RANDOM WALKS AND EVENTUAL RETURNS

THE ANONYMOUS HEROES OF MATHEMATICS

Abstract. In this paper we will take a measure-theoretic approach to address the problem of eventual returns in a random walk of several dimensions. We begin by presenting the relevant definitions along with a couple elementary but powerful results known as the Borel-Cantelli Lemmas. Next, we will show a basic application of the Borel-Cantelli Lemmas to solve the infinite monkey problem. Lastly, as promised, we will deliver the main result regarding random walks and eventual returns.

1. Introduction

An eventual return of a random walk (to be defined rigorously later) is the event that at some later time we get back to where we started. In this paper we will answer the question, "What is the probability that we will eventually return to our origin infinitely many times?" One approach to solve this problem is of course to explicitly find the probability that we will have an eventual return (not necessarily infinitely many times). This often involves a generous dose of generating functions or, even more frequently, an integral of the form (see [2]):

$$\frac{d}{(2\pi)^d}\underbrace{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}}_{d}\left(d-\sum_{k=1}^{d}\cos x_k\right)^{-1}dx_1\cdots dx_{d-1}dx_d$$

Using the above integral, one can in fact find the *exact* probability of return in a d-dimensional unbiased random walk. From this number, the probability of an infinite number of returns immediately follows. In favor of simplicity, however, this paper we will instead take a measure-theoretic approach to solve this problem. Although our methods are not robust enough to produce these exact probabilities as described above, it is nonetheless instructive and will provide some important insight into the (hopefully) surprising results that follow.

2. An Introduction to Measure Theory

Before delving into our main discussion, we must first develop the machinery that will enable us to make any useful claims about eventual returns. Recall that a *probability space* is defined as a measure space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{P}(\Omega) = 1$. In this sense, Ω can be identified with the sample space of our experiment, $\mathcal{F}$ with the family of events, and $\mathbf{P}$ as a measure on $(\Omega, \mathcal{F})$. The intuitive meaning of the above definition is that if we have an event $F \in \mathcal{F}$, the value $0 \leq \mathbf{P}(F) \leq 1$ represents the probability that a point in Ω chosen "at random" belongs to F.

Now, as hinted at in the Introduction, we are interested in the probability of a specific set of events occurring infinitely many times. The following definition taken from [4] makes this statement rigorous.

Definition 2.1. Suppose that $\{E_n \in \mathcal{F} : n \in \mathbb{N}\}$ is a sequence of events. We define:

$$\begin{aligned}(E_n,\ \text{i.o.}) &=: (E_n \text{ infinitely often}) \\ &:= \bigcap_m \bigcup_{n \geq m} E_n \\ &= \{\omega : \text{ for every } m, \exists n(\omega) \geq m \text{ such that } \omega \in E_{n(\omega)}\} \\ &= \{\omega : \omega \in E_n \text{ for infinitely many } n\}.\end{aligned}$$

We now have all of the necessary definitions to present the main results of this section. We begin with the first Borel-Cantelli Lemma.

Lemma 2.2 (Borel-Cantelli)**.** *Let $\{E_n \in \mathcal{F} : n \in \mathbb{N}\}$ be a sequence of events such that $\sum \mathbf{P}(E_n) < \infty$. Then $\mathbf{P}(E_n,\ i.o.) = 0$.*

Proof. Using Definition 2.1, we have

$$\begin{aligned}\mathbf{P}(E_n,\ \text{i.o.}) &= \mathbf{P}\left(\bigcap_m \bigcup_{n \geq m} E_n\right) \leq \mathbf{P}\left(\bigcup_{n \geq m_0} E_n\right) \\ &\leq \sum_{n \geq m_0} \mathbf{P}(E_n)\end{aligned}$$

for all $m_0 \in \mathbb{N}$. Since $\sum_{n \geq m_0} \mathbf{P}(E_n) \downarrow 0$ as $m_0 \uparrow \infty$, the result follows. ♢

The natural question that now arises is whether or not the converse of Lemma 2.2 holds, i.e. does $\mathbf{P}(E_n,\ \text{i.o.}) = 0 \implies \sum \mathbf{P}(E_n) < \infty$? It turns out that such a converse does exist as long we strengthen our hypotheses slightly. This partial converse is commonly known as the second Borel-Cantelli Lemma.

