Wave Loads on Offshore Wind Turbines

by

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Abstract

Ocean energy is one of the most important sources of alternative energy and offshore floating wind turbines are considered viable and economical means of harnessing ocean energy. The accurate prediction of nonlinear hydrodynamic wave loads and the resulting nonlinear motion and tether tension is of crucial importance in the design of floating wind turbines. A new theoretical framework is presented for analyzing hydrodynamic forces on floating bodies which is potentially applicable in a wide range of problems in ocean engineering. The total fluid force acting on a floating body is obtained by the time rate of change of the impulse of the velocity potential flow around the body. This new model called Fluid Impulse Theory is used to address the nonlinear hydrodynamic wave loads and the resulting nonlinear responses of floating wind turbine for various wave conditions in a highly efficient and robust manner in time domain. A three-dimensional time domain hydrodynamic wave-body interaction computational solver is developed in the frame work of a boundary element method based on the transient free-surface Green-function. By applying a numerical treatment that takes the free-surface boundary conditions linearized at the incident wave surface and takes the body boundary condition satisfied on the instantaneous underwater surface of the moving body, it simulates a potential flow in conjunction with the Fluid Impulse Theory for nonlinear wave-body interaction problems of large-amplitude waves and motions in time domain. Several results are presented from the application of the Fluid Impulse Theory to the extreme and fatigue wave load model: the time domain analysis of nonlinear dynamic response of floating wind turbine for extreme wave events and the time domain analysis of nonlinear wave load for an irregular sea state followed by a power spectral density analysis.

Thesis Supervisor: Paul D. Sclavounos
Title: Professor of Mechanical Engineering
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Chapter 1

Introduction

1.1 Offshore Floating Wind Turbines

Ocean energy is one of the most reliable energy alternatives for countries that have sufficiently large wind and wave sources. Given the steady and strong wind energy resources off the coastline of many countries, the offshore wind farm has become highly attractive as an ideal energy solution [6]. A floating wind turbine system is being considered a key solution to making offshore wind farms feasible from an economic standpoint, and viable as an energy resource.

The offshore floating wind turbine is a complex system that has multiple design objectives, variables and constraints, and this complexity requires an efficient and robust modeling and analysis method. A linear wave theory captures most of the leading order aspects of hydrodynamic wave loads on offshore structures in most sea states. A frequency domain analysis method based on linear wave theory provides an efficient way to explore the design space, to understand the system responses and to obtain fundamental insights into the optimization of the system’s design. Due to the stochastic characteristics of the ocean environment, a reliability-index-based design method with a spectral representation of ocean waves was adopted and has proven greatly useful as an efficient design space exploration tool for offshore floating wind turbines (Tracy [10], Lee [3] and Sclavounos, Tracy and Lee [9]). Two new design
concepts were proposed and developed for a 3-5MW wind turbine as an application for shallow and intermediate water depths, i.e. the tension leg platform (TLP) and the tension leg buoy (TLB) (Sclavounos, Lee, DiPietro, Potenza, Caramuscio and Michele [9]).

The TLP design concept which was inspired by floater designs in the oil and gas industry is based on a floating platform highly constrained with vertical tethers balanced by an excessive buoyancy force upward and it has a large but mild compliance in translational modes of motion. The motion compliance often referred to as set-down, however, is not excessive because the buoyancy effect compensates and restores the system to its mean position. By virtue of the dynamic restoring mechanism, TLP naturally acts to convert the wave energy into a form of inertia and gravity which may help to absorb or mitigate the wave loads via its compliance and to diminish the loads on tethers and anchors in severe sea states. For most cases except for extremely shallow waters, the TLP design appears to be far more attractive than the TLB in terms of the practical operation with offshore wind turbines and the dynamic loading on tethers and anchors.

In a typical design process of TLP-based (or TLB-based) offshore floating wind turbines, it is of a great interest first to determine the design pretension on the tethers given the static payload from the wind turbine. A small design pretension is preferred to minimize the anchor capacity needed on the seafloor, but it has to be large enough to take the dynamic loading from the wave and still maintain a tension on all tethers. The pretension, however, affects the required volume of the floating platform, which in turn, is proportional to the body disturbances in ambient waves, and thus the wave load as well. So an iterative method with an accurate wave load model is necessary to determine the minimum pretension on tethers, the required anchor capacity and the overall floating foundation cost.

Extreme wave environments are often observed on the ocean and they involve
steep and large amplitude waves which create significant nonlinear contributions. In steep waves, ringing loads (Sclavounos [7] and Faltinsen, Newman and Vinje [2]) may occur and excite the floating structure, which in this work is a floating wind turbine. Since the natural frequency of the wind turbine tower falls around $1.7 \ \text{rad/sec}$, this ringing load may have a substantial effect on the fatigue life of the tower, and thus lead to a system failure. Large amplitude waves cause extreme wave loads. The nonlinear hydrostatic force and nonlinear Froude-Krylov and diffraction forces are the greatest concern because they govern limits of the state of loads on tethers and anchors. For TLPs, the nonlinear extreme wave loads may lead to tether overload and tether slack which are undesirable for the foundation design. Nonlinear aspects of the hydrodynamic wave-body interaction have a significant impact on these ringing loads, extreme wave loads and the resultant system responses.

The biggest limitation of a frequency domain analysis based on the linear wave and linear dynamics theory is that the amplitudes of ambient wave and body motions have to be small compared to the ambient wavelength. With a linear theory, the amplification of wave loads could be easily predicted by a standard numerical diffraction-radiation approach. However, the linearity assumptions on the wave and motion amplitudes prevent us from investigating many of the interesting and crucial hydrodynamic interactions between waves and bodies in severe seas as addressed earlier.

Time-domain based analysis is an essential design tool to better predict the nonlinear hydrodynamic wave load and dynamic responses in extreme ocean environments. A proper design evaluation of floating wind turbines requires an accurate simulation of nonlinear system responses, such as the maximum peak value for the pull-out load on the anchors on the sea floor, the maximum dynamic tension of the tethers on the fairleads and the maximum acceleration at the nacelle of the wind turbine. These values are the critical design performance indices of the primary cost of installation and operation of the offshore floating wind turbine. A nonlinear time-domain method is
promising in terms of its computational modeling capability for those extreme events of large amplitude wave and motion.

1.2 Hydrodynamic Theory

Since Froude and Krylov first established a theoretical approach to the hydrodynamic analysis of a floating body’s motion, there has been a tremendous advancement in computational fluid dynamics technology. It has occurred through advances in analytical modeling of fluid flow and numerical representation technique. A strip theory was implemented for the study of wave loads on a body for short waves, followed by a slender body theory which was applied for the wave loads on a slender ship, but is potentially applicable to any floating body in general. A variety of numerical methods have been developed for understanding the relevant physics and accurate prediction of the interaction between fluids and bodies. Among all of these methods, the boundary element method based on the potential flow theory has become one of the most popular tools because of its efficiency, reliability and conciseness (Newman and Sclavounos [4]).

Linear frequency-domain methods are very useful in many applications. Based on linear wave theory, an offshore structure’s response to a random sea can be estimated by superposing the response to each wave frequency component in the wave spectrum. By virtue of linear wave theory, a reliability-index-based design method for a given sea spectrum provides an efficient and accurate analysis tool for hydrodynamic wave loads and system responses, and it captures most of the leading order effects in a mild sea condition.

In severe sea states or large-amplitude body motions, nonlinear effects are of great importance and interest. Examples of nonlinear effects include a nonlinear hydrostatic load by large-amplitude wave elevations, a nonlinear Froude-Krylov force and a ringing load by steep large-amplitude waves, and a nonlinear wave force by
large-amplitude body motions. A nonlinear time-domain simulation is necessary for the accurate prediction of those nonlinear wave-body interaction effects encountered with severe seas. Due to the excessive computational cost involved with the complexity of fluid and body interaction, the three-dimensional fully nonlinear numerical simulations are often quite limited. Most of the recent developments in three-dimensional time-domain methods has resulted in several useful computational methods: boundary-discretization methods which are often called boundary element method (BEM), and volume-discretization methods. Although boundary-discretization methods are much more efficient than volume-discretization methods, the computational cost substantially increases to achieve the desired accuracy and thus limits the practical applications.

