{(64.2)}
\]

The equations (64.1) and (64.2) represent the error after \((n+1)\) iterations. Now, consider the sequence of functions \( s_n = R_n g \) (multiplication here implying an integration).

Multiplying both sides of 64.2 on the right by \( g(y) \), the equation 64.2 becomes:

\* The proof of this is delayed until the second part of this thesis. See p. 52.
\[ [ g(x) - (1 - \lambda k) f_{n+1} ] \]

\[ = [ 1 - (1 - \lambda k) R_0 ]^n g(x) . \]

If the right hand side, as \( n \to \infty \) converges to zero in the sense:

\[ \lim_{n \to \infty} f_n = \lim_{n \to \infty} \max_{0 \leq x \leq 1} | [1 - (1 - \lambda k) R_0 ]^n g(\gamma) | = 0 \]

then \( f_n(x) \) converges to the solution of the equation (1) and is given by:

\[ f(x) = \lim_{n \to \infty} \int_0^1 R_n(x, \eta; \lambda) g(\eta) d\eta \]

\[ = \int_0^1 R(x, \eta; \lambda) g(\eta) d\eta . \]

Since the limit will be uniform. Now, remembering that \( 1 - \lambda k \) is \( R^{(o)} \) and letting

\[ M = \max_{0 \leq x \leq 1} | k(x, \eta) | \]

and

\[ G = \max_{0 \leq x \leq 1} | g(x) | \]

then by expanding out (66) taking absolute values if:

\[ \max_{0 \leq x \leq 1} | 2 \lambda k g - \lambda^2 k^2 g | < 1 \]

and

\[ \max_{0 \leq x \leq 1} | 2 \lambda k \cdot 1 - \lambda^2 k^2 g | < 1 \]

convergence occurs, and convergence of the \( R^{(n)} \) s to \( R \) is uniform. Hence, we have exhibited a solution occurring for at least the values of \( \lambda \) given by (69). Furthermore, the process is a second order process. This method may be applied advantageously to the procedure outlined in 4. This is so
for two reasons. Along the ray, the domains of convergence given by (69) have a maximum distance. This means that the number of jumps, $p$, can be effectively reduced. Furthermore, the error at each jump will be attained with much fewer iterations precisely because of the second order nature of the process.

It is a trivial matter to generate, using Newton's method, procedures of arbitrary order having the same region of convergence. For consider the process given by:

$$R_{n+1} = R_n \left[ \sum_{j=1}^{K} \binom{K}{j} (-1)^{j+1} \left[ R_n (1 - \lambda \kappa) \right]^{j-1} \right]$$

where $\left[ R_n (1 - \lambda \kappa) \right]^0 = 1$

order, since

$$1 - (1 - \lambda \kappa) R_n = \left[ 1 - (1 - \lambda \kappa) R_0 \right]^n.$$

However, this is merely Newton's method taken $n$ steps at a time. For $k = 2n$; for $k = 2n + 1$, it is such a process plus a first order one. Indeed, these "Newton" methods occupy a central position in such iterative procedures. For, every process of the form

$$y_{n+1} = f(y_n) = y_n g(y_n, x)$$

where $y_n g(y_n, x)$ is a polynomial in $y$, that converges to $\frac{1}{2}$, is a Newton's method of order equivalent to the degree of $yg(y)$. Furthermore, for such polynomials, no power of $y$ may be omitted because of the role the successive derivatives play in determining the order. For solving integral equations, we
are limited to such processes as these since we have no, as yet, infinitesimal or divisional operations defined on the operators involved.

6. Summary of Chapter I

Hence to conclude this chapter, we summarize by pointing out that the generation of procedures of arbitrary order and over regions which include all \( \mathbb{R} \) but those lying in the exceptional set has solved the problem of finding iterative procedures for generating solutions of the Fredholm integral equation.

With the exception of the method of 5., these algorithms bear a definite relation to the problem of obtaining the continuation of an analytic function of a complex variable and of summing divergent series. This relationship will be developed in Chapter II, where the problem will achieve its ultimate solution in terms of the above-mentioned concepts.
In this chapter, we systematize the results of the preceding one and develop that theory enabling us to construct a solution of the integral equation (1) by analytic continuation and summability methods. The relationship of the previously-described methods with the summability theory of divergent series is more than accidental, as will now be shown. This chapter is devoted to detailing a general theory of "analytic" operators; the next to a general summability theory and the relation of those methods previously discussed with the results of this chapter.

We begin by introducing the concept of a complete \( G \) normed ring. A set of elements \( G \) is called a normed ring if the following conditions are satisfied:

1. If \( K_i \) and \( K_j \in G \), then \( K_i + K_j \in G \) and \( K_i \cdot K_j \in G \).
2. If \( z \) is a complex number, then \( K \in G \) implies \( zK \in G \) and \( zK = Kz \).
3. \( G \) is an Abelian group with respect to addition, the zero element of \( G \) being the identity of the group.
4. Multiplication is associative and distributive: i.e., \( K, L, M \in G \) implies

\[
(KL)M = K(LM)
\]

and

\[
K(L + M) = KL + KM; (L + M)K = LK + MK
\]

and for \( z, w, v \in \mathbb{C} \) (the set of complex numbers):

\[
(z + w)K = zK + wK = K(z + w)
\]
and \((zw)K = z(wK)\)

and \(z(K+L) = zK + zL\); \(1 \cdot K = K, 1 = K\)

(v) To every \(K \in G\) corresponds a non-negative real number \(|K|\) called the norm of \(K\) satisfying:

\[|K| > 0 \quad K \neq \emptyset\]
\[|K| = 0 \quad K = \emptyset\]

and the inequalities \((K, L \in G, z \in Z)\):

\[|K+L| \leq |K| + |L|; \quad |KL| \leq |K||L|; \quad |zK| = |z||K|\]

A distance is defined on \(G\). For \(K, L \in G\), the distance between them is \(|K - L|\) and it satisfies:

\[|K - L| = |L - K|, \quad |K - L| = 0 \quad K = L\]

and the triangle inequality:

\[|K - L| + |L - M| > |K - M|\]

Convergence is always in norm; and a sequence of elements \(K_n \in G\) converges to an element \(K\) if and only if \(\lim_{n \to \infty} |K_n - K| = 0\).

We write this as \(\lim_{n \to \infty} K_n = K\).

The normed ring is said to be complete if the satisfying of the Cauchy convergence criterion, for an arbitrary \(\varepsilon > 0\) and an \(n, n' > N(\varepsilon), |K_{n'} - K_n| < \varepsilon\) implies the existence of a \(K \in G\) towards which the sequence \(K_n\) converges.

We now consider a complete normed ring \(G\) in which the elements \(K \in G\) are functions of a complex variable \(z \in Z\), the set of all complex numbers.