Lemma 2.3 (Borel-Cantelli)**.** *Let $\{E_n \in \mathcal{F} : n \in \mathbb{N}\}$ be a sequence of independent events such that $\sum \mathbf{P}(E_n) = \infty$. Then $\mathbf{P}(E_n,\ i.o.) = 1$.*

Proof. We proceed as we did in the proof of the first Borel-Cantelli Lemma by simply arguing from Definition 2.1. For all $m_0 \in \mathbb{N}$ we have

$$\begin{aligned}\mathbf{P}(E_n,\ \text{i.o.}) &= 1 - \mathbf{P}\left((E_n,\ \text{i.o.})^c\right) = 1 - \mathbf{P}\left(\bigcup_m \bigcap_{n \geq m} E_n^c\right) \\ &\geq 1 - \mathbf{P}\left(\bigcap_{n \geq m_0} E_n^c\right) = 1 - \prod_{n \geq m_0} (1 - \mathbf{P}(E_n))\end{aligned}$$

where the last equality follows from our independence hypothesis. Now, note that elementary inequality $1 - x \leq e^{-x}$ which may be proved by a simple derivative argument (or better yet, a picture). Using this inequality we have

$$\begin{aligned}\mathbf{P}(E_n,\ \text{i.o.}) &\geq 1 - \prod_{n \geq m_0} e^{-\mathbf{P}(E_n)} = 1 - e^{-\sum_{n \geq m_0} \mathbf{P}(E_n)} \\ &= 1\end{aligned}$$

which is still true for all $m_0 \in \mathbb{N}$. Thus $\mathbf{P}(E_n,\ \text{i.o.}) \geq 1$. Lastly, by virtue of being a probability measure, it is clear that $\mathbf{P}(E_n,\ \text{i.o.}) \leq 1$. The results follows. ♢

Remark 2.4. Notice that in the statement of the second Borel-Cantelli Lemma above, we included the extra hypothesis of independence of E_n. Observe that pairwise independence is not strong enough (why?). Furthermore, there are examples that show this is indeed a necessary condition, i.e. there exists a sequence of dependent (or even just pairwise independent) events E_n such that the statement of Lemma 2.3 does not hold!

3. Infinite Monkeys

With the Borel-Cantelli Lemmas in our arsenal, we finally possess the machinery to tackle our original problem. But first, we turn to a silly (but fun) application relating to infinite monkeys. The precise statement is that given a monkey typing on a typewriter for an infinite amount of time, they will eventually type the collected works of Shakespeare infinitely many times (or type the Bible, or construct a proof of Fermat's Last Theorem, etc...). To prove this, note that there is a positive probability ϵ such that the monkey will immediately type Moby Dick (there are a finite number of characters k in the story, and each character in turn has a positive probability of being struck). Denote this as the event E_1. Similarly, there is an equal probability ϵ that the next k characters typed will be Moby Dick. Denote this event as E_2. Proceeding, we define E_j to be the event that the j group of k characters is Moby Dick. Since each E_j has an equal positive probability of occurring, it is clear that the following sum diverges:

$$\sum_{j=1}^{\infty} \mathbf{P}(E_j) = \sum_{j=1}^{\infty} \epsilon = \infty.$$

Since each block of k characters is independent of the other blocks of characters, we know that the event E_k are independent and we may apply the Lemma 2.3. We see as an immediate consequence of this lemma that the probability of E_j occurring for infinitely many j is 1. That is, the monkey will *almost surely* type Moby Dick infinitely many times in an infinite amount of time!

4. Random Walks and Eventual Returns

Now we return to the problem of random walks. First, we will rigorously define a random walk in d dimensions.

