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Spring 2007
18.304
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Mathematical Fallacy Proofs

In world of mathematics, countless brilliant minds dedicate their lives in an effort to prove the seemingly impossible. Interestingly enough, through the plethora of established proofs which has tremendously impacted the scientific world, a few false proofs have also survived the scrutiny of mathematicians. However, rather than simply dismissing such fallacy proofs as unfortunate mistakes, equally valuable lessons can be learned through the understanding of why such fallacy proofs were able to take on the façade of a real proof. In this paper, I aim to explore a few of such fallacy proofs and the lessons that may be extracted from their presence.

1=0

1.	$(n+1)^2 = n^2+2n+1$	Expansion
2.	$(n+1)^2-(2n+1) = n^2$	Subtract from both sides
3.	$(n+1)^2-(2n+1)-n(2n+1) = n^2-n(2n+1)$	Add to both sides
4.	$(n+1)^2-(n+1)(2n+1) = n^2-n(2n+1)$	Factor
5.	$(n+1)^2-(n+1)(2n+1)+(2n+1)^2/4 = n^2-n(2n+1)+(2n+1)^2/4$	Add to both sides
6.	$[(n+1)-(2n+1)/2]^2 = [n-(2n+1)/2]^2$	Factor
7.	$(n+1)-(2n+1)/2 = n-(2n+1)/2$	Take square roots of both sides
8.	$n+1 = n$	Subtract from both sides
9.	$1 = 0$	Impossible!

The operations listed above utilize basic arithmetics to arrive at the false conclusion. Starting by simply expanding a squared equation, we can subtract 2n+1 from both sides to isolate n^2. Subtracting n(2n+1) from both sides now allows the left side to be factored. Adding $(2n+1)^2/4$ to both sides once again will enable both sides of the equation to be factored down to squared forms. By taking the square roots and then subtracting the n-(2n+1)/2, the proof is complete and 1=0.

If two numbers are equal, their squares are also equal. However, the reverse form of such a statement does not hold. In short, u = v does not imply square root of u equals square root of v due to the fact that the result of a square root is not unique. Without this fact, the above proof becomes actually legitimate.

1=2

1. Let a=b.	
2. Then $a^2 = ab$	Multiply to both sides
3. $a^2 + a^2 = a^2 + ab$	Addition to both sides
4. $2a^2 = a^2 + ab$	Contract
5. $2a^2 – 2ab = a^2 + ab – 2ab$	Addition to both sides
6. $2a^2 – 2ab = a^2 – ab$	Contract
7. $2(a^2 – ab) = 1(a^2 – ab)$	Factor
8. 1=2	Impossible!

Multiply both sides of the starting condition by a, and then add a^2 to both sides. Subtract 2ab and factor out $a^2 – ab$ and the results becomes an amazing 1=2.

Ever since the first week of math class in elementary school, we have been instructed to remember that dividing by zero results in an answer of “undefined”. However, it is unlikely that most people have evaluated the consequences of neglecting such a result. The false proof here demonstrates the disaster which may occur from the division of zero.

Curry’s Paradox

A right triangle with legs 13 and 5 can be cut into two triangles (legs 8, 3 and 5, 2, respectively). The small triangles could be fitted into the angles of the given triangle in two different ways. In one case a a 5×3 rectangle of area 15 is left over. In the other case, we get an 8×2 rectangle of area 16.

As a matter of fact, it is possible to create a right triangle with legs F_{n+1} and F_{n-1}, where F_k is the k^{th} Fibonacci Number. The two rectangles then have dimensions $F_{n-1}×F_{n-2}$ and $F_n×F_{n-3}$, with areas that always differ by 1:

$$F_n·F_{n-3} - F_{n-1}·F_{n-2} = (-1)^n,$$

This is known as *d'Ocagne's identity*.

Unfortunately, this identity applies to a scenario that may only be proved visually and not mathematically. With a keener eye, one can see that the two triangles, 3x5 and 8x2, do not have the same slope for their hypotenuse, and they do not fit together for a straight line, thus justifying the one unit difference in area.

All Acute Angles Are Right Angles

Imagine a right angle composed of one leg of height one unit and the other leg of length x units. Now attach another leg of one unit at an angle of 89 degrees at the end of leg x. Finally, complete the quadrilateral by connecting the two legs of unit one. Next find the perpendicular bisectors and their intersection inside the quadrilateral and label that point A. From point A, connect each of the four corners of the quadrilateral, forming a total of six triangles. With the current construction it can be easily proven, with the help of side-angle-side and side-side-side theorems, that the angle which had been constructed to be acute should hypothetically be a right angle.

So what went wrong? Unless there's faulty reasoning behind the trigonometric theorems, this supposed proof seems to be justified. However, the problem within this proof does not lie within the theorems used to prove it, but rather within the original assumption of the intersection being within the quadrilateral. See the figure below. The perpendicular bisectors will never intersect inside the quadrilateral and thus all the operations mentioned above are simply for a fictional scenario.

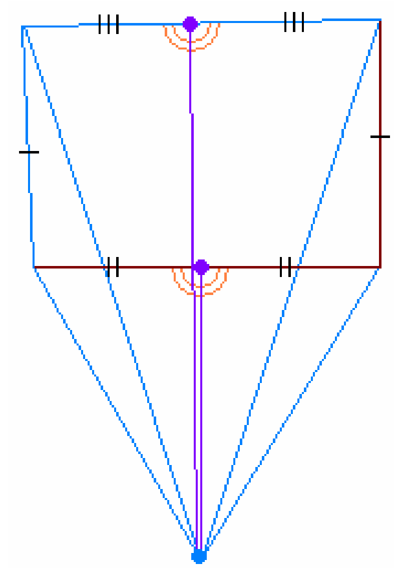

Figure by MIT OCW.

Kempe’s Four Color Theorem

The well accepted proof of a five color theorem can be found nearly anywhere in application, but when Kempe developed a proof which lowered the limit to the number of colors necessary to color any map to four, the mathematical world was so excited by its potential that its flaw was overlooked for more than 10 years.

Take any graph G with center v which has 5 neighbors in G. We can extend the coloring to v unless all 4 colors appear on its neighbors. This means that the only situation worthy of concern is when v has neighboring colors in order of A, B, A, C, D. As with the Five Color Theorem, there will be no difficulties coloring v if there is no Kempe chain linking the vertices colored B and C, by switching the colors of B and C on all vertices of the chain starting at B. Therefore, the troublesome case is a B-C colored path between the neighbors of *v* of colors B and C. Similarly, there must also be a B-D colored path between the neighbors of *v* of colors B and D. This results in two cycles isolating the two vertices of color A from each other. Inside the B-C chain, there can be an A-D chain starting with v’s second A colored neighbor that is trapped and cannot reach v’s D colored neighbor. Thus, we can switch the colors A and D on the A-D chain without changing the D colored neighbor at all. The same may be applied to the B-D chain, by forming an A-C chain. As a result, the center v is freed to take on the color A.

Although convincing, there is one fatal flaw in the reasoning of Kempe’s proof. The proof assumes that the B-D and B-C colored paths are independent paths, thus allowing the isolation of A-C and A-D chains. However, as Percy Heawood pointed out over 10 year later in 1891, there exists a situation when the B-D and B-C chains may share the same neighbor of B, thus conflicting with the argument that v may be safely colored A.

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