Quantum Non-Contact Friction in Resonant Dielectric Media

by

Michael O. Flynn

Submitted to the Department of Physics in partial fulfillment of the requirements for the degree of Bachelor of Science at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2015

© Massachusetts Institute of Technology 2015. All rights reserved.

Signature redacted

Author

Department of Physics
May 12, 2015

Signature redacted

Certified by ..............
Mehran Kardar
Professor of Physics
Thesis Supervisor

Signature redacted

Accepted by ..............
Professor Nergis Mavalvala
Senior Thesis Coordinator, Department of Physics
Quantum Non-Contact Friction in Resonant Dielectric Media

by

Michael O. Flynn

Submitted to the Department of Physics
on May 12, 2015, in partial fulfillment of the
requirements for the degree of
Bachelor of Science

Abstract

We compute the non-contact friction force induced between two isotropic planar dielectric slabs which obey a plasma model dielectric function using scalar electromagnetism. All computations are carried out at zero temperature. Strong constraints on the integration parameters which define the friction force are derived and subsequently used for a variety of numerical computations. In the limit of large dielectric separations, a saddle point approximation is used to show that the dielectric function is approximately constant in this limit. In the limit of small dielectric separations, a distinct set of integer power laws unique to the plasma model are found. These are then shown to correspond to an elastic limit for the friction force. Finally, these power laws yield a natural maximization condition for the friction which provides an interesting contrast to the behaviors observed in friction computations involving dielectric slabs with constant dielectric functions.

Thesis Supervisor: Mehran Kardar
Title: Professor of Physics
Acknowledgments

I would like to begin by thanking Professor Mehran Kardar for his guidance and support throughout the completion of this thesis. If it were not for his outstanding teaching, I doubt that I would be working in condensed matter physics at all.

I must also thank Dr. Mohammad Maghrebi, who originally suggested the line of questioning which led to this thesis. He has made himself consistently available to me for discussions, and has always kept this project on track. For being such a helpful guide, he has my deepest thanks.

The fluctuations collaboration at MIT has been a consistent source of inspiration with respect to all things Casimir. My thanks go out to all the members of the collaboration, particularly Robert Jaffe, Homer Reid, Thorsten Emig, Matthias Krueger, Giuseppe Bimonte, and Noah Graham.

Since beginning this project, I have had helpful conversations with a fair number of my fellow undergraduates. I would particularly like to thank TJ Wilkason, Nick Rivera, Eric Tomlinson, Stephen Face, and Francisco Machado for their helpful comments.

I cannot forget to thank my excellent support system. Miren Bamforth and TJ Wilkason have always helped maintain my well-being, and for that I am profoundly grateful.

Finally, to my mother: thanks for putting my education first.
Contents

1 Introduction ......................................................... 13
   1.1 Overview and Motivation ...................................... 13
   1.2 Thesis Structure ............................................. 14

2 The Fluctuation-Dissipation Theorem ............................ 17
   2.1 A Classical Example: Brownian Motion With Gaussian Noise . . . . . . . 17
   2.2 General Statement of the Fluctuation-Dissipation Theorem ........ 21
      2.2.1 Linear Response Theory and The Green-Kubo Formula ........ 21
      2.2.2 The Fluctuation-Dissipation Theorem in Electromagnetism . . . . 25

3 The Casimir Effect .................................................. 29
   3.1 Perfect Planar Conductors .................................... 29
      3.1.1 The Method of Zero-Point Energies ......................... 29
      3.1.2 A Method of Fluctuating Fields .......................... 30
      3.1.3 The Dynamic Casimir Effect .............................. 32
   3.2 Non-contact Friction .......................................... 33
      3.2.1 Origin of the Frictional Force ............................ 33
      3.2.2 The Non-Contact Friction Formula via the Fluctuation-Dissipation
           Theorem .................................................. 34

4 Non-Contact Friction: Application to Plates With a Plasma Model
   Dielectric Function .............................................. 37
   4.1 Statement of The Problem ..................................... 37
4.2 Dimensional Analysis and Scaling ........................................... 39
4.3 Constraints on Integration Parameters ...................................... 40
  4.3.1 The Perspective of Reflection Matrices ............................... 40
  4.3.2 The Perspective of Dispersion Relations .............................. 42
4.4 The Limit of Large Separations ............................................. 48
  4.4.1 The Constant Dielectric Function Approximation ................. 48
  4.4.2 Saddle Point Calculations: Corrections to the Constant Dielectric Function Approximation ..................... 50
4.5 The Limit of Small Separations ............................................. 52
  4.5.1 Asymptotic Behavior ..................................................... 52
  4.5.2 Numerical Scaling Results and Friction Maximization .......... 53
  4.5.3 Origin of the Small Separation Scaling Exponents: The Elastic Limit ..................................................... 56
  4.5.4 Corrections Associated with Finite Separation Effects ........ 59
  4.5.5 Comparison of Friction Magnitudes .................................. 62

5 Conclusions .............................................................................. 63
List of Figures

4-1 This is a plot of the spectra $\omega_1$ and $\omega_2$ against $k$, with $\omega_0 = 100, \omega_p = 200$. The light cone and resonant frequency are also plotted for reference. Note that $\omega_1$ is bounded from above by $\omega_0$, while $\omega_1$ is above the light cone. .................................................. 44

4-2 This is a plot of the spectra $\omega_1$ and $\omega_2$ against $k$, with $\omega_0 = 100, \omega_p = 200$. The light cone and resonant frequency are also plotted for reference. Note that $\omega_1$ is bounded from above by $\omega_0$, while $\omega_1$ is above the light cone. .................................................. 44

4-3 This is a plot of the spectra $\omega_1$ and $\omega'_1$, with $\omega_0 = 100, \omega_p = 500$, and $v = 0.1$. Note that the spectrum $\omega'_1$ is essentially identical to $\omega_1$, except that it has been “tilted”, acquiring a slope which is proportional to $v$ for large $|k|$. The frequency $\omega_0$ and wave vector $\frac{2\omega_0}{v}$ are also indicated. In general, we will choose the boundaries of the $k$-axis in our plots such that $k = \frac{2\omega_0}{v}$ on the boundary. .................................................. 46

4-4 This is a plot of the spectra $\omega_1$ and $\omega_1 + \omega'_1$, with $\omega_0 = 100, \omega_p = 14000$, and $v = 0.01$. The wave vectors $k_0$ and $\frac{2\omega_0}{v}$ are both indicated, along with the frequencies $\omega_0$ and $\omega_1(k_0)$. Note that $v < 2v_0$, so that $k_0$ and $\omega_1(k_0)$ have taken on nonzero values. .................................................. 47

4-5 This is a plot of the spectra $\omega_1$ and $\omega_1 + \omega'_1$, with $\omega_0 = 100, \omega_p = 14000$, and $v = 0.02$. The wave vector $\frac{2\omega_0}{v}$ and frequency $\omega_0$ are also indicated. Note that $v > 2v_0$, so that $\omega_1 + \omega'_1 < 0$ for all $k \in \left[0, \frac{2\omega_0}{v}\right]$. .................................. 47
This is a log-log plot of the friction force (per unit area) vs. velocity, with \( \omega_p = 10^4 \) and \( \omega_0 = 10 \). Below the velocity \( v = 2\nu_0 \), the friction is heavily suppressed; this can be understood by referring to figures 4-4 and 4-5, and noting that modes with large frequencies are exponentially suppressed relative to small frequency modes. This is qualitatively distinct from the case where \( \epsilon \) is a constant, as the friction vanishes identically in that case for \( v < 2\nu_0 \).

This is a log-log plot of the friction force (per unit area) vs. plate separation with \( \omega_0 = 100, v = 0.001, \) and \( \omega_p = 10^7 \). The length scales \( c/\omega_p \) and \( v/2\omega_0 \) are indicated for reference. As \( d \to 0 \), the friction becomes independent of separation, as predicted. As \( d \to \infty \), the friction scales like \( 1/d^3 \), which is also expected, since our numerical computations are done in two dimensions.

This is a log-log plot of friction vs. plasma frequency, with \( \omega_0 = 1, v = 0.003, \) and \( d = 10^{-10} \). The separation is chosen to be much smaller than the length scales \( c/\omega_p \) and \( v/\omega_0 \). We observe a sharp transition from \( \omega_p^4 \) scaling (for small \( \omega_p \)) to a regime where the friction is independent of \( \omega_p \).

This is a log-log plot of friction vs. resonant frequency, with \( \omega_p = 10^4, v = 0.003, \) and \( d = 10^{-7} \). We observe a sharp transition from a regime in which the friction scales like \( \omega_p^3 \) to a regime in which it falls off like \( 1/\omega_0 \).

This is a log-log plot of friction vs. velocity, with \( \omega_0 = 1, \omega_p = 10^6, \) and \( d = 10^{-10} \). We observe a sharp transition from a regime in which the friction scales like \( v^2 \) to one in which it falls off like \( 1/v^2 \).

This is a 3D plot of the log of the friction integrand (labeled as \( I \)) divided by \( k \) against \( \omega \) and \( k \), with \( \omega_p = 10^5, \omega_0 = 10, d = 0, \) and \( v = 0.002 \). There is a clear line in the center of the integrand along which the integrand is constant. The equation for this line is given by \( k = \frac{2\omega}{v} \).
4-12 These plots show the scaling behavior of the friction under the approximation (4.36). As indicated in the plots, the power law exponents computed in this approximation are within 10 percent of the values computed with the exact definition of the friction. The parameters for each plot are chosen to agree with their counterparts in figures 4-8 through 4-10.

4-13 These plots show how the friction scales with $\omega_p$ and $\omega_0$ near the small separation limit. The length scales $c/\omega_p$ and $v/\omega_0$ cleanly separate the regime where the power laws (4.28) and (4.29) hold and a regime where these scaling relations break down.
Chapter 1

Introduction

1.1 Overview and Motivation

Fluctuations have played an important role in fundamental physics for at least as long as statistical methods have been employed by physicists. In basic statistical and quantum mechanics, fluctuations arise in numerous contexts, where they are used to characterize a wide variety of stochastic phenomena. In the last century, fluctuations have played prominent roles in the development of field theories, perhaps most prominently in Landau's theory of symmetry breaking phase transitions. Other phenomena, such as the Casimir effect in quantum field theory, can also be understood as a macroscopic manifestation of random fluctuations.

As humanity develops technologies which span ever-shrinking length scales, an understanding of the dominant mesoscopic forces in nature is becoming increasingly important for engineering applications. In recent years, experiments in Casimir physics have achieved an unprecedented level of sensitivity, enabling the measurement of forces which were previously undetectable [12]. This has fueled a series of tremendous theoretical developments in the field, including analytical results for particular situations along with general numerical approaches [18]. A number of closely related fields, such as heat transfer in non-equilibrium Casimir physics and non-contact friction, have also received a large amount of attention [9].