The boundary element methods can be formulated in two different ways: the Rankine panel method (RPM) or the transient free surface Green function method (e.g. [1]). Each method can fall into two categories: the body-nonlinear method or the fully nonlinear method. The body-nonlinear method linearizes the dynamic and kinematic free surface conditions about the mean surface which is the z=0 plane, but it takes the exact body boundary condition imposed on the instantaneous body surface under the water, not at the mean position of body surface. The fully nonlinear method however takes the nonlinear boundary conditions on both the free surface and the body wetted surface.

In the present study, the fully nonlinear problem is approached by a weak scatterer formulation based on the transient free-surface Green-function method. Assuming a small body induced wave disturbance compared to the incident wave disturbance, the free surface boundary conditions are linearized about the incident wave surface and the nonlinear body boundary condition is applied on the instantaneous underwater surface of the moving body. By applying this numerical treatment, the nonlinear wave-body interaction problem is simulated by the Green function method to obtain the fluid flow solution.
Based upon the solution for a wave-body interaction, the hydrodynamic pressure over the body surface is obtained by the Bernoulli equation. The direct integration of the pressure over the instantaneous underwater body surface gives a fully nonlinear hydrodynamic force and moment on the body at each time. Since the Bernoulli equation involves a partial temporal derivative and spatial derivative of the velocity potential over the body surface, however, a Bernoulli-based direct integration method often becomes challenging from a computational cost perspective.

A proper application of the momentum conservation principle leads to a new way of obtaining the nonlinear hydrodynamic force and moment on the body that does not involve the temporal and spatial derivatives of the potential over the body surface (Sclavounos [8]). Based on this newly developed theoretical model called Fluid Impulse Theory, the force on the body is expressed by two distinct components: the body surface impulse and the free surface impulse. An order of magnitude analysis demonstrates that the contribution from the free surface impulse is negligible for mild wave steepnesses and body disturbances consistent with a weak-scatterer condition. This thesis focuses upon the weak-scatterer free-surface condition which leads to an accurate and efficient computational implementation: the total fluid force acting on a floating body is modeled by the time rate of change of the impulse of the velocity potential over the body surface only. It presents a new theoretical framework for analyzing the dominant nonlinear hydrodynamic forces on floating bodies which is potentially applicable in a wide range of problems in ocean engineering.

This thesis focuses on the new nonlinear wave load model in conjunction with the Green-function-based potential flow solver to address the nonlinear hydrodynamic wave loads and the resulting nonlinear responses of floating wind turbines in a highly efficient and robust manner in the time domain.
Chapter 2

Hydrodynamic Theories

2.1 Boundary Value Problem

Assuming incompressible, inviscid and irrotational fluid flow, the flow velocity can be described by the gradient of a velocity potential,

\[ \mathbf{V}(x, y, z, t) = \nabla \phi(x, y, z, t) \]  \hspace{1cm} (2.1)

Applying the principle of conservation of momentum, the pressure in the fluid can be described by the Bernoulli equation

\[ P - P_a = -\rho \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \cdot \nabla \phi + gz \right) \]  \hspace{1cm} (2.2)

By the principle of conservation of mass, the Laplace equation is satisfied over the entire fluid domain as

\[ \nabla^2 \phi = 0 \]  \hspace{1cm} (2.3)

On the free surface the mathematical function \( z - \zeta(x, y, z) \triangleq F(x, y, z, t) \) is always seen to be zero by the free-surface fluid particles, so the material derivative of \( F \) has to be zero

\[ \frac{DF}{Dt} = \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) F = 0 \text{ on } z = \zeta \]  \hspace{1cm} (2.4)
which provides the kinematic free-surface condition:

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial \varphi}{\partial x} \frac{\partial \zeta}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial \zeta}{\partial y} = \frac{\partial \varphi}{\partial z} \quad \text{on } z = \zeta
\]  

(2.5)

Applying the Bernoulli equation we can obtain the dynamic free-surface condition:

\[
\frac{\partial \varphi}{\partial t} + \frac{1}{2} \nabla \varphi \cdot \nabla \varphi + gz = 0 \quad \text{on } z = \zeta
\]  

(2.6)

The first order single free-surface condition is as follows:

\[
\frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z} = 0 \quad \text{on } z = 0
\]  

(2.7)

On the body wetted surface, a no-flux boundary condition is required:

\[
\frac{\partial \varphi}{\partial n} = \frac{\partial V}{\partial n} \quad \text{on } S_B(t)
\]  

(2.8)

### 2.2 Source Formulation

The boundary value problem described above may be represented by an integral equation based on a transient free-surface Green function. It can be expressed as derived as follows:

\[
G(P, Q, t) = G^{(0)}(P, Q) \delta(t) + H(t) \tilde{G}(P, Q, t)
\]  

(2.9)

\[
G^{(0)}(P, Q) = \frac{1}{r} - \frac{1}{r_1}
\]  

(2.10a)

\[
\tilde{G}(P, Q, t) = 2 \int_0^\infty \sqrt{g \sin(\sqrt{gkt})} e^{k(z+\zeta)} J_0(kR)dk
\]  

(2.10b)

Where \( \delta(t) \) is the dirac function, \( H(t) \) is the Heaviside step function, and \( J_0 \) a Bessel function of the first kind of order zero; \( P(x, y, z) \) denotes the field point and \( Q(\xi, \eta, \zeta) \)
the source point, both lying in the lower half space \((z \leq 0, \zeta \leq 0)\). So that we set:

\[
R = \sqrt{(x - \xi)^2 + (y - \eta)^2} \quad (2.11a)
\]

\[
\tilde{Q} = (\xi, \eta, -\zeta) \quad (2.11b)
\]

\[
r = |PQ| \quad (2.11c)
\]

\[
r_1 = |P\tilde{Q}| \quad (2.11d)
\]

The memory part of Green Function can be written as:

\[
\tilde{G}(P, Q, t) = \sqrt{gr_1^{-3/2}} F(\mu, \tau) \quad (2.12)
\]

with

\[
F(\mu, \tau) = 2 \int_0^\infty \sqrt{\lambda} \sin(\sqrt{\lambda}r)e^{-\lambda\mu} J_0(\lambda\sqrt{1 - \mu^2})d\lambda, \quad (2.13a)
\]

\[
\mu = -(z + \zeta)/r_1, \quad (2.13b)
\]

\[
\tau = t\sqrt{g/r_1} \quad (2.13c)
\]

Where the integral expression of Bessel function \(J_0(\lambda\sqrt{1 - \mu^2})\) is

\[
J_0(\lambda\sqrt{1 - \mu^2}) = \frac{1}{\pi} \int_0^\pi \cos(\lambda\sqrt{1 - \mu^2}\cos \theta)d\theta
\]

\[
= \frac{2}{\pi} \int_0^{\pi/2} \cos(\lambda\sqrt{1 - \mu^2}\cos \theta)d\theta \quad (2.14)
\]

Replacing the Bessel function \(J_0\) by its integral representation above and reversing the orders of integration gives

\[
F(\mu, \tau) = \frac{4}{\pi} \int_0^{\pi/2} \int_0^\infty \sqrt{\lambda} \sin(\sqrt{\lambda}r)e^{-\lambda\mu} \cos(\lambda\sqrt{1 - \mu^2}\cos \theta)d\lambda d\theta
\]

\[
= \frac{4}{\pi} \int_0^{\pi/2} H(\mu, \tau, \theta)d\theta \quad (2.15)
\]
with

\[
H(\mu, \tau, \theta) = \int_0^\infty \sqrt{\lambda} \sin(\sqrt{\lambda} \tau) e^{-\lambda \mu} \cos(\lambda \sqrt{1 - \mu^2 \cos \theta}) d\lambda
\]

\[
= \int_0^\infty \sqrt{\lambda} \cdot \frac{1}{2i} (e^{i\sqrt{\lambda} \tau} - e^{-i\sqrt{\lambda} \tau}) e^{-\lambda \mu} \cdot \frac{1}{2} (e^{i\lambda \sqrt{1 - \mu^2 \cos \theta}} + e^{-i\lambda \sqrt{1 - \mu^2 \cos \theta}}) d\lambda
\]

\[
= \frac{1}{4i} \int_0^\infty \sqrt{\lambda} \cdot e^{-\lambda \mu} [e^{i(\sqrt{\lambda} \tau + \lambda \sqrt{1 - \mu^2 \cos \theta})} + e^{i(\sqrt{\lambda} \tau - \lambda \sqrt{1 - \mu^2 \cos \theta})} - e^{i(-\sqrt{\lambda} \tau + \lambda \sqrt{1 - \mu^2 \cos \theta})} - e^{i(-\sqrt{\lambda} \tau - \lambda \sqrt{1 - \mu^2 \cos \theta})}] d\lambda
\]