Definition 4.1. Let X_i be i.i.d. real random variables in $\mathbb{R}^d$. The random walk is defined on X_i as the sequence of partial sums

$$S_n = \sum_{i=1}^{n} X_i.$$

The values that X_i takes on define the type of random walk being considered. For the purposes of this paper, the random walk in d dimensions will have X_i with a uniform probability distribution over $2d$ possible values as follows:

$$\begin{aligned} d = 1 \quad &: \quad X_i = \{-1\}, \{1\} \\ d = 2 \quad &: \quad X_i = \{0,1\}, \{0,-1\}, \{1,0\}, \{-1,0\} \\ d = 3 \quad &: \quad X_i = \{0,0,1\}, \{0,0,-1\}, \{0,1,0\}, \{0,-1,0\}, \{1,0,0\}, \{-1,0,0\} \end{aligned}$$

The X_i's for higher dimensions follow similarly such that each X_i moves the random walk by one in exactly one of the d dimensions.

With the definition of random walks, we can embark on our next captivating journey; we seek the answer to the question, "What is the probability of eventual return to the origin for a random walk in d dimensions?" However, as mentioned earlier, we are concerned not with the exact answer to this question but rather the answer to the question "What is the probability that we will return to the origin infinitely many times?" To answer this, we will first start with the case for $d = 1$. To relate to this mathematical problem, we will think of the random walk in one dimension as a "drunken" walk. Imagine a drunkard leaving a bar who is wandering around up and down a street; this can be roughly modeled as the random walk as defined earlier.

We would like to calculate $\mathbf{P}(S_n = 0 \text{ for some } n)$. Note that we can only return to the origin if we take an even number of steps. In $2n$ steps we must move to the right n steps and move to the left n steps. The probability of a particular sequence of $2n$ steps is $\left(\frac{1}{2}\right)^{2n}$ and the number of sequences of steps that will return to the origin is $\binom{2n}{n}$. So, we now have

$$\mathbf{P}(S_{2n} = 0) = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n}.$$

We now seek to find the probability of eventual return infinitely many times. Here, we will apply the Borel-Cantelli Lemma and calculate $\sum \mathbf{P}(E_n)$. First, recall Stirling's formula:

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

Now, we are ready to proceed with evaluating $\sum \mathbf{P}(E_n)$.

$$\begin{aligned}
\sum \mathbf{P}(E_n) &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \\
&\approx \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n} \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi (2n)}}{\left(n^n e^{-n} \sqrt{2\pi n}\right)^2}, \text{ by Stirling's formula} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n} \frac{2^{2n}}{\sqrt{\pi n}} \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} \\
&= \infty
\end{aligned}$$

From this result and Lemma 2.3, we can conclude that $\mathbf{P}(E_n, \text{ i.o.}) = 1$, or the probability of eventual return infinitely many times in a one dimensional random walk is one.

Now, we will proceed to the case of two dimensions. In our analogy of a drunkard leaving a bar, now, our drunken friend is allowed to walk freely in the neighborhood i.e. not necessarily limited to walking up and down the street. Let's now examine whether the drunkard will safely return back to the bar.

Note again that we need an even number $(2n)$ of steps, such that k steps are north, k steps are south, $(n-k)$ steps are west, and $(n-k)$ steps are east. Again, the probability of a particular path in $2n$ steps is $\left(\frac{1}{4}\right)^{2n}$. The total number of different paths in $2n$ steps is $\frac{(2n)!}{k!k!(n-k)!(n-k)!}$. So, for two dimensions,

$$\mathbf{P}(S_{2n}=0)=\sum_{k=0}^{n}\left(\frac{1}{4}\right)^{2n}\frac{(2n)!}{k!k!(n-k)!(n-k)!}.$$

