An Analysis of the Sensitivity of a Low Pressure Time Projection Chamber to a Directional Anisotropy due to WIMP Interactions

Evan M. Zayas

Submitted to the Department of Physics in partial fulfillment of the requirements for the degree of Bachelor of Science at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2015

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Abstract

The Dark Matter Time Projection Chamber (DMTPC) collaboration is a dark matter direct detection effort which develops TPCs to observe and reconstruct nuclear recoils generated by incident particles. If some of these recoils are the result of dark matter interactions, we can in theory observe an anisotropy in the direction of these recoils which is consistent with the galactic halo models of dark matter. Such an observation would serve as convincing evidence that these incident particles have an extrasolar origin. In this thesis I discuss the workings of a TPC known as the 4-shooter, the analysis used to identify nuclear recoil candidates, and the mathematics to quantify the anisotropy of a distribution. I then discuss the ways in which the pressure of the target gas in the TPC affects rejection power, and construct a framework to determine an optimal operating pressure for the 4-shooter and future DMTPC detectors.

Thesis Supervisor: Peter Fisher
Title: Professor of Physics; Department Head
Acknowledgements

First, I would like to express my gratitude to Peter Fisher for his mentorship to me during my time as an undergraduate at MIT. The advice I have taken and the knowledge I have gained from Peter will stay with me for the rest of my life. I would also like to especially thank Ross Corliss for his exceptional feedback on the presentation of this thesis, as well as Hidefumi Tomita and Cosmin Deaconu for helping to get me involved with DMTPC and making the experience unique and enjoyable.

In addition, my time at MIT has been heavily influenced by a number of faculty and students who I hope to consider lifelong colleagues and friends. I would like to particularly thank Sean Robinson, Nevin Weinberg, Joe Formaggio, Alex Siegenfeld, Marjon Moulaï, and Kevin Burdge for shaping my experience in the physics department here both academically and socially.

Finally, I would like to give a special thanks to Catherine Hrbac, whose friendship during the last four years has been second to none. It turns out you can just write a thesis!
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Chapter 1

Introduction

The dynamics and structure of galaxies is one of the most actively studied fields in modern astrophysics. For decades, scientists have studied the motion of the largest objects in the cosmos - stars, galaxies, and galactic clusters - striving to understand the mechanics of our universe at the grandest of scales. In the 1930s, physicists Jan Oort, Horace W. Babcock, and Fritz Zwicky [1] were among the first to observe a discrepancy in the masses of neighboring galaxies. Observations showed that the mass as inferred from gravitational effects was considerably greater than the mass as calculated from the density of luminous matter. This idea of "missing mass" was met with skepticism early on, but eventually gained widespread recognition in the scientific community, in large part due to Vera Rubin's 1970 study of the rotation of Andromeda [2]. It is now widely accepted that galaxies consist of both luminous matter - such as stars, gas, and interstellar dust - and dark matter, which does not interact electromagnetically and is thus invisible to our conventional telescopes. Instead, the presence of dark matter in distant galaxies can so far be inferred only from its gravitational effects.1

1The presence of dark matter can also be inferred from the perspective of theoretical cosmology. Cold dark matter is an essential component of the ΛCDM model [3], which explains (reasonably well) the large-scale structure of the universe and the formation of galaxies.
1.1 Observational Evidence for Dark Matter

Astronomers have observed for some time that the luminous matter in a galaxy is concentrated near its center. As such, early galactic models consisted of a central "bulge" containing nearly all of the mass, with stars orbiting around the periphery much like planets in a solar system. With this hypothesis, one would expect the stars in any galaxy to follow Keplerian motion:

\[ \frac{GM}{r^2} = \frac{v^2}{r} \]

Where \( G \) is the gravitational constant, \( M \) is the mass of the galaxy, \( v \) is the velocity of a star, and \( r \) is its distance from the center of the galaxy (the "galactic radius"). A relationship between \( v \) and \( r \) (of which Equation 1.2 is a specific example) is known as a rotation curve; it describes the speed of objects in a galaxy as a function of their distance from its center.

To find the rotation curve of the Milky Way, astronomers use radio telescopes which measure the intensity of 21cm radiation\(^2\) from hydrogen in the interstellar medium. The Doppler shift of the signal determines the velocity of its source, and simple geometry (the details of which are described in ref. [4]) determines its location in the galaxy as well; thus, by analyzing the signal across the galactic plane, we can plot a rotation curve and compare it to the Keplerian prediction. Figure 1.1 shows a rotation curve which was measured with a small radio telescope at MIT in the fall of 2013. While we might expect velocity to increase with distance for small \( r \) (inside the central bulge), its behavior at larger distances is certainly not supported by the Keplerian model.

In addition to rotation curves, the gravitational effects of dark matter can be seen when light from background sources is bent around a galaxy - an effect known as weak gravitational lensing. A famous example of this effect is the Bullet Cluster (shown in Figure 1.2), two galaxy clusters which collided 100 million years ago [5]. While the baryonic matter distribution looks as one might expect after a collision, a large distribution of matter appears unaffected, as if it did not interact during the collision. This suggests that a considerable

\(^2\)This radiation comes from transitions between the hyperfine states of hydrogen. The abundance of hydrogen in the interstellar medium makes it a strong signal which can be observed quite easily above background noise.
Figure 1.1: Rotation curve of the Milky Way, as measured by E. Zayas and A. Jaffe [4] with a small radio telescope at MIT in the fall of 2013. The data show an indisputable contradiction with the Keplerian model. The black dashed line shows the prediction with a dark matter sphere of uniform density, and is more consistent with the data (though still far from perfect).

fraction of matter in the Bullet Cluster interacts only very weakly with itself and with normal matter, a key characteristic which is consistent with the dark matter hypothesis.

1.2 WIMPs

With overwhelming evidence for the existence of dark matter, a natural next step is to postulate something about its properties. From observational evidence we know that dark matter interacts gravitationally; what remains to be found is the mass of individual dark matter particles and the nature of their interaction (if any) aside from gravity. If there is another force which couples to dark matter, its strength should be related to the particle mass through the relic abundance.\(^3\) With a mass of order 100 GeV (comparable to that of the weak vector bosons), the interaction strength must be similar to that of the weak nuclear force, \(\langle \sigma v \rangle \approx 3 \times 10^{-26} \text{ cm}^3 \text{ s}^{-1}\).\(^4\) This

\(^3\)The relic abundance has been measured to high precision by the WMAP collaboration [6], \(\Omega = 0.1120 \pm 0.0056\). The relationship between this quantity, the particle mass, and interaction strength is explored further in ref. [7].

\(^4\)This result is often quoted without a specific uncertainty. [7]
apparent coincidence brings forth the possibility that dark matter couples to Standard Model particles through the weak interaction. As such, the most prominent dark matter candidates are commonly called Weakly Interacting Massive Particles (WIMPs). The notion of a 'weak' interaction in this context historically referred to the weak nuclear force; however, the Standard Model as it stands today makes no predictions about WIMPs or any other potential dark matter candidates. This, in combination with a lack of concrete evidence for weak interactions with dark matter, has led most physicists today to treat the term in a more colloquial sense, meaning only that the interaction has a comparable strength to the weak force. Still, some extensions to the Standard Model - most notably supersymmetry (SUSY) - favor the existence of a particle with mass between 100 GeV and 1 TeV as a popular WIMP candidate. [8, 9]

The experimental search for WIMPs is split into three fields: production, annihilation, and direct detection. If Standard Model extensions such as SUSY are correct, experiments including ATLAS at the Large Hadron Collider might hope to detect WIMPs produced by high-energy collisions in the near future. Experiments such as the Alpha Magnetic Spectrometer (AMS-02) search for anomalies in the energy spectra of cosmic ray particles which may be explained
by the mutual annihilation of dark matter particles. On the other hand, direct detection experiments aim to observe the rare event in which a WIMP scatters elastically off a nucleus to produce a detectable signal. This is the motivation for the Dark Matter Time Projection Chamber experiment (DMTPC).

1.3 Direct Detection and DMTPC

In the context of direct detection, a common assumption is that dark matter forms a spherical, non-rotating halo around the galactic center (the halo model). The velocity of dark matter particles in the galactic frame is given by an isotropic, Maxwellian distribution with some characteristic velocity $v_0$ and a “clipping” at $v_{esc}$, the escape velocity of the galaxy (as particles exceeding this velocity would not be gravitationally bound to the halo).\(^5\) In the frame of the Earth-Sun system, the velocity along a vector tangent to the Sun’s orbit will be shifted by the orbital velocity of the Sun which we denote as $v_\odot$. The accepted values for these velocity figures are presented in Table 1.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Standard Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_0$</td>
<td>230 km/s</td>
</tr>
<tr>
<td>$v_\odot$</td>
<td>244 km/s</td>
</tr>
<tr>
<td>$v_{esc}$</td>
<td>544 km/s</td>
</tr>
</tbody>
</table>

Table 1.1: Standard accepted values for the spherical halo model of dark matter, from ref. [10]. $v_0$ is the characteristic velocity of the Maxwellian distribution of dark matter; $v_\odot$ is the tangential velocity of the Sun relative to the galactic center; $v_{esc}$ is the escape velocity of the galaxy.

The movement of the Sun through this halo produces what we call a “dark matter wind”. Dark matter particles which pass through the Earth will have an anisotropic velocity-direction distribution with a preference along the vector of the Sun’s velocity. The motivation for directional direct detection experiments such as DMTPC is to observe this preferred direction; such an observation must almost certainly come from particles of galactic origin, as any terrestrial anisotropic backgrounds would have no reason to favor this particular direction. Additionally, the direction as observed on Earth changes throughout the

\(^5\)These assumptions, while admittedly simple and likely incomplete, allow for the calculation of WIMP recoil event rates for experiments like DMTPC.
day with the Earth’s rotation, while the direction of terrestrial backgrounds will presumably remain unchanged.