(2.16)

Introducing the transformation \( \lambda = \varepsilon^2 \), the above infinite integral reduces to

\[
H(\mu, \tau, \theta) = \frac{1}{2i} \int_0^\infty \varepsilon^2 \left[e^{\varepsilon^2(-\mu+i\sqrt{1-\mu^2 \cos \theta})+\varepsilon i \tau} + e^{\varepsilon^2(-\mu-i\sqrt{1-\mu^2 \cos \theta})+\varepsilon i \tau}\right. \\
- e^{\varepsilon^2(-\mu+i\sqrt{1-\mu^2 \cos \theta})-\varepsilon i \tau} - e^{\varepsilon^2(-\mu-i\sqrt{1-\mu^2 \cos \theta})-\varepsilon i \tau}\right] d\varepsilon
\]

(2.17)

\[
= \frac{i}{2} \frac{\partial^2}{\partial \tau^2} [N_1(\mu, \tau, \theta) + N_2(\mu, \tau, \theta) - N_1(\mu, -\tau, \theta) - N_2(\mu, -\tau, \theta)]
\]

with

\[
N_1(\mu, \tau, \theta) = \int_0^\infty e^{\varepsilon^2(-\mu+i\sqrt{1-\mu^2 \cos \theta})+\varepsilon i \tau} d\varepsilon,
\]

(2.18)

\[
N_2(\mu, \tau, \theta) = \int_0^\infty e^{\varepsilon^2(-\mu-i\sqrt{1-\mu^2 \cos \theta})+\varepsilon i \tau} d\varepsilon
\]

According to Abramowitz and Stegun (1970, Section 7.4.2), introducing the notation \( a_1 = \mu - i \sqrt{1 - \mu^2 \cos \theta} \), \( a_2 = \mu + i \sqrt{1 - \mu^2 \cos \theta} \), \( b = -i \tau \). We can obtain that

\[
N_1(\mu, \tau, \theta) = \frac{1}{2} \sqrt{\frac{\pi}{a_1}} \frac{\varepsilon^2}{e^{a_1}} \text{erfc}(\frac{b}{\sqrt{a_1}}),
\]

\[
N_2(\mu, \tau, \theta) = \frac{1}{2} \sqrt{\frac{\pi}{a_2}} \frac{\varepsilon^2}{e^{a_2}} \text{erfc}(\frac{b}{\sqrt{a_2}})
\]

(2.19)
Finally, substituting the $H(\mu, \tau, \theta)$ in the equation of $F(\mu, \tau)$,

$$F(\mu, \tau) = \frac{4}{\pi} \int_0^{\pi/2} H(\mu, \tau, \theta)d\theta$$

$$= \frac{2i}{\pi} \int_0^{\pi/2} \frac{\partial^2}{\partial \tau^2} [N_1(\mu, \tau, \theta) + N_2(\mu, \tau, \theta) - N_1(\mu, -\tau, \theta) - N_2(\mu, -\tau, \theta)]d\theta$$

(2.20)

As shown above, the Green Function can be expressed as a combination of complimentary error functions $erfc(z)$, so we need to find the proper methods to calculate complimentary error function, i.e. error function. In the next chapter, the detailed evaluation of Green function will be illustrated.
Chapter 3

Fast Computation of Green Function

3.1 Error Function Approximation

The definition of different kinds of error function is shown below:

1. Error function: \( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \)

2. Complementary error function: \( \text{erfc}(z) = 1 - \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \)

3. Imaginary error function: \( \text{erfi}(z) = -i\text{erf}(iz) \)

4. Dawson’s integral: \( DAW(z) = e^{-z^2} \int_0^z e^{t^2} dt \)

3.1.1 Luke’s Approximation

Through variable change, we can obtain that:

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \overset{t \rightarrow it}{\rightarrow} \frac{2i}{\sqrt{\pi}} \int_0^{-zi} e^{t^2} dt \tag{3.1}
\]

\[
\text{erfi}(z) = -i\text{erf}(iz) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt \tag{3.2}
\]

So Dawson’s integral is:

\[
DAW(z) = e^{-z^2} \int_0^z e^{t^2} dt = \frac{\sqrt{\pi}}{2} e^{-z^2} \text{erfi}(z) \tag{3.3}
\]

\[ \text{erfi}(z) = \frac{2}{\sqrt{\pi}} z e^{-z^2} \left[ C_n\left(\frac{1}{2}, z^2\right) + V_n\left(\frac{1}{2}, z^2\right) \right] \]  

(3.4)

So we can get polynomial approximation of Dawson’s integral (when \( n \) is in large values):

\[
DAW(z) = z \left[ \frac{C_n\left(\frac{1}{2}, z^2\right)}{D_n\left(\frac{1}{2}, z^2\right)} + V_n\left(\frac{1}{2}, z^2\right) \right] 
\]

(3.5a)

\[
C_n\left(\frac{1}{2}, z^2\right) = \sum_{k=0}^{n} a_k z^{2(n-k)} , \quad a = 0
\]

(3.5b)

\[
D_n\left(\frac{1}{2}, z^2\right) = \sum_{k=0}^{n} b_k z^{2(n-k)} , \quad a = 0
\]

(3.5c)

\[
V_n\left(\frac{1}{2}, z^2\right) = \frac{(-1)^{n+1} \pi \Gamma\left(\frac{3}{2}\right) n! \Gamma(n + \frac{3}{2}) z^{2(2n+1)} \times \exp \left[-z^2 + z^2 (z^2 + 2)/4(2n + \frac{3}{2})\right]}{2^{4n+1}(2n + \frac{3}{2}) \left[ \Gamma\left(n + \frac{3}{2}\right) \Gamma\left(n + \frac{3}{4}\right) \right]^2}
\]

(3.5d)

Besides,

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} z e^{-z^2} \left[ \frac{C_n\left(\frac{1}{2}, -z^2\right)}{D_n\left(\frac{1}{2}, -z^2\right)} + V_n\left(\frac{1}{2}, z^2 e^{-i\pi}\right) \right]
\]

(3.6)

\[
\text{erfc}(z) = 1 - \text{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} z e^{-z^2} \left[ \frac{C_n\left(\frac{1}{2}, -z^2\right)}{D_n\left(\frac{1}{2}, -z^2\right)} + V_n\left(\frac{1}{2}, z^2 e^{-i\pi}\right) \right]
\]

(3.7)

where

\[
C_n\left(\frac{1}{2}, -z^2\right) = \sum_{k=0}^{n} a_k (-1)^{(n-k)} z^{2(n-k)} , \quad a = 0
\]

(3.8a)

\[
D_n\left(\frac{1}{2}, -z^2\right) = \sum_{k=0}^{n} b_k (-1)^{(n-k)} z^{2(n-k)} , \quad a = 0
\]

(3.8b)

\[
V_n\left(\frac{1}{2}, z^2 e^{-i\pi}\right) = \frac{(-1)^{n+1} \pi \Gamma\left(\frac{3}{2}\right) n! \Gamma(n + \frac{3}{2}) z^{2(2n+1)} \times \exp \left[-z^2 e^{i\pi} + z^2 e^{i\pi} (z^2 e^{i\pi} + 2)/4(2n + \frac{3}{2}) - i(2n + 1)\pi\right]}{2^{4n+1}(2n + \frac{3}{2}) \left[ \Gamma\left(n + \frac{3}{2}\right) \Gamma\left(n + \frac{3}{4}\right) \right]^2}
\]

(3.9)
Table 3.1: Difference using Luke’s approximation

<table>
<thead>
<tr>
<th>z</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>dif</td>
<td>0</td>
<td>1.83E-11</td>
<td>1.05E-11</td>
<td>-4.09E-11</td>
<td>4.67E-11</td>
<td>1.3E-11</td>
<td>4.79E-11</td>
<td>3.74E-11</td>
<td>7.78E-12</td>
</tr>
</tbody>
</table>

Then we can take \( n = 5 \) as an example: According to the table of Luke, we can calculate the \( C_n, D_n \) and \( V_n \). Substituting them into the formula of \( \text{erf}(z) \), we can obtain the values of \( \text{erf}(z) \) in the first quadrant:

\[ z = x + i*y, \ (x>0) \] and the values on the positive real axis and positive imaginary axis:

\[ z = x + i*0, \ (x>0); \ z = 0 + i*y, \ (y>0) \] Especially, we can compare the values on the real axis to the results of Abramowitz & Stegun (1972, p324).

