REVIEW Lecture 25:

- Solution of the Navier-Stokes Equations
  - Pressure Correction Methods: i) Solve momentum for a known pressure leading to new velocity, then; ii) Solve Poisson to obtain a corrected pressure and iii) Correct velocity, go to i) for next time-step.

- A Simple Explicit and Implicit Schemes
  - Nonlinear solvers, Linearized solvers and ADI solvers

- Implicit Pressure Correction Schemes for steady problems: iterate using
  - Outer iterations:
  \[
  A^{u_w} u_i^{n+1} = b_{u_w}^{n+1} - \frac{\delta p}{\delta x_i}^{m-1}
  \]
  but require
  \[
  A^{u_w} u_i^m = b_{u_w}^m - \frac{\delta p}{\delta x_i}^m
  \]
  and \[
  \frac{\delta u_i^m}{\delta x_i} = 0
  \implies \frac{\delta}{\delta x_i} \left( \frac{\delta p}{\delta x_i} \right) \approx \frac{\delta}{\delta x_i} \left( A^{u_w} u_i^m - b_{u_w}^m \right)
  \]
  - Inner iterations:
  \[
  A^{u_w^{n+1}} u_i^m = b_{u_w^{n+1}}^m - \frac{\delta p}{\delta x_i}^m
  \]

- Projection Methods: Non-Incremental and Incremental Schemes

- Fractional Step Methods:
  \[
  u_i^{n+1} = u_i^n + (C_i + D_i + P_i) \Delta t
  \]
  Example using Crank-Nicholson

- Streamfunction-Vorticity Methods: Scheme and boundary conditions
TODAY (Lecture 26): Navier-Stokes Equations and Intro to Finite Elements

• Solution of the Navier-Stokes Equations
  – Pressure Correction / Projection Methods
  – Fractional Step Methods
  – Streamfunction-Vorticity Methods: scheme and boundary conditions
  – Artificial Compressibility Methods: scheme definitions and example
  – Boundary Conditions: Wall/Symmetry and Open boundary conditions

• Finite Element Methods
  – Introduction
  – Method of Weighted Residuals: Galerkin, Subdomain and Collocation
  – General Approach to Finite Elements:
    • Steps in setting-up and solving the discrete FE system
    • Galerkin Examples in 1D and 2D
  – Computational Galerkin Methods for PDE: general case
    • Variations of MWR: summary
    • Finite Elements and their basis functions on local coordinates (1D and 2D)
    • Unstructured grids: isoparametric and triangular elements
References and Reading Assignments


• Chapters 31 on “Finite Elements” of “Chapra and Canale, Numerical Methods for Engineers, 2006.”
Artificial Compressibility Methods

• Compressible flow is of great importance (e.g. aerodynamics and turbine engine design)

• Many methods have been developed (e.g. MacCormack, Beam-Warming, etc)

• Can they be used for incompressible flows?

• Main difference between incompressible and compressible NS is the mathematical character of the equations
  – Incompressible eqns: no time derivative in the continuity eqn: \( \nabla \cdot \vec{v} = 0 \)
    • They have a mixed parabolic-elliptic character in time-space
  – Compressible eqns: there is a time-derivative in the continuity equation:
    • They have a hyperbolic character: \( \frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0 \)
    • Allow pressure/sound waves

• How to use methods for compressible flows in incompressible flows?
Artificial Compressibility Methods, Cont’d

• Most straightforward: Append a time derivative to the continuity equation
  – Since density is constant, adding a time-rate-of-change for \( \rho \) not possible
  – Use pressure instead (linked to \( \rho \) via an eqn. of state in the general case):

\[
\frac{1}{\beta} \frac{\partial p}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0
\]

• where \( \beta \) is an artificial compressibility parameter (dimension of velocity\(^2\))

• Its value is key to the performance of such methods:
  – The larger/smaller \( \beta \) is, the more/less incompressible the scheme is
  – Large \( \beta \) makes the equation stiff (not well conditioned for time-integration)

• Methods most useful for solving steady flow problem (at convergence: \( \frac{\partial p}{\partial t} = 0 \)) or inner-iterations in dual-time schemes.

