Preheating in New Higgs Inflation

Karla Guardado

Submitted to the Department of Physics in partial fulfillment of the requirements for the degree of

BACHELOR OF SCIENCE IN PHYSICS

AT THE

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

JUNE 2015

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Signature of Author:  

Department of Physics
May 8, 2015

Certified by:  

Department of Physics

Accepted by:  

Senior Thesis Coordinator
Department of Physics
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by

Karla Guardado

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Abstract

Cosmological inflation describes the phenomenon in the early universe when space-time underwent a rapid, exponential expansion right after the Big Bang. Inflation solves the so-called “horizon problem,” “flatness problem,” and “monopole problem” of standard Big Bang cosmology. Furthermore, New Inflation solves the “graceful exit problem” of the original theory. In inflation, the energy density of a patch of the early universe becomes dominated by the potential energy of a scalar field in a state of false vacuum. This particular form of energy leads to a negative pressure, creating a repulsive gravitational force, driving the region into a period of exponential expansion. Soon after the end of inflation, the field oscillates, leading to the creation of particles in a process called reheating. If reheating begins with parametric resonance, the process is called preheating. New Higgs Inflation presumes that the Higgs field is the scalar field in question, involving a characteristic non-minimal “derivative” coupling. The equation of motion for the field evolves like a damped harmonic oscillator, so we expect it to oscillate near the end of inflation. We study the dynamics of the Higgs field during and after inflation and find that preheating should be efficient in this model.

Thesis Supervisor: David Kaiser

Title: Director and Professor of Science Technology and Society, Senior Lecturer in Physics
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Acknowledgments

I would like to thank Alan Guth for giving me the opportunity to work with his team. I would especially like to thank my thesis supervisor, David Kaiser. I thank him for his patience, availability, and direction in guiding the progression of this thesis. I would also like to thank MLK post-doctoral fellow Chanda Hsu Prescod-Weinstein for her help and support throughout my years at MIT. She has been a true mentor to me. Finally, I extend my gratitude to the rest of the Density Perturbation Group. They have been a great team to work with. It has been a pleasure working with all of you.
Part I

Inflation

1 Overview of Inflation

Cosmological inflation describes the phenomenon in the early universe when spacetime underwent a rapid, exponential expansion right after the Big Bang. While the universe is still expanding today, it is doing so at a less accelerated rate. The inflationary period is speculated to have lasted from $10^{-36}$ to somewhere between $10^{-33}$ and $10^{-32}$ seconds after the Big Bang.

1.1 Reasons for Inflation

Inflation solves several problems of the original Big Bang model. Therefore, several motivations for the theory of inflation exist. Some of the more commonly discussed include, for example, the so-called “horizon problem,” the “flatness problem,” and the “monopole problem” of standard Big Bang cosmology.

1.1.1 The Horizon Problem

While it is clear from observable evidence that below certain length scales, structure formation has occurred, on the largest observable scale, our universe actually looks remarkably smooth. The best indication of this overall uniformity comes from the cosmic microwave background radiation (CMB); a picture of the CMB is shown in Fig. (1). The CMB is residual radiation that was released from a moment exceedingly early on in the history of the universe. That is, about 370,000 years after the Big Bang. This residual temperature pattern is uniform across all regions of the sky to about one part in $10^5$ [11, 12].
The horizon problem is the difficulty in explaining this large-scale uniformity of the observable universe. Of course, while any isolated system will evolve over time towards a state of thermal equilibrium, the standard processes do not suffice to explain the observed uniformity of temperature. In the standard Big Bang model, the universe evolves much too quickly to allow this uniformity to be achieved as per usual by the time the CMB was emitted. In order to understand this, we must remember that no physical process can cause matter, energy, or information to move faster than the speed of light. Thus, no process can carry energy beyond the horizon distance, $\chi$, the present distance of the furthest particles from which light has had time to reach us since the beginning of the universe. It turns out that $\chi = 3ct$ for a matter-dominated flat universe, where $c$ is the speed of light and $t$ here is the time in question. As it happens, we live in a remarkably flat universe (this will be further discussed in the next section) that became matter-dominated at about 50,000 years after the Big Bang. Therefore, the CMB radiation decoupled well after the universe became matter-dominated, so we can make the approximation

$$\chi = 3ct_d = 3c \times 370,000 \text{ yrs} = 1,100,000 \text{ ly}$$

(1)

where $t_d$ is the decoupling time of the CMB.
We can calculate the distance at the time of the CMB emission between the site of emission and our own galaxy. CMB emission occurred at a temperature of about 3,000 K; the current average temperature of the universe is about 2.7 K [11, 12]. Since $aT$ is constant as the universe expands, the redshift at the time of emission $t_d$ can be found thusly,

$$1 + z = \frac{a(t_0)}{a(t_d)} = \frac{3000 \, K}{2.7 \, K} \approx 1100$$

(2)

where $a(t)$ is known as the scale factor of the universe. For a matter-dominated flat universe, the present physical distance of an object seen at redshift $z$ is given by

$$x_0 = \frac{2cH_0^{-1}}{1 - \frac{1}{\sqrt{1 + z}}}.$$  

(3)