The difference between Luke’s results and Abramowitz & Stegun’s results is ascending when \( z \) increases. But the maximum difference in the interval \( z = [0, 2] \) is 2.1309e-04, the difference is small so that we can consider Luke’s method as accurate on the positive real axis in the interval \( z = [0, 2] \).

And when \( z = 2+i \), the difference turns to be a little large, the Re(\( \text{erf}(z) \))=1.15131087, and the Im(\( \text{erf}(z) \))=0.12729163 from nist.com, while in my calculation the Re(\( \text{erf}(z) \))=1.1863, and the Im(\( \text{erf}(z) \))=0.02715949. The difference of Im(\( \text{erf}(z) \)) is up to 0.1 when \( z = 2+i \).

### 3.1.2 Poppe’s Approximation

The purpose of Poppe’s algorithm is to evaluate the Faddeeva function:

\[
w(z) = e^{-z^2} erfc(-iz) \tag{3.10}
\]
One of the numerical recursive method can be used for the whole complex domain is:

\[
\begin{align*}
    w(z) \cong \sigma_N^{|z|}(z, h) = & \begin{cases} 
        2/\sqrt{\pi} \cdot s_{-1}, & h > 0 \ (z \text{ in } T) \\
        2/\sqrt{\pi} \cdot r_{-1}, & h = 0 \ (z \text{ in } Q)
    \end{cases} \\
\end{align*}
\]

Where:

\[
\begin{align*}
    r_v = 0, \ s_N = 0 \\
    r_{n-1} = \frac{1/2}{h - iz + (n + 1)r_n} \quad n = v, v - 1, \ldots, 0 \\
    s_{n-1} = r_{n-1}[(2h)^n + s_n], \ \text{if } n \leq N
\end{align*}
\]

For accuracy of 14 significant digits in first quadrant, the parameters are most optimally tuned in the following way:

\[
\begin{align*}
    v(z) = & \left[3 + \frac{1442}{26\rho(z) + 77}\right] \\
    N(z) = 0 \\
    h(z) = 0 \\
\end{align*}
\]

And

\[
\begin{align*}
    v(z) = & \{16 + 26s(z)\} \\
    N(Z) = & \{7 + 34s(z)\} \quad \text{for } z \text{ in } T \\
    h(z) = & 1.88s(z)
\end{align*}
\]

Where:

\[
\begin{align*}
    \rho(z) = & \sqrt{\left(\frac{x}{x_0}\right)^2 + \left(\frac{y}{y_0}\right)^2}, \ \text{with } x_0 = 6.3 \\
    & y_0 = 4.4 \\
    s(z) = & (1 - \frac{y}{y_0})\sqrt{1 - \rho^2(z)}
\end{align*}
\]

According the definition of \(w(z)\), we can obtain the expression of error function and complimentary error function:

\[
erfc(z) = w(iz)e^{-z^2}
\]
\[ \text{erf}(z) = 1 - \text{erfc}(z) \]
\[ = 1 - e^{-z^2}w(iz) \]  \hspace{1cm} (3.18)

The result of numerical calculation tells that Poppe’s method is accurate in area T (medium z) but not in area Q (large z).

3.1.3 Weideman’s Approximation

According to Weideman’s paper, Faddeeva function can be expanded as:

\[ w(z) \approx \frac{1}{\sqrt{\pi}(L - iz)} + \frac{2}{(L - iz)^2} \sum_{n=0}^{N-1} a_{n+1} Z^n \]  \hspace{1cm} (3.19)

Where:

\[ Z = \frac{L + iz}{L - iz}, \quad \text{and} \quad L = 2^{-1/4}N^{1/2} \]  \hspace{1cm} (3.20)

The coefficients \( a_n \) can be approximated by \( A_n \), where

\[ a_n \approx A_n = \frac{1}{2M} \sum_{j=-M+1}^{M-1} (L^2 + t_j^2)e^{-it_j}e^{-in\theta_j}, \quad n = 1, \ldots, N \]  \hspace{1cm} (3.21)

With

\[ t_j = L \tan\left(\frac{1}{2}\theta_j\right), \quad \theta_j = \frac{\pi j}{M}, \quad j = -M + 1, \ldots, M - 1 \]  \hspace{1cm} (3.22)

\( A_n \) is a discrete Fourier transform and computable with one FFT, rather than \( N \) separate summations.

Through the implementation of above method, the relative error is of magnitude \( 10d \), and the magnitude of \( d \) is smaller than -6 in this case. So we can get the conclusion that Weideman’s method is relatively accurate in both small \( z \) and large \( z \). Conclusion Based on the numerical results of above three approximation methods, we found that Luke’s method is accurate in small \( z \), while Poppe’s method is accurate in medium \( z \) (except for the origin), and Weideman’s method is accurate in both small \( z \) and large \( z \). So Weideman’s method is more adaptive to the calculation of Green function.
3.2 ESPRIT Method Approximation

A powerful forecasting algorithm, the ESPRIT method described by Potts-Tasche [5], has been implemented to test its forecasting capabilities. This method tries to fit the wave record to a sum of complex exponentials. The number of complex exponentials is the parameter $M$ to be determined by the algorithm given $L$ (an upper bound for $M$) that needs to be set. The objective is then finding complex coefficients $c$ and $f$ such that:

$$\zeta(t) = \Re \sum_{j=1}^{M} c_j e^{f_j t}$$  \hspace{1cm} (3.23)

Unlike other known implementations of the ESPRIT method that use the rotational invariance property, the Potts-Tasche formulation is based on standard matrix computations such as singular value decomposition (SVD), eigenvalue decomposition, pseudoinverse and standard least squares, all of them available by default in MATLAB and thus easy to implement.

The algorithm itself starts with a wave record with $2N$ samples taken at time interval $\Delta t$. These are used to form a rectangular Hankel matrix $H$, where $l$ is the row number and $m$ is the column number:

$$H_{2N-L,L+1} = \left[ \zeta(l+m)_{l,m=1}^{2N-L,L+1} \right]  \hspace{1cm} (3.24)$$

$$H_{2N-L,L+1} = \begin{bmatrix}
\zeta(1) & \zeta(2) & \ldots & \zeta(L+1) \\
\zeta(2) & \zeta(3) & \ldots & \zeta(L+2) \\
\vdots & \vdots & \ddots & \vdots \\
\zeta(2N-L) & \zeta(2N-L+1) & \ldots & \zeta(2N)
\end{bmatrix}  \hspace{1cm} (3.25)$$

Now the SVD decomposition of $H$ leads to $H = U D W$, where $D$ is a rectangular matrix of the same size as $H$ with zeros everywhere except on the main diagonal, which contains the singular values $\sigma$. The algorithm will set $M$ as the rank of $D$, by analyzing the singular values and discarding all that are below a certain threshold (the product of the biggest singular value $\sigma$ and the relative tolerance $\epsilon$) as being
irrelevant:
\[ \sigma_{M+1} < \varepsilon \sigma_1 \] (3.26)

Now matrix \( W \) needs to be truncated to discard the rows corresponding to the singular values below the desired threshold:
\[ W_{M,L+1} = W \left( 1 : M, 1 : L + 1 \right) \] (3.27)

Defined based on this matrix:
\[ W_{M,L}(s) \equiv W_{M,L+1} \left( 1 : M, 1 + s : L + s \right) \] (3.28)

Taking the cases \( s=0 \) and \( s=1 \):
\[ W_{M,L}(0) = W_{M,L+1} \left( 1 : M, 1 : L \right) \] (3.29)
\[ W_{M,L}(1) = W_{M,L+1} \left( 1 : M, 2 : L + 1 \right) \] (3.30)

These matrices are combined to form FSVDM, that has the form:
\[ F_{SVDM}^M = \left( W_{M,L}(0)^T \right)^+ W_{M,L}(1)^T \] (3.31)

where the symbol \( ^+ \) stands for the Moore-Penrose generalized inverse or pseudoinverse. The eigenvalues \( z_j \) of matrix FSVDM will lead to the exponent coefficients \( f_j \) of the exponential sum as:
\[ f_j = \frac{\log(z_j)}{\Delta t} \] (3.32)

To calculate the coefficients \( c_j \) a simple linear least-squares problem needs to be solved:
\[ \min_{c_j} \left\| \zeta(k) - \sum_{j=1}^{M} c_j e^{f_j t_k} \right\|^2 \] (3.33)

Once all the coefficients \( c_j, f_j \) of the exponential sum have been determined in this way, and if the fit to the initial wave elevation record is good, it is possible to produce
a forecast just by extending the time $t$.
The beauty of this method, unlike FFT, is that it finds the main frequency components
in the signal without forcing an arbitrary discretization of frequency, which should
lead to a more precise determination of the frequency components.