If \(K(z) \in G\) and if for all \(z \in U \subset Z\), \(K(z)\) is differentiable in \(Z\), then \(K(Z)\) is said to be analytic in \(U\); i.e.,
A function \( K(z) \in \mathbb{G} \) is analytic at \( z_0 \) if it is analytic in some open set containing \( z_0 \).

It follows from the definition of the normed ring and of analyticity that the sum and products of two analytic functions is analytic in that region \( U \subset \mathbb{Z} \) which is the intersection of the regions of analyticity of each function. Further, the product of a natural complex valued analytic function \( f(z) \) with an analytic function \( K(z) \in \mathbb{G} \) is an analytic function \( \in \mathbb{G} \).

2.

We now define the integral of \( K(z) \in \mathbb{G} \) over some region \( U \subset \mathbb{Z} \).

Let \( K(z) \in \mathbb{G} \) be a continuous function of \( z \in U \). Let \( z_0 \) and \( z \) be any two points in \( U \) connected by a rectifiable Jordan curve, \( C \), (a "path") in \( U \). Then if

\[
\Delta Z_n = Z_n - Z_{n-1}
\]

and \( \delta_n \) is an arbitrary point on \( \Delta Z_n \) which lies in the path, the integral defined by

\[
\int_{z_0}^{z} K(z) \, dz = \lim_{N \to \infty} \sum_{n=1}^{N} K(\delta_n) \Delta Z_n
\]
exists independent of the division points \( \{ z_n \} \) provided that

\[
\lim_{n \to \infty} \max |\Delta z_n| \to 0 \quad \text{and is in } G_1.
\]

The proof is exactly that outlined in Knopp\(^*\); i.e., it is possible to select two subdivisions \( \{ \Delta z_n \}, \{ \Delta z'_n \} \) of the path \((z_0, \bar z)\) such that for an arbitrary \( \epsilon > 0 \), there exists a \( S(\epsilon) \) and an \( N(\epsilon) \) and \( M(\epsilon) \) such that for

\[
\Delta = \max_{1 \leq n \leq N} |\Delta z_n| < S(\epsilon)
\]

and

\[
\Delta' = \max_{1 \leq m \leq M} |\Delta z'_m| < S(\epsilon);
\]

it follows that

\[
\left| \sum_{m=1}^{M} k(x'_m) \Delta z'_m - \sum_{n=1}^{N} k(x'_n) \Delta z_n \right| < \epsilon
\]

Every such sum is in \( G_1 \) and since \( G_1 \) is complete, the above defines a limit in \( G_1 \) called the integral. This integral has the properties:

\[\text{(i)} \quad \int_{z_0}^{\bar z} k(z) \, dz = -\int_{\bar z}^{z_0} k(z) \, dz \]

\[\text{(ii)} \quad r \int_{z_0}^{\bar z} k(z) \, dz = \int_{z_0}^{\bar z} r k(z) \, dz \]

and \( \text{(iii)} \)

\[
\left| \int_{z_0}^{\bar z} k(z) \, dz \right| \leq M L
\]

where \( M = \max_{z \in C} |k(z)| \) and \( L \) is the length of the path.

Cauchy's integral theorem follows:

If \( k(z) \) is analytic in a simply connected region \( U \subset \mathbb{C} \) for which \( C \) is an arbitrary closed path entirely within \( U \), then

\[
\int_{C} k(z) \, dz = \phi \quad \epsilon \quad G_1
\]

\[\text{---39---}\]

The proof is precisely that of Knopp*. The formation of approximating polygonal arcs leads to sequences of integrals converging in norm to $\phi \in \mathcal{G}$. The theorems 1 and 2, pp. 56 and 57* follow at once.

Furthermore, it follows that if $K(z)$ is continuous in $U$, $z_0$ arbitrary and $z \in U$ that

$$H(z) = \int_{z_0}^{z} K(w)dw$$

for any path in $U$ is in $\mathcal{G}$ and, furthermore, that $H(z)$ is an analytic function of $z$ for $z \in U$ and its derivation is $K(z)$.

Cauchy's integral representation is vital to the results of the following chapter. Briefly, if $K(z) \in \mathcal{G}$ is analytic for $z \in U \subset \mathbb{C}$, then

$$K(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{K(z)}{z - w} dw \in \mathcal{G}$$

holds for every simple, closed, positively oriented path, $\gamma$, and every $z$ in its interior if $\gamma$ and its interior lie within $U$.

Proof: \[
\frac{1}{2\pi i} \int_{\gamma} \frac{K(z)}{z - w} dw = K(z) \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{z - w} = K(z) \in \mathcal{G} \]

by the distributive and associative laws of the ring $\mathcal{G}$. Now

$$\frac{1}{2\pi i} \int_{\gamma} \frac{K(z)}{z - w} dw = K(z) \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{z - w} = K(z) \in \mathcal{G}.$$ 

By the Cauchy integral theorem, $\gamma$ may be deformed to a small path, say, a circle of radius $r_0$ about $z$, so chosen that

$$\max_{\gamma} |K(z) - K(z)| < \varepsilon$$

This follows from the analyticity of $K(z)$ for $z \in U$. Then:

$$\left| \frac{1}{2\pi i} \int_{\gamma} \frac{K(z) - K(z)}{z - w} dw \right| \leq \frac{1}{2\pi} \frac{\varepsilon}{r_0} 2\pi r_0 = \varepsilon$$

* Loc. cit. p.39
Hence
\[ \left| \frac{1}{2\pi i} \int_c \frac{K(\xi)}{\xi-z} d\xi - K(z) \right| < \varepsilon \]

\( \varepsilon \), being arbitrary, this proves the theorem. From this it follows that \( K(z) \) analytic for \( z \in \Omega \) implies that \( K(z) \) has derivations of arbitrary order which are analytic for \( z \in \Omega \), and belong to \( G \); in addition they are given by:
\[
\frac{d^n K(z)}{dz^n} = \frac{n!}{2\pi i} \int_c \frac{K(\xi)}{(\xi-z)^{n+1}} d\xi
\]
This is proved by showing that the element of \( G \) defined by
\[
\frac{1}{2\pi i} \int_c \frac{K(\xi)}{\xi-z} d\xi
\]
is analytic and differentiable any number of times for \( z \in \Omega \). But it is just \( K(z) \in G \) by the Cauchy integral theorem.

### 3. Infinite Series of Analytic Function in \( G \)

If \( \{ K_n(z) \} \in G \) and \( z \in \Omega \subset \mathbb{C} \), then the set of all \( z \) for which \( \sum K_n(z) \) converges is called the domain of convergence of the series. If the series converges, it converges to an element in \( G \). Of particular interest are power series of the form
\[
\sum K_n (z-z_0)^n
\]
where the \( K_n \) are elements of \( G \). For such series, the domain of convergence is a circle about \( z_0 \) as center. The sequence \( \sqrt[n]{|K_n|} \) is of importance. The series above is absolutely convergent for
\[
|z-z_0| < \lim_{n \to \infty} \sqrt[n]{|K_n|}
\]
and divergent for
\[
|z-z_0| > \lim_{n \to \infty} \sqrt[n]{|K_n|}
\]
The definition of uniform convergence in $z \in U$ for $\sum K_n(z)$ is the same as that for ordinary functions of a complex variable; i.e., if given $\epsilon > 0$, $\exists N(\epsilon)$ such that for any $z \in U$, any $n, \forall n(\epsilon), p > 1$ \[ \sum_{n=p}^{\infty} |K_n(z)| < \epsilon \] then $\sum K_n(z)$ converges uniformly in $U$.