These observations motivate us to take steps towards developing a more realistic
theory of non-contact friction. While the general principles of the subject are well understood, most of the worked-out applications of the theory are to cases which are somewhat unrealistic, such as objects which do not exhibit dispersion. In this thesis, we take a step towards applying the non-contact friction formalism to systems with more realistic frequency responses. In particular, we work with objects which obey a plasma model dielectric function.

The rest of this thesis will be organized as follows.

1.2 Thesis Structure

In chapter 2, we begin by discussing the fluctuation-dissipation theorem as it arises in the context of Brownian motion. This problem, first treated by Einstein [6], provides valuable physical intuition. With this example in mind, we will go on to prove the fluctuation-dissipation theorem for a generic field theory by deriving the basic results of linear response theory. To illustrate the advantages of this formalism, we will turn to a familiar field theory - electromagnetism - and discuss the electromagnetic fluctuation-dissipation theorem, as developed by Rytov [19].

Chapter 3 introduces a seemingly disparate physical phenomenon: the Casimir effect. This effect, discovered by Hendrik Casimir in 1948 [3], was originally found via the method of zero point energies. We will briefly recount Casimir’s argument. We will then promptly search for a different perspective on the Casimir force which does not require reference to zero point energies. By studying the Casimir effect with the electromagnetic fluctuation-dissipation theorem, we will be led to a method of Casimir force computation which does not incorporate zero point energies. A brief description of the dynamical Casimir effect follows, including a summary of the origin of non-contact friction.

Chapter 4 considers the non-contact friction force between planar dielectrics which obey a plasma model dielectric function. We use the methods of Maghrebi, Kardar, and Golestanian to develop similarities between this case and the case where the dielectric functions are constant. Various sets of approximations are then used to
extract the dependence of the friction force on different parameters in the theory.
Chapter 2

The Fluctuation-Dissipation Theorem

In this section we provide an overview of the fluctuation-dissipation theorem. As we will see, the theorem provides powerful physical intuition for the phenomena of interest to this thesis, so the time we are taking to develop it will be well-spent.

2.1 A Classical Example: Brownian Motion With Gaussian Noise

It was observed in 1827 by Robert Brown that pollen particles immersed in water droplets exhibit a random, jostling motion [2]. This motion is now termed Brownian motion. Einstein demonstrated via explicit calculation in 1905 that this Brownian motion arises due to random interactions of the pollen grains with the surrounding water molecules [6]. In this section, we will reproduce Einstein’s result that the fluctuations of the “noisy” force of the water molecules is related to the dissipation of energy in the medium. As we have already hinted, this is a specific case of the more general fluctuation-dissipation theorem, which will later play an important role in our understanding of the dynamic Casimir effect. This presentation of the Brownian motion example follows that of Kardar [8].
Our starting point is to write Newton's second law for the Brownian particle:

\[ m\ddot{x} = -\frac{\dot{x}}{\mu} - \frac{\partial V}{\partial x} + \bar{f}(t) \]  

(2.1)

Let us consider these contributions term by term. The first term is the standard inertial term which always appears in Newton's second law for a particle of mass \( m \).

The second term represents the viscous force of the fluid on the particle. The mobility of the particle, \( \mu \), is a function of the Reynolds number of the fluid and the shape of the Brownian particle. The precise form of \( \mu \) will be unimportant for our purposes.

The third term accounts for the contribution of some external potential, \( V \); this can be a typical force, such as gravity. However, it is interesting to note that many modern experiments, such as those which employ optical trapping methods, can create artificial potentials, leading to novel modifications of Brownian motion [1].

Finally, the force \( \bar{f}(t) \) is a random force due to the impacts of the fluid particles. We will come to understand the nature of this force quantitatively when we consider its statistical properties. For now, it is only important to note that the viscous force and the random force have the same physical origin - they are both caused by the impacts of water molecules on the Brownian particle. It is natural to anticipate, then, that the frictional and random forces must be related. The general notion that a single physical effect is responsible for fluctuations (in this case, the random force of Brownian motion) and dissipation (in this case, the energy lost due to friction) is at the core of the fluctuation-dissipation theorem [10].

Generally, the viscous term dominates the inertial term in (2.1), and we can neglect inertial contributions in our analysis. This leads to the following Langevin equation:

\[ \dot{x} = -\mu \frac{\partial V}{\partial x} + \mu \bar{f}(t) \equiv \tilde{v}(\tilde{x}) + \bar{\eta}(t) \]  

(2.2)

Let us examine the statistical properties of the stochastic term \( \bar{\eta}(t) \). The most straightforward way to do this is to write down a probability distribution for \( \bar{\eta}(t) \). Generally, it is assumed that \( \bar{\eta} \) is Gaussian distributed; this is frequently justified, as
the rate of collisions between water molecules and the Brownian particle is large. The central limit theorem then says that the probability distribution for \( \vec{\eta} \) is given by

\[
P [ \vec{\eta}(t) ] \propto \exp \left[ - \int d\tau \frac{\eta(\tau)^2}{4D} \right] = \exp \left[ - \int d\tau \frac{\eta_\beta(\tau)\eta_\beta(\tau)}{4D} \right]
\]

In the last statement in (2.3), we have assumed the Einstein summation convention, where repeated indices are summed over. Hence, by \( \eta_\alpha \) we refer to the component of \( \vec{\eta} \) in the direction \( \alpha \). We can immediately read off the cumulants of the Gaussian distribution:

\[
\langle \vec{\eta}(t) \rangle = 0
\]

\[
\langle \eta_\alpha(t)\eta_\beta(t') \rangle = 2D\delta_{\alpha\beta}\delta(t - t')
\]

From these results, it is straightforward to verify that \( D \) is the diffusion coefficient of the Brownian particle.

It is well-known that a Gaussian distribution is completely specified by its first two cumulants, so we have completely characterized the statistical properties of \( \vec{\eta}(t) \). Hence, we are in a position to use the Langevin equation (2.2) to study the statistics of the time evolution of a Brownian particle. We will now do this for the case of a harmonic potential, \( V(\vec{x}) = \frac{K}{2}x^2 \). In this case, the Langevin equation (2.2) can be written as

\[
\frac{\partial}{\partial t} \left[ e^{\mu K t} \vec{x}(t) \right] = e^{\mu K t} \vec{\eta}(t)
\]

From this, we can compute \( \vec{x}(t) \) by integrating the Langevin equation:

\[
\vec{x}(t) = \vec{x}(0)e^{-\mu K t} + \int_0^t d\tau e^{-\mu K(t-\tau)}\vec{\eta}(\tau)
\]

Using our results for the statistics of \( \vec{\eta} \), we can immediately see

\[
\langle \vec{x}(t) \rangle = \vec{x}(0)e^{-\mu K t}
\]

The variance of the position can also be computed:
\[
\langle (\vec{x}(t) - \langle \vec{x}(t) \rangle)^2 \rangle = \left\langle \int_0^t dt_1 dt_2 e^{-\mu K (2t-t_1-t_2)} \vec{\eta}(t_1) \cdot \vec{\eta}(t_2) \right\rangle \\
= \int_0^t dt_1 dt_2 e^{-\mu K (2t-t_1-t_2)} \langle \vec{\eta}(t_1) \cdot \vec{\eta}(t_2) \rangle \\
= 6D \int_0^t dt_1 dt_2 e^{-\mu K (2t-t_1-t_2)} \delta(t_1 - t_2) \\
= \frac{3D}{\mu K} (1 - e^{-2\mu K t}) 
\]

In the limit \( t \to \infty \), (2.5) converges to \( \frac{3D}{\mu K} \). Of course, for large times, the Brownian particle will come to thermal equilibrium with the fluid, and should therefore converge to a Boltzmann distribution at temperature \( T \). The Boltzmann distribution is given by

\[
P_B(\vec{x}) = \left( \frac{K}{2\pi k_B T} \right)^{3/2} \exp \left[ -\frac{K x^2}{2k_B T} \right]
\]

This distribution has variance \( \langle x^2 \rangle_B = \frac{3k_B T}{K} \). Equating this with \( \frac{3D}{\mu K} \) yields the Einstein connection

\[
D = k_B T \mu 
\]  

(2.6)

As we suspected, there is a connection between the viscous force and the random force. This connection has manifested itself as a constraint which must be obeyed by the macroscopic constants which characterize each of these forces: the diffusion coefficient associated with the action of the random force, and the mobility, which is associated with the viscous force. While we derived (2.6) for the special case of a harmonic potential, this result can be generalized to an arbitrary \( V(\vec{x}) \) [8]. Hence, (2.6) is a deep statement regarding the nature of Brownian motion.

Understanding the fluctuation-dissipation connection for this case was quite intuitive. However, there is no immediately clear way to generalize this connection to the case of a generic field theory. Part of this difficulty stems from the lack of an obvious replacement for the observables \( D \) and \( \mu \) which appear in the Brownian motion example. The role of the next section is to introduce these “replacement” observables,
and state a more complete version of the fluctuation-dissipation theorem.

\section*{2.2 General Statement of the Fluctuation-Dissipation Theorem}

We will now extract the general features of the fluctuation-dissipation theorem, of which the Einstein connection (2.6) is a special case. Precisely how to make this generalization is not immediately clear, but we have some guidance from the following two facts: the theorem should apply to an arbitrary field theory, and some description of dissipation must be included within the formalism. Experience with many systems, such as the driven harmonic oscillator or quantum mechanical scattering resonances, suggests that linear response theory offers an effective mechanism for describing dissipation via susceptibilities, or response functions \cite{111}. As we will see, introducing the linear response formalism into quantum mechanical time-dependent perturbation theory naturally incorporates correlation functions, the fundamental objects of field theories. Hence, linear response theory leads us to the generalization we desire.

The proofs which are shown here are generally well-known, and can be found through various sources. However, we are particularly indebted to the notation of Wen \cite{22}.