### 3.2.1 ESPRIT method performance

The parameters that need to be studied are the duration $T$ of the signal to be fitted
(the training period), the time step $\Delta t$, the parameter $\epsilon$ (affecting the selection of
the number of exponential components $M$) and the parameter $L$ (the upper value
of $M$, set by the user). Regarding $L$, since it determines the maximum amount of
exponential components, it should be set to be at least twice the number of sinusoidal
components in the exponential (since there will be a pair of complex poles for each
frequency component). However in general (and certainly in the case of analyzing a
real wave height record) the number of frequency components is not known. In this
case the recommendation of Potts-Tasche is to just choose $N=L$, that is, $L$ is half
the number of points in the signal. Setting $L$ in this way does seem to give the best
results, as will be shown.

**ESPRIT fitting performance of Bessel function**

Bessel function is the important component in the Green function:

$$ J_0(\lambda \sqrt{1 - \mu^2}) = \text{Re} \sum_{j=1}^{M} \gamma_j e^{-\varphi_j \lambda \sqrt{1 - \mu^2}} \quad (3.34) $$

The test fitting performance of Bessel function using ESPRIT method is shown below
in Fig. 3-1 and Fig. 3-2.

It turns out that ESPRIT algorithm fits the original dataset of Bessel Function very
well, the absolute error is less than $1.5 \times 10^{-4}$. 

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Figure 3-1: Fitted Bessel function using ESPRIT method

Figure 3-2: Absolute error between the fitted Bessel function and original Bessel function

ESPRIT fitting performance of Green function

In the fluid of infinite depth, the time-domain Green function is defined by the expression:

$$G(P, Q, t) = G^{(0)}(P, Q)\delta(t) + H(t)\tilde{G}(P, Q, t)$$

(3.35)
The memory part of above Green function is:

\[ \tilde{G}(P, Q, t) = \sqrt{gr_1^{-3/2}} F(\mu, \tau) \]  \hfill (3.36)

Where

\[ F(\mu, \tau) = 2 \int_0^\infty \sqrt{\lambda} \sin(\sqrt{\lambda} \tau) e^{-\lambda \mu} J_0(\lambda \sqrt{1 - \mu^2}) d\lambda, \]  \hfill (3.37)

\[ \mu = -(z + \zeta)/r_1, \]  \hfill (3.38)

\[ \tau = t \sqrt{g/r_1} \]  \hfill (3.39)

Green function \( F(\mu, \tau) \) can be represented in the form of a series of decaying exponentials with complex arguments and weights when \( \mu \) is fixed:

\[ F(\tau) = \sum_{j=1}^{M} \gamma_j e^{\tilde{\rho}_j \tau} \]  \hfill (3.40)

The criterion to choose fitting parameter \( M \) is that \( M \) is the least value to keep the absolute error \(< 10^4\), thus the first-order derivative of \( F(\mu, \tau) \) will have three decimal accuracy. The results about fitted Green function are plotted below with different values of \( \mu \) (the range of \( \tau \) are all the same from 0 to 8, the fitting parameter \( M \) is tuned separately):
Figure 3-3: Fitted Green function with $\mu = 0, M = 18, error = 2.48 \times 10^{-5}$

Figure 3-4: Fitted Green function with $\mu = 0.1, M = 12, error = 8.19 \times 10^{-5}$
Figure 3-5: Fitted Green function with $\mu = 0.2, M = 13, error = 2.41 \times 10^{-6}$

Figure 3-6: Fitted Green function with $\mu = 0.3, M = 13, error = 2.48 \times 10^{-6}$
Figure 3-7: Fitted Green function with $\mu = 0.4, M = 12, error = 8.16 \times 10^{-6}$

Figure 3-8: Fitted Green function with $\mu = 0.5, M = 12, error = 3.91 \times 10^{-6}$
Figure 3-9: Fitted Green function with $\mu = 0.6, M = 12, error = 2.55 \times 10^{-6}$

Figure 3-10: Fitted Green function with $\mu = 0.7, M = 11, error = 3.39 \times 10^{-5}$
Figure 3-11: Fitted Green function with $\mu = 0.8, M = 10, error = 1.76 \times 10^{-5}$

Figure 3-12: Fitted Green function with $\mu = 0.9, M = 9, error = 3.83 \times 10^{-5}$
Figure 3-13: Fitted Green function with $\mu = 1.0, M = 8, error = 4.36 \times 10^{-5}$

Then we can plot the fitted Green function with two different variables $\mu$ and $\tau$, the figures below shows the different ranges of $\tau$:

Figure 3-14: Fitted Green function with $\mu = [0, 1], \tau = [0, 8]$
3.3 Conclusion

Comparing the error function approximation and ESPRIT approximation, it is obvious that ESPRIT approximation has easier formula and takes less time to compute. So in the later part of this thesis, the fast computation of Green function will use ESPRIT approximation.
Chapter 4

Numerical Results and Analysis of Wave Loads

As illustrated in Section 2.2, a wave source formulation will be used in the present study to represent the disturbance potentials which are assumed to satisfy the proper boundary conditions and the free-surface condition on \( z = 0 \). The strength of the wave source at source point \( \xi \) at time \( t \) is now the primary unknown.

Using Green's second theorem,

\[
\varphi(\vec{x}, t) = \iint_{S_B(t)} \sigma(\vec{\xi}, t)G^{(0)}(\vec{x}; \vec{\xi})d\vec{\xi} + \int_0^t \int_{S_B(\tau)} d\tau \int_{S_B(\tau)} \sigma(\vec{\xi}, \tau)H_t(\vec{x}; \vec{\xi}, t - \tau)d\vec{\xi}
\]  

(4.1)

where

\[
H_t(\vec{x}; \vec{\xi}, t - \tau) = -\frac{1}{2\pi} \int_0^\infty \sqrt{gk} \sin[\sqrt{gk}(t - \tau)] \exp(kZ)J_0(kR)dk,
\]

(4.2)

\[
Z = z + \zeta
\]

The second term in the right hand side of (4.1) represents the transient wave part of the Green function, the so-called memory effect, which contributes a velocity potential component at time \( t \) arising from the entire time history of the source strength distribution on the body surface. The normal derivative of the velocity potential over the body surface is the key property that is imposed into the body
boundary condition formulation.

Applying the body boundary condition (4.3) by taking a derivative in a normal
direction with respect to the surface at $\vec{x}$:

$$V_n(\vec{x}, t) = \vec{n}_x \cdot \nabla_x \varphi(\vec{x}, t)$$ \hspace{1cm} (4.3)

$$V_n(\vec{x}, t) = \vec{n}_x \cdot \int_{S_B(t)} \sigma(\xi, t) \nabla_x G^{(0)}(\vec{x}; \xi) d\xi + \vec{n}_x \cdot \int_0^t \int_{S_B(\tau)} \sigma(\xi, \tau) \nabla_x H_t(\vec{x}; \xi, \tau - \tau) d\xi$$ \hspace{1cm} (4.4)

By taking the normal derivative and move the impulsive potential term to the
left, we can get:

$$\int_{S_B(t)} \sigma(\xi, t) \frac{\partial}{\partial n_x} G^{(0)}(\vec{x}; \xi) d\xi = - \int_0^t \int_{S_B(\tau)} \sigma(\xi, \tau) \frac{\partial}{\partial n_x} H_t(\vec{x}; \xi, \tau - \tau) d\xi + V_n(\vec{x}, t)$$ \hspace{1cm} (4.5)

This leads to the general source integral equation for the body disturbance velocity
potential over time. In the three dimensional wave-body hydrodynamic interaction
problem, the computation of the memory contribution becomes highly challenging.
There exists computation methods which carry out an efficient memory component
computation which is crucial for implementation in the design process. The ESPRIT
method mentioned in Section 3.2 is the way I use in the following thesis.