Using, $H_0 = 67.3 \, km - s^{-1} \, Mpc^{-1}$ (Hubble’s constant), it turns out that $x_0 \approx 2.82 \times 10^{10} \, ly$. In other words, the region of emission of the CMB that we are currently observing is a spherical shell of matter at essentially the present horizon distance. But physical distances vary with time as $a(t)$, so the physical radius of this shell of matter at the time of decoupling is given by

$$x_d = \frac{a(t_d)}{a(t_0)}x_0 \approx \frac{1}{1100} \times 2.82 \times 10^{10} \, ly \approx 2.56 \times 10^7 \, ly.$$  

(4)

Thus, at the time of emission of the cosmic background radiation, the region of emission was a spherical shell with a radius many times larger than the horizon distance. In fact, at the time of emission, the sources of two signals from opposite directions in the CMB were separated from each other by about 46 horizon distances [11, 12]. It is impossible for these two sources to have come into thermal equilibrium by any physical process. It seems then that the large-scale uniformity of the observable universe must be assumed to be a well fine-tuned initial condition.
1.1.2 The Flatness Problem

As described in the previous section, there are remarkably good empirical reasons to assume that the universe is both homogenous and isotropic, on average and at cosmic scales. The universe is homogenous because it appears uniform, i.e. there are no preferred locations in space; the universe is isotropic because there are also no preferred orientations. According to general relativity, a metric that incorporates these two symmetries is the Friedman-Robertson-Walker (FRW) spacetime. Its line element can be written as [11, 12]

\[
ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = -dt^2 + a^2(t) \left[ \frac{dr^2}{(1 - Kr^2)} + r^2(d\theta + \sin^2\theta d\phi^2) \right]
\]

where \( g_{\mu\nu} \) is the metric tensor, \( t \) is the cosmic time, and \( K \) is the spatial curvature constant that can have values of 0 or \( \pm 1 \). In these coordinates, the time component of Einstein’s field equations gives the Friedmann Equation

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2}
\]

where \( H \) is the Hubble expansion coefficient, \( G \) is the universal gravitational constant, and \( \rho \) is the energy density of the universe. From this equation, it is clear that the Hubble expansion coefficient is related to the scale factor like \( H = \frac{\dot{a}}{a} \). We also have the following equation for the acceleration of the scale factor [11, 12]:

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3\rho).
\]

Rearranging the Friedmann Eq. (6) gives Eq. (8)

\[
\frac{K}{a^2} = \frac{8\pi G}{3} \rho - H^2.
\]

From Eq. (8), there exists a critical value of the energy density, \( \rho_{cr} \), such that the spatial curvature term proportional to \( K \) vanishes. The expression for this is [11, 12]
We may also introduce a dimensionless parameter, $\Omega$, which is the ratio of the actual energy density of the universe to the critical value. We define $\Omega$ by the following:

$$\Omega = \frac{\rho}{\rho_{cr}}.$$  \hspace{1cm} (10)

A universe in which $\rho = \rho_{cr}$ is true must be spatially flat with $K = 0$. Therefore, $\Omega = 1$ corresponds to a spatially flat universe. Combining the Friedmann Equation with the definition for $\Omega$ and $\rho_{cr}$, we find that

$$\Omega = \frac{\rho}{\rho_{cr}} = \frac{1}{\rho_{cr} \frac{3}{8\pi G}} \left( H^2 + \frac{K}{a^2} \right) = 1 + \frac{K}{a^2 H^2}.$$ \hspace{1cm} (11)

And in a similar fashion, we can determine that

$$\frac{(\Omega - 1)}{\Omega} = \frac{3K}{8\pi G a^2 \rho}.$$ \hspace{1cm} (12)

For a universe filled with ordinary matter, the energy density should fall like the volume of space, $1/a^3(t)$. The energy density for radiation falls even faster like $1/a^4(t)$. Hence, from Eq. (12), a universe filled with either matter or radiation should deviate more and more from spatial flatness over time.

Herein lies the fundamental flatness problem. The above analysis suggests that a universe with the exact critical density required for spatial flatness is an unstable solution. As shown in Eq. (12), we expect the energy density to fall like $a(t)$ faster than $1/a^2(t)$, so the right hand side of Eq. (12) grows in time as the universe expands. An expanding universe should deviate away from $\Omega = 1$ over time, and any deviation from this critical density would have grown over time so that a universe that appears nearly flat today would have required fine-tuned initial conditions. Observations clearly indicate that today’s observable universe is
indeed flat to high accuracy. Furthermore, our flat universe has been experimentally verified to be extremely homogenous and isotropic to high accuracy as well. The universe could not have started out with even slightly different temperatures and compositions in different areas, even for regions that (according to the standard Big Bang model) had never been in causal contact with each other.

1.1.3 The Monopole Problem

The monopole problem was a major motivation for the initial formulation of inflation but is less critical for this thesis. A full understanding of the monopole problem requires familiarity with what physicists calls grand unified theories (GUTs), which I could not do justice in a few concise paragraphs.