\subsection*{2.2.1 Linear Response Theory and The Green-Kubo Formula}

Consider a quantum mechanical system described by a time-independent Hamiltonian $H_0$. Assume that a weak perturbation of the form $f_j(t)\mathcal{O}_j$ is turned on at a finite time; that is, $f_j(t < t_0) = 0$, for some finite $t_0$. Note that $\mathcal{O}_j$ is an arbitrary quantum mechanical operator. Our goal is to compute the response of an $H_0$ eigenstate, $|\psi_n\rangle$, to this perturbation. Starting with this eigenstate at time $t_{-\infty} = -\infty$, standard quantum mechanics says that

$$|\psi_n(t)\rangle = T \left( e^{-\frac{i}{\hbar} \int_{t_{-\infty}}^{t} dt' H(t')} \right) |\psi_n\rangle$$
where $T$ is a time-ordering operator. This naturally leads to an expansion in $f_j(t)$. We find that

$$|\psi_n(t)\rangle = \exp\left[-\frac{i}{\hbar}\int_{t-\infty}^{t} dt'H_0\right]|\psi_n\rangle - \frac{i}{\hbar}\int_{t-\infty}^{t} dt'f_j(t')e^{-\frac{1}{\hbar}H_0(t-t-)}\mathcal{O}_j(t')|\psi_n\rangle$$

We have introduced the notation $\mathcal{O}_j(t) = e^{\frac{i}{\hbar}H_0(t-t-)}\mathcal{O}_j e^{-\frac{1}{\hbar}H_0(t-t-)}$. This change in the eigenstate $|\psi_n(t)\rangle$ due to the perturbation $f_j(t)\mathcal{O}_j$ naturally induces a change in the expected value of some other operator, $\mathcal{O}_i$. This change in expected value is given by

$$\langle \psi_n(t)|\mathcal{O}_i|\psi_n(t)\rangle - \langle \psi_n|e^{\frac{i}{\hbar}H_0(t-t-)}\mathcal{O}_j e^{-\frac{1}{\hbar}H_0(t-t-)}|\psi_n\rangle$$

$$= -\frac{i}{\hbar}\int_{t-\infty}^{t} dt'f_j(t')\langle \psi_n| [\mathcal{O}_i(t), \mathcal{O}_j(t')] |\psi_n\rangle + O(f_j^2) \quad (2.7)$$

$$= \int_{-\infty}^{\infty} dt'\chi_{ij}(t, t')f_j(t') + O(f_j^2)$$

where $\chi_{ij}(t, t')$ is called a response function, and is given by

$$\chi_{ij}(t, t') = -\frac{i}{\hbar}\Theta(t-t')\langle \psi_n| [\mathcal{O}_i(t), \mathcal{O}_j(t')] |\psi_n\rangle \quad (2.8)$$

This powerful result is one of a class of relations called Green-Kubo formulas, and immediately allows one to compute changes to expected values to leading order in a perturbation:

$$\delta\langle \mathcal{O}_i(t) \rangle = \int dt'\chi_{ij}(t, t')f_j(t') \quad (2.9)$$

The form (2.9) is highly reminiscent of a Green’s function relation; in fact, the function $\chi_{ij}$ is generally associated with the retarded Green’s function. This is due to causality considerations discussed below. We also note that, as promised, (2.8) contains a sum of (two-point) correlation functions.

Let us derive the fundamental properties of $\chi_{ij}$. We begin by noting that it is generally the case that one can assume that the system is invariant under time
translations, so that $\chi_{ij}(t, t') = \chi_{ij}(t - t')$. Further, the theta function in (2.9) guarantees that the system respects causality: the system has no response (at time $t$) before the perturbation is applied (at time $t'$).

One can typically assume that the “source” $f_j(t)$ is real, and that the operators $\mathcal{O}_j$ and $\mathcal{O}_i$ are Hermitian. Since commutators of Hermitian operators are anti-Hermitian, (2.8) says that $\chi_{ij}(t - t')$ is real-valued. However, in frequency space, the response function has both real and imaginary parts, which we denote by $\chi_{ij}(\omega) = \chi'_{ij}(\omega) + i\chi''_{ij}(\omega)$. The imaginary part of the frequency-space response function can be expressed in temporal space as

$$\chi''_{ij}(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \chi''_{ij}(\omega) = -\frac{i}{2} \int \frac{d\omega}{2\pi} e^{i\omega t} [\chi_{ij}(\omega) - (\chi_{ij}(\omega))^*] \quad (2.10)$$

Since $\chi_{ij}(t)$ is real, we find

$$\chi_{ij}(\omega) = \int dt e^{-i\omega t} \chi_{ij}(t)$$

$$\Rightarrow (\chi_{ij}(\omega))^* = \int dt e^{i\omega t} \chi_{ji}(t) = \int dt e^{-i\omega t} \chi_{ji}(-t) \quad (2.11)$$

Together, (2.10) and (2.11) yield

$$\chi''_{ij}(t) = -\frac{i}{2} [\chi_{ij}(t) - \chi_{ji}(-t)] \quad (2.12)$$

This says that the imaginary part of the response function in Fourier space is sensitive to the “arrow of time”. In general, the imaginary part of the response function is responsible for the dissipative portion of the dynamics, which is reflected by the fact that it is not invariant under time-reversal transformations. Essentially identical arguments show that $\chi'_i(t)$ is insensitive to time reversal, and describes the so-called reactive portion of the system response.

We are now in a position to derive the fluctuation-dissipation theorem, as follows. The Kubo relation (2.8) allows for a rewriting of $\chi''_{ij}(t)$:

$$\chi''_{ij}(t) = -\frac{i}{2} [\chi_{ij}(t) - \chi_{ji}(-t)]$$
Time translation invariance allows us to shift the operators temporally, which leads to the elimination of the $\Theta$ functions via the identity $\Theta(t) + \Theta(-t) = 1$. Hence

$$
\chi''_{ij}(t) = \frac{1}{2\hbar} \left[ \langle O_j(0)O_i(t) \rangle - \langle O_j(t)O_i(0) \rangle \right]
$$

(2.13)

These terms can be reorganized by assuming a particular averaging procedure for computing the correlation functions. For our purposes, temperature will be fixed, so the canonical ensemble provides such a procedure:

$$
\langle O_i(t)O_j(0) \rangle = \text{Tr} \left[ e^{-\beta H_0} O_i(t)O_j(0) \right]
$$

$$
= \text{Tr} \left[ e^{-\beta H_0} O_i(t) e^{i\beta H_0} e^{-\beta H_0} O_j(0) \right]
$$

$$
= \text{Tr} \left[ e^{-\beta H_0} O_j(0) e^{-\beta H_0} O_i(t) e^{i\beta H_0} \right]
$$

$$
= \text{Tr} \left[ e^{-\beta H_0} O_j(0) O_i(t + i\beta \hbar) \right]
$$

$$
= \langle O_j(0)O_i(t + i\beta \hbar) \rangle
$$

(2.14)

Note that the unperturbed Hamiltonian $H_0$ is used to compute these correlation functions. With (2.13), we find

$$
\chi''_{ij}(t) = \frac{1}{2\hbar} \left[ \langle O_j(0)O_i(t) \rangle - \langle O_j(t)O_i(0) \rangle \right]
$$

In Fourier space, this equation becomes

$$
\chi''_{ij}(\omega) = \frac{1}{2\hbar} \left( 1 - e^{-\beta \omega} \right) S_{ij}(\omega)
$$

(2.15)

$$
\Rightarrow S_{ij}(\omega) = 2\hbar \left( 1 + n_B(\omega, T) \right) \chi''_{ij}(\omega)
$$

where $n_B(\omega, T)$ is the Bose-Einstein factor and we have defined $S_{ij}(\omega)$ to be the Fourier transform of the appropriate two-point function:

$$
S_{ij}(\omega) = \int dt e^{-i\omega t} \langle O_j(0)O_i(t) \rangle
$$

Equation (2.15) is a statement of the fluctuation-dissipation theorem. It provides
the desired correspondence between the fluctuations of a system around equilibrium (characterized by the correlation function $S_{ij}$) and the dissipative properties of the system (characterized by the imaginary part of the frequency-space response function). It is somewhat remarkable that information can be obtained about out of equilibrium behavior by computing equilibrium correlation functions.

Although we have derived (2.15) in the setting of linear response theory, it has a counterpart which can be derived more generally, and does not require the assumption that the perturbation $f(t)$ is weak. Thus, in these more general settings, the fluctuation-dissipation theorem is an exact result, and does not suffer any corrections, even when the response of the system becomes nonlinear [4]. The more general form of the fluctuation-dissipation theorem is as follows:

$$S_{ij}(\omega) = 2\hbar \left(\frac{1}{2} + n_B(\omega, T)\right) \chi''_{ij}(\omega) = \hbar \coth \left(\frac{\hbar \omega \beta}{2}\right) \chi''_{ij}(\omega) \tag{2.16}$$

The expressions (2.15) and (2.16) agree in the high temperature limit, and disagree by a factor of two when $T \to 0$.

### 2.2.2 The Fluctuation-Dissipation Theorem in Electromagnetism

For the rest of this thesis, the only field theory which will be studied is scalar electromagnetism in the presence of spatially homogeneous dielectrics (aside from a brief digression in chapter 3). Historically, the most prominent application of the fluctuation-dissipation theorem has been to electromagnetism, resulting in the theory known as fluctuational electrodynamics. The details of this theory were first worked out by Rytov and the Russian school of physicists, and have since been applied throughout the literature of statistical physics [19]. A full discussion of Rytov’s theory is unnecessary for our purposes; in this section, we will only outline and motivate its main results as they pertain to the systems studied in this thesis.

It is worth noting that the path taken by Rytov generally starts with a stochastic form of Maxwell’s equations, which incorporates both thermal and quantum fluctuations through fluctuation-induced source terms. This is essentially identical to the
approach which we took to study Brownian motion, with the additional complication that electrodynamics is a theory of vector fields. The approach presented here is less formal and focuses on analogies with classical electromagnetism and basic quantum field theory.

To begin, consider the field equation of scalar electromagnetism in the presence of a spatially uniform material with dielectric function $\varepsilon(\omega)$ and no external sources:

$$\left(\nabla^2 + \frac{\omega^2}{c^2} \varepsilon(\omega, \vec{x})\right) \Phi(\omega, \vec{x}) = 0 \quad (2.17)$$

where $\Phi$ is the scalar field. This is identical in structure to the Helmholtz equation of classical electromagnetism, which is usually obtained for either of the fields $\vec{E}$ or $\vec{H}$ by eliminating the other via Maxwell’s equations. In the presence of material sources, this equation is modified to read

$$\left(\nabla^2 + \frac{\omega^2}{c^2} \varepsilon(\omega, \vec{x})\right) \Phi(\omega, \vec{x}) = \frac{i\omega}{c} \rho(\vec{x}) \quad (2.18)$$

The scalar charge density, $\rho$, plays the same role for the scalar field $\Phi$ as an electric charge density does for the electric field. Under the assumption that the dielectrics under study do not carry a net scalar charge, the mean value of the charge density vanishes: $\langle \rho(\vec{x}) \rangle = 0$, where the average is taken in frequency space. However, the charge has nontrivial correlations which can be deduced via the fluctuation-dissipation theorem:

$$\langle \rho(\vec{x}) \rho^*(\vec{y}) \rangle = \hbar \coth \left( \frac{h\omega}{2k_B T} \right) \text{Im} \varepsilon(\omega, \vec{x}) \delta(\vec{x} - \vec{y}) \quad (2.19)$$

This takes the form of (2.16) with the identification $\chi = \varepsilon$; the only difference is the spatial $\delta$-function in (2.19). Let us justify the appearance of both of these objects, beginning with the dielectric function.

The dielectric function, $\varepsilon$, is used in basic electrodynamics to compute the polarizability of a dielectric in the presence of an electric field. The polarization, $\vec{D}$, is generally computed as
\[ \vec{D}(t, \vec{r}) = \int_{-\infty}^{\infty} \epsilon(t - t', \vec{r}) \vec{E}(t', \vec{r}) dt' \]

In this form, it is clear that \( \epsilon \) is a linear response function for the dielectric. We saw in the previous section that linear response functions can be written as correlation functions of appropriate operators, as in (2.8). In this case, the appropriate operators are spatially separated charge density operators. This is because the polarization correlates the induced charge at some location \( \vec{y} \) in the dielectric due to some source at location \( \vec{z} \). This can be shown explicitly by introducing an appropriate time-dependent perturbation and expanding the associated time-ordered exponential to extract a Green-Kubo relation.