Using ESPRIT method for memory component computation, the resulting expres-
sion in a matrix notation is as follows:

$$B_{ij} \sigma(\vec{x}_j, t) = -K_1(\vec{x}_j, t) + V_n(\vec{x}_j, t)$$

$$K_1(\vec{x}_j, t) \triangleq \int_0^t \int_{S_B(\tau)} \sigma(\xi, \tau) \frac{\partial}{\partial n_x} H_t(\vec{x}; \xi, \tau - \tau) d\xi$$ \hspace{1cm} (4.6)

$B_{ij}$ is the influence coefficient matrix which does not depend on time in the present
study. At every time step $t$ the body boundary condition $V_n(\vec{x}_j, t)$ is provided by the
body’s forced oscillation or resultant dynamic response, and the memory component
$K_1(\vec{x}_j, t)$ is evaluated carefully which may require an efficient treatment of principal
value integrals. The strength of source $\sigma(\vec{x}_j, t)$ is determined from the solution of the
linear system:
\[
\{\sigma(t)\} = [B]^{-1}\{-K_1(t) + V_n(t)\}
\]  \hspace{1cm} (4.7)

### 4.1 Added Mass and Damping

Consider a cylinder moving in the deep sea:

![Cylinder moving in the ocean](image)

Figure 4-1: Cylinder moving in the ocean

The surge motion of this cylinder can be represented as below:

\[
\begin{align*}
\xi_1(t) &= A \cos \omega t & t > 0 \\
\dot{\xi}_1(t) &= -A\omega \sin \omega t \\
\xi_1(t) &= 0, & t < 0
\end{align*}
\]  \hspace{1cm} (4.8)

The impulse response function acting on the cylinder is obtained from Bernoulli's
equation:

\[ I_1(t) = -\rho \int_s \varphi(\xi, t)n_1 d\xi \quad (4.10) \]

\[ F_1(t) = \frac{d}{dt} I_1(t) = -\rho \frac{d}{dt} \int_s \varphi(\xi, t)n_1 d\xi \quad (4.11) \]

Consider the velocity of cylinder as our input \( x(t) \), plug \( x(t) \) into Newton’s law:

\[ M \ddot{x} = F_1 = -\dot{x}A_{11} - xB_{11} \quad (4.12) \]

When \( t \to \infty \), the equation above can be rewritten in frequency domain:

\[ \frac{\|F_1\|}{A} = -\omega^2 A_{11} - i\omega B_{11} \quad (4.13) \]

\[ -\frac{\rho}{A} \int_s i\omega \varphi n_1 ds = -\omega^2 A_{11} - i\omega B_{11} \quad (4.14) \]

So we can get the added mass and damping in frequency domain (WAMIT):

\[ -\omega^2 A_{11} = -\text{Re} \left\{ \frac{i\omega \rho}{A} \int_s \varphi n_1 ds \right\} \quad (4.15) \]

\[ -\omega B_{11} = -\text{Im} \left\{ \frac{i\omega \rho}{A} \int_s \varphi n_1 ds \right\} \quad (4.16) \]

The figure below shows the numerical results in frequency domain from WAMIT:
Figure 4-2: Added mass and damping of surge in frequency domain

Back to time-domain:

\[
F_1(t) = -\ddot{x}A_{11} - xB_{11} \quad (4.17)
\]

\[
\frac{F_1(t)}{A} = \omega^2A_{11}\cos\omega t + \omega B_{11}\sin\omega t, \ t \text{ is large} \quad (4.18)
\]

Apply the Fourier analysis:

\[
\int_{T_1}^{T_2} \cos\omega t \frac{F(t)}{A} dt = \omega^2 \frac{1}{2} (T_2 - T_1) A_{11} \quad (4.19)
\]

\[
\int_{T_1}^{T_2} \sin\omega t \frac{F(t)}{A} dt = \frac{1}{2} (T_2 - T_1) B_{11} \quad (4.20)
\]

\[
A_{11} = \frac{\int_{T_1}^{T_2} \cos\omega t \frac{F(t)}{A} dt}{\omega^2 \frac{1}{2} (T_2 - T_1)}, \quad (4.21)
\]

\[
B_{11} = \frac{\int_{T_1}^{T_2} \sin\omega t \frac{F(t)}{A} dt}{\omega^2 \frac{1}{2} (T_2 - T_1)} \quad (4.22)
\]
We can get the added mass and damping in time domain by solving equation above. The time domain results are got and the comparison between time domain and frequency domain is displayed below:

![Graph of Added mass and damping in Surge](image)

Figure 4-3: Added mass and damping of surge comparison between frequency domain and time domain

From Fig. 4-3, we can see that the difference between my time-domain code and WAMIT’s frequency-domain code is fairly small, less than 8%.

### 4.2 Exciting Force

The impulse response function acting on the cylinder is obtained from Bernoulli’s equation:

\[
I_1(t) = -\rho \int_s \varphi(\xi, t)n_1 d\xi
\]

\[
F_1(t) = \frac{d}{dt} I_1(t) = -\rho \frac{d}{dt} \int_s \varphi(\xi, t)n_1 d\xi
\]

Since I have already got potential \( \varphi \) from source formulation (4.1), I can get the results of exciting force,
The convergence test from 300 panels to 4000 panels is listed below, which tells us that the proper number of panels according to convergence is around 1000, I will use this kind of meshing in my future computations.

Figure 4-4: Exciting force in surge direction

Figure 4-5: Convergence test of exciting force in surge direction
Chapter 5

Future Work

In the latter research, the second-order sea state, i.e. second-order incident wave potential will be added to the linear solution presented in above chapters during the computation of exciting forces. The second-order disturbance force due to the change of wetted surface will also be considered. The other kinds of second-order or higher-order exciting forces will be computed as well considering severe sea state and fatigue analysis.

5.1 Second Order Sea State

Perturbation expansion:

\[ \phi = \phi_1 + \phi_2 + \phi_3 + \ldots \quad (5.1) \]

\[ \zeta = \zeta_1 + \zeta_2 + \zeta_3 + \ldots \quad (5.2) \]

The second order boundary condition problem is as below:

\[
o(A^2)^{-1} \begin{cases} 
\nabla^2 \phi_2 = 0 \\
\frac{\partial^2 \phi_2}{\partial t^2} + g \frac{\partial \phi_2}{\partial z} = -\frac{\partial}{\partial t} (\nabla \phi_1 \cdot \nabla \phi_1) + \frac{1}{g} \frac{\partial \phi_1}{\partial t} \frac{\partial}{\partial z} (\frac{\partial^2 \phi_1}{\partial t^2} + g \frac{\partial \phi_1}{\partial z}) \bigg|_{z=0} \\
\zeta_2 = -\frac{1}{g} \frac{\partial \phi_2}{\partial t} + \frac{1}{2} \nabla \phi_1 \cdot \nabla \phi_1 + \zeta_1 \frac{\partial^2 \phi_1}{\partial z \partial t} \bigg|_{z=0}
\end{cases} \quad (5.3)
\]
The right-hand side of the second order free ãšsurance condition is a quadratic function of the linear solution, which can be decomposed as two components:

\[
\text{Re} \left\{ A_1 e^{i\omega_1 t} \right\} \text{Re} \left\{ A_2 e^{i\omega_2 t} \right\} = \frac{1}{2} \text{Re} \left\{ A_1 A_2 e^{i(\omega_1 + \omega_2) t} + A_1 A_2^* e^{i(\omega_1 - \omega_2) t} \right\} \tag{5.4}
\]

Thus we define the yet unknown second-order potential as follows:

\[
\Phi_2 = \text{Re} \left\{ \varphi_2^+ e^{i(\omega_1 + \omega_j) t} + \varphi_2^- e^{i(\omega_1 - \omega_j) t} \right\} \tag{5.5}
\]