Let it suffice to say that physics divides the fundamental interactions of nature into four classes: the strong\(^1\), the weak\(^2\), and electromagnetic interactions, and gravity. The Standard Model of particle physics describes the first three, but not gravity. It is what is known as a gauge theory because one can construct potentials that describe the same physics as the original field (this is a gauge transformation). A field theory can be constructed based on any gauge group \([9, 12]\). A field theory is a physical theory that describes how one or more physical fields interact with matter, such as electromagnetic field theory which is a familiar example. Therefore, people have been attempting to combine these gauge theories into GUTs that describe two or more of the four fundamental interactions to go beyond the Standard Model because it is lacking for several reasons. Gravity, however, has proven particularly tricky to incorporate.

Now consider an ordinary magnet. All magnets that we know for sure exist come in dipoles—that is, they have a both North and South pole. Even if you cut it in half, the cut end retains the same polarization. You might wonder why you have never come across such

\(^{1}\)The strong force binds quarks together inside protons, neutrons, etc. \(^{2}\)The weak force is responsible, for example, for beta decay (neutron → proton + electron + anti-electron-neutrino).
a thing as just a South or North facing magnet. According to GUTs, such a thing is actually possible. They are topological defects in the GUT Higgs fields. The monopole problem is that far too many monopoles are predicted to have been produced in the early universe in the context of conventional Big Bang cosmology, but if they should be so abundant, why is it that they have never been detected? [9, 12].

2 Inflation as the Solution

According to Alan Guth, the original inflationary universe was developed to solve the magnetic monopole problem. It then quickly became clear that the scenario would solve all three of the problems previously discussed. However, the original solution did have one major problem that came to be known as the “graceful exit problem.” The exponential expansion of inflation was terminated by a phase transition that lead to a grossly inhomogeneous universe obviously entirely unlike our own [10, 12]. It is now completely avoided by a variation known as the new inflationary universe. Its authors introduced a new mechanism by which the exponential expansion phase could be ended, putting an end to the graceful exit problem. Soon after “new inflation” was introduced, other variations became prolific—especially “chaotic inflation.” Now there exist many types of inflationary models that avoid the pitfall of Guth’s original formulation.

2.1 How Inflation Works

In many models of Inflation, the energy density of a patch of the early universe becomes dominated by the potential energy of a scalar field in a state of false vacuum. This particular form of energy leads to a negative pressure. The negative pressure creates a repulsive gravitational force, driving the region into a period of exponential expansion of inflation—hence

\(^3\) Developed independently by Andrei Linde and by Andreas Albrecht and Paul Steinhardt.

\(^4\) Proposed by Andre Linde, in chaotic inflation, inflation eternally inflates—producing an infinite multiverse.
the name of the theory [10, 12].

2.1.1 Equation of Motion for the Scalar Field

Let us look at the action for a scalar field $\phi$ that is approximately homogenous over some region of space which takes the form

$$ S = \int d^4x \sqrt{-g(x)} \left[ \frac{1}{16\pi G} R(x) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], $$

where $g(x)$ is the determinant of the metric tensor, $R(x)$ is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, and $V(\phi)$ is the potential. We can introduce the variation $\phi \to \phi + \delta \phi$ and consider how the action varies for arbitrary $\delta \phi$. We want to find conditions satisfied by the field and its derivative such that the action remains stationary [13, 3]. We can also view the action as being composed of gravity and matter-dominated Lagrangians, $\mathcal{L}^{(G)}$ and $\mathcal{L}^{(M)}$, thusly,

$$ S = \int d^4x \sqrt{-g} \left[ \mathcal{L}^{(G)} + \mathcal{L}^{(M)} \right] $$

and since $\mathcal{L}^{(G)}$ is independent of $\phi$, we can focus on the behavior of $\mathcal{L}^{(M)} = \mathcal{L}^{(M)}(\phi, \partial_\mu \phi)$ solely.

The variation of the action upon this condition is therefore

$$ \delta S = \int d^4x \sqrt{-g} \left[ \left( \frac{\partial \mathcal{L}^{(M)}}{\partial \phi} \right) \delta \phi + \left( \frac{\partial \mathcal{L}^{(M)}}{\partial (\partial_\mu \phi)} \right) \partial_\mu (\delta \phi) \right]. $$

Let us integrate the second term by parts.

$$ \delta S = \int d^4x \left[ \left( \frac{\partial (\sqrt{-g} \mathcal{L}^{(M)})}{\partial \phi} \right) \delta \phi - \partial_\mu \left( \frac{\partial (\sqrt{-g} \mathcal{L}^{(M)})}{\partial (\partial_\mu \phi)} \right) \right] \delta \phi $$

In order for the action to be stationary, $\delta S = 0$ for arbitrary $\delta \phi$, it must satisfy the Euler-Lagrange equation of motion as follows...
Comparing Eqs. (13) and (14), we see that $C(M) = \frac{1}{2} g^\mu\nu \partial_\mu \phi \partial_\nu \phi - V(\phi)$, so we can evaluate Eq. (17) more fully giving us

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \frac{\partial L^{(M)}}{\partial (\partial_\mu \phi)} \right) = 0. \tag{17}$$

Note that we solve Eq. (18) assuming that spacetime is both approximately homogenous and isotropic so that we can assume a spacetime metric like that of Eq. (5). Then the equation of motion for $\phi$ becomes simply

$$\ddot{\phi} + 3H \dot{\phi} + V,\phi = 0 \tag{18}$$

using some clever simplifications not shown here [13, 3].