Every power series converges uniformly in any circle concentric to the circle of convergence and of radius strictly less than the radius of convergence.

For $\{K_n(z)\} \in G$ and analytic for $z \in U$, we get the theorems:

(i) $\sum K_n(z)$ uniformly convergent in some $U' \subset U$ implies the sum $H(z)$ is analytic in $U'$ and is in $G$.

(ii) Integration and summation may be interchanged

$$\sum \int_c K_n(z) \, dz = \int_c H(z) \, dz$$

for every $c \subset U'$.

(iii) $\sum K_n(z)$ may be differentiated any number of times and each such differentiation yields a uniformly convergent series and an analytic function $\in G$ which is precisely the corresponding derivative of $H(z)$.

With the circle of convergence as $U$, it follows that every power series $\sum K_n z^n$ in $G$ for $z \in U'$ is an analytic function in $z \in U'$. The derived series have the same radius of convergence as does the given series and

$$H^{(k)}(z) = k! \sum \binom{n+k}{k} K_{n+k} z^n \in G$$
The Cauchy integral formula yields
\[ H^{(n)}(0) = k! \, K_n \]
and
\[ K_n = \frac{1}{2\pi i} \oint_C \frac{H(z)}{z^{n+1}} \, dz \]
and
\[ C \text{ being the circle of radius } |b|=r \]
yields the Cauchy inequality
\[ |K_n| \leq \frac{1}{2\pi} \oint_C \max_{|z|=r} \frac{|H(z)|}{r^{n+1}} = \max_{|z|=r} \frac{|H(z)|}{r^n} \]

The uniqueness theorem for analytic functions in a domain now falls out. For, let \( H(w) \) be analytic in \( U \)
\[ \frac{H(w)}{w-z} = \sum_{n=0}^{\infty} \frac{H(w)}{w^{n+1}} \, z^n \]
if \( w \) lies on a circle \( C \subset U \) and \( Z \) is interior to \( C \), the series on the right being uniformly convergent \( w.r.t. \) along \( C \) since
\[ \left| \frac{z}{w} \right| < 1 \text{ for any } w \in C. \]
Integration on \( C \) may be carried out term by term and
\[ \frac{1}{2\pi i} \oint_C \frac{H(w) \, dw}{w-z} = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_C \frac{H(w)}{w^{n+1}} \, z^n \, dw \]
Hence
\[ H(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \, H^{(n)}(0) \, z^n = \sum K_n(z) \]
is analytic in \( z \in U' \) and in \( G \). Hence every analytic function is representable by only one power series in a given domain \( U' \).

Equality of two elements of \( G \), analytic in a common domain \( U' \), over a set having one limit point in \( U' \), implies equality everywhere in \( U' \).

Let \( K(z) \in G \), analytic, and single valued for \( z \in U \) where
\[ U \text{ is a concentric annular ring about some } Z_0 \text{ as center}; \text{i.e., } K(z) \text{ is} \]
analytic for $|z-z_0| < r_1$. If now $U'$ is an annular ring about $z_0$ concentric to $U$, from the Cauchy integral formula follows the uniformly convergent "Laurent expansion":

$$K(z) = \sum_{n=-\infty}^{\infty} K_n(z-z_0)^n \in G$$

where

$$K_n = \frac{1}{2\pi i} \int_{C} \frac{K(w)}{(w-z_0)^{n+1}} dw \in G \quad n = 0, 1, 2, \ldots$$

and

$$K_n = \frac{1}{2\pi i} \int_{C} \frac{K(w)}{(w-z_0)^{-n+1}} dw \in G \quad n = 1, 2, \ldots$$

Furthermore, the expansion is unique and

$$K(z) = K_1(z) + K_2(z)$$

where $K_2(z)$ is analytic everywhere inside $|z-z_0| = r_2$ and is analytic outside $|z-z_0| = r_1$.

Of importance is the concept of analytic continuation of a function $K(z) \in G$ analytic in a region $U \subset \mathbb{C}$. Let $U_1 \subset \mathbb{C}$ be any other region such that $U \cap U_1 \neq \emptyset$. If then $K(z) = H(z) \in G$ on any set of points, having a limit point, in $U \cap U_1$ and $H(z) \in G$ is analytic in $U_1$, then $H(z)$ is the analytic continuation of $K(z)$ into $U_1$ and there is only one such continuation. They are partial representations of the same function in $G$. Of particular interest is continuation by power series. Every function $K(z) \in G$ defined by a power series has at least one singularity on its circle of convergence. We shall be concerned only with single valued functions, in $G$, of a complex variable $z$. Hence, to uniformize these functions, we introduce appropriate cuts in the complex plane. The cut plane then yields a set of $z$ values for which an analytic extension of $K(z)$ can be obtained.
which is single-valued and analytic. The totality of all such extensions defines the analytic functions, represented in some $U$ by $\kappa(z)$, in the entire cut plane. The construction of the cut plane will be dealt with later.

Let $G$ be the set of functions $K(x,y)$ continuous in $0 \leq x, y \leq 1$. The addition of two such functions and multiplication by complex numbers is defined in the ordinary way. Multiplication between the elements of $G$ is defined by:

\[ H(x,y) = \int_0^1 K(x,t) K'(t,y) dt \]

and $H(x,y) \in G$. The norm of $K(x,y)$ is defined by:

\[ \max_{0 \leq x, y \leq 1} |K(x,y)| \]

Then $G$ is a complete normed ring. The completeness follows from the uniform convergence of sequences of continuous functions to a continuous function. There is no element in $G$ having the property of being an identity under multiplication. However, the ring may be extended to include such an element with the additional hypothesis that

\[ |z + \kappa| = |z| + |\kappa| \]

for $z \in \mathbb{C}$ and $\kappa \in G$. We call this the ring $G^* = G + \mathbb{C}$.

Consider now the collection of elements $f(x)$ which are continuous on $0 \leq x \leq 1$ and for which a norm is defined as $||f(x)|| = \max_{0 \leq x \leq 1} |f(x)|$ and a metric as $|f_1 - f_2| = \max_{0 \leq x \leq 1} |f_1(x) - f_2(x)|$. Such a set forms a complete normed linear space, $B$.