The solution of second-order incident wave potential is given here:

\[
\varphi_I^\pm = \frac{Q_{II}^\pm(x, y) Z(k_{ij}^\pm z)}{-(\omega_i \pm \omega_j)^2 + g k_{ij}^\pm \tanh k_{ij}^\pm h} \tag{5.6}
\]

where \(k_{ij}^\pm = |K_i \pm K_j|\)

\[
Z(k_{ij}^\pm z) = e^{k_{ij}^\pm z} \text{ for deep water} \tag{5.7}
\]

\[
Z(k_{ij}^\pm z) = \frac{\cosh(k_{ij}^\pm (z + h))}{\cosh(k_{ij}^\pm h)} \text{ for finite depth water} \tag{5.8}
\]

\[
Q_{II}^+ = -\frac{1}{2} ig^2 A_i A_j \exp(-i(K_i + K_j) \cdot x)
\]

\[
\left[ \frac{k_i^2 - v_i^2}{2\omega_i} + \left( \frac{k_j^2 - v_j^2}{2\omega_j} \right) + \frac{(\omega_i + \omega_j)}{\omega_i \omega_j} (K_i \cdot K_j - v_i v_j) \right]
\]

\[
Q_{II}^- = \frac{1}{2} ig^2 A_i A_j^* \exp(-i(K_i - K_j) \cdot x)
\]

\[
\left[ \frac{k_i^2 - v_i^2}{2\omega_i} - \left( \frac{k_j^2 - v_j^2}{2\omega_j} \right) - \frac{(\omega_i - \omega_j)}{\omega_i \omega_j} (K_i \cdot K_j + v_i v_j) \right] \tag{5.10}
\]
5.2 Higher-Order Wave Loads

5.2.1 Froud-Krylov Force

Froude-Krylov force on mean body surface $\bar{S}_B$

$$F_{F-K}^{\bar{S}_B} = -\rho \frac{d}{dt} \int_{\bar{S}_B} \phi \vec{n}_4 ds$$

$$= -\rho \frac{d}{dt} \int_{\bar{S}_B} (\phi_1 + \phi_2) \vec{n}_4 ds$$

$$= -\rho \frac{d}{dt} \int_{\bar{S}_B} \phi_1 \vec{n}_4 + \phi_2 \vec{n}_4 ds$$

Higher-order Froude-Krylov force on $\Delta S$ (up to third order)

$$F_{F-K}^{\Delta S} = -\rho \frac{d}{dt} \int_{\Delta S} \phi \vec{n}_4 ds$$

$$= -\rho \frac{d}{dt} \int_{WL} \int_{0}^{\zeta} \phi \vec{n}_4 dz dl$$

$$= -\rho \frac{d}{dt} \int_{WL} \int_{0}^{\zeta} (\phi|_{z=0} + (z-0) \frac{\partial \phi}{\partial z}|_{z=0}) \vec{n}_4 dz dl$$

$$= -\rho \frac{d}{dt} \int_{WL} \int_{0}^{\zeta} (\phi|_{z=0} + z \frac{\partial \phi}{\partial z}|_{z=0}) dz \vec{n}_4 dl$$

$$= -\rho \frac{d}{dt} \int_{WL} (\phi|_{z=0} \zeta + \frac{\zeta^2}{2} \frac{\partial \phi}{\partial z}|_{z=0}) \vec{n}_4 dl$$

$$= -\rho \frac{d}{dt} \int_{WL} [(\phi_1 + \phi_2)|_{z=0} (\zeta_1 + \zeta_2) + \frac{(\zeta_1 + \zeta_2)^2}{2} \frac{\partial (\phi_1 + \phi_2)}{\partial z}|_{z=0})] \vec{n}_4 dl$$

$$= -\rho \frac{d}{dt} \int_{WL(z=0)} [(\phi_1 \zeta_1 + \phi_1 \zeta_2 + \phi_2 \zeta_1 + \phi_2 \zeta_2) + \frac{(\zeta_1^2 + \zeta_2^2 + 2 \zeta_1 \zeta_2) \partial (\phi_1 + \phi_2)}{2})] \vec{n}_4 dl$$

$$= -\rho \frac{d}{dt} \int_{WL(z=0)} [(\phi_1 \zeta_1 + \phi_1 \zeta_2 + \phi_2 \zeta_1 + \phi_2 \zeta_2) + \frac{1}{2} (\zeta_1^2 \frac{\partial \phi_1}{\partial z} + \zeta_1^2 \frac{\partial \phi_2}{\partial z} + \zeta_2^2 \frac{\partial \phi_1}{\partial z} + \zeta_2^2 \frac{\partial \phi_2}{\partial z})] \vec{n}_4 dl$$

$$= -\rho \frac{d}{dt} \int_{WL(z=0)} [(\phi_1 \zeta_1 + \phi_1 \zeta_2 + \phi_2 \zeta_1 + \phi_2 \zeta_2) + \zeta_2 \frac{\partial \phi_1}{\partial z} + \zeta_1 \frac{\partial \phi_2}{\partial z} + 2 \zeta_1 \zeta_2 \frac{\partial \phi_1}{\partial z} + 2 \zeta_1 \zeta_2 \frac{\partial \phi_2}{\partial z})] \vec{n}_4 dl$$

(5.14)
Neglect 4th-order and higher order term:

\[ F_{\text{F-K}}^{S_w} = -\rho \frac{d}{dt} \int_{W_L(z=0)} \frac{[\phi_1 \zeta_1 + \phi_1 \zeta_2 + \phi_2 \zeta_1 + \frac{1}{2} \zeta_1 \frac{\partial \phi_1}{\partial z}]}{o(A^2)} dl + o(A^3) \]

Higher-order Froude-Krylov force on \( S_W \) (up to third order)

The definition of Froude-Krylov force on \( S_W \) is:

\[ F_{\text{F-K}}^{S_w} = -\rho \frac{d}{dt} \int_{S_W(z=\zeta(x,y))} \phi \mathbf{n} \cdot d\mathbf{s} \]  \hspace{1cm} (5.16)

Ambient wave plane is defined as:

\[ z = \zeta(x,y) \]  \hspace{1cm} (5.17)

The corresponding surface is:

\[ G(x,y,z) = \zeta(x,y) - z = 0 \]  \hspace{1cm} (5.18)

So by geometry, the normal vector to the ambient wave plane \( S_W \) is defined as:

\[ \mathbf{n} = \frac{\nabla G}{|\nabla G|} = \frac{\nabla (\zeta(x,y) - z)}{|\nabla (\zeta(x,y) - z)|} = \left( \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}, -1 \right) \sqrt{1 + \left( \frac{\partial \zeta}{\partial x} \right)^2 + \left( \frac{\partial \zeta}{\partial y} \right)^2} \]

\[ = \left( \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}, -1 \right) (1 - \frac{1}{2} \left( \frac{\partial \zeta}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \zeta}{\partial y} \right)^2) \]

at least 4th-order when multiplied by \( \phi \)

\[ \mathbf{n} \cdot (\mathbf{x}, \mathbf{y}, -1) \]

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So the x-directional force is:

\[
F_{F_{-K,x}}^{S_w} = -\rho \frac{d}{dt} \int_{S_w(z=\zeta(x,y))} \phi \cdot n_x ds \\
= -\rho \frac{d}{dt} \int_{S_w(z=0)} \left( \phi \big|_{z=0} + \zeta \frac{\partial \phi}{\partial z} \right) \frac{\partial \zeta}{\partial x} ds \\
= -\rho \frac{d}{dt} \int_{S_w(z=0)} \left( \phi \frac{\partial \zeta}{\partial x} + \zeta \frac{\partial \phi}{\partial z} \right) ds \\
= -\rho \frac{d}{dt} \int_{S_w(z=0)} \left[ (\phi_1 + \phi_2) \left( \frac{\partial \zeta_1}{\partial x} + \frac{\partial \zeta_2}{\partial x} \right) + \zeta_1 \zeta_2 \frac{\partial (\phi_1 + \phi_2)}{\partial x} \right] ds \\
(5.20)
\]