### 2.1.2 Implications

Eq. (19) shows that the scalar field evolves in a curved spacetime like a “damped” harmonic oscillator. The “damping” term, $3H \dot{\phi}$, is critical because it comes entirely from the expansion of the universe. It turns out that for several choices of the potential, $V(\phi)$, (i.e. many models of inflation) the field $\phi$ evolves sufficiently slowly such that $H$ is nearly constant, with $|\dot{H}| \ll H^2$. When this happens, then from the Friedman Eq. (6), $\rho \simeq constant$ and so $H \simeq constant$ and hence $a(t) \simeq a_0 e^{Ht}$. Inflation implies the quasi-exponential expansion of spacetime because the energy density filling the universe changes very slowly in time.

### 2.1.3 Solving the Horizon, Flatness, and Monopole Problems

Inflation solves the horizon, flatness and monopole problems by introducing an early phase of exponential expansion post the Big Bang. First, we consider the horizon problem. In inflation, the entire observed universe evolves from a single coherent region, whose size is
much smaller than the sizes that are relevant in the standard model when the fluctuation began to grow classically. Therefore, the region had plenty of time to reach uniform temperature before the onset of inflation and exponential expansion grew this minute region of homogeneity to encompass the entire observable universe.

The flatness problem is likewise avoided by the dynamics of the exponential expansion of the coherent region. Assuming that the FRW metric approximation holds, then the scale factor evolves according to the standard Friedmann Eq. (6). Restated, the flatness problem is why the $K/a^2$ term is so small. As the coherent region expands exponentially, the mass density $\rho$ remains very nearly constant, while $K/a^2$ is largely suppressed [10, 12]. So the observation that $\Omega = 1$ makes sense. This accelerated expansion drives the energy density of the universe towards the critical density. Inflation explains the fine-tuned appearance of these parameters and how far-apart regions of space could have been in causal contact in the early universe.

In regards to the monopole problem, as long as inflation occurs after or during the process of monopole formation, the monopoles are diluted tremendously [10, 12].

3 Particle Formation After Inflation

At the stage of inflation, all energy is concentrated in the classical slow-moving inflaton field $\phi$. Soon after the end of inflation, the field oscillates with an amplitude on the order of the Planck mass. Subsequent interactions with the oscillating field lead to the creation of many ultra-relativistic particles until the oscillating field eventually decays. This process is called reheating [18, 3, 2]. Reheating can begin with a stage of parametric resonance. In this case, the energy is rapidly transferred from the inflaton field to other fields interacting with it. This process is called preheating [18, 3, 2]. The nature of preheating can be very different per model of the potential (as will be discussed in Part III). Overall, this process helps to produce the particles that make up all of the visible matter in the universe, the Standard
Model of Particle Physics, which has been experimentally tested to a high degree of precision of over the years.

Part II

Overview of Higgs Inflation

As was described in Section 2.1, most models of inflation require that the mass density of a patch of the early universe become dominated by the potential energy of a scalar field in a state of false vacuum. This field is usually referred to as the “inflaton,” but it is still subject to consideration as to what field the inflaton is actually. In Higgs Inflation, the inflaton is the Higgs field.

4 The Higgs Boson and Higgs Field

As was mentioned in Section 1.1.3, three of the four fundamental forces of nature are described by the Standard Model of particle physics. These forces are characterized by symmetries in the physical phenomena and are transmitted by particles known as gauge bosons. The gauge boson is a bosonic particle that is the force carrier; elementary particles interact with each other by exchange of these particles.

It turns out that the weak force’s symmetry implies its gauge bosons have zero mass, but experiments show that these gauge bosons are actually very massive and short-range. It proved difficult to explain their unexpected mass. The Higgs mechanism is a mathematical model devised by three groups of researchers \(^5\) that explains how and why gauge bosons could still be massive despite their governing symmetry. Apparently, the conditions for symmetry would be broken if an unusual type of field happened to exist throughout space,

\(^5\)Robert Brout and François Englert; Peter Higgs; and Gerald Guralnik, C. Richard Hagen, and Tom Kibble.
which would allow the particles to be able to have mass. This is the so-called Higgs field. The Higgs field breaks certain symmetry laws of the electroweak interaction and it triggers the Higgs mechanism that causes the weak force gauge bosons to acquire mass [17].

But the hypothetical existence of the Higgs field had to be experimentally verified. It could be confirmed by searching for the matching particle associated with it, the Higgs Boson, which must also exist if the field does (for particles are manifestations of excitations of the fields). If the Higgs Boson was detected, then that would prove the existence of the Higgs field. On July 4th, 2012, the ATLAS and CMS experiments at CERN’s Large Hadron Collider announced they had each observed a new particle consistent with the Higgs boson. Now, we are virtually certain of the existence of the Higgs boson and thus the Higgs field [1, 7].