  – To solve this new problem, many methods can be used, especially
  • All the time-marching schemes (R-K, multi-steps, etc) that we have seen
  • Finite differences or finite volumes in space
  • Alternating direction method is attractive: one spatial direction at a time
• Connecting these methods with the previous ones:
  – Consider the intermediate velocity field \((\rho u_i^*)^{n+1}\) obtained from solving momentum with the old pressure
  – It does not satisfy the incompressible continuity equation:
    \[ \frac{\delta(\rho u_i^*)^{n+1}}{\delta x_i} = \frac{\partial \rho^*}{\partial t} \]
    • There remains an erroneous time rate of change of mass flux
      \Rightarrow method needs to correct for it

• Example of an artificial compressibility scheme
  – Instead of explicit in time, let’s use implicit Euler (larger time steps)
    \[ \frac{p^{n+1} - p^n}{\beta \Delta t} + \left[ \frac{\delta(\rho u_i)}{\delta x_i} \right]^{n+1} = 0 \]
  – Issue: velocity field at \(n+1\) is not known
  – One can linearize about the old (intermediate) state and transform the above equation into a Poisson equation for the pressure or pressure correction!
Artificial Compressibility Methods: Example Scheme, Cont’d

• First, expand unknown velocity using Taylor series in pressure derivatives

\[(\rho u_i)^{n+1} \approx (\rho u_i^*)^{n+1} + \left[ \frac{\delta(\rho u_i^*)}{\delta p} \right]^{n+1} (p^{n+1} - p^n) \quad (p^{*n+1} = p^n)\]

  – Inserting \((\rho u_i)^{n+1}\) in the continuity equation leads an equation for \(p^{n+1}\)

\[\frac{p^{n+1} - p^n}{\beta \Delta t} + \frac{\delta}{\delta x_i} \left[ (\rho u_i^*)^{n+1} + \left[ \frac{\delta(\rho u_i^*)}{\delta p} \right]^{n+1} (p^{n+1} - p^n) \right] = 0\]

  – Then, take the divergence and derive a Poisson-like equation for \(p^{n+1}\)

• One could have also used directly:

\[(\rho u_i)^{n+1} \approx (\rho u_i^*)^{n+1} + \left[ \frac{\delta(\rho u_i^*)}{\delta p} \right]^{n+1} \left( \frac{\delta p^{n+1}}{\delta x_i} - \frac{\delta p^n}{\delta x_i} \right)\]

  – Then, still take divergence and derive Poisson-like equation

• Ideal value of \(\beta\) is problem dependent

  – The larger the \(\beta\), the more incompressible. Lowest values of \(\beta\) can be computed by requiring that pressure waves propagate much faster than the flow velocity or vorticity speeds
Numerical Boundary Conditions for N-S eqns.

- At a wall, the no-slip boundary condition applies:
  - Velocity at the wall is the wall velocity (Dirichlet)
  - In some cases, the tangential velocity stays constant along the wall (only for fully-developed), which by continuity, implies no normal viscous stress:
    \[
    \frac{\partial u}{\partial x}_{\text{wall}} = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial y}_{\text{wall}} = 0
    \]
    \[
    \Rightarrow \quad \tau_{yy} = 2\mu \frac{\partial v}{\partial y}_{\text{wall}} = 0
    \]

- For the shear stress:
  \[
  F_{S}^{\text{shear}} = \int_{S_{S}} \tau_{xy} \, dS = \int_{S_{S}} \mu \frac{\partial u}{\partial y} \, dS \approx \mu_{S} S_{S} \frac{u_{p} - u_{S}}{y_{P} - y_{S}}
  \]

- At a symmetry plane, it is the opposite:
  - Shear stress is null:
    \[
    \tau_{xy} = \mu \frac{\partial u}{\partial y}_{\text{sym}} = 0 \quad \Rightarrow \quad F_{S}^{\text{shear}} = 0
    \]
  - Normal stress is non-zero:
    \[
    \tau_{yy} = 2\mu \frac{\partial v}{\partial y}_{\text{sym}} \neq 0 \quad \Rightarrow \quad F_{S}^{\text{normal}} = \int_{S_{S}} \tau_{yy} \, dS = \int_{S_{S}} 2\mu \frac{\partial v}{\partial y} \, dS \approx 2\mu_{S} S_{S} \frac{v_{P} - v_{S}}{y_{P} - y_{S}}
    \]
• Wall/Symmetry Pressure BCs for the Momentum equations

  – For the momentum equations with staggered grids, the pressure is not required at boundaries (pressure is computed in the interior in the middle of the CV or FD cell)

  – With collocated arrangements, values at the boundary for \( p \) are needed. They can be extrapolated from the interior (may require grid refinement)