![Probability Density](image)

**Figure 1.3:** Probability density function of WIMP velocity along the direction of the dark matter wind, relative to Earth. This plot was generated using the figures provided in Table 1.1.

The DMTPC experiment records nuclear recoils caused by incident particles to reconstruct the momentum-vector direction of the particle. With this technique, we can theoretically observe an anisotropy in the distribution of event reconstructions which is consistent with the halo model and attributable to WIMP interactions. The experiment uses a time projection chamber (TPC, discussed further in Chapter 2) filled with low-pressure CF$_4$ gas to convert nuclear recoils into detectable signals with the use of charge-coupled devices (CCDs). The use of a gaseous target is not typical of direct detection experiments, and comes with one obvious drawback: low mass density. The number of particles which actually interact with the detector and induce a nuclear recoil is greatly diminished in comparison to similar detectors with liquid targets. However, the low mass density provides one crucial advantage necessary for directional reconstruction: the recoiling nucleus is able to travel much farther through the gas before depositing all its energy than would be possible in a liquid or solid target. This allows us to analyze the track of the nucleus and reconstruct not just scalar quantities such as its energy, but the momentum vector as well. Hence, there is a trade-off between these two properties of the detector: a higher density (i.e. pressure) yields a better effective cross section for events, but at the same time restricts our ability to analyze the events in a meaningful way. The main objective of the research presented in this thesis will be to determine an ideal operating pressure which strikes a balance between these two competing concepts and yields optimal data collection and results.
In Chapter 2 I discuss the 4-shooter detector, a TPC with a fiducial volume of roughly 20 liters which collected data from 2013 through 2014. I also describe a measurement used to calculate the capacitance of the amplification region in the 4-shooter, as a means of putting practical limits on the uniformity of the electric field. In Chapter 3 I present the data sets taken by the 4-shooter and briefly summarize the criteria used to select for likely nuclear recoil events. In Chapter 4 I examine the mathematics used to determine a preferred direction (i.e. reject isotropy) in a set of 2D vectors, which is needed to analyze the statistical significance of a theoretical WIMP signal. In Chapter 5 I study the effects of CF$_4$ pressure on the data from Chapter 3 and the analysis from Chapter 4 to determine a pressure which is best suited for current and future DMTPC detectors.
Chapter 2

The 4-Shooter Detector

This chapter describes the experimental design of a time projection chamber known as the “4-shooter”. It was built and assembled between 2008 and 2013 [11] and captures tracks with four CCDs, hence the nickname. In 2013 and 2014 it was used to collect data which helped the DMTPC group to better understand several aspects of our direct detection experiments including background event signatures, recoil reconstruction, and sparking.

2.1 TPC Theory and Operation

The basic design of a time projection chamber is broken into two parts: the drift region and the amplification region (see Figure 2.1). In the 4-shooter, these regions are separated by a thin steel mesh which is held at zero electrical potential (ground). The drift region is a cylindrical region with a uniform electric field of approximately 20,000 V/m which is filled with CF$_4$ gas. This electric field is produced by a series of copper rings within the TPC, a setup commonly referred to as the “field cage”. These copper rings are connected by a chain of resistors to establish a potential gradient (see Figure 2.2). When a particle strikes a carbon or fluorine nucleus, the nucleus is ripped away from its parent molecule and moves through the gas, losing energy to collisions with other CF$_4$ molecules. These collisions ionize the molecules, leaving behind a trail of free electrons which subsequently drift toward the amplification region due to the electric field. The current from the motion of these drifting electrons is not enough to produce a measurable signal, hence the need for amplification. The amplification region has a similarly uniform electric field to accelerate the electrons, but at a much higher magnitude ($\sim 10^6$ V/m). At this level, the electrons gain sufficient energy between collisions to liberate more electrons.
(that is, the mean free path of electrons multiplied by the electric field exceeds the work function of CF$_4$ electrons). This causes an “avalanche” which greatly amplifies the current to produce a detectable signal, as well as scintillation photons (a process which is discussed further in refs. [12, 13]) which are detected by CCDs and PMTs.

![Diagram of the 4-shooter cylindrical TPC, showing the drift and amplification regions (not to scale). Free electrons, illustrated by the blue circles, are created along the nuclear recoil track and drift towards the ground mesh. They are then accelerated through the amplification region to produce a signal which is captured by the CCDs. The labels on the left and right are the parameters used with the 4-shooter during all data collection discussed in Chapter 3.](image)

Typical TPC experiments cannot use CCDs because they require a large exposure time (~ 1 second), and with a high rate of events they would often capture multiple tracks in one frame. This makes it nearly impossible to associate other signals such as a PMT pulse to the particular track which it represents. However, DMTPC works with a very low event rate; the probability to observe two tracks in one frame (even accounting for backgrounds) is extremely small. This allows us to collect the scintillation light with CCDs.

The chamber uses CF$_4$ as both a target and a scintillator for a number of reasons, most notably:

- Protons in fluorine are highly sensitive to spin-dependent WIMP interactions, in part due to the low (1/2) nuclear spin of $^{19}$F. Discussions of these spin-dependent interactions and the nuclear form factors which
govern their strength are discussed in refs. [14, 15, 16, 17].

- The scintillation emission spectrum of CF$_4$ is bright and well-suited for readout with CCDs. [12, 13] CCDs are not typically used in TPC experiments because they only capture a 2-dimensional projection of a 3-dimensional recoil track. However, DMTPC uses photomultiplier tubes to capture information about the timing of the track signal, from which we can infer the third dimension of the track, $\Delta z$.

Figure 2.3 shows a typical readout of the current$^1$ across the anode mesh in response to an electron avalanche. First, we see a very fast ($\sim 10$ns rise time) pulse due to movement of the electrons across the mesh. The short rise time of this pulse is primarily due to the exponential nature of the cascade; most of the electrons are produced at the end of the avalanche, and thus the distance

$^1$A current-to-voltage converter is used to produce a signal in voltage instead.
Figure 2.3: Anode mesh signal from an actual nuclear recoil event recorded by the 4-shooter in 2013, adopted from ref. [11]. The signal is characterized by a sharp peak caused by the electron avalanche followed by a slower peak due to drifting ions in the amplification region. This two-peak structure is typical of nuclear recoils and helps to isolate them from less interesting background events. [18]

an electron must traverse to reach the anode is very short. Additionally, the drift velocity of the electrons in such a strong electric field is very fast, around $2 \times 10^5 \text{ m/s}$ [19] and consequently the temporal duration of the avalanche is very short. The second peak is a result of the positively charged ions which have been stripped of an electron during the avalanche, and thus also drift (in the opposite direction). This peak is much broader for essentially the same reasons: most ions must traverse close to the full length of the amplification region, and the drift velocity is slower than that of the (much lighter) electrons. This two-peak behavior helps to discriminate against background events such as $\gamma$-rays interacting with the chamber.

2.2 Capacitance of the Amplification Region

Electric field uniformity is among the most important aspects of the TPC. Ideally, scintillation photons which reach the CCDs correspond exactly to the location in $(x, y)^2$ of the energy deposition from the recoil. Transverse components in the electric field will cause the electrons to drift in $x$ and $y$, making track reconstruction unreliable. Similarly, nonuniformities in the magnitude of the field will alter the gain of the amplification, which becomes problematic for the accuracy of the reconstruction process.

$^2$Where $z$ is the direction of the electric field.
If the electric field in the amplification region is perfectly uniform, then it should be accurate to model the ground mesh and the anode mesh as a parallel-plate capacitor. The radius of the mesh is 6 inches, and the separation has been measured to be 0.43 mm. With these parameters it is straightforward to calculate the theoretical capacitance of the amplification region:

\[
C = \frac{\varepsilon A}{d} = \frac{8.854 \text{ pF/m} \times \pi (6 \text{ in})^2}{0.43 \text{ mm}} = 1.502 \text{ nF} \tag{2.1}
\]

Here \(\varepsilon\) is the permittivity of the space between the meshes, which we approximate as the electric constant \(\varepsilon_0\); \(A\) is the area of the mesh in \(x\)-\(y\) and \(d\) is the separation in \(z\). One method to check the electric field uniformity in this region is to compare the measured capacitance with this prediction. In this section, I discuss the techniques used to measure the amplification region capacitance, and compare the results to the above prediction.

### 2.2.1 Measurement setup

Since we expect the capacitance to be small, \(O(1 \text{ nF})\), it cannot be measured directly by a multimeter or similar instrument. Instead, we construct a comparator-based relaxation oscillator to measure the capacitance with high precision. This circuit outputs a square wave with frequency related to the capacitance, without the need for any frequency-specific driving voltage or input signal. It is described below and illustrated in Figure 2.4.

We can analyze the circuit to determine the frequency of oscillations in \(V_{out}\). It is worth noting that for this analysis, if we assume no current flows into the op-amp, then there is only one path it can take (around the op-amp through \(V_{out}\)). The non-inverting input to the op-amp is connected to the output with a voltage divider:

\[
V_+ = \frac{V_{out}}{2} \tag{2.2}
\]

This creates a positive feedback loop so that the output rails the supply voltage, either \(+V_s\) (which I will call HIGH) or \(-V_s\) (LOW). Suppose the output is initially HIGH and the capacitor uncharged. Then, there is no

---

\(3\) The uncertainty on this measurement is not established. I will proceed with the assumption that the measurement is exact, and later use the results of this chapter to place an uncertainty on it.

\(4\) This may seem arbitrary, but the frequency of the output does not depend on any initial conditions.
Figure 2.4: Schematic diagram of the comparator-based relaxation oscillator used to measure the capacitance of the 4-shooter amplification region. The positive feedback loop causes $V_{out}$ to reach the op-amp supply rails very quickly. The capacitor labelled $C$ is the amplification region in this context.

voltage across the capacitor and $V_\neg = 0$. The circuit evolves from this point in a series of steps:

1. With $V_\neg = 0$, the op-amp continues to rail at $+V_s$. Current runs from $V_{out}$ to $V_\neg$, charging the capacitor so that $V_\neg$ increases.