5 Higgs Inflation

The Higgs field is the only known fundamental scalar field in the Standard Model of particle physics. In the wake of the recent discovery of compelling experimental evidence of its existence, it has become an exciting candidate for the inflaton. If it is the inflaton, that would mean that the Standard Model can itself give rise to inflation.

The Higgs field can indeed lead to inflation, producing cosmological perturbations in accordance with observations. The action takes the form [5, 8]

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M^2 + \xi \phi^2}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - \nu^2)^2 \right]
\]  

(20)

where \( g \) is the determinant of the metric tensor, \( M \) is mass, \( \xi \) is a numerical parameter, \( \phi \) is the again the scalar field, \( R \) is the Ricci spacetime curvature scalar, and \( \lambda \) and \( \nu \) are parameters of the potential. What is important to note here is that, as shown in the equation, an essential requirement of Higgs inflation is the non-minimal coupling of the Higgs scalar field to gravity, i.e. the Ricci spacetime curvature scalar. However, this requirement is
modified in New Higgs Inflation, as will be described in the section to follow.

5.1 New Higgs Inflation

Because of concerns about whether the original model of Higgs Inflation remains well-behaved at arbitrarily high energies, a model known as “New Higgs Inflation” was more recently proposed [8]. In New Higgs Inflation, there is non-minimal “derivative” coupling; that is, the kinetic energy of the Higgs field couples to the Einstein tensor for spacetime. A model for a single scalar field can be described by the action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M_{pl}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2\mu^2} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \]  

where \( M_{pl} = 1/\sqrt{8\pi G} = 2.43 \times 10^{18} \text{ GeV} \) is the reduced Planck mass, \( \mu \) is some new mass scale, and \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor [14, 8]. In this derivation, we are using \( \mu \) and \( \nu \) to label spacetime indices and \( i \) and \( j \) for spatial ones. The Riemann tensor is defined as \( R^\mu_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\rho\lambda} \Gamma^\rho_{\nu\sigma} - \Gamma^\mu_{\rho\sigma} \Gamma^\rho_{\nu\lambda} \) with the Ricci tensor given by \( R_{\nu\sigma} = R^\mu_{\nu\mu\sigma} \) and the scalar by \( R = g^{\mu\nu} R_{\mu\nu} \). We may once again (as in Section 2.1.1) view the action in the form of gravity and matter-dominated Lagrangian components thusly,

\[ S = \int d^4x [\mathcal{L}_G + \mathcal{L}_M], \]  

where

\[ \mathcal{L}_G = \sqrt{-g} \frac{M_{pl}^2}{2} R \]  

(23)

and

\[ \mathcal{L}_M = \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2\mu^2} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \]  

(24)

We want to derive the equation of motion for this model like was done in Section 2.1.1 for
new inflation. Let us consider a variation of $S$ with respect to $\phi$ to get the Euler-Lagrange equation of motion for the field. As shown in Eq. (23) and Eq. (24), $\mathcal{L}_G$ is independent of $\phi$, but $\mathcal{L}_M$ depends on $\phi$ and $\dot{\phi}$. So varying $S$ with respect to $\phi$ gives the Euler-Lagrange equation of motion:

$$\frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_{\mu} \left( \frac{\partial \mathcal{L}_M}{\partial (\partial_{\mu} \phi)} \right) = 0.$$ (25)

Using Eq. (24) we find that

$$\frac{\partial \mathcal{L}_M}{\partial \phi} = -\sqrt{-g}V_\phi$$ (26)

and

$$\frac{\partial \mathcal{L}_M}{\partial (\partial_{\mu} \phi)} = -\sqrt{-g} \left( g^{\mu\nu} - \frac{1}{\mu^2} G^{\mu\nu} \right) \partial_{\nu} \phi.$$ (27)

So Eq. (25) becomes

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi \right) - \frac{1}{\mu^2} \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} G^{\mu\nu} \partial_{\nu} \phi \right) - V_\phi = 0.$$ (28)

The first term of Eq. (28) is the D’Alembertian operator acting on $\phi$ in the manner of $\Box \phi \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \phi$. Therefore, Eq. (28) may be rewritten in the form

$$\Box \phi - \frac{1}{\mu^2} \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} G^{\mu\nu} \partial_{\nu} \phi \right) - V_\phi = 0.$$ (29)

Furthermore, it turns out that $[14, 8] \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} G^{\mu\nu} \partial_{\nu} \phi \right) = G^{\mu\nu} \nabla_\mu \nabla_\nu \phi$ so we can rewrite Eq. (29) covariantly as

$$\Box \phi - \frac{1}{\mu^2} G^{\mu\nu} \nabla_\mu \nabla_\nu \phi - V_\phi = 0.$$ (30)

If we consider a scalar field in an unperturbed spatially flat FRW spacetime then $\sqrt{-g} =$
$a^3(t)$ and the only non-vanishing component of the Christoffel, Ricci, and Einstein tensors are

$$\Gamma^0_{ij} = a^2 H \delta_{ij}, \quad \Gamma^i_{j0} = H \delta^i_j, \quad (31)$$

$$R_{00} = -3 \left[ H^2 + \dot{H} \right], \quad R_{ij} = a^2 \delta_{ij} \left[ 3H^2 + \dot{H} \right], \quad R = 6 \left[ 2H^2 + \dot{H} \right] \quad (32)$$

$$G_{00} = 3H^2, \quad G_{ij} = -a^2 \delta_{ij} \left[ 3H^2 + 2\dot{H} \right], \quad (33)$$