A linear transformation of such a space, $B$, onto itself is called completely continuous when it takes all bounded sets of $B$ into compact sets in $B$. For $\kappa \in G$, we define
a linear transformation of $B$ onto itself. This transformation is completely continuous. For, consider the sequence $\{g_n(x)\}$

\[ g_n(x) = \int_0^1 K(x,y) f_n(y) \, dy \]

with $|f_n(y)| < 1$ for every $n$. Since $K \in G$ it follows that for every $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that $|x_1 - x_2| < \delta$ implies

\[ |K(x_1, y) - K(x_2, y)| < \varepsilon \quad 0 \leq y \leq 1 \]

Hence:

\[ |g_n(x_1) - g_n(x_2)| < \varepsilon \int_0^1 f_n(y) \, dy \]

so that $|g_n(x_1) - g_n(x_2)| < \varepsilon$ for all $n = 0, 1, 2$. By Arzela's theorem, this is sufficient to allow the extraction from the set of $\{g_n(x)\}$, of a subset which converges in norm (or uniformly, the two implying each other in the space $B$). From this point, we consider the set of linear transformations of $B$ onto itself of the form $E - \lambda K$, where $\lambda \in \mathbb{C}$, $K$ is the completely continuous Fredholm operator defined above and $E$ is the identity transformation. The set of such transformations form a complete normed ring, $G^*$. 

5.

Let $E, K, E \in G^*$, $\lambda \in \mathbb{C}$.

Then if $E - \lambda K \in G^*$ possesses an inverse in $G^*$, $(E - \lambda K)^{-1}$, write:

\[ (E - \lambda K)^{-1} = E + \lambda R_{\lambda} \]

where $R_{\lambda} \in G^*$ is the resolvent of $K$.

The following theorem on completely continuous operators is needed:
Either $E - \lambda K \in G^*$ possesses an inverse $E_{G^*}, E + \lambda R_{\lambda, n}$ or $\mathcal{S} = \lambda K \mathcal{S}$ has a solution $\mathcal{S} \in \mathcal{B}$ which is not identically zero.

(i) $\exists \{ A_n \} \in G^*$ converging to $K$ for which the theorem is true. We exhibit them as the polynomials previously constructed. For they reduce the theorem to one of simultaneous equations.

Hence, for a given $\lambda$, one of three cases may occur:

(i) For infinitely many $n$, there does not exist a unique inverse bounded in norm.

(ii) For infinitely many $n$, there exists a unique inverse bounded in norm.

(iii) For infinitely many $n$, there exists a unique inverse but which is unbounded in norm.

Consider (ii): $E - \lambda A_n$ for infinitely many $n$ has a unique inverse $E + \lambda R_{\lambda, n}$. But, let $E - \lambda A_n = T_n$ and $E + \lambda R_{\lambda, n} = V_n$. Then, for $\{ n_i \}$ $\lim_{n_i \to \infty} T_{n_i} = T = E - \lambda K$ and $V_n = T_n^{-1}$.

For $n_i$ large $|T V_{n_i} - E| < 1$ since $|T V_{n_i} - E| < |V_{n_i}| |I - T - T_{n_i}|$

Hence, $(T V_{n_i})^{-1}$ exists, but $V_{n_i} \to V T = E$. Hence,

$V_{n_i}^{-1} T^{-1} = T_{n_i} V_{n_i} \to E$ or $V_{n_i} \to V$. Hence if $T_{n_i} \to T$ and $V_{n_i} \to V$, then $T^{-1} = V$ and both are in $G^*$.

Consider (i): For infinitely many $n \exists \{ A_n \}$ satisfying

for $\mathcal{S}_n \in \mathcal{B}$:

$\lambda A_n \mathcal{S}_n = \mathcal{S}_n$

Now, since $A_n \in G^*$ and $\lim_{n \to \infty} A_n = \mathcal{K} \in G^*$ a subsequence $f_n$ converges to $f$ and hence $\lambda \mathcal{K} \mathcal{S} = \mathcal{S}$ for $\mathcal{S} \in \mathcal{B}$.

Consider (iii): This implies the existence of $g_n \in \mathcal{B}$ such that:
with \( |q_n| = \infty \). Let \( h_n = \frac{q_n}{|q_n|} \).

Then \( \lambda A_n h_n = h_n \in B \), which reduces to case (i). This proves the theorem. The values of \( \lambda \) for which \( R_\lambda \) does not exist are characteristic values of \( K \) and are singular points of \( R_\lambda \).

Then, the set of those \( \lambda \) for which inverses exist, for all \( f \in B \), of \( E - \lambda K \) is called the resolvent set. The complement in \( \mathbb{C} \) is called the spectrum of \( K \) and each \( \lambda \) in the spectrum is called a characteristic value of \( K \). Hence \( R_\lambda \in G^* \) for all \( \lambda \) in the resolvent set.

Then, a theorem due to Riesz states that for \( \lambda \in G^* \), the set of \( \lambda \) for which \( R_\lambda \) does not exist has no limit point in the finite \( \lambda \) plane.

Hence, \( R_\lambda \in G^* \) exists for all \( \lambda \in \mathbb{C} \) except a denumerable set of \( \lambda \)'s having no finite limit point. We then PROVE:

**Theorem:** For every \( \lambda \in G^* \) and small enough \( \lambda \), \( R_\lambda \) exists, is in \( G^* \), and is an analytic function of \( \lambda \).

Firstly, consider the series:

\[
\sum_i K^n \lambda^{n-1}
\]

which is a power series in \( \lambda \) and converges absolutely for \( |\lambda| < \frac{1}{|K|} \).

Hence, it is an analytic function for all \( \lambda \in U \subset \mathbb{C} \) where \( U \) is the open set: \( |\lambda| < \frac{1}{|K|} \). Secondly, the series above satisfies:

\[
(E - \lambda K)(E + \lambda \sum_i K^n \lambda^{n-1}) = E
\]

and, hence, may be properly called \( R_\lambda \). Since \( R_\lambda \) is the uniform limit of elements of \( G^* \), it too lies in \( G^* \) for \( \lambda \in U \).

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Further, by analytic continuation, $R_\lambda \in G^\times$ in any region into which $R_\lambda$ defined above analytically can be continued.

**Theorem:** Let now $R_\lambda$ be the resolvent of $K \in G^\times$. Let $U$ be the set of those $\lambda$ for which $R_\lambda$ exists and let $\lambda, \mu \in U$. Then

$$R_\lambda - R_\mu - (\lambda - \mu) R_\lambda R_\mu = 0$$

This follows at once from the equations

$$(E - \lambda K)(E + \lambda R_\lambda) = E = (E + \lambda R_\lambda)(E - \lambda K)$$

and

$$(E - \mu K)(E + \mu R_\mu) = E = (E + \mu R_\mu)(E - \mu K).$$

Further it is seen that

(i) $R_\lambda K = K R_\lambda$, $\lambda \in U$

(ii) From $R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu$

with $\mu = 0$ it follows that

$$(E - \lambda R_\lambda)(E + \lambda R_\lambda) = E$$

Hence $R_0 = K$.