2. After some time, the capacitor is charged such that $V_\neg > V_s/2$. At this moment, since $V_\neg > V_\pos$, the feedback loop quickly flips $V_{out}$ to LOW, and $V_\pos = -V_s/2$.

3. Once again the op-amp continues to rail, this time discharging the capacitor and decreasing $V_\neg$.

4. After some more time, $V_\neg$ passes through $-V_s/2$ and the output flips back to HIGH. The capacitor charges just as it did in step 1 above.

Thus, the output signal can be described by a square wave with half a period equal to the time between flips, i.e. when $V_\neg$ climbs from $-V_s/2$ to $+V_s/2$ or vice versa. The behavior of $V_\neg$ as a function of time can be found from simple RC circuit analysis; when charging, we have:

$$V_\neg = V_s(1 - e^{-\frac{1}{RC}t})$$

(2.3)
By substituting $V_+ \to \alpha V$, we obtain a simple expression for $t$:

$$\frac{t}{RC} = \ln \left( \frac{1}{1 - \alpha} \right)$$

(2.4)

And finally, we can write the period as:

$$T = 2 \left[ t(\alpha = 1/2) - t(\alpha = -1/2) \right]$$

(2.5)

$$= 2 \ RC \left[ \ln (2) - \ln (2/3) \right]$$

(2.6)

$$= 2 \ln (3) \ RC$$

(2.7)

Thus, a measurement of the oscillation period immediately determines the capacitance, given only that the resistance $R$ is known. We also have the freedom to choose $R$ so that the capacitance $C$ is very small; a typical oscilloscope can measure periods as small as $10$ ns, and a resistance as large as $1 \text{ M}\Omega$ is perfectly reasonable. Then, the circuit has the capability (in theory) to measure capacitances as low as $10^{-14}$ F, or $0.01$ pF. In practice, the limiting factor in the magnitude of $T$ is not the sample rate of the oscilloscope, but the time required for the op-amp output to flip between rails (the *slew time*).

The mA741C op-amp used for our measurement has a quoted slew rate of $0.5$ V/µs [20] which corresponds to a time of $24$ µs to flip between rails of ±6V. Then, for a reliable measurement, we insist that $T \gg 24$ µs. By rearranging Equation 2.7 with this constraint, we find (roughly) that for reliable results, we must have $C \geq 100$ pF; indeed, the circuit has demonstrably provided accurate capacitance measurements around this order of magnitude (see Figure 2.5).

### 2.2.2 Procedure and results

From this calibration, we are confident that the relaxation oscillator is an adequate tool to measure the capacitance of the 4-shooter amplification region. However, there is one further caveat which must be accounted for: in performing the actual measurement with the 4-shooter, the BNC wires connecting the amplification region to the oscillator circuit add a small amount of effective capacitance.\(^5\) We correct for this in two ways: first by using a $1$ nF capacitor as a reference point, and second by performing the measurement twice; once with a set of short ($\sim 1$ ft) pair of BNC wires, and again with a longer pair. Table 2.1 shows the measured period for each of these setups.

---

\(^5\)This might also explain why the data in Figure 2.5 tend to lie *slightly* above the prediction; however, the sample size is very small and all of the data points except 100 pF lie within a 5% tolerance of the quoted capacitance.
Figure 2.5: Measured period vs. capacitance with known test capacitors in the relaxation oscillator circuit, with \( R = (1 \pm 0.025) \text{ M}\Omega \). The line shows the theoretical prediction from Equation 2.7. The tolerance for each capacitor is quoted at 5%, corresponding to a fractional uncertainty of 2.5% (the tolerance is a 2\( \sigma \) interval); uncertainty in the measured period is very low, resulting primarily from the op-amp slew time. Both of these uncertainties are too small to be represented in this plot. The match between the prediction from Equation 2.7 and the actual data holds very well at \( \mathcal{O}(1 \text{ nF}) \) and only begins to break down around 100 pF.

<table>
<thead>
<tr>
<th>Capacitor</th>
<th>Short Wires</th>
<th>Long Wires</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1 \pm 0.025) \text{ nF})</td>
<td>((2.26 \pm 0.01) \text{ ms})</td>
<td>((2.50 \pm 0.01) \text{ ms})</td>
</tr>
<tr>
<td>4-sh Amplification Region</td>
<td>((3.34 \pm 0.01) \text{ ms})</td>
<td>((3.58 \pm 0.01) \text{ ms})</td>
</tr>
</tbody>
</table>

Table 2.1: Measured oscillation periods with \( R = (1 \pm 0.025) \text{ M}\Omega \) using a 1 nF capacitor for reference. The small temporal uncertainties are a consequence of op-amp slew time.

The uncertainty in the 1 nF capacitance comes from the tolerance, which is quoted as a 5% fractional uncertainty with a 95% (2\( \sigma \)) confidence interval. The small uncertainty in the periods is simply a consequence of op-amp slew time, as previously discussed.

With both capacitors, the difference between the short and long wire setup is consistently 0.24 ms. This suggests not only that the concern about added capacitance is justified, but also that it is able to be corrected. Assuming the
added capacitance due to wiring is independent of the capacitor in the circuit, we can simply subtract it from the calculation of the 4-shooter capacitance.

Proceeding with the calculation, we find:

\[
C_{4\text{-sh}} = 1.49 \pm 0.03 \text{ nF}
\] (2.8)

This result is in excellent agreement with the parallel-plate model prediction from Equation 2.1. Therefore, we can say with a great deal of confidence that any nonuniformities in the amplification region electric field are on the order of 1% or smaller.\(^6\) This is acceptably small for our event reconstruction as it will not produce a dominating source of uncertainty in the track analysis; we are confident that the electrons do not drift appreciably in any transverse direction before the signal reaches the CCDs.

\(^6\)Furthermore, with this result we can place an uncertainty on the length of the amplification region using Equation 2.1 and standard error propagation. We find that the effective separation is \((0.43 \pm 0.009)\) mm.
Chapter 3

Data Sets

Since the summer of 2013, the 4-shooter has been located in a surface lab at MIT and used for data collection. The data acquisition process is outlined as follows:

Setup

1) The chamber is pumped to near-vacuum ($< 10^{-5}$ torr) and refilled with CF$_4$ to a pressure of 60 torr. This is accomplished with the use of a Varian Turbo Pump backed with an Edwards Scroll Pump (see section 3.2 of ref. [11] for a more detailed description of the vacuum and gas system setup).

2) High voltage is applied to the PMTs, TPC cathode and anode. This must be done after the chamber is refilled to operating pressure in order to avoid damage to the PMTs caused by discharges (sparking) across the amplification region. The cathode and anode are set to $-5kV$ and 670V, respectively, as depicted in Figure 2.1.

Data collection

3) The CCD images are continuously captured and saved to a data acquisition (DAQ) dedicated machine. Additional raw data include the PMT signals, temperature of the CCDs, chamber pressure, and voltage applied to the cathode and anode - all of which are saved to the DAQ. Each ‘run’ of data consists of 100 dark frames (with the CCDs shutters closed) followed by 1000 exposed frames. The dark frames are later analyzed to
correct for biases\textsuperscript{1} in the CCD images. The exposure time of each frame is 1000 ms; accounting for dead time, a run generally takes place over the course of about 30 minutes. The chamber is regularly refilled after a predetermined number of runs, usually $\sim 100$. This is simply to ensure that gas contamination\textsuperscript{2} during runs is kept to a minimum.

4) The raw data are copied to a server where they can be accessed off-site and processed by preliminary analysis scripts. These scripts use an algorithm to identify pixel clusters which are likely the result of recoil tracks in the TPC (an example is shown in Figure 3.1). This cluster-finding algorithm was written primarily by Cosmin Deaconu, and its details are discussed in ref. [21]. The algorithm searches for clusters of “hot” pixels such as those in Figure 3.1 and estimates several parameters of the track including the head-tail confidence.

5) The voltage supplies are automatically ramped down when the preset number of runs is complete.

The data analyzed in Chapter 5 is split into two sets:

- Neutron sourced data. An Americium-Beryllium (AmBe) source was used to produce a beam of neutrons directed at the chamber. These neutrons mimic the dark matter wind because they are neutral particles with similar energies\textsuperscript{3} as we expect from WIMPs, and they are incident from a particular and known direction. The decay of $^{241}\text{Am}$ produces an $\alpha$-particle which in turn interacts with the Beryllium nucleus to produce neutrons of energy around $1 - 10$ MeV. Figure 3.2 shows a to-scale illustration of the setup. The activity of the source was calculated in ref. [11] to yield 26,000 neutrons per second.\textsuperscript{4} These data were taken during the summer of 2013.

\textsuperscript{1}The CCD pixel sensitivity varies between individual pixels; thus, a dark frame (also referred to as a ‘bias frame’) will have some pattern which characterizes this variation, so that it may be corrected in the analysis.

\textsuperscript{2}Primarily due to outgassing effects.

\textsuperscript{3}Of course, even with similar energy the moment of WIMPs depends on the WIMP mass, which is presumably very different from the neutron mass.

\textsuperscript{4}Although the uncertainty on this value is not known, it is not important in the context of our studies with the AmBe source. We only require that the rate of events is sufficient to produce a reasonable data set; the precise rate is irrelevant.
Figure 3.1: Example of a pixel cluster which is characteristic of a nuclear recoil event. The yellow outline shows the exact boundary of the cluster as identified by the analysis software. The white arrow shows the reconstructed 2D vector-direction of the recoil. Since the energy depositions are greatest in the bottom-left region of the track, the direction is most likely described by $(+X, +Y)$ as shown; however, there is some probability that this conclusion is wrong, and the true direction is antiparallel to the white arrow $(-X, -Y)$. This is an issue known as head-tail uncertainty, and it is discussed in Chapters 4 and 5. Adopted from ref. [11].

- Source free (SF) data. With no external radiation, we study only the events caused by background sources such as cosmic ray showers and natural radioactivity. These data were taken from the fall of 2013 through the spring of 2014.