The \( \delta \)-function in (2.19) can be justified by appealing to the central limit theorem. The fluctuation-induced field is generated by a tremendous number of independent, stochastically generated microscopic sources. The sum of all of their contributions generally creates a fluctuation field which is uncorrelated from point-to-point. This is precisely analogous to the situation in the case of Brownian motion, where the random force of the water molecules is uncorrelated temporally because of the large rate of interactions between the water molecules and the Brownian particle.

The Green’s function of the theory, \( G \), is defined by the usual condition

\[ \left( \nabla^2 - \frac{\omega^2}{c^2} \epsilon(\omega, \vec{x}) \right) G(\omega, \vec{x}, \vec{z}) = -\delta(\vec{x} - \vec{z}) \]

Assuming equilibrium conditions, one can then show that the field-field correlations satisfy

\[ \langle \Phi(\omega, \vec{x}) \Phi^*(\omega, \vec{y}) \rangle = \hbar \coth \left( \frac{\hbar \omega}{2k_B T} \right) \text{Im} G(\omega, \vec{x}, \vec{y}) \]

(2.20)

This relation is one of the most important computational results of fluctuational electrodynamics. In the general (that is, vector) theory laid out by Rytov, this result is simply modified to apply to the dyadic Green’s functions of electromagnetism.

These results, which can be thought of as logical consequences of the fluctuation-dissipation theorem and linear response theory, are the portions of Rytov’s theory
which will prove useful to us at a conceptual level. In our later discussions of the
dynamic Casimir effect and non-contact friction, these results will be sufficient to
construct heuristic pictures and motivate key results.
Chapter 3

The Casimir Effect

In this chapter, we introduce the Casimir effect, first described by Hendrik Casimir in 1948 [3]. The Casimir effect has garnered an unfair reputation as some sort of “spooky” manifestation of quantum mechanical zero-point energies. Part of the purpose of this chapter is to show that this interpretation is entirely unnecessary, and can be replaced with a picture of fluctuating fields. After introducing this picture, we will see that the dynamical Casimir effect arises naturally from applying the fluctuation-dissipation theorem to the normal Casimir effect. We also introduce non-contact friction, which is a particularly interesting manifestation of the dynamic Casimir effect.

3.1 Perfect Planar Conductors

3.1.1 The Method of Zero-Point Energies

In Casimir’s original computation, he considered a pair of infinitely large, perfectly conducting planes at zero temperature, with some separation $x$ in between them. Casimir solved the classical Maxwell equations in this gap, obtaining an infinite set of modes with frequencies $\omega_n(x)$. He then ascribed a quantum mechanical zero-point energy of $\frac{1}{2}\hbar\omega_n$ to each mode, so that the total energy of the system is

$$E(x) = \frac{1}{2}\hbar \sum_n \omega_n(x)$$
This quantity is divergent, and therefore, seems meaningless. However, Casimir argued that the force which arises from the change in $E$ as one varies $x$ is finite and meaningful. In this case, the force turns out to be attractive, and demonstrates that uncharged conductors exert forces on one another.

Casimir’s calculation for three dimensions is widely available, so instead of repeating it, we will provide a simple scaling argument which applies in an arbitrary number of dimensions. Let the number of spatial dimensions be $D$. The only physical constants which can arise in the description of the Casimir force are $h$ and $c$, which are both clearly present in Casimir’s original reasoning. The only remaining variables in the problem are the overall area of the plates, which we denote by $L^{D-1}$, and the plate separation $x$. It is reasonable to insist that the force is proportional to the area of the plates, so that dimensional analysis provides the unique combination of objects in the theory with dimensions of force:

$$F(x) \propto \frac{hcL^{D-1}}{x^{D+1}} \quad (3.1)$$

This produces the correct scaling behavior for the force per unit area computed by Casimir for three dimensions. However, the reasoning which enables this scaling argument is worthy of the adjective spooky. Why is it reasonable to ascribe a zero point energy to each mode of the classical electromagnetic field? The procedure introduced by Casimir is somewhat ad hoc, and offers no obvious generalization beyond the simple case of perfectly conducting parallel plates.

This motivates us to introduce a new picture to understand the origins of the Casimir force. This approach is based on the ideas of the fluctuation-dissipation theorem introduced in chapter 2, and, in particular, section 2.2.2.

### 3.1.2 A Method of Fluctuating Fields

In this section, we sketch how one can carry out Casimir force computations without appealing to zero point energies. We will not do any computation explicitly; our goal is simply to show that the results of fluctuational electrodynamics presented in section
2.2.2 are (when appropriately generalized to vector electromagnetism) sufficient for computing Casimir forces.

We begin by considering a system which is very similar to that entertained by Casimir. The only difference is that the plates in this system are not perfect conductors; they have a finite permittivity $\epsilon$ and permeability $\mu$. The materials are assumed to be spatially isotropic and non-dispersive. When the objects carry no net charges or currents, the cartesian components of the electric field $\vec{E}$ and magnetic field $\vec{H}$ vanish (in Fourier space):

$$\langle E_j(\vec{x}) \rangle \omega = \langle H_j(\vec{x}) \rangle \omega = 0$$

These are identical to the observations we made in section 2.2.2 with respect to the scalar field $\Phi$. There are nontrivial correlations between the field components, which can be related to the dyadic Green’s functions of classical electromagnetism:

$$\langle E_i(\vec{x})E_j(\vec{y}) \rangle \omega = \hbar \coth \left( \frac{\hbar \omega}{2k_BT} \right) \text{Im} G_{ij}^{EE}(\omega, \vec{x}, \vec{y})$$

$$\langle H_i(\vec{x})H_j(\vec{y}) \rangle \omega = \hbar \coth \left( \frac{\hbar \omega}{2k_BT} \right) \text{Im} G_{ij}^{MM}(\omega, \vec{x}, \vec{y})$$

(3.2)

Here, the dyadic Green’s function $G_{ij}^{EE}(\omega, \vec{x}, \vec{y})$ indicates the $i$th component of the scattered electric field at $\vec{x}$ induced by a $j$-directed electric source $\vec{y}$, and similarly for the magnetic Green’s function $G_{ij}^{MM}$. This is the natural generalization of (2.20).

With these correlations, one can construct the fluctuation-induced Maxwell stress tensor, which is defined as

$$T_{ij}(\vec{x}, \omega) = \epsilon(\vec{x}, \omega) \left[ \langle E_i(\vec{x})E_j(\vec{x}) \rangle \omega - \frac{\delta_{ij}}{2} \sum_k \langle E_k(\vec{x})E_k(\vec{x}) \rangle \omega \right] + (\epsilon \rightarrow \mu, E \rightarrow H)$$

(3.3)

The interpretation of the fluctuation-induced stress tensor is identical to the interpretation of the “standard” stress tensor. That is, one interprets $T_{ij}$ as a flow of $i$-oriented momentum in the $j$-direction. This means that $\sum_j T_{ij} \hat{n}_j$ represents an $i$-oriented pressure on a surface with normal vector $\hat{n}$. This definition allows one to
compute the force on a dielectric space by enclosing it with an arbitrary surface, and integrating the stress tensor over the boundary of the surface.

While this approach is somewhat more complicated than Casimir’s method of zero point energies, it does not require any ad hoc procedure. It is also extremely general, as this procedure, while labor intensive, can, in principal, be used to compute fluctuation-induced forces between arbitrary dielectric surfaces. Modifications must be made to incorporate spatial inhomogeneities or dispersion, but such adjustments are addressed in the literature on fluctuational electrodynamics [20].

3.1.3 The Dynamic Casimir Effect

The Casimir effect has a lesser-known cousin, the dynamic Casimir effect, which describes the force on, and radiation from, moving dielectrics. In this section, we briefly describe how this effect arises as a consequence of the fluctuation-dissipation theorem and the static Casimir effect. A general overview of the subject can be found in reference [9]. We avoid a prolonged discussion here, as we will continue exploring the dynamic Casimir effect for the rest of the thesis.

Historically, the effect began to gain attention after it was realized that moving mirrors could spontaneously create photons [16]. Spontaneous photon production even occurs for a single mirror dragged through the vacuum. This can be understood as a manifestation of the fact that the vacuum of an interacting quantum field theory (in this case, the vacuum of quantum electrodynamics) is not truly empty; it features stochastic excitations, which, in this case, leads to spontaneous photon production.

Various approaches for demonstrating the existence of this radiation can be found in the literature, including one which makes use of the fluctuation-dissipation theorem. The general picture involves considering the force fluctuations for a single dielectric, and using the fluctuation-dissipation theorem to obtain the associated response function [9]. As we discussed in chapter two, such response functions are manifestly causal, a feature which is not immediately present in other formulations of the dynamic Casimir effect.

Now that we have reviewed the physics which underlies the Casimir effect, we
will begin to consider one of its most interesting manifestations: non-contact friction. This can be thought of as a particular example of the dynamic Casimir effect.

3.2 Non-contact Friction

3.2.1 Origin of the Frictional Force

We will frame our discussion of non-contact friction in the context of a specific geometric arrangement of dielectrics. In what follows, the scalar version of electromagnetism presented in section 2.2.2 will be used in place of the full electromagnetic theory.

Consider two infinite dielectric half-spaces in $D$ dimensions with planar boundaries, which fill all of space except for a gap of width $d$ in between them. Assume they both have a spatially homogeneous dielectric function, $\epsilon(\omega)$, and that the temperature is zero. In this situation, a static Casimir force will be induced between the plates, which can be computed via the fluctuation-induced stress tensor method introduced in section 3.1.2. However, if one half-space is dragged above the other with velocity $v$, an entirely different type of force is induced.

To see what happens, we begin by noting that a wave which propagates in the dielectric with wave vector $\vec{k}$ satisfies the dispersion relation

$$\omega = \frac{c|\vec{k}|}{\sqrt{\epsilon(\omega)}}$$

The frequency $\omega$ can be extracted from this relation, which describes the spectrum of quantum field excitations in the medium. In the frame of the moving half-space, the quantum field excitations are obtained by Lorentz transforming the above dispersion relation:

$$\omega' - vk_x = \frac{c|\vec{k}|}{\sqrt{\epsilon(\omega' - vk_x)}}$$

where $k_x$ denotes the component of $\vec{k}$ which is parallel to the plate velocity. Further, we have assumed that $v \ll c$ so that the wave vector $\vec{k}$ is approximately
invariant under the Lorentz transformation. In the cases we will consider, the field spectra satisfy the relation

\[ \omega' = \omega + v k_x \]

so that the spectrum of excitations in the moving half-space is obtained from the spectrum of the resting half-space by simply adding the term \( v k_x \). The production of scalar particles in each medium becomes a spontaneous process when \( \omega + \omega' \leq 0 \). Heuristically, the creation of scalar particles drains energy from the motion of the half-space, effectively inducing a frictional force on the moving half-space. We term this frictional force non-contact friction.