• Wall/Symmetry Pressure BCs for the Poisson equation

  – When the mass flux (velocity) is specified at a boundary, this means that:
    • Correction to the mass flux (velocity) at the boundary is also zero
    • This should be implemented in the continuity equation: zero normal-velocity-correction \( \Rightarrow \) often means gradient of the pressure-correction at the boundary is then also zero

    (take the dot product of the velocity correction equation with the normal at the bnd)
Numerical BCs for N-S eqns: Outflow/Outlet Conditions

• Outlet often most problematic since information is advected from the interior to the (open) boundary

• If velocity is extrapolated to the far-away boundary, $\frac{\partial u}{\partial n} = 0$ i.e., $u_E = u_p$, then:
  – It may need to be corrected so as to ensure that the mass flux is conserved (same as the flux at the inlet)
  – These corrected BC velocities are then kept fixed for the next iteration. This implies no corrections to the mass flux BC, thus a von Neuman condition for the pressure correction (note that $p$ itself is linear along the flow if fully developed).
  – The new interior velocity is then extrapolated to the boundary, etc.
  – To avoid singularities for $p$ (von Neuman at all boundaries for $p$), one needs to specify $p$ at a one point to be fixed (or impose a fixed mean $p$)

• If flow is not fully developed: $\frac{\partial u}{\partial n} \neq 0 \Rightarrow \frac{\partial p'}{\partial n} \neq 0 \Rightarrow$ e.g. $\frac{\partial^2 u}{\partial n^2} = 0$ or $\frac{\partial^2 p'}{\partial n^2} = 0$

• If the pressure difference between the inlet and outlet is specified, then the velocities at these boundaries can not be specified.
  – They have to be computed so that the pressure loss is the specified value
  – Can be done again by extrapolation of the boundary velocities from the interior: these extrapolated velocities can be corrected to keep a constant mass flux.

• Much research in OBC in ocean modeling
FINITE ELEMENT METHODS: Introduction

• Finite Difference Methods: based on a discretization of the differential form of the conservation equations
  – Solution domain divided in a grid of discrete points or nodes
  – PDE replaced by finite-divided differences = “point-wise” approximation
  – Harder to apply to complex geometries

• Finite Volume Methods: based on a discretization of the integral forms of the conservation equations:
  – Grid generation: divide domain into set of discrete control volumes (CVs)
  – Discretize integral equation
  – Solve the resultant discrete volume/flux equations

• Finite Element Methods: based on reformulation of PDEs into minimization problem, pre-assuming piecewise shape of solution over finite elements
  – Grid generation: divide the domain into simply shaped regions or “elements”
  – Develop approximate solution of the PDE for each of these elements
  – Link together or assemble these individual element solutions, ensuring some continuity at inter-element boundaries => PDE is satisfied in piecewise fashion
Finite Elements: Introduction, Cont’d

- Originally based on the Direct Stiffness Method (Navier in 1826) and Rayleigh-Ritz, and further developed in its current form in the 1950’s (Turner and others)
- Can replace somewhat “ad-hoc” integrations of FV with more rigorous minimization principles
- Originally more difficulties with convection-dominated (fluid) problems, applied to solids with diffusion-dominated properties

Comparison of FD and FE grids

Examples of Finite elements

© McGraw-Hill. All rights reserved. This content is excluded from our Creative Commons license. For more information, see http://ocw.mit.edu/fairuse.
Finite Elements: Introduction, Cont’d

• Classic example: Rayleigh-Ritz / Calculus of variations
  – Finding the solution of \( \frac{\partial^2 u}{\partial x^2} = -f \) on \( ]0,1[ \)
    
    is the same as finding \( u \) that minimizes
    
    \[
    J(u) = \int_{0}^{1} \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 - uf \, dx
    \]

  – R-R approximation:
    • Expand unknown \( u \) into shape/trial functions
      \[
      u(x) = \sum_{i=1}^{n} a_i \phi_i(x)
      \]
      and find coefficients \( a_i \) such that \( J(u) \) is minimized

• Finite Elements:
  – As Rayleigh-Ritz but choose trial functions to be piecewise shape function defined over set of elements, with some continuity across elements
Finite Elements: Introduction, Cont’d

Method of Weigthed Residuals

• There are several avenues that lead to the same FE formulation
  – A conceptually simple, yet mathematically rigorous, approach is the Method of Weighted Residuals (MWR)
  – Two special cases of MWR: the Galerkin and Collocation Methods