Thus we get a set of equations

$E - \lambda R_\lambda$ has resolvent $R_\lambda$

$E - \mu R_\lambda$ has resolvent $R_{\lambda + \mu}$, etc.

The solution of these equations permits the analytic continuation of $R_\lambda$.

We now come to the important theorem of this section.

**Theorem:** For $K \in G^\times$, $R_\lambda$ is a meromorphic function of $\lambda$.

Firstly, the singularities of $R_\lambda$ are denumerable and isolated. Secondly, for $\lambda$ not in the spectrum of $K$, $R_\lambda$ is a regular function of $\lambda$. Hence, for $\lambda_0$ in the spectrum and $\gamma > |\lambda - \lambda_0| > \gamma > 0$, the Laurent expansion of $R_\lambda$ exists and is in $G^\times$. 
The equation $R_\lambda - R_\mu - (\lambda - \mu)R_\lambda^+ - R_\mu^-$ implies that the two equations

\[ R_\lambda^- - R_\mu^- - (\lambda - \mu)R_\lambda^+ R_\lambda^- R_\mu^- = 0 \quad \lambda, \mu \neq \lambda \]  

and

\[ R_\lambda^+ - R_\mu^+ - (\lambda - \mu)R_\lambda^- R_\mu^+ = 0 \quad \lambda, \mu \neq \lambda \]  

be true as can be shown by substituting and comparing of terms in like powers of $\lambda, \mu$.

From (65) it follows that

\[ H_{-(m+n+1)} = H_m H_n \]

Hence

\[ H_{-n} = H_{-n} H_{-1} = H_{-1} H_{-n} \quad \text{and} \quad H_{-1}^2 = H_{-1} \]

So

\[ H_{-(n+1)} = H_{-1} \]

Now $H_{-1} \in \mathbb{C}^*$ and hence makes $B$ into some sub-space of $B$.

Let it be $C$. Then $H_{-1}^2 = H_{-1}$ indicates $H_{-1}$ makes $C$ onto itself.

Since $H_{-1} \in \mathbb{C}^*$, every bounded part of $C$ is compact. Riesz* has

---

* Riesz, F. Loc cit, p. 75, Lemma 5.
shown this implies \( C \) is of finite dimension. Hence from \( H_{-(n+1)} = H^{-2} \), it follows that, \( P \) being a mapping of \( C \) onto itself:

\[
H_{-(n+1)} = P^n H^{-1}
\]

Hence, \( P \) can be represented as a triangular matrix of finite order \( k \), such that \( a_{ij} = 0 \) for \( i < j \). Then:

\[
R_\lambda = \sum_{i=1}^{\infty} \frac{H^{-i}}{(\lambda - \lambda_0)^i} = \sum_{i=1}^{\infty} \frac{P^n H^{-i}}{(\lambda - \lambda_0)^i} + \frac{H^{-1}}{\lambda - \lambda_0}
\]

since for any \( \eta > 0 \) the series must converge, it follows that the diagonal terms \( a_{ii} = 0 \). So that for \( \eta > k \), \( P^n = 0 \).

Hence \( H^{-n} = 0 \) for \( \eta > k \). Thus \( R_\lambda \) is a meromorphic function of \( \lambda \).*

Hence, we have shown \( R_\lambda \in C^*_\lambda \) to be an analytic function of \( \lambda \), representable as a power series about the origin with a non-zero radius of convergence. Indeed, \( R_\lambda \) is a meromorphic function of \( \lambda \) for all finite \( \lambda \) and has an essential singularity at \( \infty \).

* The above proof of this well-known theorem is due to Nagumo:
From the resolvent equation:

\[ R_\mu - R_\lambda - (\mu - \lambda) R_\mu R_\lambda = 0 \]

which holds where \( R_\lambda, R_\mu \) exist, we show easily that, letting \( \mu = \lambda + h \), \( \frac{dR_\lambda}{d\lambda} \) exists at least wherever \( R_\lambda \) exists, for

\[ \frac{dR_\lambda}{d\lambda} = \lim_{h \to 0} \frac{R_{\lambda+h} - R_\lambda}{h} = \lim_{h \to 0} \frac{R_{\lambda+h} R_\lambda}{h} \]

and \( \lim_{h \to 0} R_{\lambda+h} R_\lambda = R_\lambda^2 \). Hence, \( R_\lambda \) is an analytic function of \( \lambda \) in some neighborhood about the origin. Thus \( R_\lambda \) has a Taylor's series development about the origin:

\[ R_\lambda = \sum_{n=0}^{\infty} \frac{R_\lambda^{(n)}}{n!} \lambda^n \]

The uniqueness theorem for analytic functions then yields:

\[ R_\lambda^{(n)} = \frac{n! K^n}{n!} \]

Likewise, from the Taylor expansion about zero:

\[ R_\lambda = \frac{\sum_{n=0}^{\infty} R_\lambda^{(n)}}{n!} \lambda^n \]

Proof of the lemma on commutativity:

Since \( (E - \lambda K) (E + \lambda R_\lambda) = E = (E + \lambda R_\lambda K) E - \lambda K \)

it follows that

\[ K R_\lambda = R_\lambda K \]

From the equations:

\[ R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda \]

and

\[ R_\mu - \lambda = \mu R_\mu K \]

it follows that \( R_\mu = K \); i.e., that \( R_\lambda \) is the resolvent of \( R_\mu \).
Further, $E - \mu K$ commutes with $E - \mu K$; hence their inverses, where they both exist, commute also.

From:

$$(E + \lambda R_{\lambda} K E + \mu R_{\mu}) = (E + \mu R_{\mu} K E + \lambda R_{\lambda})$$

follows the commutativity of $R_{\lambda}$ with $R_{\mu}$ for any $\lambda, \mu$ for which both resolvents exist.

We get

$$R_{\lambda}^{(n)} = \sum_{n=0}^{\infty} \frac{1}{n!} R_{0}^{(k+n)} \lambda^{n}$$

Hence

$$R_{\lambda}^{(\mu)} = \frac{1}{\mu!} \sum_{n=0}^{\infty} \frac{1}{n!} (k+n)! R_{0}^{(k+n)} \lambda^{n}$$

2.

We now come to the main results of the thesis. We recall that $R_{\lambda}$ is a meromorphic function of $\lambda$, analytic in the neighborhood of $\lambda = 0$, and has a uniformly convergent power series of non-zero radius about that point.

We now construct the Mitag-Leffler star for $R_{\lambda}$. From the point $\lambda = 0$, we draw a straight line to every $\lambda_{i}$, a characteristic value of the kernel and a singularity of $R_{\lambda}$. Along the continuation of this ray from $\lambda_{i}$ to $\infty$, the $R_{\lambda}$ plane is cut. With the value of $R_{\lambda}$ at $\lambda = 0$ taken as $K$, this uniformizes $R_{\lambda}$. The entire plane minus the cuts is the star of $R_{\lambda}$ denoted by $D$.