With this picture in mind, we will now discuss the computation of non-contact friction forces for this geometry.

### 3.2.2 The Non-Contact Friction Formula via the Fluctuation-Dissipation Theorem

A general formula for computing non-contact friction forces in the scenario described in section 3.2.1 has been derived through a variety of formalisms [13, 21]. A careful derivation of this formula is unnecessary for our purposes; we will only state the result and discuss some of its features. We note that the scalar electrodynamic relations presented in section 2.2.2 are the starting point for deriving this formula.

After performing a large number of Green’s functions manipulations and applying relevant portions of scattering theory, one gets from the fluctuation-dissipation theorem (2.20) to the following formula for the friction force, \( f \):

\[
\begin{align*}
    f &= \frac{\hbar L^{D-1}}{(2\pi)^D} \int_0^\infty d\omega \int d\vec{k}_|| k_x \frac{\epsilon^{-2|\vec{k}||^d} (2 \text{Im}[R_1]) (2 \text{Im}[R_2])}{|1 - \epsilon^{-2|\vec{k}||^d} R_1 R_2|^2} \Theta(v k_x - \omega) \\
    &= \frac{\hbar L^{D-1}}{(2\pi)^D} \int_0^\infty d\omega \int d\vec{k}_|| k_x \frac{\epsilon^{-2|\vec{k}||^d} (2 \text{Im}[R_1]) (2 \text{Im}[R_2])}{|1 - \epsilon^{-2|\vec{k}||^d} R_1 R_2|^2} \Theta(v k_x - \omega) 
\end{align*}
\]

(3.4)

Here, \( L^{D-1} \) is the area of the plates, \( \vec{k}_|| \) is the \( D-1 \) dimensional vector constructed from the components of \( \vec{k} \) parallel to the boundaries of the surfaces, and
Finally, \( R_1 \) and \( R_2 \) are the reflection matrices associated with the stationary and moving objects, respectively. These are defined as

\[
R_1 (\omega, \vec{k}) = \frac{i\sqrt{\vec{k}^2 - \frac{\omega^2}{c^2} - \sqrt{\epsilon(\omega) \frac{\omega^2}{c^2} - \vec{k}^2}}}{i\sqrt{\vec{k}^2 - \frac{\omega^2}{c^2} + \sqrt{\epsilon(\omega) \frac{\omega^2}{c^2} - \vec{k}^2}}}
\]

and \( R_2 = R_1 (\omega - vk_z, \vec{k}) \) is evaluated at the Lorentz-transformed frequency. The reflection matrices give the amplitude for reflection of a wave with frequency \( \omega \) and wave vector \( \vec{k} \).

The formula (3.4) has a number of interesting features which are worth commenting on. First, since it vanishes with the imaginary parts of the reflection matrices, the modes which contribute to the friction are evanescent. In general, propagating electromagnetic modes can contribute at finite temperatures, but only evanescent modes remain in the limit \( T \to 0 \) \cite{21}. The exponential suppression of modes with large frequencies and wave vectors is also consistent with the presence of evanescent modes.

Further, the theta function in (3.4) restricts the frequency integration to the range where the Lorentz transformed frequencies are negative. It is generally the case that the onset of the dynamic Casimir effect is accompanied by the presence of “mixing” between frequency modes with opposite sign. This can be made rigorous through the input-output formalism of quantum optics \cite{13, 14}.

Given the formula (3.4), we are now in a position to “solve” a concrete non-contact friction problem. This will be the subject of the remainder of this thesis.
Chapter 4

Non-Contact Friction: Application to Plates With a Plasma Model Dielectric Function

4.1 Statement of The Problem

Our goal in this chapter is to apply the non-contact friction formalism we have discussed to study the following scenario. It is a special case of the general problem described in section 3.2. We begin by restating the problem, with appropriate changes to describe this special case.

Suppose there are two dielectric half-spaces with planar boundaries in $D$ dimensions, which fill all of space except for some gap of length $d$ in between them. Each of the half-spaces obey the following plasma model dielectric function:

$$
\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}
$$

We use a scalar model which is described by a free field theory in vacuum. Inside each of the dielectric half-spaces, the dispersive properties of the medium are described by the dielectric (or response) function of the medium. The field equation for this model is as follows:
\[
\left( \nabla^2 + \epsilon(\omega)\frac{\omega^2}{c^2} \right) \Phi(\omega, \vec{x}) = 0 \tag{4.2}
\]

where \( \epsilon = 1 \) in the gap between the spaces, and is given by the plasma model (4.1) inside the half-spaces. For each half-space, an incident plane wave of frequency \( \omega \) and wave vector \( \vec{k} \) is reflected with amplitude

\[
R(\omega, \vec{k}) = \frac{i\sqrt{\vec{k}_\parallel^2 - \frac{\omega^2}{c^2} - \sqrt{\epsilon(\omega)\frac{\omega^2}{c^2} - \vec{k}_{\parallel}^2}}}{i\sqrt{\vec{k}_\parallel^2 - \frac{\omega^2}{c^2} + \sqrt{\epsilon(\omega)\frac{\omega^2}{c^2} - \vec{k}_{\parallel}^2}}}
\tag{4.3}
\]

Here, \( \vec{k}_\parallel \) is the \( D - 1 \) dimensional vector constructed from the components of \( \vec{k} \) which are parallel to the boundaries of the half-spaces. We shall refer to \( R \) as the reflection matrix of the dielectric. One of the half-spaces is dragged parallel to the other half-space with velocity \( v \ll c \). We assume that \( v \) points in the \( x \)-direction of our coordinate system, so that \( k_x \) denotes the component of \( \vec{k} \) which is parallel to \( \vec{v} \). As we discussed in section 3.2, a frictional force is induced on the moving plate, which can be computed with the following formula:

\[
f = \frac{\hbar L^{D-1}}{(2\pi)^D} \int_0^\infty d\omega \int d\vec{k}_\parallel k_x \frac{e^{-2k_{\perp}d}(2 \text{Im}[R_1])(2 \text{Im}[R_2])}{|1 - e^{-2k_{\perp}d}R_1R_2|^2} \Theta(vk_x - \omega)
\tag{4.4}
\]

Here \( k_{\perp} = \sqrt{\frac{\omega^2}{c^2} - \vec{k}_{\parallel}^2} \), and \( L^{D-1} \) is the area of the plates. We note that the reflection matrix \( R_1 \) describes the dielectric which is at rest, while \( R_2 = R_1(\omega - vk_x, \vec{k}) \) describes the moving dielectric, and is obtained by applying a Lorentz transformation to (4.3). In general, the wave vector must also change under a Lorentz transformation, but this change is second order in \( v/c \), so we neglect it.

This completes our description of the system. Unless otherwise indicated, we assume for simplicity that we are working in two spatial dimensions. In this chapter, we will address the following questions:

- How does the friction force scale with the parameters in the theory, namely, \( \omega_0, \omega_p, d \) and \( v \)?
What are the criteria for maximizing the magnitude of the friction force? How does this magnitude compare with the maxima computed in other non-contact friction scenarios?

4.2 Dimensional Analysis and Scaling

To begin, we appeal to dimensional analysis for information regarding the scaling properties of the friction. Physically, the friction must be extensive in the area of the plates; hence, it is proportional to $L^{D-1}$. Further, the friction must be linear in $h$, for two reasons. First, the existence of charge fluctuations in the limit of zero temperature is due solely to quantum mechanical effects, which demands the presence of some power of $h$. Second, $h$ is the only object in the theory with nonzero mass dimension.

With these two constraints, it is straightforward to confirm that the most general object in this theory with dimensions of force is given by

$$\frac{h L^{D-1} \omega' \alpha + 1}{v'^{\alpha} d^{D-\alpha}}$$

(4.5)

where $\omega'$ is a frequency scale constructed from $\omega_0$ and $\omega_p$, $v'$ is a velocity scale constructed from $v$ and $c$, and $\alpha$ is a real number. The existence of an infinite family of scaling factors is an interesting result, and suggests the presence of multiple power law scaling regimes. However, in general, the scaling form in (4.5) will be modified by dimensionless factors, and from that perspective this scaling form is somewhat redundant. Hence, while (4.5) is useful for the purpose of noting that there is an infinite set of objects with dimensions of force, the following scaling form is more general:

$$\frac{h L^{D-1} v'}{d^{D+1} \bar{g} \left( \frac{v}{c}, \frac{\omega_0}{\omega_p}, \frac{\omega''}{v''} \right)}$$

(4.6)

where $\omega''$ and $v''$ are frequency and velocity scales and $\bar{g}$ is a function of all possible dimensionless parameters in the theory. Note, however, that the list of arguments of
\( \tilde{g} \) in (4.6) is exhaustive: all of the dimensionless parameters in the theory can be constructed from terms whose forms are given by one of the three arguments of \( \tilde{g} \). Further, since it is assumed throughout this problem that \( v \ll c \), we can neglect the first of these arguments, and obtain

\[
\frac{hL^{D-1}}{d^{D+1}} g \left( \frac{\omega_0}{\omega'}, \frac{\omega'' d}{\nu''} \right)
\] (4.7)

The fact that there are only two meaningfully distinct dimensionless forms in this theory strongly constrains the set of scaling relations in terms of one another, as we will see later.

4.3 Constraints on Integration Parameters

The formula (4.4) which we have introduced for computing the friction requires one to perform \( D \) integrals, each of which extends over an infinite region. In general, this is intractable analytically, and we must resort to using numerical methods to compute these integrals. Throughout this chapter, we will present numerical results to supplement and affirm our analytic findings.

However, it is foolish to blindly apply numerical integration techniques without first obtaining some intuition for the behavior of the integrand within the parameter space. In this section, we will show that the integrals in (4.4) are very strongly constrained for a number of physical reasons. In addition to providing a significant increase in efficiency for numerical calculations, these constraints will yield valuable physical intuition which will be essential for several of our analytic arguments.

4.3.1 The Perspective of Reflection Matrices

We begin by asking a very simple question: when does the integrand in (4.4) vanish? Neglecting the theta function, this can only occur when one of the reflection matrices \( R_1 \) or \( R_2 \) has zero imaginary part. Consider first \( R_1 \), which is given in two dimensions by
\[
R_1(\omega, k_x) = \frac{i \sqrt{k_x^2 - \frac{\omega^2}{c^2}} - \sqrt{\epsilon(\omega) \frac{\omega^2}{c^2} - k_x^2}}{i \sqrt{k_x^2 - \frac{\omega^2}{c^2}} + \sqrt{\epsilon(\omega) \frac{\omega^2}{c^2} - k_x^2}}
\] (4.8)

This has a nonzero imaginary part only if \(k_x^2 - \frac{\omega^2}{c^2}\) and \(\epsilon(\omega) \frac{\omega^2}{c^2} - k_x^2\) have the same sign. However, the theta function which appears in (4.4) restricts us to the regime where \(k_x^2 - \frac{\omega^2}{c^2} > 0\). Hence, the integrand is nonzero when \(\epsilon(\omega) \frac{\omega^2}{c^2} - k_x^2 > 0\). Comparing these equations, we can see that \(\epsilon(\omega) > 1\). It is worthwhile to write this out explicitly:

\[
\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} > 1
\]

\[
\Rightarrow \omega < \omega_0
\] (4.9)

This result, while simple, is powerful. Mathematically, this reduces one of the integrals which previously extended over an infinite region into one over a finite region. This is extremely helpful for performing numerical approximations. Physically, this condition carries profound significance for the limit \(d \to 0\), which we will discuss in later sections of this chapter.