### 3.1 Event Analysis and Discrimination

A number of cuts are applied to each data set in order to identify and isolate the most probable nuclear recoil events. Table 3.1 lists the number of events which pass each cut, in the order outlined below. The scripts which perform these cuts were written primarily by Cosmin Deaconu, Shawn Henderson, and Jeremy Lopez, all former graduate students of the DMTPC collaboration. The cuts are discussed in some detail in Section 5.3 of ref. [11]. I will not discuss them extensively, but the key features of the cuts are intuitive:
1) The frame must contain a track with nonzero reconstructed energy, range, and number of pixels as identified by the previously mentioned cluster-finding algorithm. Most frames captured by the CCDs will observe only noise, as recoil candidates are much less frequent than the rate of frame captures.

2) The track cannot be within 60 pixels (9.624mm) of the field cage rings or within 20 pixels (3.208mm) of the edge of the CCD field of view. This would make the reconstruction process much less reliable.

3) The track cannot be temporally near a discharge, as this triggers a very strong response from the CCDs (see Figure 3.3) which takes $\sim 10$ seconds to fully revert.

Both the AmBe and the SF data sets were reduced using these cuts prior to analysis of the reconstruction parameters, which is discussed in Chapter 5.

Figure 3.2: Diagram of the AmBe source setup used for data collection with the 4-shooter. The lead brick is used to block $\gamma$ radiation, which is produced in addition to an $\alpha$ particle in the decay of $^{241}$Am; this unfortunately introduces some uncertainty to the neutron energy spectrum as well, which we choose to neglect. Adopted from ref. [11].
Figure 3.3: (Left) A composite of the four CCD images captured during a discharge over the amplification region. The origin of the discharge can be seen near the coordinates (400, -300); the location of the field cage rings is also shown by the large circle centered near (0, 0). (Right) A series of close-up images immediately following the same spark, in 1-second intervals moving left-to-right and down the rows. A residual bulk image (RBI) persists for at least $\approx 5$ seconds after the discharge. Adopted from ref. [11].

<table>
<thead>
<tr>
<th>Cut</th>
<th>Images passed (AmBe)</th>
<th>Images passed (SF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total frames</td>
<td>1,864,000</td>
<td>13,708,000</td>
</tr>
<tr>
<td>1)  Recoil tracks</td>
<td>87,882</td>
<td>30,649</td>
</tr>
<tr>
<td>2)  Spatial fiducialization</td>
<td>19,968</td>
<td>6,990</td>
</tr>
<tr>
<td>3)  Temporal fiducialization</td>
<td>13,841</td>
<td>4,828</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>13,841 (0.7%)</strong></td>
<td><strong>4,828 (0.04%)</strong></td>
</tr>
</tbody>
</table>

Table 3.1: Reduction of AmBe and source free (SF) data sets after sequentially applying cuts AMBE-I through AMBE-VIII as described in ref. [11]. These eight cuts are condensed into the descriptions of items 1, 2, and 3 above. Additional cuts AMBE-IX through AMBE-XIII were not included, as they are application-specific.
Chapter 4

Isotropy Rejection

A WIMP-induced signal in a set of recoil tracks is expected to manifest itself as a directional preference. Tracks produced by WIMP interactions will tend to align with the dark matter wind, while background events will be isotropic. Thus, the total set of vectors (WIMP + background) will contain a small deviation from an isotropic set. To detect such a deviation, we must first understand quantitatively the expectations associated with an isotropic distribution. A deviation from these expectations which is both statistically significant and consistent with the dark matter wind would be characteristic of a WIMP-induced signal.

4.1 The Rayleigh Statistic

The direction of a DMTPC recoil track can be completely described by an angle $\theta$, as the tracks are imaged in two dimensions. Consider a set of $n$ angles $\{\theta_1 \ldots \theta_n\}$ which are independently chosen from an isotropic distribution:

$$p(\theta) = \frac{1}{2\pi}; \theta \in [0, 2\pi)$$

Where $p(\theta)$ describes the probability density of $\theta$. Each $\theta_i$ can be associated with a vector in $\mathbb{R}^2$ with unit length. If these vectors have a strong preferred direction, then their sum should point in that direction; in the case of an isotropic set, we might expect the vector sum to point anywhere and have a relatively small magnitude. This is the commonly used measure of isotropy called the Rayleigh statistic: if the sum of a set of vectors has a magnitude much greater than would be expected for an isotropic distribution, the null hypothesis is rejected and the statistic suggests a preferred direction. To compute the Rayleigh statistic, it is useful to work with the Cartesian coordinates
Using the theory of functions of a random variable, we can then find the
density functions associated with \( x \) and \( y \). I will first explore the mathematic-
ics here, as it is a worthy aside for this chapter, with an approach known
colloquially as the “\( \delta \)-function method”\(^1\).

In general, we aim to find the distribution of a variable \( z = f(\eta) \) which
depends on another variable \( \eta \) governed by the density function \( p(\eta) \).\(^2\) The
density \( p'(z) \) at a particular point \( z = z_0 \) can be interpreted as an integral over
the entire domain of \( \eta \); and at one point, or perhaps many, we integrate over
values of \( \eta \) which give \( f(\eta) = z_0 \). Then, we can generalize \( z_0 \) to all values of \( z \)
and write:

\[
p'(z) = \int_{-\infty}^{\infty} d\eta \ p(\eta) \delta(z - f(\eta)) \quad (4.4)
\]

The delta function in Equation 4.4 will be nonzero only when its argument
is zero, i.e. \( z = f(\eta) \). To express this in terms of the integration variable \( \eta \),
we write:

\[
\eta = f^{-1}(z)
\]

And thus, for any particular \( \eta_0 \) which satisfies Equation 4.5, we get a
nonzero contribution to the integral from the \( \delta \)-function:

\[
\int_{\eta_0,-}^{\eta_0,+} d\eta \ p(\eta) \delta(z - f(\eta)) = \frac{p(\eta_0)}{|\frac{df}{d\eta}|_{\eta=\eta_0}} \quad (4.6)
\]

Equation 4.4 can be evaluated as a sum over all possible values \( \eta_j \) which
satisfy Equation 4.5:

\[
p'(z) = \sum_j \frac{p(\eta_j)}{|\frac{df}{d\eta}|_{\eta=\eta_j}} \quad \forall \ j : \eta_j = f^{-1}(z) \quad (4.7)
\]

\(^1\)The \( \delta \)-function method is an approach which was taught to me by Bob Jaffe [22, 23]; I
have not seen it referenced, at least under a similar name, in literature.

\(^2\)In principle, we might encounter a case in which \( z \) depends on more than just one
variable; the same method discussed in this section is applicable to this situation equally
well by simply extending Equation 4.4 to a multi-dimensional integral, as in Equation 4.18.
We can now apply Equation 4.7 to the expression for \( x \) in Equation 4.2. We first find the values of \( \theta_j \):

\[
x = \cos \theta
\]

\[
\Rightarrow \theta_j = \begin{cases} 
\arccos x & j = 1 \\
2\pi - \arccos x & j = 2 
\end{cases}
\]

We have two solutions for \( \theta_j \) because the inverse cosine is defined only from 0 to \( \pi \). Next we find \( \partial f/\partial \theta \):

\[
\frac{\partial}{\partial \theta} (\cos \theta) = -\sin \theta
\]

With this we can substitute into Equation 4.7, noting that \( p(\theta_j) = 1/2\pi \) independent of \( \theta_j \):

\[
p(x) = \frac{1/2\pi}{|\sin (\arccos x)|} + \frac{1/2\pi}{|\sin (2\pi - \arccos x)|}
\]

The denominators are equivalent since \( \sin (\eta) = -\sin (2\pi - \eta) \). We combine the terms to get:

\[
p(x) = \frac{1}{\pi \sqrt{1 - x^2}}
\]

A similar calculation for \( y \) yields the same result:

\[
p(y) = \frac{1}{\pi \sqrt{1 - y^2}}
\]

The density function of the vector sum is now straightforward to calculate from the Central Limit Theorem, noting that \( \langle x \rangle = 0 \) and \( \langle x^2 \rangle = 1/2 \) (and the same for \( y \)):

\[
p(X) = \frac{1}{\sqrt{\pi n}} e^{-\frac{x^2}{n}} \text{ where } X \equiv \sum_i x_i
\]

\[
p(Y) = \frac{1}{\sqrt{\pi n}} e^{-\frac{y^2}{n}} \text{ where } Y \equiv \sum_i y_i
\]

Where \( n \) is the number of data points, or more specifically, the number of events. The Rayleigh statistic \( R \) is defined as:

\[
R \equiv \frac{1}{n} (X^2 + Y^2) \text{ for } R \geq 0
\]
Using the same framework of Equation 4.4 we can find \( p(R) \) with the assumption that \( X \) and \( Y \) are independent:

\[
p(R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dX \, dY}{n \pi} e^{-\frac{x^2+y^2}{n}} \delta \left( R - \frac{1}{n} (X^2 + Y^2) \right) \quad (4.18)
\]

\[
= \frac{1}{\pi} e^{-R} \int_{-\sqrt{nR}}^{\sqrt{nR}} \frac{dX}{\sqrt{nR - X^2}} \quad (4.19)
\]

\[
= e^{-R} \quad (4.20)
\]

Thus, for an isotropic distribution, the Rayleigh statistic has a very simple exponential distribution with expectation value 1 and the probability for \( R \) to be much greater than 1 is very small. Contrarily, a distribution with a direction preference will have \( \langle R \rangle \propto n \); this way, as \( n \) grows sufficiently large we have \( R \gg 1 \) which rejects the hypothesis that the initial distribution is isotropic (with some confidence depending on \( R \) of course).\(^3\)