Let now $D'$ be a bounded region interior to $D$ and $C$ a closed simply-connected contour in $D'$. Then the Cauchy
integral theorem yields
\[ R_\lambda = \frac{1}{1 - \lambda} \int_C \frac{R_\xi}{\xi - \lambda} \, d\xi \in \mathbb{C}^* \]
for \( \xi \in \mathbb{C} \) and \( \lambda \in 0' - \mathbb{C} \). In particular, let \( \lambda \) lie inside \( C \).

3.

Certain theorems and definitions in the theory of the summability of divergent series are introduced at this point. If a series \( \sum_{n=0}^{\infty} u_n \) does not necessarily converge, we consider transformations of the \( \{u_n\} \) and/or the partial sums
\[ s_n = \sum_{k=0}^{n} u_k \]
to \( \{v_n\} \) and \( t_n \). These transformations are to have the properties:

(i) They define a finite sum to \( \sum_{n=0}^{\infty} u_n \) which exists where the proper sum of \( \sum_{n=0}^{\infty} u_n \) has no value.

(ii) Whenever \( \lim_{n \to \infty} s_n = s \), then \( \lim_{n \to \infty} t_n = s \) for any \( \{s_n\} \).

Such a transformation is called a summability method.

We consider several such types of transformations. Consider the transformation \( P \), defined by the matrix \( (a_{mn}) \):
\[ t_m = \sum_{n=0}^{\infty} a_{mn} s_n \]
Then the basic theorem on such transformations is:

**Theorem:** For \( P \) to transform \( s_n \) such that \( \lim_{n \to \infty} s_n = s \) implies \( \lim_{n \to \infty} t_n = s \), it is necessary and sufficient that

\[ (1) \sum_{n=0}^{\infty} |a_{mn}| < B \quad \text{for} \quad m = 1, 2, \ldots \]

\[ (ii) \lim_{m \to \infty} a_{mn} = 0 \]
(iii) \[ \lim_{m \to \infty} \sum_{n=0}^{\infty} a_{mn} = 1 \]

For the proof, consult the fine monograph by Szasz.

We will need other types of transformations known as series to function transforms. Suppose that we have a sequence of functions \( \{ \psi_n(x) \} \) defined for some range of \( x, \ 0 < x < \Xi \) and

\[ \lim_{x \to 0} \psi_n(x) = 1 \]

If \( \varphi(x) = \sum_{n=0}^{\infty} u_n \psi_n(x) \) converges in some open interval in \( 0 < x < \Xi, < \Xi \) and approaches a limit \( S \) when \( x \to 0 \), then this defines a summability yielding \( S \) as a generalized sum of \( \sum_{n=0}^{\infty} u_n \). Then the method is regular (satisfies ii) if and only if

\[ \sum_{n=0}^{\infty} |\psi_n(x) - \psi_{n+1}(x)| < B \]

for some interval \( 0 < x < \Xi' \). This condition is equivalent to

\[ 0 \leq \psi_{n+1}(x) \leq \psi_n(x) \quad 0 < x < \Xi' \]

A third type of summability is that where the sum is defined as means of entire functions. If

\[ \frac{S(x)}{r(x)} = \sum_{n=0}^{\infty} p_n S_n X^n \]

with the coefficients \( \{ p_n \} \) non-negative and

\[ \lim_{x \to \infty} \frac{S(x)}{r(x)} = S \]

we define a summability method (J) to \( \sum_{0}^{\infty} u_n \).

A fourth method is defined by means of moment constants.
Let the sequence $\alpha_n$ be generated by
\[ \alpha_n = \int_0^\infty x^n \, d\phi(x) \quad n = 0, 1, 2, \ldots \]
where $\phi(x)$ is a bounded increasing function of $x$. Let the integral exist for all $n$. If we define:
\[ v(x) = \sum_{n=0}^\infty \left( \frac{u_n}{n!} \right) x^n \]
formal term by term integration yields:
\[ \int_0^\infty v(x) \, d\phi(x) = \sum_{n=0}^\infty \left( \frac{u_n}{n!} \right) \int_0^\infty x^n \, d\phi(x) \]
\[ = \sum_{n=0}^\infty u_n \]

We take the integral on the left as defining, where it exists, a sum for $\sum u_n$.

We are now in a position to prove that the method of back substitutions yields a sequence of functions convergent to the solution of (1) if the conditions (ii) and (iii) are satisfied. To do this, we introduce the concept of Euler summability. Consider the particular matrix $(a_{mn})$ defined by:
\[ a_{mn} = \begin{cases} \frac{1}{(q+1)^{m+1}} \binom{m+1}{n+1} q^{m-n} & n \leq m \\ 0 & n > m \end{cases} \]
This matrix defines a regular summability method.

Now, corresponding to each $\{ \lambda_i \}$ which is a singularity of $\mathcal{L}_\lambda$, we consider the set of points $\lambda \in V_i$ defined by:
\[ |\lambda \lambda_i^{-1} + i^{p-1}| < i^{p} \]
or, in terms of the $q$ above:

$$| \lambda \lambda^{-1} + q | < q + 1$$

Let $V = \cap V_i$. Condition (iii) comes from the particular value $\lambda = \infty$ which is a singular point of $R_{\lambda}$. For any $\lambda \in V$ and any $\mu$ not in the star of $R_{\lambda}$, it follows that

$$| \lambda \mu^{-1} + q | < q + 1$$

Hence, if $\lambda \in V$ and

$$| \lambda \mu^{-1} + q | > q + 1$$

then $\mu$ is in the star of $R_{\lambda}$ and, hence, a value for which $R_{\lambda}$ is regular. We point out that $V$ is not empty for every $V_i$ contains some neighborhood of $\lambda = 0$. From this, it is easy to see that, for a given $\mu \in V$, both the boundary and the interior of the circle defined by

$$| \lambda^{-1} \mu + q | > q + 1$$

contain only points of the star. If the boundary of this circle is $C$, then it is possible to enlarge $C$ to $C'$ so that the same holds true for $C'$. Hence, for $\gamma$ on $C'$, it follows that

$$| \gamma^{-1} \mu + q | < q + 1$$

Hence, uniformly in $\gamma$

$$\left| \frac{\gamma^{-1} \mu + q}{q + 1} \right| \leq \beta < 1$$

since the left hand side is a continuous function of $\gamma$.