We now move on to consider the second reflection matrix, \(R_2\). In two dimensions, it is given by

\[
R_2(\omega - vk_x, k_x) = \frac{i \sqrt{k_x^2 - (\omega - vk_x)^2} - \sqrt{\epsilon(\omega - vk_x) \frac{(\omega - vk_x)^2}{c^2} - k_x^2}}{i \sqrt{k_x^2 - (\omega - vk_x)^2} + \sqrt{\epsilon(\omega - vk_x) \frac{(\omega - vk_x)^2}{c^2} - k_x^2}}
\] (4.10)

Once again, \(R_2\) has a nonzero imaginary part only if the arguments of the square roots in \(R_2\) have the same sign. Noting that

\[
\frac{(\omega - vk_x)^2}{c^2} \leq \frac{v^2 k_x^2}{c^2} < k_x^2
\]

we can deduce the constraint \(\epsilon(\omega - vk_x) > 1\). Writing this out yields another useful bound:
Physically, both of the constraints (4.9) and (4.11) come from the fact that only evanescent modes contribute to the friction in this problem.

4.3.2 The Perspective of Dispersion Relations

The bounds which we derived in the previous section are tremendously helpful for carrying out numerical computations, in addition to refining our understanding of the overall problem. However, it is also possible to derive nontrivial lower bounds on both the allowed frequencies and wave vectors for the friction integrals (4.4). This can be done by considering the dispersion relations of the dielectric half-spaces.

To see why the dispersion relations contain the information we desire, consider the field equation (4.2). From a field-theoretic perspective, the dispersion relation of a half-space describes the spectrum of field excitations in that half-space. In particular, each frequency which is allowed by the friction integral (namely, those in the range \([0, \omega_0]\)) should appear as an allowed excitation frequency in the half-space at rest, and conversely. We emphasize that the dispersion relation of the moving body will contain excitations whose frequencies lie outside of the range \([0, \omega_0]\), as those frequencies are related to the frequencies in the rest frame via Lorentz transformation.

With this motivation, let us turn to the dispersion relation of the half-space at rest:

\[
\omega(k) = \frac{c|k|}{\sqrt{\varepsilon(\omega)}}
\]  

(4.12)

Solving this equation for \(\omega\) yields the following two spectra:
These functions (in addition to the light cone \( c|k| \) and resonant frequency \( \omega_0 \)) are plotted in figure 4-2. It is straightforward to show that \( \omega_2(\vec{k}) > c|\vec{k}| \), which implies that \( \omega_2 \) does not describe an evanescent spectrum. Hence, it cannot contribute to the friction, and we need not consider it further. In contrast, the spectrum \( \omega_1 \) is evanescent, so it does contribute to the friction. It is easy to verify that \( \omega_1 \) has the following properties:

\[
\lim_{k \to 0} \omega_1(\vec{k}) = 0 \\
\lim_{|k| \to \infty} \omega_1(\vec{k}) = \omega_0
\]

Further, since \( \omega_1 \) is a monotonically increasing function of \( |\vec{k}| \), we can conclude that the allowed frequencies are bound to the range \( [0, \omega_0] \). This is to be expected, given the constraint (4.9) and the fact that the dispersion relation completely characterizes the spectrum of field excitations in the dielectric.

We now consider the dispersion relation for the moving half-space, which is obtained by applying a Lorentz transformation to the dispersion relation of the resting half-space. After Lorentz transforming, the spectrum of excitations satisfies

\[
\omega - vk_x = \frac{c|k|}{\sqrt{\epsilon(\omega - vk_x)}}
\]

Once again, we can solve for \( \omega(\vec{k}) \), which yields the following spectra:
Figure 4-1: This is a plot of the spectra $\omega_1$ and $\omega_2$ against $k$, with $\omega_0 = 100, \omega_p = 200$. The light cone and resonant frequency are also plotted for reference. Note that $\omega_1$ is bounded from above by $\omega_0$, while $\omega_1$ is above the light cone.

\begin{align*}
\omega'_1(k) &= v k_x + \sqrt{\frac{\omega_0^2 + \omega_p^2 + c^2 k^2 - \sqrt{(\omega_0^2 + \omega_p^2 + c^2 k^2)^2 - 4c^2\omega_0^2 k^2}}{2}} \\
\omega'_2(k) &= v k_x + \sqrt{\frac{\omega_0^2 + \omega_p^2 + c^2 k^2 + \sqrt{(\omega_0^2 + \omega_p^2 + c^2 k^2)^2 - 4c^2\omega_0^2 k^2}}{2}}
\end{align*}

Neither of these spectra are bounded above by the light cone, so there is no obvious reason to disqualify either $\omega'_1$ or $\omega'_2$. However, we argued in section 3.2.1 that the onset of friction is associated with the spontaneous creation of scalar excitations in the dielectrics. In order to support this process, the spectra of the moving and resting half-spaces must sum to a non-positive value. Since the half-space at rest supports the spectrum $\omega_1$, there can only be a nonzero frictional force when at least one of the following conditions are satisfied:

\begin{align*}
\omega_1 + \omega'_1 &\leq 0 \\
\omega_1 + \omega'_2 &\leq 0
\end{align*}

It is straightforward to show that $\omega_1 + \omega'_2 > 0$, so the spectrum $\omega'_2$ does not contribute to the friction. Hence, the only spectra which we need to consider are $\omega_1$
and \( \omega'_1 \) (see figure 4-3).

Now that the spectra \( \omega_2 \) and \( \omega'_2 \) have been eliminated, we return to the first constraint in (4.16). Define \( k_0 \) by the condition \( \omega_1(k_0) + \omega'_1(k_0) = 0 \). This yields

\[
 k_0 = 2\sqrt{\frac{4c^2\omega_0^2 - v^2\omega_p^2}{4c^2v^2 - v^4}} 
\]  

(4.17)

Since \( k_0 < \frac{2\omega_0}{v} \), (4.17) serves as a lower bound on the allowed range of wave vectors. This, in turn, supplies a lower bound on the allowed range of frequencies, given by \( \omega_1(k_0) \):

\[
\omega_1(k_0) = \sqrt{\omega_0^2 - \frac{v^2\omega_p^2}{4c^2 - v^2}}
\]  

(4.18)

These lower bounds on \( \omega \) and \( k \) vanish when

\[
v \geq \frac{2c}{\sqrt{1 + \frac{\omega_p^2}{\omega_0^2}}} \equiv 2v_0
\]  

(4.19)

Figures 4-4 and 4-5 provide pictures which are useful for visualizing these constraints. We can summarize our results by writing the friction integral in two dimensions as follows (note the integration limits):

\[
f = \frac{L}{4\pi^2} \int_{\omega_1(k_0)}^{\omega_0} d\omega \int_{k_0}^{2\omega_0} dk_x h_k e^{-2|k_x|d} \frac{(2 \text{ Im}[R_1]) (2 \text{ Im}[R_2])}{|1 - e^{-2|k_x|d}R_1R_2|^2} \Theta(vk_x - \omega) 
\]  

(4.20)

### 4.4 The Limit of Large Separations

#### 4.4.1 The Constant Dielectric Function Approximation

The first situation which we will study in detail is the limit where the plate separation, \( d \), is much larger than all other length scales in the problem. In this limit, the exponential term \( \exp[-2|k_x|d] \) in (4.20) indicates that only modes with "small" values of \( k_x \) and \( \omega \) will make a sizable contribution to the friction. The Heaviside function
Figure 4-2: This is a plot of the spectra $\omega_1$ and $\omega'_1$, with $\omega_0 = 100$, $\omega_p = 500$, and $v = 0.1$. Note that the spectrum $\omega'_1$ is essentially identical to $\omega_1$, except that it has been “tilted”, acquiring a slope which is proportional to $v$ for large $|k|$. The frequency $\omega_0$ and wave vector $\frac{2\omega_0}{v}$ are also indicated. In general, we will choose the boundaries of the $k$-axis in our plots such that $k = \frac{2\omega_0}{v}$ on the boundary.

Figure 4-3: This is a plot of the spectra $\omega_1$ and $\omega_1 + \omega'_1$, with $\omega_0 = 100$, $\omega_p = 14000$, and $v = 0.01$. The wave vectors $k_0$ and $\frac{2\omega_0}{v}$ are both indicated, along with the frequencies $\omega_0$ and $\omega_1(k_0)$. Note that $v < 2v_0$, so that $k_0$ and $\omega_1(k_0)$ have taken on nonzero values.
Figure 4-4: This is a plot of the spectra $\omega_1$ and $\omega_1 + \omega_1'$, with $\omega_0 = 100$, $\omega_p = 14000$, and $v = 0.02$. The wave vector $\frac{2\omega_0}{v}$ and frequency $\omega_0$ are also indicated. Note that $v > 2v_0$, so that $\omega_1 + \omega_1' < 0$ for all $k \in [0, \frac{2\omega_0}{v}]$.

in (4.20) allows us to write this exponential as

$$\exp[-2k|d|] = \exp \left[ -2d\sqrt{\frac{k^2}{v^2} - \frac{\omega^2}{c^2}} \right] \leq \exp \left[ -2d\sqrt{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}} \right] \approx \exp \left[ -2d\frac{\omega}{v} \right]$$

This implies that a particular frequency $\omega$ can be ignored when $d >> \frac{v}{2\omega}$. Hence, when $d >> \frac{v}{2\omega}$, a series expansion of $\epsilon(\omega)$ of the following form is justified:

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} = 1 + \frac{\omega_p^2}{\omega_0^2} \left( 1 + \frac{\omega}{\omega_0} + \cdots \right) \approx 1 + \frac{\omega_p^2}{\omega_0^2} \quad (4.21)$$

In the limit of large separations, this tells us that the dielectric function is approximately constant. The existence of a regime in which the dielectric function is constant can also be inferred from figure 4-2, where the dispersion relation for the resting half-space grows linearly for $|k| \approx 0$. The result (4.21) is equivalent to saying that the slope of $\omega_1$ for $|k| \approx 0$ is given by $\frac{\partial \omega_1}{\partial k} = c/\sqrt{1 + \frac{\omega_p^2}{\omega_0^2}}$.