Evaluating $\Box \phi$ in these coordinates we get

$$\Box \phi = -\dot{\phi} - 3H \dot{\phi} + \frac{1}{a^2} \nabla^2 \phi \quad (34)$$

where $\nabla^2 \equiv \delta^{ij} \partial_i \partial_j$ is the co-moving spatial Laplacian. Then, in a similar way, we evaluate the middle term of Eq. (30) to be

$$G^\mu^\nu \nabla_\mu \nabla_\nu \phi = 3H^2 \ddot{\phi} - \frac{1}{a^2} \left( 3H^2 + 2\dot{H} \right) \nabla^2 \phi + 3H \dot{\phi} \left( 3H^2 + 2\dot{H} \right). \quad (35)$$

Combining this result with Eq. (34) we get finally the equation of motion expanded in the form

$$\ddot{\phi} \left[ 1 + \frac{3H^2}{\mu^2} \right] + 3H \dot{\phi} \left[ 1 + \frac{1}{\mu^2} \left( 3H^2 + 2\dot{H} \right) \right] - \left[ 1 + \frac{1}{\mu^2} \left( 3H^2 + 2\dot{H} \right) \right] \frac{1}{a^2} \nabla^2 \phi \right] + V_\phi = 0 \quad (36)$$

for a scalar field $\phi$. 

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Part III

New Higgs Inflation and Preheating

In this part, we solve a pair of coupled differential equations, namely, the equation of motion for the field $\phi$ and a modified form of the Friedmann equation for $H$ for the case of a single spatially homogenous background field. We consider different numerical values for a set of parameters in order to observe what happens to the field $\phi$ and $H$ given a set of initial conditions. Later, we find the time of the end of inflation for this New Higgs model to study particle production near the end of inflation.

6 Solving for $\phi$ and $H$

For this next part, we want to solve a pair of coupled differential equations for New Higgs Inflation for the case of a single spatially homogenous background field as was explored in Section 5.1. Taking the equation of motion for the field to be Eq. (36), we can thus set $\nabla^2 \phi = 0$ because of the field we are considering. This then simplifies to

$$\ddot{\phi} \left[ 1 + \frac{3H^2}{\mu^2} \right] + 3H \dot{\phi} \left[ 1 + \frac{1}{\mu^2} \left( 3H^2 + 2\dot{H} \right) \right] + V_\phi = 0. \quad (37)$$

We should now take a moment to consider the difference between Eqs. (37) and (19), which was for an action of the form of Eq. (13) as in Section 2.1.1. In order for inflation to be viable here, we must work in the regime in which $\mu^2 \ll H^2$ is true. If this holds, then the damping term, $3H \dot{\phi}$, will be significantly greater than the term coming from the potential $V_\phi$. Hence it is even easier to achieve slow-roll evolution of $\phi$ when $\mu^2 \ll H^2$ because of the added friction. (This will be discussed more in Section 7.) Furthermore, the modified Friedmann Equation for $H$ can be written as [14, 8]
Recall that $M_{pl}$ is Plank mass and that $\phi(t)$ and $H(t)$ are both functions of $t$.

In principle, the Higgs potential (even for a single field model) includes some nonzero vacuum expectation value, i.e. the $\nu$ term in Eq. (20). However, here, we are considering inflation to occur at field values $\phi$ many orders of magnitude higher than $\nu$. That means, we can ignore this $\nu$ term and write the potential $V(\phi)$ as in Eq. (39) below. Stated explicitly, it is

\[ V(\phi) = \frac{\lambda}{4} \phi^4. \]  

Therefore,

\[ V_\phi = \lambda \phi^3. \]  

We considered different ranges for the parameters $\lambda$ and $\mu$. It is important to note that, in this elaboration, we will be using Plank units, in which $M_{pl} = 1$. As this pair of coupled differential equations can only be solved numerically, we will begin by choosing to fix $\lambda$ and $\mu$ to a set of numerical values. First, we chose to set $\lambda = 10^{-2}$ and $\mu = 10^{-2}M_{pl}$, which reduces to $\mu = 10^{-2}$ in Plank units. We must also specify a set of initial conditions, so we will say that $\phi(0) = 2$ and that $\dot{\phi}(0) = 0$. From this selection, it turns out that $H(0) = \sqrt{\frac{4}{3}\lambda}$ also. If we proceed to solve this coupled set of differential equations numerically, we get a set of parametric plots of the solutions as demonstrated in Fig. (2).