### 4.2 Revisions of the Rayleigh Test for Directional Sensitivity

There is a weakness of the Rayleigh test which is perhaps not obvious, but very important in the context of WIMP directional statistics. Consider a distribution in \( \theta \) which is antipodally symmetric - that is, there exists a preferred direction \( \theta_0 \) and an equally preferred direction \( \theta_0 + \pi \) (an illustrated example is shown in Figure 4.1). In this case, the vectors near \( \theta_0 \) will cancel those at \( \theta_0 + \pi \) in the sum, leaving \( R \) with a small magnitude as if the anisotropies did not exist. Symmetries like this effectively avoid detection with the Rayleigh test. This is especially troubling because the DMTPC recoil reconstruction comes with some uncertainty on the head-tail parity; it is often very difficult to distinguish between a recoil with direction \( \theta_0 \) and another in the opposite direction \( \theta_0 + \pi \). As an example, return to Figure 3.1 and note the possibility that the true recoil direction is anti-parallel to the white arrow. If a recoil is reconstructed with angle \( \theta_0 \) and head-tail confidence \( p \) (where \( p \) is the calculated probability that \( \theta_0 \) is correct, \( 0.5 \leq p \leq 1 \)), then it is natural to describe

\(^3\)If there exists a preferred direction, then the average vector \((\langle x \rangle, \langle y \rangle)\) will be nonzero. Since the Rayleigh statistic is based on the total vector sum, it is intuitive that \( \langle R \rangle \propto n \). This can be seen more concretely from the definition in Equation 4.17; clearly, if \( X \) and \( Y \) scale with \( n \), then so does \( R \).
this data point as two vectors in the set: one with angle $\theta_0$ and length $p$, and another with angle $\theta_0 + \pi$ and length $1 - p$. Using the standard Rayleigh test, these would reduce to one vector with length $2p - 1$ which is strictly less than 1, and the Rayleigh statistic would tend to be small even in the presence of a strong anisotropy. There are two issues to overcome:

- The analysis leading to Equation 4.20 does not account for a variable vector length, which should be included to describe head-tail confidence.
- The Rayleigh test is extremely vulnerable to antipodal symmetries, which are a direct consequence of head-tail uncertainty.

Figure 4.1: Illustration of a set of vectors with antipodal symmetry. A vector in the first quadrant (shown by the blue arrow) will cancel a vector in the third quadrant (red), leaving the total vector sum very near zero. The Rayleigh test would conclude from this that the distribution is isotropic, which is clearly not true.

The first item has a straightforward resolution. The only necessary change is the application of the Central Limit Theorem in Equations 4.15 and 4.16, which depends only on the mean and variance of $x$ and $y$. By introducing the vector length $r$ which we assume is independent of $\theta$, we can compute the new mean and variance:

$$
\langle x \rangle = \langle r \cos \theta \rangle = \langle r \rangle \langle \cos \theta \rangle = 0
$$

(4.21)

$$
\langle x^2 \rangle = \langle r^2 \cos^2 \theta \rangle = \langle r^2 \rangle \langle \cos^2 \theta \rangle = \frac{1}{2} \langle r^2 \rangle
$$

(4.22)
Then the Central Limit Theorem takes $\langle x^2 \rangle \rightarrow n\langle x^2 \rangle$ so the variance is simply $\frac{1}{2}n\langle r^2 \rangle$. We can redefine the Rayleigh statistic as:

$$R \equiv \frac{1}{n\langle r^2 \rangle}(X^2 + Y^2)$$

(4.23)

This way, regardless of how the head-tail confidence is distributed (so long as it is independent of $\theta$), the Rayleigh statistic still gives an expectation value of 1 for an isotropic distribution. However, we still have the problem of antipodal symmetry. The root of the problem is illuminated by Equations 4.21 and 4.22: the Central Limit Theorem gives $p(X)$ a Gaussian distribution with zero mean and nonzero variance. With an antipodally symmetric preference, the mean of both $x$ and $y$ remains zero while the variance changes slightly, depending on the magnitude of the anisotropy. But the Rayleigh test is very sensitive to changes in the mean, as this drifts the distribution of $X$ and $Y$ away from the origin and increases $R$ much more definitively than a change in variance.

Suppose the test were built on some other quantities $\tilde{x}$ and $\tilde{y}$ such that the means $\langle \tilde{x} \rangle$ and $\langle \tilde{y} \rangle$ do in fact shift in response to a change in the variance of our original coordinates $x$ and $y$. A simple guess, requiring only that the range of $\tilde{x}/r$ extends from 0 to 1 might be:

$$\frac{\tilde{x}}{r} = 2 \left( \frac{x^2}{r^2} \right) - 1$$

(4.24)

$$\Rightarrow \tilde{x} = \frac{2x^2}{r} - r$$

(4.25)

This way, it is clear that $\langle \tilde{x} \rangle$ will be correlated with $\langle x^2 \rangle$. A similar transformation in $y$ might seem intuitive, but it is actually destructive; with the condition that $x^2 + y^2 = r^2$, we find:

$$\tilde{x} + \tilde{y} = \frac{2x^2 + 2y^2}{r} - 2r = 2r - 2r = 0$$

(4.26)

Thus, all points $(\tilde{x}, \tilde{y})$ would lie on the line $\tilde{x} = -\tilde{y}$ and the Rayleigh test becomes utterly useless. Instead, we can transform $y$ by insisting that the length of the vector is preserved,\textsuperscript{4} i.e. $\tilde{x}^2 + \tilde{y}^2 = r^2$:

$$\tilde{y} = \pm \sqrt{r^2 - \tilde{x}^2}$$

(4.27)

\textsuperscript{4}This is an important condition which must be enforced. The length of the vector represents the head-tail confidence, which is certainly invariant under an arbitrary change of coordinates.
Here we run into a predicament where \( \hat{y} \) can assume either a positive or negative value; unfortunately there is no easy correction to this. When applying these methods in our analysis, we choose to randomly assign \( \hat{y} \) to either the positive or negative square-root with equal probability. This removes some sensitivity of the test as it is then necessarily true that \( \langle \hat{y} \rangle = 0 \), but the transformation to \( \tilde{x} \) still gives the test sensitivity to changes in \( \langle x^2 \rangle \). This immediately implies an equal sensitivity to changes in \( \langle y^2 \rangle \) since it must still be true that \( \langle x^2 \rangle + \langle y^2 \rangle = \langle r^2 \rangle \).

Now the question becomes: how does the Rayleigh test operate on \( \tilde{x} \) and \( \tilde{y} \)? Once again we use functions of a random variable to find \( p(\tilde{x}) \). For simplicity, I will work with \( \tilde{x}/r \equiv \tilde{x}_r \) and \( x/r \equiv x_r \) as this is consistent with the initial definitions in Equations 4.2 and 4.3.

\[
p(\tilde{x}_r) = \int_{-\infty}^{\infty} dx_r \frac{1}{\pi \sqrt{1 - x_r^2}} \delta(\tilde{x}_r - (2x_r^2 - 1))
\]

\[
= \frac{1}{\pi \sqrt{1 - \tilde{x}_r^2}} \frac{1}{4 \sqrt{\tilde{x}_r^2 + 1}} \times 2 \tag{4.29}
\]

\[
= \frac{1}{\pi \sqrt{(1 - \tilde{x}_r)(1 + \tilde{x}_r)}} \tag{4.30}
\]

\[
= \frac{1}{\pi \sqrt{1 - \tilde{x}_r^2}} \tag{4.31}
\]

Which is exactly the same distribution as \( p(x_r) \) in Equation 4.13. It is perhaps no surprise that \( \tilde{y}_r \) also has a distribution identical to \( y_r \). Then, the Rayleigh test acting on the transformed coordinates \( (\tilde{x}, \tilde{y}) \) behaves exactly the same as on the original coordinates. We expect the same exponential distribution from the new Rayleigh statistic:

\[
\tilde{R} \equiv \frac{1}{n\langle r^2 \rangle} (\tilde{X}^2 + \tilde{Y}^2) \quad \tag{4.32}
\]

\[
p(\tilde{R}) = e^{-\tilde{R}} \quad \tag{4.33}
\]

This transformed Rayleigh statistic addresses both of the issues presented at the start of this section. However, it introduces one new problem: the statistic is not rotationally invariant. Any statistic which comments on the isotropy of a set of points should remain unchanged if the points are uniformly rotated by some angle \( \delta \theta \). To resolve this, we consider the axial direction of the vectors - i.e. the direction if all vectors are treated as axes without head-tail parity - and rotate the points such that this axis coincides with the \( x \)-axis. To
remove head-tail parity, we aim to treat vectors \( \theta_0 \) and \( \theta_0 + \pi \) as equivalent; we simply multiply each angle by a factor of 2 to accomplish this:

\[
\theta'_i \equiv 2\theta_i
\]  

(4.34)

This way, if \( \theta_i = \theta_j + \pi \) then \( \theta'_i = \theta'_j \) (modulo 2\( \pi \)). By summing the \( x \) and \( y \) coordinates of \( \theta'_i \), we obtain a vector which is invariant under the head-tail parity of any \( \theta_i \). Then, this axial direction can be expressed as an angle relative to the \( x \)-axis:

\[
\varphi \equiv \arctan \left( \frac{\sum_i \sin (\theta'_i)}{\sum_i \cos (\theta'_i)} \right)
\]  

(4.35)

To resolve the issue of rotational symmetry, we can rotate each original vector \( \theta_i \) by \(-\varphi\). This rotation obviously preserves any directional preference that may exist, and the end result - the set of rotated vectors \( \{\theta_i - \varphi \; \forall \; i\} \) - is rotationally invariant as any rotation \( \theta_i \rightarrow \theta_i + \delta \theta \) would produce the same rotation \( \varphi \rightarrow \varphi + \delta \varphi \) and cancel out.