Then the series

$$\frac{1}{q + 1} \sum_{n=0}^{\infty} \left( \frac{\gamma^{-1} \mu + q}{q + 1} \right)^n = \frac{1}{1 - \frac{\mu}{\gamma}}$$
uniformly and absolutely for \( \tau \in \mathbb{C} \). Hence

\[
R_\lambda = \frac{1}{2\pi i} \int_{\gamma} \frac{R_\tau}{\tau - \lambda} \, d\tau = \frac{1}{2\pi i} \int_{\gamma} \frac{R_\tau}{\tau} \, d\tau \frac{1}{1 - \frac{1}{\gamma_\lambda}} = \frac{1}{2\pi i} \int_{\gamma} \frac{R_\tau}{\tau} \, d\tau \sum_{n=0}^{\infty} \left( \frac{R_\tau}{\gamma_\lambda + 1} \right)^n.
\]

The uniform convergence allows interchange of summation and integration:

\[
R_\lambda = \sum_{n=0}^{\infty} (\gamma_\lambda + 1)^{n+1} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{R_\tau}{\gamma_{\lambda+k}} \left( \frac{R_\tau}{\gamma_\lambda + 1} \right)^k.
\]

Now depress \( C \) so that it lies within the circle of convergence of the Neumann series for \( R_\lambda \) and obtain:

\[
R_\lambda = \sum_{n=0}^{\infty} (\gamma_\lambda + 1)^{n+1} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{R_\tau}{\gamma_{\lambda+k}} \left( \frac{R_\tau}{\gamma_\lambda + 1} \right)^k.
\]

let \( \alpha = -\frac{1}{\gamma_\lambda + 1} \) and

\[
R_\lambda = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-\alpha)^k (1 + \alpha)^{n-k} K^k \lambda^k.
\]

which is the expansion of \( R_\lambda \) obtained by the back substitution method. Hence, we have proved that the Fredholm integral equation (1) has the solution

\[
f(x) = g(x) + \lambda \int_{0}^{1} \sum_{n=0}^{\infty} R_{\lambda,n} (x,y;\lambda) \, g(y) \, dy
\]

* For general \( \alpha \), it is defined by \( -\alpha = \gamma \) and \( -\frac{\ln \gamma}{\ln 2} = \beta \).

The branch we take is that for which \( \log 1 = 0 \).
where \( R_{\lambda, n} \) denotes the terms of the series above, the power of \( \lambda \) in each term being reduced one. The error after \( n \) steps is essentially less than \( \max_{0 \leq x \leq 1} |r_n(x)| \).

Hence

\[
E_n \leq \max_{0 \leq x \leq 1} |r_n(x)| \leq \max_{0 \leq x, y \leq 1} \sum_{k=0}^{n-1} \left( \frac{\lambda}{\lambda + 1} \right)^{n-k} \alpha^{-k} \left| \int_0^1 K(x, y) g(y) dy \right|
\]

let
\[
M = \max_{0 \leq x, y \leq 1} |K(x, y)|, \quad N = \max_{0 \leq x, y \leq 1} |g(y)|
\]

Then
\[
E_n \leq N \sum_{k=0}^{n-1} \left( \frac{\alpha}{1 + \alpha} \right)^{n-k} \left| -\alpha \lambda M \right|^k \leq N \left( |1 + \alpha| + |\alpha \lambda M| \right)^n
\]

so that, of course, \( |\lambda| \) need lie between 0 and -1.

5.

We consider now another procedure for generating solutions of the integral equation (1) for large \( \lambda \), which converges in the same domain as the above, for \( \alpha \) chosen properly.

Firstly, for the given \( \lambda \), we construct the Borel polygon of summability, \( C_\alpha \), in the following manner. After drawing a ray from \( \lambda = 0 \) to each \( \lambda_i \), a pole of \( R_{\lambda, n} \), we construct normals at \( \lambda_i \) to these rays. The set of points, \( \lambda_i \), which lie on the same side of every normal as does the origin, are interior points of the polygon of summability. We then obtain a representation of \( R_{\lambda} \) in \( Q \).

This method of summability is that of Borel, called (B') by
Szass and by Hardy. Outside of \( Q \) \( R_\lambda \) cannot be represented by this method for, the method \((B')\) has the property: if a power series is \((B')\) summable at \( \lambda = \mu \), it is a regular function of \( \lambda \) at all interior points of the circle having the ray connecting the origin to \( \mu \) as its diameter. Hence, if \( \mu \) is outside \( Q \), some \( \lambda \) lies within the circle having the ray \( 0\mu \) as a diameter which contradicts the regularity of \( R_\lambda \) inside the circle.

To prove convergence inside \( Q \), we consider the moment constant method for which \( \phi(\lambda) = 1 - e^{-\lambda} \). Then
\[
\mu_n = \int_0^\infty e^{-t} t^n dt = n!
\]
and we are concerned with the particular series
\[
\sum_{n=0}^{\infty} \lambda^n
\]
The sum of the series is defined as:
\[
\int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} dt
\]
\[
= \int_0^\infty e^{-t(1-\lambda)} dt = \frac{1}{1-\lambda} \quad \text{Re} \lambda > 1
\]

Now let \( R_\mu \) be regular inside and on a closed curve \( C \) about \( \mu = 0 \); i.e., in some region interior to the star of \( R_\lambda \). Let \( \lambda \in Q \) such that \( Q \) lies within \( C \); such that, uniformly, for \( \mu \) on \( C \):
\[
\text{Re} \left( \frac{1}{\mu} \right) \leq 1 - \delta < 1
\]
Then:
\[
R_\lambda = \frac{1}{2\pi i} \int_C \frac{R_\mu}{\mu - \lambda} d\mu = \frac{1}{2\pi i} \int_C \frac{R_\mu}{\mu} \frac{1}{1-\frac{\lambda}{\mu}} d\mu
\]
\[
= \frac{1}{2\pi i} \int_C \frac{R_\mu}{\mu} \int_0^\infty e^{-t} t \gamma_\mu dt
\]
By ( ), the second integral represents an analytic function of \( \mu \), both integrals are bounded and hence the order may be inverted to give:

\[
R_\lambda = \int_0^\infty e^{-t} \left( \frac{1}{i\pi} \int_C \frac{R_\mu}{\mu} e^{t \frac{\mu}{\mu}} d\mu \right) dt
\]

Now, contract \( C \) to a \( C' \) inside the circle of convergence of:

\[
R_\mu = \sum_{n=0}^{\infty} K^{[n]} \lambda^n
\]

Hence, for \( \mu \) on \( C' \), the above series and

\[
e^{t \lambda \mu} = \sum_{n=0}^{\infty} \frac{(t \lambda \mu)^n}{n!}
\]

converge uniformly in \( \mu \) and \( \lambda \), respectively. Hence,

\[
R_\lambda = \int_0^\infty e^{-t} dt \frac{1}{i\pi} \int_C \sum_{n=0}^{\infty} K^{[n]} \lambda^n \sum_{m=0}^{\infty} \frac{1}{m!} (t \lambda \mu)^m \frac{d\mu}{\mu}
\]

\[
= \int_0^\infty e^{-t} dt \sum_{n=0}^{\infty} K^{[n]} (t \lambda)^n
\]

\[
= \int_0^\infty e^{-t} \mathcal{V}(x, y; t\lambda) dt
\]

and the solution of (1) valid in the Borel polygon is

\[
f(x) = g(x) + \int_0^1 \left[ \int_0^\infty e^{-t} \mathcal{V}(x, y; t\lambda) dt \right] g(y) dy
\]

That \( \lambda \in V \) (the region of Euler summability) implies \( \lambda \in Q \) i.e., \( Q \supseteq V \), follows from the theorem that every series summable \( (E) \) (Euler) is summable \( (B) \). Conversely, for any \( \lambda \) inside \( Q \), there exists an \( \alpha \) (or \( p \) or \( q \)) such that
\[ \lambda \in V_\alpha, \] and hence is \((E)\) summable for that \(\lambda\). For \(t\) tends to the Borel polygon as \(q \to 0, p \to 00, \alpha \to 0\).