Next, we changed the values of $\lambda$ and $\mu$ and modified the initial conditions. We set $\lambda = 10^{-5}$ and $\mu = 10^{-6}$ and the initial conditions to be $\phi(t) = 0.3$ and $\dot{\phi}(t) = 0$. These results are illustrated in Fig. (3). For both of these results, the field $\phi$ begins to oscillate near the end of the time scale, just as we would expect for particle formation (Section 3).
Figure 2: The results of $H(t)$ and $\phi(t)$ for a single spatially homogenous background field in which $\lambda = 10^{-2}$ and $\mu = 10^{-2}$ (These are in Plank units, for which $M_{pl} = 1$).
7 The End of Inflation

Assuming that $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ means that the potential energy stored in the field dominated over the field’s kinetic energy. Furthermore, from the definition of the Hubble expansion parameter, $H \equiv \dot{a}/a$, it follows that
\[ \dot{H} = \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \]  \hspace{1cm} (41) \\

or \\

\[ \frac{\ddot{a}}{a} = H^2 + \dot{H}. \]  \hspace{1cm} (42)

At a given location in space, inflation ends when the universe stops its accelerated expansion; that is, when \( \ddot{a} = 0 \) at some time \( t_{\text{end}} \).

Looking back at Eq. (37), and recalling the behavior of a damped harmonic oscillator, we see that the \( 3H\dot{\phi} \) term contributes to a frictional drag (the Hubble drag) on the evolution of the scalar field. The larger this term, the more the field’s evolution will be overdamped. Under these conditions, the field will evolve slowly towards its equilibrium value that minimizes its potential. While the field rolls slowly down its potential from some large initial value, the kinetic energy is still \( \frac{1}{2} \dot{\phi}^2 \ll V(\phi) \). Because this effectively drives inflationary expansion, inflation models like these are called “slow-roll” models [13, 3].

Referring to Section 2.1.2, during the period of overdamped evolution, \( \dot{H} \ll H^2 \) and the scale factor grows exponentially like \( a(t) \simeq a_0 e^{Ht} \). Given the curvature of the potential, however, the field should pick up speed near the minimum; i.e., the field will not slow-roll forever. In order to estimate when the slow-roll behavior (and hence inflation) should come to an end, we introduce a couple of slow-roll parameters:

\[ \epsilon \equiv - \frac{\dot{H}}{H^2} \]  \hspace{1cm} (43) \\

\[ \eta = \epsilon - \frac{\ddot{\phi}}{H \dot{\phi}} \]  \hspace{1cm} (44)

that may be related to the curvature of the potential as well. Slow-roll evolution ends when either of these slow-roll parameters becomes of order 1. Inflation ends when \( \epsilon = 1 \). This also
implies, from Eq. (42), that at the end of inflation $H^2 = -H^2$, which is equivalent to $\ddot{a} = 0$.

7.1 The End of Inflation for the New Higgs Model

As we mentioned previously, because the scale factor evolves exponentially, we can say that between the start and end of inflation the scale factor grows by

$$\frac{a(t_{end})}{a(t_0)} = e^{H(t_{end} - t_0)} = e^N$$

where $t_{end}$ is the time of the end of inflation, $t_0$ is the time of the start of inflation, and $N$ is defined as

$$N \equiv \int_{t_0}^{t_{end}} dt H(t) \sim H(t_{end} - t_0)$$

and is known as the number of “e-folds” of inflation.

It turns out that at least $N = 65$ e-folds are required to produce sufficient inflation [13, 3]. Now we would like to proceed in evaluating the end of inflation for the New Higgs model. We ask ourselves the question, if we fix both $\lambda$ and $\mu$ to be a constants, what is the minimal value of $\phi(0)$ for which we can find at least $N = 65$ e-folds of inflation? The smaller the value of $\mu$, the larger the added Hubble drag as that effect evolves like $H^2/\mu^2$. With more frictional damping, we can initialize the field at a lower value and still achieve sufficient inflation with $N \geq 65$ e-folds. Thus, as $\mu$ decreases, so does $\phi(t_0)$. But successful inflation is still achieved. Fig. (4) is a semi-log plot of this relationship.
8 Relation to Preheating

The derivation for the equation of motion for fluctuations of the scalar field can be found in [15]. This is

$$\phi(x') = \varphi(t) + \delta\phi(x')$$

(47)

that is achieved by expanding the field and spacetime metric to first order in perturbations.

Eq. (37) describes the equation of motion for the background field $\varphi(t)$. For the fluctuations $\delta\phi(x')$ it is similarly

$$\delta\ddot{\phi}_k \left[1 + \frac{3H^2}{\mu^2}\right] + 3H\delta\dot{\phi}_k \left[1 + \frac{3H^2 + 2\dot{H}}{\mu^2}\right] + \left(\frac{k^2}{a^2(t)} \left[1 + \frac{3H^2 + 2\dot{H}}{\mu^2}\right] + V_{,\phi\phi}\right) \delta\phi_k = S_k$$

(48)
by performing a Fourier Transform. The source term $S$ is independent of the field fluctuations and depends on the metric perturbations [15].