Neglect 4th-order terms:

\[
F_{F_{-K,x}}^{S_w} = -\rho \frac{d}{dt} \int_{S_w(z=0)} \left[ \phi_1 \frac{\partial \zeta_1}{\partial x} + \phi_1 \frac{\partial \zeta_2}{\partial x} + \phi_2 \frac{\partial \zeta_1}{\partial x} + \zeta_1 \frac{\partial \phi_1}{\partial x} \right] ds \\
(5.21)
\]

The y-directional force is similar to x-directional force. The z-directional force is:

\[
F_{F_{-K,z}}^{S_w} = -\rho \frac{d}{dt} \int_{S_w(z=\zeta(x,y))} \phi \cdot (-1 + \frac{1}{2} \left( \frac{\partial \zeta}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \zeta}{\partial y} \right)^2) ds \\
= -\rho \frac{d}{dt} \int_{S_w(z=0)} \left( \phi \big|_{z=0} + \zeta \frac{\partial \phi}{\partial z} \right) \left[ (-1 + \frac{1}{2} \left( \frac{\partial \zeta}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \zeta}{\partial y} \right)^2 \right] ds \\
= -\rho \frac{d}{dt} \int_{S_w(z=0)} \left( -\phi + \frac{1}{2} \phi_1 \left( \frac{\partial \zeta_1}{\partial x} \right)^2 + \frac{1}{2} \phi_1 \left( \frac{\partial \zeta_1}{\partial y} \right)^2 - \zeta \frac{\partial \phi}{\partial z} \right) ds \\
= -\rho \frac{d}{dt} \int_{S_w(z=0)} \left[ \left( -\phi_1 + \phi_2 \right) + \frac{1}{2} \phi_1 \left( \frac{\partial \zeta_1}{\partial x} \right)^2 + \frac{1}{2} \phi_1 \left( \frac{\partial \zeta_1}{\partial y} \right)^2 - \zeta_1 \zeta_2 \frac{\partial (\phi_1 + \phi_2)}{\partial z} \right] ds \\
= -\rho \frac{d}{dt} \int_{S_w(z=0)} \left[ \left( -\phi_1 + \phi_2 \right) + \frac{1}{2} \phi_1 \left( \frac{\partial \zeta_1}{\partial x} \right)^2 + \frac{1}{2} \phi_1 \left( \frac{\partial \zeta_1}{\partial y} \right)^2 - \zeta_1 \zeta_2 \frac{\partial (\phi_1 + \phi_2)}{\partial z} + H.O.T \right] ds \\
(5.22)
\]
Neglect 4th-order and higher order terms:

\[
F_{F-Kz}^{Sw} = -\rho \frac{d}{dt} \int_{S_{W(z=0)}} \left[ -\phi_1 - \phi_2 - \zeta_1 \frac{\partial \phi_1}{\partial z} + \frac{1}{2} \phi_1 \left( \frac{\partial \zeta_1}{\partial x} \right)^2 + \frac{1}{2} \phi_1 \left( \frac{\partial \zeta_1}{\partial y} \right)^2 - \zeta_1 \frac{\partial \phi_2}{\partial z} - \zeta_2 \frac{\partial \phi_1}{\partial z} \right] |_{z=0} ds
\]

(5.23)

Total Froude-Krylov force

So the total Froude-Krylov force is the summation of all three above:

\[
F_{F-K}^{TOT} = F_{F-K}^{SB} + F_{F-K}^{ΔS} + F_{F-K}^{Sw}
\]

(5.24)

5.2.2 Diffraction Force

Diffraction force on mean body surface \( S_B \)

\[
F_{diff}^{SB} = -\rho \frac{d}{dt} \int_{S_B} \phi_D \vec{n}_+ ds
\]

\[
= -\rho \frac{d}{dt} \int_{S_B} \left( \phi_1 D + \phi_2 D \right) \vec{n}_+ ds
\]

(5.25)

\[
= -\rho \frac{d}{dt} \int_{S_B} \phi_1 D \vec{n}_+ + \phi_2 D \vec{n}_+ ds
\]

\[
\int_{o(A)} \int_{o(A^2)}
\]
Higher-order diffraction force on $\Delta S$ (up to third order)

\[
F_{aiff}^{\Delta S} = -\rho \frac{d}{dt} \int_{\Delta S} \phi \vec{n} \cdot ds
\]

\[
= -\rho \frac{d}{dt} \int_{W_L} \int_0^\zeta (\phi|_{z=0} + (z - 0) \frac{\partial \phi}{\partial z}|_{z=0}) n^x ds
\]

\[
= -\rho \frac{d}{dt} \int_{W_L} \int_0^\zeta (\phi|_{z=0} + z \frac{\partial \phi}{\partial z}|_{z=0}) dzn^x dl
\]

\[
= -\rho \frac{d}{dt} \int_{W_L} (\phi|_{z=0} \zeta + \zeta^2 \frac{\partial \phi}{\partial z}|_{z=0}) n^x dl
\]

\[
= -\rho \frac{d}{dt} \int_{W_L} [(\phi_{1D} + \phi_{2D})|_{z=0} (\zeta_1 + \zeta_2) + \frac{(\zeta_1 + \zeta_2)^2}{2} \frac{\partial (\phi_{1D} + \phi_{2D})}{\partial z}|_{z=0}] n^x dl
\]

\[
= -\rho \frac{d}{dt} \int_{W_L(z=0)} [(\phi_{1D} + \phi_{1D} \zeta_2 + \phi_{2D} \zeta_1 + \phi_{2D} \zeta_2) + \frac{(\zeta_1^2 + \zeta_2^2 + 2\zeta_1 \zeta_2) \frac{\partial (\phi_{1D} + \phi_{2D})}{\partial z}}{2}] n^x dl
\]

\[
= -\rho \frac{d}{dt} \int_{W_L(z=0)} [(\phi_{1D} \zeta_1 + \phi_{1D} \zeta_2 + \phi_{2D} \zeta_1 + \phi_{2D} \zeta_2) + \frac{1}{2} (\zeta_1^2 \frac{\partial \phi_{1D}}{\partial z} + \zeta_2^2 \frac{\partial \phi_{2D}}{\partial z} + \zeta_1^2 \frac{\partial \phi_{1D}}{\partial z} + \zeta_2^2 \frac{\partial \phi_{2D}}{\partial z}) + 2\zeta_1 \zeta_2 \frac{\partial \phi_{1D}}{\partial z} + 2\zeta_1 \zeta_2 \frac{\partial \phi_{2D}}{\partial z}] n^x dl
\]

(5.26)

Neglect 4th-order and higher order terms:

\[
F_{aiff}^{\Delta S} = -\rho \frac{d}{dt} \int_{W_L(z=0)} \left( \phi_{1D} \zeta_1 + \phi_{1D} \zeta_2 + \phi_{2D} \zeta_1 + \phi_{2D} \zeta_2 \right) n^x dl
\]

(5.27)

**Total diffraction force**

So the total diffraction force is the summation of all three above:

\[
F_{aiff}^{TOT} = F_{aiff}^{S_B} + F_{aiff}^{\Delta S}
\]

(5.28)
5.3 Partial Numerical Results of Nonlinear Wave Loads

The second-order Froud-Krylov force and second-order diffraction force on \( \Delta S \) is shown below:

![Figure 5-1: Second-order Froud-Krylov force and second-order diffraction force due to \( \Delta S \)](image)

![Figure 5-2: Second-order Froud-Krylov force and second-order diffraction force due to \( \Delta S \) compared to linear force](image)

The other nonlinear components will be computed in the future researches.
5.4 Conclusion

In the usage of the free-surface stretching method, the numerical body-surface was generated at each time step by varying the draft of the buoy in a numerical stretched domain. For extremely steep waves or for non-slender floating bodies, however, the exact mapping of the underwater body-surface into a numerical domain may be required, and investigating this effect would be needed. The validation of nonlinear responses of the bottom mounted wind turbine through model testing or full-scaled experiments would be useful. In such a validation study the viscous forces on the floater need to be added as Morrison like terms functions of the relative wave and body kinematics.
Appendix A

Sea States

The typical sea state spectrum and corresponding wave elevation in both operational and severe sea state are shown in figures below:

Figure A-1: Wave Spectrum with Significant Wave Height of 6m
Figure A-2: Wave Elevation with Significant Wave Height of 6m
Figure A-3: Wave Spectrum with Significant Wave Height of 10m
Figure A-4: Wave Elevation with Significant Wave Height of 10m
Bibliography