The transformed Rayleigh statistic in Equation 4.32 overcomes the issue of antipodal symmetry in \( x \) and \( y \), but is still vulnerable to the exact same symmetry in the transformed coordinates \( \hat{x} \) and \( \hat{y} \). By combining the two sets of coordinates into one statistic, we can avoid antipodal symmetries in both. Recall that the distributions of \( X \) and \( Y \) are Gaussian; then, the distributions of \( X + \tilde{X} \) and \( Y + \tilde{Y} \) are also Gaussian\(^5\) and we can define a “hybrid” Rayleigh statistic which incorporates both:

\[
R_{hyb} \equiv \frac{1}{2n\langle r^2 \rangle} \left( (X + \tilde{X})^2 + (Y + \tilde{Y})^2 \right)
\]  

(4.36)

The factor of 2 in the denominator arises from the increased variance when adding \( X + \tilde{X} \) and \( Y + \tilde{Y} \). For the isotropy analysis in Chapter 5, I use a modification of this statistic described by Equation A4 of Morgan and Green [24]:

\[
R_{hyb}^* = \left( 1 - \frac{1}{2n} \right) R_{hyb} + \frac{(R_{hyb})^2}{8n}
\]  

(4.37)

This reduces the error caused by the Central Limit Theorem from \( \mathcal{O}(1) \) to \( \mathcal{O}(n^{-1}) \) [25, 26]. In the following chapter, I use this statistic to examine how the TPC pressure affects the quantity of data needed to reject isotropy; ideally, the hybrid statistic should completely circumvent the problem of head-tail uncertainty.

\(^5\)The convolution of two identical Gaussians is another Gaussian with twice the variance.
Chapter 5

Pressure and Rejection Power

The hybrid Rayleigh test described in Chapter 4 establishes a powerful way to reject the isotropy of a distribution. Next, we must consider the ways in which different parameters of the reconstructed track affect this rejection power, and how the TPC pressure affects these reconstruction parameters. From this, we can describe the indirect effects of pressure on rejection power, and use the results to find a pressure which ultimately yields the best possible rejection power.

The probability for an isotropic distribution to give \( R_{hyb}^* \geq R_0 \) is:

\[
P(R_0) = \int_{R_0}^{\infty} dR \ p(R) \tag{5.1}
\]

\[
= -(e^{-\infty} - e^{-R_0}) \tag{5.2}
\]

\[
= e^{-R_0} \tag{5.3}
\]

Then, if an unknown distribution \( p(\theta) \) gives \( R_{hyb}^* = R_0 \), we can use \( e^{-R_0} \) to describe the likelihood that \( p(\theta) \) is isotropic.\(^1\) For example, a likelihood of 0.1 would imply a rejection of isotropy with 90% confidence; 0.01 would imply 99% confidence. This likelihood is the statistic that I will use to quantify the effect of several variables on rejection power.

\(^1\)I am glossing over a caveat that the true probability of an isotropic distribution is not equal to \( e^{-R_0} \). This is a simple consequence of Bayesian probability: \( p(R_0|Q) \neq p(Q|R_0) \). However, it is of course impossible to truly calculate the likelihood that a set of angles was generated from an isotropic distribution \( p(\theta) = 1/2\pi \). Consequently, it is common practice in directional statistics to use the likelihood calculated in Equation 5.3 as a quantitative measure of the rejection power.

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5.1 Rejection Power with Head-Tail Uncertainty

As discussed in the previous chapter, head-tail uncertainty is an inherent artifact of track reconstruction, and a serious obstacle for the traditional Rayleigh test. However, the hybrid test which I derived should in theory retain nearly all of the rejection power of the Rayleigh test and also show little sensitivity to head-tail uncertainty. To test this, we produce simulated data and compute the likelihood of isotropy (Equation 5.3) using both the standard and hybrid Rayleigh tests. The simulated data consist of \( n \) 2D vectors \((r, \theta)\) governed by the distributions:

\[
p(\theta) = 1 + \alpha \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{\sigma^2}{2\sigma^2}}; \quad -\pi \leq \theta < \pi
\]

\[
p(r) = e^{-20(r-0.5)} + \beta e^{20(r-1)}; \quad 0.5 \leq r \leq 1
\]

Both distributions are automatically normalized in the simulation. The anisotropy in \( \theta \) is expressed as a Gaussian peak with width \( \sigma \) and signal-to-noise ratio \( \alpha \). The head-tail confidence \( r \) is effectively given by the sum of two distributions: one with \( r \) very near 0.5 and another (with relative probability \( \beta \)) with \( r \) very near 1. The motivation for this comes from the track-finding algorithm, which tends to assign confidence values either very near 0.5 (maximally uncertain) or 1 (certain). Figure 5.1 shows \( p(r) \) for various values of \( \beta \), compared with the actual distribution of head-tail confidence from the combined AmBe + SF data set. All of the quantities \( \alpha, \beta, \sigma \), and the number of events \( n \) are free parameters in the simulation.

Figure 5.2 shows the likelihood statistic as a function of \( n \) and the average head-tail confidence \( \langle r \rangle \) using the standard (left) and hybrid (right) Rayleigh tests. The hybrid test removes nearly all dependence on head-tail confidence, as we expect. In addition, the threshold \( n \) for 90% and 95% rejection power with \( S/N \) ratio \( \alpha = 5 \) is \( \mathcal{O}(10) \) events with the hybrid test; this result is consistent with the threshold using the standard test with no head-tail uncertainty [24, 27]. From this, we can say with confidence that head-tail uncertainty does not significantly weaken the hybrid Rayleigh test, and therefore is not a major issue to consider as a consequence of the pressure.
Figure 5.1: (Left) Visual representation of $p(r)$ in Equation 5.5. The parameter $\beta$ dictates the fraction of simulated data points with head-tail confidence $r$ very near 1. This structure is motivated in part by the actual distribution (right), which clearly shows tendencies towards $r \approx 0.5$ and $r \approx 1$.

Figure 5.2: Likelihood statistic as a function of the average head-tail confidence $\langle r \rangle$ and the number of events $n$ using the standard Rayleigh test (left) and the hybrid test (right). The hybrid test appears to almost completely overcome the problem of head-tail confidence, with only a slight sacrifice in overall rejection power. For these plots, we fix $\alpha = 5$ and $\sigma = 0.2$ in Equation 5.4.

5.2 Rejection Power with Directional Uncertainty

Head-tail uncertainty is a large obstacle which is overcome by the hybrid Rayleigh statistic. However, another large obstacle which we have neglected thus far is uncertainty in the actual directions $\theta_i$. The Rayleigh test assumes implicitly that these angles are exactly known, but in practice they are derived from track reconstruction, which includes an uncertainty $\sigma_{\theta_i}$. In the 4-shooter data sets, we generally have $\sigma_{\theta_i} \sim 0.1$ radians. This creates an uncertainty which modifies (slightly weakens) the rejection power.
To find the correlation between uncertainty in the angles $\theta_i$ and the Rayleigh statistic $R$, we use the standard error propagation formula:

$$z = f(x_1, x_2, x_3 \ldots x_j)$$

$$\sigma_z^2 = \sum_{i=1}^{j} \left( \frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2$$

(5.6)  
(5.7)

From this, it is straightforward to find the uncertainties in $x$ and $y$ if we assume that $r$ has no uncertainty:

$$\sigma_x = r |\sin \theta| \sigma_\theta$$

$$\sigma_y = r |\cos \theta| \sigma_\theta$$

(5.8)  
(5.9)

We use the absolute value to ensure that $\sigma$ is strictly positive, i.e. $+\sqrt{\sigma^2}$ rather than $-\sqrt{\sigma^2}$. The Rayleigh statistic is defined in terms of the sum of every $x_i$ and $y_i$. If we assume them to be independent (that is, $x_i$ and $x_j$ have no correlation for $i \neq j$), then the uncertainty on $X$ is simply the sum of each $x_i$ in quadrature:

$$\sigma_X = \left[ \sum_{i=1}^{n} \sigma_{x_i}^2 \right]^{1/2}$$

$$= \left[ n \langle \sigma_{x}^2 \rangle \right]^{1/2}$$

$$= \left[ n \langle r^2 \rangle \langle \sin^2 \theta \rangle \langle \sigma_\theta^2 \rangle \right]^{1/2}$$

(5.10)  
(5.11)  
(5.12)

A similar calculation gives the uncertainty in $Y$:

$$\sigma_Y = \left[ n \langle r^2 \rangle \langle \cos^2 \theta \rangle \langle \sigma_\theta^2 \rangle \right]^{1/2}$$

(5.13)

Since we are once again working in the context of an isotropic distribution, $\langle \sin^2 \theta \rangle = \langle \cos^2 \theta \rangle = 1/2$. Then, we can find the resulting uncertainty in the Rayleigh statistic, $\sigma_R$. For simplicity, we first consider the traditional statistic (Equation 4.17) and later adjust the results for the hybrid statistic.

$$R = \frac{1}{n \langle r^2 \rangle} (X^2 + Y^2)$$

$$\sigma_R^2 = \frac{1}{n \langle r^2 \rangle} \left[ 4X^2 \sigma_X^2 + 4Y^2 \sigma_Y^2 \right]$$

$$= R \langle r^2 \rangle \langle \sigma_\theta^2 \rangle$$

(5.14)  
(5.15)  
(5.16)

\footnote{A valid assumption, since $r$ says nothing about the actual direction of the vector. It only represents the head-tail confidence.}
Finally, we can make two further corrections: first, the dependence on $n$ is not explicit from Equation 5.16. Indeed, with a perfectly isotropic distribution, we do not expect $\sigma_R$ to vary with the number of events, as $R$ will remain $O(1)$ for all $n$. However, if we work practically with a nonisotropic distribution, we expect $R$ to vary linearly with $n$ (a behavior which was discussed briefly in Section 4.1). Thus, we can write $R = \lambda n$ to bring out an explicit correlation between $\sigma_R$ and $n$. Of course, in practice the proportionality between $R$ and $n$ is not easily detectable due to the uncertainty in $R$. However, as $R$ grows much greater than 1 the linear relationship is more apparent (see Figure 5.3).