Summing over all n, we obtain

$$\begin{aligned}\sum \mathbf{P}(E_n) &= \sum_{n=1}^{\infty}\sum_{k=0}^{n}\left(\frac{1}{4}\right)^{2n}\frac{(2n)!}{k!k!(n-k)!(n-k)!}\\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{n}\left(\frac{1}{4}\right)^{2n}\frac{(2n)!}{n!n!}\frac{n!n!}{k!k!(n-k)!(n-k)!}\\ &= \sum_{n=1}^{\infty}\left[\binom{2n}{n}\left(\frac{1}{4}\right)^{2n}\sum_{k=0}^{n}\binom{n}{k}^2\right]\\ &= \sum_{n=1}^{\infty}\binom{2n}{n}^2\left(\frac{1}{4}\right)^{2n}\\ &\approx \sum_{n=1}^{\infty}\frac{1}{\pi n}\text{, by Stirling's formula}\\ &= \infty.\end{aligned}$$

Again, by Lemma 2.3, we can conclude that $\mathbf{P}(E_n,\ \text{i.o.})=1$, or the probability of eventual return infinitely many times in a two dimensional random walk is one. So our drunkard will certainly return back to the origin.

Now, take the case where our drunkard had several drinks containing substantial amounts of the beverage Red Bull, whose motto is "Red Bull gives you wings!" With his newfound ability to fly, this very drunk man also decides to leave the bar and walk and fly around in a three dimensional space. So, we will now evaluate the probability of eventual return to the origin infinitely many times in a three dimensional random walk.

Using similar arguments as previously, we have

$$\mathbf{P}(S_{2n}=0)=\sum_{k=0}^{n}\sum_{j=0}^{k}\left(\frac{1}{6}\right)^{2n}\frac{(2n)!}{j!j!k!k!(n-j-k)!(n-j-k)!}.$$

Summing over all n,

$$\begin{aligned}\sum \mathbf{P}(E_n) &= \sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\left(\frac{1}{6}\right)^{2n}\frac{(2n)!}{j!j!k!k!(n-j-k)!(n-j-k)!}\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{2n}\binom{2n}{n}\sum_{k=0}^{n}\sum_{j=0}^{k}\left(\frac{n!}{j!k!(n-j-k)!}\right)^2.\end{aligned}$$

Now, we can create an upper bound on $\frac{n!}{j!k!(n-j-k)!}$. Let $M = \max(\frac{n!}{j!k!(n-j-k)!})$. Note that M is maximized when $j = k = n - j - k = \frac{n}{3}$. So,

$$\begin{aligned} M &= \frac{n!}{\left[(\frac{n}{3})!\right]^3} \\ &\approx 3^{3/2}\frac{3^n}{2\pi n}, \text{ by Stirling's formula} \\ &= K\frac{3^n}{n} \end{aligned}$$

We can now proceed with our original sum:

$$\begin{aligned} \sum \mathbf{P}(E_n) &< \sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{2n}\binom{2n}{n}\sum_{k=0}^{n}\sum_{j=0}^{k}K\frac{3^n}{n}\frac{n!}{j!k!(n-j-k)!} \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\frac{K}{n}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{1}{3^n}\frac{n!}{j!k!(n-j-k)!} \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\frac{K}{n} \\ &\approx \sum_{n=1}^{\infty}\frac{K}{n\sqrt{\pi n}}, \text{ by Stirling's formula} \\ &< \infty \end{aligned}$$

Since $\sum \mathbf{P}(E_n)$ converges, then by Lemma 2.2, $\mathbf{P}(E_n, \text{ i.o.}) = 0$, or it is not certain that we will return to the origin in a three-dimensional random walk. In fact, by experimentation, the probability that a random walk in three dimensions returns to the origin is approximately 0.65 [1]. Thus we cannot ensure with certainty that our flying drunkard will return back to where he started.

References

[1] Grinstead, C. M. and Snell, J. L., "Random Walks," Introduction to Probability. American Mathematical Society, 1997.

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[3] Waliji, M., "Monkeys and Walks," `http://home.uchicago.edu/~mhwaliji/presentation.pdf`

[4] Williams, D., "Probability with Martingales," Cambridge University Press, 1991.