The asymptotic power law dependence of the friction on separation can now be extracted through a trivial rescaling of the friction integral, as follows. Beginning with (4.4), consider the rescaling $k_j \rightarrow k_j/d$, $\omega \rightarrow \omega/d$, where $k_j$ denotes the $j$th component of $\vec{k}_\parallel$. This yields
\[ f = \frac{hL^{D-1}}{(2\pi)^D d^{D+1}} \int_0^\infty d\omega \int d\vec{k}_\parallel k_x \frac{e^{-2|k_\perp|} (2 \text{Im}[R_1]) (2 \text{Im}[R_2])}{|1 - e^{-2|k_\perp| R_1 R_2|^2}|} \Theta(v k_x - \omega) \] (4.22)

The significance of this result follows from the fact that all of the separation dependence is contained within the overall factor \(1/d^{D+1}\). Referring back to our general dimensional analysis argument (4.5), we identify the exponent \(\alpha (d \to \infty) = -1\). This value of \(\alpha\) indicates that the friction is insensitive to the magnitudes of either \(\omega_0\) or \(\omega_p\), and depends only on their ratio. A plot of friction per unit area vs. velocity in the large separation limit is shown in figure 4-6.

These results are in agreement with those of Kardar, Maghrebi, and Golestanian, who studied the properties of the friction force in the case of dielectric half-spaces with a constant dielectric function [13]. The only substantial qualitative difference between these cases arises for the regime \(v < 2v_0\). In our case of a plasma model dielectric function, the friction is heavily suppressed in this regime, but nonzero. In contrast, the friction vanishes completely for \(v < 2v_0\) when the dielectric function is a constant. The study of the scaling of the friction for \(v < 2v_0\) in the plasma model case is the subject of the next section.

### 4.4.2 Saddle Point Calculations: Corrections to the Constant Dielectric Function Approximation

The method which we have presented of extracting the asymptotic separation dependence of the friction is only effective to first order. The series expansion of the dielectric function in (4.21) breaks down for \(v < 2v_0\), since the lower bounds on \(k\) and \(\omega\) (given in 4.17 and 4.18) become nonzero. This means that modes with \(\omega \approx \omega_0\) become non-negligible. Our goal in this section is to establish a systematic method of computing the functional form of the contributions to the friction beyond the first-order power law of \(1/d^{D+1}\).

In the limit \(d \to \infty\), the friction (4.4) can be approximated as follows:
Figure 4-5: This is a log-log plot of the friction force (per unit area) vs. velocity, with \( \omega_p = 10^4 \) and \( \omega_0 = 10 \). Below the velocity \( v = 2v_0 \), the friction is heavily suppressed; this can be understood by referring to figures 4-4 and 4-5, and noting that modes with large frequencies are exponentially suppressed relative to small frequency modes. This is qualitatively distinct from the case where \( \epsilon \) is a constant, as the friction vanishes identically in that case for \( v < 2v_0 \).

\[
f = \frac{hL^{D-1}}{(2\pi)^D} \int_0^\infty d\omega \int \tilde{\kappa} \int \kappa^2 e^{-2|k_\perp|d} (2 \text{Im}[R_1]) (2 \text{Im}[R_2]) \Theta (\nu k_\perp - \omega)
\] (4.23)

This is a common approximation in multiple scattering problems, and is physically justified by noting that the amplitudes of reflected waves become negligible in the limit of large separations [15]. For simplicity, we now specialize to two dimensions.

Since the integrand in (4.23) is exponentially suppressed in the limit \( d \to \infty \), a saddle point approximation is appropriate. Explanations of this technique can be found in various sources, such as references [7, 5]. In general, the technique proceeds by expanding the function in the exponential around its maximum in the integration region. This series can be truncated with little error in the limit where the argument of the exponential goes to infinity; further, the error associated with the truncation can be systematically estimated.

In this case, the argument of the exponential is \(-2d|k_\perp|\), and it must be maximized. The theta function which appears in (4.4) enforces \( k \geq \frac{\omega}{\nu} \), so that
Unfortunately, in the limit \( k \to \frac{\omega}{v} \), the rest of the integrand vanishes. This is because the reflection matrix of the moving object, \( R_2 \), is evaluated at frequency \( \omega - vk = 0 \), so that \( \text{Im} R_2 = 0 \). In order to proceed, the question which must be answered is the following: for a fixed value of the integration variable \( \omega \), what is the minimum value of \( k \) such that the integrand in (4.23) is nonzero? The constraints derived in section 4.3 provide an answer to this question, and it turns out that the exact criteria which is needed is

\[
\epsilon(\omega - vk)(\omega - vk)^2 = k^2
\]  

(4.24)

Note that this is just one of the constraints built into (4.10); it so happens that this constraint provides the best bound on \( k \) from below. It is difficult to invert (4.24) to extract a closed-form for \( k \), so two options remain for carrying out the saddle point approximation. The first is to proceed numerically, as \( k \) can be computed in a straightforward way via (4.24) given a choice of the other parameters. The other is to simply neglect the reflection matrices in the numerator of (4.23). Since these terms are not distance-dependent, neglecting them should not have an effect on the overall scaling dependence of the friction force on separation. We will proceed with the second method.

Expanding the function \(-2d|k|\) around the line \( k = \frac{\omega}{v} \) to first order in a Taylor series yields

\[
-2d|k| = -2dk
\]

Hence the integral (4.23) is given approximately by

\[
\frac{hL}{(2\pi)^2} \int_0^{\omega_0} d\omega \int_0^{2\omega_0} v k \exp\{-2dk\} dk
\]  

50
Which evaluates to

\[
v^2 (1 - \exp\left[-\frac{2\Delta\omega}{v}\right]) - \omega_0 d v \exp\left[-\frac{2\Delta\omega}{v}\right] \left(1 + \exp\left[-\frac{2\Delta\omega}{v}\right]\right) - 4d^2 \omega_0^2 \exp\left[-\frac{4\Delta\omega}{v}\right]
\]

\[
\frac{4 v d^3}{(4.25)}
\]

This result indicates that the leading order contribution to the friction as \(d \to \infty\) is given by \(1/d^3\), as we argued previously. All of the other terms, in which both the resonant frequency and different powers of \(1/d\) appear, carry factors which exponentially suppress them. While the method which we used to extract this dependence was somewhat blunt, we note that every approximation which was used served to increase our estimate of the magnitude of the friction. Hence the results in (4.25) are an upper bound for the friction, which indicates that a more careful method of approximation cannot change the result that the subleading terms are exponentially suppressed.

4.5 The Limit of Small Separations

4.5.1 Asymptotic Behavior

We now move on to consider the limit where the plate separation, \(d\), is much smaller than all other length scales in the problem. In this limit, the integral (4.4) can be written in two dimensions as

\[
f = \frac{L}{4\pi^2} \int_{\omega_0(k_0)}^{\infty} \omega \int_{k_0}^{2\omega_0} d k_x h k_x \frac{(2 \text{Im}[R_1]) (2 \text{Im}[R_2])}{|1 - R_1 R_2|^2} \Theta(v k_x - \omega)
\]

\[
(4.26)
\]

This expression has two crucial properties: first, the integrand is an analytic function in the region of interest. Since the imaginary parts of the reflection matrices are each bounded from above by 1, this implies that the integrand has no divergences. Second, the integral is carried out over a finite region of the \((k, \omega)\) plane. Hence, (4.26) is an integral of a finite function over a finite region, and is therefore finite. This indicates that there is no divergence of the friction force associated with the limit.
of small separations. In particular, since all of the separation dependence has been taken out of (4.26), the friction approaches some nonzero constant as \( d \to 0 \) (see figure 4-7). This is a consequence of the fact that the set of allowed frequencies is bounded from above by \( \omega_0 \). Generally, divergences in the small separation limit are ultraviolet in nature, so preventing arbitrarily high frequency modes from contributing to the friction eliminates these divergences.

It is clear in figure 4-7 that the friction changes from constant to decaying behavior around the length scale \( c/\omega_p \). In general, above this length scale, retardation effects become crucial, while below this scale, retardation effects are negligible. Changes in scaling associated with this length scale have been observed in other situations, including the fluctuational force induced between two atoms in relative motion [17].

Referring to the generic scaling form (4.5), we identify the exponent \( \alpha(d \to 0) = D \). This leads us to anticipate that the friction magnitude will exhibit nontrivial scaling dependencies on \( v, \omega_0, \) and \( \omega_p \). Precisely how to extract these scaling relations at this point is unclear, so we turn to numerical results for inspiration.

![Figure 4-6](image-url)

Figure 4-6: This is a log-log plot of the friction force (per unit area) vs. plate separation with \( \omega_0 = 100, v = 0.001, \) and \( \omega_p = 10^7 \). The length scales \( c/\omega_p \) and \( v/2\omega_0 \) are indicated for reference. As \( d \to 0 \), the friction becomes independent of separation, as predicted. As \( d \to \infty \), the friction scales like \( 1/d^3 \), which is also expected, since our numerical computations are done in two dimensions.
4.5.2 Numerical Scaling Results and Friction Maximization

Numerical results for the scaling of the friction with $\omega_0, \omega_p$, and $v$ are shown in figures 4-8, 4-9, and 4-10. In each case, we find that there are two unique scaling regimes which are well-described by integer power laws. Additionally, we find that the point which satisfies $v = 2v_0$ (approximately) separates these scaling regimes. This point is indicated in each plot; however, we have replaced the condition $v = 2v_0$ with $v = \frac{2\omega_c}{\omega_p}$, since

$$v = 2v_0 = \frac{2c}{\sqrt{1 + \omega_p^2/\omega_0^2}} \Rightarrow \frac{\omega_p}{\omega_0} = \frac{2c}{v} \sqrt{1 - \frac{4v^2}{c^2}} \approx \frac{2c}{v}$$ \hspace{1cm} (4.27)$$

It is not a coincidence that each of the power law exponents changes by four as one crosses between the different scaling regimes in each plot. As we noted in section 4.2, the scaling forms in this theory are strongly constrained, since there are only two unique dimensionless forms which can be constructed. This implies that the plots 4-8 through 4-10 are not independent, and it should be possible to reduce the information in these plots into two overall scaling relations.

![Figure 4-7: This is a log-log plot of friction vs. plasma frequency, with $\omega_0 = 1$, $v = 0.003$, and $d = 10^{-10}$. The separation is chosen to be much smaller than the length scales $c/\omega_p$ and $v/\omega_0$. We observe a sharp transition from $\omega_p^4$ scaling (for small $\omega_p$) to a regime where the friction is independent of $\omega_p$.](image-url)
Figure 4-8: This is a log-log plot of friction vs. resonant frequency, with $\omega_p = 10^4$, $v = 0.003$, and $d = 10^{-7}$. We observe a sharp transition from a regime in which the friction scales like $\omega_0^3$ to a regime in which it falls off like $1/\omega_0$.