The second correction accounts for the hybrid statistic. It can be shown$^3$ that the uncertainty of $\bar{X}$ is twice that of $X$, i.e. $\sigma_{\bar{X}} = 2\sigma_X$ (and the same for $Y$). Since the hybrid test adds $X$ and $\bar{X}$ linearly, the uncertainties add in quadrature and this effectively scales $\sigma_R$ by a factor of $\sqrt{1^2 + 2^2} = \sqrt{5}$. After both of these corrections, we have:

$$\sigma_{R_{hyb}} \leq \sqrt{5\lambda n \langle r^2 \rangle \langle \sigma_{\bar{X}}^2 \rangle}$$

(5.17)

![Figure 5.3](image.png)

Figure 5.3: Typical example plots showing the evolution of the hybrid Rayleigh statistic with increasing event number $n$ for low (left) and high (right) signal-to-noise ratio. When $R = O(1)$ the behavior is chaotic; however, for large $n$ we see a behavior which suggests that $R \propto n$, as we expect.

To find the effect of $\sigma_R$ on the rejection power, we treat $R$ as a normally distributed random variable and find the expected value of the likelihood statistic $e^{-R}$. This treatment may seem misguided since we have already established that $p(R) = e^{-R}$ is very non-gaussian, but it is still justified to simply place a Gaussian error bar on $R$ as a consequence of directional uncertainty. The

$^3$I omit the calculations, but they are similar to those which I have already done for $\sigma_{X,Y}$. 

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exponential distribution is not derived from this directional uncertainty in any way. Still, introducing an uncertainty $\sigma_R$ becomes problematic if the Gaussian tail extends below $R = 0$, as we have established by definition that $R \geq 0$. I will assume that $\sigma_R \ll R$ so that the domain $R < 0$ can be ignored.

\[
\langle e^{-R} \rangle = \int_0^\infty d\rho \ e^{-R} \frac{1}{\sqrt{2\pi \sigma_R^2}} e^{-\frac{(\rho - R)^2}{2\sigma_R^2}}
\]

\[
\approx \exp \left[ -R + \frac{\sigma_R^2}{2} \right]
\]

\[
= \exp \left[ -R \left( 1 - \frac{5}{2} \langle r^2 \rangle \langle \sigma_\theta^2 \rangle \right) \right]
\]

Thus, the rejection power is given in terms of an effective Rayleigh statistic:

\[
R_{\text{eff}} = \lambda n (1 - \gamma)
\]

with $\gamma \equiv \frac{5}{2} \langle r^2 \rangle \langle \sigma_\theta^2 \rangle$

Interestingly, it would seem that if $\langle \sigma_\theta^2 \rangle$ is sufficiently large we have $\gamma \rightarrow 1$ and $R_{\text{eff}} \rightarrow 0$, implying that rejection is impossible. This is simply a consequence of the Gaussian uncertainty on $R$, which breaks down if $\sigma_R$ is of comparable magnitude to $R$. Still, it should come as no surprise that as $\sigma_\theta$ grows to a considerable fraction of $2\pi$, the rejection power becomes very small (and my assumption that $\sigma_R \ll R$ breaks down).

Equation 5.21 describes the rejection power with the following considerations: first, we have accounted for both head-tail uncertainty and directional uncertainty. It was determined in Section 5.1 that with the hybrid Rayleigh test, head-tail uncertainty does not appreciably affect the magnitude of $R$, but in Section 5.2 we have found that it does affect the uncertainty, as $\sigma_R$ has some dependence on $\langle r^2 \rangle$. Secondly, the effective statistic $R_{\text{eff}}$ depends only on the number of events $n$, the average directional uncertainty squared $\langle \sigma_\theta^2 \rangle$, the average head-tail confidence squared $\langle r^2 \rangle$, and the parameter $\lambda$ which is related to the signal-to-noise ratio. In the following section, I will study the relationship between TPC pressure and $\sigma_\theta^2$ in order to determine the rejection power as a function of pressure. I will leave $\lambda$ as a free parameter as it depends on the specific experimental setup (for example, $\lambda$ for the AmBe data set would be much greater than for the SF data set).
5.3 Pressure Optimization

5.3.1 Dependence on $n$

First, we can address the low-hanging fruit: the relationship between pressure $P$ and number of events $n$. At constant volume and temperature, the ideal gas law gives:

$$N = \frac{PV}{kT}$$  \hspace{1cm} (5.23)

Where $N$ is the number of particles, $P$ is the pressure of the TPC, $V$ is the fiducial volume of the chamber, $T$ is the temperature and $k$ is Boltzmann's constant. The number of events $n$ in a time $t$ is:

$$n = \sigma_{x}JNt$$  \hspace{1cm} (5.24)

Where $\sigma_{x}$ is the interaction cross section and $J$ is the incident flux. Combining these two equations, we find a constant proportionality between $n$ and $P$:

$$n = \frac{\sigma_{x}JtV}{kT}P$$ \hspace{1cm} (5.25)

5.3.2 A general model for $\langle \sigma_{0}^{2} \rangle$

Next, we consider the expectation values $\langle \sigma_{0}^{2} \rangle$ and $\langle r^{2} \rangle$. Each of these depends heavily on the recoil reconstruction process, which is subject to change over time and may vary wildly between different experiments similar to DMTPC. We postulate their dependence on certain aspects of the recoil tracks - for example, the physical length and width - in an effort to model their behavior with variable pressure. However, the 4-shooter has not collected appreciable data sets at varying pressure, so any such model will be based primarily on theory and a number of assumptions, rather than experimental results. Later in this section I will construct such a model using the 4-shooter data sets, but first we can consider the more general case in which $\langle \sigma_{0}^{2} \rangle$ and $\langle r^{2} \rangle$ remain arbitrary functions of the pressure $P$. For simplicity, we can reduce the number of free parameters by evaluating $\langle r^{2} \rangle$ for the combined (AmBe + SF) data set.\footnote{I choose to evaluate $\langle r^{2} \rangle$ because (a) it has already been discussed at some length, (b) it depends almost entirely on the reconstruction process; to model its behavior with pressure is a topic much too involved for this work, and (c) it has a well-defined range between 0.25 and 1, and is easy to adjust if needed.} Using the numerical result $\frac{5}{2}(r^{2}) = 1.135$, we rewrite Equation
5.21 as:
\[ R_{\text{eff}}(P) = \frac{\lambda \sigma_n J t V}{kT} P \left[ 1 - 1.135 \langle \sigma_\theta^2 \rangle(P) \right] \] (5.26)

For simplicity, we call the leading constant \( K \equiv \frac{\lambda \sigma_n J t V}{kT} \) and \( \eta \equiv 1.135 \):
\[ R_{\text{eff}}(P) = K P \left[ 1 - \eta \langle \sigma_\theta^2 \rangle(P) \right] \] (5.27)

This is the quantity we wish to maximize for the best possible rejection power (note that \( K \) becomes irrelevant). Figure 5.4 illustrates how \( R_{\text{eff}} \) varies with \( P \) and \( \langle \sigma_\theta^2 \rangle \); unsurprisingly, the rejection power grows with \( P \) (more events) and diminishes with \( \langle \sigma_\theta^2 \rangle \) (less directional certainty). To calculate an optimal pressure, we must construct a model which predicts \( \langle \sigma_\theta^2 \rangle \) as a function of \( P \) which can then be substituted into Equation 5.27. This can be understood visually with Figure 5.4; a model relating \( \langle \sigma_\theta^2 \rangle \) and \( P \) represents a 2-dimensional curve superimposed on the figure, and the point on the curve which maximizes \( R_{\text{eff}} \) corresponds to the optimal pressure. Some arbitrary examples, with maximum points indicated, are shown.

![Effective Rayleigh Statistic vs. \( \langle \sigma_\theta^2 \rangle \) vs. Pressure](image)

Figure 5.4: Behavior of \( R_{\text{eff}} \) with both \( P \) and \( \langle \sigma_\theta^2 \rangle \) as free parameters. The black lines represent models of \( \langle \sigma_\theta^2 \rangle \) vs. \( P \), with maximum \( R_{\text{eff}} \) indicated by the stars. These models are hand-drawn and purely hypothetical, motivated only by the idea that we expect \( \langle \sigma_\theta^2 \rangle \) to be positively correlated with pressure (an idea explored further in the remainder of this section).
5.3.3 A specific model for the combined 4-shooter data set

Finally, we consider a model to describe the relationship between $\langle \sigma^2_\theta \rangle$ and pressure. At the most fundamental level, directional uncertainty arises from a nonzero width of the track - that is, energy depositions which deviate from the overall trajectory of the track. A recoil track is the aggregation of thousands of collisions between nuclei, each of which scatter in a direction which may be very different from the single direction which the reconstruction algorithm assigns to the track. To account for this width, the spatial dimensions of the track can be described as a 2-dimensional box, as illustrated in Figure 5.5.

![Figure 5.5: Illustration of directional uncertainty $\sigma_\theta$ using a rectangular box. $\sigma_\theta$ represents the maximum deviation from $\theta$ that is contained within a box with length $l$ and width $w$, where $l$ and $w$ describe the dimensions of the recoil track.](image)

If a vector which points directly through the box is described by the angle $\theta$, then the angle $\theta + \sigma_\theta$ could describe a vector which connects opposite corners, i.e. the maximum deviation from $\theta$ which still extends to the entire length of the box. In terms of the dimensions of the box $w$ and $l$, we can write:

$$\sigma_\theta = \arctan \left( \frac{w}{l} \right)$$

(5.28)

We must also consider the event reconstruction process, which is complicated and unique to any particular experiment. Any number of factors including CCD resolution, electric field uniformity, CCD bias/noise, etc. can play an important role in the directional uncertainty. To account for this, we introduce the following modifications to Equation 5.28:
• A scaling factor $B$ inside the arctangent. The rectangular box is certainly a crude model, and the true directional uncertainty might not depend on exactly the ratio $w/l$, but some constant multiple of it.

• A scaling factor $A$ outside the arctangent. This serves primarily to account for other facets of the reconstruction process which scale with the arctangent described in Equation 5.28.

• An offset $C$ outside the arctangent. This accounts for other sources of dispersion/uncertainty in the reconstruction process which are independent of $w$ and $l$.