• In the MWR, the desired function $u$ is replaced by a finite series approximation into shape/basis/interpolation functions:

$$\tilde{u}(x) = \sum_{i=1}^{n} a_i \, \phi_i(x)$$

  – $\phi_i(x)$ chosen such they satisfy the boundary conditions of the problem
  – But, they will not in general satisfy the PDE:  \[ L(u) = f \]
  $\Rightarrow$ they lead to a residual:  \[ L(\tilde{u}(x)) - f(x) = R(x) \neq 0 \]
  – The objective is to select the undetermined coefficients $a_i$ so that this residual is minimized in some sense
Finite Elements: Method of Weighted Residuals, Cont’d

- One possible choice is to set the integral of the residual to be zero. This only leads to one equation for \( n \) unknowns

\[ \int_{0}^{L} \int_{V} R(x) w_i(x) \, dx \, dt = 0, \quad i = 1, 2, \ldots, n \]

- In 3D, this becomes:

\[ \int_{V} \int_{V} R(x) w_i(x) \, dx \, dt = 0, \quad i = 1, 2, \ldots, n \]

- A variety of FE schemes arise from the definition of the weighting functions and of the choice of the shape functions
  - Galerkin: the weighting functions are chosen to be the shape functions
  - Subdomain method: the weighting function is chosen to be unity in the sub-region over which it is applied
  - Collocation Method: the weighting function is chosen to be a Dirac-delta
Finite Elements: Method of Weighted Residuals, Cont’d

• **Galerkin:** \[ \int_V \int_t R(x) \phi_i(x) \, dx \, dt = 0, \quad i = 1, 2, \ldots, n \]
  - Basis functions formally required to be complete set of functions
  - Can be seen as “residual forced to zero by being orthogonal to all basis functions”

• **Subdomain method:**
  \[ \int_V \int_t R(x) \, dx \, dt = 0, \quad i = 1, 2, \ldots, n \]
  - Non-overlapping domains \( V_i \) often set to elements
  - Easy integration, but not as accurate

• **Collocation Method:**
  \[ \int_V \int_t R(x) \delta_{x_i}(x) \, dx \, dt = 0, \quad i = 1, 2, \ldots, n \]
  - Mathematically equivalent to say that each residual vanishes at each collocation points \( x_i \) \( \Rightarrow \) Accuracy strongly depends on locations \( x_i \).
  - Requires no integration.
General Approach to Finite Elements

1. Discretization: divide domain into “finite elements”
   - Define nodes (vertex of elements) and nodal lines/planes

2. Set-up Element equations
   i. Choose appropriate basis functions $\phi_i(x)$: $\tilde{u}(x) = \sum_{i=1}^{n} a_i \phi_i(x)$
      - 1D Example with Lagrange’s polynomials: Interpolating functions $N_i(x)$
        $\tilde{u} = a_0 + a_1 x = u_1 N_1(x) + u_2 N_2(x)$
        where $N_1(x) = \frac{x_2 - x}{x_2 - x_1}$ and $N_2(x) = \frac{x - x_1}{x_2 - x_1}$
        - With this choice, we obtain for example the 2nd order CDS and Trapezoidal rule:
          $\frac{d \tilde{u}}{dx} = a_1 = \frac{u_2 - u_1}{x_2 - x_1}$ and
          $\int_{x_1}^{x_2} \tilde{u} \, dx = \frac{u_1 + u_2}{2} (x_2 - x_1)$
   ii. Evaluate coefficients of these basis functions by approximating the solution in an optimal way
      - This develops the equations governing the element’s dynamics
      - Two main approaches: Method of Weighted Residuals (MWR) or Variational Approach

⇒ Result: relationships between the unknown coefficients $a_i$ so as to satisfy the PDE in an optimal approximate way
2. **Set-up Element equations, Cont’d**

   - Mathematically, combining i. and ii. gives the element equations: a set of (often linear) algebraic equations for a given element \( e \):

   \[
   K_e u_e = f_e
   \]

   where \( K_e \) is the element property matrix (stiffness matrix in solids), \( u_e \) the vector of unknowns at the nodes and \( f_e \) the vector of external forcing.

3. **Assembly**:

   - After the individual element equations are derived, they must be assembled: i.e. impose continuity constraints for contiguous elements.

   - This leads to:

   \[
   Ku = f
   \]

   where \( K \) is the assemblage property or coefficient matrix, \( u \) and \( f \) the vector of unknowns at the nodes and \( f_e \) the vector of external forcing.

4. **Boundary Conditions**: Modify “\( Ku = f \)” to account for BCs.

5. **Solution**: use LU, banded, iterative, gradient or other methods.

6. **Post-processing**: compute secondary variables, errors, plot, etc.