Figure 4-9: This is a log-log plot of friction vs. velocity, with $\omega_0 = 1$, $\omega_p = 10^6$, and $d = 10^{-10}$. We observe a sharp transition from a regime in which the friction scales like $v^2$ to one in which it falls off like $1/v^2$. 
We will construct these scaling relations presently. For \( v < 2v_0 \), the numerical results indicate that \( f \propto \omega_0^2 \). We can put this into the form (4.5) by writing

\[
f \propto \frac{\hbar \omega_p^4 v^2}{c^4 \omega_0} = \frac{\hbar \omega_p^3}{\omega_0/c} \times \frac{\omega_p/v}{\omega_0/v} \times \frac{v}{c} \tag{4.28}
\]

In contrast, for \( v > 2v_0 \), the numerical results indicate that \( f \propto \omega_0^3 \). Putting this into a form consistent with (4.5) yields

\[
f \propto \frac{\hbar \omega_0^3}{v^2} \tag{4.29}
\]

It is now natural to note that one of the most striking features of figures 4-9 and 4-10 is the transition from growing behavior to decaying behavior. This signifies that the point where the transition occurs (namely, where \( v = 2v_0 \)) maximizes the friction. This straightforward condition provides a robust method of estimating the maximum value of the friction force for a fixed set of material parameters.

### 4.5.3 Origin of the Small Separation Scaling Exponents: The Elastic Limit

The results (4.28) and (4.29) are powerful, but it is unclear how these scaling exponents arise. An important clue comes from comparing the friction integral in two dimensions (with \( v > 2v_0 \)) to the scaling form in (4.29):

\[
f = \frac{L}{4\pi^2} \int_0^{\omega_0} d\omega \int_0^{2\omega_0} dk_x k_x \frac{(2 \text{ Im}[R_1]) (2 \text{ Im}[R_2])}{|1 - R_1 R_2|^2} \Theta(vk_x - \omega) \propto \frac{\hbar L \omega_0^3}{v^2} \tag{4.30}
\]

The exact same scaling form is obtained if we neglect all of the terms in the integrand except for \( \hbar L k_x \Theta(vk_x - \omega) \). That is,

\[
\hbar L \int_0^{\omega_0} d\omega \int_0^{2\omega_0} dk_x k_x \Theta(vk_x - \omega) \propto \frac{\hbar L \omega_0^3}{v^2}
\]

where it is understood that the integral is only carried out over the region where
Im $R_1 \text{Im} R_2 \neq 0$. The effects of this restriction will be neglected throughout this section, as it ultimately amounts to an overall multiplicative factor in each case. Hence, the appropriate scaling form (4.29) can be recovered by assuming that

$$\frac{\text{Im}[R_1]\text{Im}[R_2]}{|1 - R_1 R_2|^2} \approx \text{constant} \quad (4.31)$$

Can the scaling form (4.28) be recovered through the same approximation? To answer this question, one must evaluate the following:

$$f = \hbar L \int_{\omega_1(k_0)}^{\omega_0} d\omega \int_{k_0}^{2\omega_0} dk_x k_x \Theta (vk_x - \omega) = \frac{\hbar L}{2} (\omega_0 - \omega_1(k_0)) \left( \frac{4\omega_0^2}{v^2} - k_0^2 \right) \quad (4.32)$$

Analytic forms for $k_0$ and $\omega_1(k_0)$ are provided in (4.17) and (4.18), respectively. Expanding each of these in powers of velocity yields

$$k_0^2 = \frac{4 \left( 4c^2 - v^2 \omega_0^2 - v^2 \omega_p^2 \right)}{4c^2 v^2 - v^4} \approx \frac{4\omega_0^2}{v^2} - \frac{\omega_p^2}{c^2} + \mathcal{O} (v^4) \quad (4.33)$$

$$\omega_1(k_0) = \sqrt{\omega_0^2 - \frac{v^2 \omega_p^2}{4c^2 - v^2}} \approx \omega_0 - \frac{\omega_p^2 v^2}{8c^2 \omega_0} + \mathcal{O} (v^4)$$

Hence (4.32) yields the scaling relation in (4.29), with corrections proportional to $v^4$.

At this point, it has not been definitively established that it is reasonable to assume the approximation (4.31). However, it is possible to explicitly check the robustness of this approximation by the following method. Denote the friction integrand by $I$. The condition (4.31) requires that $I/k$ is approximately constant over the integration region. We will take a step towards justifying this by showing that $I/k$ is exactly constant on the line defined by the equation $k = \frac{2\omega}{v}$. First note that the reflection matrices $R_1$ and $R_2$ are equal on this line, since

$$|\omega - vk| = |\omega - 2\omega| = \omega$$

Hence, the reflection matrices $R_1$ and $R_2$ are evaluated at the same frequency, and
are therefore equal. We find it useful to write the reflection matrix as follows:

\[ R = R_1 = R_2 = \frac{ip - q}{ip + q} \]

where \( p \) and \( q \) are the same functions that appear in (4.3), evaluated at \( k = \frac{2\omega}{v} \).

Explicitly, these are

\[ p = \sqrt{\frac{4\omega^2 - \omega^2}{v^2 - c^2}} \quad q = \sqrt{\frac{\epsilon(\omega)\omega^2 - 4\omega^2}{v^2}} \]

One can then show that

\[ (\text{Im } R)^2 = \frac{4p^2q^2}{(p^2 + q^2)^2} \]

\[ |1 - R^2|^2 = \frac{16p^2q^2}{(p^2 + q^2)^2} \]

So we find

\[ \frac{(\text{Im } R)^2}{|1 - R^2|^2} = \frac{1}{4} \] (4.35)

Figure 4-11 is a plot of \( I/k \). The line \( k = \frac{2\omega}{v} \) is clearly visible along the center of the integrand. By inspection, it is clear that \( I/k \) is maximized on this line, which can be verified by straightforward differentiation. Since the line \( k = \frac{2\omega}{v} \) maximizes \( I/k \), we expect the contribution from this line to dominate the friction integral. This expectation is reinforced by the observation that \( I/k \) varies “slowly” away from the boundaries of the integration region. This implies that the approximation (4.31) should be reasonable if we evaluate the reflection matrices on the line \( k = \frac{2\omega}{v} \). Using the result (4.35) and the friction integral (4.26), we have

\[ f \approx \frac{L}{4\pi^2} \int_{\omega_1(k_0)}^{\omega_0} d\omega \int_{k_0}^{\frac{2\omega_0}{v}} dk_x h(k_x) \Theta(vk_x - \omega) \] (4.36)

Plots of the scaling data extracted using this approximation are provided in figure 4-12. Comparison with figures 4-8 through 4-10 indicates that (4.31) is a reasonable approximation, as all of the previously quoted power law exponents are reproduced.
to within 10 percent of their original values.

The expression (4.36) indicates that the limit of small plate separations is an approximately elastic limit, where the friction is given by a simple, unweighted integral over all the momenta transferred between the dielectric slabs. Corrections to (4.36) can be systematically computed by expanding the integrand around the line $k = \frac{2\omega}{v}$.

![Figure 4-10: This is a 3D plot of the log of the friction integrand (labeled as $I$) divided by $k$ against $\omega$ and $k$, with $\omega_p = 10^5, \omega_0 = 10, d = 0$, and $v = 0.002$. There is a clear line in the center of the integrand along which the integrand is constant. The equation for this line is given by $k = \frac{2\omega}{v}$.]

4.5.4 Corrections Associated with Finite Separation Effects

When the separation between the dielectric slabs becomes non-negligible compared to other length scales in the problem, the power laws observed in the previous sections break down. Figure 4-13 illustrates how the power law dependence of $f$ on $\omega_0$ and $\omega_p$ changes as the plate separation is increased. The significance of the length scales $v/\omega_0$ and $c/\omega_p$ has already been identified: the former is associated with the exponential suppression of modes with large frequencies, and the latter determines whether or not retardation effects are relevant. Both of these length scales appear in figure 4-13. They cleanly divide the plots into a region where the power laws (4.28) and (4.29) are obeyed, and a region where there are significant deviations from these power laws.

For plate separations which are above the aforementioned length scales, the friction

58
Figure 4-11: These plots show the scaling behavior of the friction under the approximation \((4.36)\). As indicated in the plots, the power law exponents computed in this approximation are within 10 percent of the values computed with the exact definition of the friction. The parameters for each plot are chosen to agree with their counterparts in figures 4-8 through 4-10.
is not expected to exhibit power law dependencies on either $\omega_0$ or $\omega_p$. This is because, as noted in section 4.2, there is a continuous set of allowed scaling exponents for this problem. While these exponents behave nicely and become integers in the limits $d \to 0$ and $d \to \infty$, the intermediate regime is expected to be an interpolating region, where the scaling exponents continuously vary between their asymptotic values. Indeed, our numerical work indicates that this regime is not well-described by power laws.

Figure 4-12: These plots show how the friction scales with $\omega_p$ and $\omega_0$ near the small separation limit. The length scales $c/\omega_p$ and $v/\omega_0$ cleanly separate the regime where the power laws (4.28) and (4.29) hold and a regime where these scaling relations break down.
4.5.5 Comparison of Friction Magnitudes

Part of the motivation for choosing a plasma model dielectric function for this research was the intuitive idea that friction might be enhanced when a resonance is inserted in the dielectric function. Now that a large amount of scaling information for the plasma model case is available, it is worthwhile to compare the magnitude of this force with the force which arises when the dielectric function is a constant.

The saddle point arguments presented in section 4.4.2 make it clear that the two cases would produce roughly the same friction force if the constant dielectric function $\epsilon$ is chosen to coincide with the value $1 + \frac{\omega^2}{\omega_0^2}$. However, in the limit of small separations, it is clear that the frictional force associated with the constant dielectric function will always have a larger maximum than the plasma model, since the former scales like $1/d^{D+1}$ for arbitrarily small separations. Hence, plates with constant dielectric functions will generally experience a larger friction force in the limit of small separations.

There is, however, one important regime where the plasma model is expected to experience a much larger frictional force. This regime is the limit of small separations with $v < 2\nu_0$. For this velocity regime, the constant dielectric function experiences zero friction, while the plasma model only experiences $v^2$ suppression (assuming that $d > v/\omega_0$). Since it is presumably difficult to achieve the velocities necessary to measure the friction force for plates with constant dielectric functions, the plasma model offers an alternative system for which it is more experimentally feasible to measure the non-contact friction force.

The plasma model also has the advantage of applying to many different materials. Since it is derived from a damped oscillator approximation, many systems are described approximately by the plasma model.
Chapter 5

Conclusions

We have applied the established non-contact friction formalism to study planar dielectrics which obey a plasma model. This has led to the discovery of several interesting scaling regimes. In the limit of large dielectric separations, we showed that the plasma model is well-approximated by a constant dielectric function. Saddle point methods were used to show that this is an extremely robust approximation. In the opposite regime of small dielectric separations, the dielectric function is highly nonlinear, and the friction exhibits nontrivial power law dependencies on each of the parameters in the theory. These exponents arise from the fact that the friction approaches an elastic limit as the dielectric separation vanishes. Further, a simple criterion was identified in the limit of small separations for maximizing the friction force.

These results suggest that materials which obey a plasma model could be ideal for experimental measurements, since the friction for these materials does not vanish unless \( v \to 0 \). This means that difficulties associated with achieving a sufficiently large velocity to have a nonzero frictional force can be circumvented with this system.
Bibliography