With these considerations, we obtain the following expression for $\sigma_\theta$ (including explicitly the dependence of $w$ and $l$ on the pressure $P$):

$$\sigma_\theta = A \arctan \left( B \frac{w(P)}{l(P)} \right) + C$$  \hspace{1cm} (5.29)

All of $\sigma_\theta$, $w$, and $l$ are quantities which are calculated in the reconstruction. Then, we can fit the combined data set to Equation 5.29 to obtain constants $A$, $B$, and $C$; the results are presented in Table 5.1 and illustrated in Figure 5.6.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Fit Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>2.39 ± 0.5</td>
</tr>
<tr>
<td>$B$</td>
<td>0.62 ± 0.13</td>
</tr>
<tr>
<td>$C$</td>
<td>-0.17 ± 0.01</td>
</tr>
</tbody>
</table>

Table 5.1: Fit results of Equation 5.29 using the combined AmBe + SF data set and a standard least-squares fit.

Next, we must model the dependence of $w$ and $l$ on $P$. Distributions of each are shown in Figure 5.7. Certainly, it stands to reason that the length of the track $l$ is closely related to the mean free path of particles in the gas, and therefore negatively correlated with pressure. One might expect that the width $w$ is also negatively correlated with pressure, perhaps in the same way as $l$, but I will instead assume that it is independent of the pressure for the following reasons:

1. The distributions of $w$ and $l$ are very different. If the two quantities were related to pressure in a similar way, we should expect similar distributions.
2. The correlation between \( w \) and \( l \) is minimal. A density map of the two is shown in Figure 5.8; the correlation coefficient evaluates to 0.35.

3. The distribution of \( w \) is fairly normal. This might suggest that it has an underlying distribution which is washed out by some Gaussian dispersion that is independent of other recoil track parameters. Contrarily, if the distribution of \( l \) is subject to a similar dispersion, it can likely maintain some non-Gaussian structure as the length scale of \( l \) is large in comparison to that of \( w \).

![Density Map of Directional Uncertainty vs. Width-Length Ratio](image)

Figure 5.6: Heatmap showing density of events vs. \( \sigma_\theta \) vs. track width-length ratio for the combined data set. A fit to Equation 5.29 is shown with the black line, with an envelope of 1 standard error enclosed by the grey lines.

![Distribution of Track Width](image) ![Distribution of Track Length](image)

Figure 5.7: Distributions of track width (left) and length (right) for the combined data set, with Gaussian and Gamma distribution fits, respectively. Both dimensions are quoted in CCD pixels.

Then, we can evaluate \( w \) with a simple Gaussian fit, and define the uncertainty \( \sigma_w \) to be the standard deviation of the fit:

\[
w(P) = 5.26 \pm 1.57
\]  (5.30)
Figure 5.8: Density map showing a negligible correlation between track length and width. Both quantities are quoted in CCD pixels, and once again we use the combined AmBe + SF data set.

To model the distribution of the length $l$, I choose a Gamma distribution:

$$p(l) = \frac{1}{\Gamma(\kappa)\xi^\kappa}l^{\kappa-1}e^{-\frac{l}{\xi}}$$

(5.31)

The motivation for this is not physical, but the Gamma distribution (a) is necessarily zero for $l = 0$, and (b) decays exponentially for large $l$. Both of these are characteristics which we hope to observe in a distribution of track length. In addition, the Gamma distribution fits the data quite well (see Figure 5.7); for the purposes of this model, specific to the 4-shooter data sets, this fit makes the choice of distribution satisfactory, even if we do not consider carefully its motivation. Since the distribution provides an accurate model of the data, we can trust its ability to predict the directional uncertainty given by Equation 5.29.

The parameter $\kappa$ dictates the shape of the Gamma distribution in the domain of $l$ where it is not dominated by the exponential decay, and $\xi$ dictates the scale of the exponential decay. To model the dependence of $l$ on pressure, we postulate the following:

1. The shape parameter $\kappa$ depends primarily on reconstruction, and not the pressure of the gas.

2. The mean value $\langle l \rangle$ is proportional to the mean free path of a particle in the gas.
The mean value of the Gamma distribution is $\langle l \rangle = \kappa \xi$. Furthermore, in an ideal gas the mean free path is inversely proportional to the pressure:

$$d = \frac{1}{\sigma n} = \frac{kT}{\sigma P} \quad (5.32)$$

Where $d$ is the mean free path, $n$ is the number of particles per unit volume, $\sigma$ is the effective cross section, $k$ is the Boltzmann constant, and $T$ is the temperature. Thus, with the above assumptions, we assert that $\kappa$ does not change with pressure and therefore write:

$$\xi = \frac{\langle l \rangle}{\kappa} \quad (5.33)$$

$$= \text{const} \times \frac{d}{\kappa} \quad (5.34)$$

$$= \text{const} \times \frac{kT}{\kappa \sigma P} \quad (5.35)$$

$$\Rightarrow \xi \propto \frac{1}{P} \quad (5.36)$$

Fitting Equation 5.31 to the combined data set, we obtain $\kappa = 5.89$ and $\xi = 3.35$. Figure 5.7 shows this fit superimposed on the data as well as the Gaussian fit for $p(w)$. Then, noting that the data set was collected at a pressure of 60 torr, we can establish the proportionality factor between $\xi$ and $1/P$:

$$\xi = 3.35 \times \frac{60 \text{ torr}}{P} \quad (5.37)$$

### 5.3.4 Numerical calculations and discussion

Finally, we have all the necessary parameters to calculate $\langle \sigma_\theta^2 \rangle$ from Equation 5.29. With numerical integration, we evaluate $\langle \sigma_\theta^2 \rangle$ for many values of $\xi$ to construct a two-dimensional curve in the space of $(\langle \sigma_\theta^2 \rangle, P)$. Figure 5.9 shows this curve superimposed on Figure 5.4, the map of $R_{\text{eff}}$.

A numerical calculation yields a maximum $R_{\text{eff}}$ at the point:

$$P = (80.4 \pm 16.9) \text{ torr} \quad (5.38)$$

$$\langle \sigma_\theta^2 \rangle = 0.27 \pm 0.13 \quad (5.39)$$

The large uncertainties in this calculation come primarily from the uncertainty in the scaling parameter, $A = 2.39 \pm 0.5$. This is likely due to the high variability of the reconstruction process, as was discussed earlier. The reconstruction process introduces significant dispersion in $\sigma_\theta$ (Figure 5.6 provides a good visualization of this dispersion and the resulting uncertainty on $A$).
Figure 5.9: Plot of $R_{\text{eff}}$ as a function of $P$ and $\langle \sigma_{\theta}^2 \rangle$, superimposed with a numerically computed curve relating $\langle \sigma_{\theta}^2 \rangle$ to $P$. The star marks the location of maximum $R_{\text{eff}}$, at a pressure of 80.4 torr.

which makes it very difficult to model effectively. In addition, the uncertainty is highly asymmetric, but this is not obvious from any equations or figures. It would seem that the numerical maximization of $R_{\text{eff}}$ varies nonlinearly with the scaling parameter $A$; consequently, the result is more sensitive to a decrease in $A$ - corresponding to better directional certainty and a higher optimal pressure - than it is to an increase in $A$.

There is one concession I must include which is not considered even in the general model (Equation 5.27): sparking across the amplification region. As was shortly discussed in Chapter 3, the light produced by a discharge across the amplification region will completely bury any recoil tracks for up to 10 seconds. These sparks are the motivation for the temporal fiducialization cut, which reduces the number of track candidates by roughly 30% according to Table 3.1. Thus, the number of events $n$ is perhaps more accurately described by:

$$n = \frac{K}{\lambda} P(1 - s)$$

(5.40)

Where $s$ describes the fraction of detector live time which is cut by the
temporal fiducialization. The behavior of $s$ with variable pressure is a question which is explored to some extent in ref. [11]; the analysis is only qualitative at this point, but it is clear that the sparking rate is negatively correlated with pressure.\textsuperscript{5} This implies that the optimal pressure could be slightly higher than suggested by Equation 5.38, to allow for greater live time. If the relationship between sparking rate and pressure can be modelled well, it could provide a valuable modification to the model presented in this chapter.

Still, with these concessions, the primary issue of reconstruction variability is largely accounted for by the uncertainties in Equations 5.38 and 5.39. The calculated pressure of 80.4 torr is certainly achievable with DMTPC detectors, and the same order of magnitude as the pressures which have been used since the project's inception. Furthermore, the general model presented in Section 5.3.2 is adaptable to changes in the reconstruction process, which will certainly evolve with future DMTPC experiments.

\textsuperscript{5}See Figure 5-4 of ref. [11] on page 151.
Chapter 6

Conclusions

The rejection of isotropy is a very important aspect of data analysis in directional dark matter direct detection. In DMTPC experiments, the primary obstacles to isotropy rejection are head-tail uncertainty and directional uncertainty; in this work, I have explored both of these aspects in the context of event reconstructions produced by DMTPC. I have developed a modification to the Rayleigh test which largely overcomes the problem of head-tail uncertainty without sacrificing overall rejection power. I have also studied the effects of directional uncertainty on rejection power, and constructed a general model which predicts the rejection power as a function of directional uncertainty and TPC pressure. The relationship between directional uncertainty and pressure is complicated, and subject to change as DMTPC experiments and data analysis techniques evolve. The general model discussed in Section 5.3.2 is adaptable to these types of changes, and even applicable to future detectors. A specific model which satisfactorily predicts directional uncertainty is difficult to achieve, in large part because of highly variable event reconstruction; however, with such a model, we can calculate an operating pressure which yields the best possible rejection power. Using such a model, I calculated the optimal pressure for the 4-shooter to be ($80.4^{+16.9}_{-9.6}$) torr, which is slightly higher than the current operating pressure of 60 torr. The problem of variable reconstruction is largely accounted for by the uncertainty in this result. I am hopeful that the research presented in this thesis will be a valuable contribution to current and future DMTPC experiments.
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