In [6, 19, 16], it is demonstrated that particle production is related to the violation of the adiabatic condition. We will define the adiabatic parameter to be

$$A_k \equiv \frac{\dot{\omega}_k}{\omega_k^2}$$

(49)

where $\omega_k(t)$ is the natural frequency of the fluctuations. In the specific model under consideration, the natural frequency for the fluctuations may be written as

$$\omega_k^2(t) = \left( \frac{k^2}{a^2(t)} \right) \left[ 1 + \frac{2\dot{H}}{3H^3 + \mu^2} \right] + \frac{\mu^2 V_{,\phi\phi}}{3H^2 + \mu^2},$$

(50)

which came from dividing Eq. (48) through by $[1 + (3H^2/\mu^2)]$. The breakdown of the adiabatic condition for a given mode $k$ requires that $|A| \gg 1$ be true. The number density of particles produced during such non-adiabatic evolution is proportional to $A_k^2$. Thus, we can expect significant particle production when $|A| \gg 1$ is true [6, 19].

9 Violation of the Adiabatic Condition

With this information, we can now check to see if the adiabaticity parameter constructed from Eq. (49) for the natural frequency ever satisfies the condition for significant particle production at the end of inflation. To proceed, we normalize $a(t_{end}) = 1$ defined by $\epsilon(t_{end}) = 1$ for the end of inflation.

Fixing the parameters to be $\lambda = 0.01$ and $\mu = 10^{-6}$ and the initial conditions to be $\phi(0) = 0.1$ and $\dot{\phi}(0) = 0$, we found that $t_{end} = 584,997$, for $N = 108.6$ e-folds of inflation. Starting with this $t_{end}$ as a base, the scale factor grew like
Table 1: The growth of the scale factor $a(t)$ for multiples of $t_{\text{end}}$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$a(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{\text{end}}$</td>
<td>1.00</td>
</tr>
<tr>
<td>$1.1t_{\text{end}}$</td>
<td>2.26</td>
</tr>
<tr>
<td>$1.2t_{\text{end}}$</td>
<td>3.37</td>
</tr>
<tr>
<td>$1.3t_{\text{end}}$</td>
<td>4.38</td>
</tr>
<tr>
<td>$1.4t_{\text{end}}$</td>
<td>5.33</td>
</tr>
<tr>
<td>$1.5t_{\text{end}}$</td>
<td>6.27</td>
</tr>
</tbody>
</table>

We used this data to interpolate and plot the resulting function against the relevant actual numerical solution of $H(t)$ as shown in Fig. (5).

Figure 5: $H(t)$ was solved numerically in the same manner as in Section 6 (orange) and via interpolation using the values of Table (1) (blue).

As we saw in the Section 7, given a numerical solution for $H(t)$, we may construct an approximate function for $a(t)$ as
So now, we have enough information to construct $A_k$ for $k = 1$. If we do that, we see that $|A_k| \gg 1$ when $\varphi^2 = 0$, as in Fig. (6). The background field oscillates at the end of inflation, meaning that this occurs cyclically, implying quasi-periodic bursts of particle production.

![Diagram](image-url)

Figure 6: The adiabaticity parameter $|A_k|$ (blue) for $k = 1$, $\lambda = 0.01$, and $\mu = 10^{-6}$ against $10^4\varphi^2(t)$ (red). (The factor of 10,000 was added for visibility.)

Therefore, this model undergoes significant particle production at the end of inflation. With these parameters, $H(t_{end}) \sim 10^{-5}$, as shown in Fig. (6). We have also normalized $a(t_{end}) = 1$, so $k = 1$ corresponds to a mode that is well within the horizon at the end of inflation: $k \gg aH$. It remains to study the efficiency of particle production in this model as $k$ varies, that is, looking at the momentum (or wavenumber) dependence of the resonant particle production during preheating. Then, likewise, to study how the spectrum of produced particles varies with the model parameter $\mu$. It is an interesting question that
deserves further research using the process employed here.

10 Conclusion

New Higgs Inflation poses a candidate for the scalar field that drives inflation and particle production near the end of inflation by introducing a characteristic non-minimal “derivative” coupling. Considering a potential for the field in the form of $V(\phi) = \frac{\lambda}{4} \phi^4$, when we chose different values for $\lambda$ and $\mu$, the field, $\phi$, in both cases, behaved as expected towards the end of the time scale. The equation of motion for the field evolves like a damped harmonic oscillator, so we expect it to oscillate near the end. This was indeed the case; we further explored this oscillation as an indicative of preheating, and thus, particle formation.

In order to ensure sufficient inflation, it is required that $N \geq 65$ e-folds. Once the couplings of the model, $\lambda$ and $\mu$, are chosen, one may always find initial conditions for the field, $\phi(0)$ and $\dot{\phi}(0)$, that will satisfy this condition. Given those parameters and initial conditions for the field, we were able to construct a function for the scale factor, $a(t)$, at the end of inflation, which we used to define the adiabaticity parameter, $A_k$. Violation of the adiabatic condition, $|A_k| \gg 1$, is what leads to particle production. In our case, as the background field oscillates at the end of inflation, this occurs cyclically, implying quasi-periodic bursts of particle production.

"The process developed here may be used to study the spectrum of produced particles, by studying how $A_k$ varies across momentum states, $k$. One may also study particle production as one varies the new mass-scale of the model, $\mu$. The parameter-dependence of the spectrum of produced particles in this model is an interesting question that deserves further research."
References