That \(C'\) can be so chosen follows from the following argument: \(\lambda \in Q,\) then \(R_\lambda\) is regular in a circle having \(0\lambda\) as diameter. Then \(\exists \lambda \in Q,\) say \(\lambda',\) such that \(0\lambda',\) as diameter, generates a circle, \(C',\) concentric to \(OQ\) and still containing only regular values of \(R_\lambda\).

Now fix \(\lambda \in Q\). Then for \(\mu\) on \(C',\) \(\Re \left( \frac{\mu}{\lambda} \right) < 1\) for all \(\mu\) on \(C';\) and being continuous takes on a maximum. Hence, for some

\[ \Re \left( \frac{\mu}{\lambda} \right) \leq 1 - \delta < 1 \]

So for \(\lambda\), fixed, in \(Q\) take \(C'\) as the contour.

6.

Applying the same approach, we now exhibit solutions of the integral equation; i.e., representations of \(R_\lambda\), valid in every bounded region of the star. Firstly, consider real functions \(A_n(t)\) of the real variable \(t\), the so-called Abelian means and variants of them, satisfying

\[ (1) \quad \lim_{t \to 0} A_n(t) = 1 \quad n = 1, 2, \ldots \]

and the series \(\sum u_n\) is given the generalized sum

\[ \lim_{t \to 0} \sum_{n=0}^{\infty} A_n(t) u_n \]

Then if \(\{A_n(\lambda)\}\) are restricted so that \((1)\) \(\sum_{n=0}^{\infty} A_n(t) \lambda^n\) is an entire function of \(\lambda\) for \(t \to 0\) and that \((ii)\) if

\[ S_t(\lambda) = \sum_{n=0}^{\infty} A_n(t) \lambda^n \]
then \( \lim_{t \to 0} S_{\lambda}(t) = \frac{1}{1-\lambda} \) uniformly in every bounded region of the star of \( \frac{1}{1-\lambda} \). Then in every bounded region \( Q \) of the \( \lambda \) plane interior to the star

\[
R_{\lambda} = \lim_{t \to 0} \sum_{n=0}^{\infty} A_n(t) K^n \lambda^n
\]

uniformly in \( \lambda \).

We may take \( Q \) to be star-shaped. Then we can strictly enclose \( Q \) in a star-shaped region \( Q' \) still interior to the star of \( R_{\lambda} \). Hence, let \( C \) be the boundary of \( Q' \). Then:

\[
R_{\lambda} = \frac{1}{2\pi i} \int_C \frac{R_{\lambda}}{\mu-\lambda} \, d\mu
\]

but for \( \lambda \in Q \), \( \mu \) on \( C \), no \( \frac{A}{\mu} \) lies outside the star of \( \frac{1}{1-\lambda} \). Hence,

\[
R_{\lambda} = \frac{1}{2\pi i} \int_C \frac{R_{\lambda}}{\mu} \left[ \lim_{t \to 0} S_{\lambda}(\frac{A}{\mu}) \right] d\mu
\]

The limit being uniform, we write:

\[
R_{\lambda} = \lim_{t \to 0} \frac{1}{2\pi i} \int_C \frac{R_{\lambda}}{\mu} \sum_{n=0}^{\infty} A_n(t) \left( \frac{A}{\mu} \right)^n d\mu
\]

Now contract \( C \) within \( |\lambda| = |\lambda| \) and hence,

\[
R_{\lambda} = \lim_{t \to 0} \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} A_n(t) \left( \frac{A}{\mu} \right)^n \sum_{m=0}^{\infty} K^{m} \mu^m d\mu
\]

\[
= \lim_{t \to 0} \sum_{n=0}^{\infty} A_n(t) K^n \lambda^n
\]

since the uniformly convergent series may be integrated term
by term. The result patently holds for every \( \lambda \in \mathbb{Q} \).

Hence, (1) has the solution:

\[
f(x) = g(x) + \int_0^1 \lim_{t \to 0} \sum_{n=0}^{\infty} A_n(t) K^n(x, y) \lambda^n g(y) dy
\]

Three such sets \( \{ A_n(t) \} \) are given by:

(i) \[ A_n(t) = \frac{\Gamma(nt)}{\Gamma(n+1)} \quad (12) \]

(ii) \[ A_n(t) = \begin{cases} 0 & n = 0 \\ e^{-nt \log n} & n = 1, 2, \ldots \end{cases} \quad (12) \]

(iii) \[ A_n(t) = \frac{1}{\Gamma(1+nt)} \quad (12) \]

For proofs that these actually sum \( \sum_{n=0}^{\infty} \lambda^n \) uniformly interior in the star of \( \frac{1}{1-\lambda} \) to \( \frac{1}{1+\lambda} \), consult the above references.

Finally, we point out that the method of stepwise continuation introduced in Chapter I is precisely the procedure by which a Taylor's series is continued throughout the star of \( R_\lambda \).
Bibliography


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Biography

Alan Jay Perlis was born in Pittsburgh, Pennsylvania, on April 1, 1922. He was enrolled in the first of a succession of schools, the Colfax Public School of Pittsburgh, in 1927. Then followed, from 1933 to 1939, study at Taylor Allderdice High School. He entered Carnegie Institute of Technology in September, 1939, studied chemistry, and graduated with honor on December 20, 1942. The morning of December 22, 1942, found him in an air cadet school for meteorologists of the U. S. Army Air Force. Immediately upon being commissioned a 2nd lieutenant in the meteorology service, he was dispatched to Air Intelligence School and forthwith sent to Europe where he served eighteen months as an intelligence officer at the operational headquarters of the 9th U. S. A. A. F., and as a squadron weather officer with a reconnaissance squadron. He was honorably discharged in September, 1945, and entered graduate school at the California Institute of Technology. After one-and-one-quarter years of graduate study in chemistry, he transferred to the Massachusetts Institute of Technology, entering in September, 1947. He received his M. S. in mathematics in February, 1949. During the summers of 1948 and 1949, he held a position with the digital computer project, Whirlwind, at M. I. T.